Worksheet 3

1 Norms Equivalency

Two norms in a finite-dimensional linear space X (e.g.: $\mathbb{R}^n$), $\|\cdot\|_a$ and $\|\cdot\|_b$, are called equivalent if there exists constant c and C, such that for all x in X,

$$c\|x\|_a \leq \|x\|_b \leq C\|x\|_a \tag{1}$$

(1.a) Suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent, and we know that an algorithm produces a sequence of vectors $\{e_n\}_{n\geq 1}$, $\|e_n\|_a \to 0$ as $n \to \infty$. What could we conclude about $\|e_n\|_b$'s behavior for $n \to \infty$?

(1.b) We first show that the vector norms on $\mathbb{R}^n$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$, are equivalent. To do this prove the inequality:

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty.$$

(1.c) The induced matrix norm on $\mathbb{R}^{n\times n}$: $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent as well. Prove the inequality

$$\|A\|_\infty \leq \sqrt{n}\|A\|_2,$$
$$\|A\|_2 \leq \sqrt{n}\|A\|_\infty.$$

2 Inverse matrix computation

Let us use the LU-decomposition to compute the inverse of a matrix.

(2.a) Describe an algorithm that uses the LU-decomposition of an $n \times n$ matrix A for computing A^{-1} by solving n systems of equations (one for each unit vector).

(2.b) Calculate the floating point operation count of this algorithm.

(2.c) Improve the algorithm by taking advantage of the structure (i.e., the many zero entries) of the right-hand side. What is the new algorithm's floating point operation count? (Hint: consider splitting the solution vector for the k-th equation from part (a) into two pieces, and solve for each piece separately, on paper or using forward/backward substitution.)

3 Stability of the Gaussian elimination

Consider the linear system

$$Ax = b,$$

where A is an $n \times n$ matrix that has ones on the diagonal, minus ones below the diagonal, and ones in the last column, with all other entries zero. For example, when $n = 5$, we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

(3.a) Consider the matrix A for some arbitrary n. Perform Gaussian elimination on A to obtain the upper triangular matrix U appearing in the LU-factorization $A = LU$. What is $\max_{i,j} |u_{i,j}|$ as a function of n?

(3.b) For large n, e.g., $n = 2000$, what problems can you envision if you try to solve the system using Gaussian elimination on a computer?