# GSOE9210 Engineering Decisions

## Problem Set 07

1. Show that an irreflexive and transitive relation is asymmetric.

2. An *equivalence relation* on a set $A$ is any binary relation which is: a) reflexive; b) symmetric; and c) transitive

   Show that for any fixed $m \in \mathbb{N}$, the relation $R_m \subseteq \mathbb{Z} \times \mathbb{Z}$ such that $xR_my$ iff $x - y = km$ for some $k \in \mathbb{Z}$, is an equivalence relation.

   Define $[n]_m = [n]_{R_m}$. Describe the equivalence class $[3]_0$, $[3]_1$, and $[3]_2$, $[3]_3$. In general, describe the equivalence classes $[n]_m$? Show that $[m]_4 \subseteq [m]_2$, for any $m \in \mathbb{Z}$. More generally, show that if for some $k, n, p \in \mathbb{Z}$, $n = kp$, then $[m]_n \subseteq [m]_p$

3. Verify that for any finite (or indeed infinite) sets $A$ and $B$, the relation $A \simeq B$ iff $|A| = |B|$, where $|A|$ is the *cardinality* of $A$ (*i.e.*, the number of elements in $A$) is an equivalence relation.

4. A *partial order* is any relation which is reflexive, antisymmetric, and transitive.

   Define the relation $| \subseteq \mathbb{N} \times \mathbb{N}$ by $x|y$ iff $x$ divides $y$ (or $x$ is a factor of $y$, or $y$ is a multiple of $x$). Show that $|$ is a partial order (*i.e.*, that it is reflexive, antisymmetric, and transitive).

5. For a weak preference relation $\succsim$, verify the following:

   (a) If an agent's preferences are consistent then $\sim$ is an equivalence relation
   (b) The corresponding strict preference relation $\succ$ is a strict total order
   (c) Strict preference satisfies an 'indifference version' of the trichotomy law; *i.e.*, exactly one of the following holds between any $x, y \in A$: $x \succ y$ or $x \sim y$ or $y \succ x$.

6. Verify that the following properties hold from the axiomatisation of $\succsim$ given in lectures.

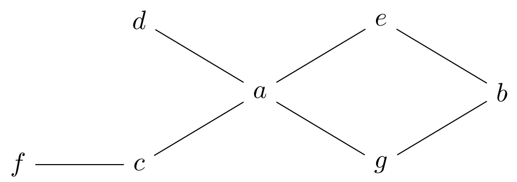
   - Strict preference properties:
     - if $x \succ y$, then it should be that $y \succ x$
     - if $x \succ y$ and $y \succ z$, then it should not be that $z \succ x$
   - Indifference properties:
     - if $x \sim y$, then $y \sim x$
     - if $x \sim y$ and $y \sim z$, then $x \sim z$
     - $x \sim x$ holds for any $x \in A$
   - Combined properties:

- if $x \sim y$ and $z \succ x$, then $z \succ y$
- if $x \sim y$ and $x \succ z$, then $y \succ z$
- for any $x, y$ *either* $x \succ y$ or $x \sim y$ or $y \succ x$

7. Let $[x]$ be an abbreviation for $[x]_\sim$, show that:

   (a) if $x \sim y$, then $[x] = [y]$
   (b) if $[x] \cap [y] \neq \varnothing$, then $[x] = [y]$
   (c) if $x \succ y$, then if $a \in [x]$ and $b \in [y]$, then $a \succ b$

8. Left the left-to-right edges in the Hasse diagram below represent $\succ$.




   In terms of $\succ$ what is the relationship between:

   (a) $d$ and $a$
   (b) $a$ and $e$
   (c) $a$ and $b$
   (d) $f$ and $d$

9. Consider the following preferences on the set $A = \{a, b, c, d, e\}$:

$$c \succsim a \quad b \succsim d \quad e \succsim d \quad d \succsim a \quad d \succsim e \quad a \succsim c$$

   (a) What additional instances of $\succsim$ can be inferred from the axioms given in lectures?
   (b) Assume that any inference not present above, or inferred from them, is false. From the definition of $\succ$ in terms of $\succsim$, what are the instances of $\succ$?
   (c) For an equivalence relation $(A, \sim)$, denote the set of all equivalence classes of $A$ by $A/\sim$. (Sometimes $A/\sim$ is called the *quotient class* of $A$.) List the indifference classes in $A/\sim$?
   (d) Draw the Hasse diagram for $\succ$.
   (e) Draw the Hasse diagram for $\succ_I$: the preference relation on indifference classes.
   (f) Define an ordinal function $V$ on the members of $A/\sim$ (*i.e.*, $V : A/\sim \to \mathbb{R}$) and hence, one on $A$ ($v : A \to \mathbb{R}$).

10. Show that the weak preference ordering $\succsim_I$ on indifference classes is anti-symmetric.

11. Show that for the weak preference relation $\succsim_I$ on indifference classes:

    (a) for any $X, Y \in A/\sim$, $X \succsim_I Y$ iff for *every* $x \in X$ and $y \in Y$, $x \succsim y$
    (b) $\succsim_I$ is a weak total order

12. Show that for any ordinal value function $v$:

   (a) $v(x) > v(y)$ iff $x \succ y$.
   (b) $v(x) = v(y)$ iff $x \sim y$.