GSOE9210 Engineering Decisions

Problem Set 05

1. Consider the river problem described in lectures:

	p	$1-p$	
	f	$\overline{f}$	V_B
A	4	0	$4p$
B	3	1	$2p+1$

(a) For $p = \frac{3}{4}$, what is the slope of the *Bayes* indifference line through A?
(b) Draw the *Bayes* indifference curves for $p = \frac{1}{4}$ and $\frac{3}{4}$ through A and B.
(c) Draw the *Bayes* indifference curve for which an agent would be indifferent between A and B, respectively. What is the slope of the line?
(d) For which probability (*i.e.*, value of p) would an agent be indifferent between A and B under the *Bayes* decision rule?
(e) What is the *Bayes* value associated with the indifference curve through A and B?
(f) For which values of p would an agent prefer A to B?

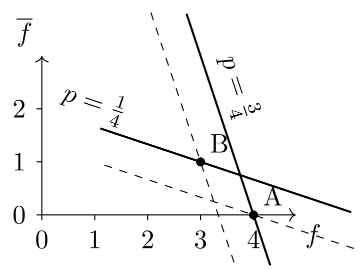
Solution

(a) The indifference curves are given by the points (v_1, v_2) which, for fixed $u \in \mathbb{R}$, satisfy:

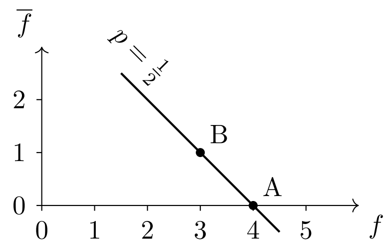
$$pv_1 + (1-p)v_2 = u$$

In gradient-intercept form, $v_2 = \frac{u}{1-p} - \frac{p}{1-p}v_1$, where $m = -\frac{p}{1-p}$; *e.g.*, for $p = \frac{3}{4}$, $m = -\frac{3}{4}/\frac{1}{4} = -\frac{3}{1}$.

(b)



(c)



The line AB places A and B on the same indifference curve. The slope of the line is given by:

$$\begin{aligned} m_{\mathrm{AB}} &= \frac{3-4}{1-0} \\ &= -1 \end{aligned}$$

(d) We saw above that $m_{\mathrm{AB}} = -\frac{p}{1-p}$; *i.e.*, $-\frac{p}{1-p} = -1$. Hence $p = 1 - p$; *i.e.*, $2p = 1$. Therefore $p = \frac{1}{2}$.
Alternatively, $p = \frac{\Delta y}{\Delta x + \Delta y} = \frac{1}{1+1} = \frac{1}{2}$.
Alternatively, where m is the gradient of the line, $p = \frac{m}{m-1} = \frac{-1}{-1-1} = \frac{-1}{-2} = \frac{1}{2}$.

(e) Because the indifference line AB goes through A (and B), we can associate with it the *Bayes* value of A (or B); *i.e.*, $u_{\mathrm{A}} = V_B(\mathrm{A}) = 4p = 4 \times \frac{1}{2} = 2$.

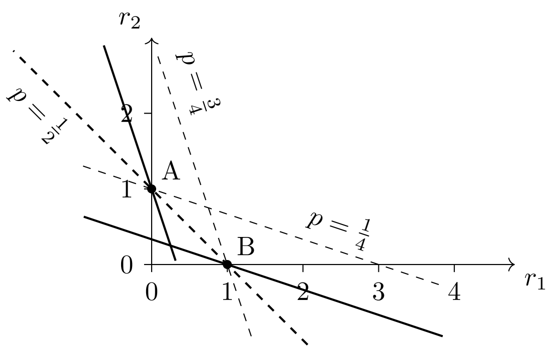
(f) From the graph, for values $p > \frac{1}{2}$, the slope is steeper ($m < -1$) than that of line AB, and hence B is below the indifference line through A; *i.e.*, A would be preferred to B.
Alternatively, analytically:

$$\begin{aligned} V_B(\mathrm{A}) > V_B(\mathrm{B}) &\text{ iff } 4p > 2p + 1 \\ &\text{ iff } 2p > 1 \\ &\text{ iff } p > \tfrac{1}{2} \end{aligned}$$

2. Repeat the above exercises for regret. What can you infer about the *Bayes* decision rule when applied to the original values versus regrets?

Solution
The regrets—in regret space—are shown in the graph below.

Since we want to minimise regret under the *miniMax Regret* rule, lower-left (regret) indifference lines are preferred (*i.e.*, correspond to lower—more preferred—*Bayes* regret values).



The *Bayes* regret value for a strategy A is given by the *Bayes* value of A—written $V_{BR}(\text{A})$—with A situated in regret space.

Bayes regrets are calculated in the same way, using regrets instead of the original values.

Indifference lines for given p are obtained by fixing the *Bayes* regret value:

$$pr_1 + (1-p)r_2 = u$$

A is at $(0, 1)$ in regret space. The *Bayes* value along the indifference line through A for $p = \frac{3}{4}$ is given by setting $r_1 = 0, r_2 = 1$ in the expression for $V_{BR}(\text{A})$ above:

$$u_{\text{A}} = (1-p) = (1 - \tfrac{3}{4}) = \tfrac{1}{4}$$

B is at $(1, 0)$, so for $p = \frac{1}{4}$, the *Bayes* value of the indifference line through B is given by $u_{\text{B}} = p = \frac{1}{4}$.

AB has slope $m = -1$, hence it corresponds to $p = \frac{1}{2}$. Moreover, $V_{BR}(\text{B}) = u_{\text{B}} = p = 1 - p = \frac{1}{2}$.

When considering regret, strategy A is preferred to B when its *Bayes* regret value is lesser, which is the case for probabilities that produce lines steeper than gradient -1 ($m < -1$); *i.e.*, $V_{BR}(\text{A}) < V_{BR}(\text{B})$ iff $m < -1$; *i.e.*, $p > 1 - p$ iff $p > \frac{1}{2}$.

Note that as comparison of *Bayes* values and *Bayes* regret values, $V_B(\text{A})$ and $V_{BR}(\text{B})$, depend, in both cases, only on the slope of their indifference curves. It follows that the *Bayes* decision rule is invariant under original values and regrets, in the sense that $V_B(\text{A}) > V_B(\text{B})$ iff $V_{BR}(\text{A}) < V_{BR}(\text{B})$. That is, A is preferred to B under the *Bayes* decision rule for the original values if an only if it is also preferred under the *Bayes* decision rule for regrets.

3. Consider the generic two-strategy problem below:

	p	$1-p$
	s_1	s_2
A	a_1	a_2
B	b_1	b_2

Assume neither strategy dominates the other.

(a) Prove that an agent will be indifferent between A and B under *Bayes* when:

$$p = \frac{\Delta y}{\Delta x + \Delta y}$$

where

$$\Delta y = |a_2 - b_2|$$
$$\Delta x = |a_1 - b_1|$$

(b) Prove that:

$$p = \frac{m}{m-1}$$

where $m = -\frac{\Delta y}{\Delta x}$ is the slope of the line joining A and B in the Cartesian plane.

Solution

(a) If neither strategy is dominated then $(b_2 - a_2)(b_1 - a_1) < 0$; *i.e.*, $b_2 - a_2 < 0$ iff $b_1 - a_1 > 0$.

$$V_B(\text{A}) = pa_1 + (1-p)a_2$$
$$V_B(\text{B}) = pb_1 + (1-p)b_2$$

Setting $V_B(\text{A}) = V_B(\text{B})$:

$$\begin{aligned} pa_1 + (1-p)a_2 &= pb_1 + (1-p)b_2 \\ p(a_1 - a_2) + a_2 &= p(b_1 - b_2) + b_2 \\ p(a_1 - b_1 + b_2 - a_2) &= b_2 - a_2 \\ p &= \frac{b_2 - a_2}{(a_1 - b_1) + (a_2 - b_2)} \\ &= \frac{\Delta y}{\Delta x + \Delta y} \end{aligned}$$

(b) From lectures:

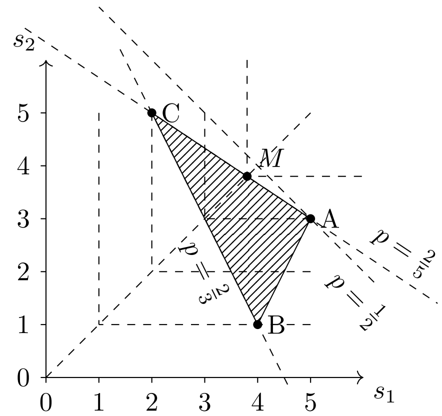
$$\begin{aligned} \frac{p}{1-p} &= -m \\ p &= mp - m \\ m &= p(m-1) \\ \therefore\ p &= \frac{m}{m-1} \end{aligned}$$

4. Consider the decision table below, with $P(s_1) = p$:

	p	$1-p$
	s_1	s_2
A	5	3
B	4	1
C	2	5

(a) For which value of p would the agent be indifferent between A and C?
(b) Plot the *Bayes* values for the strategies as p varies from 0 to 1.
(c) For which values of p are A, B, and C preferred, respectively, under the *Bayes* decision rule?

Solution

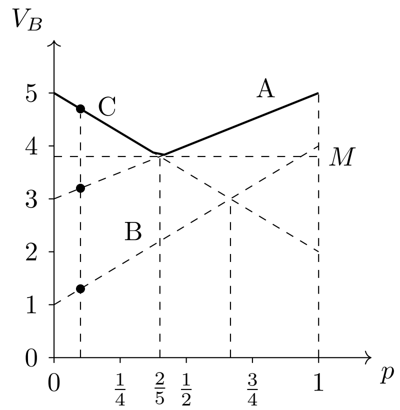


(a) Slope of AC: $m = \frac{5-3}{2-5} = -\frac{2}{3}$.
Hence:

$$\begin{aligned} \frac{p}{1-p} &= \frac{2}{3} \\ 3p &= 2 - 2p \\ 5p &= 2 \\ \therefore\ p &= \frac{2}{5} \end{aligned}$$

Hence for $p < \frac{2}{5}$, C is preferred. For $p > \frac{2}{5}$, A is preferred.
Note that B is (strongly) dominated, hence is not admissible, and therefore is never preferred.

(b) Consider the plot of the *Bayes* values of the strategies against p:



(c) From the graph it is clear that for $0 < p < \frac{2}{5}$, C is preferred. For $\frac{2}{5} < p < 1$, A is preferred.

5. Alice sells drinks at a local market once every month. She can order stock to sell several drink types: a) hot chocolate; b) iced tea; c) lemonade; d) orange juice.

   From past experience she knows that when she sells only one type of drink, on warm days her sales total for each type are: \$10 on hot chocolate, \$40 on iced tea, \$30 on lemonade, and \$40 on orange juice. On cool days, however, her sales totals are: \$30 on hot chocolate, \$0 on iced tea, \$20 on lemonade, and \$10 on orange juice.

   She has to order her stock weeks in advance, long before she can predict the temperature on the day of the market.

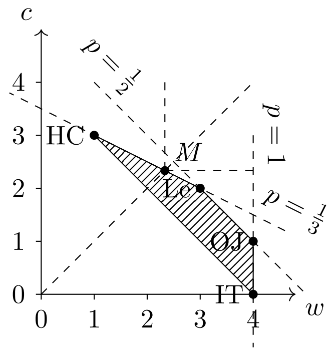
   (a) Produce a decision table for this problem.
   (b) What proportion of drinks should she stock to maximise her guaranteed (*i.e.*, minimum) sales total regardless of the temperature?
   (c) Find the *Bayes* strategies for $p = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.
   (d) What is the least favourable probability distribution on warm and cool (not warm) days?
   (e) Repeat the above analysis for the *miniMax Regret* rule.
   (f) Define the admissibility frontier for this problem.

   Solution

   (a) Consider the decision table below, with $P(s_1) = p$. Values are expressed in tens of dollars. The associated graph is also shown.

	w	c
HC	1	3
IT	4	0
Le	3	2
OJ	4	1

where: w warm day
c cold day



   (b) She would maximise her guaranteed sales by having the mixture of stock which maximises the minimum sales irrespective of whether the day is warm or cold.
   It is clear from the graph that the optimal mixture should comprise hot chocolate and lemonade only.
   Let m_w be the average sales of the relevant mixture of drinks on a warm day and m_c the mixture's average sales on a cool day.

If μ is the desired proportion of hot chocolate in the mixture, then $M = (m_w, m_c) = (3,2) + \mu[(1,3) - (3,2)]$; *i.e.*,

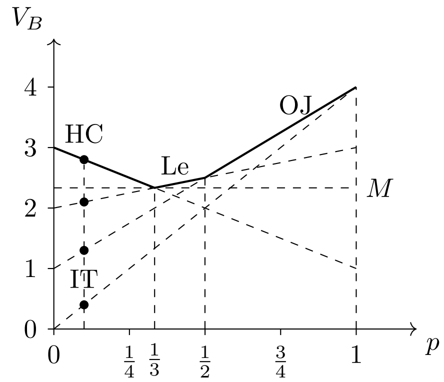
$$\begin{aligned} m_w &= 3 + (1-3)\mu = 3 - 2\mu \\ m_c &= 2 + (3-2)\mu = 2 + \mu \end{aligned}$$

Setting $m_w = m_c$ to find the *Maximin* mixed strategy:

$$\begin{aligned} 3 - 2\mu &= 2 + \mu \\ 1 &= 3\mu \\ \therefore\ \mu &= \tfrac{1}{3} \end{aligned}$$

That is, she should have a mixture consisting of one third of the units on sale being hot chocolate and the other two thirds lemonade. That is, a ratio of two units of lemonade per unit of hot chocolate.

(c) Consider the plot of the *Bayes* values of the strategies against p:



From the graph:

p	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
Bayes strategy	HC	HC	Le & OJ	OJ	IT & OJ

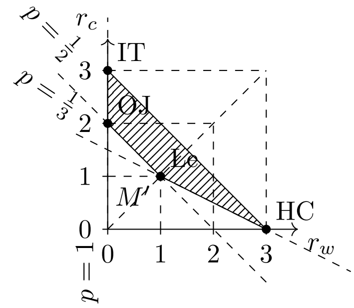
For probabilities for which multiple pure strategies are *Bayes* strategies, mixtures of those strategies involved would also be *Bayes* strategies; *e.g.*, for $p = \frac{1}{2}$, any mixture of Le and OJ would also be a *Bayes* strategy.

(d) The least favourable probability distribution is the one that minimises the value of the *Bayes* strategies, and corresponds to the probability associated with the indifference curve on which the *Maximin* strategy lies.
This is obtained from the slope of the segment on which M lies; *i.e.*, the segment joining HC and Le. Since this slope is $m = -\frac{1}{2}$, the probability is $p = \frac{1}{1+2} = \frac{1}{3}$. This is verified by inspection of the above graph of the *Bayes* values against p.

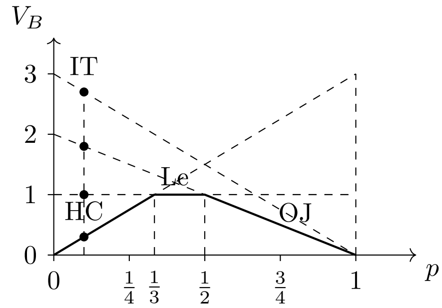
(e) The maximum regret indifference curves are shown on the graph below (right). Since *miniMax Regret* seeks to minimise the maximum regret, preference is for curves to the lower left (instead of upper right, which would correspond to preference under *Maximin*).

	p	$1-p$
	w	c
HC	3	0
IT	0	3
Le	1	1
OJ	0	2

where: w warm day
c cold day

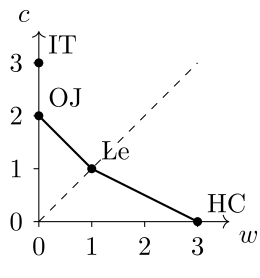


Notice that the *miniMax Regret* mixed strategy is the pure strategy Le, and that this does not agree with the *Maximin* strategy—which is a mixture of HC and Le.
Consider the plot of the *Bayes* regret values of the strategies against p:



Notice that this graph resembles the other one but is inverted, and the values at $p=1$ have been shifted by 1. Because of the similarity, the graphs of the lines for the strategies relative to each other are preserved, and hence the *Bayes* strategies remain unaffected for every value of p; *i.e.*, *Bayes* strategies are invariant under regret.
This can also be seen from the graph in regret space; the strategies are rotated (double reflection) in the same relative positions relative to each other, so the slopes (*i.e.*, probabilities) will still produce the same strategies under the *Bayes* decision rule when minimising *Bayes* regret rather than maximising the original *Bayes* values.

(f) Consider:



Notice that iced tea (IT) is weakly dominated by OJ, and hence is not on the admissible frontier; in fact, the entire set of non-degenerate

mixtures of IT with OJ (the segment joining IT and OJ, excluding OJ itself) are inadmissible.
When minimising regret, the admissibility frontier has the same shape, but is inverted (rotated).

6. Show that a strategy is admissible iff it is a *Bayes* strategy for some probability distribution.

   Solution
   Consider an arbitrary inadmissible strategy A; *i.e.*, there exists some strategy B such that for each of A's payoffs, a_i, for the corresponding payoff b_i under B, we have $b_i > a_i$. For an arbitrary probability distribution, let p_i be the probability of payoffs a_i and b_i. It follows that:

$$\begin{aligned} b_i > a_i \quad & \text{iff} \quad p_i b_i > p_i a_i \\ & \text{iff} \quad \sum_i p_i b_i > \sum_i p_i a_i \\ & \text{iff} \quad V_B(\mathrm{B}) > V_B(\mathrm{A}) \end{aligned}$$

   Therefore, B will be preferred over A under the *Bayes* decision rule for any probability distribution, and hence A will not be a *Bayes* strategy.

   Conversely, suppose A is admissible, then for any other strategy B, for some i, $a_i \geqslant b_i$. So for any probability distribution such that $p_i = 1$ (*i.e.*, $p_j = 0$ for all $j \neq i$), $V_B(\mathrm{A}) = \sum_k p_k a_k = p_i a_i \geqslant p_i b_i = \sum_k p_k b_k = V_B(\mathrm{B})$. It follows that for some probability distribution, A is a *Bayes* strategy.

   The two paragraphs above conclude the proof.

7. Show that a *Maximin* strategy is always a *Bayes* strategy for some probability distribution.

   Solution
   A proof sketch is outlined for the case of two states.

   Let M $= (m_1, m_2)$ be a *Maximin* strategy. (Does there always exist a *Maximin* strategy?) There are two cases to consider:a) M is a pure strategy; or b) M is a mixture.

   If M is a pure strategy then there must be some state s_i in which $m_i \geqslant a_i$ for any other strategy A. In this case M is admissible, and hence, by the result above, a *Bayes* strategy for some probability distribution.

   If M is a mixture then we saw that for the least favourable probability distribution P, M will receive a *Bayes* value no less than any admissible mixture. So M will be a *Bayes* strategy for P.

   In both cases M is a *Bayes* strategy, which completes the proof.

8. Prove that for any two actions A and B, if A weakly dominates B, and all state probabilities are non-zero, then the *Bayes* decision rule will *strictly* prefer A over B.

   Solution
   Suppose A weakly dominates B; *i.e.*, for all i, $a_i \geqslant b_i$ and for some j, $a_j > b_j$. Since for all i, $p_i > 0$, then it follows that for all i, $p_i a_i \geqslant p_i b_i$ and $p_j a_j > p_j b_j$. But then $V_B(\mathrm{A}) = \sum_i p_i a_i = \sum_{i \neq j} p_i a_i + p_j a_j > \sum_{i \neq j} p_i b_i + p_j b_j = V_B(\mathrm{B})$.