Math 578 Assignment 2

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Fall 2016

Question 1

Question 1.1

Question 1.1.1

Plug the exact solution into the scheme, expand out the Taylor series, cancel and collect terms. Throughout, we denote u_{j+i}^{n+k} as u_i^k, and u_j^n as u. Let $\hat{u}$ be the exact solution.

$$
\begin{aligned}
\text{LTE} &= |\hat{u}^{n+1} + u^{n+1}| \\
&= |u + u_t\Delta t + u_{tt}\frac{\Delta t}{2} + O(\Delta t^3) - [u + \Omega[(1-\theta)(u_{xx}\Delta x^2 + O(\Delta x^4))) + \theta(u_{xx}^{n+1}\Delta x^2 + O(\Delta x^4))]]| \\
&= |u_{tt}\frac{\Delta t^2}{2} + O(\Delta t^3) - \Delta t O(\Delta x^2)\text{TODO} \\
&= |(1-2\theta)u_{tt}\frac{\Delta t^2}{2} + O(\Delta t^3) + \Delta t O(\Delta x^2)|
\end{aligned}
$$

Question 1.1.2

Let $u = e^{ikx}$, $\Omega = \frac{\Delta t}{\Delta x^2}$ sub into the scheme:

$$
\begin{aligned}
G - 1 &= \Omega[(1-\theta)(e^{ikx} + e^{-ikx} - 2) + \theta G(e^{ikx} + e^{-ikx} - 2)] \\
G &= 1 + 2\Omega(1-\theta) + 2\Omega G\theta y
\end{aligned}
$$

After letting $y = e^{ikx} + e^{-ikx} - 2$. Solving for G gets us:

$$
G = \frac{2\Omega(1-\theta)y}{1 - 2\Omega\theta y}
$$

And our stability restriction is $|G| \leq 1$, which does not have a convenient form in terms of θ, Δx, Δt.

Question 1.1.3

Picking $\theta = \frac{1}{2}$ is obvious: it gets us $max(O(\Delta t^3), \Delta t O(\Delta x^2))$ accuracy for no extra cost compared to any θ value not equal to 0.

Now, considering the balance of δt and δx, we derive an expression for the product of computer runtime, $C = \Delta t \Delta x$, and global error, $E = \frac{\text{LTE}}{\Delta t}$, which seems to be as good a metric as any.

$$\begin{aligned} CE &= (\Delta x \Delta t)\frac{\Delta t^3 + \Delta t \Delta x^2}{\Delta t} \\ &= \Delta x(\Delta t^3 + \Delta t \Delta x^2) \\ &= \Delta x \Delta t^3 + \Delta t \Delta x^3 \end{aligned}$$

We wind up with $\theta = \frac{1}{2}$, and $\delta t = (\delta x)^2$

Question 1.1.4

Mess around with solutions of the form u=sincos TODO

Question 1.2

1 get D again

2 Hideously ugly LTE.

3 Even uglier stability.

4 "modified equation approach…"? Somewhere in notes. Scheme cannot be second order, because linear schemes can't be.

Question 1.3

1 Assume periodic conditions. Then, we apply TV norm to $u^n + 1_i$, crunch out a nice expression for it. Then, summing over all spatial indices, we have glorious cancellation and TVD.

2 Follows almost immediately from the linearity of the TV norm.

3 Is it a convex combination of Euler steps? Yes. So, yes.

4 TODO

Question 2

Question 2.1

1 Let $u = sin(x)$ and $f = ...TODO$.

Does $sin(x)$ count as trivial? Maybe. Regardles, the error is nearly 0 from the get go, since a sine wave is well-represented by a truncated Fourier series.

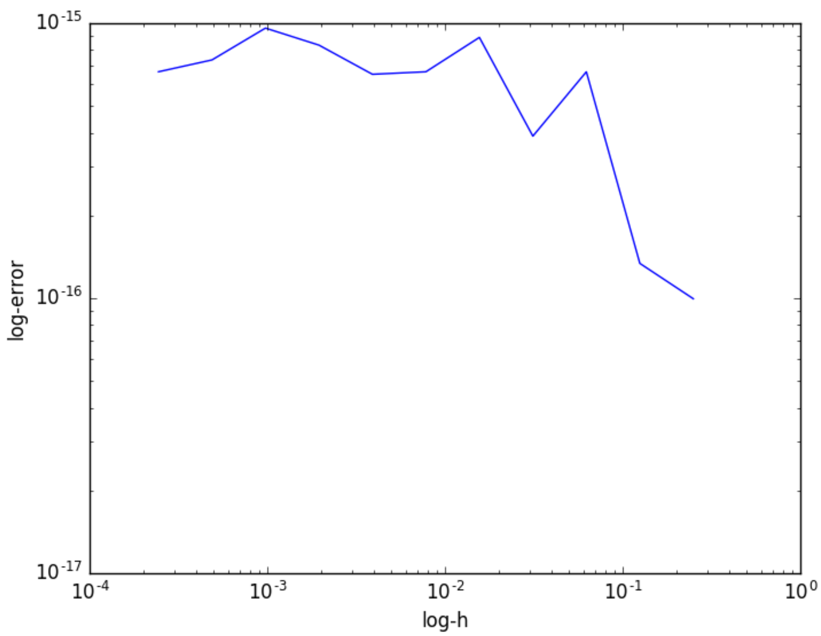

Figure 1: Question 2.1, L^∞ error vs. h

Question 2.2

Gibbs phenomenon.

Question 3

Question 3.1

$u = -\omega^2 sin(\omega x)cos(\omega y)$

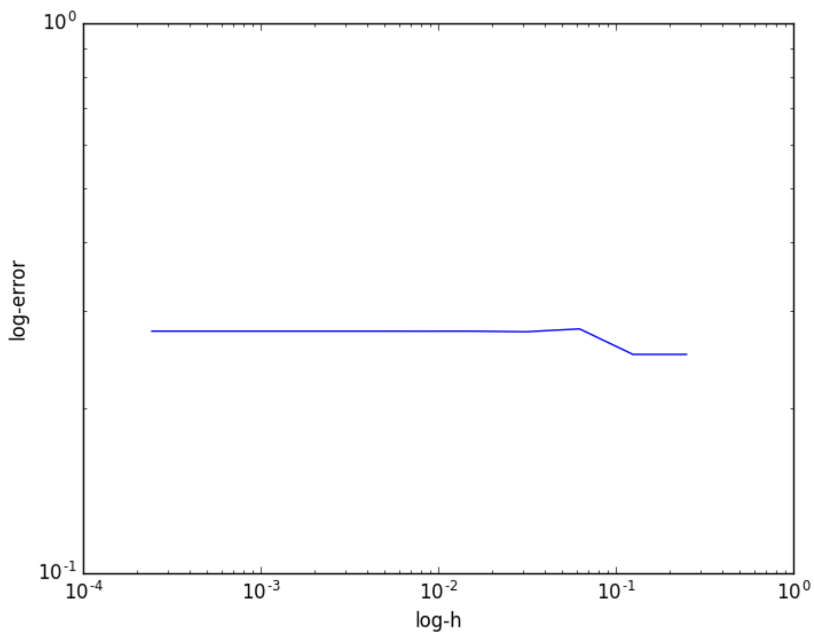

Figure 2: Question 2.2 L^∞ error vs. h

The error is small since the frequency of our input function f is below the Nyquist limit.

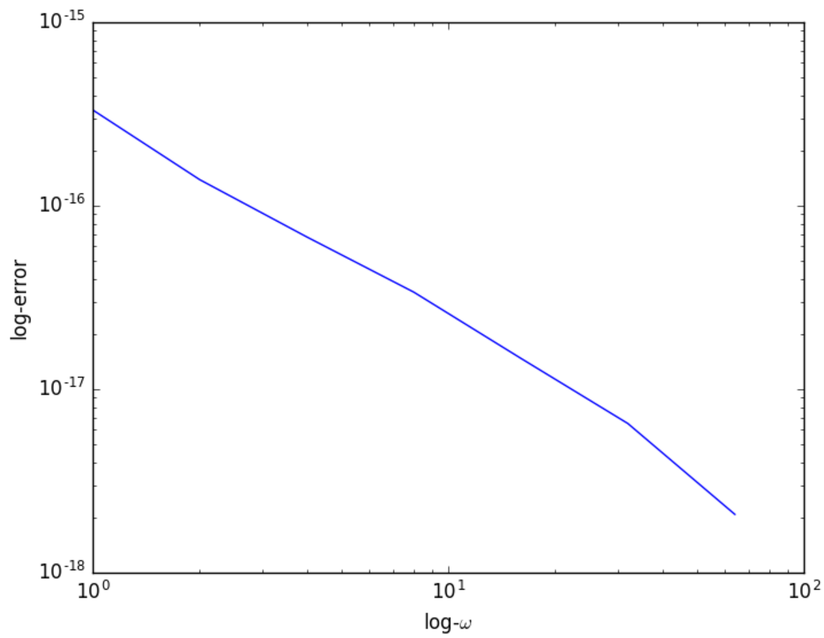

Figure 3: Question 3.1 $L^{\infty} error$ vs. ω

Question 3.2

Damned if I know. been bashing my head against this one for a while.

Question 3.3