Motivic Zeta Function

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Definiton 1. In the following the term *k-variety* always means a separated, integral scheme of finite type over a field k. We will write $\mathcal{V}_k$ for the category of k-varieties.

Definiton 2. Let k be a field. Consider the group of formal linear combinations of isomorphism-classes in $\mathcal{V}_k$. Setting $[X] \times [Y] := [X \times Y]$ makes this into a ring. The *Grothendieck ring of varieties* $K_0[\mathcal{V}_k]$ is then obtained by modding out relations of the form

$$[X] - [Y] = [X \setminus Y]$$

where Y is closed in X.

A *motivic measure* is a ring homomorphism $\mu : K_0[\mathcal{V}_k] \to A$ into a ring A. The identity function $id : K_0[\mathcal{V}_k] \to K_0[\mathcal{V}_k]$ is called the *universal motivic measure.*

Let us now make some remarks about this ring.

Remark 3. The Grothendieck ring of varieties is commutative, as

$$X \times Y \cong Y \times X$$

for two schemes X and Y.

Remark 4. By [Har77, Proposition 10.1 (d)] the product of two smooth varieties over k is again smooth. Hence the isomorphism classes of smooth irreducible complete varieties form a multiplicative monoid, in the following denoted by $\mathcal{M}$.

Example 5. *Using the decomposition $\mathbb{P}_k^n = \mathbb{P}_k^{n-1} \coprod \mathbb{A}_k^n$ where $\mathbb{P}_k^{n-1}$ is a (closed) hyperplane in $\mathbb{P}_k^n$ we get $[\mathbb{P}_k^n] = [\mathbb{P}_k^{n-1}] + [\mathbb{A}_k^n]$ in the Grothendieck ring. Inductively this yields the identity*

$$[\mathbb{P}_k^n] = \sum_{k=0}^{n} [\mathbb{A}_k^1]^k$$

We also denote the isomorphism class of the affine line as $\mathbb{L}$.

Definiton 6. Let $\mu : K_0[\mathcal{V}_k] \to A$ be a motivic measure and X a variety. Then we define the zeta function $\zeta_\mu(X, t) \in A[[t]]$ as

$$\zeta_\mu(X, t) := \sum_{n=0}^{\infty} \mu([Sym^n(X)])t^n$$

Example 7. *We can now compute a first zeta function, namely* $\zeta_{id}(\mathbb{P}^1_k, t)$*. Using the identity and notation from example 5 we calculate*

$$\begin{aligned}
\zeta_{id}(\mathbb{P}^1_k, t) &= \sum_{n=0}^{\infty} [Sym^n(\mathbb{P}^1_k)] t^n \\
&= \sum_{n=0}^{\infty} \big(\sum_{k=0}^{n} \mathbb{L}^k \big) t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (\mathbb{L}t)^k t^{n-k} \\
&= \Big(\sum_{n=0}^{\infty} t^n \Big) \Big(\sum_{n=0}^{\infty} (\mathbb{L}t)^n \Big) \\
&= \frac{1}{(1-t)(1-\mathbb{L}t)}
\end{aligned}$$

Hence $\zeta_{id}(\mathbb{P}^1_k, t)$ *is in fact a rational function.*

Kapranov proves in [Kap00] that the zeta function of a curve with coefficents in a field is in fact always rational. In their paper [LL03] Larsen and Lunts prove that in the case of surfaces this is false in general. Later the same authors gave a more precise characterisation when the zeta function of a complex surface is rational (namely if and only if the Kodaira dimension is $-\infty$. See [LL04]).

We will proof the following statement from [LL03]

Theorem 8. *Assume that* $k = \mathbb{C}$*. There exists a field* $\mathcal{H}$ *and a motivic measure* $\mu : K_0[\mathcal{V}_k] \to \mathcal{H}$ *with the following property: if* X *is a smooth complex projective surface such that* $P_g(X) = h^{2,0}(X) \geq 2$*, then the zeta-function* $\zeta_\mu(X, t)$ *is not rational.*

The first important result on the way to prove this is a structure theorem for $K_0[\mathcal{V}_\mathbb{C}]$.

Theorem 9. *Set* $k = \mathbb{C}$*. Let* G *be an abelian commutative monoid and* $\mathbb{Z}[G]$ *be the corresponding monoid ring. As above, denote by* $\mathcal{M}$ *the multiplicative monoid of irreducible smooth complete varieties. Let*

$$\psi : \mathcal{M} \to G$$

be a homomorphism of monoids such that

(i) $\psi([X]) = \psi([Y])$ *if* X *and* Y *are birational;*

(ii) $\psi([\mathbb{P}^n]) = 1$ *for all* $n \geq 0$*.*

Then ψ *can be uniquely extended to a ring homomorphism*

$$\phi : K_0[\mathcal{V}_\mathbb{C}] \to \mathbb{Z}[G]$$

To prove this result we will use a result by Bittner.

Theorem 10 (Bittner, [Bit04, Theorem 3.1])**.** *The Grothendieck group $K_0[\mathcal{V}_{\mathbb{C}}]$ is generated by classes of smooth complete varieties subject to relations of the form*

$$[X] - [f^{-1}(Z)] = [Y] - [Z]$$

where X,Y are smooth complete varieties and $f : X \to Y$ is a morphism which is a blowup with a smooth center $Z \subset Y$.

Proof of theorem 9. We have to check that ψ preserves the above relations, i.e. that $\psi([X]) - \psi([f^{-1}(Z)]) = \psi([Y]) - \psi([Z])$. But $[X]$ and $[Y]$ are birational since f is a blowup. (WHY?) Now $f^{-1}(Z)$ is birational to $Z \times \mathbb{P}^n$ (WHY??) and thus

$$\psi([f^{-1}(Z)]) = \psi([Z \times \mathbb{P}^n]) = \psi([Z][\mathbb{P}^n]) = \psi([Z])\psi([\mathbb{P}^n) = \psi([Z])$$

Hence we can linearly extend ψ to define the morphism $\phi : K_0[\mathcal{V}_{\mathbb{C}}] \to \mathbb{Z}[G]$ $\square$

References

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