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Recurrence quantification analysis of the logistic equation with transients

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Abstract

Recurrence quantification analysis (RQA) detects state changes in drifting dynamical systems without necessitating any *a priori* constraining mathematical assumptions. Study of the logistic equation with transients posits that RQA may be ideal for analyzing complex biological systems whose equations are unknown and whose dynamics are characteristically non-linear and non-stationary.

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1. Introduction

Typically, the aperiodicities, nonstationarities and system drifts of integrated physiological processes are very complex, defying strict mathematical description, except in a statistical sense. Whereas linear multivariate data analysis (MDA) has been employed to extract information from complex biological systems [1], recurrence plot analysis (RPA) has the demonstrated utility of detecting hidden rhythms and deterministic structuring of systems in higher dimensional space [2]. Recurrence quantification analysis (RQA) is a non-biased extension of RPA which allows recurrent structures to be followed over time [3]. In order to mimic dynamical biological systems, this paper studies the logistic equation in which the transients are fully preserved in the time series. The results show that RQA can localize bifurcation behavior in this system without making any a priori assumptions regarding the underlying equations of motion. Such an approach to dynamical systems may be critically significant for analyzing biological systems whose complex mathematics are unknown, but whose state-dependent fluctuations are essential for a complete systemic characterization [4].

2. Logistic difference equation

The logistic equation is one example of an iterated map on the interval serving as a dynamical system [5] which has a demonstrated utility in the field of

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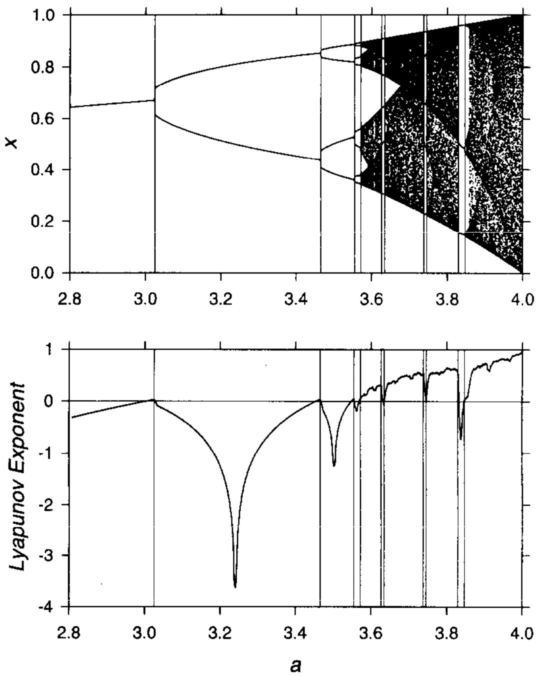

Fig. 1. Bifurcation diagram and Lyapunov exponents (λ) of the logistic equation with transients. On each iteration of the equation, parameter a was step-incremented by 0.00001. The drifting time series contains 120001 points (upper panel) which were blocked into 800-point epochs and shifted by 10 points for the computation of 11920 λ values (lower panel). Periodic windows ($\lambda < 0.0$) and chaotic windows ($\lambda > 0.0$) are clearly distinguished by ten vertical lines positioned at points of bifurcation (specific a values).

population biology [6]. The bifurcation characteristics of this equation are usually examined in its so-called steady-state regime. The long transients are allowed to decay away by holding the control parameter, a, constant and by discarding the first several hundred iterations. In order to model complex biological systems characterized by nonstationary drifts and state changes, however, we studied the bifurcation behavior of the logistic equation in which the dynamical transients were retained.

A transient time series consisting of 120001 points, x, was generated from the first-order logistic difference equation [7] by consistently incrementing parameter a in steps of 0.00001 on each iteration (parameter $k = 1$),

$$x_{n+1} = ax_n(1 - x_n/k). \tag{1}$$

Thus, over the range of $a = 2.8$ to 4.0, the dynamic was consistently kept away from the logistic attractor, within the larger basin of attraction where motion is known to be very complex [8]. The bifurcation diagram for this drifting time series is depicted in Fig. 1 (top panel). Ten vertical lines mark off eleven distinct dynamical regimes in the transient time series: period 1, period 2, period 4, period 8, chaotic, period 6, chaotic, period 5, chaotic, period 3, chaotic.

The transient dynamic was further characterized by computing the largest Lyapunov exponents (λ) of the drifting logistic time series according to the following analytical equation [9],

$$\lambda = \frac{1}{N}\sum_{n=1}^{N} \log_2 |a(1 - 2x_n)|. \tag{2}$$

Computations were performed on time series data within episodic windows consisting of 800 consecutive points, each point having its own unique (increasing) a value. Sequential windows were shifted by 10 points, granting a total of 11920 λ values. As shown in Fig. 1 (lower panel), Lyapunov exponents remained negative within periodic windows ($\lambda < 0.0$), but became positive in chaotic zones ($\lambda > 0.0$). Compared with steady-state solutions to the logistic equation [10], it can be shown that the logistic equation with transients has: (1) more abrupt bifurcations which are shifted to larger a values; and (2) Lyapunov exponents which are smoothed (filtered) over the range of explored a values.

3. Recurrence plot analysis

RPA was performed on specific regions of the drifting time series of the logistic equation to exactly localize points of bifurcation. RPA is based on the computation of a distance matrix between rows of embedded points in the time series of interest [2]. For the present analysis, an embedding dimension of 3 was selected with a lag of 1 for 800 consecutive points. Whenever i, j coordinates in the distance matrix were within a predetermined cutoff distance (local neighborhood: $d_{ij} \leqslant 0.5$ relative units), pixels were darkened at corresponding i, j coordinates in the recurrence plot. By this definition, recurrence plots are perfectly symmetrical ($d_{ij} = d_{ji}$) and characterized by a main diagonal line of identity. The darkened points individuate the recurrences of the dynamical process and the recurrence plot itself can be considered to be the global representation of the autocorrelation structure of the system under investigation [2]. The use of the distance-matrix formalization permits the utilization of any kind of data irrespective of its statistical distribution, nonstationarity, or nonlinearity.

Two recurrence plots generated from different 800 point segments of the logistic equation with transients are presented in Fig. 2. The top diagram represents the state change from chaos to the three-state oscillation [7]. The exact transition boundary is delineated by the large increase in recurrences starting at the middle of the plot (a = 3.82907). The bottom diagram locates a period-4 oscillation isolated by a square island of heavier recurrences (a = 3.90585 to 3.90723). Although this 139-point periodic dynamic is too short to be seen in the bifurcation plot of the original time series, it is readily detectable by RPA.

4. Recurrence quantification analysis

RQA has been used to quantify the recurrent structure of the dynamical time series with transients [3]. Five RQA variables are usually examined: (1) *%recurrence* (percentage of darkened pixels in recurrence plot); (2) *%determinism* (percentage of recurrent points forming diagonal line structures); (3) *entropy* (Shannon entropy of line segment distributions); (4) *trend* (measure of the paling of recurrent points away from the central diagonal; (5) $1/line_{\max}$ (reciprocal of the longest diagonal line segment within each epoch). Since $1/line_{\max}$ relates directly to largest positive Lyapunov exponent [2], short diagonal line structuring in recurrence plots is indicative of diverging trajectories (mathematical chaos).

5. RQA of the logistic equation with transients

RQA was performed on the logistic system with transients embedded in 3-space. RQA was repeatedly performed on 800-point epochs, generating 11920 values for each of the five RQA variables and two linear metrics, *mean* and *standard deviation*. Neighboring epochs were shifted by 10 points (98.75% overlap).

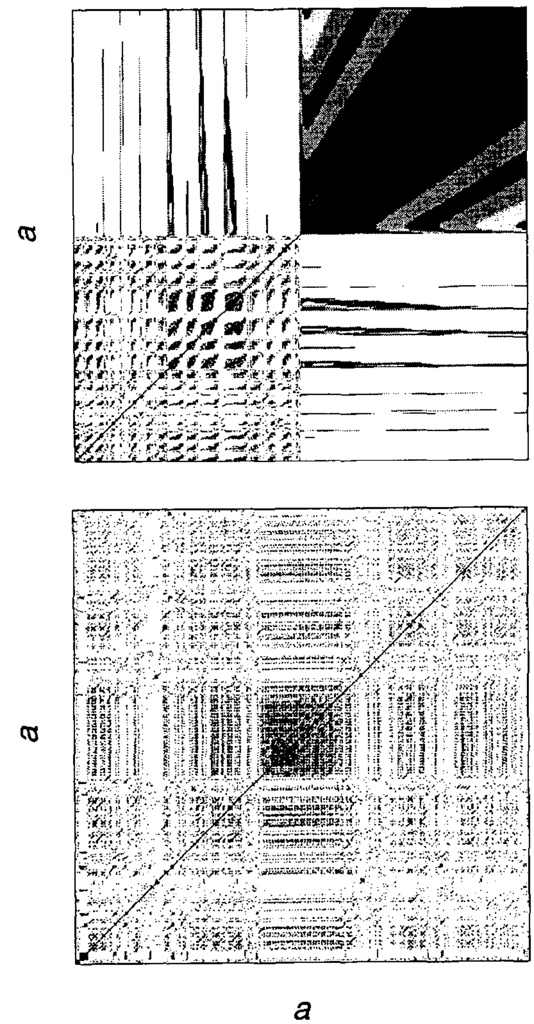

Fig. 2. Recurrence plot analysis (RPA) of the logistic equation with transients. The transition from chaos to the period-3 window (a = 3.82907) is marked by abrupt saturation in *%recurrence* (upper panel). Likewise, the period-4 window (central square, a = 3.90585 to 3.90723) has a higher *%determinism* than the surrounding chaos (lower panel). RQA parameters: point lag = 1; embedding dimension = 3; distance cutoff = 0.5 relative units; line definition = 2 points. Recurrence plot calibration of 800-point epochs: a = 3.82508 to 3.83307 (upper panel); a = 3.90254 to 3.91053 (lower panel). The various shades of gray in the plots distinguish differing bands of recurrence distances (d_{ij}) below the cutoff.

Mean and *standard deviation* variables are plotted in Fig. 3 as functions of parameter a. It is evident that bifurcations in the logistic dynamic (vertical lines) are poorly detected by shifts in these two measured

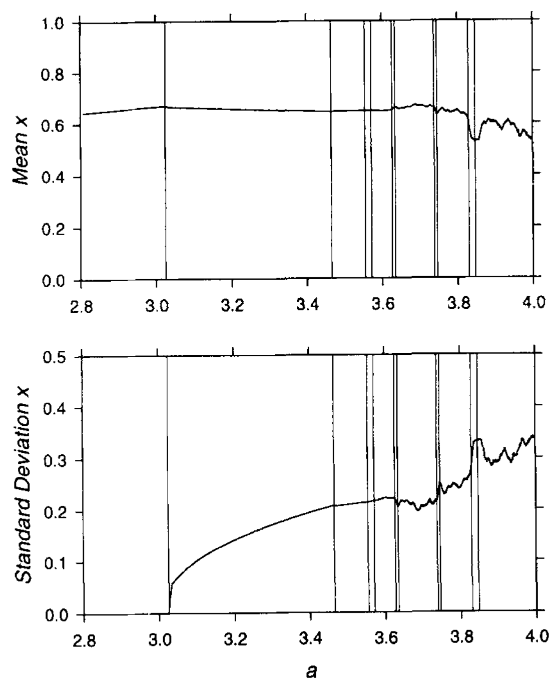

Fig. 3. Linear metrics of the logistic equation with transients including the running *mean* (upper panel) and *standard deviation* (lower panel). Values are computed from an 800-point window (epoch) moving down the drifting logistic equation. Adjacent epochs are shifted by 10 points. RQA parameters: point lag = 1; embedding dimension = 3; distance cutoff = 0.5 relative units; line definition = 2 points.

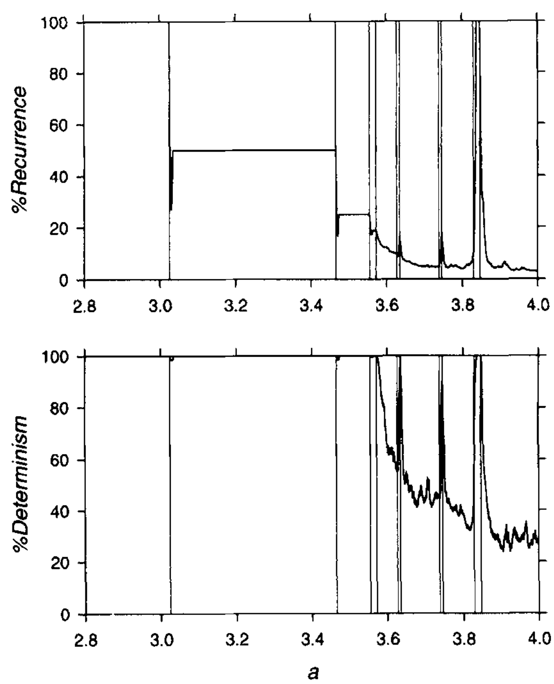

Fig. 4. Nonlinear metrics of the logistic equation with transients including *%recurrence* (upper panel) and *%determinism* (lower panel). Values are computed from an 800-point window (epoch) moving down the drifting logistic equation data shifted 10 points between epochs. RQA parameters: point lag = 1; embedding dimension = 3; distance cutoff = 0.5 relative units; line definition = 2 points.

variables, except at the transition from chaos to the period-3 dynamic ($a = 3.8291$). More subtle changes in the *standard deviation* do occur for the period-2 dynamic ($a = 3.0249$), the period-5 dynamic ($a = 3.7380$), and the period-4 dynamic ($a = 3.9058$).

The nonlinear variables *%recurrence* and *%determinism* are plotted in Fig. 4 as functions of parameter *a*. As illustrated there are strong alterations in both of these RQA variables at most of the bifurcation points. As expected, dynamical transitions from chaos to period-*n* oscillations are marked by sharp rises in these variables ($a = 3.6265, 3.7380, 3.8291$). Increases are more blunted for the period-4 dynamic ($a = 3.9059$) since the periodic window is only 139 points, a mere 17% of the 800-point epoch (see Fig. 2 bottom). It is interesting that the general aspects of these two plots remained substantially invariant under different RQA parameter selections.

Two additional nonlinear variables, *entropy* and *trend*, are plotted in Fig. 5 as functions of parameter *a*. In general, *entropy* values were high within periodic windows (large diversity in diagonal line lengths), but low within chaotic windows (small diversity in diagonal line lengths), indicative of loss of recurrence complexity. *Trend* values, however, were significantly better at detecting bifurcation shifts in the logistic dynamic, either between adjacent periodic/periodic or periodic/chaotic windows. For example, the *trend* exhibits two negative spikes for the period-3 window, one upon entry ($a = 3.8291$) and a second upon exit ($a = 3.8465$), a sensitivity not observed with any of the other RQA variables.

Finally, $1/line_{max}$ is plotted in Fig. 6 as functions of parameter *a* and of the Lyapunov exponent. Again, characteristic changes are detected by this RQA variable at bifurcation points (top panel). Periodic and chaotic dynamics are associated with, respectively, small or large $1/line_{max}$ values. As illustrated (lower

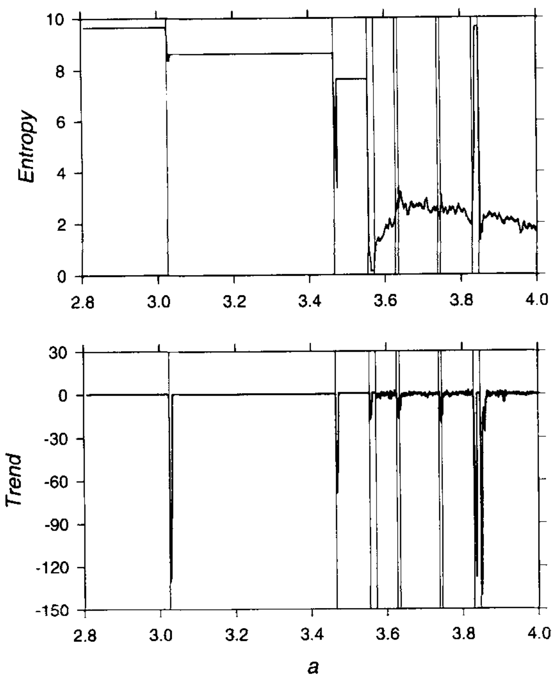

Fig. 5. Nonlinear metrics of the logistic equation with transients including *entropy* (upper panel) and *trend* (lower panel). Values are computed from an 800-point window (epoch) moving down the drifting logistic equation data. Adjacent epochs are shifted by 10 points. RQA parameters: point lag = 1; embedding dimension = 3; distance cutoff = 0.5 relative units; line definition = 2 points. *Trend* is measured in *%local recurrence*/1000 points (see Ref. [12]) and *entropy* is measured in Shannon bits of information (see Ref. [3]).

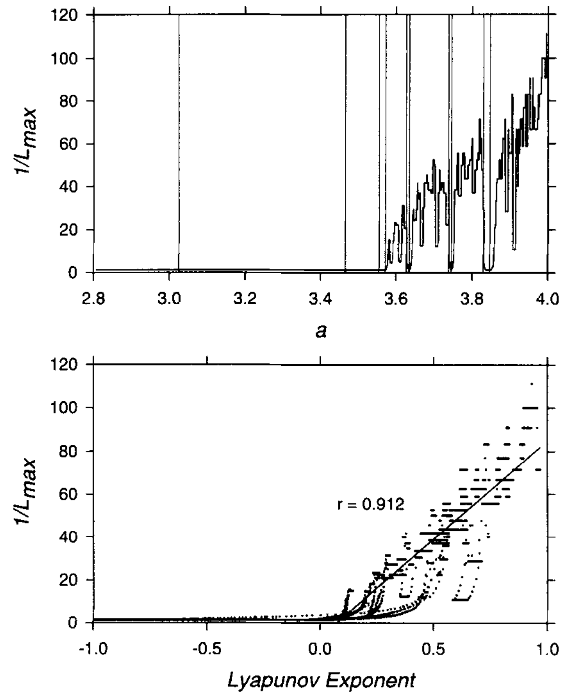

Fig. 6. Reciprocal of the maximum line length of the logistic equation with transients as a function of parameter a (upper panel) and logistic Lyapunov exponent (lower panel). Values are computed from an 800-point window (epoch) moving down the drifting logistic equation. Adjacent epochs are shifted by 10 points. RQA parameters: point lag = 1; embedding dimension = 3; distance cutoff = 0.5 relative units; line definition = 2 points. The regression r value and line of best fit were computed by the method of least squares for all positive Lyapunov exponents.

panel), $1/line_{\max}$ is directly correlated with positive Lyapunov exponents at the larger parameter values (e.g. $a > 3.5688$), but the correlation collapses within periodic regions where Lyapunov exponents are negative.

6. Discussion

Employing the logistic equation with transients as a model system, we have demonstrated the utility of RQA in diagnosing points of bifurcation extant in the drifting time series. This was accomplished independent from any knowledge of the specific operating variables and parameter values driving the process. Such distinctive information cannot similarly be extracted by the single-dimensional linear metrics (e.g. *mean* and *standard deviation*). These results may be critically important for biological systems in which the equations of motion (if any) are unknown and where participant variables and their interactions are constantly changing. To the extent that biological processes are high-dimensional entities, living on transients amidst a field of relatively weak attractors [11], inspires the implementation of sophisticated modes of analysis.

We posit that RQA may be useful in diagnosing complex biological processes which are characterized by non-stationary drifts and state changes. In fact, RQA variables have already been shown to detect changes in biological states [3,12,13]. Insofar that RQA methodology still requires standardization and verification, biological applications of RQA necessitates, indeed invites, concerted research efforts in the future. Nevertheless, it is intriguing that significant insight to system dynamics can be accrued from short epochs of time-series data (e.g. 800 points). Interestingly, the methodology is also fully applicable to continuous time series data [14].

7. Conclusions

This paper highlights the optimal features of RQA in deriving useful information about the dynamics of complex systems without necessitating mathematical assumptions a priori of any kind. Since that RQA variables have the ability to delineate exact points of bifurcations in the logistic map with transients, it is hypothesized that biological complexities can similarly be sorted. Many biological pathologies involve analogous bifurcations in their altered dynamics, and the ability of RQA to detect such sudden state changes has obvious applicative appeal. We suggest that it is not unreasonable to envision that multidimensional, nonlinear systems are best analyzed by multidimensional, nonlinear tools, of which RQA may be the best candidate to date.

8. Recurrence quantification analysis software

RPA graphs and RQA variables were computed with C-language programs written by one of us (CLW). Copyrighted DOS programs and example data sets are available on the World Wide Web: http://orion.it.luc.edu/~cwebber/.

References

[1] R. Benigni and A. Giuliani, Am. J. Physiol. 266 (1994) R1697.
[2] J.-P. Eckmann, S.O. Kamphorst and D. Ruelle, Europhys. Lett. 4 (1987) 973.
[3] C.L. Webber Jr. and J.P. Zbilut, J. Appl. Physiol. 76 (1994) 965.
[4] R. Lydic, FASEB J. 1 (1987) 6.
[5] P. Collet and J.-P. Eckmann, Iterated maps on the interval as dynamical systems (Birkhäuser, Basel, 1980).
[6] R.M. May, Nature 261 (1976) 459.
[7] T.-Y. Li and J.A. Yorke, Am. Math. Monthly 82 (1975) 985.
[8] H.E. Nusse and J.A. Yorke, Science 271 (1996) 1376.
[9] M. Markus and B. Hess, Comp. Graphics 13 (1989) 553.
[10] R. Shaw, Z. Naturforsh. 36a (1981) 80.
[11] J.P. Zbilut, in: Rhythms in physiological systems, eds. H. Haken and H.-P. Koepchen (Springer, Berlin, 1991) p. 139.
[12] C.L. Webber Jr., M.A. Schmidt and J.M. Walsh, J. Appl. Physiol. 78 (1995) 814.
[13] C.L. Webber Jr. and J.P. Zbilut, in: Bioengineering approaches to pulmonary physiology and medicine, ed. M.C.K. Khoo (Plenum, New York, 1996) p. 137.
[14] J.P. Zbilut and C.L. Webber Jr., Phys. Lett. A 171 (1992) 199.