GSOE9210 Engineering Decisions

Victor Jauregui

vicj@cse.unsw.edu.au
www.cse.unsw.edu.au/~gs9210

Updating belief

Outline

1. Bayesian updating
   - Airline case study
2. Value of information
3. Revision of Bayesian beliefs
   - Incorporating additional information
   - Updating reliability likelihood
4. Sensitivity analysis

Case study: capital purchase

Example (To purchase or not)

You're the chief engineer of a small commercial airline which, due to increased demand, is considering adding to its fleet by buying (B) a used airliner. Another company is offering to sell one of its airliners for $\$400{,}000$. Used airlines range in reliability, which is hard to evaluate without a detailed inspection.

Question: should you purchase?

Problem modelling

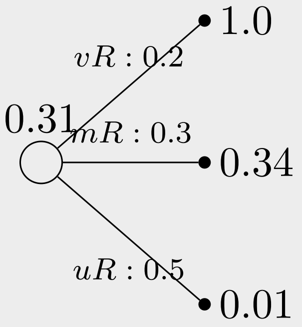
- Problem 1: how to measure reliability? Operating hours
- Simplification 1: classify airliners as either: very reliable (vR) ($> 90\%$), moderately reliable (mR), or unreliable (uR) ($< 50\%$)
  Beliefs about reliability:

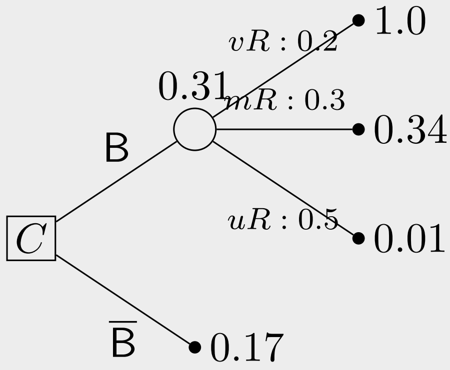
	Reliability		
	vR	mR	uR
Probability	0.2	0.3	0.5
Utility	1.0	0.34	0.01

- Simplification 2: assume a very reliable airliner makes $\$1M$ profit (best outcome); an unreliable one makes $\$200K$ loss (worst)
- Simplification 3: utility of not buying airliner—*status quo*: 0.17

Decision C (buy or not)

Buy:



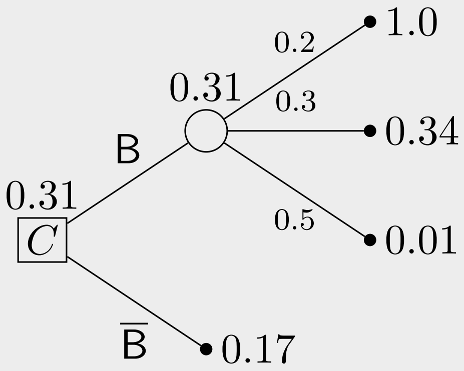


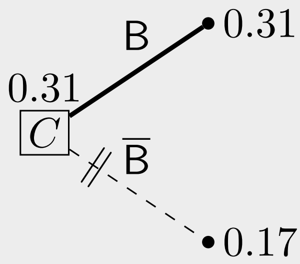
$$U(\text{B}) = 0.2(1.0) + 0.3(0.34) + 0.5(0.01) = 0.31$$
$$U(\overline{\text{B}}) = 0.17$$

	0.2	0.3	0.5	
	vR	mR	uR	U
B	1.00	0.34	0.01	0.31
$\overline{\text{B}}$	0.17	0.17	0.17	0.17

Decision C

- Evaluate decision points/nodes by maximal utility of alternatives (*i.e.*, actions/strategies)
- The value of node C is 0.31, because $0.31 > 0.17$; *i.e.*, $0.31 = \max\{0.17, 0.31\}$





Outline

Get more information?

Example (Additional information)

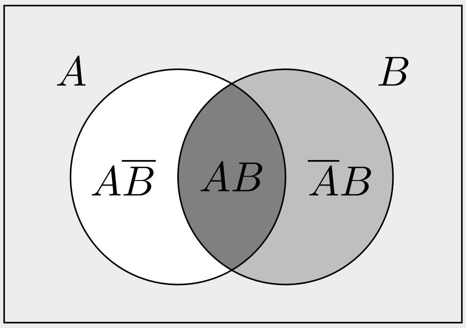
You have the option to consult an aeronautical engineering firm to conduct an assessment of the airliner for $\$10K$. The report's will be either favourable (f) or unfavourable (u) as to whether or not to purchase.

- Firm's assessment reliable?
- Guess/estimate that 90% of very reliable planes receive favourable assessment; *i.e.*, $P(f|vR) = 0.9$

	... conditional on:		
Probability of:	vR	mR	uR
f	0.9	0.6	0.1
u	0.1	0.4	0.9

Outline

Conditional probability



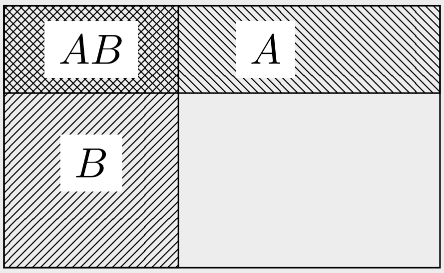
Definition

The *conditional probability* of event A conditional on B (provided B is possible; *i.e.*, $P(B) \neq 0$), written $P(A|B)$, is defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

In the diagram above, $P(A|B)$ represents the ratio of (the area of) the region AB (the dark region) to that of the whole of B.

Conditional independence



Definition

Event A is *(conditionally) independent* of event B if:

$$P(A|B) = P(A).$$

Event A is *(conditionally) dependent* on B if A is not (conditionally) independent of B.

For example, if B is a random sample of a population.

Bayes's rule

- Rearranging the definition of conditional probability:

$$P(A \cap B) = P(A|B)P(B)$$

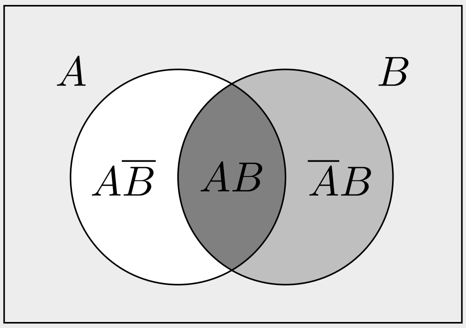
- By symmetry $P(A \cap B) = P(B \cap A)$;
  therefore: $P(A|B)P(B) = P(B|A)P(A)$.
  Rearranging gives:

Theorem (Bayes's Theorem I)

If A and B are any two events (such that $P(A) \neq 0$), then:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Bayes's Venn diagram



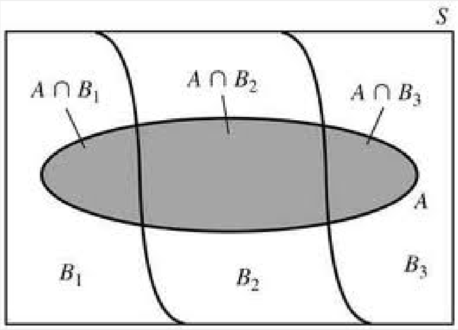
But $A = AB \cup A\overline{B}$. So we get the following:

Theorem (Bayes's Theorem I')

If A and B are any two events ($P(A) \neq 0$), then:

$$P(B|A) = \frac{P(AB)}{P(AB) + P(A\overline{B})}$$

Extending Bayes's rule



$$P(B_1|A) = \frac{P(AB_1)}{P(AB_1) + P(AB_2) + P(AB_3)}$$
$$P(B_2|A) = \frac{P(AB_2)}{P(AB_1) + P(AB_2) + P(AB_3)}$$
$$P(B_3|A) = \frac{P(AB_3)}{P(AB_1) + P(AB_2) + P(AB_3)}$$

Bayes's rule generalised

Events $B_1, \ldots, B_n$ are said to be *universally exhaustive* (of Ω) if $\bigcup_{i=1}^{n} B_i = \Omega$.

Theorem (Bayes's Theorem II)

If events $B_1, \ldots, B_k, \ldots, B_n$ are mutually exclusive and universally exhaustive, and A is a possible event ($P(A) \neq 0$), then:

$$P(B_k|A) = \frac{P(AB_k)}{\sum_{i=1}^{n} P(AB_i)}$$

Theorem (Bayes's Theorem II')

If $B_1, \ldots, B_k, \ldots, B_n$ are mutually exclusive and universally exhaustive events and A is a possible event ($P(A) \neq 0$) then:

$$P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

Example: Bayes's rule

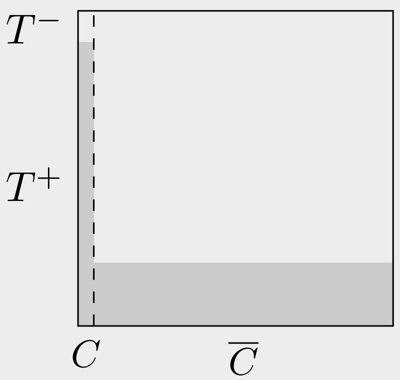
Example (Medical diagnostics)

In a given population of people, one in every thousand have hypo-cytocitic cancer. A certain pathology test is used to detect the disease. The test is 'good' but not perfect; it returns a positive result in 98% of persons with the disease, and registers a *false positive* (*i.e.*, gives a positive result for a person free of the disease) 5% of the time.

Exercise

A random person comes in to get tested and the test returns a positive result. What is the probability that the person has cancer?

Example: solution



Given information:

$$
\begin{array}{ll}
P(C) = \frac{1}{1000} & P(\overline{C}) = \frac{999}{1000} \\
P(T^+|C) = \frac{98}{100} & P(\overline{T^+}|C) = \frac{2}{100} \\
P(T^+|\overline{C}) = \frac{5}{100} & P(\overline{T^+}|\overline{C}) = \frac{95}{100}
\end{array}
$$

What is $P(C|T^+)$?

$$
P(C|T^+) = \frac{P(CT^+)}{P(T^+)} = \frac{P(CT^+)}{P(CT^+ \cup \overline{C}T^+)} = \frac{P(CT^+)}{P(CT^+) + P(\overline{C}T^+)}
$$

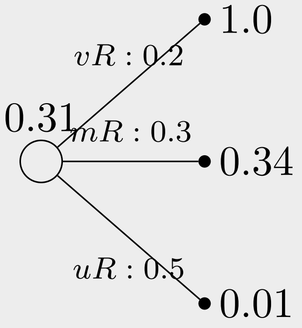
$$
= \frac{P(T^+|C)P(C)}{P(T^+|C)P(C) + P(T^+|\overline{C})P(\overline{C})}
$$

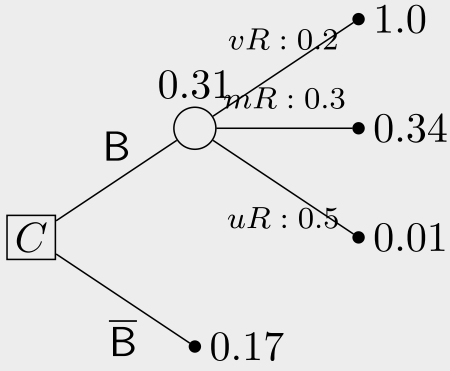
$$
= \frac{\frac{98}{100} \times \frac{1}{1000}}{\frac{98}{100} \times \frac{1}{1000} + \frac{5}{100} \times \frac{999}{1000}} = \frac{98}{98 + 5 \times 999} \approx \frac{100}{5000} = 0.02
$$

Patient only has 2% chance of having cancer despite testing positive?!

Decision C (buy or not)

Buy:





$$U(\text{B}) = 0.2(1.0) + 0.3(0.34) + 0.5(0.01) = 0.31$$
$$U(\overline{\text{B}}) = 0.17$$

	0.2	0.3	0.5	
	vR	mR	uR	U
B	1.00	0.34	0.01	0.31
$\overline{\text{B}}$	0.17	0.17	0.17	0.17

Post-report (posterior) probabilities

- If report favourable (f):

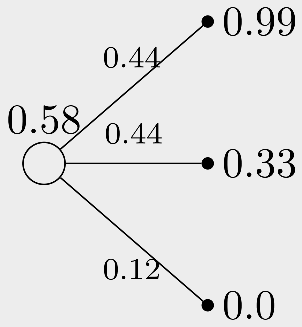
$$\begin{aligned} P(vR|f) &= \frac{P(f|vR)P(vR)}{P(f|vR)P(vR)+P(f|mR)P(mR)+P(f|uR)P(uR)} \\ &= \frac{0.9(0.2)}{0.9(0.2)+0.6(0.3)+0.1(0.5)} \\ &= \frac{0.18}{0.41} \approx 0.44 \end{aligned}$$

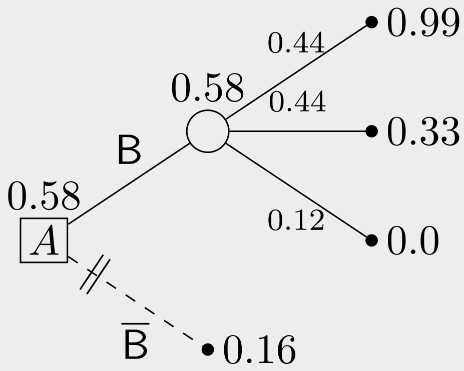
  Similarly: $P(mR|f) \approx 0.44$ and $P(uR|f) \approx 0.12$

- If report unfavourable (u):

$$\begin{aligned} P(vR|u) &= \frac{0.02}{0.59} \approx 0.04 \\ P(mR|u) &\approx 0.20 \\ P(uR|u) &\approx 0.76 \end{aligned}$$

Decision A (report favourable)

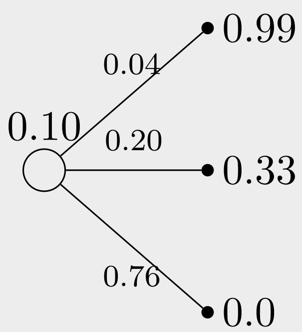


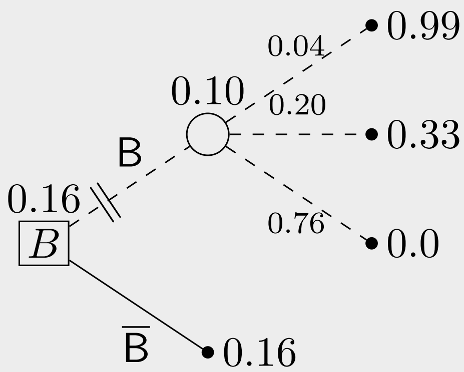


- The revised expected utility of buying the airliner is $U(\mathsf{B}) = 0.44(0.99) + 0.44(0.33) + 0.12(0.0) = 0.58$
- The utility of not buying it is $U(\overline{\mathsf{B}}) = 0.16$.

	0.44	0.44	0.12	
	vR	mR	uR	U
B	0.99	0.33	0.0	0.58
$\overline{\mathsf{B}}$	0.16	0.16	0.16	0.16

Decision B (report unfavourable)





- The revised expected utility of buying the airliner is $U(\mathsf{B}) = 0.04(0.99) + 0.20(0.33) + 0.76(0.0) = 0.10$
- The utility of not buying it is $U(\overline{\mathsf{B}}) = 0.16$.

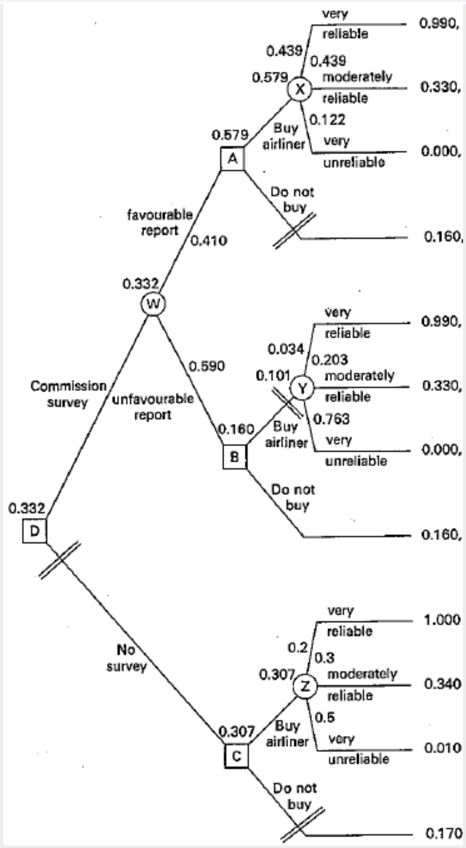
	0.04	0.20	0.76	
	vR	mR	uR	U
B	0.99	0.33	0.0	0.10
$\overline{\mathsf{B}}$	0.16	0.16	0.16	0.16

Utility adjustments

- Problem: cost associated with report?
  Question: How does report's cost ($\$10K$) affect utility?
- Observation: report cost small relative to other monetary quantities: potential profit $\$1M$; *i.e.*, $\$10K \ll \$1M$
- Simplification 3: model effect by constant shift; *i.e.*, for report costing $\$x$ $(x \ll 1M)$, change of utility is $\Delta u = \frac{x}{1M}$; $\$1M$
- That is, every $\$10K$ is worth 0.01 *utiles*

Combined decision

- Combine all three possible cases into one big decision problem
- Introduce new decision: commission survey/report, and no survey/report
- Introduce new event: report outcome (f or u)
- If consultant good, report likely to be good predictor of (*i.e.*, correlated to) aircraft reliability
- Consultant's increased predictive accuracy is *valuable* in making decision

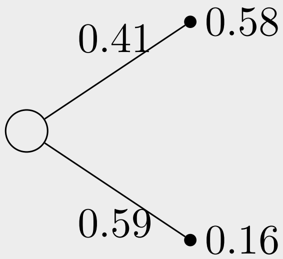


Combined decision

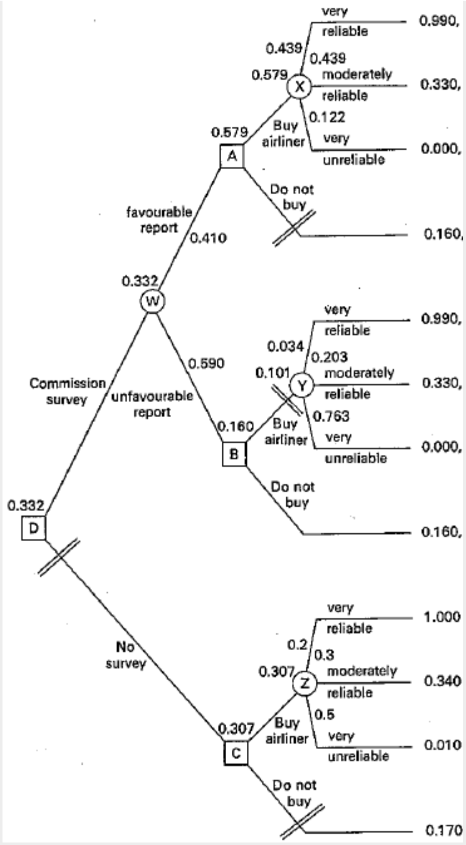
- From the denominators in the earlier calculations:

$$P(f) = 0.41$$
$$P(u) = 0.59$$

- Therefore, if report commissioned:



- Utility of report: 0.33



Decision table

	f, vR	f, mR	f, uR	u, vR	u, mR	u, uR	U
A_1	1.0	0.34	0.01	1.0	0.34	0.01	0.31
A_2	0.17	0.17	0.17	0.17	0.17	0.17	0.17
A_3	0.99	0.33	0	0.99	0.33	0	
$\vdots$	$\vdots$	$\vdots$		$\dots$			
A_6							

where

A_1 no survey; buy airliner

A_2 no survey; don't buy airliner

A_3 commission survey; buy airliner

A_4 commission survey; don't buy

A_5 commission survey; if favourable, buy airliner; else don't buy

A_6 commission survey; if favourable, don't buy airliner; else buy

Value of information

- Optimal policy if report commissioned:

Policy for C: report commissioned

If report favourable, buy airliner, if not don't buy.

- Value of policy is $U(\mathsf{C}) = 0.33$, inclusive of the 0.01 fee
- Optimal policy if report not commissioned:

Policy for $\overline{\mathsf{C}}$: report not commissioned

Buy the airliner.

- $U(\overline{\mathsf{C}}) = 0.31$
- How much is report worth?
- $U(\mathsf{C}) = 0.34 - u_r \geqslant 0.31 = U(\overline{\mathsf{C}})$; *i.e.*, should commission report for fee up to $u_r = 0.03$; *i.e.*, for any fee up to $\$30K$

Outline

Production and demand

Example (Production)

Alice is the CTO at a company and Bob is the CFO. They're considering two possible production processes for a product. Process A is expected to net $\$40K$ if demand increases, $\$30K$ if demand remains stable, and $\$20K$ if demand falls. Process B requires a greater initial capital expenditure; it will only net $\$10K$ if demand drops, and $\$40K$ otherwise.
Future estimates of demand are: 20% of an increase, 30% chance of staying level, and 50% of a decrease.

Which process should Alice implement?

Example

The decision table is:

	$\frac{5}{10}$ $\downarrow$	$\frac{3}{10}$ $-$	$\frac{2}{10}$ $\uparrow$	$V_\$$
A	$20	$30	$40	$27
B	$10	$40	$40	$25

$$\begin{aligned} V_\$(\mathsf{A}) &= \tfrac{5}{10}(20) + \tfrac{3}{10}(30) + \tfrac{2}{10}(40) \\ &= 10 + 9 + 8 = \$27 \\ V_\$(\mathsf{B}) &= \tfrac{5}{10}(10) + \tfrac{3}{10}(40) + \tfrac{2}{10}(40) \\ &= 5 + 12 + 8 = \$25 \end{aligned}$$

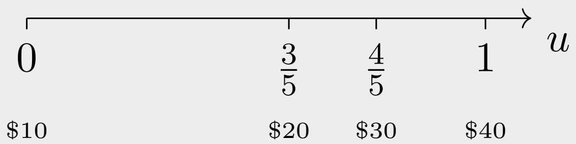
Alternative A has greater expected monetary value

Example

Alice consults Bob who advises her that, under its current financial position, the company's preferences are:

$$\$20 \sim \left[\tfrac{3}{5} : \$40 | \tfrac{2}{5} : \$10\right]$$
$$\$30 \sim \left[\tfrac{4}{5} : \$40 | \tfrac{1}{5} : \$10\right]$$

The company's utility for money is:



The utility table:

	$\frac{5}{10}$ $\downarrow$	$\frac{3}{10}$ $-$	$\frac{2}{10}$ $\uparrow$	U
A	$\frac{3}{5}$	$\frac{4}{5}$	1	$\frac{74}{100}$
B	0	1	1	$\frac{50}{100}$

$$\begin{aligned} U(\mathsf{A}) &= \tfrac{5}{10}(\tfrac{3}{5}) + \tfrac{3}{10}(\tfrac{4}{5}) + \tfrac{2}{10}(1) \\ &= \tfrac{1}{50}\,(15 + 12 + 10) = \tfrac{74}{100} \\ U(\mathsf{B}) &= \tfrac{5}{10}(0) + \tfrac{3}{10}(1) + \tfrac{2}{10}(1) \\ &= \tfrac{1}{50}\,(0 + 15 + 10) = \tfrac{50}{100} \end{aligned}$$

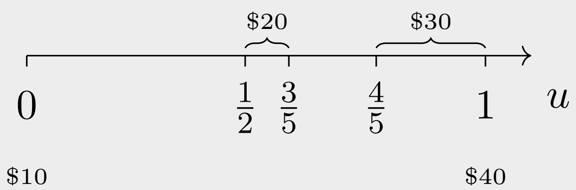
Therefore, A will also have greater utility

Sensitivity analysis

Suppose Bob cannot give precise assessments on values of \$20 and \$30, only bounds:

$$\left[\tfrac{3}{5}\$40\right] \succ \$20 \succ \left[\tfrac{1}{2}\$40\right]$$
$$\$40 \succ \$30 \succ \left[\tfrac{4}{5}\$40\right]$$

The utility for money is:



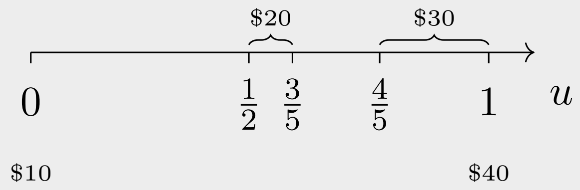
Lower bound for A:

	$\frac{5}{10}$ $\downarrow$	$\frac{3}{10}$ $-$	$\frac{2}{10}$ $\uparrow$	U
A	$\frac{1}{2}$	$\frac{4}{5}$	1	$\frac{69}{100}$
B	0	1	1	$\frac{50}{100}$

Upper bound for A:

	$\frac{5}{10}$ $\downarrow$	$\frac{3}{10}$ $-$	$\frac{2}{10}$ $\uparrow$	U
A	$\frac{3}{5}$	1	1	$\frac{80}{100}$
B	0	1	1	$\frac{50}{100}$

Sensitivity analysis



Bounds on A:

$$
\begin{aligned}
U(\mathsf{A}) &> \tfrac{5}{10}(\tfrac{1}{2}) + \tfrac{3}{10}(\tfrac{4}{5}) + \tfrac{2}{10}(1) \\
&= \tfrac{1}{100}(25 + 24 + 20) \\
&= \tfrac{69}{100} \\
U(\mathsf{A}) &< \tfrac{5}{10}(\tfrac{3}{5}) + \tfrac{3}{10}(1) + \tfrac{2}{10}(1) \\
&= \tfrac{1}{100}(30 + 30 + 20) \\
&= \tfrac{80}{100}
\end{aligned}
$$

That is:

$$\tfrac{69}{100} < U(\mathsf{A}) < \tfrac{80}{100}$$

Conclusion:
A is guaranteed to be preferred to B ($U(\mathsf{B}) = \frac{50}{100}$) regardless of the uncertainty over the precise preference for \$20 and \$30.

Summary

- Explored decision problems in greater depth:
  - actions that affect epistemic state (value of information-gathering actions)
  - dealing with uncertainty in preferences (sensitivity analysis)
- Updating beliefs (epistemic state) via *Bayes*'s theorem
- Value of information: cost of gathering more information versus increase in expected utility due to new information
- Sensitivity analysis:
  - decisions under imprecise preferences
  - how might preference uncertainty affect a decision?