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Classification
Physics Abstracts
02.50 — 05.40 — 05.45

Short Communication

Complexity and $1/f$ noise. A phase space approach

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(Received 1 May 1991, accepted 13 May 1991)

Abstract. — A complexity measure is introduced which is the integral of the coarse-grained, scale dependent entropies. As an application we analyze a Gaussian distribution model for noise signals. We find that among all possible distributions the one having the $1/f$ power spectrum maximizes the complexity measure.

Our experience in statistical mechanics gives us the impression that both regular and completely disordered configurations are not really *complex*. Several recent studies [1] suggest that there should exist a complexity measure which attains its maximum value for configurations between the above two cases. However, there is so far no generally accepted definition of the complexity measure.

The $1/f$ noise problem refers to the low frequency divergence of the power spectra. Noise signals in many different scientific disciplines were found to have approximately the $1/f$ power spectrum [2]. Its ubiquity may suggest that there should exist some general principle which would make the appearance of $1/f$ noise unavoidable. However most work in this field concentrate on the search for specific mechanisms in specific systems.

In this work we introduce a complexity measure based on the entropy of various scales. It is a thermal dynamical quantity and is uniquely defined whenever the entropy exists. The traditional entropy characterizes the disorderness of the system. We generalize this to consider the scale dependent entropy which is defined on the coarse-grained variables of the original system. Our complexity measure (*complexity* for short) is an integral of all these scale dependent entropies. Below through a calculable model we show that the above two problems (complexity and $1/f$ noise) are closely related, with $1/f$ noise in Nature being the most complex.

Let us consider a continuous stochastic signal $x(t)$, in the time interval $0 \leq t \leq N$. For most of our purposes, it is sufficient to consider only the two point correlation function,

$$C(t-t') = < x(t) x(t') >, \tag{1}$$

from it one can obtain the power spectrum

$$D(f) = \int_0^N \mathrm{d}t \cos 2\pi f t C(t). \tag{2}$$

In this work we use the simplifying assumption that a signal $x(t)$ is completely specified by its two point correlation function $C(t)$. This means that we ignore all the higher correlations. We introduce a Gaussian ansatz [3] for the distribution functional of $x(t)$,

$$P[x(t)] = \frac{1}{Z} \exp - \frac{1}{2} \int \mathrm{d}t\mathrm{d}t' x(t) x(t')\, g(t - t'), \tag{3}$$

where the normalization factor is

$$Z = \int \mathcal{D}x \exp - \frac{1}{2} \int \mathrm{d}t\mathrm{d}t' x(t) x(t')\, g(t - t'), \tag{4}$$

and $g(t)$ is related to $C(t)$ through

$$\int \mathrm{d}t'' g(t - t'')\, C(t'' - t') = \delta(t - t'). \tag{5}$$

Thus by construction, equation (1) can be reproduced using the above distribution functional $P[x(t)]$ (distribution for short). For our phase space analysis we may treat a noise signal as a purely geometric, static curve of length N, ignoring its dynamic origin. We can further assume the periodic boundary condition and the kernel $g(t)$ is taken to be symmetrical. The effects of boundary conditions vanish when $N \to \infty$.

The Shannon-like entropy (entropy for short) of the distribution can be defined by the functional integral

$$S = - \int \mathcal{D}x P[x(t)] \ln P[x(t)] = N/2 + \ln Z. \tag{6}$$

This entropy is simply related to the phase space volume Z in equation (4). Note that if $x(t)$ takes on continuous values, as in our case, the entropy is not positive definite.

Due to the simple Gaussian form, it is convenient to work in the Fourier space. Let us write

$$x(t) = \sqrt{\frac{2}{N}} \sum_{k=1}^{N} \cos\frac{2\pi k t}{N} x_k,\ g(t) = \frac{2}{N} \sum_{k=1}^{N} \cos\frac{2\pi k t}{N} g_k. \tag{7}$$

Equations (3,4) are now in particularly simple form,

$$P[x(t)] = \frac{1}{Z} \exp - \frac{1}{2} \sum_{k=1}^{N} g_k x_k^2,\ \ Z = \prod_{k=1}^{N} \int_{-\infty}^{\infty} \mathrm{d}x_k \exp - \frac{1}{2} \sum_{k=1}^{N} x_k^2. \tag{8}$$

Without loss of generality, we consider $g(t)$ being defined by g_k through equation (7), g_k thus completely specifies the distribution $P[x(t)]$. We want now to compare different signal distributions characterized by different g_k's To make such a comparison meaningful, we have to impose a common normalization condition on the signal $x(t)$, for all the distributions. We consider the mean square fluctuation (or the amplitude),

$$W \equiv \frac{1}{N} \int_0^N \mathrm{d}t < x^2(t) > = \frac{1}{N} \sum_{k=1}^{N} < x_k^2 > = \frac{1}{N} \sum_{k=1}^{N} 1/g_k \simeq \int_{1/N}^{1} D(f), \tag{9}$$

where $< \cdot >$ denotes the average using $P[x(t)]$ in equation (8), $< x(t) >= 0$, the approximation in the last step is due to the replacement of the discrete sum by an integral. We require that W remains constant for all distributions,

$$W = \sum_{k=1}^{N} w_k = \text{const}, \quad w_k \equiv \frac{1}{N g_k}. \tag{10}$$

Other normalization condition can be likewise considered. The above condition is particularly attractive if we identify $x(t)$ as the current fluctuation in a circuit, W has the meaning of the average power of the circuit. We may regard the amplitude W as the energy needed to create the signal $x(t)$.

The Gaussian integrals in equation (8) can be easily carried out, from equations (6,8) we obtain the entropy,

$$S = \frac{N}{2} + \frac{1}{2} \sum_{k=1}^{N} \ln 2\pi N w_k. \tag{11}$$

For what distribution does the above entropy attains its maximal value? It is straight-forward to see that the white noise solution $w_k = W/N$ ($w_k \sim D(f)$) is optimal, when $S = \frac{N}{2}(1 + \ln 2\pi W)$. The conclusion is hardly surprising that white noise has the largest entropy.

It is instructive to see how correlation reduces the entropy. For this purpose we consider the power-law ansatz

$$g_k = q k^{\sigma}, \quad \sigma > 0, \tag{12}$$

which corresponds to the power spectrum $D(f) \approx 1/f^{\sigma}$ ($\sigma = 0$: white noise), and $C(t) \approx t^{\sigma-1}$, and q is a normalization constant. The condition equation (10) becomes,

$$W = \sum_{k=1}^{N} \frac{1}{N q k^{\sigma}} = \frac{1}{Nq} \theta_N(\sigma), \quad \theta_N(\sigma) \equiv \sum_{k=1}^{N} \frac{1}{k^{\sigma}} \tag{13}$$

We find $q = \theta_N(\sigma)/NW$, thus $w_k = W/\left(\theta_N(\sigma) k^{\sigma}\right)$. Substituting the latter into equation (11), and taking the asymptotic forms of $\theta_N(\sigma)$ for various σ, we obtain

$$S = N/2[1 + \ln 2\pi W + \ln(1-\sigma) + \sigma/2], \quad \text{for } 0 < \sigma < 1 \tag{14}$$

$$S = N/2[3/2 + \ln 2\pi W - \ln \ln N], \quad \text{for } \sigma = 1 \tag{15}$$

$$S = N/2[1 + \ln 2\pi W - \ln\zeta(\sigma) + \sigma/2 - (\sigma - 1)\ln N], \quad \text{for } \sigma > 1, \tag{16}$$

where $\zeta(\sigma)$ is the Riemann function. Note that in all cases the entropy is smaller than that of white noise. For weak correlation $0 < \sigma < 1$, ($C(t) \to 0$, $t \to \infty$), we see that there is a constant entropy *per* unit length. $\sigma = 1$ is the borderline case and for $\sigma > 1$, let us call it the strong correlation region, the entropy is no longer additive. There is a logarithmic reduction for the entropy *per* unit length. This fact has been noted previously in other model systems [4].

The Shannon-like entropy is traditionally regarded as the information content of the signal $x(t)$. However, this view implicitly assumes that the resolution of our measurement is sufficient to reveal all the information. It does not give indications about how much information one can obtain if the resolution length is not so good.

Suppose that our measurement can only reveal structures of length (time) scales τ or larger. We want to know how much entropy the system has. Let us take $1 \leq \tau < N$, where the lower bound 1 should coincide the original resolution length. We consider the coarse-grained variable

$$y(t) = \frac{1}{\tau} \int_{t-\tau/2}^{t+\tau/2} \mathrm{d}t' x(t'), \tag{17}$$

$y(t)$ is the signal $x(t)$ with its fluctuations on scales smaller than τ eliminated.

Repeating the previous steps we obtain the entropy of resolution scale τ, denoted by $S(\tau)$, from the effective distribution for $y(t)$. Z in equation (8) now becomes

$$Z(\tau) = \prod_{1}^{[N/\tau]} \int_{-\infty}^{\infty} \mathrm{d}y_k \exp -\frac{1}{2} \sum_{1}^{[N/\tau]} \tau g_k y_k^2 \tag{18}$$

$$S(\tau) = \frac{N}{2\tau} + \ln Z(\tau) = \frac{N}{2\tau} + \frac{1}{2} \sum_{k=1}^{[N/\tau]} \ln(2\pi N w_k/\tau), \tag{19}$$

where the ratio $[N/\tau]$ is the integer part of N/τ. $S(\tau)$ can be regarded as the total information content if the resolution length is τ. $S(\tau)$ bears resemblance to the Komolgorov entropy. Note that $S(\tau)$ is a strickly decreasing function of τ. By definition, we have $S(1) = S$.

It is useful to consider the *per* unit length quantity.

$$I(\tau) = S(\tau)\tau/N. \tag{20}$$

By varying τ, $I(\tau)$ can be regarded as the distribution spectrum of the information content, in analogy to the much more familar power spectrum $D(f)$. Since $I(\tau)$ is scale independent, we want to plot it against the scaleless quantity $\ln \tau$. A generic curve is shown in figure 1. It can be easily shown that for white noise, we have $I(\tau) = \frac{1}{2}(1 + \ln 2\pi W - \ln \tau)$, so in figure 1 it would be a straight line with the slope $-1/2$. This implies that white noise loses rapidly information upon coarse-graining. For a random walker ($\sigma = 2$), on the other hand, we can likewise show that it should be represented by a straight line of the slope 1/2. The borderline case $1/f$ noise ($\sigma = 1$) would be represented by a horizontal line in figure 1.

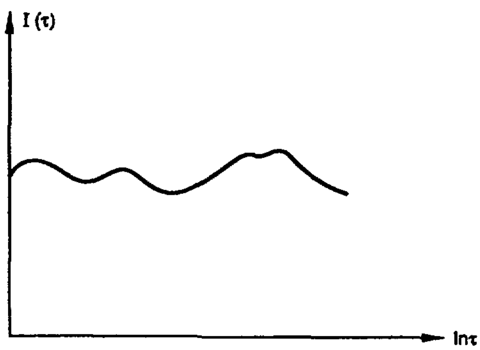

Fig. 1. — A generic curve of the information distribution spectrum.

To measure the information content on different resolution scales, it calls for a quantity that comprises entropy of all the resolution scales. We introduce the quantity *complexity K*, which is the integral of all the scale dependent entropies

$$K = \int_{1}^{N} \mathrm{d}\tau S(\tau). \tag{21}$$

In a sense our complexity K reflects the disorderness on all scales, very much in the same sense that the entropy does on the smallest resolution scale. In principle there should be an arbitrary measure in the above integral. Our choice is based on the purely dimensional consideration: we *want* K to have the same dimensional dependence of the Shannon-like entropy S, this uniquely fixes the measure in (21) (up to logarithmic factors). Therefore the above two considerations uniquely define the complexity measure: 1) it should contain entropy on different resolution scales; 2) it should be an extensive quantity (having dimension of N) just as the entropy itself. Note however K is not additive, due to logarithmic contributions. Had we chosen K to be proportional to another power of N, K would be maximized by a different type of noise than $1/f$. Our complexity is a legitimate thermal dynamical quantity, which is built on top of the entropy and has the same dimension. K has also the meaning of the area beneath the $I(\tau)$ curve in figure 1.

Substituting $S(\tau)$ of equation (19) into equation (21), we obtain

$$K = \frac{N}{2}\ln N + \frac{1}{2}\int_1^N \mathrm{d}\tau \sum_{k=1}^{[N/\tau]} \ln(2\pi N w_k/\tau) \tag{22}$$

$$\simeq \frac{N}{2}\left[\ln N(1+\ln 2\pi) + 1/2(\ln N)^2 + \sum_{k=1}^{N}\left(\frac{1}{k}-\frac{1}{N}\right)\ln w_k\right], \tag{23}$$

where the approximation is caused by the interchange between the integrals and the sums. We may ask what distribution maximizes the complexity K. It follows from the Kullback inequality in information theory [5] that among all the functions w_k subject to the condition (10),

$$w_k \simeq W/k \ln N \tag{24}$$

is the optimal solution, for asymptotically large N. Therefore we conclude that the distribution which has the $1/f$ power spectrum maximizes the complexity K.

As an example let us consider the power law distribution characterized by σ in equation (12).

Using equation (19) we obtain the complexity for the power law case

$$K = \frac{N\ln N}{2}\left[1 + \frac{\sigma}{2} + \ln 2\pi W + \frac{1-\sigma}{2}\ln N - \ln\theta_N(\sigma)\right]. \tag{25}$$

We plot the per unit length quantity K/N against σ in figure 2. We see that for the power law parametrization the complexity is a well defined smooth function of σ with the extremum attained at the point $\sigma = 1$. Note that $K(\sigma)$ is not a symmetrical function of σ, in a sense we may say that a random walker carries more information than white noise.

It is also instructive to consider constant complexity K attained by different power law distributions. The energy expenditure W is now taken to be a variable. We solve $\ln W$ from equation (25), plot it against σ and keep K constant, it is also shown in figure 2. We interprete that $1/f$ noise requires the least energy input to achieve the same amount of complexity.

The complexity measure introduced in this work can be easily extended to higher dimensions and discrete systems. Firstly we construct the scale dependent entropy $S(\tau)$ from the coarse-grained variable in the volume τ^d, as that done in equation (17), then on dimensional grounds the complexity measure for a $d-$dimensional system is

$$K = \int_1^N \tau^{d-1}\mathrm{d}\tau\, S(\tau). \tag{26}$$

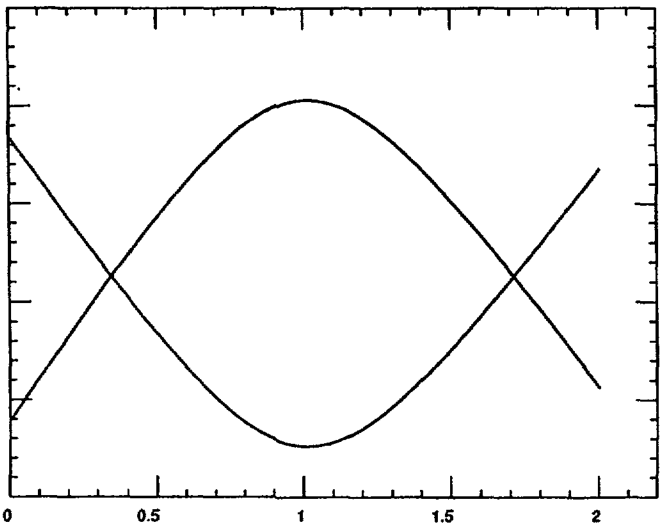

Fig. 2. — The upper curve is the complexity K, the lower curve is the energy expenditure ln W, on arbitrary scales. The horizontal axis is σ. The quantities are for a chain of $N = 10^6$ units.

$d = 2$ is a case of special interest, where the complex configurations have direct visual effects. We mention in passing another application of the complexity: take the Boltzmann factor of the 2d nearest neighbor ferromagnetic Ising model, in place of equation (3). One can show that the complexity measure is a function of the temperature $K(T)$, with its maximal value attained at T_c. Elsewhere we will address this and similar problems.

The main point of this work is to introduce a complexity measure K for probabilistic and statistic mechanical models. Under two rather weak conditions — 1) K being a linear superposition of the resolution scale dependent entropies, 2) being an extensive thermal dynamic variable — we arrive at a unique expression. For the 1d Gaussian model we have analyzed in details we find that K reaches its maximum for $1/f$ noise. From practical point of view, we can say that we find an attractive quantity which can characterize complex systems, and one may draw tantalizing interpretations. However, from conceptual point of view, we cannot yet explain *why* and *when* K in a system *should* be maximized. One feels that the approach presented in this work might be elevated into some sort of first principle for strongly constrained, self-organized [6] and open systems, like many sub-systems in Nature. Should we postulate that, for some of these systems, the intrinsic spatial temporal fluctuations are characterized by a maximal complexity principle, and $1/f$ noise is just one of *its* consequences?

Acknowledgements.

I thank Nordita for sponsoring my visit to Aarhus Univ. and I appreciate the discussions with H.C. Fogedby, I. Procaccia and D. Stauffer.

References

[1] BACHAS C.P. and HUBERMAN B.A., *Phys. Rev. Lett.* **57** (1986) 1965;
HOGG T. and HUBERMAN B.A., *Physica* **D22** (1986) 376;
for a recent review see GRASSBERGER P., Randomness, Information, and Complexity, Wuppertal Preprint (1989).
[2] PRESS W.H., *Comments Astrophys.* **7** (1978) 103.
[3] The Gaussian model for stochastic noise first appeared in FEYNMAN R.P. and HIBBS A.R., Quantum Mechanics and Path Integrals (New York, McGraw-Hill, 1965) pp.331.
[4] EBELING W. and NICOLIS G., *Europhys. Lett.* **14** (1991) 191;
GRASSBERGER P., in Ref. [1].
[5] KULLBACK S., Information Theory and Statistics (New York, John Wiley, Sons Co, 1959).
[6] BAK P., TANG C. and WIESENFELD K., *Phys. Rev.* **A38** (1987) 364.