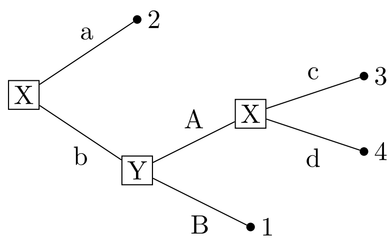
GSOE9210 Engineering Decisions

Problem Set 10

1. Consider the following game tree for a zero-sum game (payoffs shown are for player X):



(a) What are the possible strategies for players X and Y?
(b) Draw the corresponding normal (matrix) form from the perspective of player X.
(c) Simplify the game by eliminating strategies which aren't best responses.

Solution

Interpret the code 'b;A/c' as follows: "do c at the node arrived at by doing b followed by A". Note that the implied grouping is (b;A)/c. However, because all action names are unique in this example the interpretation b;(A/c) would also be appropriate.

More generally, α/A means "do A at the decision point reached by path α".

(a) For X: a, b;A/c, b;A/d
For Y: b/A, b/B
(b) The game matrix is shown below:

		Y	
		b/A	b/B
X	a	2	2
	b;A/c	3	1
	b;A/d	4	1

		Y	
		~~b/A~~	b/B
X	a	~~2~~	2
	~~b;A/c~~	~~3~~	~~1~~
	~~b;A/d~~	~~4~~	~~1~~

(c) See above right, for the simplification via iterated dominance.
The solution after simplification is (a,b/B).

2. Find all saddle points for the zero-sum game with the following matrix:

	b_1	b_2	b_3	b_4
a_1	6	-1	1	0
a_2	5	1	7	-2
a_3	3	2	4	3
a_4	-1	0	0	6

Solution
Evaluating the *Maximin* strategies of the two players (recall that the column player is effectively playing *miniMax*):

	b_1	b_2	b_3	b_4	
a_1	6	-1	1	0	-1
a_2	5	1	7	-2	-2
a_3	3	${}_{*}2^{*}$	4	3	2
a_4	-1	0	0	6	-1
	6	2	7	6	

3. Find all saddle points for the following zero-sum game:

	b_1	b_2	b_3	b_4
a_1	4	2	5	2
a_2	2	1	-1	-2
a_3	3	2	4	2
a_4	-1	0	6	1

Solution
Evaluating the *Maximin* strategies of the two players (recall that the column player is effectively playing *miniMax*):

	$\cancel{b_1}$	b_2	$\cancel{b_3}$	$\cancel{b_4}$	
a_1	4	${}_{*}2^{*}$	5	${}_{*}2^{*}$	2
$\cancel{a_2}$	$\cancel{2}$	$\cancel{1}$	$\cancel{-1}$	$\cancel{-2}$	-2
$\cancel{a_3}$	$\cancel{3}$	${}_{*}\cancel{2}^{*}$	$\cancel{4}$	${}_{*}\cancel{2}^{*}$	2
$\cancel{a_4}$	$\cancel{-1}$	$\cancel{0}$	$\cancel{6}$	$\cancel{1}$	-1
	4	2	6	2	

	$\cancel{a_1}$	a_2	$\cancel{a_3}$	a_4	
a_1	4	${}_{*}2^{*}$	5	${}_{*}2^{*}$	2
$\cancel{a_2}$	$\cancel{2}$	$\cancel{1}$	$\cancel{-1}$	$\cancel{-2}$	-2
a_3	3	${}_{*}2^{*}$	4	${}_{*}2^{*}$	2
$\cancel{a_4}$	$\cancel{-1}$	$\cancel{0}$	$\cancel{6}$	$\cancel{1}$	-1
	4	2	6	2	

Here there are four saddle points. Note that each has the same value (2). In general all saddle points of a zero-sum game must have the same value.

The left matrix shows the simplification steps when using weak dominance, which is the default assumption we use in this course. Notice that already after the first simplification step some of the saddle points (*i.e.*, equilibrium plays) are part of a (weakly) dominated strategy.

In general, if in the process of simplification strategies are eliminated using weak dominance some equilibria may be eliminated. However, at least one of the saddle points will always survive simplification.

The right matrix has been simplified by eliminating only based on strict dominance. Notice that exactly the four equilibria survive.

4. Show that for any zero-sum game in normal (strategic) form the column minimax value is no less than the row maximin.

   Consequently, verify that the column maxima are all greater than the row minima.

   Solution

   Consider a $k \times n$ matrix (*i.e.*, k rows and n columns):

	c_1	$\dots$	c_j	$\dots$	c_q	$\dots$	c_n	m
r_1	$\dots$	$\dots$	a_{1j}	$\dots$	a_{1q}	$\dots$	$\dots$	m_1
$\vdots$				$\vdots$				$\vdots$
r_i	a_{i1}	$\dots$	a_{ij}	$\dots$	a^*_{iq}	$\dots$	a_{in}	m^*_i
$\vdots$				$\vdots$				$\vdots$
r_k	$\dots$	$\dots$	a_{kj}	$\dots$	a_{kq}	$\dots$	$\dots$	m_k
	M_1	$\dots$	M^*_j	$\dots$	M_q	$\dots$	M_n	

   Define $m_i = \min_j\{a_{ij}\}$ and $M_j = \max_i\{a_{ij}\}$; *i.e.*, m_i is the minimum value of row i. Similarly, M_j is the maximum of column j.

   It follows that $m_i \leqslant a_{ij}$ for each $1 \leqslant j \leqslant n$, and $M_j \geqslant a_{ij}$ for each $1 \leqslant i \leqslant k$.

   Define the row player's maximin value as $m^* = \max_i\{m_i\}$, and the column player's minimax value as $M^* = \min_j\{M_j\}$.

   We want to show that $m^* \leqslant M^*$.

   Assume that $m^* = m_i$, for some i. Moreover, assume that $m_i = a_{iq}$.

   Assume that $M^* = M_j$, for some j. Now $M^* = M_j \geqslant a_{ij} \geqslant a_{iq} = m^*$. This concludes the proof.

   Moreover, it follows, for any M_j and m_i, that $M_j \geqslant M* \geqslant m^* > m_i$. That is, all the column player's minimax values are no less than any of the row player's maximin values.

   This result shows that in a zero-sum game, by playing a *Maximin* strategy neither player can guarantee winning more than what the other player can guarantee losing.

5. Consider the game matrix below for a non zero-sum game in which the row player is R and the column player is C.

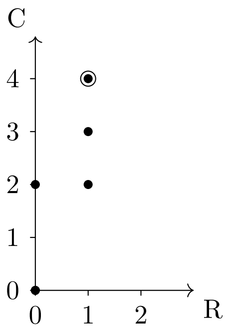
	b_1	b_2	b_3
a_1	$0, 0$	$1, 2$	$0, 2$
a_2	$1, 3$	$1, 4$	$0, 0$

(a) Identify the equilibrium plays.
(b) Which plays survive simplification?
(c) Which plays are Pareto optimal?

Solution

	b_1	b_2	b_3
a_1	$0, 0$	$1, 2$	$0, 2$
a_2	$1, 3$	$1, 4$	$0, 0$

(a) From the matrix above, equilibrium plays are: $(a_1, b_2), (a_1, b_3), (a_2, b_2)$.
(b) Only (a_2, b_2) survives. Note that it is an equilibrium play.
(c)



The only Pareto optimal play is the equilibrium which survives dominance elimination: (a_2, b_2). Note that in this case there is a Pareto dominant play; *i.e.*, a play that weakly Pareto dominates all other plays.

6. Show that in a zero-sum game every outcome is Pareto optimal.

Solution
Geometrically, all outcomes in a zero-sum game lie on the off-diagonal; *i.e.*, the line $y = -x$. Intuitively, this shows that no outcome can lie above another in the Pareto sense.

Formally, consider any two arbitrary outcomes in a game with payoffs $(u, -u)$ and $(v, -v)$.

If the two outcomes are equivalent (*i.e.*, $u = v$) the result is trivial.

Consider any two non-equivalent (*i.e.*, distinct) outcomes of a zero-sum game: *e.g.*, $(u, -u)$ and $(v, -v)$, where $u \neq v$. Assume the latter 'Pareto dominates' the former, and that $v > u$, making the two distinct. It follows that $-u > -v$. But Pareto dominance would require that $-v \geqslant -u$; *i.e.*, $u \geqslant v$, which is impossible, since $v > u$. It follows that no outcome can 'Pareto dominate' another distinct outcome; *i.e.*, all distinct outcomes must be Pareto optimal.

Therefore, combining the two cases above, it follows that *all* outcomes must be Pareto optimal.

7. The game 'Matching Pennies' is a two-player zero-sum game in which each player places a covered coin on a table. Each player chooses the face on the coin but keeps it covered so that the other player doesn't know which face is up. The players uncover their coins simultaneously. The 'matching' player (M) wins if the faces on the coin match, and the 'opposites' player (O) wins if the faces are opposite. The winner takes both coins.
   (a) Represent this game in normal form, showing M's payoffs.
   (b) Reduce the game using dominance.
   (c) Find the equilibrium points of this game.
   (d) If M believes that O is twice as likely to play heads than tails, what would be his best response?
   (e) What should be M's best response if he believes O will play heads and tails with equal likelihood?
   (f) Repeat the above for O.
   (g) If the game is repeated many times, what strategy should the players play?

Solution

(a) The game matrix is shown below:

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{\text{O}} \\ & & \text{h} & \text{t} \\ \hline \text{M} & \text{H} & 1 & -1 \\ & \text{T} & -1 & 1 \end{array}$$

(b) Neither strategy, for either player, is dominated.

(c) The game has no equilibrium plays/points among the pure strategies, however see below.

(d) Suppose M's belief about O is represented by the probability distribution over O's strategies $\beta \sim \frac{2}{3}\text{h}\frac{1}{3}\text{t}$. Because belief is modelled by a probability distribution, we can use *Bayes* values to evaluate actions. The *Bayes* values for each of M's strategies are:

$$\begin{aligned} V_B^\beta(\text{H}) &= \tfrac{2}{3}(1) + \tfrac{1}{3}(-1) \\ &= \tfrac{2}{3} - \tfrac{1}{3} = \tfrac{1}{3} \\ V_B^\beta(\text{T}) &= \tfrac{2}{3}(-1) + \tfrac{1}{3}(1) \\ &= -\tfrac{2}{3} + \tfrac{1}{3} = -\tfrac{1}{3} \end{aligned}$$

The best response is H.

(e) Suppose M's belief about O is represented by the probability distribution over O's strategies $\beta \sim \frac{1}{2}\text{h}\frac{1}{2}\text{t}$. The *Bayes* values for each of

M's strategies are:

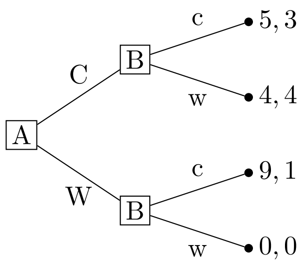
$$\begin{aligned} V_B^\beta(\mathrm{H}) &= \tfrac{1}{2}(1) + \tfrac{1}{2}(-1) \\ &= \tfrac{1}{2} - \tfrac{1}{2} = 0 \\ V_B^\beta(\mathrm{T}) &= \tfrac{1}{2}(-1) + \tfrac{1}{2}(1) \\ &= -\tfrac{1}{2} + \tfrac{1}{2} = 0 \end{aligned}$$

Both H and T are best responses.

(f) The results are the same for O as the game is symmetric.

(g) If the game is repeated many times then it makes sense for players to also play mixed strategies. In this case, from the previous question, if O plays the mixture $\frac{1}{2}\mathrm{h}\frac{1}{2}\mathrm{t}$, then $V_B^\beta(\mathrm{H}) = V_B^\beta(\mathrm{T}) = 0$. Similarly for any mixture $M(\mu) = \mu\mathrm{H} + (1-\mu)\mathrm{T}$ of M's strategies, $V_B^\beta(M(\mu)) = 0$. Hence, all of M's mixtures are best responses to O's $\frac{1}{2}\mathrm{h}\frac{1}{2}\mathrm{t}$. This holds in particular for M's mixture $M(\frac{1}{2}) = \frac{1}{2}\mathrm{H}\frac{1}{2}\mathrm{T}$.
But if M plays $\frac{1}{2}\mathrm{H}\frac{1}{2}\mathrm{T}$, then, by the symmetry of the game/matrix, any mixed strategy of O's will be a best response to this strategy. In particular, $m(\frac{1}{2}) = \frac{1}{2}\mathrm{h}\frac{1}{2}\mathrm{t}$, will be a best response for O.
It follows that the play $(\frac{1}{2}\mathrm{H}\frac{1}{2}\mathrm{T}, \frac{1}{2}\mathrm{h}\frac{1}{2}\mathrm{t})$, is a 'mixed strategy equilibrium' of the game.
In general, by a similar reasoning, for any zero-sum game there exists at least one mixed-strategy equilibrium play.

8. The following game tree represent the example in lectures for the case in which Alice the gorilla moves first and Bob the monkey moves second.



(a) What is Bob's best response to Alice if she waits? If Alice climbs?

(b) Are there any equilibrium points? If so, which are they?

(c) Rationalise the game using dominance.

Solution

(a) Note that Bob's strategies must specify what to do at both the C-node and the W-node.
From the tree it is evident that Bob's best responses to Alice's W are both $\genfrac{}{}{0pt}{}{\mathrm{C/c}}{\mathrm{W/c}}$ and $\genfrac{}{}{0pt}{}{\mathrm{C/w}}{\mathrm{W/c}}$. Note that, in this case (Alice chooses W), the response to C in Bob's strategy is irrelevant.
For Alice's C, Bob's best response involves waiting; *i.e.*, both $\genfrac{}{}{0pt}{}{\mathrm{W/w}}{\mathrm{C/w}}$ and $\genfrac{}{}{0pt}{}{\mathrm{W/c}}{\mathrm{C/w}}$ are best responses. In this case, Bob's response to W is irrelevant.

(b) Alice's best response to either of $\genfrac{}{}{0pt}{}{\mathrm{C/c}}{\mathrm{W/c}}$ and $\genfrac{}{}{0pt}{}{\mathrm{C/w}}{\mathrm{W/c}}$ is W. So $\left(\mathrm{W}, \genfrac{}{}{0pt}{}{\mathrm{W/c}}{\mathrm{C/w}}\right)$ and $\left(\mathrm{W}, \genfrac{}{}{0pt}{}{\mathrm{W/c}}{\mathrm{C/c}}\right)$ are both equilibrium plays.

For Bob's $\genfrac{}{}{0pt}{}{\mathrm{W/c}}{\mathrm{C/w}}$, Alice's best response is W. So of these two only $\left(\mathrm{C}, \genfrac{}{}{0pt}{}{\mathrm{W/w}}{\mathrm{C/w}}\right)$ is an equilibrium play.

Therefore the equilibrium plays are:
$\left(\mathrm{W}, \genfrac{}{}{0pt}{}{\mathrm{W/c}}{\mathrm{C/w}}\right)$, $\left(\mathrm{W}, \genfrac{}{}{0pt}{}{\mathrm{W/c}}{\mathrm{C/c}}\right)$, and $\left(\mathrm{C}, \genfrac{}{}{0pt}{}{\mathrm{W/w}}{\mathrm{C/w}}\right)$, with values $(9,1), (9,1), (4,4)$ respectively.

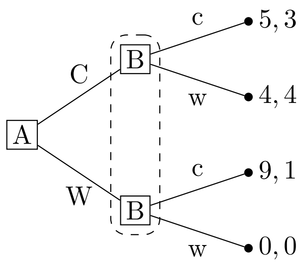
(c) The only (sub-game perfect) equilibrium is $\left(\mathrm{W}, \genfrac{}{}{0pt}{}{\mathrm{C/w}}{\mathrm{W/c}}\right)$. This can be verified from the game matrix below:

		B			
		$\genfrac{}{}{0pt}{}{\mathrm{C/w}}{\mathrm{W/w}}$	$\genfrac{}{}{0pt}{}{\mathrm{C/w}}{\mathrm{W/c}}$	$\genfrac{}{}{0pt}{}{\mathrm{C/c}}{\mathrm{W/w}}$	$\genfrac{}{}{0pt}{}{\mathrm{C/c}}{\mathrm{W/c}}$
A	~~C~~	4, 4	~~4, 4~~	5, 3	5, 3
	W	0, 0	9, 1	0, 0	9, 1

Note that because Bob's $\genfrac{}{}{0pt}{}{\mathrm{C/c}}{\mathrm{W/c}}$ is a sub-optimal response to Alice's C (it is dominated by Bob's $\genfrac{}{}{0pt}{}{\mathrm{C/c}}{\mathrm{W/w}}$) then $\left(\mathrm{W}, \genfrac{}{}{0pt}{}{\mathrm{C/c}}{\mathrm{W/c}}\right)$ can be ignored as a solution to the game (it is said to not be *sub-game perfect*). A similar argument would eliminate $\left(\mathrm{C}, \genfrac{}{}{0pt}{}{\mathrm{C/w}}{\mathrm{W/w}}\right)$, as Bob should not choose it in preference to $\genfrac{}{}{0pt}{}{\mathrm{C/w}}{\mathrm{W/c}}$.

This example show cases in which some equilibrium points may be dominated. Such equilibria may be disregarded as solutions.

9. If Alice and Bob move simultaneously we have the game tree below.



(a) Draw the game matrix.
(b) Are there any equilibrium plays/points? If so, which are they?
(c) If Bob had an injured arm, 'signalling' that he would be less likely to climb than to wait, which would be the rational solution?

Solution

(a) Neither Bob nor Alice have the information to make strategies available to them that allow them to react to each other's move.

	c	w
C	$5, 3$	$4^*, 4^*$
W	$9^*, 1^*$	$0, 0$

(b) Equilibrium plays are those in which the players to opposites: (C,w) and (W,c).

(c) Suppose Alice believed that Bob is less likely to climb than to wait; *i.e.*, $P(\mathrm{c}) < P(\mathrm{w})$. Let $p = P(\mathrm{c})$. Then Alice's belief about Bob would be $\beta = (P(\mathrm{c}), P(\mathrm{w})) = (p, 1-p)$, where $p < \frac{1}{2}$. Now:

$$\begin{aligned} V_B^\beta(\mathrm{C}) &= p(5) + (1-p)(4) \\ &= 4 + p \\ V_B^\beta(\mathrm{W}) &= p(9) + (1-p)(0) \\ &= 9p \end{aligned}$$

It follows that Alice should climb if:

$$\begin{aligned} V_B^\beta(\mathrm{C}) &\geqslant V_B^\beta(\mathrm{W}) \\ 4 + p &\geqslant 9p \\ 4 &\geqslant 8p \\ p &\leqslant \tfrac{1}{2} \end{aligned}$$

In this case simplification doesn't eliminate any strategies (*i.e.*, assign them a belief of 0), but we have some information about relative likelihoods; *i.e.*, $P(\mathrm{c}) < P(\mathrm{w})$. This implies that $p < 1 - p$, and therefore that $p < \frac{1}{2}$.
Because Alice prefers to climb in response to her belief that Bob is less likely to do so (*i.e.*, given $P(\mathrm{c}) < P(\mathrm{w})$), and because Bob knows this, the equilibrium (C,w) is the rational solution.

10. Show that if a zero-sum game has a dominant row and column, then the two determine an equilibrium play.

Solution

A row that dominates must have values that are maximal in each column. Similarly, for a dominant column all values in that column must be minimal in each row. Hence the corresponding pair of strategies must be minimal in their row and maximal in its column. This makes the outcome a saddle point, and hence the play an equilibrium.