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Worksheet 1

We review methods for finding roots of nonlinear functions such as the fixed-point method and Newton's method. We will do a mix of coding and theory exercises.

1 Fixed point methods

We want to find the roots of the nonlinear equation

$$f(x) := x^2 - x - 2 = 0$$

using the fixed point method.

We will try to find the positive root (x^*) through the fixed point iteration of the form $x_{k+1} = g(x_k)$. We investigate two choices:

- $g_1(x) = x^2 - 2$
- $g_2(x) = \sqrt{x+2}$

(1.a) Verify that x^* is indeed fixed points for the two functions. That is, $x^* = g(x^*)$.

1. You could verify this with a plot. (If there are two lines in a plot, put legends on them.)

(1.b) Will both choices work in the fixed point algorithm to find the root x^*? (Hint: consider the stability of the fixed point.)

1. Could you determine the stability using the plot you made in (1.a)?

(1.c) Implement the fixed point method for both choices of $g(x)$ above.

1. Set $x_0 = 5$.
2. What would be some good termination criteria for the fixed point algorithm?
3. Try to "generalize" your code. For example, what if we want to pick a new function $g(x)$?

Now consider the convergence behavior of the stable fixed point algorithm.

(1.d) From theory alone, how fast do you expect the convergence? (linear/superlinear/quadratic)? What is the expected convergence rate?

(1.e) Is this what your numerical results show? How do you verify the convergence behavior numerically? Try to show this with a plot.

1. Plot a representation of the error $|x_k - x^*|$ against iteration numbers. Which methods of plotting should we use? `plot`, `semilogy`, or `loglog`?
2. Did you remember to add a title and axis label to your plot?

(1.f) Could you diagnose the convergence rate numerically? Does it match the theoretical expectation?

2 Newton's method and roots with higher multiplicity

The convergence theorem for Newton's method requires the first derivatives at the root x^* to be nonzero. We will explore what happens when $f'(x^*) = 0$.

(2.a) Show that if $f'(x^*) = 0$, then Newton's method converges linearly.

To further study this, we first define multiplicity: $x^* \in \mathbb{R}$ is a root of multiplicity m for the equation $f(x) = 0$ if there is a function $h(x)$ such that $h(x^*) \neq 0$ and $f(x) = (x - x^*)^m h(x)$.

(2.b) Suppose that a function f has m continuous derivative on the interval (a, b) containing c (i.e. $f(x) \in C^m_{(a,b)}$). Show that f has a zero of multiplicity m at x^* if and only if

$$0 = f(x^*) = f'(x^*) = \cdots = f^{(m-1)}(x^*)$$

and

$$f^m(x^*) \neq 0.$$

(2.c) Suppose x^* is a zero of multiplicity m of f, and $f(x) \in C^m_{(a,b)}$, $x^* \in (a, b)$. Show that the following fixed-point method has $g'(x^*) = 0$:

$$g(x) = x - m\frac{f(x)}{f'(x)}.$$

What can you say about the convergence behavior of this fixed-point method?

(2.d) Code this modified Newton's method to solve $f(x) = x^2 = 0$. Check the convergence numerically. How do you plan to show this?