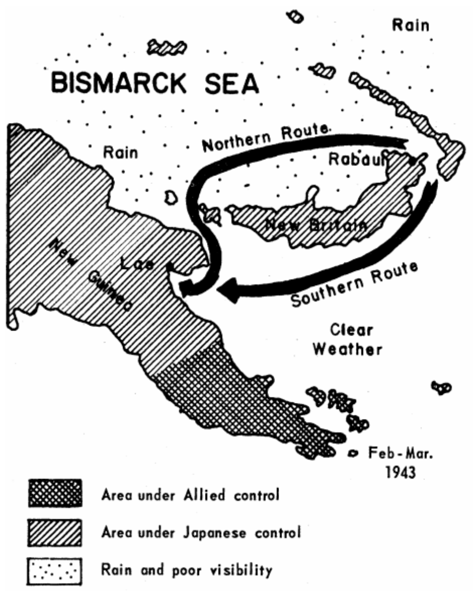
GSOE9210 Engineering Decisions

Problem Set 09

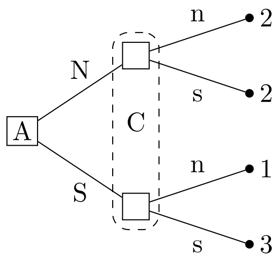
1. Consider the 'Battle of the Bismarck Sea' discussed in lectures.



(a) Can this be represented as a zero-sum game?
(b) Represent the game in extensive form (*i.e.*, as a game tree).
(c) Represent the game in normal (strategic) form (*i.e.*, as a game matrix).
(d) Simplify the problem by eliminating dominated strategies.
(e) Which, if any, are the rational 'solutions' to the game?

Solution

(a) The game is strictly competitive, so, yes, the Convoy Captain's (C) payoffs can be considered as the complement of the Allies General's (A).
(b) The game tree is shown below:



Note that the Convoy Captain does not know about the Allies General's search strategy, and vice versa (*i.e.*, this is not a game of perfect information), and hence, when modelling the problem by the tree above, both of the Captain's decision nodes must belong to the same information set.

Note also that if the Allies General chooses to search North, then the Convoy Captain's actions lead to different outcomes which happen to have the same payoff (in contradistinction to the same outcome).

(c) The normal form is shown below:

		C	
		N	S
A	n	2	2
	s	1	3

		C	
		N	~~S~~
A	n	2	~~2~~
	~~s~~	~~1~~	~~3~~

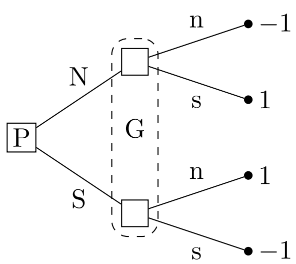
(d) The reduction is shown above (right).
(e) The game's rational(ised) solution is (n,N).

2. Consider the 'Jailbreak game' from lectures. Suppose that neither the prisoner (P) nor the guard (G) know the other's move.

   (a) Is the game zero-sum?
   (b) Draw the game tree for this game.
   (c) Convert this to extensive form, with the prisoner as the row player.
   (d) Simplify the game using dominance.
   (e) Repeat the above for the game in which the prisoner knows which wing the guard will patrol (*i.e.*, N or S).

   Solution

   (a) The game can be represented as a zero sum game, at least with some reasonable assumptions.
   (b) The game tree is shown below:

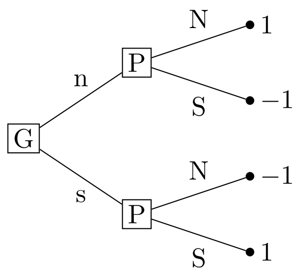


		G	
		n	s
P	N	-1	1
	S	1	-1

(c) The game can be represented by the matrix above (right).

(d) No strategy is dominated, for either player. The game cannot be rationalised (*i.e.*, simplified) further by eliminating dominated strategies.

(e) The tree is shown below, with payoffs to the guard:



Note that the prisoner has different epistemic states at his two nodes (after n and s), at which point, he knows for certain the guard's strategy.

The corresponding game matrix is shown below:

G \ P	~~n/N s/N~~	~~n/N s/S~~	n/S s/N	~~n/S s/S~~
n	~~1~~	~~1~~	−1	~~−1~~
s	~~−1~~	~~1~~	−1	~~1~~

G \ P	n/S s/N
n	−1
s	−1

For P the third column ('do the opposite of what the guard does') weakly dominates the others. The reduced game is shown above (right).

This sub-game cannot be simplified further.

This is a lose-lose outcome for the guard: regardless of what the guard does, the prisoner will escape.

3. Use dominance to solve the following zero-sum game (payoffs are for the row player):

	b_1	b_2	b_3	b_4
a_1	0	1	7	7
a_2	4	1	2	10
a_3	3	1	0	25
a_4	0	0	7	10

Solution

The elimination steps are shown, beginning with b_4, which is dominated by b_1, b_2, and b_3.

	b_1	b_2	b_3	b_4
a_1	0	1	7	7
a_2	4	1	2	10
a_3	3	1	0	25
a_4	0	0	7	10

The solution, arrived at by iteratively eliminating dominated strategies, is (a_2, b_2).

4. Use dominance to reduce the following zero-sum games:

	b_1	b_2	b_3
a_1	3	8	3
a_2	0	1	10
a_3	3	6	5

	b_1	b_2	b_3	b_4
a_1	1	2	3	3
a_2	1	5	0	0
a_3	1	6	4	1

Solution

On the left, b_2 and b_3 are dominated by b_1. Then a_2 is dominated by a_1 and a_3. So the solutions are the plays (a_1, b_1) and (a_3, b_1).

	b_1	b_2	b_3
a_1	3	8	3
a_2	0	1	10
a_3	3	6	5

	b_1	b_2	b_3	b_4
a_1	1	2	3	3
a_2	1	5	0	0
a_3	1	6	4	1

On the right, a_2 is dominated by a_3, and b_2 and b_3 by b_1 and b_4 respectively. Then a_3 is dominated by a_1 and b_4 by b_1, leaving only play (a_1, b_1).

5. Consider the following zero-sum game, in which mixed strategies are allowed.

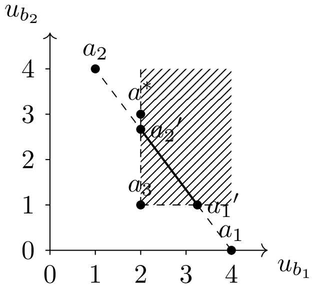
		B	
		b_1	b_2
A	a_1	4	0
	a_2	1	4
	a_3	2	1

(a) Which, if any, strategies can be eliminated by using dominance?

(b) Show that if player A had a possible strategy a^*, with payoffs 2 and 3 in response to player B's strategies b_1 and b_2 respectively, then a^* would not be dominated.

Solution

(a) Since there are only two pure strategies for B, and neither dominates the other, then neither can be eliminated by dominance, using mixtures or otherwise.
For player A, strategy a_3 is dominated by a mixture of a_1 and a_2.



B

A	b_1	b_2
a_1	4	0
a_2	1	4
~~a_3~~	~~2~~	~~1~~

The reduced matrix is shown above (right).

(b) Consider mixtures of a_1 and a_2 with parameter μ representing the proportion of a_1 in the mixture; *i.e.*, $M(\mu) = \mu a_1 + (1 - \mu)a_2$.
It follows that $M(\mu) = (3\mu + 1, 4 - 4\mu)$. For any mixture $M(\mu)$ to dominate a^* we require that $3\mu + 1 \geqslant 2$ and $4 - 4\mu \geqslant 3$; *i.e.*, $\mu \geqslant \frac{1}{3}$ and $\mu \leqslant \frac{1}{4}$. Since there is no $\mu \in [0, 1]$ that can simultaneously satisfy both conditions, it follows that there can be no mixture of a_1 and a_2 that dominates a^*.

6. Use dominance to solve the following matrix representation of a two-player non strictly competitive game.

Y

X	b_1	b_2	b_3
a_1	0, 0	1, 2	0, 2
a_2	1, 3	1, 4	0, 0

Solution

Y's strategy b_2 dominates b_1 and b_3, and X's strategy a_2 dominates a_1. The reduction steps are shown below:

Y

X	~~b_1~~	b_2	~~b_3~~
~~a_1~~	~~0, 0~~	~~1, 2~~	~~0, 2~~
a_2	~~1, 3~~	1, 4	~~0, 0~~

7. Two companies, X and Y, produce a similar product which earns a profit of \$1 per unit sold. The two companies compete for a total annual market of 4000 units. However, if either company (or both) advertises, the total annual market will increase by 50%.

If neither or both companies advertise then they split the market evenly. If only one advertises, then the one that advertises gains two-thirds of the market.

Company X is deciding whether to close production (exit this market), or continue, and if so, whether to advertise or not.

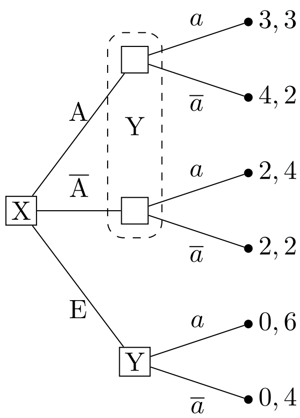
Company Y is committed to this market (*i.e.*, it won't leave), but is monitoring whether ot not company X stays in the market before deciding whether ot not to advertise.

If X stays, both companies must decide whether or not to advertise this year before they know whether the other will.

(a) Draw the extensive form of this game.
(b) Draw the corresponding game matrix from the perspective of player X.
(c) Reduce this game to identify possible solutions.
(d) Repeat the above for the case where the annual cost of advertising for each company is \$1000.

Solution

(a) The possible strategies for X are:
(E) Exit the market; (A) Advertise; ($\overline{\text{A}}$) Don't advertise.
For Y:
(a) Advertise; ($\overline{a}$) Don't advertise.
Note that player Y does not know at its first decision point whether or not Y has decided to advertise or not, but does know if X has exited the market or not.
The game tree is shown below:



Note that it would be possible to prune Y's $\overline{a}$ branch (in path E), because the outcome is dominated. But the branches in the combined information set (A and $\overline{\text{A}}$ branches) cannot as the branches exiting any nodes in an information set must agree, otherwise the two nodes would not be indistinguishable (*i.e.*, the possible actions would give information to differentiate the two).

(b) The game matrices (reduced on the right) are shown below:

		Y	
		a	$\overline{a}$
	E	0, 6	0, 4
X	A	3, 3	4, 2
	$\overline{\text{A}}$	2, 4	2, 2

		Y	
		a	~~$\overline{a}$~~
	~~E~~	~~0, 6~~	~~0, 4~~
X	A	3, 3	~~4, 2~~
	~~$\overline{\text{A}}$~~	~~2, 4~~	~~2, 2~~

(c) See above.

(d) See below:

		Y	
		a	$\overline{a}$
	E	0, 5	0, 4
X	A	2, 2	3, 2
	$\overline{\text{A}}$	2, 3	2, 2

		Y	
		a	~~$\overline{a}$~~
	~~E~~	~~0, 5~~	~~0, 4~~
X	A	2, 2	~~3, 2~~
	~~$\overline{\text{A}}$~~	~~2, 3~~	~~2, 2~~

Note that $\overline{\text{A}}$ is (weakly) dominated in the first round, but would not be in the second. It should be eliminated in the first round. If it is allowed to remain, then it will survive the second round, which will change the answer by admitting a second solution.

Extension exercise: what would happen if the cost of advertising was \$2000? \$3000?