VII. *On a Method of Investigating Periodicities in Disturbed Series, with special reference to Wolfer's Sunspot Numbers.*

By G. Udny Yule, *C.B.E., M.A., F.R.S., University Lecturer in Statistics, and Fellow of St. John's College, Cambridge.*

(Received December 11, 1926.—Revised February 17, 1927.)

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I. Introductory: Superposed Fluctuations and Disturbances.

If we take a curve representing a simple harmonic function of the time, and superpose on the ordinates *small* random errors, the only effect is to make the graph somewhat irregular, leaving the suggestion of periodicity still quite clear to the eye. Fig. 1 (*a*) shows such a curve, the random errors having been determined by the throws of dice. If the errors are increased in magnitude, as in fig. 1 (*b*), the graph becomes more irregular, the suggestion of periodicity more obscure, and we have only sufficiently to increase the "errors" to mask completely any appearance of periodicity. But, however large the errors, periodogram analysis is applicable to such a curve, and, given a sufficient number of periods, should yield a close approximation to the period and amplitude of the underlying harmonic function.

 [Published April 28, 1927.

When periodogram analysis is applied to data respecting any physical phenomenon in the expectation of eliciting one or more true periodicities, there is usually, as it seems to me, a tendency to start from the initial hypothesis that the periodicity or periodicities are masked solely by such more or less random *superposed fluctuations*—fluctuations which do not in any way disturb the steady course of the underlying periodic function or functions. It is true that the periodogram itself will indicate the truth or otherwise of the hypothesis made, but there seems no reason for assuming it to be the hypothesis most likely *a priori*.

If we observe at short equal intervals of time the departures of a simple harmonic pendulum from its position of rest, errors of observation will cause superposed fluctuations of the kind supposed in fig. 1. But by improvement of apparatus and automatic methods of recording, let us say, errors of observation are practically eliminated. The recording

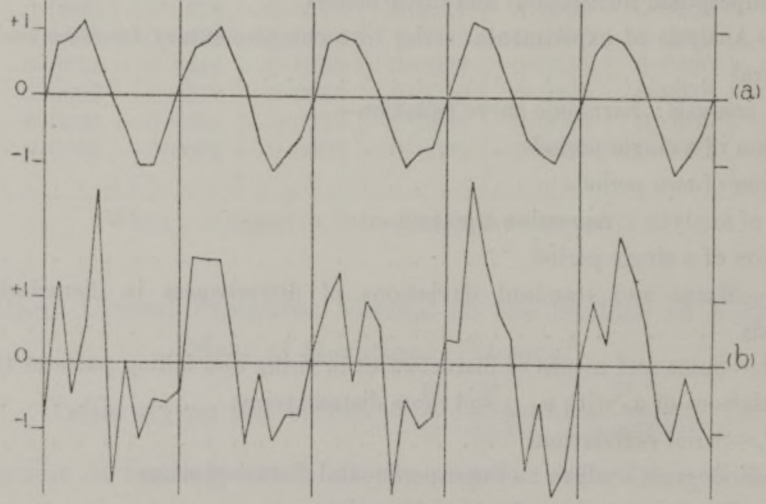

FIG. 1.—Graphs of simple harmonic functions of unit amplitude with superposed random fluctuations: (*a*) smaller fluctuations, (*b*) larger fluctuations.

apparatus is left to itself, and unfortunately boys get into the room and start pelting the pendulum with peas, sometimes from one side and sometimes from the other. The motion is now affected, not by *superposed fluctuations* but by true *disturbances*, and the effect on the graph will be of an entirely different kind. The graph will remain surprisingly smooth, but amplitude and phase will vary continually.

Working with finite in lieu of infinitesimal intervals, we may construct an approximation to a curve of the kind supposed. Let the terms of the trigonometrical series be

$$\left.\begin{aligned} u_0 &= \mathrm{A} \sin 2\pi \frac{t+\tau}{\mathrm{T}} \\ u_1 &= \mathrm{A} \sin 2\pi \frac{t+\tau+h}{\mathrm{T}} \\ u_2 &= \mathrm{A} \sin 2\pi \frac{t+\tau+2h}{\mathrm{T}} \end{aligned}\right\}, \qquad (1)$$

etc., where A is the amplitude, t the time, T the period, h the interval between successive terms, and τ gives the phase. Then, with a little reduction, we have for the second difference

$$\Delta^2 (u_0) = -\left(4 \sin^2 \pi \frac{h}{\mathrm{T}}\right) u_1 = -\mu u_1, \quad \ldots\ldots\ldots\ldots (2)$$

where, if θ is the angle corresponding to the interval,

$$\mu = 4 \sin^2 \pi \frac{h}{\mathrm{T}} = 2(1 - \cos\theta). \quad \ldots\ldots\ldots\ldots (3)$$

But, in terms of the u's, (2) gives

$$u_2 = (2 - \mu) u_1 - u_0, \quad \ldots\ldots\ldots\ldots (4)$$

where it may be noted

$$2 - \mu = 2\cos\theta. \quad \ldots\ldots\ldots\ldots (5)$$

If there are no disturbances, (4) gives u_x generally in terms of u_{x-1} and u_{x-2}. Provided the interval were infinitesimal, (4) would still give u_x correctly, even if the velocity in the interval $x-2$ to $x-1$ were affected by an impulse, so long as the interval $x-1$ to x were undisturbed. But if a disturbance occurred also in the latter interval, we would have, say,

$$u_x = (2 - \mu) u_{x-1} - u_{x-2} + \varepsilon, \quad \ldots\ldots\ldots\ldots (6)$$

where ε is an "error" varying with the impulse or disturbance.

Fig. 2 shows a graph constructed in the following way from this equation:—The period was taken as 10 intervals, and the first two ordinates as 0 and sin 36° (0·588). Thereafter all the ordinates were calculated in succession by (6), the errors or "disturbances" ε being given by dice-throwing. Four dice were thrown together: the divergence of the sum of the pips from the mean number (that is, 14) was divided by 20, and this was taken as the value of ε. The values so determined fluctuate round zero with a standard deviation 0·1708, and are thus fairly considerable, ranging up to $\pm$ 0·5. Inspection of the figure shows that there are now no abrupt variations in the graph, but the amplitude varies within wide limits, and the phase is continually shifting. Increasing the magnitude of the disturbances simply increases the amplitude: the graph remains smooth. At one point, in fact, the "disturbance" was inadvertently magnified by an error of calculation, but there was no appreciable kink in the graph to direct attention to the blunder. It is, of course, true that the graph may be made to pass through any assigned series of points, however irregular, but to introduce such irregularities appropriate large and erratic disturbances must be given: abrupt irregularities do not naturally occur with random disturbances.

It is of interest to look a little more closely into the question why the graph does present such a smooth appearance. An undisturbed harmonic function may be regarded as the solution of the difference equation

$$\Delta^2 (u_x) + \mu (u_{x+1}) = 0. \quad \ldots\ldots\ldots\ldots (7)$$

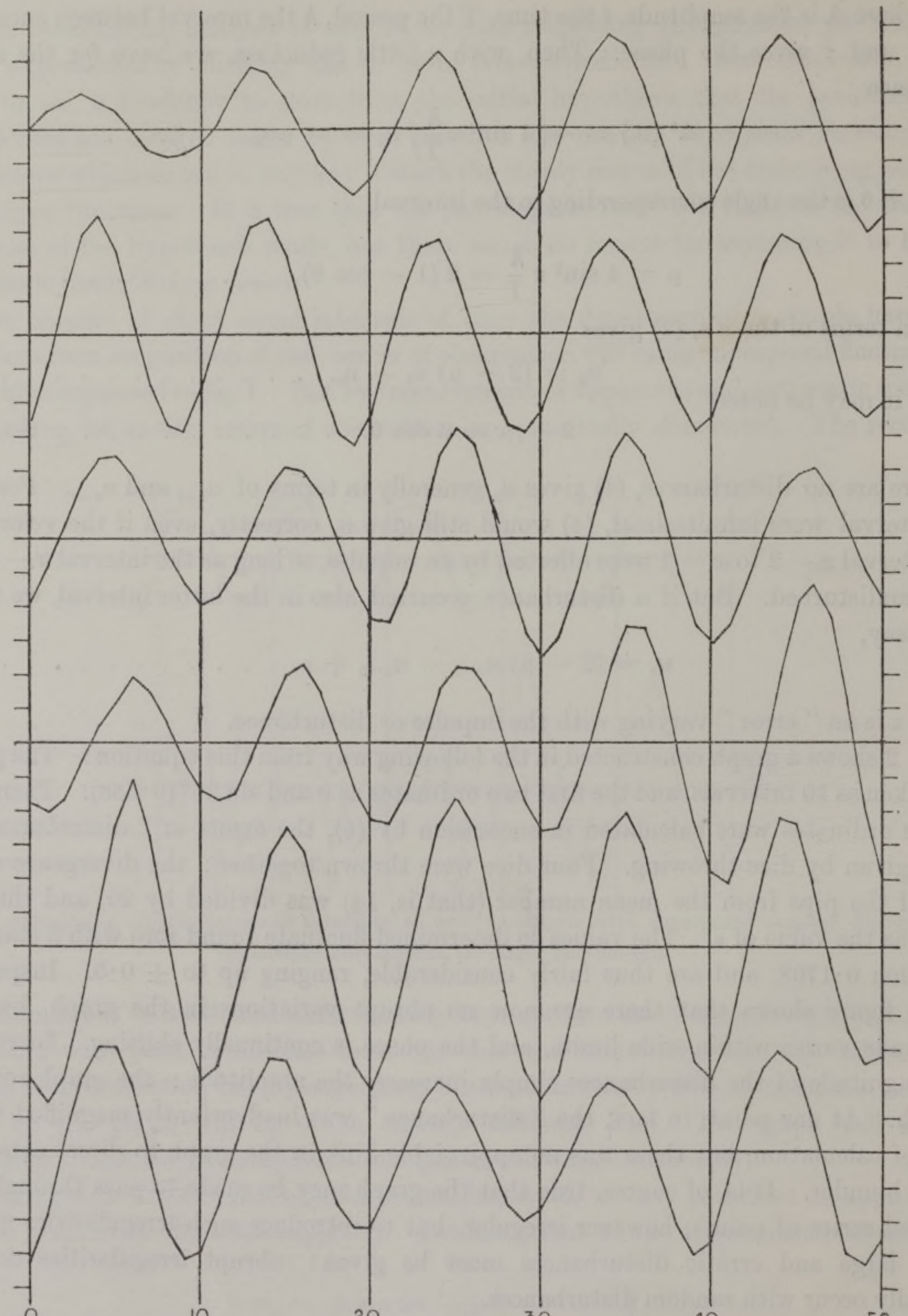

FIG. 2.—Graph of a disturbed harmonic function, experimental series.

If the motion is disturbed, on the other hand, we have, say,

$$\Delta^2 (u_x) + \mu (u_{x+1}) = \phi (t), \quad \ldots \ldots \ldots \ldots \quad (8)$$

where $\phi(t)$ may be regarded as a "disturbance function." Looking at the matter from this standpoint, the simple harmonic function is the complementary function in the solution of (8), and the difference between the simple harmonic function and the oscillatory function of the time represented by the graph is the particular integral.

Consider the formation of the series by equation (6), given initial terms u_0 and u_1 and the disturbances ε_2, ε_3, ε_4, etc., appropriate to u_2 onwards. Writing for brevity

$$2 - \mu = k, \qquad (9)$$

the series will be

0. u_0
1. u_1
2. $ku_1 - u_0 + \varepsilon_2$
3. $(k^2 - 1)\,u_1 - ku_0 + k\varepsilon_2 + \varepsilon_3$
4. $\{k(k^2 - 1) - k\}\,u_1 - (k^2 - 1)\,u_0 + (k^2 - 1)\,\varepsilon_2 + k\varepsilon_3 + \varepsilon_4$
5. $\{k[k(k^2 - 1) - k] - (k^2 - 1)\}\,u_1 - \{k(k^2 - 1) - k\}\,u_0 + \{k(k^2 - 1) - k\}\,\varepsilon_2 + (k^2 - 1)\,\varepsilon_3 + k\varepsilon_4 + \varepsilon_5,$

etc. Examining the ε terms, we see that the coefficients $1, k, k^2 - 1, k(k^2 - 1) - k$, etc., are related by an equation of the form

$$A_m = k\,(A_{m-1}) - A_{m-2}. \qquad (10)$$

But this is simply an equation of the form (4), and the coefficients of the ε's are therefore the terms of a sine series, of the same period as the complementary function, with initial terms 1 and k. For the experimental series they are

$$\begin{array}{r} +\ 1 \\ +\ 1\cdot618034 \\ +\ 1\cdot618034 \\ +\ 1 \\ 0 \\ -\ 1 \\ -\ 1\cdot618034 \end{array}$$

etc. Table I shows the first 30 terms of the experimental series, the complementary function, the particular integral, and the values of ε. Thus, on line 10 of the table we have

$$\begin{aligned} -\ 0\cdot54272 = &+ 1\,(-\ 0\cdot10) \\ &+ 1\cdot618034\,(-\ 0\cdot10) \\ &+ 1\cdot618034\,(+\ 0\cdot20) \\ &+ 1\,(-\ 0\cdot05) \\ &+ 0 \\ &- 1\,(+\ 0\cdot30) \\ &- 1\cdot618034\,(-\ 0\cdot10) \\ &- 1\cdot618034\,(+\ 0\cdot35) \\ &- 1\,(-\ 0\cdot15) \end{aligned}$$

TABLE I.—Analysis of first 30 terms of experimental series used for fig. 2 into complementary function (simple harmonic function) and particular integral (function of the disturbances alone).

Term.	Observed series.	Complementary function.	Particular integral.	Disturbance ε
0	0	0	0	0
1	+ 0·58779	+ 0·58779	0	0
2	+ 0·80106	+ 0·95106	− 0·15	− 0·15
3	+ 1·05835	+ 0·95106	+ 0·10729	+ 0·35
4	+ 0·81139	+ 0·58779	+ 0·22360	− 0·10
5	+ 0·55451	0	+ 0·55451	+ 0·30
6	− 0·01417	− 0·58779	+ 0·57362	− 0·10
7	− 0·62744	− 0·95106	+ 0·32362	− 0·05
8	− 0·80105	− 0·95106	+ 0·15001	+ 0·20
9	− 0·76869	− 0·58779	− 0·18090	− 0·10
10	− 0·54272	0	− 0·54272	− 0·10
11	+ 0·14155	+ 0·58779	− 0·44624	+ 0·25
12	+ 1·12175	+ 0·95106	+ 0·17069	+ 0·35
13	+ 1·87348	+ 0·95106	+ 0·92242	+ 0·20
14	+ 1·65960	+ 0·58779	+ 1·07181	− 0·25
15	+ 0·96181	0	+ 0·96181	+ 0·15
16	− 0·10336	− 0·58779	+ 0·48443	0
17	− 0·92905	− 0·95106	+ 0·02201	+ 0·20
18	− 1·54987	− 0·95106	− 0·59881	− 0·15
19	− 1·92869	− 0·58779	− 1·34090	− 0·35
20	− 1·47082	0	− 1·47082	+ 0·10
21	− 0·50115	+ 0·58779	− 1·08894	− 0·05
22	+ 0·55994	+ 0·95106	− 0·39112	− 0·10
23	+ 1·75715	+ 0·95106	+ 0·80609	+ 0·35
24	+ 2·23319	+ 0·58779	+ 1·64540	− 0·05
25	+ 1·70623	0	+ 1·70623	− 0·15
26	+ 0·37755	− 0·58779	+ 0·96534	− 0·15
27	− 1·09534	− 0·95106	− 0·14428	0
28	− 2·24985	− 0·95106	− 1·29879	− 0·10
29	− 2·69499	− 0·58779	− 2·10720	− 0·15
30	− 1·86074	0	− 1·86074	+ 0·25

The series tends to be oscillatory, since, if we take adjacent terms, most of the periodic coefficients of the ε's are of the same sign, and consequently the adjacent terms are positively correlated; whereas if we take terms, say, 5 places apart, the periodic coefficients of the ε's are of opposite sign, and therefore the terms are negatively correlated. The series also tends to be smooth—*i.e.*, adjacent terms highly correlated—since adjacent terms represent simply differently weighted sums of ε's, all but one of which are the same. [*Added, February* 17.—It may be noted that if, in constructing an empirical series by equation (6), we put $u_0 = u_1 = 0$, there would be no true harmonic component, and the series would reduce to the particular integral alone; but the graph would present to the eye an appearance hardly different from that of fig. 2. The case would correspond to that of a pendulum initially at rest, but started into movement by the disturbances.]

It is evident that the problem of determining with any precision the period of the fundamental undisturbed function from the data of such a graph as fig. 2 is a much more difficult one than that of determining the period when we have only to deal with superposed fluctuations. It is doubtful if any method can give a result that is not subject to an unpleasantly large margin of error if our data are available for no more than, say, 10 or 15 periods. Determining the epochs of minima on the graph of fig. 2 by interpolation, I find periods for individual waves ranging from 8·57 to 11·29 intervals, the true period being 10. The first 15 waves give an average of 10·185, the second 15 an average of 9·850, and it takes the whole 30 to give an average, 10·026, near the truth. The question of the applicability of periodogram analysis to a series of this type is further discussed in Section IV below; but from mere inspection of the graph it is, I think, clear that it must give results subject to a much larger margin of error than is usually supposed—results, consequently, which must be interpreted with the greatest caution, and that if applied to data covering only a few periods it may easily give results which are apparently absurd or highly paradoxical.

Inspection of a graph of WOLFER's annual sunspot numbers, the upper curve in fig. 8 (p. 296) suggests quite definitely to my eye that we have to deal with a graph of the type of fig. 2, not of the type of fig. 1, at least as regards its principal features. It is true that there are minor irregularities, which may represent superposed fluctuations, probably in part of the nature of errors of observation; for the sunspot numbers can only be taken as more or less approximate "index-numbers" to sunspot activity. But in the main the graph is wonderfully smooth, and its departures from true periodicity, which have troubled all previous analysts of the data, are precisely those found in fig. 2—great variation in amplitude and continual changes of phase.

If this interpretation is correct, it seems desirable to break away from the periodogram method: the problem is, in fact, no longer one merely of determining the period, but also of determining the values of ε, the "disturbances," as I term them for short. It is natural, then, to approach the problem from the standpoint of the equation relating u_x to u_{x-1}, u_{x-2}, etc. Starting also, as I did, with the conception of periodogram analysis and harmonic periodicities in my mind, it was natural to assume an equation of the form (6), and an equation of corresponding form for two periodicities. This gave the first method tried.

It only occurred to me later that the method started from an unnecessarily limited assumption; that it would be better simply to find the linear regression equation of u_x on u_{x-1}, u_{x-2}, and more terms if necessary, and solve this as a finite difference equation. This gave my second method. As the results of the first method were interesting, I give both methods below.

On doing the work I was puzzled by the fact that the equation first found suggested a period obviously too short. A little consideration suggested that this was probably due to the presence of superposed fluctuations: as already noted, the graph of sunspot numbers suggests the presence of minor irregularities due to this cause, notwithstanding that a certain amount of smoothing has already been introduced by employing the annual average and not the monthly numbers. I therefore desired to repeat the work on graduated figures, assuming that graduation would largely eliminate such irregularities, and used the following method. The u's were first summed in overlapping sets of three, thus :—

$$\left.\begin{aligned} w_1 &= u_0 + u_1 + u_2 \\ w_2 &= u_1 + u_2 + u_3 \\ w_3 &= u_2 + u_3 + u_4 \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots (11)$$

Here $w_2/3$ is evidently a first approximation to a graduated value of u_2. As the second approximation was taken the corrected value

$$\bar{u}_2 = \frac{w_2}{3} - \frac{1}{9}\Delta^2(w_1). \quad \ldots \ldots \ldots \ldots (12)$$

The results were not very good, as will be seen from the graph, the second curve in fig. 8 (p. 296), and from a discussion below in Section III, p. 282. A better result could probably have been obtained by graphic smoothing on a large-scale chart; but a "mechanical" method of graduation gave directly figures for calculation, and they served the purpose of comparison. The graduated figures, together with WOLFER's numbers,* are given in Table A at the end of the paper. A test of the graduation gives for the mean difference (actual number less graduated number) $-0 \cdot 04$; standard deviation of differences, $6 \cdot 04$. Over 90 per cent. of the differences lie within $\pm$ 10 points.

II. First Method of Analysis: Harmonic Curve Equation.

A.—*Assumption of a Single Period only.*—In equation (6) the average value of ε is assumed to be zero. Hence, if we form an equation

$$u_x = ku_{x-1} - u_{x-2}, \quad \ldots \ldots \ldots \ldots (13)$$

and determine k by the method of least squares, we have

$$k = 2 - \mu = 2\cos\theta \quad \ldots \ldots \ldots \ldots (14)$$

by (5). But

$$S\,(\overline{u_x - ku_{x-1} - u_{x-2}})^2 = S\,(\overline{u_x + u_{x-2} - ku_{x-1}})^2,$$

and hence we can most readily determine k by finding the correlation between $u_x + u_{x-2}$ and u_{x-1}, and forming the regression equation of the former on the latter. In the case of the sunspot numbers we would have to work, of course, with the deviations of the u's from the general mean, and subsequently transform to zero as origin, so that there would be a constant on the right of (13).

Since I could see no valid method of determining probable errors in cases of the present

* 'Terrestrial Magnetism and Atmospheric Electricity,' Baltimore (June, 1925).

kind, where we are not dealing with random samples in the ordinary sense but with samples from series all the terms of which are highly correlated with one another,* a practical test of the method on the data of fig. 2 seemed to be of interest. The series of 300 terms was divided into two series of 150 terms each, which gave the following results—accents to the u's denote deviations :—

Empirical Series.

First 150 terms :

$$u'_x = 1\cdot62438\, u'_{x-1} - u'_{x-2},$$

$$\cos\theta = 0\cdot81219\ ;\ \theta = 35^\circ\cdot69\ ;\ \text{period} = 10\cdot087.$$

Second 150 terms :

$$u'_x = 1\cdot60636\, u'_{x-1} - u'_{x-2}$$

$$\cos\theta = 0\cdot80318\ ;\ \theta = 36^\circ\cdot56\ ;\ \text{period} = 9\cdot845.$$

The periods thus found are not far from those obtained from the interpolated minima (p. 273), viz., 10·185 and 9·850 : the coefficient of u'_{x-1} for a period 10 should be 1·61803. The respective equations give values of the disturbances ε having correlations + 0·997 and + 0·987 with the true disturbances, and even in the latter case give quite a fair picture of the true state of affairs. On the whole, I think the result may be regarded as reasonably satisfactory.

Turning now to Wolfer's sunspot numbers, the series was used as a whole (1749–1924) : the deviations of the individual numbers from the general mean were written down to the nearest unit, and the correlation worked without further grouping. The results were :—

Wolfer's *Sunspot Numbers*, 1749–1924 :

s.d. of whole series = 34·66 points

$$u_x = 1\cdot62373\, u_{x-1} - u_{x-2} + 16\cdot99 \quad \ldots\ldots\ldots\ldots \quad (15)$$

$$\cos\theta = 0\cdot81187\ ;\ \theta = 35^\circ\cdot72\ ;\ \text{period} = 10\cdot08 \text{ years}.$$

s.d. of disturbances = 17·05 points.

As, in view of the subsequent work, I judge that the disturbances calculated from (15) have no special importance, it has not been thought worth while to tabulate them, but they are shown in the third graph in fig. 8 : the graph is to double the scale of the graphs of sunspot numbers, and the line drawn gives quinquennial averages. It will be seen that the disturbances are very variable, running up to over ± 50 points. But the course of affairs is rather curious. From 1751 to 1792, or thereabouts, the disturbances are mainly positive and highly erratic ; from 1793 to 1834 or thereabouts, when the

* *Cf.* the general discussion of the nature of time-series in "Why do we sometimes get nonsense correlations between time-series? A study in sampling and the nature of time-series." Presidential Address, G. U. Yule, 'Journ. Stat. Soc.,' vol. 89 (1926).

sunspot curve was depressed, they are mainly negative and very much less scattered; from 1835 to 1875, or thereabouts, they are again mainly positive and highly erratic; and finally, from 1876 to 1915, or thereabouts, once more mainly negative and much less erratic. It looks as if the "disturbance function" had itself a period of somewhere about 80 to 84 years, alternate intervals of 40 to 42 years being highly disturbed and relatively quiet. This characteristic appears in whatever way the disturbances are calculated, whether from the graduated or ungraduated numbers, and is returned to below (p. 283).

But it is evident that the period, 10·08 years, given by equation (15) is markedly too low: it ought, one would expect, to be in fair agreement with the usual estimate of rather over 11 years—11·125 years (SCHUSTER*) or 11·21 years (LARMOR and YAMAGA†). As already mentioned, the divergence might be due to the presence of superposed fluctuations. If such fluctuations are present, our two variables $u_x + u_{x-2}$ and u_{x-1} are, as it were, affected by errors of observation, which would have the effect of reducing the correlation and also the regression. Reducing the regression means reducing the value of cos θ—that is, increasing θ or reducing the apparent period. It was therefore attempted to eliminate superposed fluctuations by graduating the numbers, using the method already described (p. 274), and doing the work again on these graduated figures, which will be found in Table A at the end of the paper. The results were as follows:—

Graduated Sunspot Numbers, 1753–1920:

s.d. of whole series = 33·75 points.

$$u_x = 1\cdot 68426\, u_{x-1} - u_{x-2} + 14\cdot 13 \quad \ldots\ldots\ldots\ldots \quad (16)$$

$$\cos\theta = 0\cdot 84213\,;\ \theta = 32^\circ\cdot 63\,;\ \text{period} = 11\cdot 03 \text{ years.}$$

s.d. of disturbances = 11·43 points.

The estimate of the period is now much closer to that usually given, and I think it may be concluded that the reason assigned for the low value obtained from the ungraduated numbers is correct.

For lack of space on the plate a graph has not been given of the disturbances as calculated from (16). The scatter is greatly reduced (s.d. of disturbances 11·43 against 17·05), but the general course of affairs is very similar to that shown by the graph for the ungraduated numbers.

The graphic test was applied to see whether the regression of $u_x + u_{x-2}$ on u_{x-1} was, in fact, appreciably linear or no. Figs. 3 and 4 show dot-diagrams for the ungraduated and the graduated numbers respectively. It will be seen that over the greater part of the range the regression is effectively linear. But for the larger negative deviations (low values of the sunspot numbers) there is an appreciable, though small, divergence

* 'Phil. Trans.,' A, vol. 206, pp. 69–100 (1906).

† 'Roy. Soc. Proc.,' A, vol. 93, pp. 493–506 (1917).

from linearity affecting some 10 per cent. of the points. On the whole, however, divergence from linearity does not look as if it would be a serious trouble.

If we ask ourselves the question how much of the variance of u_x has been accounted for by u_{x-1} and u_{x-2}, the answer is not afforded directly in the usual way by the

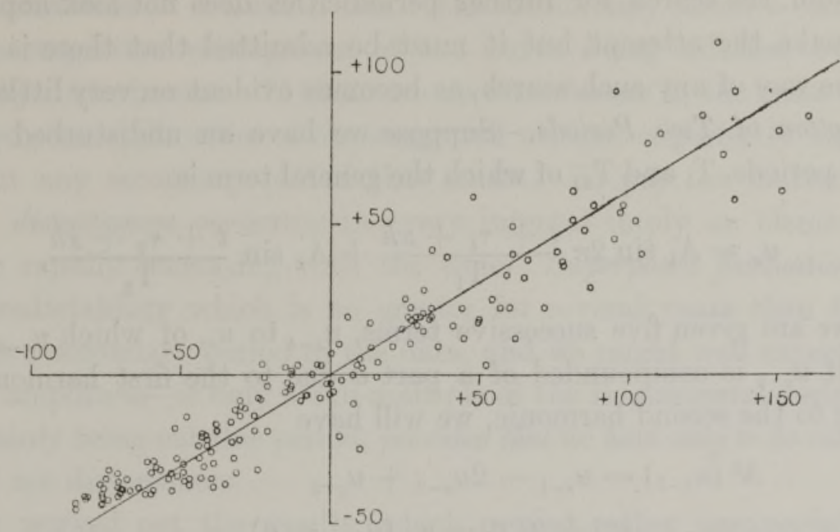

FIG. 3.—Graph to test approximation to linearity of regression of $u'_x + u'_{x-2}$ (horizontal) on u'_{x-1} (vertical) : WOLFER's sunspot numbers.

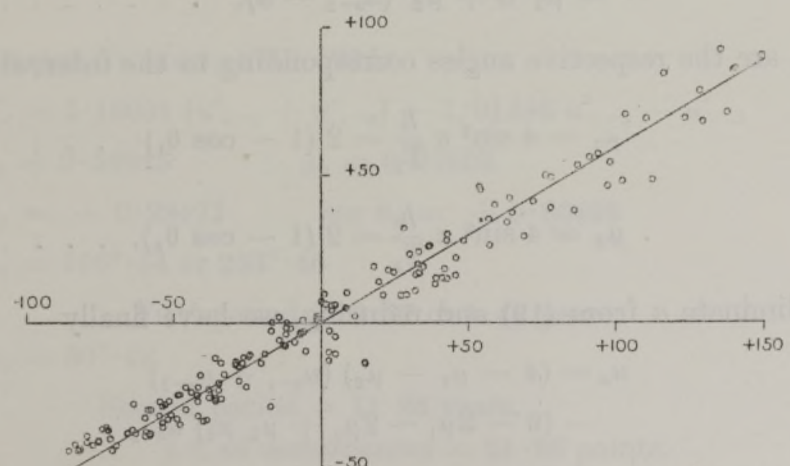

FIG. 4.—Graph to test approximation to linearity of regression of $u'_x + u'_{x-2}$ (horizontal) on u'_{x-1} (vertical) : graduated sunspot numbers.

correlation calculated, which is the correlation between $u_x + u_{x-2}$ and u_{x-1}, not the correlation between the right and left-hand sides of (13). We must take the standard deviations given. We have :—

WOLFER's numbers.		Graduated numbers.	
$34 \cdot 66^2$	$1201 \cdot 3$	$33 \cdot 75^2$	$1139 \cdot 1$
$17 \cdot 05^2$	$290 \cdot 7$	$11 \cdot 43^2$	$130 \cdot 6$
	$910 \cdot 6$		$1008 \cdot 5$

and have therefore accounted for some 76 per cent. of the variance in the case of WOLFER'S numbers and some 89 per cent. in the case of the graduated figures. On the present lines we cannot account for superposed fluctuations, and, bearing in mind the look of the chart and its suggestion to the eye that a great part of the variation of the disturbances is merely random, the search for further periodicities does not look hopeful. It seems desirable to make the attempt, but it must be admitted that there is a very serious difficulty in the way of any such search, as becomes evident on very little consideration.

B.—*Assumption of Two Periods.*—Suppose we have an undisturbed periodic series involving two periods, T_1 and T_2, of which the general term is

$$u_x = A_1 \sin 2\pi \frac{t + \tau_1 + xh}{T_1} + A_2 \sin \frac{t + \tau_2 + xh}{T_2}, \quad \ldots \ldots (17)$$

and suppose we are given five successive terms, u_{x-4} to u_x, of which u_{x-2} is the central term. Then if u_{x-2} is compounded of a part a due to the first harmonic and a part $(u_{x-2} - a)$ due to the second harmonic, we will have

$$\begin{aligned} \Delta^2 (u_{x-3}) &= u_{x-1} - 2u_{x-2} + u_{x-3} \\ &= - \mu_1 a - \mu_2 (u_{x-2} - a) \quad \ldots \ldots \ldots \ldots (18) \\ \Delta^4 (u_{x-4}) &= u_x - 4u_{x-1} + 6u_{x-2} - 4u_{x-3} + u_{x-4} \\ &= \mu_1^2 a + \mu_2^2 (u_{x-2} - a), \quad \ldots \ldots \ldots \ldots (19) \end{aligned}$$

where, if θ_1, θ_2 are the respective angles corresponding to the interval h,

$$\mu_1 = 4 \sin^2 \pi \frac{h}{T_1} = 2 (1 - \cos \theta_1) \quad \ldots \ldots \ldots (20)$$

$$\mu_2 = 4 \sin^2 \pi \frac{h}{T_2} = 2 (1 - \cos \theta_2). \quad \ldots \ldots \ldots (21)$$

Using (18) to eliminate a from (19) and reducing, we have finally

$$\begin{aligned} u_x = &(4 - \mu_1 - \mu_2) (u_{x-1} + u_{x-3}) \\ &- (6 - 2\mu_1 - 2\mu_2 + \mu_1 \mu_2) u_{x-2} \\ &- u_{x-4}. \quad \ldots \ldots \ldots \ldots \ldots \ldots (22) \end{aligned}$$

If the series is completely undisturbed, (22) can be used to calculate in succession all the following terms, when the first four are given.

But now comes the problem. What happens if the series is "disturbed" in the sense of the previous work? Can we, exactly as before, assume

$$u_x = k_1 (u_{x-1} + u_{x-3}) - k_2 u_{x-2} - u_{x-4} + \varepsilon, \quad \ldots \ldots \ldots (23)$$

where ε is a deviation, of the nature of an error of observation, varying with the disturbance? It seems clear that we cannot legitimately make any such assumption. It was going far enough to treat the single interval as if it were infinitesimal: we cannot possibly stretch the assumption to cover three intervals. If, admitting this, we nevertheless assume a relation of the form (23), and proceed to determine k_1 and k_2 by the method of least squares, regarding $u_x + u_{x-4}$, $u_{x-1} + u_{x-3}$, and u_{x-2} as our three variables, and forming the regression equation for the first on the last two, can this give us any useful information ? I think it can. The results may afford a certain criterion as between the respective conceptions of the curve being affected by superposed fluctuations or by disturbances. If there are no *disturbances* in the sense in which the term is here used, the application of the suggested method is perfectly legitimate, and should bring out any secondary period that exists. To put the matter in a rather different way : *disturbances* occurring in every interval imply an element of unpredictability very rapidly increasing with the time. *Superposed fluctuations* imply an element of unpredictability which is no greater for several years than for one year. If, then, there is a secondary period in the data, and we might well expect a period of relatively small amplitude—if only a sub-multiple of the fundamental period—equation (23) should certainly bring out this period, *provided that we have only to do with superposed fluctuations and not disturbances.*

I accordingly worked out the results, which proved rather unexpected. I simply give the equations and the resulting values of the μ's and θ's. Accented u's denote deviations as before : it was not worth while working out the constant term in view of the results.

WOLFER'S *Sunspot Numbers*, 1749–1924 :

$$u'_x = 1\cdot16051\,(u'_{x-1} + u'_{x-3}) - 1\cdot01486\,u'_{x-2} - u'_{x-4} \quad \ldots\ldots \quad (24)$$

$$\mu_1 = 2\cdot56945 \qquad \mu_2 = 0\cdot27003$$

$$\cos\theta_1 = -\,0\cdot28473 \qquad \cos\theta_2 = +\,0\cdot86498$$

$$\theta_1 = 106^\circ\cdot54 \text{ or } 253^\circ\cdot46$$

First period $= 3\cdot37$ or $1\cdot42$ years.

$$\theta_2 = 30^\circ\cdot12$$

Second period $= 11\cdot95$ years.

s.d. of disturbances $= 21\cdot95$ points.

Graduated Numbers, 1753–1920 :

$$u'_x = 1\cdot65539\,(u'_{x-1} + u'_{x-3}) - 1\cdot83955\,u'_{x-2} - u'_{x-4} \quad \ldots\ldots \quad (25)$$

$$\mu_1 = 2\cdot09183 \qquad \mu_2 = 0\cdot25278$$

$$\cos\theta_1 = -\,0\cdot04592 \qquad \cos\theta_2 = +\,0\cdot87361$$

$$\theta_1 = 92^\circ\cdot63 \text{ or } 267^\circ\cdot37$$

First period $= 3\cdot89$ or $1\cdot35$ years.

$$\theta_2 = 29^\circ\cdot12$$

Second period $= 12\cdot36$ years.

s.d. of disturbances $= 17\cdot47$ points.

Since the values of the μ's give cos θ and not θ itself, the value of θ is not strictly determinate: the longer period is naturally taken as approximate to the fundamental, but the choice of the shorter period is quite uncertain. So far as the results go then, they at first sight suggest the existence of two periods, one a year or more longer than the value which anyone, on a mere inspection of the graph, would be inclined to take for the fundamental, and the other much shorter. On the face of it the result looks odd, and the last figures given for the ungraduated and graduated numbers respectively show that it is really of no meaning. *The standard deviations found for the disturbances are in both cases larger than when we assumed the existence of a single period only:* 21·95 *against* 17·05, *and* 17·47 *against* 11·43. So far from having improved matters by the assumption of a second period, we have made them very appreciably worse: we get a worse and not a better estimate of u_x when u_{x-3} and u_{x-4} are brought into account than when we confine ourselves to u_{x-1} and u_{x-2} alone. To put it moderately, there is at least no evidence that any secondary period exists—a conclusion in entire accord with that of LARMOR and YAMAGA (*loc. cit.*). The result also bears out the assumption that it is disturbances rather than superposed fluctuations which are the main cause of the irregularity, the element of unpredictability, in the data.

The fact that we get a worse and not a better estimate, although that estimate is based on a larger number of variables, will naturally seem paradoxical to those who are accustomed to the ordinary theory of correlation. It is simply due to the fact that we have insisted on the regression equation being of a particular form, the coefficients of u_{x-1} and u_{x-3} being identical, and the coefficient of u_{x-4} unity. The result tells us merely that, if we insist on this, such and such values of the coefficients are the best, but even so they cannot give as good a result as the equation of form (13) with only two terms on the right.

III. SECOND METHOD OF ANALYSIS: REGRESSION EQUATION.

A.—*Assumption of a Single Period only.*—We form the ordinary regression equation for u_x on u_{x-1} and u_{x-2}:

$$u_x = b_1 u_{x-1} - b_2 u_{x-2}. \qquad (26)$$

With a curve that fluctuates round zero as base-line there will be no constant term on the right; with the sunspot numbers there must be a constant as before, but this is immaterial. If the curve is of periodic form, the roots of the equation*

$$E^2 - b_1 E + b_2 = 0 \qquad (27)$$

must be imaginary. Let the roots be

$$\alpha \pm i\beta,$$

* I follow BOOLE, 'Finite Differences,' 2nd edition, chap. xi, pp. 208–212.

and let
$$\alpha^2 + \beta^2 = b_2 = e^{2\lambda} \quad \ldots \ldots \ldots \ldots \quad (28)$$
$$\tan \theta = \beta/\alpha. \quad \ldots \ldots \ldots \ldots \quad (29)$$

Then the general solution of the difference equation (26) is of the form

$$u_x = e^{\lambda x}(\mathrm{A} \cos \theta x + \mathrm{B} \sin \theta x). \quad \ldots \ldots \ldots \quad (30)$$

Here θ is, as before, the angle corresponding to the interval h. For a real physical phenomenon one would in general expect λ to be negative, the solution representing damped harmonic vibrations; or zero, the solution being simple harmonic vibrations. The condition for the latter solution is that b_2 shall be unity.

This method also was tested on the empirical data of fig. 2, using two series of 150 terms each as before. The correlation between u_{x-1} and u_{x-2} was assumed to be the same as that between u_x and u_{x-1}, both representing correlations between adjacent terms: we cannot get correlations between terms one apart, two apart, etc., that involve precisely the same terms, and results are, in so far, approximate; but the closeness of approximation will be the greater the longer the series.*

Empirical Series.

First 150 terms.

$$u_x = 1\cdot6117\, u_{x-1} - 0\cdot9867\, u_{x-2}.$$

Roots of (27): $0\cdot80585 \pm 0\cdot58078\, i$.

$\tan \theta = 0\cdot72070$: $\theta = 35°\cdot78$: Period $= 10\cdot06$: $\lambda = -0\cdot0067$.

Second 150 terms.

$$u_x = 1\cdot5975\, u_{x-1} - 0\cdot9875\, u_{x-2}$$

Roots of (27): $0\cdot79875 \pm 0\cdot59119\, i$.

$\tan \theta = 0\cdot74014$: $\theta = 36°\cdot51$: Period $= 9\cdot86$: $\lambda = -0\cdot0063$.

The values found for the period, $10\cdot06$ and $9\cdot86$, are close to the values given by the harmonic curve equation, viz., $10\cdot087$ and $9\cdot845$. The values found for λ, which should be zero, are, in fact, numerically less than $0\cdot01$ in each case. The agreement seems quite satisfactory.

Proceeding now to the work on the sunspot numbers, the following are the results:—

WOLFER'S *Sunspot Numbers*, 1749–1924:

$$u_x = 1\cdot34254\, u_{x-1} - 0\cdot65504\, u_{x-2} + 13\cdot854. \quad \ldots \ldots \quad (31)$$

Roots of (27): $0\cdot67127 \pm 0\cdot45215\, i$.

$\tan \theta = 0\cdot67358$: $\theta = 33°\cdot963$: Period $= 10\cdot600$ years:

$\lambda = -0\cdot21154$:

s.d. of disturbances $= 15\cdot41$ points.

* The correlations required are the first two *serial correlations*, as I have termed them. *Cf. Address*, already cited, 'Journ. Stat. Soc.,' vol. 89 (1926).

Graduated Sunspot Numbers, 1753–1920:

$$u_x = 1\cdot51527\, u_{x-1} - 0\cdot80245\, u_{x-2} + 12\cdot854. \quad \ldots\ldots \quad (32)$$

Roots of (27): $0\cdot75764 \pm 0\cdot47795\, i$.

$\tan\theta = 0\cdot63085$: $\theta = 32°\cdot246$: Period $= 11\cdot164$ years.

$\lambda = -0\cdot11004$:

s.d. of disturbances $= 10\cdot79$ points.

The magnitudes of the disturbances as calculated from equations (31) and (32) respectively are given in Table A at the end of the paper: the regressions were cut down to three decimal places for these calculations. The disturbances are also shown, together with the quinquennial averages as indicated by the lines, in the fourth and fifth graphs in fig. 8 (p. 296).

The period derived from WOLFER's numbers is now higher than that given by the harmonic formula ($10\cdot60$ against $10\cdot08$), but still too low; that derived from the graduated data is also a little higher ($11\cdot16$ against $11\cdot03$), and now lies between the values suggested by SCHUSTER, and by LARMOR and YAMAGA, respectively.

But the solution of both equations is a *heavily damped* and not a simple harmonic movement. The damping given by the graduated data is, however, only about half that given by WOLFER's numbers: (31) gives a vibration reduced to $0\cdot106$ of the original amplitude in the duration of a period, (32) a vibration reduced to $0\cdot293$ only. This is at first sight a very puzzling result, and precisely the reverse of what was to be expected by the elimination or reduction of superposed fluctuations. For let x_1, x_2, x_3 be three variables with the same standard deviation σ, and let the correlations between x_1 and x_2 and between x_2 and x_3 be r_1, and the correlation between x_1 and x_3 be r_2. Then, with a little reduction, we have for the partial regressions in the usual notation,

$$\left.\begin{aligned} b_{12\cdot3} &= \frac{r_1(1-r_2)}{1-r_1^2} \\ b_{13\cdot2} &= \frac{r_2 - r_1^2}{1-r_1^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (33)$$

Now suppose all three variables to have random errors of the same standard deviation—errors completely uncorrelated with each other—superposed on them. Then both correlations will be reduced in the same proportion and become, say, pr_1 and pr_2. Whence, for the partial regressions in this case, we have

$$\left.\begin{aligned} b'_{12\cdot3} &= \frac{pr_1(1-pr_2)}{1-p^2r_1^2} \\ b'_{13\cdot2} &= \frac{pr_2 - p^2r_1^2}{1-p^2r_1^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (34)$$

The respective ratios of the second coefficient to the first are :

$$\left.\begin{aligned} \frac{b_{13\cdot 2}}{b_{12\cdot 3}} &= \frac{r_2 - r_1^2}{r_1(1 - r_2)} \\ \frac{b'_{13\cdot 2}}{b'_{12\cdot 3}} &= \frac{r_2 - pr_1^2}{r_1(1 - pr_2)} \end{aligned}\right\} \quad \text{. . . . . (35)}$$

The condition that the second ratio shall be greater than the first (r_2 not equal to r_1) reduces simply to $p < 1$, which is necessarily true. That is to say, where superposed random errors occur, we would expect the ratio of the second partial regression to the first to be greater than when such errors are eliminated or reduced. We have found precisely the contrary, for the ratio is greater for the graduated than for the ungraduated numbers.

An examination of the chart and of the figures suggests that the explanation may lie in an unexpected and unintended effect of the graduation. The occurrence of a damping term in the solution of the empirical finite difference equation may conceivably be due to an attempt of the equation to represent the asymmetry of the waves in the sunspot curve, which is a marked feature of the sunspot curve in waves of large amplitude. A careful inspection of the graphs and of the figures of Table A suggests that the graduation has tended to lessen this asymmetry, owing presumably to second differences only having been taken into account. As definite features in Table A, it may be noted that graduation has pushed forward the maximum from 1769 to 1770, has greatly lessened the difference between the ordinates at 1778 and 1779, and has advanced the maximum again from 1787 to 1788, from 1870 to 1871, from 1905 to 1906, and from 1917 to 1918. If we take two damped sine curves with the above respective values of λ, but, for fair comparison, the same period—say, 11·164 years—the first with the greater damping factor would have its first maximum at 2·15 years, the second with the lower damping factor at 2·44 years—*i.e.*, the maximum would be advanced by roundly 0·3 of a year. The graduation seems to have had, unintentionally, the effect of producing an average advance of this order of magnitude, and therefore of reducing the apparent damping.

The question whether in fact the damping factor represents a physical reality or merely an attempt of the empirical formula to adjust itself to the asymmetry of the waves is for the present postponed. In the first place, a more detailed examination of the disturbances is desirable.

First let us examine more closely the apparent alternation of disturbed and quiet periods, each some 40 to 42 years in duration, which was already noted in the graph of the disturbances as calculated from the harmonic formula and is equally evident in the present graphs. It is not possible to assign the beginnings and ends of such periods with precision; and, as it happens, doing the work independently, I did not take precisely the same years for WOLFER's and the graduated numbers. There is also a difficulty at the commencement of the data. As I judge it, the magnitude of the disturbance in 1751 indicates this year as within a disturbed period, and it is the first year for which we can calculate a disturbance, so that it must be taken as the beginning of the period for WOLFER's numbers. But 1753 is the first year for which we can calculate a disturbance for the graduated numbers, and this must be taken as the opening of the period. Table II gives the mean values of the disturbances for the periods finally adopted, and also the standard deviations. It will be seen that they completely confirm the impression given by the graph. Alternate periods give positive and negative mean values* of the disturbance: the periods with positive mean give a high value of the standard deviation, the periods with negative mean a low value of the standard deviation. At

TABLE II.—Means and Standard Deviations of disturbances in disturbed and quiet periods of 40 to 42 years. Disturbances from Table A at the end of the paper.

Period.	Number of years.	Mean disturbance.	Standard deviation.
WOLFER's numbers.			
1751–1792	42	+ 6·74	17·80
1793–1834	42	− 5·80	7·32
1835–1875	41	+ 4·43	17·85
1876–1915	40	− 2·61	10·59
Graduated numbers.			
1753–1793	41	+ 4·41	11·62
1794–1834	41	− 5·42	5·30
1835–1875	41	+ 3·95	12·49
1876–1915	40	− 2·95	6·61

the same time the last quiet period, taken as 1876 to 1915 inclusive, was more disturbed than the very conspicuously quiet period from 1793 to 1834 or thereabouts. While a much longer experience will be necessary to confirm the result, this alternation seems a rather conspicuous feature of the existing data.

Further inspection of the graphs suggests another feature which is at least not obvious in the first graph for the disturbances calculated from the harmonic curve formula. In the lines showing quinquennial averages, on the two lower graphs, there are distinct "humps" more or less consilient with the waves in the sunspot graph, but a little earlier

* It may be noted that positive or negative disturbances as calculated from (31) or (32) are what is meant. But the constant term in each of these equations may be understood as a steady positive "disturbance," and if added to the disturbances of Table A, from which the graphs are plotted, would render positive the bulk of the negative disturbances there given.

in phase. Examination of the figures of Table A suggests, in fact, that positive disturbances tend to begin at or just after the minimum, and continue till the maximum or a year or two before, disturbances from the maximum to the minimum, or a year or two after the minimum, being preponderantly negative. Preceding the maximum there is often a group of two or more large positive disturbances. Table III gives a summary

TABLE III.—Sums and means of disturbances in rising and preponderantly falling parts of the graph of sunspot numbers: "rising" implying from the minimum or a year beyond to the maximum or a year or two before.

WOLFER's numbers.				Graduated numbers.			
Years.	Rise + or fall —	Sum of disturbances.	Mean.	Years.	Rise + or fall —	Sum of disturbances.	Mean.
1751–1756	—	— 37·5	— 6·2	1753–1755	—	— 10·8	— 3·6
1757–1761	+	+ 36·0	+ 7·2	1756–1760	+	+ 13·2	+ 2·6
1762–1766	—	— 38·4	— 7·7	1761–1765	—	— 16·3	— 3·3
1767–1769	+	+ 58·2	+ 19·4	1766–1768	+	+ 36·9	+ 12·3
1770–1775	—	— 25·3	— 4·2	1769–1774	—	— 7·6	— 1·3
1776–1778	+	+102·5	+ 34·2	1775–1778	+	+ 71·4	+ 17·8
1779–1784	—	— 27·5	— 4·6	1779–1784	—	— 4·8	— 0·8
1785–1787	+	+ 77·5	+ 25·8	1785–1787	+	+ 59·5	+ 19·8
1788–1799	—	— 10·7	— 0·9	1788–1799	—	— 0·2	— 0·0
1800–1802	+	— 5·7	— 1·9	1800–1802	+	— 9·0	— 3·0
1803–1810	—	— 64·5	— 8·1	1803–1810	—	— 56·7	— 7·1
1811–1816	+	— 37·7	— 6·3	1811–1816	+	— 36·9	— 6·1
1817–1823	—	— 68·2	— 9·7	1817–1823	—	— 58·9	— 8·4
1824–1830	+	+ 6·9	+ 1·0	1824–1829	+	— 6·4	— 1·1
1831–1833	—	— 32·3	— 10·8	1830–1833	—	— 23·4	— 5·8
1834–1836	+	+ 76·7	+ 25·6	1834–1836	+	+ 61·7	+ 20·6
1837–1843	—	— 7·7	— 1·1	1837–1843	—	+ 3·5	+ 0·5
1844–1848	+	+ 66·0	+ 13·2	1844–1848	+	+ 39·6	+ 7·9
1849–1856	—	— 23·5	— 2·9	1849–1855	—	— 1·4	— 0·2
1857–1859	+	+ 42·0	+ 14·0	1856–1859	+	+ 25·6	+ 6·4
1860–1867	—	— 27·5	— 3·4	1860–1866	—	— 15·9	— 2·3
1868–1870	+	+ 89·4	+ 29·8	1867–1871	+	+ 62·0	+ 12·4
1871–1879	—	— 53·5	— 5·9	1872–1879	—	— 24·7	— 3·1
1880–1884	+	+ 16·1	+ 3·2	1880–1884	+	+ 3·1	+ 0·6
1885–1890	—	— 49·8	— 8·3	1885–1889	—	— 32·7	— 6·5
1891–1892	+	+ 32·3	+ 16·1	1890–1892	+	+ 17·8	+ 5·9
1893–1901	—	— 48·2	— 5·4	1893–1902	—	— 42·9	— 4·3
1902–1907	+	+ 6·9	+ 1·1	1903–1908	+	+ 0·5	+ 0·1
1908–1913	—	— 54·0	— 9·0	1909–1913	—	— 41·4	— 8·3
1914–1917	+	+ 48·1	+ 12·0	1914–1918	+	+ 20·9	+ 4·2
1918–1923	—	— 51·4	— 8·6	1919–1922	—	— 19·6	— 4·9

of the disturbances over such alternate rising or preponderantly falling periods. Owing to the small shift noted as a secondary effect of the graduation, the years taken are not quite the same for WOLFER's numbers and for the graduated numbers. In both cases the tendency to alternation in sign is clear, though the closer for the ungraduated data. During the earlier part of the " very quiet " period that ended with 1834, all the sums of disturbances are negative, whether for a rising or falling part of the graph, but the sums tend to be higher during a fall than during a rise. During the rather abnormal long fall from 1788 to 1799 there is also an irregularity. The numbers seem to have been maintained during these years by a succession of positive disturbances up to 1792 or 1793, and the negative disturbances of the following years only just overbalance these and leave a small negative total.

This distribution of the disturbances seems to me to have some bearing on the question whether we may perhaps tentatively regard the damped harmonic formula at which we have empirically arrived as being something more than merely empirical, and representing some physical reality. As it seems to me, the disturbances do occur just in the kind of way that would be necessary to maintain a damped vibration, and this suggests that broadly the conception fits the facts.*

Clearly, however, a simple damped vibration, varying round zero, is not quite what is wanted. One would rather expect a function of the form of the square of a damped harmonic vibration, say,

$$y = Ae^{-at}(1 - \cos \theta t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The form of this function is shown in fig. 5, and it would look as though a train of such functions superposed on each other would give a graph not unlike that of the sunspot numbers (*cf.* below, fig. 6). But the difference equation of this function is of the third order, and would therefore have to be extended to include u_{x-3}. This raises again the serious theoretical difficulties briefly mentioned in Section II, B, p. 278. Even if the difference equation is in fact of the form supposed, it is doubtful if it can be determined. The question does render it necessary, however, to examine the correlations of u_x with u_{x-3} and more distant terms, and see what information they give us.

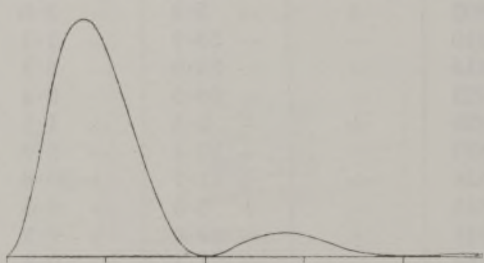

FIG. 5.—Graph of the function equation (36).

B.—*The Correlations of u_x with u_{x-3} and more Distant Terms, and the Information given thereby.*—On the left of Table IV are given the *serial correlations*, as I have termed them, for WOLFER's sunspot numbers and the graduated numbers respectively: r_1 is the correlation between u_x and u_{x-1}, r_2 the correlation between u_x and u_{x-2}, and so on. From these all the partial correlations are calculated on the assumption that the series is indefinitely long, so that we may assume that the correlation between u_{x-1} and u_{x-2} is the same as that between u_x and u_{x-1}, and so forth—an assumption which implies

* But I fail to find any relation between the disturbances and TURNER's dates of discontinuity of phase (' Monthly Notices,' R.A.S., vol. 74, p. 82).

corresponding equalities between partial correlations. The serial correlations are given as far as r_5, which is the negative maximum, r_6 being slightly smaller numerically.

TABLE IV.—Serial correlations for WOLFER's and the graduated sunspot numbers, and the deduced partial correlations, etc. In the serial correlations, 1 denotes the correlation between u_x and u_{x-1}, 2 the correlation between u_x and u_{x-2}, and so on. In the partial correlations 13·2 denotes the correlation between u_x and u_{x-2}, u_{x-1} constant, and so on.

Serial correlations.		Partial correlations.		$1 - r^2$.	Continued product of $1 - r^2$.
WOLFER's sunspot numbers.					
1	+ 0·811180	12	+ 0·811180	0·341987	0·341987
2	+ 0·433998	13·2	− 0·655040	0·570923	0·195248
3	+ 0·031574	14·23	− 0·101043	0·989790	0·193255
4	− 0·264463	15·234	+ 0·013531	0·999817	0·193219
5	− 0·404119	16·2345	− 0·050001	0·997500	0·192736
Graduated sunspot numbers.					
1	+ 0·840670	12	+ 0·840670	0·293274	0·293274
2	+ 0·471388	13·2	− 0·802451	0·356072	0·104427
3	+ 0·047038	14·23	+ 0·037840	0·998568	0·104277
4	− 0·264147	15·234	+ 0·351917	0·876154	0·091363
5	− 0·404327	16·2345	+ 0·325556	0·894013	0·081680

In the case of the partial correlations, $r_{13\cdot2}$ denotes the correlation between u_x and u_{x-2}, u_{x-1} constant ; $r_{14\cdot23}$ the correlation between u_x and u_{x-3}, u_{x-1} and u_{x-2} constant ; and so on. Only those partial correlations are given which are necessary to show how far we can improve the estimate of u_x by taking into account the successive terms beyond u_{x-1}. The continued products of $(1 - r^2)$ are given in the last column on the right, and we may fix our attention on these, considering first the figures for WOLFER's numbers. It will be seen that after the first two terms all the correlations are so small that the continued product of $(1 - r^2)$ hardly falls at all, the variance of the disturbances—that is, the errors made in estimating u_x from the preceding terms—only falling from some 19·5 per cent. to some 19·3 per cent. of the total variance of the numbers themselves. It seems quite clear that in the case of the ungraduated numbers it would be an entire waste of time to take into account any terms more distant from u_x than u_{x-2} for purposes of estimation. As regards the idea suggested that the difference equation should be of the form required for such a function as (36), it may be noted that $r_{14\cdot23}$ is of the wrong sign : a positive correlation would be required. The correlations give no evidence at all of any periodicity other than the fundamental, nor of any other exponential function. They strongly emphasise the increase of the element of unpredictability with the time.

When we turn to the correlations for the graduated numbers, matters are not altogether so clear. The last two partial correlations given in Table IV rise to markedly higher values than are found for the ungraduated figures, both exceeding $+0 \cdot 3$. The total correlations are based on 167 to 171 observations, and if we calculated the standard error by the ordinary formula, it would be under $0 \cdot 08$, and we would reckon both correlations as significant. Here, however, we have to do with a correlated series, not a random sample; the standard error is probably higher, and personally I am inclined to doubt whether either correlation is really significant. The discrepancies of sign as compared with the partial correlations for the ungraduated numbers in the case of the third and fifth partials may alone suffice to raise doubts. In any case, the effect of these correlations on the continued product of $(1 - r^2)$ is extremely small, only reducing the variance of disturbances from $10 \cdot 4$ to $8 \cdot 2$ per cent. of the total. In this case also there is very little to be gained by taking into account any terms beyond u_{x-2}, even if that little be significant.

This result, that terms beyond u_{x-2} hardly come into account if we attempt to estimate u_x by means of the preceding terms is, as it seems to me, what ought to be expected if the series is in fact "disturbed." But, in a sense, it is rather disappointing. Fig. 6

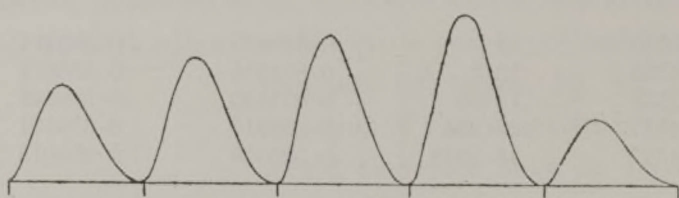

Fig. 6.—Graph of a series of superposed functions of the form of fig. 5, each one starting when the one before reaches its first minimum.

shows a graph formed by superposing a series of functions of the form (36), or fig. 5, of varying amplitudes, a new one starting when the one before reaches its first minimum. It will be seen that the graph is very like that of the sunspot numbers. It may be that this is, or is a close approximation to, the actual function, but when disturbances affect the movement, it does not seem possible to determine the constants, at least by the present method. The road seems to be blocked at the first approximation given by the regression equations of the second order (31) and (32).

The objection may be raised that the suggested function is not "anti-symmetrical," as required by the result of Larmor and Yamaga's investigation, so that $F(t) = -F(-t)$ when the origin is taken on the periodic curve at the mean height, and the Fourier series consists of sine terms only. But the divergence of the suggested function from anti-symmetry is quite small, and Larmor and Yamaga only state that the cosine terms found for the mean sunspot wave had amplitudes less than unity. To get a comparison with their results I have worked out the harmonic analysis of the first period of the function (36) with the following numerical values, being those used for fig. 5 :—

$$y = 100e^{-0\cdot 23026t}\left(1 - \cos\frac{2\pi}{10}t\right), \quad \ldots \ldots \ldots \ldots \quad (37)$$

where $0\cdot23026 = 1/10 \log e$. Integration has first to be effected with zero as time-origin, and when the sine and cosine amplitudes have been determined the origin can be shifted to the epoch at which the function attains its mean value, $t = 1\cdot6802$. The final result is

$$\left.\begin{aligned} y = 34\cdot459 + 36\cdot10 \sin \theta x + 4\cdot07 \sin 2\theta x + 1\cdot06 \sin 3\theta x \\ + \ 0\cdot63 \cos \theta x - 1\cdot61 \cos 2\theta x + 0\cdot47 \cos 3\theta x \end{aligned}\right\} \quad \ldots \ldots \quad (38)$$

The expansion LARMOR and YAMAGA find for the unmodified sunspot curve is

$$y = 44\cdot5 + 35\cdot4 \sin kt + 6\cdot6 \sin 2kt, \quad \ldots \ldots \ldots \ldots \quad (39)$$

and for the same curve as modified by equalising the amplitudes of all the waves in the sunspot curve over the period considered

$$y = 45 + 37\cdot8 \sin k't + 6\cdot5 \sin 2k't + 1\cdot4 \sin 3k't. \quad \ldots \ldots \quad (40)$$

I chose the amplitude in (37) so as to make the amplitude in the first sine term approximately the same in (38) as in (39) and (40), so that the three expansions are fairly comparable. It will be seen that in (38) two of the three cosine amplitudes are less than unity, and would have been ignored on LARMOR and YAMAGA's criterion : the third is only $1\cdot61$. It does not seem to me that (38) differs very materially from (39) or (40) : the main difference seems to lie in the relatively low value obtained in (38) for the amplitude of the second sine term, rather than in the amplitudes of the cosine terms, and the value taken for the damping coefficient, $a = 0\cdot23026$, is high. In the above case a/θ is $0\cdot37$ roundly : from the position of the maximum in LARMOR and YAMAGA's fig. 4 I should estimate a at about $0\cdot146$, a/θ (the year as unit) at about $0\cdot26$.

IV. SOME TRIALS OF PERIODOGRAM ANALYSIS ON THE EXPERIMENTAL DISTURBED SERIES.

The opinion was expressed in Section I that the application of periodogram analysis to "disturbed" functions must yield results subject to a much larger margin of error than is usually supposed. I can see no direct way of tackling the problem and finding the standard error of the amplitude of a period found from any number n of observations, given the standard deviation of the disturbances, even assuming for simplicity that these are random, as in my experimental case. For the u's—*i.e.*, the terms of the observed series—are not terms of a random but of an oscillatory series, in which u_n is correlated with u_{n+x} and the sign of the correlation changes as x increases.* The problem is too complex for my abilities at least. Hence I thought it worth while to carry out some trials on the data used for fig. 2, and for testing the methods developed in Sections II and III. To avoid decimal places in the analysis, the original figures were cut down to two decimals and then multiplied by 100, so that one unit in fig. 2 corresponds to an amplitude of 100, or an intensity of 10,000.

A.—For the first test four groups of observations were used, covering roughly the first, second, third and fourth quarter of the observations respectively. For each of these groups the intensities of the periods 8, 9, 10, 11 and 12 alone were determined in the first instance. Subsequently, periods 9·5 and 10·5 were added. Since for these periods it was necessary to take observations covering an even number of periods, 8 periods were used, so that 84 observations were required for the period 10·5. The original intention was to use only 70 observations or as few more as could be helped; hence only 70 observations were employed to get the intensity of the period 10. The figures, however, are reasonably comparable, and are given in Table V: the second (italicised) line of figures for each group gives the calculated intensity for a simple harmonic function of period 10 and the intensity shown in the table.

It will be seen that while in every group the intensity for period 10 is the greatest, the relative intensities of the respective periods in the different groups vary largely. In Group II the intensity of 10·5 is nearly equal to the intensity of 10, while the intensity of 9·5 is less than a third of the intensity of 10. In Group I the intensities of 9·5 and 10·5 are not far from equal, while in Groups III and IV the intensity of 9·5 is much greater than that of 10·5. In Groups I and II the intensity of 9·5 is less than the calculated figure, in Groups III and IV substantially greater, while for the period 10·5 matters go just the other way.

If we look at the figures for periods diverging more largely from the fundamental, the variation is almost more striking. In the case of periods 8 and 9, observed intensities vary roundly from one-fourth to four times the calculated intensity for a simple harmonic function, or more. For period 11 the range is from under a fourth to about three times the calculated figure. For period 12 Group IV shows an intensity well over four times the calculated figure: Group III gives an almost vanishingly small intensity against a calculated intensity of nearly 3,000.

* Any series that is worth analysing at all must be an oscillatory series in this sense, the sense in which the term is used in the *Address* already cited ('Journ. Stat. Soc.' (1926)). SCHUSTER's exponential formula renders great service by enabling us to exclude at once terms which might arise even in the analysis of a random series; but fluctuations of sampling in intensities based on samples from an oscillatory series may be much larger than the corresponding fluctuations in samples from a random series, and at present one is thrown back on empirical tests of significance in such cases. It is true that they may also be lower, but SCHUSTER's formula then leaves one on the safe side.

TABLE V.—Periodogram analysis, for seven periods only, of four groups of 70 to 80 observations each in the experimental series. The first line in the table for each group gives the intensity found, the second line in italic type the calculated intensity for a simple harmonic function of period 10 and the intensity shown. Original ordinates multiplied by 100.

Group.	Period : intervals.						
	8	9	9·5	10	10·5	11	12
I	880 *673*	9,139 *3,410*	35,606 *35,708*	62,341 —	40,233 *35,708*	12,480 *8,437*	708 *1,516*
II	2,344 *659*	771 *3,338*	18,030 *34,949*	61,015 —	60,047 *34,949*	26,399 *8,258*	245 *1,483*
III	5,649 *1,255*	23,961 *6,356*	92,193 *66,562*	116,206 —	51,695 *66,562*	14,511 *15,727*	54 *2,825*
IV	3,738 *982*	6,591 *4,974*	65,395 *52,082*	90,927 —	33,457 *52,082*	2,912 *12,306*	9,476 *2,210*
Periods covered by analysis.	9	8	8	7	8	7	6

As might be expected from an examination of the graph, fig. 2, the phase of the fundamental period varies largely from group to group. The relative phases are :—

Group I—310°
II—276°
III—252°
IV—295°

It is evident that, for curves of this type, identity of phase in successive sections of the observations cannot serve as an empirical test of the reality of a period. The periodicity —or, rather, the fundamental tendency to the given period—may be absolutely real, but phase may shift backwards and forwards over quite a large fraction of the period.

B.—The second test carried out was a detailed periodogram analysis of the series as a whole (273 to 327 observations) between periods 9 and 11. The results are shown in Table VI and the graph fig. 7.

One effect of the shifting phase of the fundamental period is immediately noticeable. The intensity of period 10 comes out at 67,600 roundly, a figure little higher than the intensities for Groups I and II in Table V, and much less than those shown by Groups III and IV. The average for Groups I to IV would be 82,600 ; but even this figure is

TABLE VI.—Periodogram analysis of experimental series (273–327 observations) for periods between 9 and 11 intervals. The values of A and B for periods 9 and 11 have been adjusted to observation 1 as origin. Original ordinates multiplied by 100.

Period: intervals.	Observations used.	A.—Sine amplitude.	B.—Cosine amplitude.	Intensity: $I = A^2 + B^2$	Calculated I for harmonic function.
9·0	2 – 298	− 16·58	− 23·60	832	0
9·1	1 – 273	− 56·37	− 27·26	3,921	615
$9\frac{1}{8}$	1 – 275	− 10·79	− 52·63	2,886	1,096
9·2	1 – 276	+ 15·07	− 43·46	2,116	1,076
9·3	1 – 279	+ 14·33	+ 33·57	2,398	148
$9\frac{1}{3}$	1 – 280	− 17·66	+ 45·50	2,382	0
9·4	1 – 282	− 85·16	+ 9·00	7,333	730
9·5	1 – 285	− 51·52	− 117·14	16,376	3,045
9·6	1 – 288	+ 94·98	− 84·57	16,173	1,644
$9\frac{2}{3}$	1 – 290	+ 99·77	+ 24·52	10,555	0
9·7	1 – 291	+ 58·12	+ 67·24	7,899	808
9·8	1 – 294	− 113·40	+ 14·63	13,074	17,215
9·9	1 – 297	− 55·21	− 193·85	40,618	49,829
10·0	*1 – 300*	*+ 190·71*	*− 176·79*	*67,625*	*67,625*
10·1	1 – 303	+ 238·48	+ 57·14	60,138	49,829
10·2	1 – 306	+ 77·83	+ 156·29	30,484	17,215
10·3	1 – 309	− 5·73	+ 86·89	7,583	808
$10\frac{1}{3}$	1 – 310	− 4·92	+ 67·45	4,574	0
10·4	1 – 312	+ 2·91	+ 58·58	3,440	1,644
10·5	1 – 315	− 22·01	+ 62·44	4,383	3,045
10·6	1 – 318	− 33·74	+ 21·46	1,599	730
$10\frac{2}{3}$	1 – 320	− 12·88	+ 5·55	14	0
10·7	1 – 321	− 3·97	+ 9·43	10	148
10·8	1 – 324	− 17·27	+ 31·30	1,278	1,076
$10\frac{5}{8}$	1 – 325	− 29·41	+ 27·62	1,628	1,096
10·9	1 – 327	− 38·06	− 0·67	1,449	615
11·0	2 – 298	− 6·51	− 8·70	118	0

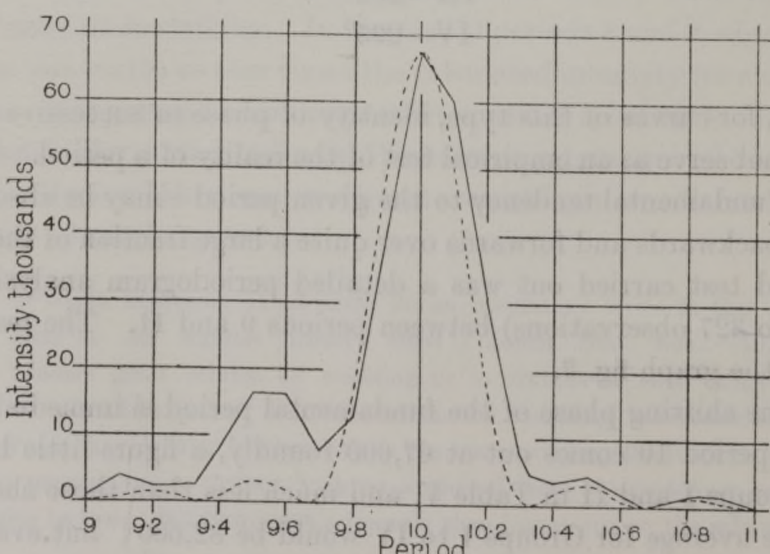

FIG. 7.—Periodogram for the experimental disturbed series on which fig. 2 is based: data in Table VI.

misleadingly low, for the figure for each group is also lowered by the shifting phase. If the graph represented a simple harmonic function, the intensity would be given by $A^2 = 2\sigma^2$, where σ is the standard deviation of the series about the zero base-line, and for the series in question this would show an intensity of 92,700. The periodogram intensity is therefore too low by some 25 per cent.

It is clear, then, that periodogram analysis applied to functions of the present kind tends to give much too low* an intensity for the fundamental. But this at once raises the question, what would be the magnitude of such reduction if the series were indefinitely long ? If we refer back to Section I and the analysis of the series into its component parts, the complementary function which is a simple harmonic function and the particular integral which is a fluctuating series, the answer is, I think, clear. The intensity of the complementary function alone would remain at its true value : the intensity of the particular integral would vanish. But if disturbances are sufficiently large, the particular integral contributes nearly the whole of the intensity : even in the present case it contributes roundly eight-ninths. Hence in a heavily disturbed series the intensity would in the long run tend almost to vanish, and the result given by the periodogram might tend to mislead the incautious interpreter. Probably, however, the result would not mislead in practice, because only a modest number of periods is usually available.

Turning now either to the last two columns of the table or to the graph, the main general effect of the varying phase is evidently to broaden the band that would be given by a simple harmonic function of the same period. At the points where the intensity should drop to zero for a simple harmonic function, the observed intensities show a tendency to drop to minima but remain above zero. The two sides of the periodogram are, however, strikingly unlike each other : the graph is markedly asymmetrical, and one cannot help wondering whether the same characteristic would have remained had the series been longer—3,000 observations, say, instead of 300. Outside limits of about 9·7 to 10·3, intensities are much higher for periods below the fundamental than for periods correspondingly above it.

It is of interest to compare fig. 7 with Graph A in fig. 2 of SCHUSTER's paper, in which the periodogram for the sunspot numbers between 1826 and 1900 is compared with the calculated periodogram for a simple harmonic function. It shows something of the same characteristics, the intensities of shorter periods being much higher than the calculated values, while the intensities of longer periods, round about 14 years, are actually *lower*. The graph of the periodogram for the entire series, from 1750 to 1900, SCHUSTER's fig. 1, somewhat similarly but even more markedly shows much higher intensities for periods from 8·25 to 10·25 years than for periods from 12·25 to 14·25, the average intensities over these ranges being 1,061 and 394 respectively.

* [*Added, February* 17.—Too low, I hope it is clear, only in the sense in which the whole of the intensity may be regarded as due to the tendency to follow the fundamental period. Actually the intensity is much higher than that of the complementary function, which is the strictly periodic component.]

It must be borne in mind that this section has been concerned solely with the effect of a certain series of random disturbances. In data referring to natural phenomena it is unlikely that the disturbances will be strictly random with respect to time, and the effect on the per iodogram may be much more marked.

V. Summary and Conclusions.

The sunspot numbers, it is suggested, should be regarded as analogous to the data that would be given by observations of a disturbed periodic movement, such as that of a pendulum subjected to successive small random impulses.

The graph of such a movement presents the principal features of the graph of sunspot numbers, viz., a surprising degree of smoothness accompanied by a continual change of amplitude and shift of phase.

It is suggested that in this case the application of the periodogram method gives results subject to a large margin of error, and may be misleading. Trial on an empirical disturbed series (Section IV) showed, in fact, that with only 7 or 8 periods results are highly erratic; with a larger number of periods, about 30, the main effect is a broadening of the band due to the fundamental and a reduction of the apparent intensity.

The problem of determining the period and the disturbances, in the case of the sunspot numbers, was attacked in the first instance (Section II) by finding the best (least square) linear equation relating $u_x + u_{x-2}$ to u_{x-1}, this giving the form of difference equation required for a simple harmonic function. The equation gave a period which was obviously too low. It is suggested that this result is due to the presence of superposed fluctuations in addition to disturbances, a suggestion borne out by applying the same method to graduated values of the numbers. This yielded a much closer approximation to the period suggested by the graph.

Applying an extension of the same method in an endeavour to determine whether or no there was any secondary period in addition to the fundamental, the paradoxical result was reached that (with the particular form of equation corresponding to two simple harmonic functions) u_x cannot be so closely estimated in terms of u_{x-1} to u_{x-4} as in terms of u_{x-1} and u_{x-2} alone. There is thus no evidence of the existence of any secondary period. The result also suggests the existence of disturbances (as distinct from superposed fluctuations), since only disturbances can give the required element of unpredictability rapidly increasing with the time.

The better and more general method was then applied (Section III) of determining the regression equation for u_x on u_{x-1} and u_{x-2}, and solving this as a finite difference equation. The solution is a rapidly damped harmonic function.

The "disturbances" deduced from the equation show two principal features: (1) a tendency to give preponderantly positive and highly variable disturbances, and preponderantly negative and less variable disturbances, in alternate intervals of 40 to 42 years (*cf.* Table II); (2) a tendency for positive disturbances during the approach to the maximum of the sunspot numbers, negative disturbances during the approach to minimum (*cf.* Table III). It is suggested that the second feature accords with the necessity for maintaining a damped periodic function.

A damped harmonic function of the time is, however, clearly not the mathematical form required: a form that would suit the data very well would be the square of a damped harmonic function. The difference equation of this function is of the third degree, and would therefore require u_{x-3} to be brought into the regression equation as well as u_{x-1} and u_{x-2}.

But investigation of the correlations (Table IV) shows that for the ungraduated numbers it would be no use whatever, for the graduated numbers very little use, to bring in further terms beyond u_{x-2}, once more emphasising the rapid increase of the element of unpredictability with the time.

Further work on this line seems, therefore, to be blocked—a rather disappointing conclusion, since the form of function suggested otherwise looks hopeful.

The correlations, like the method of Section II, equally fail to suggest the presence of any period other than the fundamental, a conclusion entirely in accord with the work of LARMOR and YAMAGA.

I do not put forward the methods used in the present paper as necessarily the best, nor even in all cases applicable. I was attacking a problem which, to me at least, was a new one, and used the methods that seemed best at the moment; but experience may suggest better methods. With the present experience, indeed, it seems clear that the method of Section III is not a good method for determining the period, for it tends to give too low a value when superposed fluctuations are present in addition to disturbances, as in all probability they nearly always are. It might be better, for example, to determine the period first by the simple and straightforward method of taking the interval between the first and last maxima (or minima), and dividing by the number of intervening periods, leaving only the damping factor to be found by least squares or otherwise. But while this is quite a possible method for the sunspot numbers, it would not be possible with a variable largely affected by superposed fluctuations: maxima and minima would be too indefinite. Variables affected largely both by disturbances and by superposed fluctuations present a very difficult problem for analysis.

And I would like in conclusion to suggest that many series which have been or might be subjected to periodogram analysis may be subject to "disturbance" in the sense in which the term is here used,* and that this may possibly be the source of some rather odd results which have been reached. Disturbance will always arise if the value of the variable is affected by external circumstance and the oscillatory variation with time is wholly or partly self-determined, owing to the value of the variable at any one time being a function of the immediately preceding values. Disturbance, as it seems to me,

* [*Added, February* 17.—A number of the graphs in HEDGES and MYERS' 'The Problem of Physico-Chemical Periodicity' (Arnold, 1926) look obviously of the "disturbed" type, the waves being very smooth, but varying in phase and amplitude.]

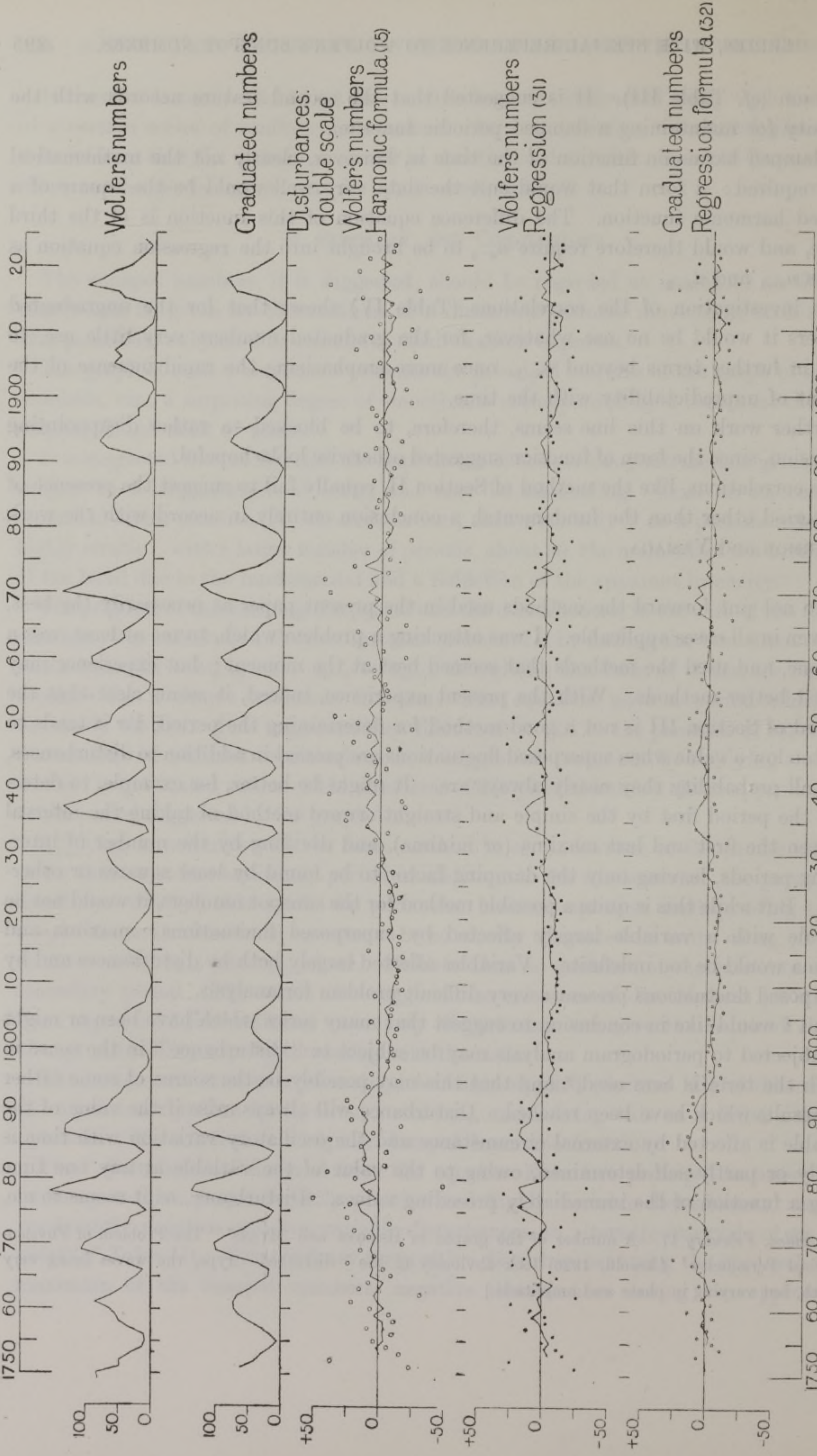

FIG. 8.—Graphs of WOLFER's numbers and the graduated numbers, and of the disturbances by equations (15), (31) and (32): the lines on the graphs of disturbances show quinquennial averages.

can only be excluded if either (1) the variable is quite unaffected by external circumstance, or (2) we are dealing with a forced vibration and the external circumstances producing this forced vibration are themselves undisturbed.

TABLE A.—WOLFER'S sunspot numbers and graduated numbers, and the disturbances calculated by equations (31) and (32) respectively. The graduated numbers are given solely because they were used for part of the preceding work: they are not a good graduation (*cf.* text, pp. 282–283).

Year.	WOLFER'S number.	Disturbance.	Graduated number.	Disturbance.	Year.	WOLFER'S number.	Disturbance.	Graduated number.	Disturbance.
1749	80·9	—	—	—	1793	46·9	− 3·9	50·8	+ 9·8
1750	83·4	—	—	—	1794	41·0	+ 3·5	35·5	− 9·5
1751	47·7	− 25·2	61·8	—	1795	21·3	− 16·9	26·5	+ 0·6
1752	47·8	+ 24·5	40·2	—	1796	16·0	+ 0·4	12·6	− 11·9
1753	30·7	− 16·1	30·5	+ 6·3	1797	6·4	− 15·0	7·9	− 2·8
1754	12·2	− 11·6	15·5	− 11·3	1798	4·1	− 7·9	3·8	− 10·9
1755	9·6	− 0·5	6·1	− 5·8	1799	6·8	− 8·4	6·0	− 6·3
1756	10·2	− 8·6	15·4	+ 5·7	1800	14·5	− 5·8	17·5	− 1·4
1757	32·4	+ 11·1	29·4	− 1·9	1801	34·0	+ 5·1	32·2	− 2·4
1758	47·6	− 3·1	46·1	+ 1·1	1802	45·0	− 5·0	42·4	− 5·2
1759	54·0	− 2·6	54·0	− 5·1	1803	43·1	− 8·9	47·0	− 4·3
1760	62·9	+ 7·7	71·1	+ 13·4	1804	47·5	+ 5·2	45·6	− 4·5
1761	85·9	+ 22·9	72·8	− 4·5	1805	42·2	− 7·2	41·8	− 2·4
1762	61·2	− 26·8	67·6	+ 1·5	1806	28·1	− 11·3	26·4	− 13·2
1763	45·1	+ 5·3	46·5	− 10·4	1807	10·1	− 13·9	14·5	− 4·8
1764	36·4	+ 2·1	33·4	+ 4·3	1808	8·1	− 0·9	5·2	− 8·4
1765	20·9	− 12·3	19·0	− 7·2	1809	2·5	− 15·6	3·2	− 5·9
1766	11·4	− 6·7	18·1	+ 3·3	1810	0·0	− 11·9	0·3	− 13·2
1767	37·8	+ 22·3	34·6	+ 9·6	1811	1·4	− 10·8	1·1	− 9·6
1768	69·8	+ 12·6	74·8	+ 24·0	1812	5·0	− 10·7	6·2	− 8·1
1769	106·1	+ 23·3	97·9	− 0·5	1813	12·2	− 7·5	8·4	− 13·0
1770	100·8	− 9·8	101·9	+ 0·7	1814	13·9	− 13·1	20·1	− 0·5
1771	81·6	+ 1·9	85·9	− 2·8	1815	35·4	+ 10·9	32·4	− 4·2
1772	66·5	+ 9·1	59·3	− 2·0	1816	45·8	− 6·5	44·3	− 1·5
1773	34·8	− 14·9	44·9	+ 11·1	1817	41·1	− 11·1	41·0	− 13·0
1774	30·6	+ 13·6	19·2	− 14·1	1818	30·4	− 8·7	32·3	− 7·1
1775	7·0	− 25·2	10·6	+ 4·7	1819	23·9	− 3·9	23·2	− 5·7
1776	19·8	+ 16·6	30·3	+ 16·8	1820	15·7	− 10·3	15·0	− 7·1
1777	92·5	+ 56·6	93·5	+ 43·2	1821	6·6	− 12·7	8·1	− 8·9
1778	154·4	+ 29·3	136·9	+ 6·7	1822	4·0	− 8·4	2·4	− 10·7
1779	125·9	− 34·7	130·4	− 14·9	1823	1·8	− 13·1	3·6	− 6·4
1780	84·8	+ 3·0	93·1	− 7·5	1824	8·5	− 5·2	6·5	− 9·9
1781	68·1	+ 22·8	61·0	+ 11·7	1825	16·6	− 7·5	19·7	− 0·1
1782	38·5	− 11·3	42·7	+ 12·1	1826	36·3	+ 5·7	33·7	− 3·8
1783	22·8	+ 1·8	19·0	− 9·6	1827	49·7	− 2·0	51·2	+ 3·1
1784	10·2	− 9·1	10·8	+ 3·4	1828	62·5	+ 5·7	60·8	− 2·6
1785	24·1	+ 11·5	32·2	+ 18·2	1829	67·0	+ 1·8	70·8	+ 6·9
1786	82·9	+ 43·4	81·3	+ 28·3	1830	71·0	+ 8·4	64·7	− 6·7
1787	132·0	+ 22·6	123·2	+ 13·0	1831	47·8	− 17·5	51·3	− 2·8
1788	130·9	− 5·9	135·6	+ 1·3	1832	27·5	− 3·8	24·8	− 13·9
1789	118·1	+ 15·2	115·4	− 4·1	1833	8·5	− 11·0	9·3	0·0
1790	89·9	+ 3·2	90·8	+ 11·9	1834	13·2	+ 5·9	16·9	+ 9·9
1791	66·6	+ 9·4	70·5	+ 12·6	1835	56·9	+ 30·9	62·5	+ 31·5
1792	60·0	+ 15·6	55·9	+ 9·1	1836	121·5	+ 39·9	114·3	+ 20·3

TABLE A (continued).

Year.	WOLFER'S number.	Disturbance.	Graduated number.	Disturbance.	Year.	WOLFER'S number.	Disturbance.	Graduated number.	Disturbance.
1837	138·3	− 1·5	130·1	− 5·8	1881	54·3	+ 1·0	51·2	+ 0·4
1838	103·2	− 16·8	113·5	− 4·8	1882	59·7	− 5·9	61·7	− 4·2
1839	85·8	+ 23·9	83·1	+ 2·6	1883	63·7	+ 5·2	64·2	− 1·1
1840	63·2	+ 1·7	61·4	+ 13·7	1884	63·5	+ 3·2	63·2	+ 2·6
1841	36·8	− 5·7	40·4	+ 1·2	1885	52·2	− 5·2	48·4	− 8·7
1842	24·2	+ 2·3	20·5	− 4·3	1886	25·4	− 17·0	29·7	− 5·8
1843	10·7	− 11·6	12·4	+ 0·9	1887	13·1	− 0·7	12·2	− 6·8
1844	15·0	+ 2·6	18·1	+ 2·8	1888	6·8	− 8·0	7·3	− 0·2
1845	40·1	+ 13·1	35·2	+ 4·9	1889	6·3	− 8·1	2·9	− 11·2
1846	61·5	+ 3·6	66·6	+ 14·9	1890	7·1	− 10·8	12·1	+ 0·7
1847	98·5	+ 28·3	100·3	+ 14·8	1891	35·6	+ 16·3	37·3	+ 8·4
1848	124·3	+ 18·4	113·6	+ 2·2	1892	73·0	+ 16·0	68·4	+ 8·7
1849	95·9	− 20·4	98·7	− 5·8	1893	84·9	− 3·7	84·3	− 2·3
1850	66·5	+ 5·3	73·6	+ 2·3	1894	78·0	− 2·1	79·4	− 6·3
1851	64·5	+ 24·2	60·2	+ 15·0	1895	64·0	+ 1·0	62·2	− 3·3
1852	54·2	− 2·7	54·4	+ 9·4	1896	41·8	− 6·9	42·4	− 1·0
1853	39·0	− 5·4	38·3	− 8·7	1897	26·2	− 1·9	30·7	+ 3·5
1854	20·6	− 10·1	20·7	− 6·5	1898	26·7	+ 5·0	20·2	− 5·2
1855	6·7	− 9·3	6·4	− 7·1	1899	12·1	− 20·5	16·9	− 1·9
1856	4·3	− 5·1	6·2	+ 0·3	1900	9·5	− 3·1	6·2	− 16·1
1857	22·8	+ 7·6	22·7	+ 5·6	1901	2·7	− 16·0	3·3	− 5·4
1858	54·8	+ 13·1	59·0	+ 16·7	1902	5·0	− 6·3	8·0	− 4·9
1859	93·8	+ 21·3	87·0	+ 3·0	1903	24·4	+ 5·6	21·7	− 0·6
1860	95·7	− 8·2	95·2	− 2·1	1904	42·0	− 1·3	46·5	+ 7·2
1861	77·2	− 3·7	79·2	− 8·1	1905	63·5	+ 9·2	54·1	− 11·8
1862	59·1	+ 4·2	57·7	+ 1·2	1906	53·8	− 17·8	63·7	+ 6·2
1863	44·0	+ 1·3	49·9	+ 13·2	1907	62·0	+ 17·5	54·2	− 11·8
1864	47·0	+ 12·8	40·4	− 1·8	1908	48·5	− 13·4	55·2	+ 11·3
1865	30·5	− 17·7	32·6	− 1·4	1909	43·9	+ 5·5	36·9	− 16·1
1866	16·3	− 7·7	12·9	− 16·9	1910	18·6	− 22·4	22·5	− 2·0
1867	7·3	− 8·5	14·7	+ 8·5	1911	5·7	− 4·4	6·7	− 10·6
1868	37·3	+ 24·3	31·2	+ 6·4	1912	3·6	− 5·7	1·2	− 3·8
1869	73·9	+ 14·7	89·9	+ 41·6	1913	1·4	− 13·6	0·4	− 8·9
1870	139·1	+ 50·4	113·2	− 10·8	1914	9·6	− 3·8	18·1	+ 5·6
1871	111·2	− 41·1	128·5	+ 16·3	1915	47·4	+ 21·6	33·7	− 6·3
1872	101·7	+ 29·6	92·4	− 24·3	1916	57·1	− 14·1	76·3	+ 26·9
1873	66·3	− 11·3	72·9	+ 23·1	1917	103·9	+ 44·4	83·5	− 17·9
1874	44·7	+ 8·4	39·4	− 9·8	1918	80·6	− 35·4	90·8	+ 12·6
1875	17·1	− 13·4	21·9	+ 7·8	1919	63·6	+ 9·6	59·3	− 24·1
1876	11·3	+ 3·8	11·5	− 2·9	1920	37·6	− 8·9	41·9	+ 12·0
1877	12·3	− 5·5	8·1	− 4·6	1921	26·1	+ 3·4	24·0	− 4·8
1878	3·4	− 19·6	4·4	− 11·5	1922	14·2	− 10·1	12·9	− 2·7
1879	6·0	− 4·4	10·5	− 2·5	1923	5·8	− 10·0	—	—
1880	32·3	+ 12·6	30·6	+ 5·4	1924	16·7	+ 4·4	—	—