

Math 578 Assignment 2

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Question 1.1

Question 1.1.1

Plug the exact solution into the scheme, expand out the Taylor series, cancel and collect terms. Throughout, we denote u_{j+i}^{n+k} as u_i^k , and u_j^n as u . Let \hat{u} be the exact solution.

$$\begin{aligned}\text{LTE} &= |\hat{u}^{n+1} - u^{n+1}| \\ &= |u + u_t \Delta t + u_{tt} \frac{\Delta t^2}{2} + O(\Delta t^3) - [u + \Omega[(1 - \theta)(u_{xx} \Delta x^2 + O(\Delta x^4))] + \theta(u_{xx}^{n+1} \Delta x^2 + O(\Delta x^4))]| \\ &= |u_{tt} \frac{\Delta t^2}{2} + O(\Delta t^3) - \Delta t O(\Delta x^2)| \\ &= |(1 - 2\theta)u_{tt} \frac{\Delta t^2}{2} + O(\Delta t^3) + \Delta t O(\Delta x^2)|\end{aligned}$$

Question 1.1.2

Let $u = e^{ikx}$, $\Omega = \frac{\Delta t}{\Delta x^2}$ sub into the scheme:

$$\begin{aligned}G - 1 &= \Omega[(1 - \theta)(e^{ikx} + e^{-ikx} - 2) + \theta G(e^{ikx} + e^{-ikx} - 2)] \\ G &= 1 + 2\Omega(1 - \theta) + 2\Omega G \theta y\end{aligned}$$

After letting $y = e^{ikx} + e^{-ikx} - 2$. Solving for G gets us:

$$G = \frac{2\Omega(1 - \theta)y}{1 - 2\Omega\theta y}$$

And our stability restriction is $|G| \leq 1$, which does not have a convenient form in terms of θ , Δx , Δt .

Question 1.1.3

Picking $\theta = \frac{1}{2}$ is obvious: it gets us $\max(O(\Delta t^3), \Delta t O(\Delta x^2))$ accuracy for no extra cost compared to any θ value not equal to 0.

Now, considering the balance of Δt and Δx , we derive an expression for the product of computer runtime, $C = \Delta t \Delta x$, and global error, $E = \frac{LTE}{\Delta t}$, which seems to be as good a metric as any.

$$\begin{aligned} CE &= (\Delta x \Delta t) \frac{\Delta t^3 + \Delta t \Delta x^2}{\Delta t} \\ &= \Delta x (\Delta t^3 + \Delta t \Delta x^2) \\ &= \Delta x \Delta t^3 + \Delta t \Delta x^3 \end{aligned}$$

We wind up with $\theta = \frac{1}{2}$, and $\Delta t = (\Delta x)^2$

Question 1.1.4

Let $f(x, t) = \sin x$. Then, taking the Fourier series of both sides gets us:

$$\begin{aligned} d_t \hat{u}_n &= -n^2 \hat{u}_n + \mathbb{K}_{n=1} \\ d_t \hat{u}_n &= -n^2 \hat{u}_n + \mathbb{K}_{n=1} \end{aligned}$$

So, assuming that $u_0 = 0$, we run the scheme to $t = 1$. We wind up with a plot of our “efficiency surface”, (efficiency = $(CE)^{-1}$).

Question 1.2

Question 1.2.1

We need fourth derivative, so we need at least five points. Since we are taking a centered difference, we pick up second-order at no extra cost. Solve:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ -2h^2 & -h^2 & 0 & h^2 & 2h^2 \\ -2h^3 & -h^3 & 0 & h^3 & 2h^3 \\ -2h^4 & -h^4 & 0 & h^4 & 2h^4 \end{pmatrix} \begin{pmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

And obtain:

$$\begin{pmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 6 \\ -4 \\ 1 \end{pmatrix}$$

Question 1.2.2

2 Hideously ugly LTE.

3 Even uglier stability.

4 “modified equation approach...”? Somewhere in notes. Scheme cannot be second order, because linear schemes can’t be.

Question 1.3

1 Assume periodic conditions. Then, we apply TV norm to $u^n + 1_i$, crunch out a nice expression for it. Then, summing over all spatial indices, we have glorious cancellation and TVD.

2 Follows almost immediately from the linearity of the TV norm.

3 Is it a convex combination of Euler steps? Yes. So, yes.

4 TODO

Question 2

Question 2.1

1 Let $u = \sin(x)$ and $f = \dots$ *TODO*.

Does $\sin(x)$ count as trivial? Maybe. Regardless, the error is nearly 0 from the get go, since a sine wave is well-represented by a truncated Fourier series.

Question 2.2

Gibbs phenomenon.

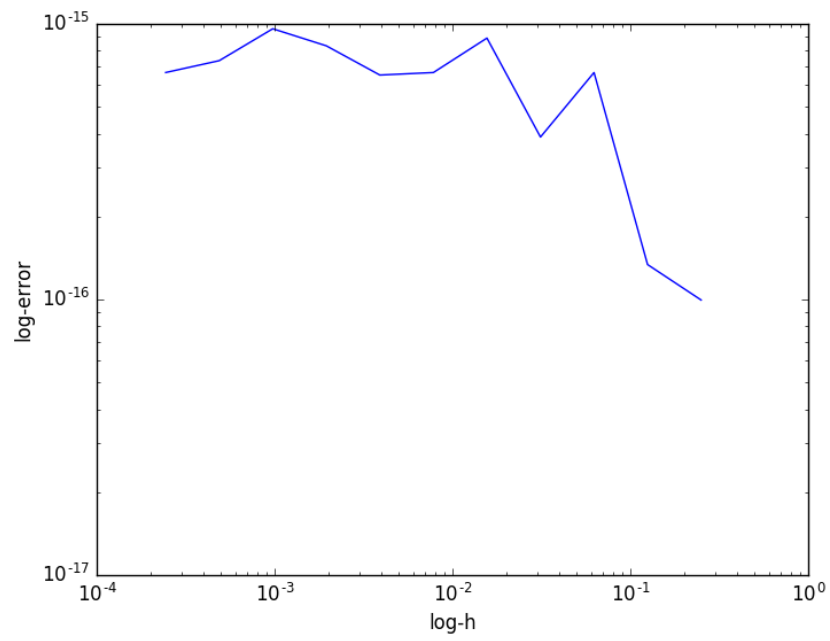


Figure 1: Question 2.1, L^∞ error vs. h

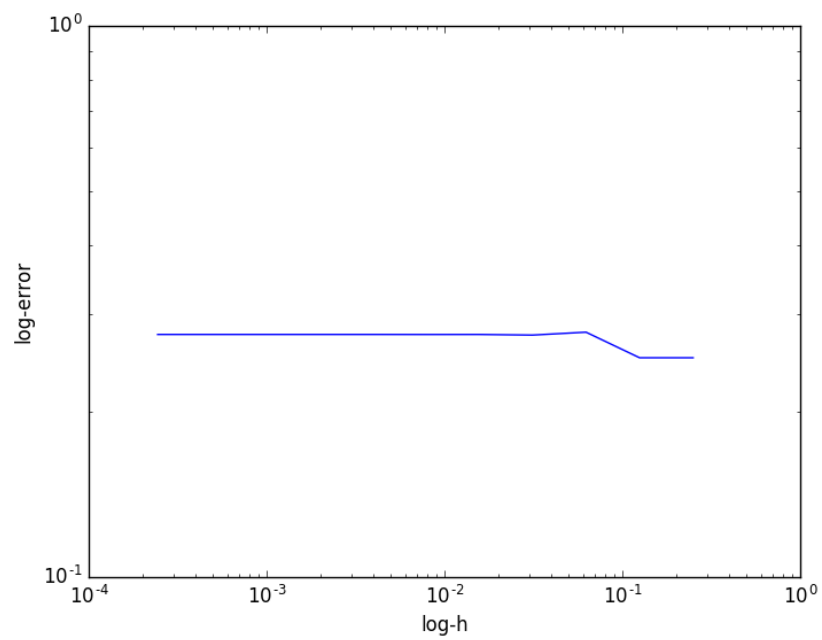


Figure 2: Question 2.2 L^∞ error vs. h

Question 3

Question 3.1

$$u = -\omega^2 \sin(\omega x) \cos(\omega y)$$

The error is small since the frequency of our input function f is below the Nyquist limit.

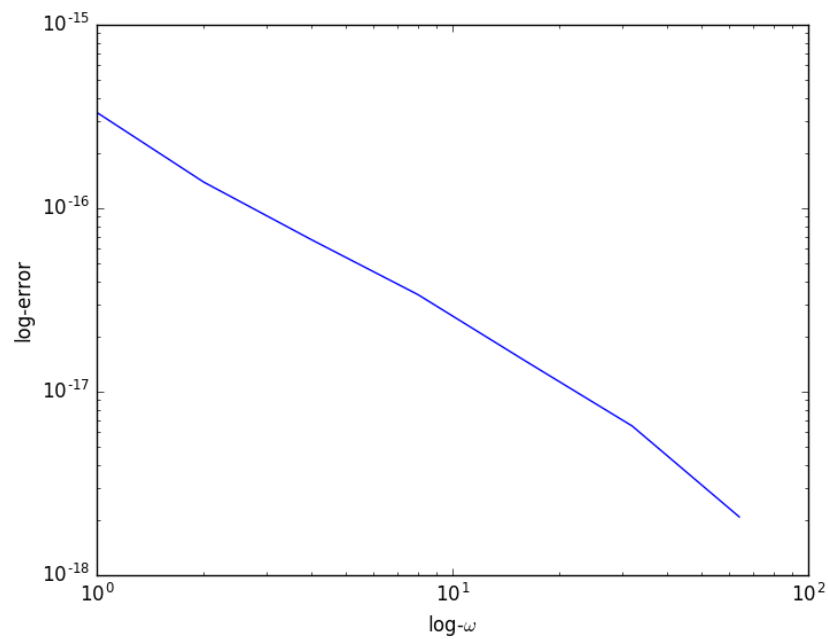


Figure 3: Question 3.1 $L^\infty error$ vs. ω

Question 3.2

Damned if I know. been bashing my head against this one for a while.

Question 3.3