## **Motivic Zeta Function**

## Maximilian Rath

Geboren am 2. Januar 1995 in Reutlingen 2.2.2016

Bachelorarbeit Mathematik

Prof. Dr. Daniel Huybrechts

MATHEMATISCHES INSTITUT

MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT DER RHEINISCHEN FRIEDRICH-WILHELMS-UNIVERSITÄT BONN

**Definition 1.** In the following the term k-variety always means a separated, integral scheme of finite type over a field k. We will write  $\mathcal{V}_k$  for the category of k-varieties.

**Definition 2.** Let k be a field. Consider the group of formal linear combinations of isomorphism-classes in  $\mathcal{V}_k$ . Setting  $[X] \times [Y] := [X \times Y]$  makes this into a ring. The *Grothendieck ring of varieties*  $K_0[\mathcal{V}_k]$  is then obtained by modding out relations of the form

$$[X] - [Y] = [X \setminus Y]$$

where Y is closed in X.

A motivic measure is a ring homomorphism  $\mu: K_0[\mathcal{V}_k] \to A$  into a ring A. The identity function  $id: K_0[\mathcal{V}_k] \to K_0[\mathcal{V}_k]$  is called the universal motivic measure.

Let us now make some remarks about this ring.

**Remark 3.** The Grothendieck ring of varieties is commutative, as

$$X \times Y \cong Y \times X$$

for two schemes X and Y.

**Remark 4.** By [Har77, Proposition 10.1 (d)] the product of two smooth varieties over k is again smooth. Hence the isomorphism classes of smooth irreducible complete varieties form a multiplicative monoid, in the following denoted by  $\mathcal{M}$ .

**Example 5.** Using the decomposition  $\mathbb{P}^n_k = \mathbb{P}^{n-1}_k \coprod \mathbb{A}^n_k$  where  $\mathbb{P}^{n-1}_k$  is a (closed) hyperplane in  $\mathbb{P}^n_k$  we get  $[\mathbb{P}^n_k] = [\mathbb{P}^{n-1}_k] + [\mathbb{A}^n_k]$  in the Grothendieck ring. Inductively this yields the identity

$$[\mathbb{P}_k^n] = \sum_{k=0}^n [\mathbb{A}_k^1]^k$$

We also denote the isomorphism class of the affine line as  $\mathbb{L}$ .

**Definition 6.** Let  $\mu: K_0[\mathcal{V}_k] \to A$  be a motivic measure and X a variety. Then we define the zeta function  $\zeta_{\mu}(X,t) \in A[[t]]$  as

$$\zeta_{\mu}(X,t) := \sum_{n=0}^{\infty} \mu([Sym^{n}(X)])t^{n}$$

**Example 7.** We can now compute a first zeta function, namely  $\zeta_{id}(\mathbb{P}^1_k, t)$ . Using the identity and notation from example 5 we calculate

$$\zeta_{id}(\mathbb{P}_k^1, t) = \sum_{n=0}^{\infty} [Sym^n(\mathbb{P}_k^1)]t^n$$

$$= \sum_{n=0}^{\infty} (\sum_{k=0}^n \mathbb{L}^k)t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^n (\mathbb{L}t)^k t^{n-k}$$

$$= (\sum_{n=0}^{\infty} t^n)(\sum_{n=0}^{\infty} (\mathbb{L}t)^n)$$

$$= \frac{1}{(1-t)(1-\mathbb{L}t)}$$

Hence  $\zeta_{id}(\mathbb{P}^1_k,t)$  is in fact a rational function.

Kapranov proves in [Kap00] that the zeta function of a curve with coefficents in a field is in fact always rational. In their paper [LL03] Larsen and Lunts prove that in the case of surfaces this is false in general. Later the same authors gave a more precise characterisation when the zeta function of a complex surface is rational (namely if and only if the Kodaira dimension is  $-\infty$ . See [LL04]).

We will proof the following statement from [LL03]

**Theorem 8.** Assume that  $k = \mathbb{C}$ . There exists a field  $\mathcal{H}$  and a motivic measure  $\mu : K_0[\mathcal{V}_k] \to \mathcal{H}$  with the following property: if X is a smooth complex projective surface such that  $P_g(X) = h^{2,0}(X) \geq 2$ , then the zeta-function  $\zeta_{\mu}(X,t)$  is not rational.

The first important result on the way to prove this is a structure theorem for  $K_0[\mathcal{V}_{\mathbb{C}}]$ .

**Theorem 9.** Set  $k = \mathbb{C}$ . Let G be an abelian commutative monoid and  $\mathbb{Z}[G]$  be the corresponding monoid ring. As above, denote by  $\mathcal{M}$  the multiplicative monoid of irreducible smooth complete varieties. Let

$$\psi: \mathcal{M} \to G$$

be a homomorphism of monoids such that

(i)  $\psi([X]) = \psi([Y])$  if X and Y are birational;

(ii) 
$$\psi([\mathbb{P}^n]) = 1$$
 for all  $n \geq 0$ .

Then  $\psi$  can be uniquely extended to a ring homomorphism

$$\phi: K_0[\mathcal{V}_{\mathbb{C}}] \to \mathbb{Z}[G]$$

To prove this result we will use a result by Bittner.

**Theorem 10** (Bittner, [Bit04, Theorem 3.1]). The Grothendieck group  $K_0[\mathcal{V}_{\mathbb{C}}]$  is generated by classes of smooth complete varieties subject to relations of the form

$$[X] - [f^{-1}(Z)] = [Y] - [Z]$$

where X,Y are smooth complete varieties and  $f:X\to Y$  is a morphism which is a blowup with a smooth center  $Z\subset Y$ .

Proof of theorem 9. We have to check that  $\psi$  preserves the above relations, i.e. that  $\psi([X]) - \psi([f^{-1}(Z)]) = \psi([Y]) - \psi([Z])$ . But [X] and [Y] are birational since f is a blowup. (WHY?) Now  $f^{-1}(Z)$  is birational to  $Z \times \mathbb{P}^n$  (WHY??) and thus

$$\psi([f^{-1}(Z)]) = \psi([Z \times \mathbb{P}^n]) = \psi([Z][\mathbb{P}^n]) = \psi([Z])\psi([\mathbb{P}^n) = \psi([Z])$$

Hence we can linearly extend  $\psi$  to define the morphism  $\phi: K_0[\mathcal{V}_{\mathbb{C}}] \to \mathbb{Z}[G]$ 

## References

- [Bit04] Franziska Bittner. The universal euler characteristic for varieties of characteristic zero. *Compos. Math.*, 140(4):1011–1032, 2004.
- [Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
- [Kap00] M. Kapranov. The elliptic curve in the s-duality theory and eisenstein series for kac-moody groups, 2000.
- [LL03] Michael Larsen and Valery A. Lunts. Motivic measures and stable birational geometry. *Mosc. Math. J.*, 3(1):85–95, 259, 2003.
- [LL04] Michael Larsen and Valery A. Lunts. Rationality criteria for motivic zeta functions. *Compos. Math.*, 140(6):1537–1560, 2004.