Midterm Review

1. Solution of equations

   Solve for $f(x)=0$ for $x \in [a,b]$

   ① Existence ② Fixed pt Iter. ③ Convergence ④ iterative methods

① if $f(a)\cdot f(b) < 0$, then $\exists\, x \in [a,b]$ s.t. $f(x)=0$

② Solving for $f(x)=0 \iff g(x)=x$ for suitable choice of g.

existence?

if $g(x) \in [a,b]$ $\forall x \in [a,b]$, exist!

Fixed point iteration:

If $g(x)$ is $\mathcal{C}[a,b]$, given $x_0 \in [a,b]$, define $x_{k+1} = g(x_k)$ for $k=0,1,2,\cdots$

③ Convergence

· (strongest)

If $\exists L<1$, s.t. $\forall x,y \in [a,b]$, $|g(x)-g(y)| \le L \cdot |x-y|$, then converges regardless of choice of starting point.

$\max_{x\in[a,b]} |g'(x)|$ if $g \in \mathcal{C}^1$

· (weaker)

Let ξ be the fixed point of g. If $g \in \mathcal{C}^1$ and $|g'(\xi)|<1$, then converges if we start sufficiently close to ξ.

Stability of fixed point

- ξ is stable if fixed point iteration converges to ξ whenever x_0 close to ξ. $\iff |g'(\xi)| < 1$
- ξ is unstable if no fixed point iteration started close to ξ converges, unless $x_0=\xi$. $\iff |g'(\xi)| > 1$
- $|g'(\xi)|=1$, neither .. or ..

Eg: Let $f(x) = xe^x - 1$. Show that $f(x)$ has unique root on $\mathbb{R}$.

1° existence. $f(0) = -1$, $f(1) = e-1 > 0$ $\Rightarrow \exists$ root on $[0,1]$.

2° uniqueness. For $x \le 0$, $xe^x \le 0$, so $f(x) \le -1$

For $x>0$, $f'(x) = xe^x + e^x = (x+1)e^x > e^x > 1$

$\Rightarrow f$ is strictly increasing $\forall x>0$.

$\Rightarrow$ the solution we found in $[0,1]$ must be unique.

Eg: $x_{k+1} = \varphi(x_k) = x_k - \lambda f'(x_k)$, $\lambda > 0$, $f \in C^2$

① What are the fixed points of this iteration?

Sln: If x is a fixed point, then $x = \varphi(x) = x - \lambda f'(x) \Rightarrow f'(x) = 0$.

② Assume $f''(x) > c > 0$. What choice of λ will the fixed points iteration converges for any initial condition x_0?

Sln: need $|\varphi'(x)| < 1$

$\varphi'(x) = 1 - \lambda f''(x) < 1 - \lambda c \quad \forall x$

Want $|\varphi'(x)| < 1 \Rightarrow \underbrace{1 - \lambda c < 1}_{\text{true}}$ and $1 - \lambda c > -1$

$-\lambda c > -2$

$\lambda c < 2$

$\boxed{\lambda < \frac{2}{c}}$

· Speed of convergence

The sequence $\{x_k\}$ converges linearly if

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = \mu \in \mathbb{R} \qquad \begin{cases} \mu = 0 & \text{super-linear} \\ \mu \in (0,1) & \text{linear} \\ \mu = 1 & \text{sub-linear} \end{cases}$$

$\mu \in (0, \infty)$

converges with order q.

④ Iterative methods for rooting finding

① Newton's method.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \cdots, \quad f'(x_k) \neq 0.$$

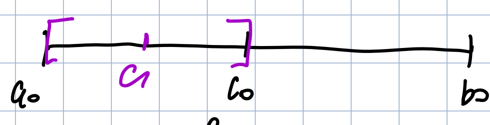
If converges, $f'(\xi) \neq 0$, quadratic — ξ is the root of f, i.e. $f(\xi) = 0$.

— - — $f'(\xi) = 0$, linear

② Secant Method. $f'(x_k) \approx \dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$

③ Bisection Method.

$f(a_0) \cdot f(b_0) < 0$



$f(c_0)$.

· if $f(a_0) f(c_0) < 0$, the next interval $[a_0, c_0]$

· if $f(b_0) f(c_0) < 0$, —— —— $[c_0, b_0]$

Eg: $f(x) = xe^x - 1$. Use Newton's method w/ $x_0 = 0$, find x_1 and x_2.

$$x - \frac{f(x)}{f'(x)} = x - \frac{xe^x - 1}{xe^x + e^x} = \frac{x^2 e^x - 1}{(x+1)e^x}$$

$$\Rightarrow x_1 = 0 - \frac{0e^0 - 1}{0e^0 + e^0} = 1.$$

$$x_2 = 1 - \frac{1 \cdot e - 1}{1 \cdot e + e} = 1 - \frac{e-1}{2e} = \frac{2e - e + 1}{2e} = \frac{e+1}{2e}.$$

2. Solutions of systems of linear equations

Solve the system $Ax=b$

① existence of solution ② LU decomposition ③ Pivoting.

unique sln if A is invertible

② LU decomposition

$$A=LU, \quad L=\begin{pmatrix} 1 & & & \\ \times & \ddots & & 0 \\ \times & \times & \ddots & \\ \times & \cdots & - & 1 \end{pmatrix}, \quad U=\begin{pmatrix} \times & \times & \cdots & \times \\ & \times & \ddots & \vdots \\ & 0 & \ddots & \\ & & & \times \end{pmatrix}$$

L: obtained while doing Gaussian elimination on A (row operations)

U: result of A after Gaussian elimination.

Given $A=LU$, $Ax=LUx=b \Rightarrow \begin{cases} Ly=b & \text{forward} \\ Ux=y & \text{backward.} \end{cases}$

Cost of LU: $\frac{2}{3}n^3-\frac{1}{2}n^2$

--- forward / backward: $2n^2$

Eg: $A=\begin{pmatrix} 4 & 2 & 0 & 0 \\ 2 & 5 & 4 & 0 \\ 0 & 4 & 6 & 3 \\ 0 & 0 & 3 & 6 \end{pmatrix}$ $\quad L=\begin{pmatrix} 1 & & & \\ 1/2 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 0 & 3/2 & 1 \end{pmatrix}$

$$\frac{a_{21}}{a_{11}}=\frac{2}{4}=\frac{1}{2}$$

$$a_{21}-\frac{a_{21}}{a_{11}}a_{11}=0$$

$$\begin{pmatrix} 4 & 2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 4 & 6 & 3 \\ 0 & 0 & 3 & 6 \end{pmatrix} \rightarrow \frac{a_{32}}{a_{22}}=\frac{4}{4}=1$$

$$\begin{pmatrix} 4 & 2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 6 \end{pmatrix} \rightarrow \frac{a_{43}}{a_{33}}=\frac{3}{2}$$

$$\underbrace{\begin{pmatrix} 4 & 2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3/2 \end{pmatrix}}_{U} \qquad 6-\frac{3}{2}\times 3=6-\frac{9}{2}=\frac{12-9}{2}=\frac{3}{2}$$

③ Pivoting. $PA = LU \iff A = P^T LU$ (b/c $P^T = P^{-1}$)

↳ get the largest entry in the diagonal

: dividing by 0 is dangerous, and dividing by a small number causes large error.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

To switch the 2nd & 3rd row of A, multiply A by $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

3. Norms and Conditioning

Given $Ax = b$, how sensitive is x to changes in A and/or b?

① Conditioning ② norms ③ Condition #.

↳ Small change in b may cause large change in x

⇒ Want a bound on the change of x

$A(x + \delta x) = b + \delta b$

$\Rightarrow \|\delta x\| \le K(A) \cdot \|\delta b\|$ or $\frac{\|\delta x\|}{\|x\|} \le K(A) \cdot \frac{\|\delta b\|}{\|b\|}$.

② Norms.

- $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$
- $\|\lambda v\| = |\lambda| \cdot \|v\|$, $\forall v \in V$.
- $\|v + w\| \le \|v\| + \|w\|$.

Vector Norms: $\|v\|_2 = \left(\sum_{i=1}^{n} |v_i|^2\right)^{1/2}$

$\|v\|_1 = \sum_{i=1}^{n} |v_i|$

$\|v\|_\infty = \max_{1 \le i \le n} |v_i|$

Matrix Norms: $\|A\|_\infty = \max_{x \ne 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$ (largest row sum)

$\|A\|_1 = \max_{x \ne 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$ (largest column sum)

$\|A\|_2 = \max_{x \ne 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{1 \le i \le n} \sqrt{\lambda_i (A^T A)}$

③ Condition numbers. $K(A) = \|A\| \cdot \|A^{-1}\|$

Thm: $\frac{\|\delta x\|}{\|x\|} \le K(A) \cdot \frac{\|\delta b\|}{\|b\|}$

Eg: (a) For two invertible matrices, show that $K(AB) \le K(A) \cdot K(B)$

Sln: $K(AB) = \|AB\| \cdot \|B^{-1}A^{-1}\|$

$\le \|A\| \cdot \|B\| \cdot \|B^{-1}\| \cdot \|A^{-1}\|$

$= K(A) \cdot K(B)$.

(b) Compute the condition number of the matrices using $\|\cdot\|_1$ and $\|\cdot\|_\infty$ norm:

(i) $\begin{pmatrix} a+1 & a \\ a & a-1 \end{pmatrix} = A$ (ii) $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} = B$

$$A^{-1} = \frac{1}{a^2-1-a^2}\begin{pmatrix} a-1 & -a \\ -a & a+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-a & a \\ a & -1-a \end{pmatrix}$$

$\|A\|_1 = \max\{|a+1|+|a|,\ |a|+|a-1|\}.$

$\|A^{-1}\|_1 = \|A\|_1$

$\Rightarrow K_1(A) = \max\{|a+1|+|a|,\ |a|+|a-1|\}^2$

$K_\infty(A) = K_1(A).$

(ii) $B = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$ $B^{-1} = \frac{1}{0+2}\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/2 \\ 1 & 0 \end{pmatrix}$

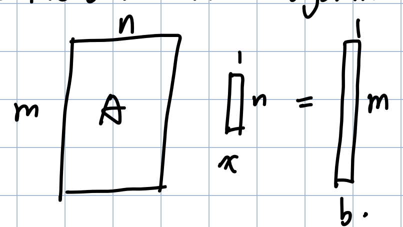
$\Rightarrow \|B\|_1 = 2,\ \|B^{-1}\|_1 = 1 \quad \Rightarrow K_1(B) = 2$

$\|B\|_\infty = 2,$ - - - - - - - - $K_\infty(B) = 2.$

4. Least Squares.

Solve the overdetermine system $Ax = b$.

(m>n)



① Least square ② normal equations ③ QR factorization.

$\min_x \|Ax-b\|_2$

$A^TAx = A^Tb$

if A is full rank,

rank(A) = n.

then A^TA invertible

$\Rightarrow$ unique solution x

Eg: $\left.\begin{aligned} x_1+2x_3&=1 \\ x_2+3x_3&=0 \\ -x_1+x_2+x_3&=0 \\ -x_2-3x_3&=1 \end{aligned}\right\}$ $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 1 & 1 \\ 0 & -1 & -3 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\min_{x \in \mathbb{R}^3} \|Ax-b\|_2 \iff A^TAx = A^Tb.$

$$A^TA = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 3 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 1 & 1 \\ 0 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 3 & 7 \\ 1 & 7 & 23 \end{pmatrix}$$

NOT invertible!

$$A^Tb = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 3 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ -1 & 3 & 7 & -1 \\ 1 & 7 & 23 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 7 & 23 & -1 \\ -1 & 3 & 7 & -1 \\ 2 & -1 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 7 & 23 & -1 \\ 0 & 10 & 30 & -2 \\ 0 & -15 & -45 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 7 & 23 & -1 \\ 0 & 10 & 30 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3: free

- $10x_2 + 30 \cdot x_3 = -2$

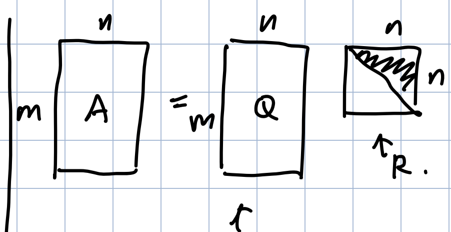
  $x_2 = -\frac{1}{5} - 3x_3.$
- $x_1 + 7x_2 + 23x_3 = -1$

  $x_1 + 7(-\frac{1}{5} - 3x_3) + 23x_3 = -1$

  $x_1 = \frac{2}{5} - 2x_3.$

$\Rightarrow$ All x satisfying $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2/5 - 2x_3 \\ -1/5 - 3x_3 \\ x_3 \end{pmatrix}$ Solves the least square problem.

③ QR factorization:



$QQ^T = Q^TQ = I.$

Q has orthonormal columns

Gram-Schmidt

Using $A = QR$, we can solve the least square $\min_x \|Ax - b\|$ by

$Rx = Q^Tb.$