

# Motivic Zeta Function

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Geboren am 2. Januar 1995 in Reutlingen

2.2.2016

Bachelorarbeit Mathematik

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## 1 Introduction

In 1949 André Weil, in the paper [Wei49], formulated his famous conjectures which predicted certain deep properties of the Hasse–Weil zeta function  $Z(X, t)$  for a variety  $X$  over a finite field  $k = \mathbb{F}_q$ . The zeta function is defined as

$$Z(X, t) = \exp \left( \sum_{m=0}^{\infty} \frac{|X(\mathbb{F}_{q^m})|}{m} t^m \right) \in \mathbb{Q}[[t]].$$

Dwork first established the rationality of the zeta function in [Dwo60], the most basic of the three conjectures. Later, using his new scheme theoretic approach to algebraic geometry, Grothendieck gave another proof for the special case of smooth projective varieties ([Gro66]).

In [Kap00], Kapranov generalized the Hasse–Weil zeta function, defining it not only for varieties over finite fields but varieties over arbitrary ground fields. This *motivic zeta function* has coefficients in the so called *Grothendieck ring of varieties*  $K_0[\mathcal{V}_k]$  over some fixed field  $k$ , which is the free abelian group generated by isomorphism classes of varieties subject to additional ”cut-and-paste” relations (see Definition 3.1), and is given as

$$\zeta_{\mu}(X, t) := \sum_{n=0}^{\infty} \mu([\mathrm{Sym}^n(X)]) t^n$$

where  $\mu$  is a ring homomorphism from  $K_0[\mathcal{V}_k]$  to some fixed ring  $A$  (also called a *motivic measure*), and  $\mathrm{Sym}^n(X)$  is the  $n$ -fold symmetric product of  $X$  (see Definition 4.8). It is a natural question to ask if this zeta function is still a rational function (at least if  $\mu$  takes values in a field). Kapranov proved this for smooth projective curves ([Kap00, Thm 1.1.9]) and conjectured that it is true in general. But it turns out that in the motivic setting the zeta function is not as well behaved in terms of rationality as in the classical case: Larsen and Lunts showed in [LL03] that the zeta function fails to be rational for complex surfaces.

**Theorem 1.1** ([LL03, Thm. 1.6]). *Assume that  $k = \mathbb{C}$ . There exists a field  $\mathcal{H}$  and a motivic measure  $\mu_h: K_0[\mathcal{V}_k] \rightarrow \mathcal{H}$  with the following property: if  $X$  is a smooth complex projective surface such that the geometric genus  $P_g(X) = h^{2,0}(X)$  is at least 2, then the zeta-function  $\zeta_{\mu_h}(X, t)$  is not a rational function.*

The goal of this thesis is to give a simplified proof of this result, using results about the Grothendieck ring that were not available to Larsen and Lunts. In this thesis we will first introduce the Grothendieck ring in Section 3 and investigate its structure. This culminates in a structure theorem due to Bittner ([Bit04, Thm 3.1]) which will allow us to substantially simplify the original proof. The second section is devoted to the definition of the motivic zeta function, which entails defining and giving examples for the symmetric product. There we will also see the connection between the classical Hasse–Weil zeta function and the motivic one. Construction of the motivic measure  $\mu_h$  under which  $Z_{\mu_h}(X, t)$  becomes irrational is content of Section 5 while irrationality is then finally proven in the last section.

## 2 German Summary

In dieser Bachelorarbeit soll ein Resultat von Larsen und Lunts ([LL03]) über die Rationalität der motivischen Zetafunktion bewiesen werden. Die Motivische Zetafunktion ist eine Verallgemeinerung der Hasse–Weil Zetafunktion für Varietäten über endlichen Körpern mit Koeffizienten im Grothendieckring der Varietäten  $K_0[\mathcal{V}_k]$  der von Isomorphieklassen von algebraischen Varietäten erzeugt wird zwischen denen gewissen zusätzliche ”cut-and-paste” Relationen bestehen (Siehe 3.1 für die genaue Definition). Diese motivische Zetafunktion ist, im Gegensatz zur Hasse–Weil Zetafunktion, im allgemeinen nicht rational. Konkret werden wir folgendes Resultat zeigen.

**Theorem 2.1** ([LL03, Thm. 1.6]). *Sei  $k = \mathbb{C}$ . Dann existiert ein Körper  $\mathcal{H}$  und ein Ringhomomorphismus  $\mu_h: K_0[\mathcal{V}_k] \rightarrow \mathcal{H}$  mit der folgenden Eigenschaft: Falls  $X$  eine glatte, projektive, komplexe Fläche mit geometrischem Genus  $P_g(X) = h^{2,0}(X)$  von mindestens 2 ist, dann ist die motivische Zetafunktion  $\zeta_{\mu_h}(X, t)$  nicht rational.*

Wir werden ein Theorem von Bittner ([Bit04]) verwenden um den ursprünglichen Beweis deutlich kompakter darstellen zu können.

## 3 Motivic Measures on the Grothendieck Ring of Varieties

In the following the term *k-variety* always means a separated, reduced scheme of finite type over a field  $k$ . We will write  $\mathcal{V}_k$  for the category of  $k$ -varieties.

**Definiton 3.1.** Let  $k$  be a perfect field. Consider the abelian group of formal linear combinations of isomorphism classes of varieties, subject to relations of the form

$$[X \setminus Y] = [X] - [Y]$$

where  $Y$  is closed in  $X$ . With multiplication given by

$$[X][Y] = [X \times Y]$$

this datum forms a ring, called the *Grothendieck ring of varieties* and denoted by  $K_0[\mathcal{V}_k]$ .

A *motivic measure* with values in a ring  $A$  is a ring homomorphism  $\mu: K_0[\mathcal{V}_k] \rightarrow A$ . The identity function  $\text{id}: K_0[\mathcal{V}_k] \rightarrow K_0[\mathcal{V}_k]$  is called the *universal motivic measure*.

To see that this is really a ring we actually have to check that multiplication is compatible with the relations, but we have

$$\begin{aligned} [Z]([X \setminus Y]) &= [Z \times (X \setminus Y)] = [(Z \times X) \setminus (Z \times Y)] \\ &= [Z \times X] - [Z \times Y] = [Z]([X] - [Y]) \end{aligned}$$

**Remark 3.2.** The reason that we require  $k$  to be perfect is that for nonperfect  $k$  the product of two varieties need not be reduced. Take for example  $k = k_0(t)$  with  $k_0$  of characteristic  $p > 0$  and  $X = \text{Spec}(k[x]/(x^p - t))$ ,  $Y = \text{Spec}(K) := \text{Spec}(k(t^{\frac{1}{p}}))$ . Then  $X$  is a variety since  $x^p - t$  is irreducible in  $k_0(t)$ . The product, on the other hand, can be described as

$$X \times Y = \text{Spec}(K[x]/(x^p t)) = \text{Spec}(K[x]/((x - t^{\frac{1}{p}})^p)$$

which is not reduced. We might adapt the definition to this case by changing the definition of the product to  $[X][Y] := [(X \times Y)_{\text{red}}]$ , taking the induced reduced scheme structure on  $X \times Y$ , but since we are only working in characteristic zero or over finite fields we will not bother with this technical issue.

Since the product of varieties is commutative up to isomorphism this is a commutative ring with 1 (equal to  $[\text{Spec}(k)]$ ). The cut and paste relation furthermore gives us

$$0 = [\emptyset] - [\emptyset] = [\emptyset \setminus \emptyset] = [\emptyset]$$

**Example 3.3.** For a finite field  $k$ , consider the function <sup>1</sup>

$$\begin{aligned} \psi: \mathcal{V}_k &\rightarrow \mathbb{Z} \\ X &\mapsto |X(k)| \end{aligned}$$

Observe that  $\psi$  has the following properties:

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<sup>1</sup>not bothering with any set-theoretic issues

1. If  $X$  and  $Y$  are isomorphic  $k$ -varieties, then  $\psi(X) = \psi(Y)$ .
2. If  $Y \subset X$  is closed subvariety, then  $\psi(X \setminus Y) = \psi(X) - \psi(Y)$ .
3. If  $X$  and  $Y$  are two varieties, then  $\psi(X \times Y) = \psi(X)\psi(Y)$ .

Hence  $\psi$  factors over  $K_0[\mathcal{V}_k]$  and gives rise to a motivic measure.

**Example 3.4.** Using the decomposition  $\mathbb{P}_k^n = \mathbb{P}_k^{n-1} \coprod \mathbb{A}_k^n$  where  $\mathbb{P}_k^{n-1}$  is a hyperplane in  $\mathbb{P}_k^n$  we get  $[\mathbb{P}_k^n] = [\mathbb{P}_k^{n-1}] + [\mathbb{A}_k^n]$  in the Grothendieck ring. Inductively this yields the identity

$$[\mathbb{P}_k^n] = \sum_{m=0}^n \mathbb{L}^m$$

where we write  $\mathbb{L}$  for the isomorphism class of the affine line.

This calculation also gives rise to an example of two irreducible varieties that have the same equivalence class in  $K_0[\mathcal{V}_k]$  but are not isomorphic:  $\mathbb{P}^n \times (\mathbb{A}^1 \setminus \{0\})$  and  $\mathbb{A}^{n+1}$ . These are not isomorphic since the global section of the former are  $k[x]_x$ , but in the Grothendieck ring we get

$$\begin{aligned} [\mathbb{P}^n \times (\mathbb{A}^1 \setminus \{0\})] &= [\mathbb{P}^n] ([\mathbb{A}^1] - 1) \\ &= \sum_{m=0}^n \mathbb{L}^{m+1} - \sum_{m=0}^n \mathbb{L}^m \\ &= \mathbb{L}^{n+1} \end{aligned}$$

We now investigate the structure of the Grothendieck ring. First we note that it suffices to take irreducible varieties to generate  $K_0[\mathcal{V}_k]$ . To see this, take a variety  $X$  with irreducible components  $X_1, \dots, X_n$  and set  $U_i := X_i \setminus \bigcup_{j \neq i} X_j$  and  $U := \bigcup_i U_i$ . By construction, the last union is a disjoint union of closed subsets of  $U$ , and hence  $[U] = \sum_i [U_i]$ . Together this gives

$$[X] = [X \setminus U] + \sum_i [U_i].$$

Now the  $U_i$ , as open subsets of irreducible sets, are irreducible and  $[X \setminus U]$  has dimension strictly smaller than the dimension of  $X$  since we removed a proper open subset from each irreducible component. Hence we can inductively rewrite  $[X \setminus U]$ , losing at least one dimension in each step, with the base case of  $\dim X = 0$  being trivial, as  $X$  is then just a collection of points.

In characteristic zero we can use a weak form of Hironaka's Theorem on the resolution of singularities, namely that for every irreducible, projective variety  $X$  there is a smooth, projective variety that is birational to  $X$  to restrict the set of needed generators even more. This observation culminates in a structure theorem about the Grothendieck ring proven by Bittner in [Bit04].

**Theorem 3.5** ([Bit04, Thm. 3.1]). *Let  $k$  be a field of characteristic zero. Then  $K_0[\mathcal{V}_k]$ , as an abelian group, is generated by smooth, integral, complete varieties. Furthermore, if  $k$  is algebraically closed, it suffices to consider relations of the form*

$$[X] - [f^{-1}(Z)] = [Y] - [Z]$$

where  $X$  and  $Y$  are smooth complete varieties and  $f: X \rightarrow Y$  is a morphism which is a blowup with a smooth center  $Z \subset Y$ .

*Proof.* We will only proof the first part of the statement, for a proof of the second part see [Bit04]. As remarked before, we can restrict our attention to irreducible varieties. We once again argue by induction on the dimension. Let  $X$  be an irreducible variety of dimension  $n$ . By choosing some nonempty affine subvariety and passing to its projective closure we find  $X'$  projective, irreducible and birational to  $X$ . Resolving singularities we find  $\tilde{X}$  smooth, projective, irreducible birational to  $X$ . Hence we find isomorphic open subset  $U \subset X$ ,  $V \subset \tilde{X}$  which gives us

$$[X \setminus U] - [\tilde{X} \setminus V] = [X] - [U] - [\tilde{X}] + [V] = [X] - [\tilde{X}].$$

Thus  $[X]$  can be written as the sum of  $[\tilde{X}]$  and some varieties of lower dimension for which we can invoke the induction hypothesis.  $\square$

We are going to use Theorem 3.5 in the following form we proved in the induction step.

**Corollary 3.6.** *For each (irreducible) variety  $X$  over a field of characteristic zero there exists a smooth, irreducible, projective variety  $Y$  birational to  $X$  such that  $[X] = [Y] + \sum_i m_i [W_i]$  where the  $[W_i]$  are smooth, irreducible, projective varieties of dimension strictly smaller than the dimension of  $X$ .*  $\square$

## 4 Symmetric Products and the Motivic Zeta Function

We now want to introduce the concept of the symmetric product, i.e. how it can be defined as the quotient of the (regular) product by the action of the symmetric group. This section follows the presentation in [Mus11, Appendix A] and [Har92, Lecture 10].

**Definition 4.1.** Let  $X$  be an  $S$ -Scheme,  $G$  a group acting on  $X$  (from the left) via  $S$ -automorphisms. Then a variety  $Y$  together with a morphism  $\pi: X \rightarrow Y$  over  $S$  is called the *categorical quotient* (also denoted by  $X/G$ ) of  $Y$  by  $G$  if  $\pi$  is  $G$ -invariant which means that for all  $g \in G$  we have  $\pi \circ g = \pi$

and is universal in this respect, i.e. for every other  $G$ -invariant morphism  $f: X \rightarrow Z$  there exists a unique  $G$ -invariant morphism  $g: Z \rightarrow Y$  that makes the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X/G \\ \downarrow f & \swarrow g & \\ Z & & \end{array} .$$

Then  $\pi: X \rightarrow Y$  is the categorical quotient of  $X$  by the action of  $G$ .

**Remark 4.2.** Since it is defined via an universal property, the quotient is unique up to unique isomorphism if it exists. In contrast to other settings the scheme theoretic quotient does not always exist and can be quite subtle to construct. In the following we construct the quotient in the case that  $G$  is finite and the scheme  $X$  is a quasiprojective variety.

We first construct the quotient in the affine case. If we were actually taking the quotient in the category of affine schemes things would be quite simple: we could as well work in the category of rings and construct something that might be called a co-quotient, i.e. an object that satisfies the conditions of the definition, just with all arrows reversed (now  $G$  is a right-action). This is simply the  $G$  invariant ring  $A^G := \{a \in A \mid ga = a \ \forall g \in G\} \subset A$ . Hence in the category of affine schemes the quotient would be given by  $\text{Spec}(A) \rightarrow \text{Spec}(A^G)$ . But since we would like to take quotients in the full category of schemes we have to do some more work. First we slightly rephrase the definition for the case of schemes.

**Lemma 4.3.** *Let  $\pi: X \rightarrow Y$  be a surjective morphism of schemes together with a group  $G$  acting on  $X$  via automorphism such that the following conditions hold:*

- i) *The fibres of  $\pi$  are in one to one correspondence to the orbits of the  $G$ -action.*
- ii)  *$Y$  carries the quotient topology.*
- iii) *There exists a unique isomorphism  $\mathcal{O}_X = \pi_* \mathcal{O}_Y^G$*

*Proof.* The first two properties ensure that  $\pi$  is a quotient of topological spaces, i.e.  $\pi$  is a universally  $G$ -invariant continuous map. By the discussion above, the third condition ensures that now actually all these morphisms are also morphism of locally ringed spaces.  $\square$

**Remark 4.4.** This shows that the quotient is in fact a local notion in the sense that if  $U$  is an open subset of the quotient  $X/G$  then  $\pi|_{\pi^{-1}(U)}: \pi^{-1}(U) \rightarrow U$  is the quotient of  $\pi^{-1}(U)$  by the action of  $G$ .



**Lemma 4.5.** *Let  $X = \operatorname{Spec} A$  be an affine variety with a finite group  $G$  acting on it. Then the categorical quotient exists, is itself a variety and is given by  $X/G = \operatorname{Spec}(A^G)$ , the spectrum of the  $G$ -invariant subring of  $A$ .*

*Proof.* We first show that  $\operatorname{Spec}(A^G)$  is again a variety. As a subring of a reduced  $k$ -algebra  $A^G$  is again reduced, so what is left to check is that it is of finite type over  $k$ . Let  $T_1, \dots, T_n$  be generators of  $A$  as  $k$ -algebra. By passing to the union of the orbits of the  $T_i$  we might assume that  $G$  acts on the set of generators via permutations and hence can be seen as a subgroup of the symmetric group  $S_n$ . Hence we get the following chain of ring inclusions:  $k \subset A^{S_n} \subset A^G \subset A$ . Now  $A^{S_n}$  is generated by the elementary symmetric polynomials in the  $T_i$  and hence is of finite type over  $k$ . It is now enough to show that  $A$  is integral over  $A^{S_n}$  because then  $A$  is of finite type and integral over  $A^{S_n}$  and hence finite, and since  $A^{S_n}$  is noetherian  $A^G$ , as a submodule of  $A$ , is again a finite module over  $A^{S_n}$  hence of finite type over  $k$ . So let be  $a \in A$ , then we find a  $f_a \in k[X_1, \dots, X_n]$  such that  $f_a(T_1, \dots, T_n) = a$ , and we find that  $a$  is a zero of the polynomial

$$\prod_{g \in S_n} (X - g(a)) = \prod_{g \in G} (X - f_a(g(T_1), \dots, g(T_n))) \in A^{S_n}[X]$$

hence it is integral over  $A^{S_n}$  (and also over  $A^G$ ). This proves that  $\operatorname{Spec}(A^G)$  is again a variety. To show that it is also the quotient of  $X$  by the action of  $G$  we check the properties i) to iii) of 4.3. Let us prove the first property. Since we have  $g\mathfrak{p} \cap A^G = \mathfrak{p} \cap A^G$  each orbit is contained in some fibre. For the converse assume we have prime ideals  $\mathfrak{p}, \mathfrak{q} \subset A$  such that  $\mathfrak{p} \cap A^G = \mathfrak{q} \cap A^G$ . For  $a \in \mathfrak{p}$  we have  $\prod_{g \in G} ga \in \mathfrak{p} \cap A^G = \mathfrak{q} \cap A^G$ , and since  $\mathfrak{q} \cap A^G$  is prime we find some  $g \in G$  such that  $ga \in \mathfrak{q} \cap A^G$  or equivalently  $a \in g^{-1}\mathfrak{q} \cap A^G \subset \mathfrak{q}$ , hence we find  $\mathfrak{p} \subset \bigcup_{g \in G} g\mathfrak{q}$  and can apply the prime avoidance lemma to find that  $\mathfrak{p}$  had to be contained in some  $g\mathfrak{q}$ . Because the situation is symmetric in  $\mathfrak{p}$  and  $\mathfrak{q}$  we find  $h \in G$  such that  $\mathfrak{q} \subset h\mathfrak{p}$  hence  $\mathfrak{p} \subset g\mathfrak{q} \subset hg\mathfrak{p}$ . Now we use the fact that for a integral extension two prime ideals with the same contraction with one subset of another must be equal. Applied to this situation and again using that  $\mathfrak{p} \cap A^G = hg\mathfrak{p} \cap A^G$  we obtain  $hg\mathfrak{p} = \mathfrak{p}$  and by the same argument finally  $\mathfrak{p} = g\mathfrak{q}$ . Hence we have shown that fibres and orbits coincide. Furthermore this shows that  $\pi$  is surjective and finite, hence  $\operatorname{Spec}(A^G)$  has the quotient topology. For the last property we note that the action of  $G$  extends to an action on  $A_f$  for some  $f \in A^G$  and we have  $A_f^G = (A_f)^G$  for some  $f \in A^G$  since  $\frac{a}{f^n} = \frac{ga}{f^n}$  if and only if  $a = ga$ . This means that the natural map  $A_f^G = \mathcal{O}_{\operatorname{Spec}(A^G)}(D(f)) \rightarrow \pi_* \mathcal{O}_{\operatorname{Spec}(A)}(D(f))^G = (A_f)^G$  is an isomorphism, and since the  $D(f)$  form a basis of the topology we have proven property iii).  $\square$

We now turn to a relevant special case: the quotient of an quasiprojective variety  $X$  by the action of a finite group  $G$ , using the following general result.

**Lemma 4.6** ([Gro71, Prop. 1.8]). *Let  $X$  be a scheme with a finite group  $G$  acting on it via automorphisms. If there exists an open cover by  $G$ -invariant affine sets then the quotient  $X/G$  exists. We call such an action admissible*

*Proof.* Say  $X$  is covered by the  $G$ -invariant affine sets  $U_1, \dots, U_i$  for which we may construct the quotient  $\pi_i: U_i \rightarrow U_i/G$  as by the previous lemma. Now we would like to glue these. This is possible since by Remark 4.4  $\pi_i: U_i \cap U_j \rightarrow \pi_i(U_i \cap U_j)$  and  $\pi_j: U_i \cap U_j \rightarrow \pi_j(U_i \cap U_j)$  are both quotients of  $U_i \cap U_j$  by  $G$  and therefore naturally isomorphic. Since we may check the conditions of Lemma 4.3 locally gluing gives a quotient of  $X$  by  $G$ .  $\square$

**Corollary 4.7.** *If  $X$  is a quasiprojective variety, then the quotient by the action of a finite group always exists and is itself a quasiprojective variety.*

*Proof.* Let  $X \subset \mathbb{P}_k^n$  be locally closed with a finite group  $G = \{g_1, \dots, g_m\}$  acting on it. Consider the orbit  $\{g_1(x), \dots, g_m(x)\} \subset X$  of a point  $x \in X$ , and denote by  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$  the corresponding graded prime ideals corresponding to  $\{x\}$ , and by  $\mathfrak{q}$  the graded prime ideal corresponding to  $\bar{X} \setminus X$  (taken to be the irrelevant ideal if this set is empty). Hence the ideal  $\mathfrak{q}$  is not contained in any of the  $\mathfrak{p}_i$ s hence, by the graded version of the prime avoidance lemma we find an element with positive degree  $y \in \mathfrak{q}$  that is contained in none of the  $\mathfrak{p}_i$  and hence the hypersurface  $V_+(y)$  contains  $\bar{X} \setminus X$  but not the  $g_1(x), \dots, g_m(x)$ , so  $U := D_+(y)$  is an affine set that lies in  $X$  and contains the orbit of  $x$  under  $G$ . Now consider  $U' := \bigcap_{g \in G} g(U)$ . This is clearly  $G$ -invariant, and since  $U$  contains the orbit of  $y$  we still find that  $y$  lies in  $U'$ . Hence we have found a open cover of  $X$  by  $G$ -invariant open affines, and thus the quotient exists by the above lemma.  $\square$

**Definiton 4.8.** Let  $X$  be a quasiprojective variety, then the  $n$ -fold *symmetric product*  $\text{Sym}^n(X)$  is defined as the quotient of  $X^n$  (which is again quasiprojective via the Segre embedding) by the natural action of the symmetric group in  $n$  letters.

**Example 4.9.**  $\text{Sym}^n(\mathbb{P}^1) = \mathbb{P}^n$ . (TODO: write up explanation)

**Definiton 4.10.** Let  $\mu: K_0[\mathcal{V}_k] \rightarrow A$  be a motivic measure and  $X$  a quasiprojective variety. Then we define the *motivic zeta function*  $\zeta_\mu(X, t) \in A[[t]]$  as

$$\zeta_\mu(X, t) := \sum_{n=0}^{\infty} \mu([\text{Sym}^n(X)]) t^n$$

**Example 4.11.** We can now compute a first zeta function, namely  $\zeta_{id}(\mathbb{P}_k^1, t)$ .

Using the identity and notation from example 3.4 we calculate

$$\begin{aligned}
\zeta_{id}(\mathbb{P}_k^1, t) &= \sum_{n=0}^{\infty} [\text{Sym}^n(\mathbb{P}_k^1)] t^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \mathbb{L}^k \right) t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n (\mathbb{L}t)^k t^{n-k} \\
&= \left( \sum_{n=0}^{\infty} t^n \right) \left( \sum_{n=0}^{\infty} (\mathbb{L}t)^n \right) = \frac{1}{(1-t)(1-\mathbb{L}t)}.
\end{aligned}$$

Hence  $\zeta_{id}(\mathbb{P}_k^1, t)$  is in fact a rational function.

## 5 Constructing Motivic Measures

We will now construct the measure  $\mu_h$  of Theorem 1.1.

The first important result on the way is a lemma that helps us to construct such measures by extending maps from the monoid of smooth, irreducible and complete varieties. For this we quickly recall the notion of a monoid ring.

**Definiton 5.1.** Let  $G$  be a (multiplicative) monoid. Write  $\mathbb{Z}[G]$  for the free abelian group over  $G$ , i.e. the set of formal sums  $\sum_{g \in G} n_g g$  with the  $n_g$  being integers, only finitely many non zero. Defining multiplication as

$$\left( \sum_{g \in G} n_g g \right) \left( \sum_{g \in G} m_g g \right) = \sum_{g \in G} \left( \sum_{hl=g} n_h m_l \right) g$$

this forms a ring, the *monoid ring* over  $G$ .

**Theorem 5.2.** Set  $k = \mathbb{C}$ . Let  $G$  be an abelian monoid and  $\mathbb{Z}[G]$  the corresponding monoid ring. As above, denote by  $\mathcal{M}$  the multiplicative monoid of irreducible smooth complete varieties. Let

$$\psi: \mathcal{M} \rightarrow G$$

be a homomorphism of monoids such that

- i)  $\psi([X]) = \psi([Y])$  if  $X$  and  $Y$  are birational;
- ii)  $\psi([\mathbb{P}^n]) = 1$  for all  $n \geq 0$ .

Then  $\psi$  can be uniquely extended to a ring homomorphism

$$\phi: K_0[\mathcal{V}_{\mathbb{C}}] \rightarrow \mathbb{Z}[G]$$

*Proof.* By Theorem 3.5 we know that the Grothendieck ring (as an abelian group) is generated by  $\mathcal{M}$ , so after extending  $\psi$  to the free abelian group over  $\mathcal{M}$  we have to check that  $\psi$  preserves the relations of the blowup presentation of  $K_0[\mathcal{V}_{\mathbb{C}}]$  and hence factors over it, i.e. given a blow-up  $f: X \rightarrow Y$  with smooth center  $Z \subset X$  then  $\psi([X]) - \psi([f^{-1}(Z)]) = \psi([Y]) - \psi([Z])$ . Note that  $[X]$  and  $[Y]$  are birational, since  $f$  is a blowup, thus  $\psi([X]) = \psi([Y])$  by property i).

Now, since  $X$  and  $Z$  are nonsingular, the exceptional divisor  $f^{-1}(Z)$  is isomorphic to the projective space bundle  $\mathbb{P}(\mathcal{I}/\mathcal{I}^2)$  where  $\mathcal{I}$  is the ideal sheaf associated to  $Z$  (see for example [Har77, II.8, Thm 8.24]), which is a  $\mathbb{P}^n$ -bundle, i.e. locally trivializes to  $\mathbb{P}_k^n \times Z$  and hence is birational to  $\mathbb{P}_k^n \times Z$ . We can thus write  $\psi([f^{-1}(Z)])$  as

$$\psi([f^{-1}(Z)]) = \psi([Z \times \mathbb{P}^n]) = \psi([Z][\mathbb{P}^n]) = \psi([Z])\psi([\mathbb{P}^n]) = \psi([Z])$$

where the last equation holds because of property ii). Hence, we can extend  $\psi$  to define a morphism  $\phi: K_0[\mathcal{V}_{\mathbb{C}}] \rightarrow \mathbb{Z}[G]$   $\square$

Before constructing a suitable monoid homomorphism that will yield our desired measure, we recall the definition of the Hodge numbers of a variety as well as two basic results (all these are exercises in [Har77, Ch. II]).

**Definiton 5.3.** Let  $X$  be a projective variety. The  $i$ -th *Hodge number*  $h^{i,0}$  is given as the dimension of  $H^0(X, \Omega_X^i)$ , which is a finite dimensional  $k$ -vector space by a theorem of Serre ([Ser55, §3 Prop. 7]). If  $d$  is the dimension of a nonsingular variety  $X$  we will also write  $P_g(X)$  for  $h^{d,0}(X)$ , called the *geometric genus* of  $X$ .

**Lemma 5.4** ([Har77, II Ex. 8.8]). *The Hodge numbers  $h^{i,0}$  are birational invariants of a variety: let  $X, Y$  be birationally equivalent, smooth, projective varieties then  $h^{i,0}(X) = h^{i,0}(Y)$ .*

**Lemma 5.5** (Chow, [Har77, II Ex. 4.10]). *Every proper variety is birational to a projective variety.*

**Remark 5.6.** By [Har77, Prop. 10.1 (d)] the product of two smooth varieties over  $k$  is again smooth. Hence the isomorphism classes of smooth irreducible complete varieties form a multiplicative monoid, in the following denoted by  $\mathcal{M}$ .

**Definiton 5.7.** Denote by  $C \subset \mathbb{Z}[t]$  the multiplicative monoid of polynomials with positive leading coefficient. For a smooth projective complex variety  $Z$  of dimension  $d$  define

$$\Psi_h(X) := 1 + h^{1,0}(X)t + \cdots + h^{d,0}(X)t^d \in C.$$

(Note that the Hodge numbers might be zero, but  $\Psi_h(X)$  still has a positive leading coefficient, just its degree might be smaller than the dimension of

X) By Chow's lemma we can also define  $\Psi_h$  for a smooth complete variety  $Z$  by choosing a smooth projective variety  $X$  which is birational to  $Z$  and setting  $\Psi_h(Z) := \Psi_h(X)$ . This is well defined by Lemma 5.4.

We now check that  $\Psi_h$  satisfies all conditions of Theorem 5.2. Independence of birational equivalence class was the content of Lemma 5.4.

To check multiplicativity, we use the following lemma.

**Lemma 5.8.** *Let  $X, Y$  be smooth, irreducible, projective  $k$ -varieties. Then the following equality holds:*

$$h^{p,0}(X \times_k Y) = \sum_{i+j=p} h^{i,0}(X) h^{j,0}(Y)$$

*Proof.* TODO □

With this we directly calculate

$$\begin{aligned} \Psi_h(X) \Psi_h(Y) &= \left( \sum_n h^{n,0}(X) t^n \right) \left( \sum_n h^{n,0}(Y) t^n \right) = \sum_n \left( \sum_{i+j=n} h^{i,0}(X) h^{j,0}(Y) \right) t^n \\ &= \sum_n h^{n,0}(X \times_k Y) t^n = \Psi_h(X \times_k Y) \end{aligned}$$

## 6 Irrationality of the Motivic Zeta Function

The constructed motivic measure does not yet take values in a field, so we would like to pass to the quotient field of  $\mathbb{Z}[\mathcal{M}]$ . That we are able to do so is the content of the next lemma.

**Lemma 6.1.** *Let  $A$  be a factorial ring, and  $S \subset A$  a multiplicative submonoid such that 1 is the only unit in  $S$ . Then the monoid ring  $\mathbb{Z}[S]$  is a polynomial ring (in possibly infinitely many variables), and hence an integral domain.*

*Proof.* Since  $A$  is factorial, every  $s \in S$  has a unique factorization, and since 1 is the only unit in  $S$ ,  $s$  can be uniquely written as product of prime elements, hence if we take  $B$  to be the polynomial ring over the formal variables  $\{x_s | s \in S, s \text{ prime}\}$  we get an isomorphism of rings

$$B \rightarrow \mathbb{Z}[S], \quad x_s \mapsto s$$

□

**Definiton 6.2.** Denote by  $\mathcal{H}$  the quotient field of  $\mathbb{Z}[C]$  where, as above,  $C$  denotes the submonoid of  $\mathbb{Z}[t]$  consisting of polynomials with positive leading coefficients. Since  $-1$  is not contained in  $C$  it satisfies the conditions of the previous lemma. We define the motivic measure  $\mu_h: K_0[\mathcal{V}_k] \rightarrow \mathcal{H}$  as the measure obtained extending  $\Psi_h$  as by Theorem 5.2

**Lemma 6.3.** *Let  $Y_1, \dots, Y_s, Z$  be irreducible varieties of dimension  $d$  over a field of characteristic zero such that  $\mu_h([Z]) = \sum_i n_i \mu_h([Y_i])$  for some  $n_i \in \mathbb{Z}$  and  $P_g(Z) \neq 0$  then  $P_g(Z) = P_g(Y_i)$  for some  $i$*

*Proof.* By Corollary 3.6, we find smooth, irreducible projective varieties  $\overline{Z}, \overline{Y}_1, \dots, \overline{Y}_s$  in the same birational class as the original varieties allowing us to rewrite the original equality as

$$\mu_h([\overline{Z}]) = \sum n_i \mu_h([\overline{Y}_i]) + \sum l_i \mu_h([X_i])$$

where the  $X_i$  are smooth irreducible varieties of dimension  $< d$ . Now, since  $\mu_h$  was obtained as an extension of  $\Psi_h$  which was defined on smooth, irreducible, projective varieties, this is actually an equation in  $\mathbb{Z}[C]$ , namely

$$\Psi_h(\overline{Z}) = \sum n_i \Psi_h(\overline{Y}_i) + \sum l_i \Psi_h(X_i)$$

Since there are no (additive) relations between elements of  $C$  in the monoid ring, one of the polynomials  $\Psi_h(\dots)$  on the right hand side, must actually be the polynomial on the left hand side, but with the dimension of  $X_i$  being strictly smaller than that of  $Z$  and the assumption that  $P_g(\overline{Z}) = h^{d,0}(\overline{Z}) \neq 0$  it cannot be one of the  $\Psi_h(X_i)$  because all these polynomials have strictly smaller degree. Hence we have  $\Psi_h(\overline{Z}) = \Psi_h(\overline{Y}_i)$  for some  $i$  and in particular, since they are of the same dimension, the genus of  $\overline{Z}$  and  $Y_i$  must agree.  $\square$

Now let  $X$  be a smooth projective surface with  $P_g(X) \geq 2$ . We will show that  $\zeta_{\mu_h}(X, t) \in \mathcal{H}[[t]]$  is not rational, thus proving Theorem 1.1. We need one technical lemma about the genus of symmetric powers of  $X$  whose proof is out of scope of this thesis.

**Lemma 6.4** ([LL03, Lem. 3.8]). *Let  $X$  be a smooth, projective surface over  $\mathbb{C}$ . Then*

$$P_g(\text{Sym}^n(X)) = \binom{P_g(X) + n - 1}{P_g(X) - 1}$$

We will use the following rationality criterion for power series.

**Lemma 6.5** ([Bru63, Lem.3.1]). *Let  $k$  be a field and suppose the power series  $\sum a_i t^i \in k[[t]]$  is a rational function (i.e. an element in  $k(t)$ ) then there exist  $n, n_0 > 0$  such that for each  $m > n_0$  the determinant*

$$\begin{vmatrix} a_m & a_{m+1} & \dots & a_{m+n} \\ a_{m+1} & a_{m+2} & \dots & a_{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+n} & a_{m+n+1} & \dots & a_{m+2n} \end{vmatrix}$$

*vanishes.*

*Proof.* Denote the above matrix by  $A$ . Since  $\sum a_i t^i$  is rational, we may write it as

$$\sum a_i t^i = \frac{p_0 + \cdots + p_e t^e}{q_0 + \cdots + q_d t^d}$$

with not all  $q_i$  being zero. Hence  $p_0 + \cdots + p_e t^e = (\sum a_i t^i) (q_0 + \cdots + q_d t^d) = \sum_{i=0}^{\infty} (\sum_{k+l=i} a_k q_l) t^i$ . This means that for  $m > e$  we get  $0 = \sum_{k+l=m} a_k q_l$ . now note that  $q_l = 0$  for  $l > n$  hence we may rewrite this as

$$0 = \sum_{l=0}^n q_l a_{m-l} = \sum_{l=0}^n q_{n-l} a_{m-n+l}.$$

Thus we have shown that

$$\begin{pmatrix} a_m & a_{m+1} & \cdots & a_{m+n} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+n} & a_{m+n+1} & \cdots & a_{m+2n} \end{pmatrix} \begin{pmatrix} q_n \\ \vdots \\ q_1 \\ q_0 \end{pmatrix} = \begin{pmatrix} \sum_{l=0}^n a_{m+l} q_{n-l} \\ \vdots \\ \sum_{l=0}^n a_{m+n+l} q_{n-l} \end{pmatrix} = 0$$

for  $m$  big enough. Since one of the  $q_l$  is non zero this means that  $A$  has nonempty kernel and hence its determinant vanishes.  $\square$

*Proof of Theorem 1.1.* Assume that  $\zeta_{\mu_h}(X, t)$  is rational, hence by the above criterion there is an  $n$  such that for  $m$  big enough

$$\begin{aligned} & \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma) \mu_h \left( \left[ \prod_{i=1}^{n+1} \text{Sym}^{m+i+\sigma(i)-2}(X) \right] \right) = 0 \\ \Leftrightarrow & \mu_h \left( \left[ \prod_{i=0}^n \text{Sym}^{m+2i}(X) \right] \right) = - \sum_{\substack{\sigma \in S_{n+1} \\ \sigma \neq id}} \text{sgn}(\sigma) \mu_h \left( \left[ \prod_{i=0}^n \text{Sym}^{m+i+\sigma(i+1)-1}(X) \right] \right) \end{aligned}$$

Now we can apply Lemma 6.3 to conclude that there is a permutation  $\sigma$ , not the identity permutation, such that

$$P_g \left( \prod_{i=0}^n \text{Sym}^{m+2i}(X) \right) = P_g \left( \prod_{i=0}^n \text{Sym}^{m+i+\sigma(i+1)-1}(X) \right)$$

By Lemma 6.4 (and using the fact that the genus is multiplicative), we get

$$\prod_{i=0}^n \binom{P_g(X) + m + 2i - 1}{P_g(X) - 1} - \prod_{i=0}^n \binom{P_g(X) + m + i + \sigma(i+1) - 2}{P_g(X) - 1} = 0.$$

By assumption  $P_g(X) \geq 2$  hence the left hand side, considered as a polynomial in  $m$ , is not the zero polynomial. So by taking  $m$  large enough we obtain a contradiction.  $\square$

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