Motivic Zeta Function

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1 Introduction

In 1949 André Weil, in the paper [Wei49], formulated his famous Weil-Conjectures which predicted certain deep properties of the Hasse–Weil zeta function for a variety X over a finite field $k = \mathbb{F}_q$ defined as

$$Z(X,t) = \exp\Big(\sum_{m=0}^{\infty} \frac{|X(\mathbb{F}_{q^m})|}{m} t^m \Big).$$

The rationality of the zeta function, the most basic of the three conjectures, was first established by Dwork. Grothendieck later gave another proof for the case of smooth projective varieties using his new scheme-theoretic approach to algebraic geometry.

Grothendieck's new perspective was fundamental for the work of Deligne, who used these new methods to finally establish the last part of the conjectures, an analogue of the Riemann hypothesis, in 1974. In [Kap00] Kapranov generalized the Hasse-Weil zeta function, defining it not only for varieties over finite fields but varieties in general. It turns out that in this more general setting the zeta function is not as well behaved in terms of rationality as the "classical" Hasse–Weil zeta function and we will investigate this phenomena more closely in the case of complex surfaces, following the work of Larsen and Lunts [LL03].

We now want to motivate what a possible generalization of the Hasse-Weil zeta function for varieties over a arbitrary field k could be. This part therefore, in some places, sacrifices mathematical precision for the sake of brevity. We still give citations for most of the results and will of course develop everything that is important for the presentation and proof of our main result in later sections in full detail.

Two things do not generalize easily: the passing to bigger and bigger extensions of the ground field k (what if k is algebraically closed, for example?), and counting k-rational points (again, if $k = \bar{k}$ there are, in general, a lot of k-rational points).

To see what could be generalizations for these phenomena, we give another equivalent definition of the Hasse–Weil zeta function in terms of effective 0-cycles. Recall that a 0-cycle on a variety X is an element of the free abelian group over the closed points of X. Such a cycle $\alpha = \sum_i n_i x_i$ is called *effective* if the n_i are greater than or equal to zero. The *degree* of α is given by $\deg(\alpha) = \sum_i n_i \deg(x_i)$ (with $\deg(x_i)$ being the degree of the field extension $\kappa(x_i)/k$).

Using this notation we can rewrite the zeta function as follows

$$Z(X,t) = \sum_{\alpha} t^{\deg(\alpha)}. \tag{1}$$

Where the sum ranges over all effective 0-cycles on X. (For a proof of this (elementary) identity see for example Mustaţă's great notes on zeta functions in algebraic geometry [Mus11, Rem. 2.9])

This definition still does not easily generalize, since for example $\mathbb{A}^1_{\mathbb{C}}$ has infinitely many 0-cycles of degree n.

But counting degree zero cycles of a given degree n, is the same as counting k-rational points of the n-fold symmetric product of the variety X (denoted by $Sym^n(X)$). We will not formally introduce the symmetric product until later, but thinking of it as the quotient of the n-fold product of X with itself by the natural action of the symmetric group in n letters gives the right intuition. For k algebraically closed, the bijection is obvious, since then the k-rational points are exactly the closed points. Dividing out by the action of the symmetric group hence introduces the commutativity-relations we have if we consider formal sums of such closed points. This intuition carries over for the case of finite fields, hence we get

$$Z(X,t) = \sum_n |Sym^n(X)(k)| t^n. \tag{2}$$

So, in some rough sense, we are "measuring" the size of bigger and bigger symmetric products of X, so we expect the symmetric product to make an apearance in our generalized zeta function, taking the role of counting solutions over bigger and bigger field-extensions.

We now collect some properties of counting rational points of a variety over a finite field, that will guide our definition for "measures" of the symmetric product.

Remark 1. For a finite field k, consider the function [1]

$$\begin{aligned} \psi : \mathcal{V}_k &\to \mathbb{Z} \\ X &\mapsto |X(k)| \end{aligned}$$

1. If X and Y are isomorphic k-varieties, then $\psi(X) = \psi(Y)$
2. If $Y \subset X$ is closed subvariety, then $\psi(X \setminus Y) = \psi(X) - \psi(Y)$ (This essentialy gives the equation $Z(X \setminus Y, t) = \frac{Z(X,t)}{Z(Y,t)}$)
3. If X and Y are two varieties, then $\psi(X \times Y) = \psi(X)\psi(Y)$ (This gives, for example, the identity $Z(X \times \mathbb{A}^n_k, t) = Z(X, q^n t)$)

2 The Grothendieck Ring of Varieties and Motivic Measures

We now construct the Grothendieck ring of varieties as the universal ring over which any map with these properties factors as a ring homomorphism.

[1] not bothering with any set-theoretic issues

Then our generalized measures will simply be ring homomorphisms out of this universal ring.

Definiton 2. In the following the term *k-variety* always means a separated, reduced scheme of finite type over a field k. We will write $\mathcal{V}_k$ for the category of k-varieties.

Definiton 3. Let k be a perfect field.[2] Consider the abelian group of formal linear combinations of isomorphism classes of varieties, subject to relations of the form

$$[X \setminus Y] = [X] - [Y]$$

where Y is closed in X. With multiplication given by

$$[X][Y] = [X \times Y]$$

this forms a ring, called the *Grothendieck ring of varieties* and denoted by $K_0[\mathcal{V}_k]$.

A *motivic measure* with values in a ring A is a ring homomorphism $\mu\colon K_0[\mathcal{V}_k] \to A$. The identity function $id\colon K_0[\mathcal{V}_k] \to K_0[\mathcal{V}_k]$ is called the *universal motivic measure.*

Since the product is commutative up to isomorphism this is a commutative ring with 1 (equal to [Spec(k)]). The cut and paste relation furthermore gives us

$$0 = [\emptyset] - [\emptyset] = [\emptyset \setminus \emptyset] = [\emptyset]$$

Example 4. *Using the decomposition $\mathbb{P}_k^n = \mathbb{P}_k^{n-1} \coprod \mathbb{A}_k^n$ where $\mathbb{P}_k^{n-1}$ is a (closed) hyperplane in $\mathbb{P}_k^n$ we get $[\mathbb{P}_k^n] = [\mathbb{P}_k^{n-1}] + [\mathbb{A}_k^n]$ in the Grothendieck ring. Inductively this yields the identity*

$$[\mathbb{P}_k^n] = \sum_{m=0}^{n} [\mathbb{A}_k^1]^m$$

We also denote the isomorphism class of the affine line as $\mathbb{L}$.

Remark 5. By [Har77, Prop. 10.1 (d)] the product of two smooth varieties over k is again smooth. Hence the isomorphism classes of smooth irreducible complete varieties form a multiplicative monoid, in the following denoted by $\mathcal{M}$.

We now investigate the structure of the Grothendieck ring. First we note that it suffices to take irreducible varieties to generate $K_0[\mathcal{V}_k]$ To

[2]The definition can be altered to make sense even if k is not perfect, in this case we just have to account for the fact that the product of two varieties need not be reduced. But since all our applications are either over finite fields or in characteristic zero this slight loss of generality will not affect us.

see this, take a variety X with irreducible componets $Y_1, \cdots, Y_k$ and set $U_i := Y_i \setminus \bigcup_{i \neq j} Y_j$ and $U := \bigcup_i U_i$. By construction, the last union is actually a disjoint union, and hence $[U] = \sum_i [U_i]$. Together this gives

$$[X] = [X \setminus U] + \sum_i [U_i].$$

Now the U_i are irreducible as open subsets of irreducible sets, and $[X \setminus U]$ has dimension strictly smaller than the dimension of X since we removed a proper open subset from each irreducible component. Hence we can inductively rewrite $[X \setminus U]$, loosing at least one dimension in each step, with the base case of $\dim X = 0$ being trivial, as X is then just a collection of points.

In characteristic zero we can use a weak form of Hironaka's Theorem on the resolution of singularites, namely that for every irreducible, projective variety X there is a smooth, projective variety that is birational to X to restrict the set of needed generators even more.

Lemma 6. *Let k be a field of charakteristic zero. Then $K_0[\mathcal{V}_k]$ is generated by smooth, integral, projective varieties.*

Proof. As remarked before, we can restrict our attention to irreducible varietes. We once again argue by induction on the dimension. So let X be a irreducible variety of dimension n, by choosing some nonempty affine subvariety and passing to its projective closure we find X' projective, irreducible and birational to X. Resolving singularities we find $\widetilde{X}$ smooth, projective, irreducible birational to X. Hence we find isomorphic open subset $U \subset X$, $V \subset \widetilde{X}$ hand hence

$$[X \setminus U] - [\widetilde{X} \setminus V] = [X] - [U] - [\widetilde{X}] - [V] = [X] - [\widetilde{X}]$$

Hence $[X]$ can be written as the sum of $[\widetilde{X}]$ and some varieties of lower dimension for which we can invoke the induction hypothesis. $\square$

We will use the slightly stronger conclusion we just used in the induction, hence we state it as a separate corollary.

Corollary 7. *For each (irreducible) variety X there exists a smooth, irreducible, projective variety Y birational to X such that $[X] = [Y] + \sum_i^N m_i[W_i]$ where the $[W_i]$ are smooth, irreducible, projective varieties of dimension strictly smaller than the dimension of X.*

A deeper structure theorem, which we will not prove here, was given by Bittner.

Theorem 8 (Bittner, [Bit04, Thm. 3.1])**.** *The Grothendieck group $K_0[\mathcal{V}_\mathbb{C}]$ is generated by classes of smooth complete varieties subject to relations of the form*

$$[X] - [f^{-1}(Z)] = [Y] - [Z]$$

where X,Y are smooth complete varieties and $f \colon X \to Y$ is a morphism which is a blowup with a smooth center $Z \subset Y$.

3 Symmetric Products and the Motivic Zeta Function

Definiton 9. Let $\mu\colon K_0[\mathcal{V}_k] \to A$ be a motivic measure and X a variety. Then we define the *motivic zeta function* $\zeta_\mu(X,t) \in A[[t]]$ as

$$\zeta_\mu(X,t) := \sum_{n=0}^{\infty} \mu([Sym^n(X)])t^n$$

Example 10. *We can now compute a first zeta function, namely $\zeta_{id}(\mathbb{P}^1_k, t)$. Using the identity and notation from example 4 we calculate*

$$\begin{aligned}
\zeta_{id}(\mathbb{P}^1_k, t) &= \sum_{n=0}^{\infty} [Sym^n(\mathbb{P}^1_k)]t^n = \sum_{n=0}^{\infty} \big(\sum_{k=0}^{n} \mathbb{L}^k\big)t^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} (\mathbb{L}t)^k t^{n-k} \\
&= \Big(\sum_{n=0}^{\infty} t^n\Big)\Big(\sum_{n=0}^{\infty} (\mathbb{L}t)^n\Big) = \frac{1}{(1-t)(1-\mathbb{L}t)}
\end{aligned}$$

Hence $\zeta_{id}(\mathbb{P}^1_k, t)$ is in fact a rational function.

Kapranov proves in [Kap00] that the zeta function of a curve with coefficents in a field is in fact always rational. In their paper [LL03] Larsen and Lunts prove that in the case of surfaces this is false in general. Later the same authors gave a more precise characterisation when the zeta function of a complex surface is rational (namely if and only if the Kodaira dimension is $-\infty$. See [LL04]).

We will proof the following statement from [LL03]

Theorem 11. *Assume that $k = \mathbb{C}$. There exists a field $\mathcal{H}$ and a motivic measure $\mu\colon K_0[\mathcal{V}_k] \to \mathcal{H}$ with the following property: if X is a smooth complex projective surface such that $P_g(X) = h^{2,0}(X) \geq 2$, then the zeta-function $\zeta_\mu(X,t)$ is not rational.*

4 Constructing Motivic Measures

We will now construct a measure μ for which we will show irrationality of the zeta function.

The first important result on the way is a lemma that helps us to construct such measures by extending maps from the monoid of smooth, irreducible and complete varieties.

Theorem 12. *Set $k = \mathbb{C}$. Let G be an abelian commutative monoid and $\mathbb{Z}[G]$ be the corresponding monoid ring. As above, denote by $\mathcal{M}$ the multiplicative monoid of irreducible smooth complete varieties. Let*

$$\psi : \mathcal{M} \to G$$

be a homomorphism of monoids such that

1. *$\psi([X]) = \psi([Y])$ if X and Y are birational;*
2. *$\psi([\mathbb{P}^n]) = 1$ for all $n \geq 0$.*

Then ψ can be uniquely extended to a ring homomorphism

$$\phi : K_0[\mathcal{V}_{\mathbb{C}}] \to \mathbb{Z}[G]$$

Proof of Theorem 12. We have to check that ψ preserves the relations of the blowup presentation of $K_0[\mathcal{V}_{\mathbb{C}}]$, i.e. that $\psi([X]) - \psi([f^{-1}(Z)]) = \psi([Y]) - \psi([Z])$. But $[X]$ and $[Y]$ are birational since f is a blowup. Now $f^{-1}(Z)$ is birational to $Z \times \mathbb{P}^n$ (WHY??) and thus

$$\psi([f^{-1}(Z)]) = \psi([Z \times \mathbb{P}^n]) = \psi([Z][\mathbb{P}^n]) = \psi([Z])\psi([\mathbb{P}^n) = \psi([Z])$$

Hence we can linearly extend ψ to define a morphism $\phi\colon K_0[\mathcal{V}_{\mathbb{C}}] \to \mathbb{Z}[G]$ □

Before constructing a suitable monoid homomorphism that will extend to our desired measure, we recall the definition of the Hodge numbers of a variety as well as two basic results (all these are excercises in [Har77, Ch. II]).

Definiton 13. Let X be a projective variety. The i-th *Hodge number* $h^{i,0}$ is given as the dimension of $H^0(X, \Omega^i_X)$, which is a finite dimensional k-vector space by a theorem of Serre ([Ser55, §3 Prop. 7]). We will also write $P_g(X)$ for $h^{d,0}(X)$, called the geometric genus of a nonsingular variety X, with d being the dimension of X.

Lemma 14 ([Har77, II Ex. 8.8])**.** *The Hodge numbers $h^{i,0}$ are a birational invariant of a variety.*

Lemma 15 (Chow, [Har77, II Ex. 4.10])**.** *Every proper variety is birational to a projective variety.*

Definiton 16. Denote by $C \subset \mathbb{Z}[t]$ the multiplicative monoid of polynomials with positive leading coefficient. For a smooth projective complex variety Z of dimension d define

$$\Psi_h(X) := 1 + h^{1,0}(X)t + \cdots + h^{d,0}(X)t^d \in C.$$

By Chow's lemma we can also define Ψ_h for a smooth complete variety Z by choosing a smooth projective variety X which is birational to Z and setting $\Psi_h(Z) := \Psi_h(X)$. This is well defined by Lemma 14.

We now check that Ψ_h satisfies all conditions of Theorem 12. Independence of birational equivalence class was the content of Lemma 14.

To check multiplicativity, we use the following Lemma.

Lemma 17. *Let X, Y be smooth, irreducible, projective k-varieties. Then the following equality holds:*

$$h^{p,0}(X \times_k Y) = \sum_{i+j=p} h^{i,0}(X) h^{j,0}(Y)$$

Proof. TODO □

With this we directly calculate

$$\begin{aligned} \Psi_h(X \times_k Y) &= \sum_p h^{p,0}(X \times_k Y) = \sum_p \sum_{i+j=p} h^{i,0}(X) h^{j,0}(Y) \\ &= \Big(\sum_p h^{p,0}(X) \Big) \Big(\sum_p h^{p,0}(Y) \Big) = \Psi_h(X) \Psi_h(Y) \end{aligned}$$

5 Irrationality

The constructed motivic measure does not yet yield values in a field, so we would like to pass to the quotient field of $\mathbb{Z}[\mathcal{M}]$. That we are able to do so is the content of the next lemma.

Lemma 18. *Let A be a factorial ring, and $S \subset A$ a multaplicative submonoid such that 1 is the only unit in S. Then the monoid ring $\mathbb{Z}[S]$ is a polynomial ring (in possibly infinitely many variables), and hence an integral domain.*

Proof. Since A is factorial, every $s \in S$ has a unique factorization, and since 1 is the only unit in S, s can be uniquely written as product of prime elements, hence if we take B to be the polynomial ring over the formal variables $\{x_s | s \in S, sprime\}$ we get an isomorphism of rings

$$\begin{aligned} B &\to \mathbb{Z}[S] \\ x_s &\mapsto s \end{aligned}$$

□

Definiton 19. Denote by $\mathcal{H}$ the quotient field of $\mathbb{Z}[C]$ (with C being the submonoid of $\mathbb{Z}[t]$ consisting of polynomials p with $p(0) = 1$). We define the motivic measure $\mu_h \colon K_0[\mathcal{V}_k] \to \mathcal{H}$ as the measure obtained extending Ψ_h as by Theorem 12

Lemma 20. *Let $Y_1, \cdots, Y_s, Z$ be irreducible varieties of dimension d over a field of characteristic zero such that $\mu_h([Z]) = \sum_i n_i \mu_h([Y_i])$ for some $n_i \in \mathbb{Z}$ and $P_g(Z) \neq 0$ then $P_g(Z) = P_g(Y_i)$ for some i*

Proof. By Corollary 7, we find smooth, irreducible projective varieties $\overline{Z}, \overline{Y_1}, \cdots, \overline{Y_s}$ in the same birationial class as the original varieties allowing us to rewrite the original equality as

$$\mu_h([\overline{Z}]) = \sum n_i \mu_h([\overline{Y}_i]) + \sum l_i \mu_h([X_i])$$

With the X_i being smooth irreducible varieties of dimension $< d$. But now, since μ_h was obtained as an extension of Ψ_h which was defined on smooth, irreducible, projective varieties, this is actually an equation in $\mathbb{Z}[C]$, namely

$$\Psi_h(\overline{Z}) = \sum n_i \Psi_h(\overline{Y}_i) + \sum l_i \Psi_h(X_i)$$

Since there are no (additive) relations between elements of C in the monoid ring, one of the polynomials $\Psi_h(\dots)$ must actually be the polynomial on the left hand side, but with the dimension of X_i being strictly smaller than that of Z and the assumption that $P_g(\overline{Z}) = h^{d,0}(\overline{Z}) \neq 0$ it can not be one of the $\Psi_h(X_i)$ because all these polynomials have strictly smaller degree. Hence we have $\Psi_h(\bar{(Z)}) = \Psi_h(\overline{Y}_i)$ and in particular, since they are of the same dimension, the genus of $\overline{Z}$ and Y_i must agree. □

Now let X be a smooth projective surface with $P_g(X) \geq 2$. We will show that $\zeta_{\mu_h}(X,t) \in \mathcal{H}[[t]]$ is not rational, thus proving Theorem 11. We need one technical lemma about the genus of symmetric powers of X, which we can not proof here.

Lemma 21 ([LL03, Lem. 3.8])**.** *Let X be a smooth, projective surface over $\mathbb{C}$. Then*

$$P_g(Sym^n(X)) = \binom{P_g(X) + n - 1}{P_g(X) - 1}$$

We will use the following rationality criterion for power series.

Lemma 22. *Let k be a field. Then a power series $\sum a_i t^i \in k[[t]]$ is a rational function (i.e. an element in $k(t)$) if and only if there exist $n, n_0 > 0$ such that for each $m > n_0$ the determinant*

$$\begin{vmatrix} a_m & a_{m+1} & \dots & a_{m+n} \\ a_{m+1} & a_{m+2} & \dots & a_{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+n} & a_{m+n+1} & \dots & a_{m+2n} \end{vmatrix}$$

vanishes.

Proof. See [Bru63, Section 5.3, Lem. 1] □

Proof of Theorem 11. Assume that $\zeta_{\mu_h}(X,t)$ is rational, hence by the above criterion there is a n such that for m big enough

$$\begin{aligned} & \sum_{\sigma \in S_{n+1}} \mu_h(\prod_{i=1}^{n+1} Sym^{m+i+\sigma(i)-2}) = 0 \\ \Leftrightarrow \mu_h\Big(\prod_{i=0}^{n} Sym^{m+2i}\Big) = - & \sum_{\substack{\sigma \in S_{n+1} \\ \sigma \neq id}} \mu_h\Big(\prod_{i=0}^{n} Sym^{m+i+\sigma(i+1)-1}(X)\Big) \end{aligned} \tag{3}$$

Now we can apply Lemma 20 to conclude that there is a permutation σ, not the identity permutation, such that

$$P_g\Big(\prod_{i=0}^{n} Sym^{m+2i}(X)\Big) = P_g\Big(\prod_{i=0}^{n} Sym^{m+i+\sigma(i+1)-1}(X)\Big)$$

By Lemma 21 (and using the fact that the genus is multiplicative), we get

$$\prod_{i=0}^{n} \binom{P_g(X)+m+2i-1}{P_g(X)-1} - \prod_{i=0}^{n} \binom{P_g(X)+m+i+\sigma(i+1)-2}{P_g(X)-1} = 0$$

By assumption $P_g(X) \geq 2$ hence the left hand side, considered as an polynomial in m, is not the zero polynomial. So by taking m large enough we obtain a contradiction. □

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