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1 Introduction

2 Topological Spaces

Problem 2-5: For each of the following properties, give an example consisting of two subsets $X, Y \subseteq \mathbb{R}^2$, both considered as topological spaces with their Euclidian topologies, together with a map $f : X \rightarrow Y$ that has the indicated property.

- f is open but neither closed nor continuous.
- f is closed but neither open nor continuous.
- f is continuous but neither open nor closed.
- f is continuous and open but not closed.
- f is continuous and closed but not open.
- f is open and closed but not continuous.

Proof: Define the following topological spaces:

- D is the discrete topology on $\{x, y\}$
- T is the trivial topology on $\{x, y\}$
- X is the one-point topology on the set $\{x, y, z\}$ on x

Then we can proceed like so:

- Consider the inclusion map $f : T \rightarrow X$. This map is not continuous because the preimage $f^{-1}(\{x\}) = \{x\}$ of the open set in X is not open in T . The map is also not closed because the image $f(\{x, y\}) = \{x, y\}$ is not closed in X . The image of the lone non-null open set $f(\{x, y\}) = \{x, y\}$ is open, however, implying that the map is open.
- Consider the map $f : T \rightarrow X$ mapping $x \mapsto y$ and $y \mapsto z$. This map is not continuous because the preimage of the open set $\{y\}$ in X is not open in T . The map is also not open because the image $f(\{x, y\}) = \{y, z\}$ is not open in X . The image of the lone non-null closed set $f(\{x, y\}) = \{y, z\}$ is closed, however, implying that the map is closed.
- Consider the identity map $f : D \rightarrow T$. This map is continuous (since any map from a discrete space is continuous), but neither open nor closed, since the image of the clopen set $\{x\}$ in D is neither closed nor open in T .
- Consider the constant map $f : D \rightarrow X$ sending all elements of T to x . This map is continuous since any map from a discrete space is continuous. The image of any set in D is either \emptyset or $\{x\}$, both of which are open in X . In particular, the image of any open set is open. But the image of the closed set $\{x\}$ in D is $\{x\}$, which is not closed in X .
- Consider the constant map $f : D \rightarrow X$ sending all elements of T to y . Similar to (d), this map is continuous, but the image of any set in T is either \emptyset or $\{y\}$, both of which are closed. Similar to the reasoning in (d), this map is closed but not open.
- Consider the identity map $f : T \rightarrow D$. This map is open and closed, since the image of the lone non-null clopen set $\{x, y\}$ is clopen in D , but it is also not continuous because the preimage $f^{-1}(\{x\}) = \{x\}$ of the open set in D is not open in T .

□

Problem 2-6: Suppose X and Y are topological spaces, and $f : X \rightarrow Y$ is a continuous map.

- f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$.
- f is closed if and only if $f(\overline{A}) \supseteq \overline{f(A)}$ for all $A \subseteq X$.
- f is continuous if and only if $f^{-1}(\text{Int } B) \subseteq \text{Int } f^{-1}(B)$ for all $B \subseteq Y$.
- f is open if and only if $f^{-1}(\text{Int } B) \supseteq \text{Int } f^{-1}(B)$ for all $B \subseteq Y$.

Proof: We will use repeatedly the following set-theoretic identities that hold for any function $f : X \rightarrow Y$ and subsets $A \subseteq X, B \subseteq Y$:

- $f(A) \subseteq B \iff A \subseteq f^{-1}(B)$ (adjunction identity)
- $f^{-1}(f(A)) \supseteq A$ (unit identity)
- $f(f^{-1}(B)) \subseteq B$ (co-unit identity)

- (\implies) By the “unit identity” above, we have,

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)})$$

From which it follows that,

$$\overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$$

Where the last equality follows from the fact that f is continuous, so that $f^{-1}(\overline{f(A)})$ is closed. Then this is equivalent to,

$$f(\overline{A}) \subseteq \overline{f(A)}$$

By the “adjunction identity” above.

(\impliedby) Suppose $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$, and U is a closed set in Y . We wish to show that $f^{-1}(U)$ is closed in X . First note that by the “co-unit identity” above:

$$f(f^{-1}(U)) \subseteq U$$

So,

$$\overline{f(f^{-1}(U))} \subseteq \overline{U} = U$$

Where the last equality follows from U closed. Then by the assumption,

$$f(\overline{f^{-1}(U)}) \subseteq \overline{f(f^{-1}(U))} \subseteq U$$

So that $\overline{f^{-1}(U)} \subseteq f^{-1}(U)$, implying that $f^{-1}(U)$ is closed.

- (\implies) Suppose f is closed. Then $f(\overline{A})$ is closed since \overline{A} is closed so $\overline{f(\overline{A})} \subseteq f(\overline{A})$. Then $A \subseteq \overline{A} \implies f(A) \subseteq f(\overline{A}) \implies \overline{f(A)} \subseteq f(\overline{A})$, and thus,

$$\overline{f(A)} \subseteq \overline{f(\overline{A})} \subseteq f(\overline{A})$$

(\impliedby) Suppose $\overline{f(A)} \subseteq f(\overline{A})$ for all $A \subseteq X$ and U is closed in Y . Since U is closed $\overline{U} = U$, so $\overline{f(U)} \subseteq f(\overline{U}) = f(U)$. It follows that $f(U)$ is closed.

- First note that,

$$\begin{aligned} f^{-1}(\text{Int } B) &\subseteq \text{Int } f^{-1}(B) \quad \forall B \subseteq Y \\ \iff X - \text{Int } f^{-1}(B) &\subseteq X - f^{-1}(\text{Int } B) = f^{-1}(X - \text{Int } B) \quad \forall B \subseteq Y \\ \iff \overline{X - f^{-1}(B)} &\subseteq f^{-1}(\overline{X - \text{Int } B}) \quad \forall B \subseteq Y \\ \iff \overline{f^{-1}(X - B)} &\subseteq f^{-1}(\overline{X - B}) \quad \forall B \subseteq Y \\ \iff \overline{f^{-1}(B)} &\subseteq f^{-1}(\overline{B}) \quad \forall B \subseteq Y \end{aligned}$$

Next we can show that,

$$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \quad \forall B \subseteq Y \iff f(\overline{A}) \subseteq \overline{f(A)} \quad \forall A \subseteq X$$

Which is equivalent to,

$$f(\overline{f^{-1}(B)}) \subseteq \overline{B} \quad \forall B \subseteq Y \iff \overline{A} \subseteq f^{-1}(\overline{f(A)}) \quad \forall A \subseteq X$$

Since for any subsets $A \subseteq X$ and $B \subseteq Y$ we have $f(A) \subseteq B \iff A \subseteq f^{-1}(B)$. To show this we have:

$$\begin{aligned} (\implies) \quad \overline{f(A)} &\supseteq f(\overline{f^{-1}(f(A))}) \supseteq f(\overline{A}) \\ (\impliedby) \quad \overline{f^{-1}(B)} &\subseteq f^{-1}(\overline{f(f^{-1}(B))}) \subseteq f^{-1}(\overline{B}) \end{aligned}$$

Where in either case, the leftmost inclusion follows from substitution of either $\overline{f(A)}$ or $\overline{f^{-1}(B)}$ into the antecedent, and the rightmost inclusion follows from the “(co-)unit” identities.

Therefore the subset condition in (c) is equivalent to the one in (a).

- (\implies) We have,

$$\begin{aligned} \text{Int } f^{-1}(B) &\subseteq f^{-1}(B) \\ \implies f(\text{Int } f^{-1}(B)) &\subseteq f(f^{-1}(B)) \subseteq B \end{aligned}$$

But since f is open by assumption, that makes $f(\text{Int } f^{-1}(B))$ a subset not just of B but of $\text{Int } B$. So:

$$\begin{aligned} f(\text{Int } f^{-1}(B)) &\subseteq \text{Int } B \\ \implies \text{Int } f^{-1}(B) &\subseteq f^{-1}(\text{Int } B) \end{aligned}$$

(\impliedby) Let U be an open set in X . Then,

$$\text{Int } f^{-1}(f(U)) \subseteq f^{-1}(f(U)) \subseteq U$$

So,

$$\text{Int } f(U) \subseteq f(\text{Int } f^{-1}(f(U))) \subseteq f(U)$$

Where the leftmost inclusion follows from the assumption applied to $f(U) \subseteq Y$. It follows that $f(U)$ is open.

□