GSOE 9210 Midterm Summary

LN01 Decision Problems

1. Actions, Possible States (Events), Outcomes
2. Decision Trees: leaf node (outcome), branch (action or chance event), internal node (decision nodes -> boxes, chance nodes -> circles)
3. Decision Tables: row = action, column = state
4. Multiple trees may correspond to the same table; Going from tables (normal form) to trees (extensive form) is straight forward, but the converse can be tricky
5. Decisions depend on: preferences & epistemic state

LN02 Decision under Certainty & Ignorance

1. Decision problem classes

   Decision problems can be classified based on an agent's epistemic state:
   - Decisions under *certainty*: the agent knows the actual state
   - Decisions under *uncertainty*:
     - Decisions under *ignorance* (full uncertainty): the agent believes multiple states/outcomes are possible; likelihoods unknown
     - Decisions under *risk*: the agent believes multiple states/outcomes are possible; likelihood information available

2. Rational Decisions under Certainty: Given a value function V : A → R over actions, rational agents should prefer action A to B iff V (A) > V (B).
3. Lottery:

   Definition (Lottery)

   A *lottery* over a finite set of states $\mathcal{S}$, and outcomes, or *prizes*, Ω, is a function $\ell : \mathcal{S} \to \Omega$. The lottery ℓ is written:

   $$\ell = [s_1 : \omega_1 | s_2 : \omega_2 | \ldots | s_n : \omega_n]$$

   where for each $s_i \in \mathcal{S}$, $\omega_i = \ell(s_i)$.

4. Decision under Ignorance:

(1) Maximax (for extreme risk takers)

(2) Maximin (best in the worst case)

(3) Hurwicz's optimism index (V(A) = aM(A) + (1 - a)m(A))

(4) minimax Regret

(5) Laplace's insufficient reason

$$V(\mathsf{A}) = \frac{1}{n}v(\omega(\mathsf{A}, s_1)) + \cdots + \frac{1}{n}v(\omega(\mathsf{A}, s_n))$$

5. Better Actions, Best Response:

   Definition (Better actions)

   An action A is *strictly (weakly) better* than action B in a given state s iff A's outcome in s strictly (weakly) preferred to B's.

   Definition (Best response)

   An action A is a strictly (weakly) *best response* in state s iff A is strictly (weakly) better than every other action in state s.

LN03 Multi-stage Decisions

1. The Maximin Principle (the main decision principle used under complete uncertainty): Assume that only the minimally preferred outcomes will occur and choose those actions that lead to the most preferred among these.
2. Multi-stage Normalisation:

Encoding:

- α/A says:
  At the decision node reached via path α choose action A.
- Example: Au;s/S:
  If development has been authorised (Au) and has succeeded (s), choose to build a small factory (S).
- Strategies for this problem:

A_1	Au;s/L
A_2	Au;s/S
A_3	Ab

3. Indifference: If two actions A and B are equally preferred then the agent is said to be indifferent between A and B.
   Indifference class: An indifference class is a non-empty set of all actions/outcomes between which an agent is indifferent.

$$I(\mathrm{A}) = \{a \in \mathcal{A} \mid V(a) = V(\mathrm{A})\}$$

4. Graphing
5. Dominance: Strict dominance, weak dominance

Definition (Strict dominance)

Strategy A *strictly dominates* B iff every outcome of A is more preferred than the corresponding outcome of B.

Definition (Weak dominance)

Strategy A *weakly dominates* B iff every outcome of A is no less preferred than the corresponding outcome of B, and some outcome is more preferred.

Corollary

Strategy A strictly dominates B iff A is a better response than B in each possible state.

Corollary

Strategy A weakly dominates B iff A is a better response than B in some possible state and B is not a better response than A in any state.

6. Admissible: An action is admissible iff it is not dominated by any other action. An action which is not admissible is said to be inadmissible. The set of all admissible actions is called the admissible frontier.
7. Dominance Elimination: A decision rule is said to satisfy (strict/weak) dominance elimination if it never chooses actions that are (strictly/weakly) dominated.

Rules that satisfy strict/weak dominance elimination.

Rule	Strict	Weak
MaxiMax	$\surd$	$\times$
Maximin	$\surd$	$\times$
Hurwicz's	$\surd$	$\times$
miniMax Regret	$\surd$	$\times$
Laplace's	$\surd$	$\surd$

LN04 Mixed Strategies

1. Mixed Strategy: mixed strategy (or mixture) is a strategy in which the basic strategies are distributed in proportions. A strategy in which the entire proportion is from one basic strategy is called a pure strategy.

Mixed strategies

In general:

- For basic strategies $\mathcal{A} = \{a_1, \dots, a_k\}$, mixed strategies determined by mixtures $(\mu_{a_1}, \dots, \mu_{a_k})$ of basic strategies
- Value of mixed strategy $M(\mu_{a_1}, \dots, \mu_{a_k})$ in state $s \in \mathcal{S}$ is expected value of basic strategies:

$$\begin{aligned} V(M, s) &= \mu_{a_1} v(a_1, s) + \cdots + \mu_{a_k} v(a_k, s) \\ &= \sum_{a \in \mathcal{A}} \mu_a v(a, s). \end{aligned}$$

where

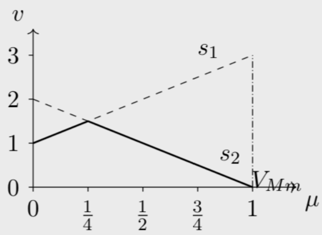
$$\sum_{a \in \mathcal{A}} \mu_a = 1 \quad \text{and} \quad \mu_a \geqslant 0$$

- Think of mixtures as many independent decisions in a single unknown state

2. Mixture Plots & Many States:

Consider mixtures M, where $\mu_{\mathsf{A}} = \mu$:

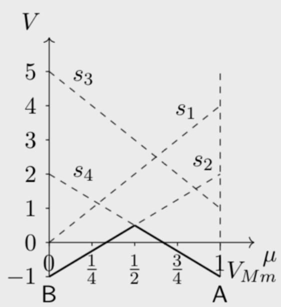
	s_1	s_2
A	3	0
B	1	2
$M(\mu)$	$2\mu + 1$	$2 - 2\mu$



Consider a problem with four states, two basic strategies, and mixtures, M, where $\mu_{\mathsf{A}} = \mu$:

	s_1	s_2	s_3	s_4
A	4	2	1	-1
B	0	-1	5	2
$M(\mu)$	4μ	$3\mu - 1$	$5 - 4\mu$	$2 - 3\mu$

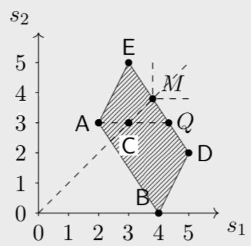
Maximin values for mixed strategies $M(\mu)$ lie on solid line.
Maximin mixed strategy M^* given by $\mu^* = \frac{1}{2}$ which maximises *Maximin* values; *i.e.*, $V_{Mm}(M^*) = \frac{1}{2}$.



3. Mixing Many Strategies

Mixed strategies: many basic strategies

	s_1	s_2
A	2	3
B	4	0
C	3	3
D	5	2
E	3	5



- Can mix more than two strategies: *e.g.*, C $= \mu_A \mathrm{A} \mu_E \mathrm{E} \mu_D \mathrm{D}$
- Mixtures lie inside (or on boundary) of shaded region. Why? 看边界

Note: miniMax Regret mixed action R′ doesn't correspond to Maximin mixed action M.

4. Generalised Dominance:

Definition (Strict dominance)
Strategy A *strictly dominates* B iff every outcome of A is more preferred than the corresponding outcome of B.

Definition (Weak dominance)
Strategy A *weakly dominates* B iff every outcome of A is no less preferred than the corresponding outcome of B, and some outcome is more preferred.

LN05 Bayes Decisions

1. Decisions under risk: likelihood information available
2. Probabilistic Lottery:

Definition (Probabilistic lottery)
A *probabilistic lottery* over a finite set of outcomes, or *prizes*, Ω, is a pair $\ell = (\Omega, P)$, where $P : \Omega \to \mathbb{R}$ is a probability function. The lottery ℓ is written:

$$\ell = [p_1 : c_1 | p_2 : c_2 | \ldots | p_n : c_n]$$

where for each $s_i \in \mathcal{S} \subseteq \mathbb{P}(\Omega)$, $p_i = P(s_i) = P(c_i)$.

Value of a Lottery:

Definition (Value of a lottery)
The value of a probabilistic lottery (Ω, P, v) is the expected value over its outcomes:

$$V_v(\ell) = E(v) = \sum_{\omega \in \Omega} P(\omega) v(\omega)$$

3. Bayes Values, Bayes Strategy, Bayes Decision Rule:

Definition (Bayes value)
Given a probability distribution over states, the *Bayes value*, V_B, of a strategy is the expected value of its outcomes.

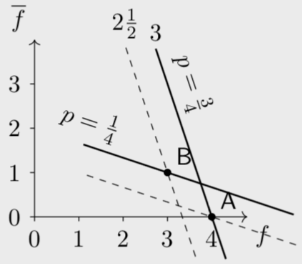
Definition (Bayes strategy)
A *Bayes strategy* is a strategy with maximal *Bayes* value.

Definition (Bayes decision rule)
The *Bayes decision rule* is the rule which selects all the *Bayes* strategies.

4. Bayes Indifference Curves:

What do *Bayes* indifference curves look like?

	p	$1-p$	
	f	$\bar{f}$	V_B
A	4	0	$4p$
B	3	1	$2p+1$
a	v_1	v_2	$pv_1 + (1-p)v_2$



Indifference curves:

$V_B(a) = pv_1 + (1-p)v_2 = u$

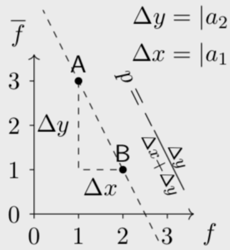
- In gradient-intercept form, $v_2 = \frac{u}{1-p} - \frac{p}{1-p}v_1$, where $m = -\frac{p}{1-p}$; *e.g.*, for $p = \frac{3}{4}$, $m = -\frac{3}{4}/\frac{1}{4} = -\frac{3}{1}$
- Because $v_2 \propto u$; *i.e.*, 'higher' lines receive greater *Bayes* values

In general, for two actions:

	p	$1-p$
	s_1	s_2
A	a_1	a_2
B	b_1	b_2

$$p = \frac{\Delta y}{\Delta x + \Delta y} = \frac{m}{m-1}$$

where m is the gradient of line AB.



$\Delta y = |a_2 - b_2|$

$\Delta x = |a_1 - b_1|$

For example: if A is $(1,3)$ and B is $(2,1)$ then:

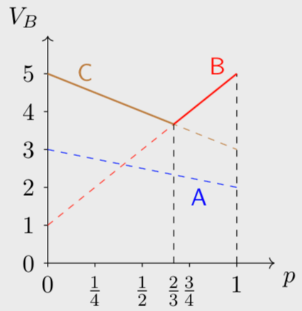
$$p = \frac{3-1}{(2-1)+(3-1)} = \frac{2}{1+2} = \frac{2}{3}$$

5. Probability Plots:

For the pure actions below with $P(s_1) = p$:

	s_1	s_2	V_B
A	2	3	$3-p$
B	5	1	$1+4p$
C	3	5	$5-2p$

For $p = \frac{2}{3}$, the value of the *Bayes* action(s) is least.



Definition

The *least favourable probability distribution* on the states/outcomes is the probability distribution for which *Bayes* strategies have minimal values.

6. Theorem:

Results about Bayes decision rule:

(1) Mixing can improve upon the Maximin value of pure strategies, but it does not improve upon the Bayes value of pure strategies.

(2) Bayes strategies are invariant/preserved under regret; i.e., the same strategy is chosen under regret as otherwise.

7. A strategy is admissible iff it is a Bayes strategy for some probability distribution.
8. A Maximin strategy is always a Bayes strategy for some prob- ability distribution.
9. For any two actions A and B, if A weakly dominates B, and all state probabilities are non-zero, then the Bayes decision rule will strictly prefer A over B.

LN06 Risk Attitude and Utility

1. Fair Bet:

> **Definition (Fair bet)**
>
> A two-way gamble/bet is *fair* for an agent if the expected monetary value for the corresponding lottery is no less than the value of not gambling; *i.e.*,
>
> $$V_\$(\ell_{\mathsf{G}}) = E(v_\$) \geqslant V_\$(\ell_{\overline{\mathsf{G}}})$$

> **Theorem (Fair bets)**
>
> *Let a be agent A's stake and b be B's stake in a bet in which p is A's probability of winning. The bet is fair iff:*
>
> $$\frac{a}{b} = \frac{p}{1-p}$$

(A win: a + b, A loss: 0 …)

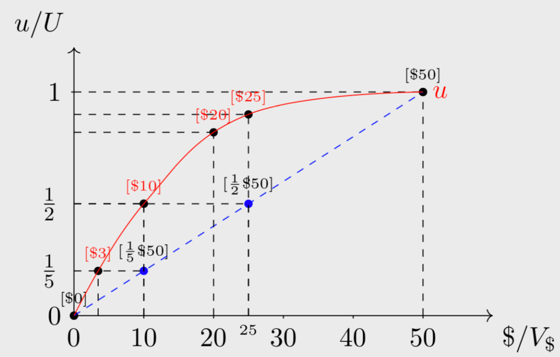
2. Favorable Bet:

> **Definition (Favourable bet)**
>
> A bet is *favourable* to an agent if the value of the corresponding lottery for the agent is greater than that of not gambling. It is *unfavourable* if it is neither fair nor favourable.

3. Utility of Money

Fix u scale:

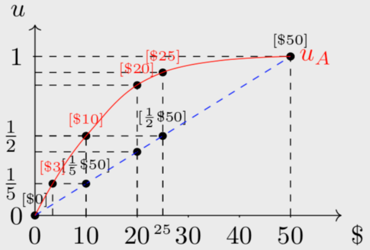
$$u(\$0) = 0$$
$$u(\$50) = 1$$



Possible gambles lie on diagonal:

$$U([\tfrac{1}{2} : \$50|\tfrac{1}{2} : \$0]) = \tfrac{1}{2}u(\$50) + \tfrac{1}{2}u(\$0) = \tfrac{1}{2}$$
$$U([p : \$50|(1-p) : \$0]) = p$$

4. Certainty Equivalent: xc
   Risk Premium: V$(l) - xc



How much money is $[\frac{1}{2}\$50]$ worth to Alice? $\$10$ xc

The EMV of $[\frac{1}{2}\$50]$ is $\$25$. How much of that amount is Alice willing to give up for a certain $\$10$? Up to $\$25 - \$10 = \$15$

Definition (Certainty equivalent)

An agent's *certainty equivalent* for a lottery is the value x_c for which the agent would be indifferent between it and the lottery; *i.e.*, $u(x_c) = U(\ell)$.

Definition (Risk premium)

The *risk premium* of an agent for lottery ℓ is the difference between the EMV of the lottery and the certainty equivalent: $V_{\$}(\ell) - x_c$.

5. Repeated Trials
6. Risk Attitude

Definition (Risk attitudes)

An agent is:

- *risk averse* iff its certainty equivalent is less than the lottery's expected value; *i.e.*, it values the lottery to be worth less than the expected value.
- *risk seeking* (*risk prone*) iff its certainty equivalent is greater than the lottery's expected value.
- *risk-neutral* otherwise.

Definition (Risk averse)

An agent is *risk averse* if its utility function is concave down.

Definition (Risk seeking)

An agent is *risk seeking* if its utility function is concave up (convex).

Definition (Risk neutral)

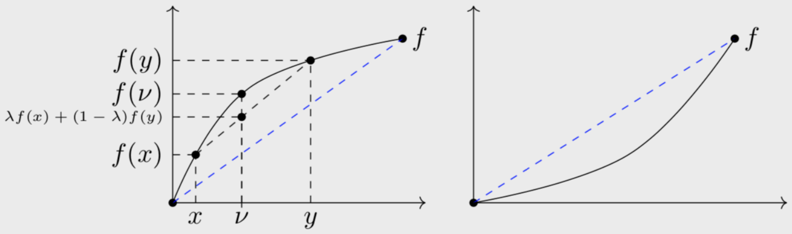
An agent is *risk neutral* if its utility function both concave down and up; *i.e.*, linear.

7. Concave (down) and convex (concave up) functions

Definition (Concave and convex)

A function $f : \mathbb{R} \to \mathbb{R}$ is *concave down* in the interval $[a, b]$ if for all $x, y \in [a, b]$, and all $\lambda \in [0, 1]$, $f(\lambda x + (1-\lambda)y) \geqslant \lambda f(x) + (1-\lambda)f(y)$, and *concave up* (or *convex*) if $f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y)$.

concave convex



where $\nu = \lambda x + (1-\lambda)y$