1 Induced Matrix Norms

Let $A, B \in \mathbb{R}^{n\times n}$ and let the matrix norm $\|\cdot\|$ be induced by/subordinate of a vector norm $\|\cdot\|$.

(1.a) Show that $\|AB\| \leq \|A\|\|B\|$.

(1.b) For the identity matrix $I \in \mathbb{R}^{n\times n}$, show that $\|I\| = 1$.

(1.c) For A invertible, show that $\kappa(A) \geq 1$, where $\kappa(A)$ is the condition number of that matrix A corresponding to the norm $\|\cdot\|$. Use the above two properties with $B := A^{-1}$ for your argument.

(1.d) Argue that the Frobenius matrix norm $\|A\|_F := \left(\sum_{i,j=1}^n a_{ij}^2\right)^{1/2}$ cannot be induced by a suitable vector norm. *Hint*: Use one of the points above.

2 Eigenvalue/vector Properties

Prove the following statements, using the basic definition of eigenvalues and eigenvectors, or give a counterexample showing the statement is not true. Assume $A \in \mathbb{R}^{n\times n}$, $n \geq 1$.

(2.a) If λ is an eigenvalue of A and $c \in \mathbb{R}$, then $\lambda + c$ is an eigenvalue of $A + cI$, where I is the identity matrix.

(2.b) If λ is an eigenvalue of A and $c \in \mathbb{R}$, then $c\lambda$ is an eigenvalue of cA.

(2.c) If λ is an eigenvalue of A, then for any positive integer p, λ^p is an eigenvalue of A^p.

(2.d) Every matrix with $n \geq 2$ has at least two *distinct* eigenvalues.

(2.e) Every real matrix has a real eigenvalue.

(2.f) If A is singular, then it has an eigenvalue equal to zero.

(2.g) If all the eigenvalues of a matrix A are zero, then $A = 0$.

(2.h) If $\widetilde{A}$ is "similar" to A, which means that there is a nonsingular matrix P such that $A = P\widetilde{A}P^{-1}$, then if λ is an eigenvalue of A, it is also an eigenvalue of $\widetilde{A}$. How do the eigenvectors of $\widetilde{A}$ relate to the eigenvectors of A?

3 Square Root of Matrix

The eigenvalue decomposition can be used to generalize scalar functions to matrix valued functions. For this problem, it is OK to assume that A is normal or that it is Hermitian/symmetric, although some parts can be proven by only assuming A is non-defective.

(3.a) Relate the eigenvalue decomposition of A^{-1} to that of A.

(3.b) Given a square matrix A and real number t, the *matrix exponential* e^{tA} is defined via the Taylor series for the exponential function:

$$e^{tA} = 1 + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + + \frac{(tA)^4}{4!} + \dots \tag{1}$$

Can you relate the eigenvalue decomposition of e^{tA} (as defined above) with that of A?

(3.c) The square root of a Hermitian/symmetric positive semidefinite matrix A (i.e., a matrix with all *non-negative* eigenvalues) is a matrix X such that $X^*X = A$, where star denotes complex conjugate transpose. If you know the eigenvalue decomposition of A, can you find at least one such matrix X? Often the matrix X is restricted to be Hermitian also (i.e., $X^2 = A$). Can you find a Hermitian square root of A from the eigenvalue decomposition of A? Is such a square matrix unique? If not, can you think of some way to make it unique (just like we make $\sqrt{4} = 2$ be unique instead of the more general $\sqrt{4} = \pm 2$)?

(3.d) Using what you found above, can you think of how you may apply any scalar function (say sine) to a symmetric matrix?