Motivic Zeta Function

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In 1949 Andrè Weil, in his Paper "Number of solutions of equations in finite fields", formulated his famous Weil-Conjectures which predicted certain deep properties of the Hasse-Weil zeta function for a Variety X over a finite field $k = \mathbb{F}_q$ defined as

$$Z(X,t) = exp\big(\sum_{m=0}^{\infty} \frac{|X(\mathbb{F}_{q^m})|}{m} t^m\big)$$

The rationality of the zeta function, the most basic of the three conjectures, was first established by Dwork. Grothendieck later gave another proof using his new scheme-theoretic approach to algebraic geometry, an approach that was in large parts motivated by the desire to at some point proof all the Weil conjectures. Grothendieck's revolutionary new perspective was fundamental for the work of Deligne, who used these new methods to finally establish the last part of the conjectures, an analouge of the Riemann hypothesis, in 1974. In this thesis we want to generalize the Hasse-Weil zeta function to a broader setting, not only talking about varieties over finite fields but varieties in general. It turns out that in this more general setting the zeta function isn't as well behaved in terms of rationality as the "classical" Hasse-Weil zetafunction and we will investigate this phenomen more closely in the case of complex surfaces, following the work of Larsen and Lunts.

Definiton 1. In the following the term *k-variety* always means a separated, integral scheme of finite type over a field k. We will write $\mathcal{V}_k$ for the category of k-varieties.

To get a better feeling for what exactly would be a good generalisation, we give another equivalent definition of the Hasse-Weil zeta function in terms of effective 0-cycles. Recall that a 0-cycle on a Variety X is an element of the free abelian group over the closed points of X. Such a cycle $\alpha = \sum_i n_i x_i$ is called *effective* if the n_i are greater than or equal to zero. The *degree* of α is given by $deg(\alpha) = \sum_i n_i deg(x_i)$ ($deg(x_i)$ being the degree of the field extension $k(x_i)/k$).

Using this notation we can rewrite the zeta function as

$$Z(X,t) = \sum_{\alpha} t^{deg(\alpha)} \tag{1}$$

(For a proof of this (elementary) identity see for example Mustaţǎs great notes on zeta functions in algebraic geometry [Mus, Remark 2.9]

So the definition comes down to counting effective 0-cycles of a given degree.

Definiton 2. Let k be a perfect field.[1] Now consider the abelian group of formal linear combinations of isomorphism classes of varieties, subject to relations of the form

$$[X \setminus Y] = [X] - [Y]$$

where Y is closed in X. With multiplication given by

$$[X][Y] = [X \times Y]$$

this forms a ring, called the *Grothendieck ring of varieties* and denoted by $K_0[\mathcal{V}_k]$. A *motivic measure* is a ring homomorphism $\mu : K_0[\mathcal{V}_k] \to A$ into a ring A. The identity function $id : K_0[\mathcal{V}_k] \to K_0[\mathcal{V}_k]$ is called the *universal motivic measure.*

Since the fibre product is commutative up to isomorphism this ring is commutative. The cut and paste relation furthermore gives us

$$0 = [\emptyset] - [\emptyset] = [\emptyset \setminus \emptyset] = [\emptyset]$$

and the multiplicative neutral element is given by $1 = [Spec(k)]$

Remark 3. By [Har77, Proposition 10.1 (d)] the product of two smooth varieties over k is again smooth. Hence the isomorphism classes of smooth irreducible complete varieties form a multiplicative monoid, in the following denoted by $\mathcal{M}$.

Example 4. *Using the decomposition* $\mathbb{P}_k^n = \mathbb{P}_k^{n-1} \coprod \mathbb{A}_k^n$ *where* $\mathbb{P}_k^{n-1}$ *is a (closed) hyperplane in* $\mathbb{P}_k^n$ *we get* $[\mathbb{P}_k^n] = [\mathbb{P}_k^{n-1}] + [\mathbb{A}_k^n]$ *in the Grothendieck ring. Inductively this yields the identity*

$$[\mathbb{P}_k^n] = \sum_{k=0}^{n} [\mathbb{A}_k^1]^k$$

We also denote the isomorphism class of the affine line as $\mathbb{L}$.

Definiton 5. Let $\mu : K_0[\mathcal{V}_k] \to A$ be a motivic measure and X a variety. Then we define the *motivic zeta function* $\zeta_\mu(X,t) \in A[[t]]$ as

$$\zeta_\mu(X,t) := \sum_{n=0}^{\infty} \mu([Sym^n(X)])t^n$$

[1] The definition makes sense even if k isn't perfect, we just have to adjust to the fact that the product of two varieties need not be reduced. But since all our applications are either over finite fields or in characteristic zero we make our life a little easier by assuming k as perfect

Example 6. *We can now compute a first zeta function, namely $\zeta_{id}(\mathbb{P}^1_k, t)$. Using the identity and notation from example 4 we calculate*

$$\begin{aligned}\zeta_{id}(\mathbb{P}^1_k, t) &= \sum_{n=0}^{\infty}[Sym^n(\mathbb{P}^1_k)]t^n \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\mathbb{L}^k\Big)t^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(\mathbb{L}t)^k t^{n-k} \\ &= \Big(\sum_{n=0}^{\infty}t^n\Big)\Big(\sum_{n=0}^{\infty}(\mathbb{L}t)^n\Big) \\ &= \frac{1}{(1-t)(1-\mathbb{L}t)}\end{aligned}$$

Hence $\zeta_{id}(\mathbb{P}^1_k, t)$ is in fact a rational function.

Kapranov proves in [Kap00] that the zeta function of a curve with coefficents in a field is in fact always rational. In their paper [LL03] Larsen and Lunts prove that in the case of surfaces this is false in general. Later the same authors gave a more precise characterisation when the zeta function of a complex surface is rational (namely if and only if the Kodaira dimension is $-\infty$. See [LL04]).

We will proof the following statement from [LL03]

Theorem 7. *Assume that $k = \mathbb{C}$. There exists a field $\mathcal{H}$ and a motivic measure $\mu : K_0[\mathcal{V}_k] \to \mathcal{H}$ with the following property: if X is a smooth complex projective surface such that $P_g(X) = h^{2,0}(X) \geq 2$, then the zeta-function $\zeta_\mu(X, t)$ is not rational.*

The first important result on the way to prove this is a structure theorem for $K_0[\mathcal{V}_{\mathbb{C}}]$.

Theorem 8. *Set $k = \mathbb{C}$. Let G be an abelian commutative monoid and $\mathbb{Z}[G]$ be the corresponding monoid ring. As above, denote by $\mathcal{M}$ the multiplicative monoid of irreducible smooth complete varieties. Let*

$$\psi : \mathcal{M} \to G$$

be a homomorphism of monoids such that

(i) $\psi([X]) = \psi([Y])$ if X and Y are birational;

(ii) $\psi([\mathbb{P}^n]) = 1$ for all $n \geq 0$.

Then ψ can be uniquely extended to a ring homomorphism

$$\phi : K_0[\mathcal{V}_{\mathbb{C}}] \to \mathbb{Z}[G]$$

To prove this result we will use a result by Bittner.

Theorem 9 (Bittner, [Bit04, Theorem 3.1])**.** *The Grothendieck group $K_0[\mathcal{V}_{\mathbb{C}}]$ is generated by classes of smooth complete varieties subject to relations of the form*

$$[X] - [f^{-1}(Z)] = [Y] - [Z]$$

where X,Y are smooth complete varieties and $f : X \to Y$ is a morphism which is a blowup with a smooth center $Z \subset Y$.

Proof of theorem 8. We have to check that ψ preserves the above relations, i.e. that $\psi([X]) - \psi([f^{-1}(Z)]) = \psi([Y]) - \psi([Z])$. But $[X]$ and $[Y]$ are birational since f is a blowup. (WHY?) Now $f^{-1}(Z)$ is birational to $Z \times \mathbb{P}^n$ (WHY??) and thus

$$\psi([f^{-1}(Z)]) = \psi([Z \times \mathbb{P}^n]) = \psi([Z][\mathbb{P}^n]) = \psi([Z])\psi([\mathbb{P}^n) = \psi([Z])$$

Hence we can linearly extend ψ to define the morphism $\phi : K_0[\mathcal{V}_{\mathbb{C}}] \to \mathbb{Z}[G]$ $\square$

References

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