Motivic Zeta Function

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1 Introduction

In 1949 André Weil, in the paper "Number of solutions of equations in finite fields" [Wei49], formulated his famous Weil-Conjectures which predicted certain deep properties of the Hasse-Weil zeta function for a variety X over a finite field $k = \mathbb{F}_q$ defined as

$$Z(X,t) = \exp\Big(\sum_{m=0}^{\infty} \frac{|X(\mathbb{F}_{q^m})|}{m} t^m \Big).$$

The rationality of the zeta function, the most basic of the three conjectures, was first established by Dwork. Grothendieck later gave another proof for the case of projective varieties using his new scheme-theoretic approach to algebraic geometry.

Grothendieck's new perspective was fundamental for the work of Deligne, who used these new methods to finally establish the last part of the conjectures, an analogue of the Riemann hypothesis, in 1974. In [Kap00] Kapranov generalized the Hasse-Weil zeta function, defining it not only for varieties over finite fields but varieties in general. It turns out that in this more general setting the zeta function isn't as well behaved in terms of rationality as the "classical" Hasse-Weil zetafunction and we will investigate this phenomena more closely in the case of complex surfaces, following the work of Larsen and Lunts in [LL03].

Definiton 1. In the following the term *k-variety* always means a separated, integral scheme of finite type over a field k. We will write $\mathcal{V}_k$ for the category of k-varieties.

We now want to motivate what a possible generalization of the Hasse-Weil zeta function for varieties over a arbitrary field k should be. This part therefore, in some places, sacrifices mathematical precision for the sake of brevity. We still give citations for most of the results and will of course develop everything that is important for the presentation and proof of our main result in later sections in full detail.

Two things do not generalize easily: the passing to bigger and bigger extensions of the ground field k (what if k is algebraically closed, for example?), and counting k-rational points (again, if $k = \bar{k}$ there are, in general, a lot of k-rational points).

To see what could be generalizations for these phenomena, we give another equivalent definition of the Hasse-Weil zeta function in terms of effective 0-cycles. Recall that a 0-cycle on a Variety X is an element of the free abelian group over the closed points of X. Such a cycle $\alpha = \sum_i n_i x_i$ is called *effective* if the n_i are greater than or equal to zero. The *degree* of α is given by $deg(\alpha) = \sum_i n_i deg(x_i)$ ($deg(x_i)$ being the degree of the field extension $\kappa(x_i)/k$).

Using this notation we can rewrite the zeta function as follows

$$Z(X,t) = \sum_{\alpha} t^{deg(\alpha)} \tag{1}$$

Where the sum ranges over all 0-cycles on X. (For a proof of this (elementary) identity see for example Mustaţă's great notes on zeta functions in algebraic geometry [Mus11, Remark 2.9])

This definition still does not easily generalize, since for example $\mathbb{A}^1_{\mathbb{C}}$ has infinitely many 0-cycles of degree n.

But counting degree zero cycles of a given degree n, is the same as counting k-rational points of the n-fold symmetric product of the variety X (denoted by $Sym^n(X)$). We will not formally introduce the symmetric product until later, but thinking of it as the quotient of the n-fold product of X with itself by the natural action of the symmetric group in n letters gives the right intuition. For k algebraically closed, the bijection is obvious, since then the k-rational points are exactly the closed points. Dividing out by the action of the symmetric product hence introduces the commutativity-relations we have if we consider formal sums of such closed points. This intuition carries over for the case of finite fields, hence we get

$$Z(X,t) = \sum_{n} |Sym^n(X)(k)| t^n \tag{2}$$

So, in some rough sense, we are "measuring" the size of bigger and bigger symmetric products of X, so we expect the symmetric product to make an apearance in our generalized zeta function, taking the role of counting solutions over bigger and bigger field-extensions.

We now collect some properties of counting rational points of a variety over a finite field, that will guide our definition for "measures" of the symmetric product.

Remark 2. For a finite field k, consider the function [1]

$$\begin{aligned} \psi : \mathcal{V}_k &\to \mathbb{Z} \\ X &\mapsto |X(k)| \end{aligned}$$

1. If X and Y are isomorphic k-varieties, then $\psi(X) = \psi(Y)$
2. If $Y \subset X$ is closed subvariety, then $\psi(X \setminus Y) = \psi(X) - \psi(Y)$ (This essentialy gives the equation $Z(X \setminus Y, t) = \frac{Z(X,t)}{Z(Y,t)}$))
3. If X and Y are two varieties, then $\psi(X \times Y) = \psi(X)\psi(Y)$ (This gives, for example, the identity $Z(X \times \mathbb{A}^n_k, t) = Z(X, q^n t)$)

[1] not bothering with any set-theoretic issues

2 The Grothendieck Ring of Varieties and Motivic Measures

We now construct the Grothendieck ring of varieties as the universal ring over which any map with these properties factors as a ring homomorphism. Then our generalized measures will simply be ring homomorphisms out of this universal ring.

Definiton 3. Let k be a perfect field.[2] Consider the abelian group of formal linear combinations of isomorphism classes of varieties, subject to relations of the form

$$[X \setminus Y] = [X] - [Y]$$

where Y is closed in X. With multiplication given by

$$[X][Y] = [X \times Y]$$

this forms a ring, called the *Grothendieck ring of varieties* and denoted by $K_0[\mathcal{V}_k]$.

A *motivic measure* with values in a ring A is a ring homomorphism $\mu : K_0[\mathcal{V}_k] \to A$. The identity function $id : K_0[\mathcal{V}_k] \to K_0[\mathcal{V}_k]$ is called the *universal motivic measure.*

Since the product is commutative up to isomorphism this is a commutative ring with 1 (equal to [Spec(k)]). The cut and paste relation furthermore gives us

$$0 = [\emptyset] - [\emptyset] = [\emptyset \setminus \emptyset] = [\emptyset]$$

Example 4. *Using the decomposition $\mathbb{P}^n_k = \mathbb{P}^{n-1}_k \coprod \mathbb{A}^n_k$ where $\mathbb{P}^{n-1}_k$ is a (closed) hyperplane in $\mathbb{P}^n_k$ we get $[\mathbb{P}^n_k] = [\mathbb{P}^{n-1}_k] + [\mathbb{A}^n_k]$ in the Grothendieck ring. Inductively this yields the identity*

$$[\mathbb{P}^n_k] = \sum_{m=0}^{n} [\mathbb{A}^1_k]^m$$

We also denote the isomorphism class of the affine line as $\mathbb{L}$.

Remark 5. By [Har77, Proposition 10.1 (d)] the product of two smooth varieties over k is again smooth. Hence the isomorphism classes of smooth irreducible complete varieties form a multiplicative monoid, in the following denoted by $\mathcal{M}$.

A deep structure theorem, which we will not prove here, was given by Bittner.

[2] The definition can be altered to make sense even if k isn't perfect, in this case we just have to account for the fact that the product of two varieties need not be reduced. But since all our applications are either over finite fields or in characteristic zero this slight loss of generality won't affect us.

Theorem 6 (Bittner, [Bit04, Theorem 3.1])**.** *The Grothendieck group $K_0[\mathcal{V}_{\mathbb{C}}]$ is generated by classes of smooth complete varieties subject to relations of the form*

$$[X] - [f^{-1}(Z)] = [Y] - [Z]$$

where X,Y are smooth complete varieties and $f : X \to Y$ is a morphism which is a blowup with a smooth center $Z \subset Y$.

3 Symmetric Products and the Motivic Zeta Function

Definiton 7. Let $\mu : K_0[\mathcal{V}_k] \to A$ be a motivic measure and X a variety. Then we define the *motivic zeta function* $\zeta_\mu(X,t) \in A[[t]]$ as

$$\zeta_\mu(X,t) := \sum_{n=0}^{\infty} \mu([Sym^n(X)])t^n$$

Example 8. *We can now compute a first zeta function, namely $\zeta_{id}(\mathbb{P}^1_k, t)$. Using the identity and notation from example 4 we calculate*

$$\begin{aligned}
\zeta_{id}(\mathbb{P}^1_k, t) &= \sum_{n=0}^{\infty} [Sym^n(\mathbb{P}^1_k)]t^n \\
&= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \mathbb{L}^k\Big)t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (\mathbb{L}t)^k t^{n-k} \\
&= \Big(\sum_{n=0}^{\infty} t^n\Big)\Big(\sum_{n=0}^{\infty} (\mathbb{L}t)^n\Big) \\
&= \frac{1}{(1-t)(1-\mathbb{L}t)}
\end{aligned}$$

Hence $\zeta_{id}(\mathbb{P}^1_k, t)$ is in fact a rational function.

Kapranov proves in [Kap00] that the zeta function of a curve with coefficents in a field is in fact always rational. In their paper [LL03] Larsen and Lunts prove that in the case of surfaces this is false in general. Later the same authors gave a more precise characterisation when the zeta function of a complex surface is rational (namely if and only if the Kodaira dimension is $-\infty$. See [LL04]).

We will proof the following statement from [LL03]

Theorem 9. *Assume that $k = \mathbb{C}$. There exists a field $\mathcal{H}$ and a motivic measure $\mu : K_0[\mathcal{V}_k] \to \mathcal{H}$ with the following property: if X is a smooth* *complex projective surface such that $P_g(X) = h^{2,0}(X) \geq 2$, then the zeta-function $\zeta_\mu(X,t)$ is not rational.*

4 Constructing Motivic Measures

We will now construct this measure μ for which we will show irrationality of the zeta function.

The first important result on the way is a lemma that helps us to construct such measures by extending maps from the monoid of smooth, irreducible and complete varieties.

Theorem 10. *Set $k = \mathbb{C}$. Let G be an abelian commutative monoid and $\mathbb{Z}[G]$ be the corresponding monoid ring. As above, denote by $\mathcal{M}$ the multiplicative monoid of irreducible smooth complete varieties. Let*

$$\psi : \mathcal{M} \to G$$

be a homomorphism of monoids such that
(i) $\psi([X]) = \psi([Y])$ if X and Y are birational;
(ii) $\psi([\mathbb{P}^n]) = 1$ for all $n \geq 0$.
Then ψ can be uniquely extended to a ring homomorphism

$$\phi : K_0[\mathcal{V}_\mathbb{C}] \to \mathbb{Z}[G]$$

Proof of theorem 10. We have to check that ψ preserves the relations of the blowup presentation of $K_0[\mathcal{V}_\mathbb{C}]$, i.e. that $\psi([X]) - \psi([f^{-1}(Z)]) = \psi([Y]) - \psi([Z])$. But $[X]$ and $[Y]$ are birational since f is a blowup. Now $f^{-1}(Z)$ is birational to $Z \times \mathbb{P}^n$ (WHY??) and thus

$$\psi([f^{-1}(Z)]) = \psi([Z \times \mathbb{P}^n]) = \psi([Z][\mathbb{P}^n]) = \psi([Z])\psi([\mathbb{P}^n) = \psi([Z])$$

Hence we can linearly extend ψ to define the morphism $\phi : K_0[\mathcal{V}_\mathbb{C}] \to \mathbb{Z}[G]$ □

Before constructing a suitable monoid homomorphism that will extend to our desired measure, we recall the definition of the Hodge numbers of a variety as well as two basic results (all these are excercises in [Har77, Chapter II]).

Definiton 11. Let X be a projective variety. The i-th *Hodge number* $h^{i,0}$ is given as the dimension of $H^0(X, \Omega^i_X)$, which is a finite dimensional k-vectorspace by a theorem of Serre ([Ser55, §3 Prop. 7])

Lemma 12 ([Har77, II Ex. 8.8])**.** *The Hodge numbers are a birational invariant of a variety.*

Lemma 13 (Chow, [Har77, II Ex. 4.10])**.** *Every proper variety is birational to a projective variety.*

Definiton 14. Denote by $C \subset \mathbb{Z}[t]$ the multiplicative monoid of polynomials with a positive leading coefficient. For a smooth projective complex variety Z of dimension d define

$$\Psi_h(X) := 1 + h^{1,0}(X)t + \cdots + h^{d,0}(X)t^d \in C.$$

Where the $h^{i,0} := \dim_k(H^0(X, \Omega_X^i)$ are the *Hodge numbers* of X. By Chow's lemma we can also define Ψ_h for a smooth complete variety Z by choosing a smooth projective variety X which is birational to Z and setting $\Psi_h(Z) := \Psi_h(X)$. This is well defined by lemma 12.

We now check that Ψ_h satisfies all conditions of theorem 10. Independence of birational equivalence class was content of lemma 12.

5 Irrationality

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