Math 578 Assignment 2

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Question 1.1

Question 1.1.1

Plug the exact solution into the scheme, expand out the Taylor series, cancel and collect terms. Throughout, we denote u_{j+i}^{n+k} as u_i^k, and u_j^n as u. Let $\hat{u}$ be the exact solution.

$$\begin{aligned}
\text{LTE} &= |\hat{u}^{n+1} + u^{n+1}| \\
&= |u + u_t\Delta t + u_{tt}\frac{\Delta t}{2} + O(\Delta t^3) - [u + \Omega[(1-\theta)(u_{xx}\Delta x^2 + O(\Delta x^4))) + \theta(u_{xx}^{n+1}\Delta x^2 + O(\Delta x^4))]]| \\
&= |u_{tt}\frac{\Delta t^2}{2} + O(\Delta t^3) - \Delta t O(\Delta x^2)\text{TODO} \\
&= |(1-2\theta)u_{tt}\frac{\Delta t^2}{2} + O(\Delta t^3) + \Delta t O(\Delta x^2)|
\end{aligned}$$

Question 1.1.2

Let $u = e^{ikx}$, $\Omega = \frac{\Delta t}{\Delta x^2}$ sub into the scheme:

$$\begin{aligned}
G - 1 &= \Omega[(1-\theta)(e^{ikx} + e^{-ikx} - 2) + \theta G(e^{ikx} + e^{-ikx} - 2)] \\
G &= 1 + 2\Omega(1-\theta) + 2\Omega G\theta y
\end{aligned}$$

After letting $y = e^{ikx} + e^{-ikx} - 2$. Solving for G gets us:

$$G = \frac{2\Omega(1-\theta)y}{1 - 2\Omega\theta y}$$

And our stability restriction is $|G| \leq 1$, which does not have a convenient form in terms of θ, Δx, Δt.

Question 1.1.3

Picking $\theta = \frac{1}{2}$ is obvious: it gets us $max(O(\Delta t^3), \Delta t O(\Delta x^2))$ accuracy for no extra cost compared to any θ value not equal to 0.

Now, considering the balance of Δt and Δx, we derive an expression for the product of computer runtime, $C = \Delta t \Delta x$, and global error, $E = \frac{\text{LTE}}{\Delta t}$, which seems to be as good a metric as any.

$$\begin{aligned} CE &= (\Delta x \Delta t)\frac{\Delta t^3 + \Delta t \Delta x^2}{\Delta t} \\ &= \Delta x(\Delta t^3 + \Delta t \Delta x^2) \\ &= \Delta x \Delta t^3 + \Delta t \Delta x^3 \end{aligned}$$

We wind up with $\theta = \frac{1}{2}$, and $\Delta t = (\Delta x)^2$

Question 1.1.4

Let $f(x, t) = \sin x$. Then, taking the Fourier series of both sides gets us:

$$\begin{aligned} d_t \hat{u}_n &= -n^2 \hat{u}_n + \mathbf{1_{n=1}} \\ d_t \hat{u}_n &= -n^2 \hat{u}_n + \mathbf{1_{n=1}} \end{aligned}$$

So, assuming that $u_0 = 0$, we run the scheme to $t = 1$. We wind up with a plot of our "efficiency surface", (efficiency $= (CE)^{-1}$).

Question 1.2

Question 1.2.1

We need fourth derivative, so we need at least five points. Since we are taking a centered difference, we pick up second-order at no extra cost. Solve:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ -2h^2 & -h^2 & 0 & h^2 & 2h^2 \\ -2h^3 & -h^3 & 0 & h^3 & 2h^3 \\ -2h^4 & -h^4 & 0 & h^4 & 2h^4 \end{pmatrix} \begin{pmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

And obtain:

$$\begin{pmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{pmatrix} = \frac{1}{24h^4} \begin{pmatrix} 1 \\ -4 \\ 6 \\ -4 \\ 1 \end{pmatrix}$$

Question 1.2.2

$$\begin{aligned}
\text{LTE} &= |\hat{u}^{n+1} + u^{n+1}| \\
&= |u_t \Delta t + u_{tt}\frac{\Delta t^2}{2} + O(\Delta t^3) - \frac{\Delta t}{2\Delta x}(u_{j+1} - u_{j-1}) - \theta \Delta t[u_{xxxx} + O(\Delta x^2)]| \\
&= |u_t \Delta t + u_{tt}\frac{\Delta t^2}{2} + O(\Delta t^3) - \Delta t[u_t + u_{xx}\Delta x + u_{xxx}\Delta x^2 + u_{xxxx}\Delta x^3 + O(\Delta x^4)] - \theta \Delta t[u_{xxxx} + O(\Delta x^2) \\
&= |u_{tt}\frac{\Delta t^2}{2} + O(\Delta t^3) - \Delta t[u_{xx}\Delta x + u_{xxx}\Delta x^2 + u_{xxxx}\Delta x^3 + O(\Delta x^4)] - \theta \Delta t[u_{xxxx} + O(\Delta x^2)]|
\end{aligned}$$

And we choose θ to match the coefficent of u_{xxxx} from the first derivative stencil, so $\theta = -1$. However, this will not do much to change the asymptotic behaviour, since Δx^5 was pretty darn small already.

Question 1.2.3

$$G - 1 =$$

Question 1.2.4

Scheme cannot be second order and stable, because linear schemes can't be. (Godunov)

Question 1.3

Question 1.3.1

Assume periodic conditions. Derive a nice expression for $u_{j+1}^{n+1} - u^{n+1}$. Then, summing over all spatial indices, we have glorious cancellation and TVD.

We use the same index convention as the first question: we denote u_{j+i}^{n+k} as u_i^k, and u_j^n as u.

Begin by rearranging the scheme:

$$u^{n+1} = u + D^-(u_{j-1} - u) + D^+(u_{j+1} - u)$$

Where we have defined $D^i \equiv \Delta t C^i$.

Now, consider $u_{j+1}^{n+1} - u^{n+1}$

$$\begin{aligned}
u_{j+1}^{n+1} - u^{n+1} &= u_{j+1} + D^-(u_j - u_{j+1}) + D^+(u_{j+2} - u_{j+1}) - [u + D^-(u_{j-1} - u) + D^+(u_{j+1} - u)] \\
&= (1 - D^- - D^+)(u_{j+1} - u) + D^+(u_{j+2} - u_{j+1}) + D^-(u - u_{j-1}) \\
|u_{j+1}^{n+1} - u^{n+1}| &\leq (1 - D^- - D^+)|u_{j+1} - u| + D^+|u_{j+2} - u_{j+1}| + D^-|u - u_{j-1}|
\end{aligned}$$

If we now sum over all j, the D terms cancel with the preceding and proceding terms, leaving the inequality:

$$\sum_j |u_{j+1}^{n+1} - u^{n+1}| \leq \sum_j |u_{j+1} - u|$$

Question 1.3.2

This follows almost immediately from the linearity of the TV norm. We consider the case of only two terms; the extension is easy. Let $v^{n+1} \equiv C^-(u_{j-1} - u) + C^+(u_{j+1} - u)$ $w^{n+1} \equiv D^-(u_{j-1} - u) + D^+(u_{j+1} - u)$, $u = \theta v + (1 - \theta)w$.

$$\begin{aligned}
&TV(u^{n+1}) = \theta TV(v^{n+1}) + (1 - \theta)TV(w^{n+1}) \\
&TV(u^{n+1}) \leq \theta \max TV(v^{n+1}), TV(w^{n+1}) + (1 - \theta) \max TV(v^{n+1}), TV(w^{n+1}) \\
&TV(u^{n+1}) \leq \max TV(v^{n+1}), TV(w^{n+1}) \\
&TV(u^{n+1}) \leq TV(u)
\end{aligned}$$

Question 1.3.3

Is it a convex combination of Euler steps? Yes. So, yes.

Question 2

Question 2.1

Let $u = \sin x$. Then, $f(x) = \cos 2x - 2\sin x$.

Does $sin(x)$ count as trivial? Maybe. Regardless, the error is nearly 0 from the get go, since a sine wave is well-represented by a truncated Fourier series.

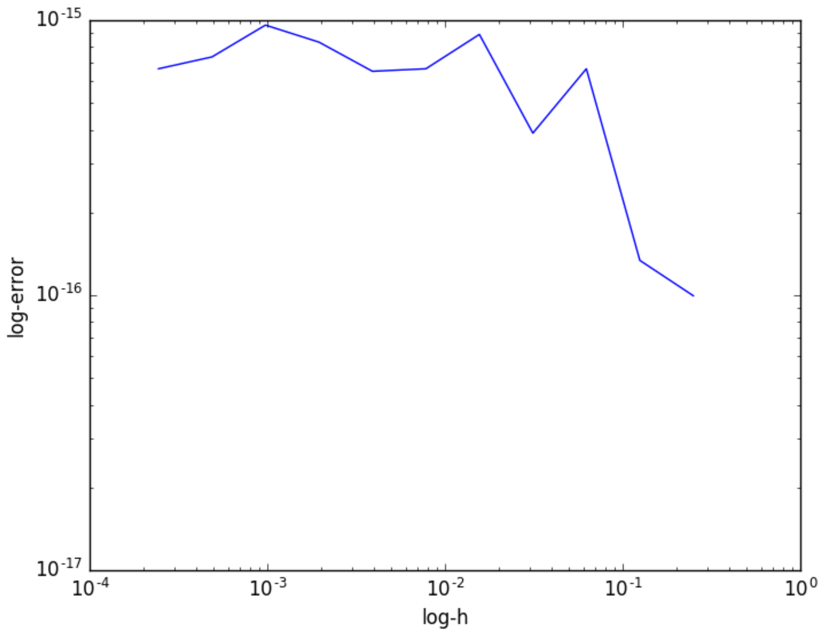

Figure 1: Question 2.1, L^∞ error vs. h

Question 2.2

Gibbs phenomenon: the L^∞ error approaches a finite, non-zero amount when attempting to represent an discontinous function with a truncated Fourier series.

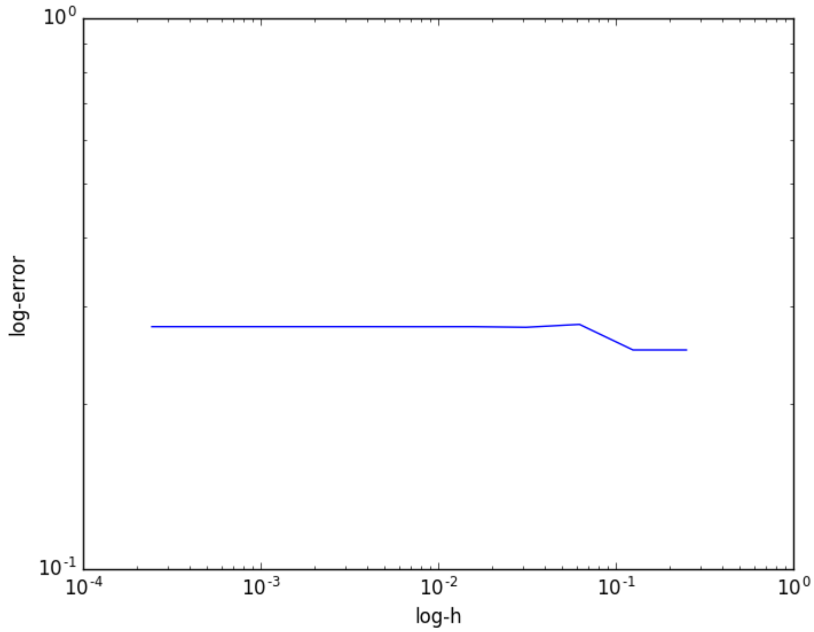

Figure 2: Question 2.2 L^∞ error vs. h

Question 3

Question 3.1

$u = -\omega^2 sin(\omega x)cos(\omega y)$

The error is small since the frequency of our input function f is below the Nyquist limit.

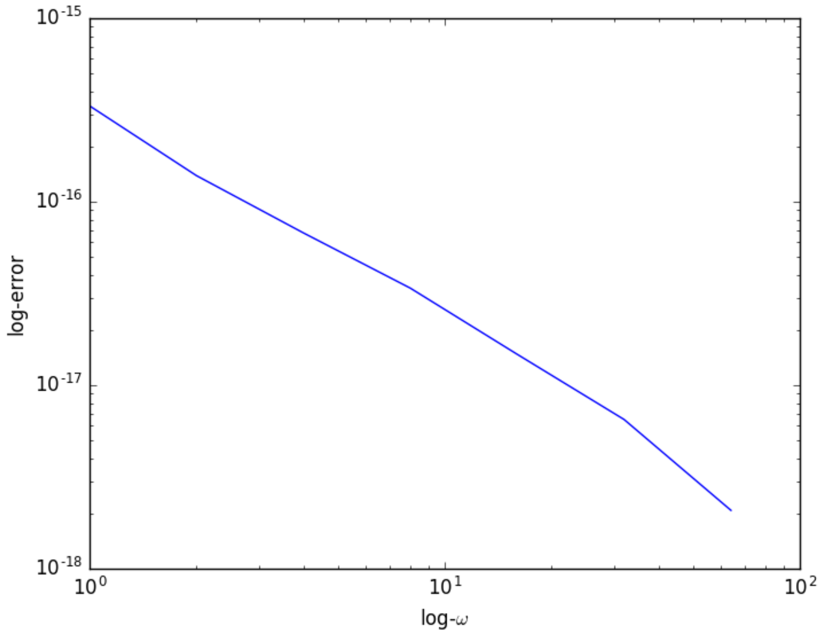

Figure 3: Question 3.1 $L^{\infty} error$ vs. ω

Question 3.2

Damned if I know. been bashing my head against this one for a while.

Question 3.3

Similarly.