

Motivic Zeta Function

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Definiton 1. In the following the term *k-variety* always means a separated, integral scheme of finite type over a field k . We will write \mathcal{V}_k for the category of k -varieties.

Definiton 2. Let k be a field. Consider the group of formal linear combinations of isomorphism-classes in \mathcal{V}_k . Setting $[X] \times [Y] := [X \times Y]$ makes this into a ring. The *Grothendieck ring of varieties* $K_0[\mathcal{V}_k]$ is then obtained by modding out relations of the form

$$[X] - [Y] = [X \setminus Y]$$

where Y is closed in X .

A *motivic measure* is a ring homomorphism $\mu : K_0[\mathcal{V}_k] \rightarrow A$ into a ring A . The identity function $id : K_0[\mathcal{V}_k] \rightarrow K_0[\mathcal{V}_k]$ is called the *universal motivic measure*.

Let us now make some remarks about this ring.

Remark 3. The Grothendieck ring of varieties is commutative, as

$$X \times Y \cong Y \times X$$

for two schemes X and Y .

Remark 4. By [Har77, Proposition 10.1 (d)] the product of two smooth varieties over k is again smooth. Hence the isomorphism classes of smooth irreducible complete varieties form a multiplicative monoid, in the following denoted by \mathcal{M} .

Example 5. Using the decomposition $\mathbb{P}_k^n = \mathbb{P}_k^{n-1} \amalg \mathbb{A}_k^n$ where \mathbb{P}_k^{n-1} is a (closed) hyperplane in \mathbb{P}_k^n we get $[\mathbb{P}_k^n] = [\mathbb{P}_k^{n-1}] + [\mathbb{A}_k^n]$ in the Grothendieck ring. Inductively this yields the identity

$$[\mathbb{P}_k^n] = \sum_{k=0}^n [\mathbb{A}_k^1]^k$$

We also denote the isomorphism class of the affine line as \mathbb{L} .

Definiton 6. Let $\mu : K_0[\mathcal{V}_k] \rightarrow A$ be a motivic measure and X a variety. Then we define the zeta function $\zeta_\mu(X, t) \in A[[t]]$ as

$$\zeta_\mu(X, t) := \sum_{n=0}^{\infty} \mu([Sym^n(X)]) t^n$$

Example 7. We can now compute a first zeta function, namely $\zeta_{id}(\mathbb{P}_k^1, t)$. Using the identity and notation from example 5 we calculate

$$\begin{aligned}
\zeta_{id}(\mathbb{P}_k^1, t) &= \sum_{n=0}^{\infty} [\text{Sym}^n(\mathbb{P}_k^1)] t^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \mathbb{L}^k \right) t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n (\mathbb{L}t)^k t^{n-k} \\
&= \left(\sum_{n=0}^{\infty} t^n \right) \left(\sum_{n=0}^{\infty} (\mathbb{L}t)^n \right) \\
&= \frac{1}{(1-t)(1-\mathbb{L}t)}
\end{aligned}$$

Hence $\zeta_{id}(\mathbb{P}_k^1, t)$ is in fact a rational function.

Kapranov proves in [Kap00] that the zeta function of a curve with coefficients in a field is in fact always rational. In their paper [LL03] Larsen and Lunts prove that in the case of surfaces this is false in general. Later the same authors gave a more precise characterisation when the zeta function of a complex surface is rational (namely if and only if the Kodaira dimension is $-\infty$. See [LL04]).

We will prove the following statement from [LL03]

Theorem 8. Assume that $k = \mathbb{C}$. There exists a field \mathcal{H} and a motivic measure $\mu : K_0[\mathcal{V}_k] \rightarrow \mathcal{H}$ with the following property: if X is a smooth complex projective surface such that $P_g(X) = h^{2,0}(X) \geq 2$, then the zeta-function $\zeta_{\mu}(X, t)$ is not rational.

The first important result on the way to prove this is a structure theorem for $K_0[\mathcal{V}_{\mathbb{C}}]$.

Theorem 9. Set $k = \mathbb{C}$. Let G be an abelian commutative monoid and $\mathbb{Z}[G]$ be the corresponding monoid ring. As above, denote by \mathcal{M} the multiplicative monoid of irreducible smooth complete varieties. Let

$$\psi : \mathcal{M} \rightarrow G$$

be a homomorphism of monoids such that

- (i) $\psi([X]) = \psi([Y])$ if X and Y are birational;
- (ii) $\psi([\mathbb{P}^n]) = 1$ for all $n \geq 0$.

Then ψ can be uniquely extended to a ring homomorphism

$$\phi : K_0[\mathcal{V}_{\mathbb{C}}] \rightarrow \mathbb{Z}[G]$$

To prove this result we will use a result by Bittner.

Theorem 10 (Bittner, [Bit04, Theorem 3.1]). *The Grothendieck group $K_0[\mathcal{V}_{\mathbb{C}}]$ is generated by classes of smooth complete varieties subject to relations of the form*

$$[X] - [f^{-1}(Z)] = [Y] - [Z]$$

where X, Y are smooth complete varieties and $f : X \rightarrow Y$ is a morphism which is a blowup with a smooth center $Z \subset Y$.

Proof of theorem 9. We have to check that ψ preserves the above relations, i.e. that $\psi([X]) - \psi[f^{-1}(Z)] = \psi([Y]) - \psi([Z])$. But $[X]$ and $[Y]$ are birational since f is a blowup. (WHY?) Now $f^{-1}(Z)$ is birational to $Z \times \mathbb{P}^n$ (WHY??) and thus

$$\psi([f^{-1}(Z)]) = \psi([Z \times \mathbb{P}^n]) = \psi([Z][\mathbb{P}^n]) = \psi([Z])\psi([\mathbb{P}^n]) = \psi([Z])$$

Hence we can linearly extend ψ to define the morphism $\phi : K_0[\mathcal{V}_{\mathbb{C}}] \rightarrow \mathbb{Z}[G]$ \square

References

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