Motivic Zeta Function

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Definiton 1. In the following the term k-*variety* always means a separated, integral scheme of finite type over a field k. Denote by $\mathcal{V}_k$ the category of k-varieties.

Definiton 2. Let k be a Field. Consider the group of formal linear combinations of isomorphism-classes in $\mathcal{V}_k$. Setting $[X] \times [Y] := [X \times Y]$ makes this into a ring. The *Grothendieck ring of varieties* $K_0[\mathcal{V}_k]$ is then obtained by modding out relations of the form

$$[X] - [Y] = [X \setminus Y]$$

Where Y is closed in X.

A *motivic measure* is a ringhomomorphism $\mu : K_0[\mathcal{V}_k] \to A$ into a ring A. The identity function $\mathrm{id} : K_0[\mathcal{V}_k] \to K_0[\mathcal{V}_k]$ is called the *universal motivic measure*.

Let us now make some remarks about this ring.

Remark 1. The Grothendieck ring of varieties is commutative as $X \times Y \cong Y \times X$ for two schemes X and Y.

Remark 2. By [Har77, Proposition 10.1 (d)] the product of two smooth varieties over k is again smooth. Hence the isomorphism classes of smooth irreducible complete varieties form a multiplicative monoid, in the following denoted by $\mathcal{M}$.

Example 1. *Using the decomposition* $\mathbb{P}^n_k = \mathbb{P}^{n-1}_k \coprod \mathbb{A}^n_k$ *where* $\mathbb{P}^{n-1}_k$ *is closed in* $\mathbb{P}^n_k$ *we get* $[\mathbb{P}^n_k] = [\mathbb{P}^{n-1}_k] + [\mathbb{A}^n_k]$ *in the Grothendieck ring. Inductively this yields the identity*

$$[\mathbb{P}^n_k] = \sum_{k=0}^{n} [\mathbb{A}^1_k]^k$$

We also denote the isomorphism class of the affine line as $\mathbb{L}$.

In their paper [LL03] Larsen and Lunts prove the following result:

Theorem 1. *Assume that* $k = \mathbb{C}$. *There exists a field* $\mathcal{H}$ *and a motivic measure* $\mu : K_0[\mathcal{V}_k] \to \mathcal{H}$ *with the following property: if X is a smooth complex projective surface such that* $P_g(X) = h^{2,0}(X) \geqslant 2$, *then the zeta-function* $\zeta_\mu(X, t)$ *is not rational.*

The first important theorem on the way to prove this result is this:

Theorem 2. *Set* $k = \mathbb{C}$. *Let* G *be an abelian commutative monoid and* $\mathbb{Z}[G]$ *be the corresponding monoid ring. As above, denote by* $\mathcal{M}$ *the multiplicative monoid of irreducible smooth complete varieties. Let*

$$\psi : \mathcal{M} \to G$$

be a homomorphism of monoids such that

(i) $\psi([X]) = \psi([Y])$ *if* X *and* Y *are birational;*

(ii) $\psi([\mathbb{P}^n]) = 1$ *for all* $n \geqslant 0$.

Then ψ *can be uniquely extended to a ring homomorphism*

$$\phi : K_0[\mathcal{V}_{\mathbb{C}}] \to \mathbb{Z}[G]$$

To prove this result we will use a result by Bittner (see [Bit04, Theorem 3.1]).

Theorem 3. *The Grothendieck group* $K_0[\mathcal{V}_{\mathbb{C}}]$ *is generated by classes of smooth complete varieties subject to relations of the form*

$$[X] - [f^{-1}(Z)] = [Y] - [Z]$$

where X, Y *are smooth complete varieties and* $f : X \to Y$ *is a morphism which is a blowup with a smooth center* $Z \subset Y$.

Proof of theorem 2. We have to check that ψ preserves the above relations, i.e. that $\psi([X]) - \psi[f^{-1}(Z)] = \psi([Y]) - \psi([Z])$. But $[X]$ and $[Y]$ are birational since f is a blowup. (WHY?) Now $f^{-1}(Z)$ is birational to $Z \times \mathbb{P}^n$ (WHY??) and thus

$$\psi([f^{-1}(Z)]) = \psi([Z \times \mathbb{P}^n]) = \psi([Z][\mathbb{P}^n]) = \psi([Z])\psi([\mathbb{P}^n]) = \psi([Z])$$

Hence we can linearly extend ψ to define the morphism $\phi : K_0[\mathcal{V}_{\mathbb{C}}] \to \mathbb{Z}[G]$ $\square$

References

[Bit04] Franziska Bittner. The universal Euler characteristic for varieties of characteristic zero. *Compos. Math.*, 140(4):1011–1032, 2004.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[LL03] Michael Larsen and Valery A. Lunts. Motivic measures and stable birational geometry. *Mosc. Math. J.*, 3(1):85–95, 259, 2003.