QCI402 - Mathmatical Foundations for Quantum Computing

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The underpinnings of all scientific advancements is the ability to express natural phenomena with the art of Mathmatics; this is no different for the subject of Quantum Computing. While the boundaries of quantum computing have been pushed beyond limits in theoretical terms on university blackboards, it has become of great interest to realize the theoretical computational power with the advances of hardware and technology.

However, these notes mainly concerns itself with the mathmatical underpinnings of quantum computing that the course surrounds itself with. Mathmatical Foundations for Quantum Computingtakes a scaffolding approach designed to efficiently convey the required theoretical understanding of mathmatics in order to able to learn quantum computing. As of writing, we are basing the notes on verison one of the textbook published in March 2025. In this text, we will primarily be using dirac notation for the expression of vectors, operators, and their interactions.

Contents

1 Summation and Product Notations

This section primarily focuses on the common notations applied across mathmatics to denote and shorten addition and product notation.

1.1 Summation over a single Variable

The sigma notation is defined as follows

$$\sum_{i=1}^{n} f(i)$$

where we use sigma $\sum$ to represent the sum of a series. For example, the sum of all numbers in a series beginning with m and ending at index n is written as:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

Sums can also be infinite, commonly seen when Sigma looks as follows: $\sum_{i=m}^{\infty}$. Infinite sums are either convergent or divergent. A few of the most common converging infinite sums are as follows:

$$\sum_{i=0}^{\infty} \frac{1}{2^i} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$$

$$\sum_{i=0}^{\infty} \frac{1}{i^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

The first example is an infinite geometric series, and the sum of the first n terms is given by:

$$S_n = \sum_{i=0}^{n} \frac{1}{2^i} = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}}$$

As $n \to \infty$, $\frac{1}{2^n} \to 0$. Consequently, $S_n \to \frac{1}{1-\frac{1}{2}} = 2$. A rigorous proof of the second example requires extensive calculus and is not immediately obvious. While any mathmatical symbol can be used for the index of a summation, it is more practical to use something other than i as in the context of complex numbers, i commonly denotes the complex number$\sqrt{-1}$. moreover, sume can also be specified using descriptions. For example,

$$\sum_{p \in P} f(p) \qquad P \in \mathbb{N}'$$

where $\mathbb{N}'$ is the set of all prime numbers. Summations can also contain parameters other than the index, which results in functions of those parameters. For example the discrete Fourier transform (DFT) is given by

$$\tilde{x}_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} kn}, \quad k = 0, 1, \cdots N-1$$

where x_n represents the N values index by n and $\tilde{x}_k$ are the Fourier coefficients. Here, i is the imaginary numebr and N is a positive integer representing the dimension fo the DFT, of which we will cover in greater depth in Chapter 3. The following are some useful summation forumae commonly encountered in quantum computing:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=0}^{n}(a_0+id)=(n+1)\left(a_0+\frac{nd}{2}\right)\quad\text{(arithmetic series)}$$

$$\sum_{i=0}^{n}a^i=\frac{1-a^{n+1}}{1-a}\quad\text{(geometric series)}$$

$$(a+b)^n=\sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i\quad\text{(binomial theorem)}$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots\quad(|x|<1)$$

$$\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n-1}=1+2x+3x^2+4x^3+\cdots\quad(|x|<1)$$

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\quad(|x|<1)$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$$

Below are also a list of the common summations rules and manupulations:

$$\sum_{i=m}^{n}a_i=\sum_{j=m}^{n}a_j\quad\text{(change of index variable)}$$

$$\sum_{i=s}^{t}f(i)=\sum_{n=s}^{t}f(n)\quad\text{(change of index variable)}$$

$$\sum_{n=s}^{t}f(n)=\sum_{n=s}^{j}f(n)+\sum_{n=j+1}^{t}f(n)\quad\text{(splitting a sum)}$$

$$\sum_{n=s}^{t}f(n)=\sum_{n=0}^{t-s}f(t-n)\quad\text{(reverse order)}$$

$$\sum_{n=s}^{t}f(n)=\sum_{n=s+p}^{t+p}f(n-p)\quad\text{(index shift)}$$

$$\sum_{n=s}^{t}a\cdot f(n)=a\cdot\sum_{n=s}^{t}f(n)\quad\text{(distributivity)}$$

$$\sum_{n=s}^{t}f(n)\pm\sum_{n=s}^{t}g(n)=\sum_{n=s}^{t}(f(n)\pm g(n))\quad\text{(commutativity)}$$

1.2 Products and other Notations

Similar to the $\sum$ notation for addition, the $\prod$ (Pi) symbol is also more commonly used to dentoe the product of a series of terms. In this

$$\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \cdots \cdot a_{n-1} \cdot a_n$$

for example, the factorial of n is expressed as

$$\prod_{i=0}^{n} i = n!$$

and the relationship between $\sum$ and $\prod$, which are

$$b^{\sum_{n=s}^{t} f(n)} = \prod_{n=s}^{t} b^{f(n)}$$

$$\sum_{n=s}^{t} \log_b f(n) = log_b \prod_{n=s}^{t} f(n)$$

It is worth noting that in quantum computing and linear algebra, there are a few special notations such as the modulo-2 sum (bitwise XOR), or in other contexts the direct sum of linear spaces, represented by $\oplus$, and the tensor product represented by $\otimes$.

1.3 Summation over Multiple Variables

The double summation over a rectangular array is given by

$$\begin{aligned}\sum_{i=1,j=1}^{n_1,n_2} a_{i,j} = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} a_{i,j} = \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} a_{i,j} \\ &= a_{1,1} + a_{1,2} + a_{1,3} + a_{1,4} + \cdots + a_{1,n_2} \\ &+ a_{2,1} + a_{2,2} + a_{2,3} + a_{2,4} + \cdots + a_{2,n_2} \\ &+ a_{3,1} + a_{3,2} + a_{3,3} + a_{3,4} + \cdots + a_{3,n_2} \\ &+ a_{4,1} + a_{4,2} + a_{4,3} + a_{4,4} + \cdots + a_{4,n_2} \\ &+ \cdots \\ &+ a_{n_1,1} + a_{n_1,2} + a_{n_1,3} + a_{n_1,4} + \cdots + a_{n_1,n_2}\end{aligned}$$

Here, $\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}$ represents summing over each row first and then summing the results, while $\sum_{j=1}^{n_2}\sum_{i=1}^{n_1}$ will represent summing over the columns and then summing those results. The term $\sum_{i=1,j=1}^{n_1,n_2} a_{i,j}$ represents the summation over the rectangular array, irrespecive of the order. The product of two sums can be expanded into a double sum as follows:

$$\begin{aligned}\left(\sum_{i=1}^{m} a_i\right)\left(\sum_{j=1}^{n} b_j\right) &= (a_1 + a_2 + \cdots + a_m)(b_1 + b_2 + \cdots + b_n) \\ &= \ a_1b_1 + a_1b_2 + a_1b_3 + a_1b_4 + \cdots + a_1b_n \\ &+ a_2b_1 + a_2b_2 + a_2b_3 + a_2b_4 + \cdots + a_2b_n \\ &+ a_3b_1 + a_3b_2 + a_3b_3 + a_3b_4 + \cdots + a_3b_n \\ &+ \cdots \\ &+ a_mb_1 + a_mb_2 + a_mb_3 + a_mb_4 + \cdots + a_mb_n \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_ib_j = \sum_{i=1}^{m} a_i \sum_{j=1}^{n} b_j\end{aligned}$$

which is actually rather intuitive given how the expansion of the standard expansion of the term $(a+b)^2$ plays out, a more elementary application of the distributive property which the above equation generalizes over. For a triangular matrix, in this case the lower triangular matrix, the sum is given by

$$\sum_{1\leq j\leq n} a_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{i} a_{i,j} = \sum_{j=1}^{n}\sum_{i=j}^{n} a_{i,j} = \sum_{j=0}^{n-1}\sum_{j=1}^{n-j} a_{i+j,i}$$
$$\begin{aligned}
&= a_{1,1} \\
&+ a_{2,1} + a_{2,2} \\
&+ a_{3,1} + a_{3,2} + a_{3,3} \\
&+ a_{4,1} + a_{4,2} + a_{4,3} + a_{4,4} \\
&+ \cdots \\
&+ a_{n,1} + a_{n,2} + a_{n,3} + a_{n,4} + \cdots + a_{n,n}
\end{aligned}$$

where the term $\sum_{1\leq j\leq n} a_{i,j}$ denotes the summation over all elements in a lower triangular array including the diagonal. The first notation variation will sum up each row to the ith element then aggregate while the second notation sums each column starting from the jth element downwards then aggregate the sums. The final expression will sum along the diagonal where $j = 0$ represents the main diagonal and $j = n - 1$ is the first off-diagonal, which is a single term.

Example. *Say we would like to expand the product of* $(1 + x_i)$ *from* 1 *to* n*. We have*

$$\prod_{i=1}^{n}(1 + x_i) = 1 + \sum_{k=1}^{n}\left(\sum_{1\leq i_1<\cdots<i_k\leq n}\prod_{j=1}^{k} x_{i_j}\right)$$

This formula represents the *multinomial expansion* of a product. When you expand the equation by hand, you get the product

$$\prod_{i=1}^{n}(1 + x_i) = (1 + x_1)(1 + x_2)\cdots(1 + x_n)$$

If we break this down, we see that the outer summation $\sum_{k=1}^{n}$ will go through each possible summation size in terms of the variables in question, and that the inner summation $\sum_{1\leq i_1<\cdots<i_k\leq n}$ will iterate through each possible unique product of the variables. while ensuring that they are unique. Not sure how this works, but if all x_i are the same, then wesee that the equation actually simplifies to a subset of the binomial theorem

$$(1 + x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

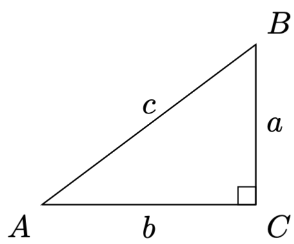
where $\binom{n}{k}$ is the binomial coefficient representing the numebr of ways to choose k elements from a set of n distinct elements.

2 Trigonometry

You can't escape this.

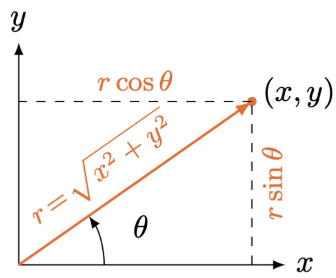
2.1 Definitions

I mean, where do I start? The basic trigonometric functions are defined as the ratios between the angles of a right triangle. I will not show how these ratios remain the same given the same angle, nor will I go into great mathmatical detail of how to prove these items. However, we still have to go over this. Don't ask me why.



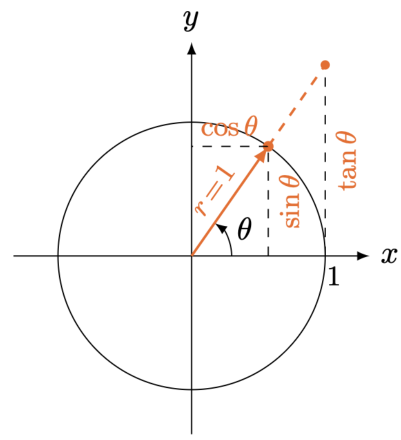
$$\sin A = \frac{a}{c},\ \cos A = \frac{b}{c}$$

$$\tan A = \frac{a}{b},\ \cot A = \frac{b}{a}$$

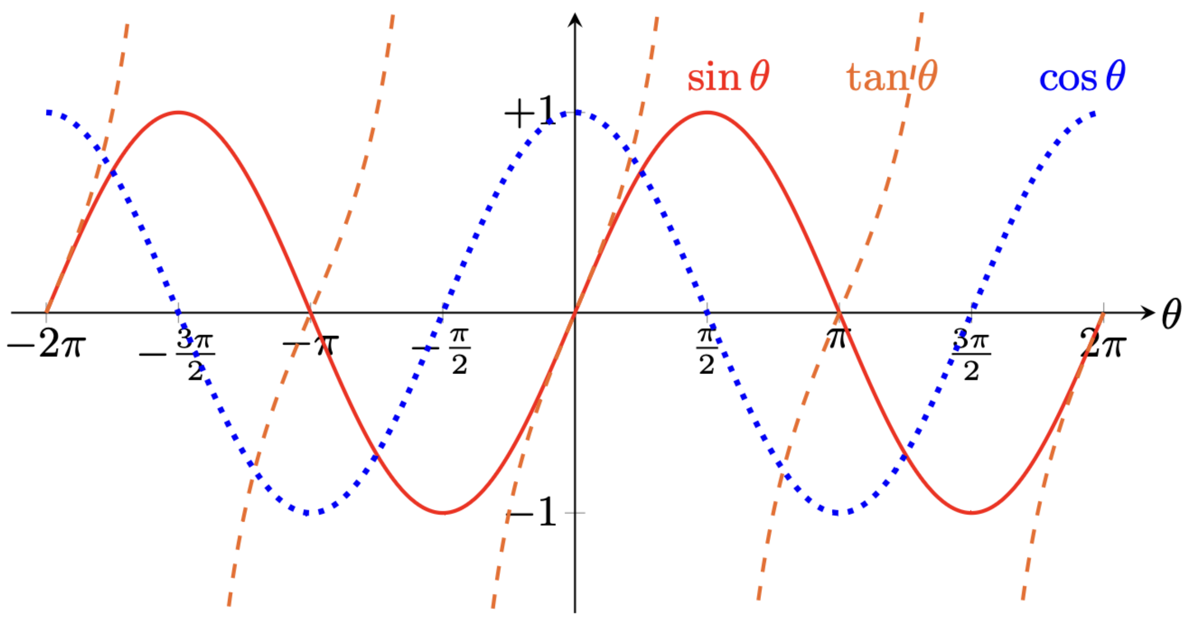


$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x},\ \cot\theta = \frac{x}{y}$$



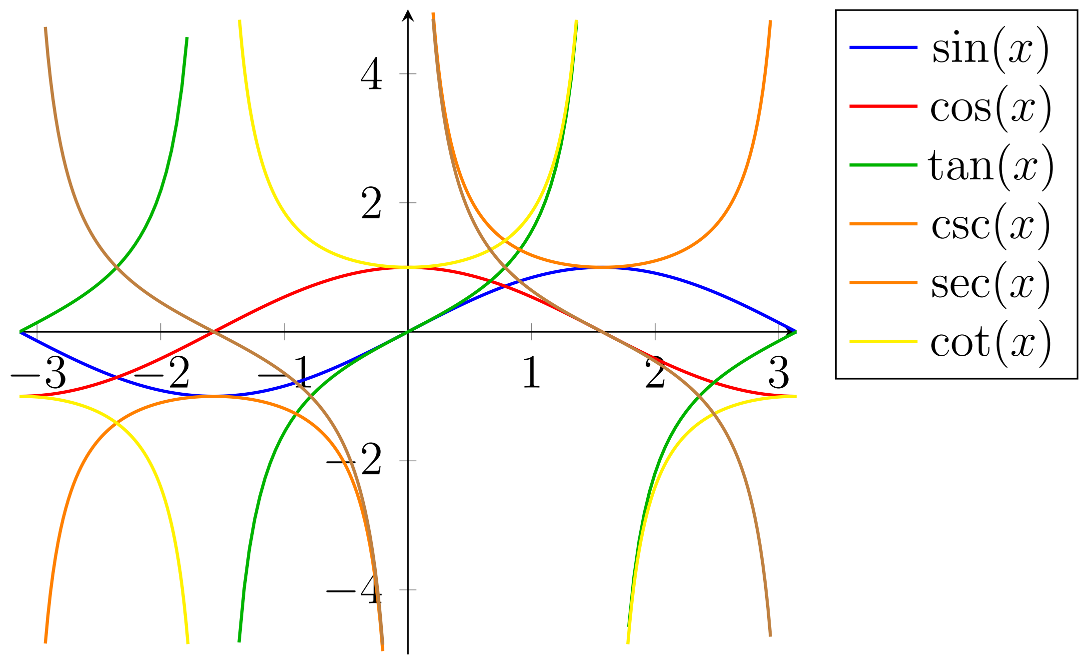
The functions of trigonometric functions can also be plotted out as follows:



2.2 Basic Properties and Inverse Functions

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
Definition	y/r	x/r	y/x	r/y	r/x	x/y
Period	2π	2π	π	2π	2π	π
Range	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$	$(-\infty, -1] \cup [1, \infty)$		$(-\infty, \infty)$
Zeros	$n\pi$	$(n + \frac{1}{2})\pi$	$n\pi$			$(n + \frac{1}{2})\pi$
Poles			$(n + \frac{1}{2})\pi$	$n\pi$	$(n + \frac{1}{2})\pi$	$n\pi$

Note: n is an integer.



We can also see that there are certain useful symmetric properties of the trigonometric functions

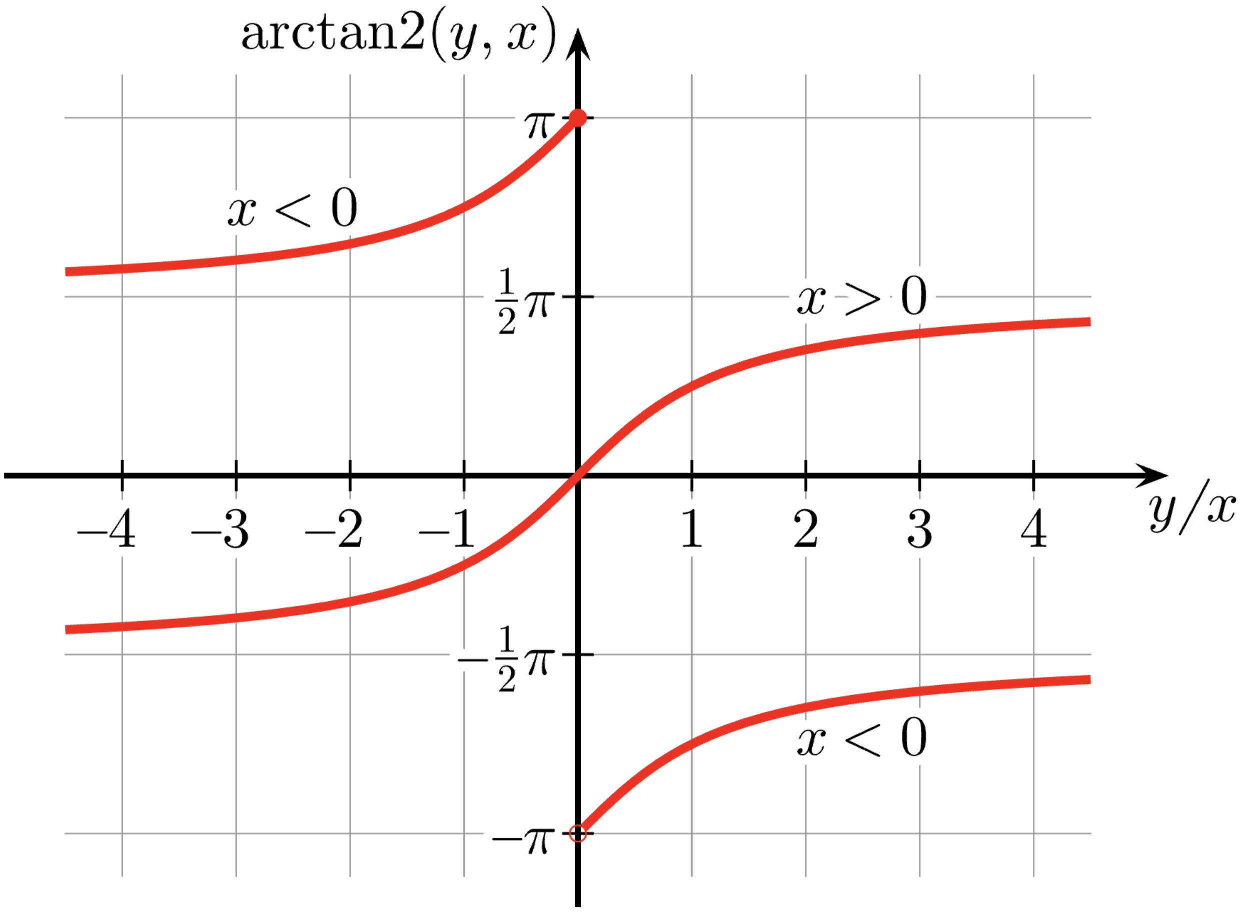
$$
\begin{aligned}
\sin(-\theta) &= -\sin(\theta) & \sin(\pi-\theta) &= \sin(\theta) & \sin(\pi+\theta) &= -\sin(\theta) \\
\cos(-\theta) &= \cos(\theta) & \cos(\pi-\theta) &= -\cos(\theta) & \cos(\pi+\theta) &= -\cos(\theta) \\
\tan(-\theta) &= -\tan(\theta) & \tan(\pi-\theta) &= -\tan(\theta) & \tan(\pi+\theta) &= \tan(\theta)
\end{aligned}
$$

There are also some common inverse functions associated with the functions.

Function	**sin**	**cos**	**tan**	**csc**	**sec**	**cot**
Inverse	$\sin^{-1}$ arcsin	$\cos^{-1}$ arccos	$\tan^{-1}$ arctan	$\csc^{-1}$	$\sec^{-1}$	$\cot^{-1}$
Domain	$[-1,1]$	$[-1,1]$	$(-\infty,\infty)$	$(-\infty,-1]\cup[1,\infty)$		$(-\infty,\infty)$
Range	$[-\frac{\pi}{2},\frac{\pi}{2}]$	$[0,\pi]$	$(-\frac{\pi}{2},\frac{\pi}{2})$	$[-\frac{\pi}{2},\frac{\pi}{2}]\backslash\{0\}$	$[0,\pi]\backslash\{\frac{\pi}{2}\}$	$(-\frac{\pi}{2},\frac{\pi}{2})$

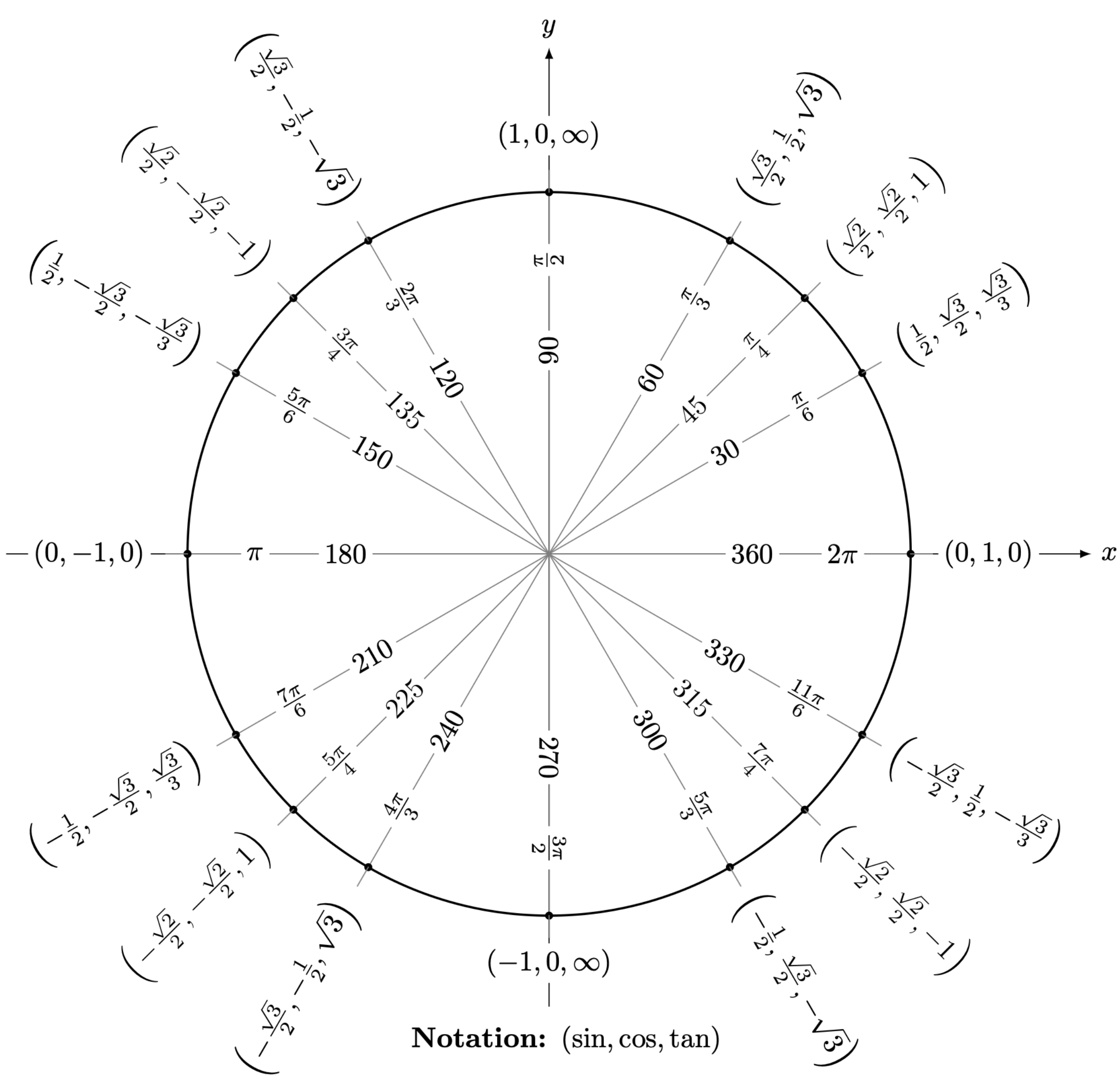
But also some interesting extensions of the commonly known inverse functions, with the example being arctan2, a function that effectively doubles the domain of the function while preserving its properties for the purpose of, say, converting values from cartesian to spherical coordinates for the azimuthal angle ϕ.

$$
\arctan2(y,x) = \begin{cases}
\arctan\left(\frac{y}{x}\right) & \text{if } x>0, \\
\arctan\left(\frac{y}{x}\right)+\pi & \text{if } x<0 \text{ and } y\geq 0, \\
\arctan\left(\frac{y}{x}\right)-\pi & \text{if } x<0 \text{ and } y<0, \\
+\frac{\pi}{2} & \text{if } x=0 \text{ and } y>0, \\
-\frac{\pi}{2} & \text{if } x=0 \text{ and } y<0, \\
0 & \text{if } x=0 \text{ and } y=0.
\end{cases}
$$



2.3 Special Angles and Function Values

There are a few special angles that are worth remembering for the trigonometric functions mentioned above, given by the wheel below.



2.4 Trigonometric Identities

Reciprocal and Quotient Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Cofunction Identities

$$\begin{aligned} \sin\left(\frac{\pi}{2}-\theta\right) &= \cos\theta, & \cos\left(\frac{\pi}{2}-\theta\right) &= \sin\theta \\ \tan\left(\frac{\pi}{2}-\theta\right) &= \cot\theta, & \cot\left(\frac{\pi}{2}-\theta\right) &= \tan\theta \\ \sec\left(\frac{\pi}{2}-\theta\right) &= \csc\theta, & \csc\left(\frac{\pi}{2}-\theta\right) &= \sec\theta \end{aligned}$$

Pythagorean Identities

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}$$

Even–Odd Symmetry

$$\begin{aligned} \sin(-\theta) &= -\sin\theta, & \cos(-\theta) &= \cos\theta, & \tan(-\theta) &= -\tan\theta \\ \csc(-\theta) &= -\csc\theta, & \sec(-\theta) &= \sec\theta, & \cot(-\theta) &= -\cot\theta \end{aligned}$$

Sum and Difference Formulas

$$\begin{aligned} \sin(\alpha\pm\beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha\pm\beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\ \tan(\alpha\pm\beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \end{aligned}$$

Double Angle Formulas

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ \tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} \end{aligned}$$

Half Angle Formulas

$$\begin{aligned} \sin^2\left(\frac{\theta}{2}\right) &= \frac{1-\cos\theta}{2} \\ \cos^2\left(\frac{\theta}{2}\right) &= \frac{1+\cos\theta}{2} \\ \tan\left(\frac{\theta}{2}\right) &= \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta} \end{aligned}$$

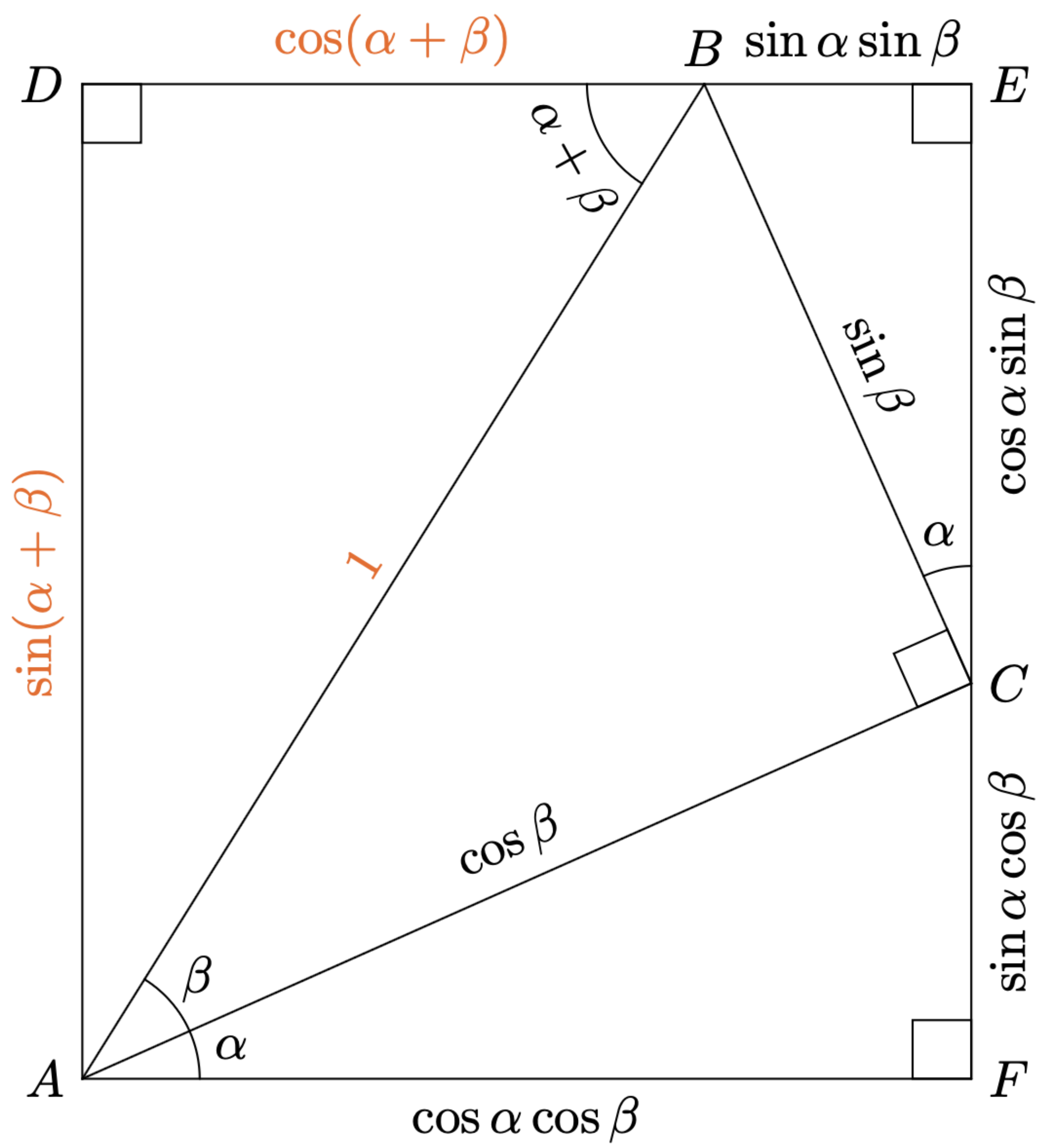
Product-to-Sum Identities

$$\sin\alpha\sin\beta = \frac{1}{2}\big[\cos(\alpha-\beta) - \cos(\alpha+\beta)\big]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\big[\cos(\alpha-\beta) + \cos(\alpha+\beta)\big]$$
$$\sin\alpha\cos\beta = \frac{1}{2}\big[\sin(\alpha+\beta) + \sin(\alpha-\beta)\big]$$

Sum-to-Product Identities

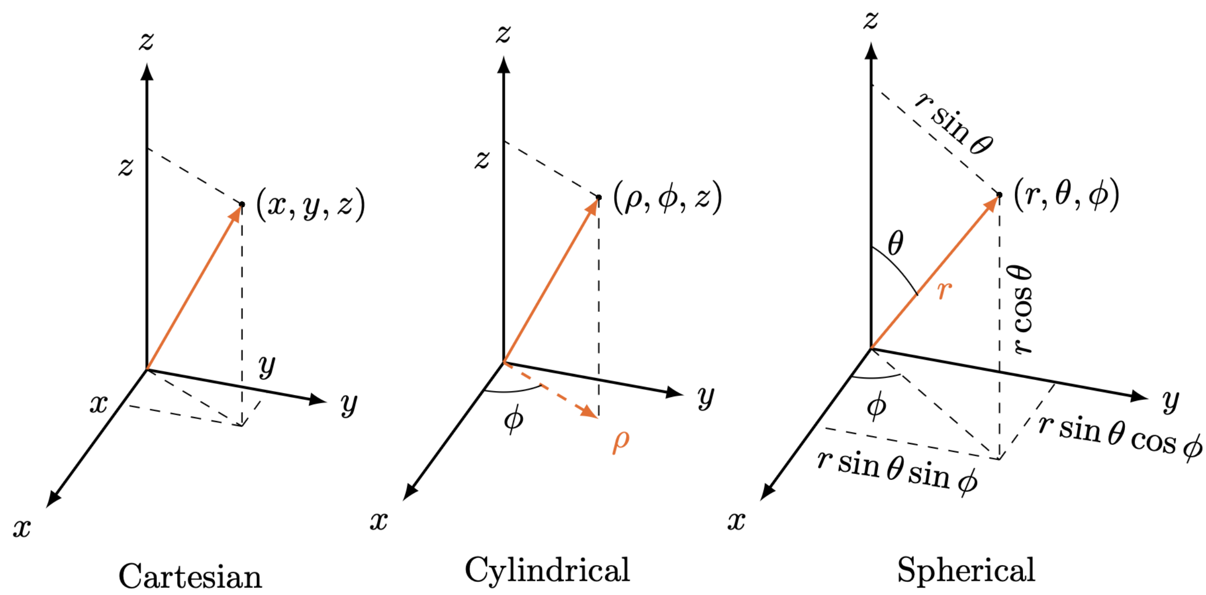
$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

A rather nice photo to sum the secion up is by relating the angles to each other using the following image



2.5 The Spherical Coordinate System

While the expansion of the cartesian coordinate system into three dimensions is the logical linear expansion to take, some interesting basis begin to form if we consider the angles as a unit of measurement in three dimensions



The cylindrical coordinate system elevates the polar coordinate system into three dimensions with the addition of the z axis, while the spherical system is a more organic translation of the polar coordinate system into three dimensions. While the former is best suited for describing not only cylinder-like structures yet also helices, while the latter is best suited for rotations in three-dimensional space, which is particularly potent in the field of quantum computing.

For example, the spherical coordinate system is commonly used to represent qubit states on the Bloch sphere, employing a radius (r), a polar angle (θ), and an azimuthal angle (ϕ) to represent a point in three-dimensional space. The azimuthal angle ϕ is measured in the xy-plane from the positive x-axis with common values ranging from $(-\pi, \pi]$ or $(0, 2\pi]$. The polar angle is commonly measured from the positive z axis towards the xy-plane, with values ranging from $[0, \pi]$. Note that $r \in \mathbb{R}$, meaning that we can cover the other half of the range simply by flipping the sign around.

Conversion is relatively simple, with conversion to and from spherical to cartesian being as follows

$$\begin{aligned} x &= r \sin\theta \cos\phi & \qquad r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin\theta \sin\phi & \qquad \phi &= \operatorname{arctan2}(y, x) \\ z &= r \cos\theta & \qquad \theta &= \arccos \frac{z}{r} \end{aligned}$$

where arctan2 was previously defined as a optimal inverse mapping onto the range of $[-\pi, \pi]$. We also have a few definitions

Definition 2.1. *The Law of Sines is defined as*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Definition 2.2. *The Law of Cosines is defined as*

$$a^2 = b^2 + c^2 - 2bc \cos A$$

which can be rewritten as

$$\cos A = \frac{b^2 + c^2 - 2bc}{a^2}$$

Definition 2.3. *The Law of Tangents is defined as*

$$\frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}$$

3 Complex Numbers

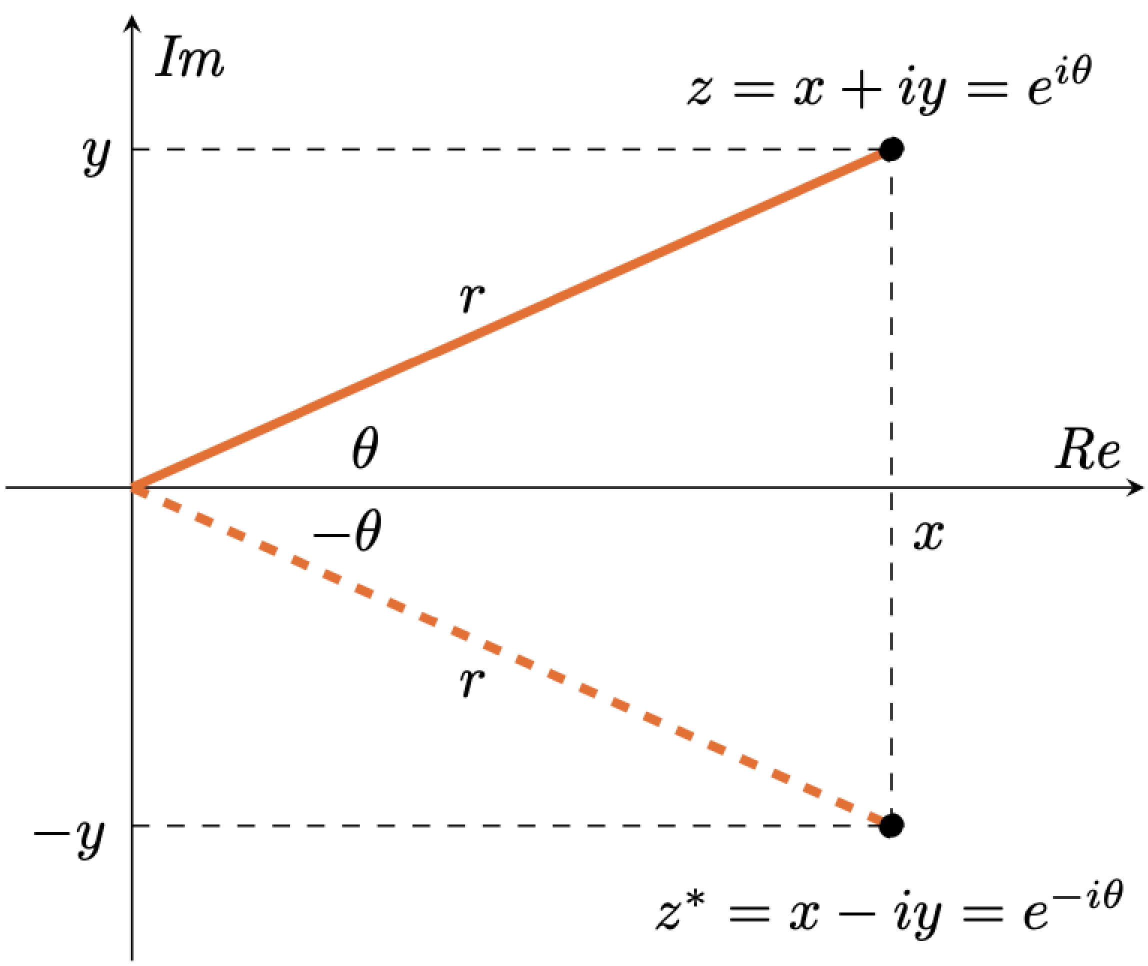
We consider numbers to be complex when they compose of a real and imaginary part, and they are not only fundamental to a complete understanding of algebra and mathmatics as a whole, but also form the backbone of quantum mechanics, and, by extension, quantum computing. Mastering complex numbers is like Rosie mastering the rivet gun, so we have to study it.

3.1 Cartesian Form

Definition 3.1. *A complex number z is defined as*

$$z = x + iy, \qquad x, y \in \mathbb{R}, \quad i^2 = -1$$

This is called the **cartesian form** of the complex number z and corresponds to a point in the two-dimensional complex plane. We commonly refer to i as the imaginary unit. It may seem ironic that we need imaginary numbers in quantum computing, or that we really need the imaginary number. Take it as you may.



Complex numbers not motivated by quantum computing. In the numbers system, we have the real numbers $\mathbb{N}$, the integer numbers $\mathbb{Z}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. the set incursions go this way $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$, all supersets of the preceeding set. The set of all complex numbers $\mathbb{C}$ is closed over all algabraic operations, which include addition, substraction, multiplication, division, power, and root, and is considered the superset of all numbers.

Definition 3.2. *The basic components of a complex number are defined as follows.*

$$\Re(z) = x, \Im(z) = y$$

which are the real and imaginary components of a complex number z. Of course the complex number itself has a few interesting properties, such as $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. The complex conjugate of a complex number z is defined as

$$z^* = x - iy$$

defined as inverting the sign of the imaginary component. We can express the modulus (vector length) and argument (angle with respect to the real axis) (which are r and θ in polar coordinates), as follows:

$$r = |z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$$

A very convenient property derived from algebra is that $zz^* = x^2 + y^2$.

$$\theta = arg(z)$$

For the angle, we note that

$$\tan \frac{y}{x} \implies \text{arctan2}(x, y)$$

where arctan2 *has been defined in the previous section.*

Example. *Given $z = 1 + \sqrt{3}i$, we have*

$$z^* = 1 - \sqrt{3}\, i.$$

$$|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2.$$

$$zz^* = (1 + \sqrt{3}\, i)(1 - \sqrt{3}\, i) = 1 - (\sqrt{3}\, i)^2 = 1 + 3 = 4 = |z|^2.$$

$$\theta = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}.$$

3.2 Exponential Form

Now it is worth noting that while we commonly write $z = x + iy$ to represent a complex numnber, we like to use the following definitions of the complex number in polar form to represent a complex number itself, defined as $z = r(\cos\theta + i\sin\theta)$. However, multiplication and its inverse operation, division, becomes unnecessarily difficult givne the presence of another notation, namely **exponential form**.

Definition 3.3. *The exponential/euler forms of the complex numbers can be thought of as a circular form of the function $z = x + iy$. In polar coordinates, we can rewrite this number as*

$$z = r\cos\theta + i\sin\theta, \qquad r \in \mathbb{R}$$

Conversely, the conversion between cartesian and polar are

$$x = r\cos\theta \qquad y = r\sin\theta$$

The formula for z above can be rewritten as

$$z = re^{i\theta}$$

Theorem 3.4. *Euler's formula states that for any complex number $z = r\cos\theta + i\sin\theta$, we have:*

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Proof. Euler's formula can be proven using the Taylor series expansion for the functions:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

if we replace x in the formula for e^x with e^{ix}, we then have

$$\begin{aligned}
e^{ix} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \cdots\right) \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

which sums up essentially what mathmaticians call the most beautiful proof man has known. If this proof were a female robot, mathematicians would compose harmonic waves and produce digital flowers in LaTeXto please Euler's genius. □

As noted before, we know that the set of all algebraic operations is well defined and closed on the set of all complex numbers $\mathbb{C}$. Addition will be easier in cartesian form, while multiplication will be considerably simpler in exponential form. Conversion between the two is also not difficult:

	Cartesian Form	**Exponential Form**
	$z = x + iy$	$z = re^{i\theta}$
Conjugate	$z^* = x - iy$	$z^* = re^{-i\theta}$
Modulus	$\lvert z\rvert = \sqrt{zz^*} = \sqrt{x^2 + y^2}$	$\lvert z\rvert = r$
Conversion	$x = r\cos\theta$	$r = \sqrt{x^2 + y^2}$
	$y = r\sin\theta$	$\theta = \text{arctan2}(y, x)$

3.3 Basic Operations

As we touched upon earlier, the set of all complex numbers are closed on operations of addition and subtraction:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

As well on multiplication and division. For

$$z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2}$$

We have:

$$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

We also have the following properties for the conjugates of complex numbers

$$\begin{aligned}
|z| &= |z^*| \\
(z_1 \pm z_2)^* &= z_1^* \pm z_2^* \\
(z_1 \cdot z_2)^* &= z_1^* \cdot z_2^* \\
(z_1 / z_2)^* &= z_1^* / z_2^* \\
(z^x)^* &= (z^*)^x \quad x \in \mathbb{R} \\
(x^z)^* &= x^{z^*} \quad x \in \mathbb{R}
\end{aligned}$$

Where the last two are not immediately obvious. To prove that $(z^x)^* = (z^*)^x$, it is useful to write out z using the complex notation $re^{i\theta}$, and the last property is best proven using the identity $a^b = e^{b \ln a}$. As for powers and roots of complex numbers, we have

Theorem 3.5. *De Moivre's theorem states that*

$$(\cos\theta + i \sin\theta)^s = \cos s\theta + i \sin s\theta$$

which is conveniently derived from the fact that $z^s = r^s e^{is\theta}$.

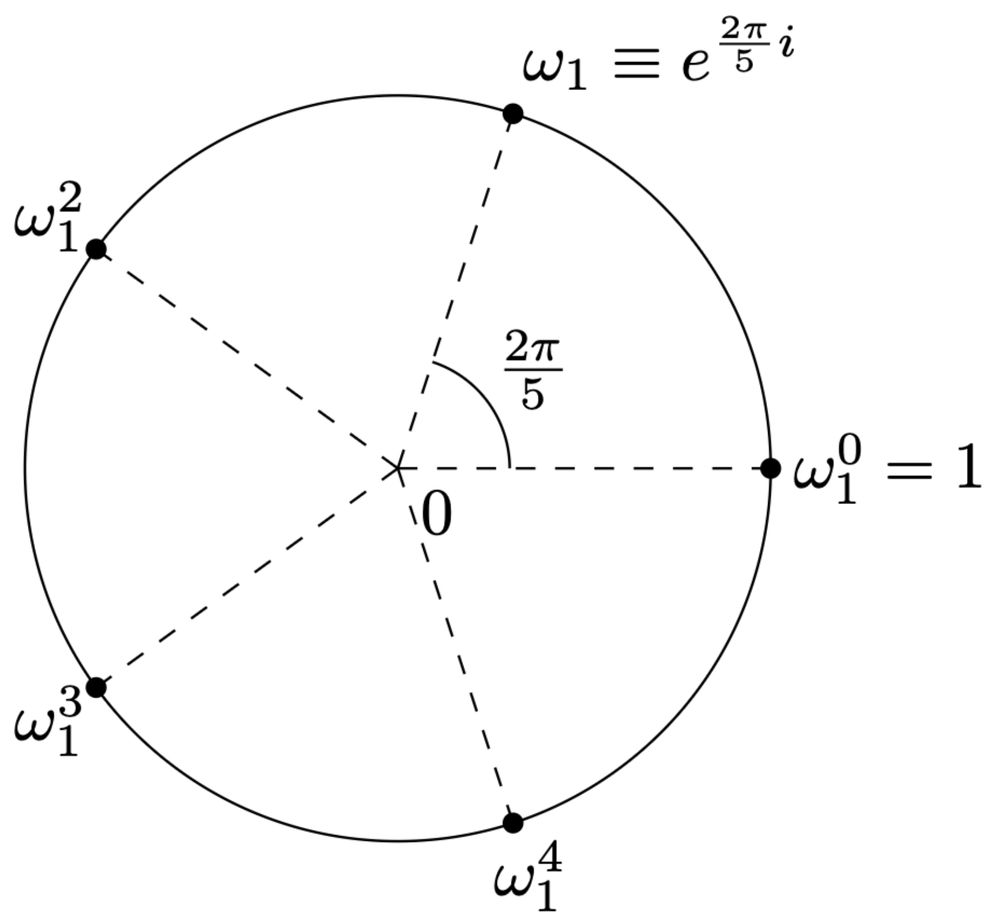
In particular, an application of this theorem is where we describe the roots of unity. Any root of unity can be descibed by a power of the first root of unity,

$$\omega_1 = e^{\frac{2\pi i}{n}}$$

the n-th roots of unity ($n \in \mathbb{N}$) are given by

$$\omega_k = e^{\frac{2\pi i}{n}k} = w_1^k \qquad k = 1, 2, \ldots, n-1$$

which essentially says that there are n roots to the complex polynomial. For $n = 5$, we have the 5 roots of unity given by



In general, we say that there are n values of k that satisfy the equation $\omega_1^n = e^{(\frac{2\pi i}{n})^n} = 1$. From this, we can generalize what we know into the summations over ω_k, which is any k-th root of unity except for $\omega = 1$.

$$\sum_{k=0}^{n-1} \omega_k = \sum_{k=0}^{n-1} \omega_1^k = 0$$

This formula can be conviniently proven by applying the formula for summing a geometric sequence to the summation. From this, we can conviniently derive a useful mathmatical condition, being

Example. *The DFT Orthonormality condition depends on two parameters k and l, and is stated as follows*

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{-\frac{2\pi i}{N}kn} e^{\frac{2\pi i}{N}ln} = \delta_{k-l \bmod N}$$

where $\delta_{k-l} \pmod N = 1$ if and only if $k \equiv l \pmod N$, else 0. It is saying that when k is congruent to l, equivalent to $k - l = mN$, where the difference between k and l is divisible by some integer m. The $\delta_{k-l \bmod N}$ term is a Kronecker delta of $k \equiv l \pmod N$, where the result is 1 if $k \equiv l \pmod N$ holds and 0 in the case of $k \not\cong l \pmod N$. If we define $\omega = e^{i\frac{2\pi}{N}}$ as a primitive Nth root of unity (satisfying $\omega^N = 1$), we have the derivation

$$\begin{aligned}
\frac{1}{N}\sum_{n=0}^{N-1} e^{-\frac{2\pi i}{N}kn}\, e^{\frac{2\pi i}{N}ln} &= \frac{1}{N}\sum_{n=0}^{N-1} \omega^{-kn}\omega^{ln} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} \omega^{n(l-k)} \\
&= \begin{cases} \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} 1 = 1, & \text{if } l \equiv k \pmod N, \\ \dfrac{1}{N}\dfrac{1-\omega^{(l-k)N}}{1-\omega^{(l-k)}} = 0, & \text{if } l \not\equiv k \pmod N \end{cases} \\
&= \delta_{k-l \bmod N},
\end{aligned}$$

where we used the fact that $\omega^{n^N} = \left(\omega^N\right)^n = 1^n = 1$ for $n \in \mathbb{N}$.

3.4 Advanced Operations

It's probably best to illustrate more advanced operations on complex numbers with the help of some examples

Example. *Evaluating $\sqrt{i}$ or $\sqrt{\sqrt{1}}$ gives:*

$$\sqrt{i} = \left(e^{\frac{\pi i}{2}}\right)^{\frac{1}{2}} = e^{\frac{\pi i}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+i)$$

The inverse is given by

$$\left(\frac{1}{\sqrt{2}}(1+i)\right)^2 = \frac{1}{2}(1+2i+i^2) = \frac{(2i)}{2} = i$$

Example. *Evaluating*

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{50}$$

gives

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{50} = \left(e^{\frac{\pi i}{3}}\right)^{50} = e^{\frac{50\pi i}{3}} = e^{\left(16+\frac{2}{3}\right)\pi i} = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Example. *Evaluating*

$$2^{3+4i}$$

gives us

$$2^{3+4i} = 2^3 \cdot 2^{4i} = 8 \cdot e^{4\ln(2)\,i} = 8\cos\big(4\ln 2\big) + i\,8\sin\big(4\ln 2\big).$$

Example. *Evaluating*

$$\cos(3+4i)$$

gives us

$$\begin{aligned}
\cos(3+4i) &= \frac{1}{2}\left(e^{i(3+4i)} + e^{-i(3+4i)}\right) \\
&= \frac{1}{2}\left(e^{-4+3i} + e^{4-3i}\right) \\
&= \frac{1}{2}e^{-4}(\cos 3 + i\sin 3) + \frac{1}{2}e^{4}(\cos 3 - i\sin 3) \\
&= \frac{1}{2}(e^{-4} + e^4)\cos 3 + i\,\frac{1}{2}(e^{-4} - e^4)\sin 3
\end{aligned}$$

Example. *If we have the equation* $z^5 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$*, solving for* z *gives*

$$z_k = e^{\frac{\pi i}{15}} e^{\frac{2k\pi i}{5}} \qquad k = 0, 1, 2, 3, 4.$$

It is worth noting that there exists a way to express the trigonometric functions sin and cos as a function of euler's number. We know that

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

from this, we can derive that

$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta \\ e^{i\theta} + e^{-i\theta} &= \cos\theta + \cos\theta \\ \cos\theta &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \end{aligned}$$

and that

$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) \\ e^{i\theta} - e^{-i\theta} &= i\sin\theta - (i\sin\theta) \\ \sin\theta &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \end{aligned}$$

Another way of expressing this is by saying

$$\begin{aligned} \cos x = \Re(e^{i\theta}) &= \frac{e^{ix} + e^{-ix}}{2} \\ \sin x = \Im(e^{i\theta}) &= \frac{e^{ix} - e^{-ix}}{2i} \end{aligned}$$

One final yet very important item to rememebr throughout the curriculum is that powers for complex numbers are **rotations**.

4 Sets, Groups, and Functions