1 Introduction

2 Topological Spaces

Problem 2-5: For each of the following properties, give an example consisting of two subsets $X, Y \subseteq \mathbb{R}^2$, both considered as topological spaces with their Euclidian topologies, together with a map $f : X \to Y$ that has the indicated property.

(a) f is open but neither closed nor continuous.
(b) f is closed but neither open nor continuous.
(c) f is continuous but neither open nor closed.
(d) f is continuous and open but not closed.
(e) f is continuous and closed but not open.
(f) f is open and closed but not continuous.

Proof: Define the following topological spaces:

- D is the discrete topology on $\{x, y\}$
- T is the trivial topology on $\{x, y\}$
- X is the one-point topology on the set $\{x, y, z\}$ on x

Then we can proceed like so:

(a) Consider the inclusion map $f : T \to X$. This map is not continuous because the preimage $f^{-1}(\{x\}) = \{x\}$ of the open set in X is not open in T. The map is also not closed because the image $f(\{x, y\}) = \{x, y\}$ is not closed in X. The image of the lone non-null open set $f(\{x, y\}) = \{x, y\}$ is open, however, implying that the map is open.
(b) Consider the map $f : T \to X$ mapping $x \mapsto y$ and $y \mapsto z$. This map is not continuous because the preimage of the open set $\{y\}$ in X is not open in T. The map is also not open because the image $f(\{x, y\}) = \{y, z\}$ is not open in X. The image of the lone non-null closed set $f(\{x, y\}) = \{y, z\}$ is closed, however, implying that the map is closed.
(c) Consider the identity map $f : D \to T$. This map is continuous (since any map from a discrete space is continuous), but neither open nor closed, since the image of the clopen set $\{x\}$ in D is neither closed nor open in T.
(d) Consider the constant map $f : D \to X$ sending all elements of T to x. This map is continuous since any map from a discrete space is continuous. The image of any set in D is either $\emptyset$ or $\{x\}$, both of which are open in X. In particular, the image of any open set is open. But the image of the closed set $\{x\}$ in D is $\{x\}$, which is not closed in X.
(e) Consider the constant map $f : D \to X$ sending all elements of T to y. Similar to (d), this map is continuous, but the image of any set in T is either $\emptyset$ or $\{y\}$, both of which are closed. Similar to the reasoning in (d), this map is closed but not open.
(f) Consider the identity map $f : T \to D$. This map is open and closed, since the image of the lone non-null clopen set $\{x, y\}$ is clopen in D, but it is also not continuous because the preimage $f^{-1}(\{x\}) = \{x\}$ of the open set in D is not open in T.

□

Problem 2-6: Suppose X and Y are topological spaces, and $f : X \to Y$ is a continuous map.

(a) f is continuous if and only if $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for all $A \subseteq X$.
(b) f is closed if and only if $f\left(\overline{A}\right) \supseteq \overline{f(A)}$ for all $A \subseteq X$.
(c) f is continuous if and only if $f^{-1}(\operatorname{Int} B) \subseteq \operatorname{Int} f^{-1}(B)$ for all $B \subseteq Y$.
(d) f is open if and only if $f^{-1}(\operatorname{Int} B) \supseteq \operatorname{Int} f^{-1}(B)$ for all $B \subseteq Y$.

Proof: We will use repeatedly the following set-theoretic identities that hold for any function $f : X \to Y$ and subsets $A \subseteq X$, $B \subseteq Y$:

- $f(A) \subseteq B \iff A \subseteq f^{-1}(B)$ (adjunction identity)
- $f^{-1}(f(A)) \supseteq A$ (unit identity)
- $f(f^{-1}(B)) \subseteq B$ (co-unit identity)

(a) ($\Longrightarrow$) By the "unit identity" above, we have,

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\left(\overline{f(A)}\right)$$

From which it follows that,

$$\overline{A} \subseteq \overline{f^{-1}\left(\overline{f(A)}\right)} = f^{-1}\left(\overline{f(A)}\right)$$

Where the last equality follows from the fact that f is continuous, so that $f^{-1}\left(\overline{f(A)}\right)$ is closed. Then this is equivalent to,

$$f\left(\overline{A}\right) \subseteq \overline{f(A)}$$

By the "adjunction identity" above.

($\Longleftarrow$) Suppose $f\left(\overline{A}\right) \subseteq \overline{f(A)}$ for all $A \subseteq X$, and U is a closed set in Y. We wish to show that $f^{-1}(U)$ is closed in X. First note that by the "co-unit identity" above:

$$f(f^{-1}(U)) \subseteq U$$

So,

$$\overline{f(f^{-1}(U))} \subseteq \overline{U} = U$$

Where the last equality follows from U closed. Then by the assumption,

$$f\left(\overline{f^{-1}(U)}\right) \subseteq \overline{f(f^{-1}(U))} \subseteq U$$

So that $\overline{f^{-1}(U)} \subseteq f^{-1}(U)$, implying that $f^{-1}(U)$ is closed.

(b) ($\Longrightarrow$) Suppose f is closed. Then $f\left(\overline{A}\right)$ is closed since $\overline{A}$ is closed so $\overline{f\left(\overline{A}\right)} \subseteq f\left(\overline{A}\right)$. Then $A \subseteq \overline{A} \Longrightarrow f(A) \subseteq f\left(\overline{A}\right) \Longrightarrow \overline{f(A)} \subseteq f\left(\overline{A}\right)$, and thus,

$$\overline{f(A)} \subseteq \overline{f\left(\overline{A}\right)} \subseteq f\left(\overline{A}\right)$$

($\Longleftarrow$) Suppose $\overline{f(A)} \subseteq f\left(\overline{A}\right)$ for all $A \subseteq X$ and U is closed in X. Since U is closed $\overline{U} = U$, so $\overline{f(U)} \subseteq f\left(\overline{U}\right) = f(U)$. It follows that $f(U)$ is closed.

(c) First note that,

$$\begin{aligned}
&f^{-1}(\operatorname{Int} B) \subseteq \operatorname{Int} f^{-1}(B) \quad \forall B \subseteq Y \\
\iff &X - \operatorname{Int} f^{-1}(B) \subseteq X - f^{-1}(\operatorname{Int} B) = f^{-1}(X - \operatorname{Int} B) \quad \forall B \subseteq Y \\
\iff &\overline{X - f^{-1}(B)} \subseteq f^{-1}\left(\overline{X - B}\right) \quad \forall B \subseteq Y \\
\iff &\overline{f^{-1}(X - B)} \subseteq f^{-1}\left(\overline{X - B}\right) \quad \forall B \subseteq Y \\
\iff &\overline{f^{-1}(B)} \subseteq f^{-1}\left(\overline{B}\right) \quad \forall B \subseteq Y
\end{aligned}$$

Next we can show that,

$$\overline{f^{-1}(B)} \subseteq f^{-1}\left(\overline{B}\right) \quad \forall B \subseteq Y \iff f\left(\overline{A}\right) \subseteq \overline{f(A)} \quad \forall A \subseteq X$$

Which is equivalent to,

$$f\left(\overline{f^{-1}(B)}\right) \subseteq \overline{B} \quad \forall B \subseteq Y \iff \overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right) \quad \forall A \subseteq X$$

Since for any subsets $A \subseteq X$ and $B \subseteq Y$ we have $f(A) \subseteq B \iff A \subseteq f^{-1}(B)$. To show this we have:

$$\begin{aligned}
(\Longrightarrow) \quad & \overline{f(A)} \supseteq f\left(\overline{f^{-1}(f(A))}\right) \supseteq f\left(\overline{A}\right) \\
(\Longleftarrow) \quad & \overline{f^{-1}(B)} \subseteq f^{-1}\left(\overline{f(f^{-1}(B))}\right) \subseteq f^{-1}\left(\overline{B}\right)
\end{aligned}$$

Where in either case, the leftmost inclusion follows from substitution of either $\overline{f(A)}$ or $\overline{f^{-1}(B)}$ into the antecedent, and the rightmost inclusion follows from the "(co-)unit" identities.

Therefore the subset condition in (c) is equivalent to the one in (a).

(d) ($\Longrightarrow$) We have,

$$\begin{aligned}
&\operatorname{Int} f^{-1}(B) \subseteq f^{-1}(B) \\
\Longrightarrow\ &f(\operatorname{Int} f^{-1}(B)) \subseteq f(f^{-1}(B)) \subseteq B
\end{aligned}$$

But since f is open by assumption, that makes $f(\operatorname{Int} f^{-1}(B))$ a subset not just of B but of $\operatorname{Int} B$. So:

$$\begin{aligned}
&f(\operatorname{Int} f^{-1}(B)) \subseteq \operatorname{Int} B \\
\Longrightarrow\ &\operatorname{Int} f^{-1}(B) \subseteq f^{-1}(\operatorname{Int} B)
\end{aligned}$$

($\Longleftarrow$) Let U be an open set in X. Then,

$$\operatorname{Int} f^{-1}(f(U)) \subseteq f^{-1}(f(U)) \subseteq U$$

So,

$$\operatorname{Int} f(U) \subseteq f(\operatorname{Int} f^{-1}(f(U))) \subseteq f(U)$$

Where the leftmost inclusion follows from the assumption applied to $f(U) \subseteq Y$. It follows that $f(U)$ is open.

□