Motivic Zeta Function

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1 Introduction

In 1949 André Weil, in the paper [Wei49], formulated his famous conjectures which predicted certain deep properties of the Hasse–Weil zeta function $Z(X,t)$ for a variety X over a finite field $k = \mathbb{F}_q$ which is defined as

$$Z(X,t) = \exp\left(\sum_{m=0}^{\infty} \frac{|X(\mathbb{F}_{q^m})|}{m} t^m\right) \in \mathbb{Q}[[t]].$$

Dwork first established the rationality of the zeta function in [Dwo60], the most basic of the three conjectures. Later, using his new scheme theoretic approach to algebraic geometry, Grothendieck gave another proof for the special case of smooth projective varieties ([Gro95]).

In [Kap00], Kapranov generalized the Hasse–Weil zeta function, defining it not only for varieties over finite fields but varieties in general. This *motivic zeta function* has coefficents in the so called *Grothendieck ring of varieties* $K_0[\mathcal{V}_k]$ over some fixed field k, which consists of isomorphism classes of varieties subject to some additional relations (see Definition 3.2), and is given as

$$\zeta_\mu(X,t) := \sum_{n=0}^{\infty} \mu\left([\mathrm{Sym}^n(X)]\right) t^n$$

where μ is a ring homomorphism from $K_0[\mathcal{V}_k]$ to some arbitrary ring (also called a *motivic measure*), and $Sym^n(X)$ is the n-fold symmetric product of X (see Definition 4.5). It is a natural question to ask if this zeta function is still a rational function (at least if μ takes values in a field). Kapranov proved this for smooth projective curves ([Kap00, Thm 1.1.9]) and conjectured that it is true in general. But it turns out that in the motivic setting the zeta function is not as well behaved in terms of rationality as in the classical case: Larsen and Lunts showed in [LL03] that the zeta function fails to be rational for complex surfaces.

Theorem 1.1 ([LL03, Thm. 1.6])**.** *Assume that $k = \mathbb{C}$. There exists a field $\mathcal{H}$ and a motivic measure $\mu_h \colon K_0[\mathcal{V}_k] \to \mathcal{H}$ with the following property: if X is a smooth complex projective surface such that the geometric genus $P_g(X) = h^{2,0}(X)$ is at least 2, then the zeta-function $\zeta_{\mu_h}(X,t)$ is not a rational function.*

In this thesis we will first introduce the Grothendieck ring in Section 3 and investigate its structure. This culminates in a structure theorem due to Bittner ([Bit04, Thm 3.1]) which will allow us to substantially simplify the original proof by Larsen and Lunts. The second section is devoted to the definition of the motivic zeta function, which entails defining and giving examples for the symmetric product. There we will also see the connection between the classical Hasse–Weil zeta function and the motivic one. Construction of the motivic measure μ_h under wich $Z_{\mu_h}(X,t)$ becomes irrational is content of Section 5 while irrationality is then finally proven in the last section.

2 German Summary

In dieser Bachelorarbeit soll ein Resultat von Larsen und Lunts ([LL03]) über die rationalität der motivischen Zetafunktion bewiesen werden. Die Motivische Zetafunktion ist eine Verallgemeinerung der Hasse–Weil Zetafunktion für Varietäten über endlichen Körpern im Kontext des Grothendieckrings der Varietäten $K_0[\mathcal{V}_k]$ über Isomorphieklassen von Varietäten zwischen denen gewissen zusätzliche Relationen bestehen (Siehe 3.2 für die genaue Definition). Diese motivische Zetafunktion ist, im Gegensatz zur Hasse-Weil Zetafunktion, im allgemeinen nicht rational. Konkret werden wir folgendes Resultat zeigen.

Theorem 2.1 ([LL03, Thm. 1.6])**.** *Sei $k = \mathbb{C}$. Dann existiert ein Körper $\mathcal{H}$ und ein Ringhomomorphismus $\mu_h \colon K_0[\mathcal{V}_k] \to \mathcal{H}$ mit der folgenden Eigenschaft: Falls X eine glatte, projektive, komplexe Fläche mit geometrischem Genus $P_g(X) = h^{2,0}(X)$ von mindestens 2 ist, dann ist die motivische Zetafunktion $\zeta_{\mu_h}(X,t)$ nicht rational.*

Wir werden ein Theorem von Bittner ([Bit04]) verwenden um den ursprünglichen Beweis deutlich kompakter darstellen zu können.

3 Motivic Measures on the Grothendieck Ring of Varieties

Definiton 3.1. In the following the term *k-variety* always means a separated, reduced scheme of finite type over a field k. We will write $\mathcal{V}_k$ for the category of k-varieties.

Definiton 3.2. Let k be a perfect field. Consider the abelian group of formal linear combinations of isomorphism classes of varieties, subject to relations of the form

$$[X \setminus Y] = [X] - [Y]$$

where Y is closed in X. With multiplication given by

$$[X][Y] = [X \times Y]$$

this forms a ring, called the *Grothendieck ring of varieties* and denoted by $K_0[\mathcal{V}_k]$.

A *motivic measure* with values in a ring A is a ring homomorphism $\mu \colon K_0[\mathcal{V}_k] \to A$. The identity function $\mathrm{id} \colon K_0[\mathcal{V}_k] \to K_0[\mathcal{V}_k]$ is called the *universal motivic measure.*

The reason that we require k to be perfect is that for nonperfect k the product of two varieties need not be reduced. Take for example $k = k_0(t)$ with k_0 of characteristic $p > 0$ and $X = \mathrm{Spec}(k[x]/(x^p - t))$, $Y = \mathrm{Spec}(K) \coloneqq \mathrm{Spec}(k(t^{\frac{1}{p}}))$. Then X is a variety since $x^p - t$ is irreducible in $k_0(t)$. The product, on the other hand, can be described as

$$X \times Y = \mathrm{Spec}(K[x]/(x^p - t)) = \mathrm{Spec}(K[X]/((x - t^{\frac{1}{p}})^p)$$

which is not reduced. We might adapt the defintion to this case by changing the defintion of the product to $[X][Y] \coloneqq [(X \times Y)_{\mathrm{red}}]$, taking the induced reduced scheme structure on $X \times Y$, but since we are only working in characteristic zero or over finite fields we will not bother with this technical issue.

Since the product of varities is commutative up to isomorphism this is a commutative ring with 1 (equal to $[\mathrm{Spec}(k)]$). The cut and paste relation furthermore gives us

$$0 = [\emptyset] - [\emptyset] = [\emptyset \setminus \emptyset] = [\emptyset]$$

Example 3.3. For a finite field k, consider the function [1]

$$\begin{aligned} \psi \colon \mathcal{V}_k &\to \mathbb{Z} \\ X &\mapsto |X(k)| \end{aligned}$$

Observe that ψ has the following properties:

1. If X and Y are isomorphic k-varieties, then $\psi(X) = \psi(Y)$.
2. If $Y \subset X$ is closed subvariety, then $\psi(X \setminus Y) = \psi(X) - \psi(Y)$.
3. If X and Y are two varieties, then $\psi(X \times Y) = \psi(X)\psi(Y)$.

[1] not bothering with any set-theoretic issues

Hence ψ actually factors over $K_0[\mathcal{V}_k]$ and gives rise to a motivic measure.

Example 3.4. Using the decomposition $\mathbb{P}_k^n = \mathbb{P}_k^{n-1} \coprod \mathbb{A}_k^n$ where $\mathbb{P}_k^{n-1}$ is a (closed) hyperplane in $\mathbb{P}_k^n$ we get $[\mathbb{P}_k^n] = [\mathbb{P}_k^{n-1}] + [\mathbb{A}_k^n]$ in the Grothendieck ring. Inductively this yields the identity

$$[\mathbb{P}_k^n] = \sum_{m=0}^{n} \mathbb{L}^m$$

Where we write $\mathbb{L}$ for the isomorphism class of the affine line.

This calculation also gives rise to an example of two irreducible varieties that have the same equivalence class in $K_0[\mathcal{V}_k]$ but are not isomorphic: $\mathbb{P}^n \times (\mathbb{A}^1 \setminus \{0\})$ and $\mathbb{A}^{n+1}$. These are isomorphic since the global section of the former are $k[x]_x$, but in the Grothendieck ring we get

$$\begin{aligned}
[\mathbb{P}^n \times (\mathbb{A}^1 \setminus \{0\})] &= [\mathbb{P}^n] \left([\mathbb{A}^1] - 1\right) \\
&= \sum_{m=0}^{n} \mathbb{L}^{m+1} - \sum_{m=0}^{n} \mathbb{L}^m \\
&= \mathbb{L}^{n+1}
\end{aligned}$$

Remark 3.5. By [Har77, Prop. 10.1 (d)] the product of two smooth varieties over k is again smooth. Hence the isomorphism classes of smooth irreducible complete varieties form a multiplicative monoid, in the following denoted by $\mathcal{M}$.

We now investigate the structure of the Grothendieck ring. First we note that it suffices to take irreducible varieties to generate $K_0[\mathcal{V}_k]$ To see this, take a variety X with irreducible componets $X_1, \dots, X_n$ and set $U_i \coloneqq X_i \setminus \bigcup_{i \neq j} X_j$ and $U \coloneqq \bigcup_i U_i$. By construction, the last union is actually a disjoint union of closed subsets of U, and hence $[U] = \sum_i [U_i]$. Together this gives

$$[X] = [X \setminus U] + \sum_i [U_i].$$

Now the U_i, as open subsets of irreducible sets, are irreducible and $[X \setminus U]$ has dimension strictly smaller than the dimension of X since we removed a proper open subset from each irreducible component. Hence we can inductively rewrite $[X \setminus U]$, loosing at least one dimension in each step, with the base case of $\dim X = 0$ being trivial, as X is then just a collection of points.

In characteristic zero we can use a weak form of Hironaka's Theorem on the resolution of singularites, namely that for every irreducible, projective variety X there is a smooth, projective variety that is birational to X to restrict the set of needed generators even more. This observation culminates in a structure theorem about the Grothendieck ring proven by Bittner in [Bit04].

Theorem 3.6 ([Bit04, Thm. 3.1])**.** *Let k be a field of characteristic zero. Then $K_0[\mathcal{V}_k]$, as an abelian group, is generated by smooth, integral, complete varieties. Furthermore, if k is algebraically closed, it suffices to consider relations of the form*

$$[X] - [f^{-1}(Z)] = [Y] - [Z]$$

where X,Y are smooth complete varieties and $f\colon X \to Y$ is a morphism which is a blowup with a smooth center $Z \subset Y$.

Proof. We will only proof the first part of the statement. As remarked before, we can restrict our attention to irreducible varietes. We once again argue by induction on the dimension. Let X be an irreducible variety of dimension n. By choosing some nonempty affine subvariety and passing to its projective closure we find X' projective, irreducible and birational to X. Resolving singularities we find $\widetilde{X}$ smooth, projective, irreducible birational to X. Hence we find isomorphic open subset $U \subset X$, $V \subset \widetilde{X}$ which gives us

$$[X \setminus U] - [\widetilde{X} \setminus V] = [X] - [U] - [\widetilde{X}] + [V] = [X] - [\widetilde{X}].$$

Thus $[X]$ can be written as the sum of $[\widetilde{X}]$ and some varieties of lower dimension for which we can invoke the induction hypothesis. □

We will use the slightly stronger conclusion we just shown, hence we state it as a separate corollary.

Corollary 3.7. *For each (irreducible) variety X over a field of characteristic zero there exists a smooth, irreducible, projective variety Y birational to X such that $[X] = [Y] + \sum_i m_i[W_i]$ where the $[W_i]$ are smooth, irreducible, projective varieties of dimension strictly smaller than the dimension of X.* □

4 Symmetric Products and the Motivic Zeta Function

We now want to make the concept of the symetric product precise, i.e. how it can be defined as the quotient of the (regular) product by the action of the symmetric group.

Definiton 4.1. Let X be a S-Scheme, G a group acting on X via S-automorphisms (i.e there is a group homomorphism $\sigma\colon G \to Aut_S(X)$, we also write σ_g for the image of g under σ.) Then a variety Y together with a morphism $\pi\colon X \to Y$ over S is called the quotient (also denoted by X/G) of Y by G if π is G-invariant which means that for all $g \in G$ we have $\pi \circ g = \pi$ and is universal in this respect, i.e. for every other G-invariant morphism

$f\colon X \to Z$ there exists a unique G-invariant morphism $g\colon Z \to Y$ that makes the following diagram commute

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & X/G \\ Z \arrow[ur, dashed, "g"] & \end{tikzcd} .$$

We first construct the quotient in the affine case.

Lemma 4.2. *Let $X = \operatorname{Spec} A$ be an affine variety with a group G acting on it. Then the categorial quotient exists, is itself a variety and is given by $X/G = \operatorname{Spec}(A^G)$, the spectrum of the G-invariant subring of A.*

Proof. TODO □

We now turn to a relevant special case: the quotient of an quasiprojective variety X by the action of a finite group G, using the following general result from [Gro71]

Lemma 4.3 ([Gro71, Prop. 1.8])**.** *Let X be a scheme with a finite group G acting on it via automorphisms. If there exists an open cover by G-invariant affine sets then the quotient X/G exists. We call such an action* admissible

Corollary 4.4. *If X is quasiprojective, then the quotient by the action of a finite group always exists.*

Proof. Let $X \subset \mathbb{P}^n_k$ be locally closed with a finite group $G = \{g_1, \cdots, g_m\}$ acting on it. Consider the orbit $\{g_1(x), \cdots, g_m(x)\} \subset X$ of a point $x \in X$, and denote by $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ the corresponding graded prime ideals corresponding to $\overline{\{x\}}$, and by $\mathfrak{q}$ the graded prime ideal corresponding to $\overline{X} \setminus X$ (taken to be the irrelevant ideal if this set is empty). Hence the ideal $\mathfrak{q}$ is not contained in any of the $\mathfrak{p}_i$s hence, by the graded version of the prime avoidance lemma we find an element with positive degree $y \in \mathfrak{q}$ that is contained in none of the $\mathfrak{p}_i$ and hence the hypersurface $V_+(y)$ contains $\bar{X} \setminus X$ but not the $g_1(x), \cdots, g_m(x)$, so $U := D_+(y)$ is an affine set that lies in X and contains the orbit of x under G. Now consider $U' := \bigcap_{g \in G} g(U)$. This is clearly G-invariant, and since U contains the orbit of y we still find that y lies in U'. Hence we have found a open cover of X by G-invariant open affines, and thus the quotient exists by the above lemma. □

Definiton 4.5. Let X be a quasiprojective variety, then the n-fold *symmetric product* $\operatorname{Sym}^n(X)$ is defined as the quotient of X^n (which is again quasiprojective via the Segre embedding) by the natural action of the symmetric group in n letters.

Example 4.6.

Definiton 4.7. Let $\mu\colon K_0[\mathcal{V}_k] \to A$ be a motivic measure and X a quasiprojective variety. Then we define the *motivic zeta function* $\zeta_\mu(X,t) \in A[[t]]$ as

$$\zeta_\mu(X,t) := \sum_{n=0}^{\infty} \mu([\mathrm{Sym}^n(X)])t^n$$

Example 4.8. We can now compute a first zeta function, namely $\zeta_{id}(\mathbb{P}^1_k, t)$. Using the identity and notation from example 3.4 we calculate

$$\begin{aligned}
\zeta_{id}(\mathbb{P}^1_k, t) &= \sum_{n=0}^{\infty} \left[\mathrm{Sym}^n(\mathbb{P}^1_k)\right] t^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathbb{L}^k \right) t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (\mathbb{L}t)^k \, t^{n-k} \\
&= \left(\sum_{n=0}^{\infty} t^n \right) \left(\sum_{n=0}^{\infty} (\mathbb{L}t)^n \right) = \frac{1}{(1-t)(1-\mathbb{L}t)}.
\end{aligned}$$

Hence $\zeta_{id}(\mathbb{P}^1_k, t)$ is in fact a rational function.

5 Constructing Motivic Measures

We will now construct the measure μ_h of Theorem 1.1.

The first important result on the way is a lemma that helps us to construct such measures by extending maps from the monoid of smooth, irreducible and complete varieties. For this we quickly recall the notion of a monoid ring.

Definiton 5.1. Let G be a (multiplicative) monoid. Write $\mathbb{Z}[G]$ for the free abelian group over G, i.e. the set of formal sums $\sum_{g\in G} n_g g$ with the m_g being integers, only finitely many non zero. Defining multiplication as

$$\Big(\sum_{g\in G} n_g g\Big)\Big(\sum_{g\in G} m_g g\Big) = \sum_{g\in G} \Big(\sum_{hl=g} n_h m_l\Big) g$$

this forms a ring: the *monoid ring* over G.

Theorem 5.2. *Set $k = \mathbb{C}$. Let G be an abelian monoid and $\mathbb{Z}[G]$ the corresponding monoid ring. As above, denote by $\mathcal{M}$ the multiplicative monoid of irreducible smooth complete varieties. Let*

$$\psi\colon \mathcal{M} \to G$$

be a homomorphism of monoids such that

i) *$\psi([X]) = \psi([Y])$ if X and Y are birational;*

ii) $\psi([\mathbb{P}^n]) = 1$ *for all* $n \geq 0$.

Then ψ *can be uniquely extended to a ring homomorphism*

$$\phi\colon K_0[\mathcal{V}_{\mathbb{C}}] \to \mathbb{Z}[G]$$

Proof. By Theorem 3.6 we know that the Grothendieck ring (as an abelian group) is generated by $\mathcal{M}$, so after extending ψ to the free abelian group over $\mathcal{M}$ we have to check that ψ preserves the relations of the blowup presentation of $K_0[\mathcal{V}_{\mathbb{C}}]$ and hence factors over it, i.e. given a blow-up $f\colon X \to Y$ with smooth center $Z \subset X$ then $\psi([X]) - \psi([f^{-1}(Z)]) = \psi([Y]) - \psi([Z])$. Note that $[X]$ and $[Y]$ are birational, since f is a blowup, thus $\psi([X]) = \psi([Y])$ by property i).

Now, since X and Z are nonsingular, the exceptional divisor $f^{-1}(Z)$ is isomorphic to the projective space bundle $\mathbb{P}(\mathcal{I}/\mathcal{I}^2)$ where $\mathcal{I}$ is the ideal sheaf associated to Z (see for example [Har77, II.8, Thm 8.24]), which is a $\mathbb{P}^n$-bundle, i.e. locally trivializes to $\mathbb{P}^n_k \times Z$ and hence is birational to $\mathbb{P}^n_k \times Z$. We can thus write $\psi([f^{-1}(Z)])$ as

$$\psi([f^{-1}(Z)]) = \psi([Z \times \mathbb{P}^n]) = \psi([Z][\mathbb{P}^n]) = \psi([Z])\psi([\mathbb{P}^n]) = \psi([Z])$$

where the last equation holds because of property ii). Hence, we can extend ψ to define a morphism $\phi\colon K_0[\mathcal{V}_{\mathbb{C}}] \to \mathbb{Z}[G]$ □

Before constructing a suitable monoid homomorphism that will yield our desired measure, we recall the definition of the Hodge numbers of a variety as well as two basic results (all these are excercises in [Har77, Ch. II]).

Definiton 5.3. Let X be a projective variety. The i-th *Hodge number* $h^{i,0}$ is given as the dimension of $H^0(X, \Omega^i_X)$, which is a finite dimensional k-vector space by a theorem of Serre ([Ser55, §3 Prop. 7]). If d is the dimension of a nonsingular variety X we will also write $P_g(X)$ for $h^{d,0}(X)$, called the *geometric genus* of X.

Lemma 5.4 ([Har77, II Ex. 8.8])**.** *The Hodge numbers* $h^{i,0}$ *are birational invariants of a variety: let* X, Y *be birationally equivalent, smooth, projective varieties then* $h^{i,0}(X) = h^{i,0}(Y)$.

Lemma 5.5 (Chow, [Har77, II Ex. 4.10])**.** *Every proper variety is birational to a projective variety.*

Definiton 5.6. Denote by $C \subset \mathbb{Z}[t]$ the multiplicative monoid of polynomials with positive leading coefficient. For a smooth projective complex variety Z of dimension d define

$$\Psi_h(X) \coloneqq 1 + h^{1,0}(X)t + \cdots + h^{d,0}(X)t^d \in C.$$

(Note that the Hodge numbers might be zero, but $\Psi_h(X)$ still has a positive leading coefficient, just its degree might be smaller than the dimension of X) By Chow's lemma we can also define Ψ_h for a smooth complete variety Z by choosing a smooth projective variety X which is birational to Z and setting $\Psi_h(Z) \coloneqq \Psi_h(X)$. This is well defined by Lemma 5.4.

We now check that Ψ_h satisfies all conditions of Theorem 5.2. Independence of birational equivalence class was the content of Lemma 5.4.

To check multiplicativity, we use the following lemma.

Lemma 5.7. *Let X, Y be smooth, irreducible, projective k-varieties. Then the following equality holds:*

$$h^{p,0}(X \times_k Y) = \sum_{i+j=p} h^{i,0}(X)h^{j,0}(Y)$$

Proof. TODO □

With this we directly calculate

$$\begin{aligned}\Psi_h(X)\Psi_h(Y) &= \left(\sum_n h^{n,0}(X)t^n\right)\left(\sum_n h^{n,0}(Y)t^n\right) = \sum_n \left(\sum_{i+j=n} h^{i,0}(X)h^{j,0}(Y)\right) t^n \\ &= \sum_n h^{n,0}(X \times_k Y)t^n = \Psi_h(X \times_k Y)\end{aligned}$$

6 Irrationality of the Motivic Zeta Function

The constructed motivic measure does not yet take values in a field, so we would like to pass to the quotient field of $\mathbb{Z}[\mathcal{M}]$. That we are able to do so is the content of the next lemma.

Lemma 6.1. *Let A be a factorial ring, and $S \subset A$ a multiplicative submonoid such that 1 is the only unit in S. Then the monoid ring $\mathbb{Z}[S]$ is a polynomial ring (in possibly infinitely many variables), and hence an integral domain.*

Proof. Since A is factorial, every $s \in S$ has a unique factorization, and since 1 is the only unit in S, s can be uniquely written as product of prime elements, hence if we take B to be the polynomial ring over the formal variables $\{x_s | s \in S, s \text{ prime}\}$ we get an isomorphism of rings

$$B \to \mathbb{Z}[S],\ x_s \mapsto s$$

□

Definiton 6.2. Denote by $\mathcal{H}$ the quotient field of $\mathbb{Z}[C]$ where, as above, C denotes the submonoid of $\mathbb{Z}[t]$ consisting of polynomials with positive leading coefficients. Since -1 is not contained in C it satisfies the conditions of the previous lemma. We define the motivic measure $\mu_h \colon K_0[\mathcal{V}_k] \to \mathcal{H}$ as the measure obtained extending Ψ_h as by Theorem 5.2

Lemma 6.3. *Let $Y_1, \dots, Y_s, Z$ be irreducible varieties of dimension d over a field of characteristic zero such that $\mu_h([Z]) = \sum_i n_i \mu_h([Y_i])$ for some $n_i \in \mathbb{Z}$ and $P_g(Z) \neq 0$ then $P_g(Z) = P_g(Y_i)$ for some i*

Proof. By Corollary 3.7, we find smooth, irreducible projective varieties $\overline{Z}, \overline{Y_1}, \cdots, \overline{Y_s}$ in the same birational class as the original varieties allowing us to rewrite the original equality as

$$\mu_h([\overline{Z}]) = \sum n_i \mu_h([\overline{Y}_i]) + \sum l_i \mu_h([X_i])$$

where the X_i are smooth irreducible varieties of dimension $< d$. Now, since μ_h was obtained as an extension of Ψ_h which was defined on smooth, irreducible, projective varieties, this is actually an equation in $\mathbb{Z}[C]$, namely

$$\Psi_h(\overline{Z}) = \sum n_i \Psi_h(\overline{Y}_i) + \sum l_i \Psi_h(X_i)$$

Since there are no (additive) relations between elements of C in the monoid ring, one of the polynomials $\Psi_h(\dots)$ on the right hand side, must actually be the polynomial on the left hand side, but with the dimension of X_i being strictly smaller than that of Z and the assumption that $P_g(\overline{Z}) = h^{d,0}(\overline{Z}) \neq 0$ it cannot be one of the $\Psi_h(X_i)$ because all these polynomials have strictly smaller degree. Hence we have $\Psi_h(\overline{Z}) = \Psi_h(\overline{Y}_i)$ for some i and in particular, since they are of the same dimension, the genus of $\overline{Z}$ and Y_i must agree. □

Now let X be a smooth projective surface with $P_g(X) \geq 2$. We will show that $\zeta_{\mu_h}(X, t) \in \mathcal{H}[[t]]$ is not rational, thus proving Theorem 1.1. We need one technical lemma about the genus of symmetric powers of X whose proof is out of scope of this thesis.

Lemma 6.4 ([LL03, Lem. 3.8])**.** *Let X be a smooth, projective surface over $\mathbb{C}$. Then*

$$P_g(\mathrm{Sym}^n(X)) = \binom{P_g(X) + n - 1}{P_g(X) - 1}$$

We will use the following rationality criterion for power series.

Lemma 6.5. *Let k be a field and suppose the power series $\sum a_i t^i \in k[[t]]$ is a rational function (i.e. an element in $k(t)$) then there exist $n, n_0 > 0$ such that for each $m > n_0$ the determinant*

$$\begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+n} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+n} & a_{m+n+1} & \cdots & a_{m+2n} \end{vmatrix}$$

vanishes.

Proof. Denote the above matrix by A. Since $\sum a_i t^i$ is rational, we may write it as

$$\sum a_i t^i = \frac{p_0 + \cdots + p_e t^e}{q_0 + \cdots + q_d t^n}$$

with not all q_i being zero. Hence $p_0 + \cdots + p_e t^e = \left(\sum a_i t^i\right)\left(q_0 + \cdots + q_d t^d\right) = \sum_{i=0}^{\infty}\left(\sum_{k+l=i} a_k q_l\right) t^i$. This means that for $m > e$ we get $0 = \sum_{k+l=m} a_k q_l$. now note that $q_l = 0$ for $l > n$ hence we may rewrite this as

$$0 = \sum_{l=0}^{n} q_l a_{m-l} = \sum_{l=0}^{n} q_{n-l} a_{m-n+l}.$$

Thus we have shown that

$$\begin{pmatrix} a_m & a_{m+1} & \cdots & a_{m+n} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+n} & a_{m+n+1} & \cdots & a_{m+2n} \end{pmatrix} \begin{pmatrix} q_n \\ \vdots \\ q_1 \\ q_0 \end{pmatrix} = \begin{pmatrix} \sum_{l=0} a_{m+l} q_{n-l} \\ \vdots \\ \vdots \\ \sum_{l=0} a_{m+n+l} q_{n-l} \end{pmatrix} = 0$$

for m big enough. Since one of the q_l is non zero this means that A has nonempty kernel and hence its determinant vanishes. □

Proof of Theorem 1.1. Assume that $\zeta_{\mu_h}(X,t)$ is rational, hence by the above criterion there is an n such that for m big enough

$$\sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma)\mu_h\Big(\Big[\prod_{i=1}^{n+1} \operatorname{Sym}^{m+i+\sigma(i)-2}(X)\Big]\Big) = 0$$

$$\Leftrightarrow \mu_h\Big(\Big[\prod_{i=0}^{n} \operatorname{Sym}^{m+2i}(X)\Big]\Big) = -\sum_{\substack{\sigma \in S_{n+1} \\ \sigma \neq id}} \operatorname{sgn}(\sigma)\mu_h\Big(\Big[\prod_{i=0}^{n} \operatorname{Sym}^{m+i+\sigma(i+1)-1}(X)\Big]\Big)$$

Now we can apply Lemma 6.3 to conclude that there is a permutation σ, not the identity permutation, such that

$$P_g\Big(\prod_{i=0}^{n} \operatorname{Sym}^{m+2i}(X)\Big) = P_g\Big(\prod_{i=0}^{n} \operatorname{Sym}^{m+i+\sigma(i+1)-1}(X)\Big)$$

By Lemma 6.4 (and using the fact that the genus is multiplicative), we get

$$\prod_{i=0}^{n} \binom{P_g(X) + m + 2i - 1}{P_g(X) - 1} - \prod_{i=0}^{n} \binom{P_g(X) + m + i + \sigma(i+1) - 2}{P_g(X) - 1} = 0.$$

By assumption $P_g(X) \geq 2$ hence the left hand side, considered as an polynomial in m, is not the zero polynomial. So by taking m large enough we obtain a contradiction. □

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