Student ID: ______________________________

Student Name: ______________________________

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The University of New South Wales
Session 2, 2018

# GSOE9210 Engineering Decisions

## sample final exam

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Instructions:

- Time allowed: 2 hours
- Reading time: 10 minutes
- This paper has 43 pages
- Total number of questions: 53 (multiple choice)
- Total marks available: 60 (not all questions are of equal value)
- Allowed materials: UNSW approved calculator, pencil (2B), pen, ruler, language dictionary (paper)

  This exam is closed-book. No books, study notes, or other study materials may be used
- Provided materials: generalised multiple choice answer sheet, graph paper (1 page), working out booklet
- Answers should be marked in pencil (2B) on the accompanying multiple choice answer sheet
- The exam paper may not be retained by the candidate

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Start of exam

*Questions 1 to 7 refer to the problem below.*

Recall the school fund-raiser example from lectures in which Alice, the principal of a local school, is planning to hold a single-day fund-raiser. There are two options for the fund-raising activity: a fête (F) or a sports day (S). Proceeds of each activity depend on the (unpredictable) weather: on a dry day ($d$) a fête will make a profit of \$150 and a sports day only \$120; however, on a wet day ($w$) the sports day will net \$85 and the fête only \$75.

Suppose Alice has no information about the likelihood of whether any given day will be dry or wet.

1. (*1 mark*) On any given day, which of the two activities (S or F) will ensure the greatest lower bound on profit? [1]

   a) S only
   b) F only
   c) both S and F
   d) neither S nor F
   e) a mixture of S and F

Solution
a)—S only.

The basic (pure) actions correspond to the two activities, S and F. The uncertainty surrounds the weather: dry ($d$) or wet ($w$).

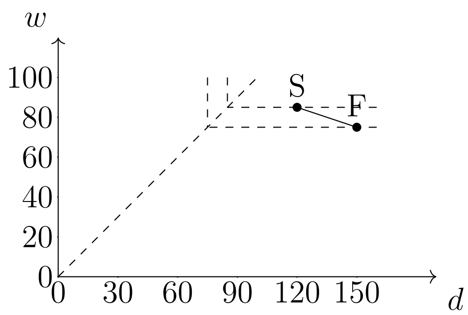
Because no information is available about the likelihoods of dry and wet days, this is a decision made under ignorance.

The event is once-off on a single day, so mixtures can be disregarded; *i.e.*, only pure actions should be considered.

We're asked for the greatest minimum (lower bound) profit, so *Maximin* is the relevant decision rule.

It is clear from the decision table and accompanying plot below that S is the only *Maximin* pure action when considering profits.

| | $d$ | $w$ | min |
|---|---|---|---|
| S | 120 | 85 | 85 |
| F | 150 | 75 | 75 |




2. (*1 mark*) Suppose Alice is more concerned about limiting the maximum regret—she doesn't like to miss out on opportunities. Which activity would Alice prefer? [2]

   a) S only
   b) F only
   c) both S and F
   d) neither S nor F
   e) a mixture of S and F

   Solution

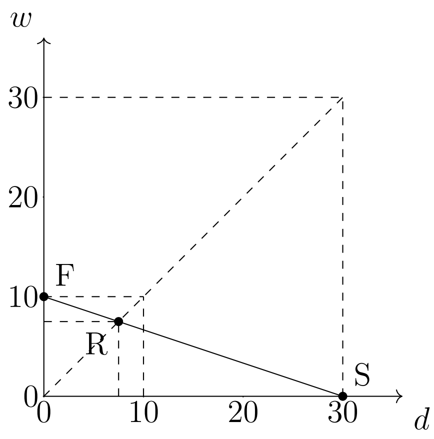
   b)—F only.

   This question is asking for the action with least upper bound on regret. Given that this is a once-off event, the aim is to determine the pure actions selected under *miniMax Regret*.

   In the decision table and plot below the entries correspond to regrets (on profits) rather than the actual profits themselves. Because mixtures aren't being considered the mixture line SF can be ignored at this stage.

   Notice that the *miniMax* indifference curve (the 'down-left' quadrants) for the regret of F intersects the diagonal at a lower point than that of S; *i.e.*, the *miniMax Regret* pure action is F only.

| | $d$ | $w$ | max |
|---|---|---|---|
| S | 30 | 0 | 30 |
| F | 0 | 10 | 10 |




For the following questions assume the following:

Imagine Alice works for the local branch of the Government's education department. She is in charge of twelve local schools, and is planning to hold a single-day fund-raiser in each school on the same day. She can hold different activities in different schools, if she wishes.

3. (*1 mark*) In how many schools should Alice hold a sports day if she wants to ensure the greatest minimum profit? [3]

   a) in none of them
   b) in four of them
   c) in six of them
   d) in eight of them
   e) in all twelve of them

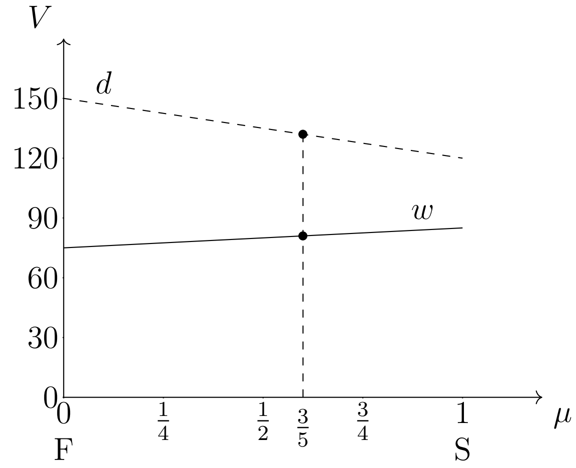
Solution
e)—all twelve.

From the earlier plot, the *Maximin* mixed action on profits (in contradistinction to regrets) is also S; *i.e.*, all schools should hold a sports day.

The worst possible case for both actions is if the day of the fund-raiser is wet. The best action in this case is S. Hosting a fête on a wet day will only reduce the profit.

This can also be seen by plotting the profits in each state as functions of the mixture proportions of the two actions (see below).

Let $\mu = \mu_S$ be the proportion of schools that host sports days. The plots below shows the (combined) profits of mixtures of the two actions in each of the different states ($d$ and $w$). Mixtures correspond to values on the horizontal axis, where F is on the left-most extreme ($\mu = 0$) and S is right-most ($\mu = 1$). (The pure actions below the horizontal axis label the mixture extremes.)




Clearly the minimum value for each mixture is determined by its value in wet days (the solid line labelled $w$). The greatest minimum is obtained for $\mu = 1$ (the right-most ordinate on the horizontal axis); *i.e.*, a full mixture of S. This corresponds to hosting sports days in all schools.

4. (*1 mark*) In how many schools should a sports day be hosted if limiting the maximum regret is the main consideration? [4]

   a) in none of them
   b) in three of them
   c) in four of them
   d) in six of them
   e) in all twelve of them

Solution

b)—three out of twelve.

The regret plot shown earlier reveals that the *miniMax Regret* mixed action (R) is a mixture of the pure actions. Moreover, R is closer to F than to S, hence it must have a lower proportion of S than F.

More precisely, let $\mu = \mu_S$ be the proportion of S in mixture R:

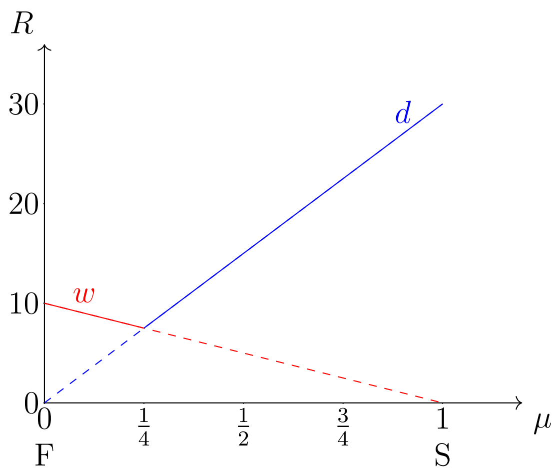
$$\begin{aligned} R = (r, r) &= \mu \mathrm{S} + (1-\mu)\mathrm{F} \\ &= \mu \cdot (30, 0) + (1-\mu) \cdot (0, 10) \\ &= (30\mu, 10 - 10\mu) \end{aligned}$$

Equating R's coordinates (the mixed *miniMax Regret* point must be on the diagonal):

$$\begin{aligned} 30\mu &= 10 - 10\mu \\ 40\mu &= 10 \\ \mu &= \tfrac{1}{4} \end{aligned}$$

The *miniMax Regret* mixed action is $\frac{1}{4}\mathrm{S}\frac{3}{4}\mathrm{F}$. Moreover, the regret associated with $R$ is $r = 30\mu = \frac{30}{4} = \frac{15}{2} = 7\frac{1}{2}$.

The plots below (dahed lines) show the (combined) regrets of mixtures of the two actions in each state ($d$ and $w$). The maximum regret for each mixture corresponds to the solid line(s).




Note that, in this case, mixtures balance the pure actions' regrets across the two states; as regret increases in one state it decreases in the other.

There is a mixture ($\mu = \frac{1}{4}$) for which the actions' average regret in each state coincide. This mixture bounds the maximum regret over all states to the least possible value; any other mixture would yield a greater regret in some state. Therefore, this mixture is the *miniMax Regret* mixed action.

It follows that a quarter of the schools (three of the twelve) should host a sports day.

For the following question, suppose that fund-raising events are held in one day of each week of every month.

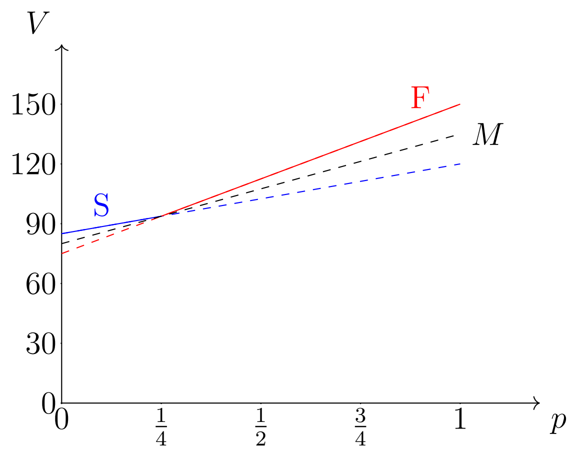
5. (*1 mark*) Let $p = P(d)$ be the probability that any given day is dry. Which is the *Bayes* action for probability $p = \frac{1}{2}$? [5]

   a) S only
   b) F only
   c) both S and F
   d) neither S nor F
   e) a mixture of S and F

Solution
b)—F only.

Below are plots of the pure actions' *Bayes* values against $p$:

| | $d$ | $w$ | $V_B$ |
|---|---|---|---|
| S | 120 | 85 | $85 + 35p$ |
| F | 150 | 75 | $75 + 75p$ |




Notice that any mixture of the two actions (*e.g.*, $M = \frac{1}{2}\text{S}\frac{1}{2}\text{F}$) will be inside the pure actions' 'envelope', so that for each value of $p$ there will always a be some pure action with equal or greater *Bayes* value—not necessarily the same pure action for all $p$. Therefore, mixed actions can be all but ignored when *Bayes* actions are being determined.

From the plot, the *Bayes* action for any $p > \frac{1}{4}$, including $p = \frac{1}{2}$, is F only.

More specifically, for $p = \frac{1}{2}$:

$$V_B(\text{S}) = 85 + 35(\tfrac{1}{2}) = 85 + 17\tfrac{1}{2} = 102\tfrac{1}{2}$$
$$V_B(\text{F}) = 75 + 75(\tfrac{1}{2}) = 75 + 37\tfrac{1}{2} = 112\tfrac{1}{2}$$

This confirms that F is the unique *Bayes* action for $p = \frac{1}{2}$.

Records kept over the last ten years indicate that, on average, the number of dry days per month in Alice's geographic area are as follows:[1]

| Month | Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sep | Oct | Nov | Dec |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Dry days | 15 | 13 | 10 | 8 | 6 | 5 | 5 | 7 | 11 | 13 | 14 | 16 |

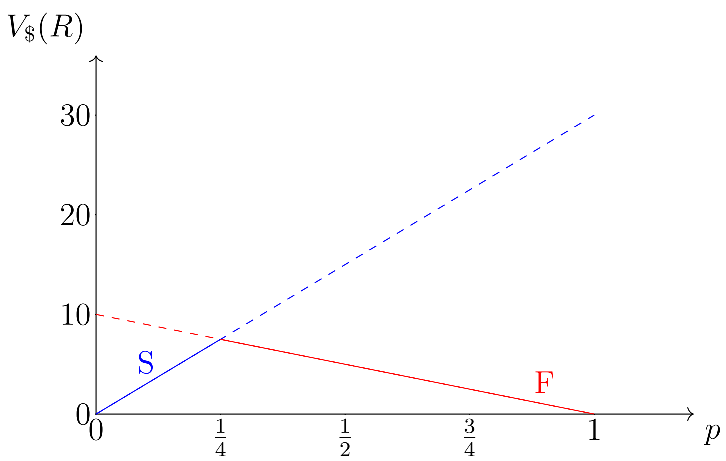
6. (*2 marks*) Alice holds her fund-raisers every month except the one month in which she takes her annual holidays. If Alice is concerned with limiting the maximum regret, which of the options below would be best time for Alice to take her holidays? [7]

   a) Jan or Feb
   b) Feb or Sep
   c) June or July
   d) Apr or Aug
   e) Jan or Dec

Solution

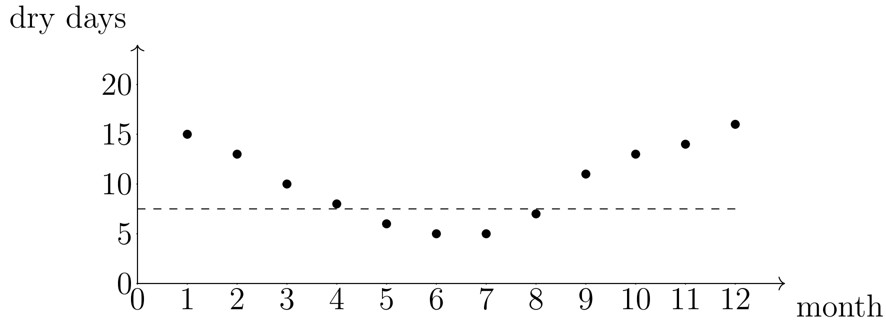
d)—April or August.

The plot below shows *Bayes* values for regret for each of the two actions. Note that for values of $p = P(d)$ other than $\frac{1}{4}$, some action (S for $p < \frac{1}{4}$ and F for $p > \frac{1}{4}$) will have a lesser average (*Bayes*) regret than for $p = \frac{1}{4}$. The least favourable probability distribution of dry and wet days is, therefore, $\frac{1}{4}d\frac{3}{4}w$; for that distribution Alice, regardless of the action she chooses, can only limit the regret to \$7.50. If, for example, $p > \frac{1}{4}$ then by choosing F she would be able to limit the expected regret to a lesser value.




[1]Note that Alice lives in a very wet area; perhaps a mountain valley.

The plot below shows the average number of dry days per month.




The dashed line corresponds to the number of dry days of the least favourable distribution (*i.e.*, $\approx \frac{1}{4} \times 30 = 7.5$ days/month). It follows that in months Apr and Aug (months 4 and 8 respectively—those months with values closest to the dashed line) both action(s) will have the greatest average regret.

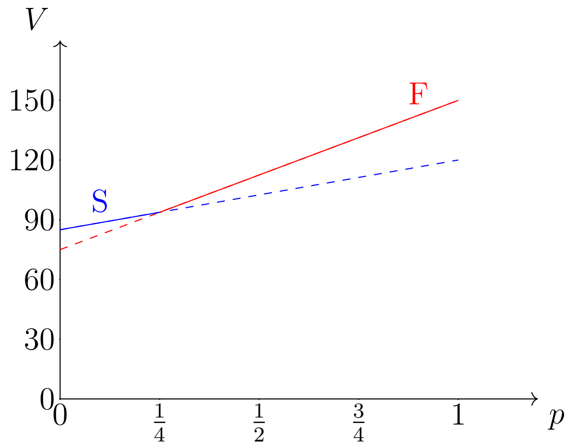
By taking her holidays in one of those months she will avoid a period of greatest average regret ($7.50).

7. (*1 mark*) If Alice were concerned with securing the greatest minimum profit, in which months should she schedule her holidays? [8]

   a) Jan or Feb
   b) Feb or Sep
   c) June or July
   d) Apr or Aug
   e) Jan or Dec

Solution
c)—June or July.

The plot below shows that months with more dry days raise greater funds (as both lines represent increasing functions).




The least favourable probability distribution in this case is one which has the fewest dry days (most wet days); *i.e.*, months Jun or Jul.

Note that in those months there are, on average, 5 wet days. In those months $p \approx \frac{5}{30} = \frac{1}{6} < \frac{1}{4}$, hence the best action is S.

Moreover, observe that if the minimum average number of dry days (say in Jun or Jul) never fell below eight, then the minimum $p$ would be $\frac{4}{15} > \frac{1}{4}$, and hence F would always be preferred over S. Sometimes the constraints on the problem (*e.g.*, here regarding the frequency of wet days) may reduce the choices among the best possible actions. We've seen something similar in games, where this is often due to beliefs about other players' strategies.

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*Questions 8 to 22 refer to decision table below.*

Consider the following decision table for a problem in which the outcomes are measured in dollars ($).

| | $s_1$ | $s_2$ |
|---|---|---|
| $a_1$ | 10 | 50 |
| $a_2$ | 40 | 20 |

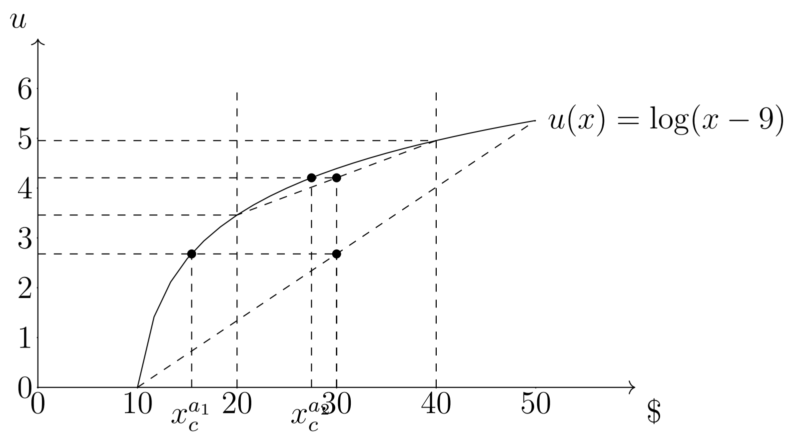
There are two agents, A and B, who are making independent decisions on which of the possible actions ($a_1$ and $a_2$) to take—note that this is *not* a game: both agents are choosing separate decisions at different times.

Consider agent A first. Agent A's utility function for money is logarithmic (with base 2); *i.e.*, $u(x) = \log_2(x - a)$, where $a \in \mathbb{R}$ is a parameter to be determined.

8. (*1 mark*) If $u(10) = 0$, which alternative below best describes the utility function $u(x)$? [9]

a) $\log(x)$
b) $\log(x-1)$
c) $\log(x+9)$
d) $\log(x-9)$
e) none of the above

Solution




$$\begin{aligned} \text{Set} \qquad u(10) &= 0 \\ \log(10-a) &= 0 = \log(1) \\ 10 - a &= 1 \\ a &= 9 \end{aligned}$$

That is, $u(x) = \log(x-9)$.

9. (*1 mark*) Let $p = P(s_1)$. If $p = \frac{1}{2}$, which of the following statements is correct? [10]

a) $a_1$ has greater expected dollar value than $a_2$
b) $a_2$ has greater expected dollar value than $a_1$
c) both actions have the same expected dollar value
d) $a_1$ is dominated
e) none of the above

Solution

c)—both actions have the same expected dollar value.

The expected monetary value for $a_1$ is:

$$\begin{aligned} V_{\$}(a_1) &= \tfrac{1}{2}(10) + \tfrac{1}{2}(50) \\ &= \tfrac{1}{2}(10 + 50) \\ &= 30 \end{aligned}$$

The expected monetary amount for $a_2$ is:

$$\begin{aligned} V_{\$}(a_2) &= \tfrac{1}{2}(40) + \tfrac{1}{2}(20) \\ &= \tfrac{1}{2}(40 + 20) \\ &= 30 \end{aligned}$$

The two actions have the same expected dollar value.

10. (*2 marks*) For $p = \frac{1}{2}$, which of the following statements is true? [12]

a) A prefers $a_1$ to $a_2$
b) A prefers $a_2$ to $a_1$
c) A is indifferent between the two actions
d) A prefers neither action
e) none of the above

Solution

b)—A prefers $a_2$ to $a_1$.

The agent's preferences are determined by its utilities, not the dollar values. Actions are measured according to their *Bayes* utility values (*i.e.*, expected utilities).

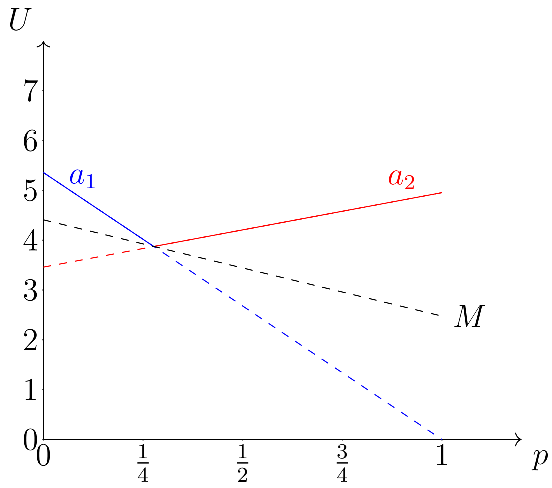
The decision table with approximate utilities is shown below (note that calculating the precise utilities is not required at this stage):

| | $s_1$ | $s_2$ |
|---|---|---|
| $a_1$ | 0 | 5.4 |
| $a_2$ | 5 | 3.5 |

where, for example, $3.5 \approx u(20) = \log_2(20 - 9) = \log_2(11)$.

Below are plots of the pure actions' *Bayes* values against $p$:

| | $s_1$ | $s_2$ | $U$ |
|---|---|---|---|
| $a_1$ | 0 | 5.4 | $5.4 - 5.4p$ |
| $a_2$ | 5 | 3.5 | $3.5 + 1.5p$ |




Inspection of the plots shows that for $p = \frac{1}{2}$, action $a_2$ has greater utility.

Analytically:

The utility for $a_1$ is:

$$\begin{aligned} U(a_1) &= \tfrac{1}{2}\log_2(1) + \tfrac{1}{2}\log(41) \\ &= \tfrac{1}{2}\log_2(41) \end{aligned}$$

The utility of $a_2$ is:

$$\begin{aligned} U(a_2) &= \tfrac{1}{2}\log_2(11) + \tfrac{1}{2}\log_2(31) \\ &= \tfrac{1}{2}(\log_2(11) + \log_2(31)) \\ &= \tfrac{1}{2}\log_2(11 \times 31) \end{aligned}$$

Because $11 \times 31 > 41$ and the log function is an increasing function, this suffices to ensure that $U(a_2) > U(a_1)$. The above calculations confirm that for $p = \frac{1}{2}$ agent A prefers $a_2$ to $a_1$.

11. (*1 mark*) For which value(s) of $p$ would A be indifferent between the two actions? [13]

   a) $p = 0$
   b) $0 < p \leqslant \frac{1}{4}$
   c) $\frac{1}{4} < p \leqslant \frac{1}{2}$
   d) $\frac{1}{2} < p < \frac{3}{4}$
   e) $\frac{3}{4} \leqslant p$

Solution

c)—$\frac{1}{4} < p \leqslant \frac{1}{2}$

Set $U(a_1) = U(a_2)$

$$\begin{aligned} p\log(1) + (1-p)\log(41) &= p\log(31) + (1-p)\log(11) \\ \log(41) - p\log(41) &= \log(11) + p(\log(31) - \log(11)) \\ \log(41) - \log(11) &= p(\log(41) + \log(31) - \log(11)) \\ \log(41/11) &= p\log(41/11 \times 31) \\ p &= \frac{\log(41/11)}{\log(41/11 \times 31)} \approx 0.28 > \tfrac{1}{4} \end{aligned}$$

That is $\frac{1}{4} < p \leqslant \frac{1}{2}$.[2]

12. (*1 mark*) For $p = \frac{1}{2}$, the certainty equivalent of $a_1$ is closest to ... [14]

a) \$0
b) \$10
c) \$15
d) \$25
e) \$45

Solution

c)—\$15.

The certainty equivalent in this case is the dollar amount for which the agent would be indifferent to the action; *i.e.*, dollar amount $\$x_c^{a_1}$ such that $u(x_c^{a_1}) = U(a_1)$.

$$\begin{aligned} \log(x_c^{a_1} - 9) &= \tfrac{1}{2}(0) + \tfrac{1}{2}\log(41) \\ \log(x_c^{a_1} - 9) &= \tfrac{1}{2}\log(41) = \log(\sqrt{41}) \\ x_c^{a_1} - 9 &= \sqrt{41} \\ x_c^{a_1} &= \sqrt{41} + 9 \end{aligned}$$

But $6^2 = 36 < 41 < 49 = 7^2$, which gives $\sqrt{41} \approx 6.4$. Therefore, $15 < x_c^{a_1} < 16$. More precisely, $x_c^{a_1} \approx \$15.40$.

[2] Because $\log_2 x = \log_2 10 \times \log_{10} x$, a ratio of logarithms is independent of the base, so any base (*e.g.*, 10) may be used for this calculation.

13. (*1 mark*) For $p = \frac{1}{2}$, the certainty equivalent of $a_2$ is closest to ... [15]

a) \$0
b) \$10
c) \$15
d) \$25
e) \$45

Solution

d)—\$25.

Setting $u(x_c^{a_2}) = U(a_2)$:

$$\begin{aligned}
\log(x_c^{a_2} - 9) &= \tfrac{1}{2}\log(31) + \tfrac{1}{2}\log(11) \\
\log(x_c^{a_2} - 9) &= \tfrac{1}{2}\log(31 \times 11) \\
\log(x_c - 9) &= \log(\sqrt{31 \times 11}) \\
x_c^{a_2} - 9 &= \sqrt{31 \times 11} \\
x_c^{a_2} &= \sqrt{31 \times 11} + 9
\end{aligned}$$

Because the utility curve is concave down, the graph makes it clear that $x_c^{a_2} < V_\$(a_2) = 30$. Moreover, $121 = 11^2 < 31 \times 11$. Therefore, $\sqrt{31 \times 11} + 9 > 11 + 9 = 20$. It follows that $20 < x_c^{a_2} < 30$, which is closer to \$25 than any other value listed.

More precisely, $x_c^{a_2} \approx \$27$.

14. (*1 mark*) For $p = \frac{1}{2}$, what is the approximate value of the risk premium of $a_1$? [16]

a) \$0
b) $-$\$10
c) $-$\$6
d) \$15
e) \$20

Solution

d)—\$15.

Since $x_c^{a_1} \approx \$15$ and $V_\$(a_1) = \$30$. The risk premium is the difference between the expected monetary value of the lottery/action and the agent's dollar value of the lottery/action; *i.e.*, the agent's certainty equivalent of the lottery. That is, $rp_{a_1} = V_\$(a_1) - x_c^{a_1} \approx \$30 - \$15 = \$15$.

Note that because the curve is concave down the risk premium must be positive.

15. (*1 mark*) For $p = \frac{1}{2}$, what is the approximate value of the risk premium of $a_2$? [17]

a) \$0
b) $-\$10$
c) $-\$3$
d) \$3
e) \$10

Solution
d)—\$3.

Since $x_c^{a_2} \approx \$27$ and $V_\$(a_2) = \$30$. That is, $rp_{a_1} = V_\$(a_1) - x_c^{a_1} = \$30 - \$27 = \$3$.

For agent B all we know is that she is indifferent between a certain \$20 and 10% chance of \$50 and 90% of \$10. She is also indifferent between \$40 and the lottery $[\frac{6}{10} : \$50 | \frac{4}{10} : \$10]$.

Assume in the following questions that $p = P(s_1) = \frac{1}{2}$.

16. (*1 mark*) Which of the following statements is true? [18]

a) B prefers $a_1$ to $a_2$
b) B prefers $a_2$ to $a_1$
c) B is indifferent between the two actions
d) B prefers neither action
e) none of the above

Solution
a)—B prefers $a_1$ to $a_2$.

The agent's preferences are determined by its utilities, not the dollar values. Actions are measured according to their *Bayes* value, which in this case is the expected utility.

We could use normalised utilities in the interval [0,1], but any affine (linearly scaled) transformation will also do. We scale by a factor of 10 to give utilities in the interval [0,10] which allow us to work with whole numbers.

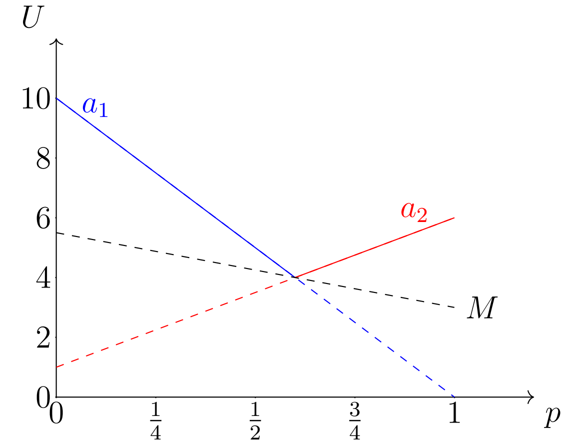
In terms of utility values we have the following decision table:

| | $s_1$ | $s_2$ |
|---|---|---|
| $a_1$ | 0 | 10 |
| $a_2$ | 6 | 1 |

where, for example, $1 \approx u(20) = 10 \times \frac{1}{10}$.

Below are plots of the pure actions' *Bayes* values against $p$:

| | $s_1$ | $s_2$ | $V_B$ |
|---|---|---|---|
| $a_1$ | 0 | 10 | $10 - 10p$ |
| $a_2$ | 6 | 1 | $1 + 5p$ |




Inspection of the plot shows that for $p = \frac{1}{2}$, action $a_1$ has greater utility.

Analytically:

The utility for $a_1$ is:

$$\begin{aligned} U(a_1) &= \tfrac{1}{2}(0) + \tfrac{1}{2}(10) \\ &= \tfrac{1}{2}(10) \\ &= 5 \end{aligned}$$

The utility of $a_2$ is:

$$\begin{aligned} U(a_2) &= \tfrac{1}{2}(6) + \tfrac{1}{2}(1) \\ &= \tfrac{1}{2}(7) \\ &= 3.5 \end{aligned}$$

Therefore, because $U(a_1) > U(a_2)$, agent A prefers $a_1$ to $a_2$.

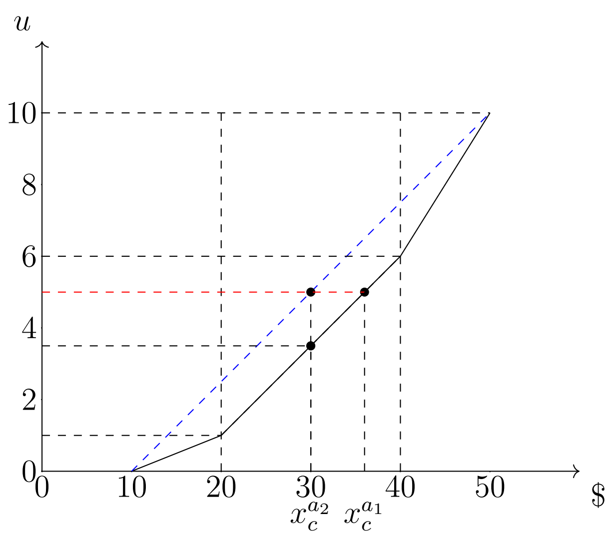
17. (*2 marks*) Assume that utilities for dollar values other than those given can be linearly interpolated. For a utility scale in the range $[0, 10]$, which expression below best represents $u(x)$ for $\$20 \leqslant x \leqslant \$40$? [20]

a) $x - 10$
b) $\frac{1}{10}x - 1$
c) $\frac{2}{5}x - 10$
d) $4 - 4x$
e) $\frac{1}{4}x - 4$

Solution

e)—$\frac{1}{4}x - 4$.

Agent B's utility is plotted below:




For $x \leqslant 10$, $u(x) = 0$.

For $10 < x \leqslant 20$:

$$
\begin{aligned}
\frac{u(x) - 0}{x - 10} &= \frac{1 - 0}{20 - 10} \\
\frac{u(x)}{x - 10} &= \frac{1}{10} \\
u(x) &= \tfrac{1}{10}(x - 10) \\
&= \tfrac{1}{10}x - 1
\end{aligned}
$$

For $20 < x \leqslant 40$:

$$\begin{aligned}
\frac{u(x)-1}{x-20} &= \frac{6-1}{40-20} \\
\frac{u(x)-1}{x-20} &= \frac{5}{20} \\
u(x) &= \tfrac{5}{20}(x-20)+1 \\
u(x) &= \tfrac{1}{4}(x-20)+1 \\
&= \tfrac{1}{4}x-4
\end{aligned}$$

For $40 < x \leqslant 50$:

$$\begin{aligned}
\frac{u(x)-6}{x-40} &= \frac{10-6}{50-40} \\
\frac{u(x)-6}{x-40} &= \frac{4}{10} \\
u(x) &= \tfrac{4}{10}(x-40)+6 \\
u(x) &= \tfrac{2}{5}(x-40)+6 \\
&= \tfrac{2}{5}x-10
\end{aligned}$$

For $x \geqslant 50$, $u(x) = 10$.

In summary:

$$u(x) = \begin{cases} 0 & \text{if } x \leqslant 10 \\ \frac{1}{10}x-1 & \text{if } 10 < x \leqslant 20 \\ \frac{1}{4}x-4 & \text{if } 20 < x \leqslant 40 \\ \frac{2}{5}x-10 & \text{if } 40 < x \leqslant 50 \\ 10 & \text{if } x \geqslant 50 \end{cases}$$

18. (*1 mark*) The certainty equivalent of $a_1$ is closest to ... [21]

a) \$20
b) \$25
c) \$30
d) \$35
e) \$40

Solution

d)—\$35.

The certainty equivalent in this case is the dollar amount with which the agent would be indifferent to the action; *i.e.*, dollar amount $\$x_c^{a_1}$ such that $u(x_c^{a_1}) = U(a_1)$. Note that this value must be in the range \$20 to \$40. So $u(x_c^{a_1}) = \frac{1}{4}x_c^{a_1} - 4$.

$$\begin{aligned} \tfrac{1}{4}x_c^{a_1} - 4 &= 5 \\ \tfrac{1}{4}x_c^{a_1} &= 9 \\ x_c^{a_1} &= 36 \end{aligned}$$

Therefore, $x_c^{a_1} = \$36 \approx \$35$.

19. (*1 mark*) The certainty equivalent of $a_2$ is closest to ... [22]

    a) \$0
    b) \$10
    c) \$15
    d) \$25
    e) \$45

    Solution
    d)–\$25.

    As the plot of $u(x)$ suggests, in this case the expected utility, or *Bayes* value, of $a_2$, $U(a_2) = u(V_\$(a_2))$. Setting $u(x_c^{a_2}) = U(a_2) = u(V_\$(a_2))$ yields $x_c^{a_2} = V_\$(a_2) = \$30$.

20. (*1 mark*) What is the approximate value of the risk premium of $a_1$? [23]

    a) \$0
    b) $-\$10$
    c) $-\$6$
    d) \$15
    e) \$20

    Solution
    c)—$-\$6$.

    The risk premium is the amount of the expected dollar value of the lottery/action and the agent's dollar value of the lottery/action; *i.e.*, the agent's certainty equivalent of the lottery. That is, $rp_{a_1} = V_\$(a_1) - x_c^{a_1}$. We've determined that $x_c^{a_1} = \$36$ and $V_\$(a_1) = \$30$. Therefore $rp_{a_1} = \$30 - \$36 = -\$6$.

    Notice that B's utility curve is concave up; *i.e.*, she is risk seeking. The risk premium must be non-positive.

21. (*1 mark*) What is the approximate value of the risk premium of $a_2$? [24]

a) \$0
b) $-\$10$
c) $-\$3$
d) \$3
e) \$10

Solution
a)—\$0.

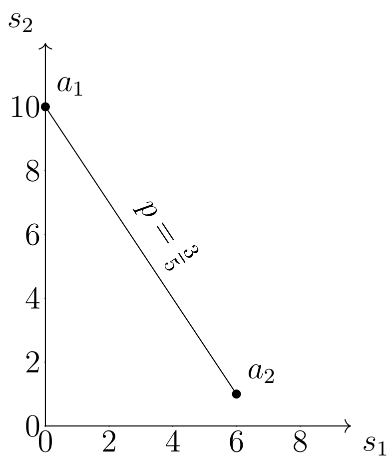
Since $x_c^{a_2} = \$30$ and $V_\$(a_2) = \$30$. $rp_{a_2} = \$0$. The linear interpolation means that over interval [20, 40] agent B is locally risk neutral, even though she is risk seeking over the interval [10, 50].

22. (*1 mark*) For which value of $p = P(s_1)$ would B be indifferent between the two actions? [25]

a) $\frac{1}{10}$
b) $\frac{1}{5}$
c) $\frac{2}{5}$
d) $\frac{3}{5}$
e) $\frac{7}{10}$

Solution
d)—$p = \frac{3}{5}$.

| | $s_1$ | $s_2$ | $V_B$ |
|---|---|---|---|
| $a_1$ | 0 | 10 | $10 - 10p$ |
| $a_2$ | 6 | 1 | $1 + 5p$ |




The plot above shows that the two action sit on the same *Bayes* indifference curve for $p = \frac{3}{5}$.

Analytically:

$$\begin{aligned} U(a_1) &= U(a_2) \\ 10 - 10p &= 1 + 5p \\ 9 &= 15p \\ p &= \tfrac{3}{5} \end{aligned}$$

---

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*Questions 23 to 26 refer to the problem below.*

Two friends agree to "meet at *the* park", but subsequently each realises that there are two identical parks (A and B) nearby, to which the other might've referred. Each friend has to decide, independently, to which park to go to meet their friend. The game is modelled by the matrix below.

| | A | B |
|---|---|---|
| A | 1, 1 | 0, 0 |
| B | 0, 0 | 1, 1 |

23. (*1 mark*) How many plays survive simplification by elimination of dominated strategies? [26]

    a) none
    b) one
    c) two
    d) three
    e) four

    Solution
    e)—all four.

    No strategies are dominated for either player; all four plays survive simplification.

24. (*1 mark*) How many equilibrium points does this game have? [27]

    a) none
    b) one
    c) two
    d) three
    e) four

Solution

c)—two.

Best responses for the column (*) and row (*) players are shown below:

| | A | B |
|---|---|---|
| A | $_{*}1, 1^{*}$ | $0, 0$ |
| B | $0, 0$ | $_{*}1, 1^{*}$ |

There are two equilibrium plays, (A, A) and (B, B), as shown above. To see why (A, A) is an equilibrium play notice that if the two friends had met at park A on several previous occasions, and hence believe that the other will meet them there next time, then each has no independent individual incentive to change their plan on subsequent occasions; *i.e.*, if one were to switch to B and the other remained with A, then the one who changed would fare no better (worse, in this case). A similar argument applies to (B, B).

25. (*1 mark*) How many Pareto optimal plays are there in this game? [28]

   a) none
   b) one
   c) two
   d) three
   e) four

Solution

c)—the two equilibrium plays are the only Pareto optimal plays.

26. (*1 mark*) Suppose Alice believes that the probability of Bob going to park A is $p = P_B(\text{A})$. Which value of $p$ would leave Alice indifferent between going to either park? [29]

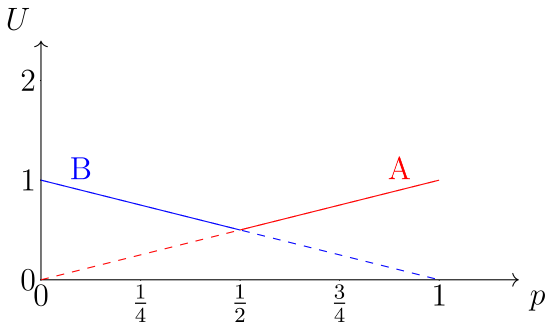
   a) $p = 0$
   b) $p = \frac{1}{4}$
   c) $p = \frac{1}{3}$
   d) $p = \frac{1}{2}$
   e) for any $p \in [0, 1]$

Solution

d)—$p = \frac{1}{2}$.

Let $p = P_B(\text{A})$.

| | A | B | $U$ |
|---|---|---|---|
| A | $1,1$ | $0,0$ | $p$ |
| B | $0,0$ | $1,1$ | $1-p$ |




From the plot, or analytically:

$$
\begin{aligned}
U(\mathrm{A}) &= U(\mathrm{B}) \\
p &= 1-p \\
p &= \tfrac{1}{2}
\end{aligned}
$$

---

---

*Questions 27 to 30 refer to problem below.*

Alice and Bob have agreed to meet for lunch. Alice prefers restaurant A and Bob prefers restaurant B. Unfortunately, they didn't specify at which restaurant they were to meet. This 'game' is modelled by the following game matrix.

| | a | b |
|---|---|---|
| A | $2,1$ | $0,0$ |
| B | $0,0$ | $1,2$ |

27. (*1 mark*) How many plays survive simplification by elimination of dominated strategies? [30]

    a) none
    b) one
    c) two
    d) three
    e) four

**Solution**

e)—all four.

No strategies are dominated for either player; all four plays survive.

28. (*1 mark*) How many equilibrium points does this game have? [31]

    a) none
    b) one
    c) two
    d) three
    e) four

**Solution**

c)—two.

Best responses for the column (*) and row (*) players are shown below:

| | a | b |
|---|---|---|
| A | $\overset{*}{2}, 1^{*}$ | $0, 0$ |
| B | $0, 0$ | $\overset{*}{1}, 2^{*}$ |

There are two equilibrium plays, (A, a) and (B, b), as shown above. The first is better for Alice and the second for Bob.

29. (*1 mark*) How many Pareto optimal plays are there in this game? [32]

    a) none
    b) one
    c) two
    d) three
    e) four

**Solution**

c)—the two equilibrium plays are the Pareto optimal plays.

30. (*1 mark*) Suppose Alice believes that the probability of Bob going to restaurant A is $p = P_B(\mathrm{a})$. Which value of $p$ would leave Alice indifferent between going to either restaurant? [33]

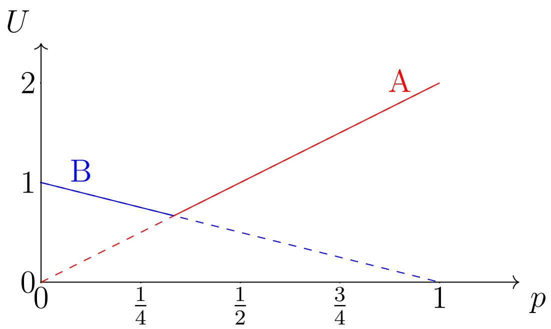
a) $p = 0$
b) $p = \frac{1}{4}$
c) $p = \frac{1}{3}$
d) $p = \frac{1}{2}$
e) for any $p \in [0, 1]$

Solution

c)—$p = \frac{1}{3}$.

Let $p = P_B(\mathrm{a})$.

| | a | b | $U$ |
|---|---|---|---|
| A | $2, 1$ | $0, 0$ | $2p$ |
| B | $0, 0$ | $1, 2$ | $1 - p$ |




From the plot, or analytically:

$$\begin{aligned} U(\mathrm{A}) &= U(\mathrm{B}) \\ 2p &= 1 - p \\ p &= \tfrac{1}{3} \end{aligned}$$

---

*Questions 31 to 33 refer to the problem below.*

Alice and Bob, who are tennis partners, agreed to play this weekend. There are two tennis courts near them, A and B, but they didn't specify at which court they would play. Court A is closer to both. This 'game' is modelled by the following game matrix.

| | a | b |
|---|---|---|
| A | $2, 2$ | $0, 0$ |
| B | $0, 0$ | $1, 1$ |

31. (*1 mark*) How many equilibrium points does this game have? [34]

    a) none
    b) one
    c) two
    d) three
    e) four

Solution
c)—two.

Best responses for the column (*) and row (*) players are shown below:

| | a | b |
|---|---|---|
| A | $\overset{*}{}2, 2^{*}$ | $0, 0$ |
| B | $0, 0$ | $\overset{*}{}1, 1^{*}$ |

There are two equilibrium plays, (A, a) and (B, b), as shown above.

32. (*1 mark*) How many Pareto optimal plays are there in this game? [35]

    a) none
    b) one
    c) two
    d) three
    e) four

Solution
b)—one.

Play (A, a)—an equilibrium play—is the unique Pareto optimal play.

33. (*1 mark*) Suppose Alice believes that the probability of Bob going to court A is $p = P_B(\mathrm{a})$. Which value of $p$ would leave Alice indifferent between going to either court? [36]

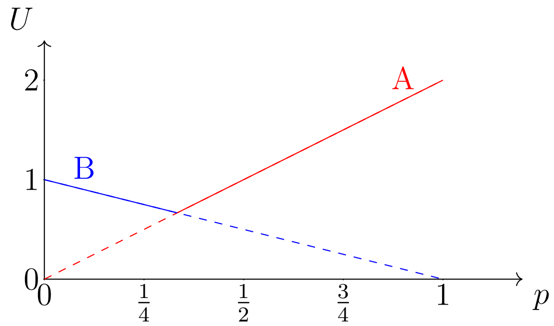
a) $p = 0$
b) $p = \frac{1}{4}$
c) $p = \frac{1}{3}$
d) $p = \frac{1}{2}$
e) for any $p \in [0, 1]$

Solution

c)—$p = \frac{1}{3}$.

Let $p = P_B(\mathrm{a})$.

| | a | b | $U$ |
|---|---|---|---|
| A | $2, 2$ | $0, 0$ | $2p$ |
| B | $0, 0$ | $1, 1$ | $1 - p$ |




From the plot, or analytically:

$$
\begin{aligned}
U(\mathrm{A}) &= U(\mathrm{B}) \\
2p &= 1 - p \\
p &= \tfrac{1}{3}
\end{aligned}
$$

---

---

*Questions 34 to 36 refer to problem below.*

Alice sells magazines. She advertises her business by sending out promotional leaflets to her customers. She has printed three types of leaflet (A, B, or C), but she can only afford to send one leaflet per customer. Her market—the customers to which she sells her magazines—is segmented into two categories, $s_1$ and $s_2$.

Her average sales, per 100 leaflets sent, are shown in the table below.

| | $s_1$ | $s_2$ |
|---|---|---|
| A | 0 | 19 |
| B | 15 | 5 |
| C | 10 | 12 |

34. (*1 mark*) For the decision problem described by the table above, Alice's guaranteed minimum average sales per hundred leaflets, if she didn't know to which segment her customers belong when she sent out her leaflets, is: [37]

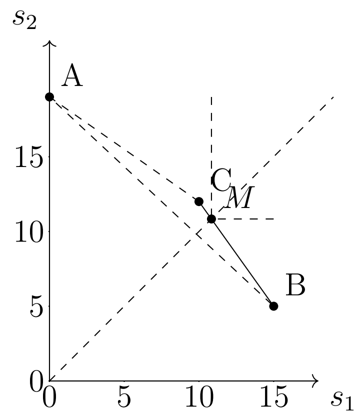
   a) $\frac{65}{12}$
   b) $\frac{75}{12}$
   c) $\frac{85}{12}$
   d) $\frac{95}{12}$
   e) none of the above

Solution

e)—none of the above.

The decision table looks as follows, where the entries represent average sales per hundred leaflets:

| | $s_1$ | $s_2$ | min |
|---|---|---|---|
| A | 0 | 19 | 0 |
| B | 15 | 5 | 5 |
| C | 10 | 12 | 10 |




The *Maximin* action is a mixture of B and C. Note that the *Maximin* value must be greater than 10, as the *Maximin* (mixed) action must lie to the right of C.

Analytically, using basic linear algebra:

$$
\begin{aligned}
\frac{y-5}{x-15} &= \frac{12-5}{10-15} = \frac{7}{-5} \\
5(5-y) &= 7(x-15) \\
25-5x &= 7x-105 \qquad \text{setting } x=y \\
12x &= 130 \\
x &= \tfrac{130}{12}
\end{aligned}
$$

35. (*1 mark*) Let $p = P(s_1)$ be the probability that a customer belongs to segment $s_1$. If $p = \frac{7}{10}$, which leaflet would be most profitable? [38]

a) A
b) B
c) C
d) a non-pure mixture of A and C
e) none of the above

Solution
b)—B

$$
\begin{aligned}
V_B(\mathrm{A}) &= p(0) + (1-p)(19) = 19(1-p) \\
V_B(\mathrm{B}) &= p(15) + (1-p)(5) = 10p + 5 \\
V_B(\mathrm{C}) &= p(10) + (1-p)(12) = 12 - 2p
\end{aligned}
$$

For $p = \frac{7}{10}$, these give:

$$
\begin{aligned}
V_B(\mathrm{A}) &= 19(1-\tfrac{7}{10}) = 19(\tfrac{3}{10}) = \tfrac{57}{10} = 5.7 \\
V_B(\mathrm{B}) &= 10(\tfrac{7}{10}) + 5 = 12 \\
V_B(\mathrm{C}) &= 12 - 2(\tfrac{7}{10}) = 12 - \tfrac{7}{5} = \tfrac{60-7}{5} = 10.6
\end{aligned}
$$

It follows that B has the highest *Bayes* value.

Alternatively, the same result can be obtained geometrically: setting $p = \frac{7}{10}$ would give steep gradients to the *Bayes* indifference curves. This would place B on the highest indifference curve.

36. (*2 marks*) Assume $p = \frac{7}{10}$, as in the previous question. Suppose Alice could hire an oracle who could predict to which segment each customer belongs with complete accuracy. If each unit sold makes a profit of \$10, what is the highest rate, in dollars per 100 leaflets/customers, which Alice should pay for the oracle's service? [40]

a) \$29
b) \$42
c) \$23
d) \$37
e) none of the above

Solution
b)—\$42.

If Alice doesn't know which customers belong to which segment the best she can do is choose some pure or mixed strategy. This strategy will, for example, sometimes send leaflet A to customers in $s_1$, which will lose her sales (the best response in $s_1$ would be B).

Because the oracle is able to correctly identify to which segment each customer belongs, by using this information, Alice would have access to a strategy which chooses the best response for each customer (state); that is, choose B for the 70% of customers which belong to $s_1$, and A for the remaining 30%. The payoff for the best response in $s_1$ would be the maximum value in the first column: *i.e.*, $\max\{0, 15, 10\} = 15$. Similarly, the payoff in $s_2$ would be $\max\{19, 5, 12\} = 19$.

The two cases are represented in Figure 1. The tree on the right represents the

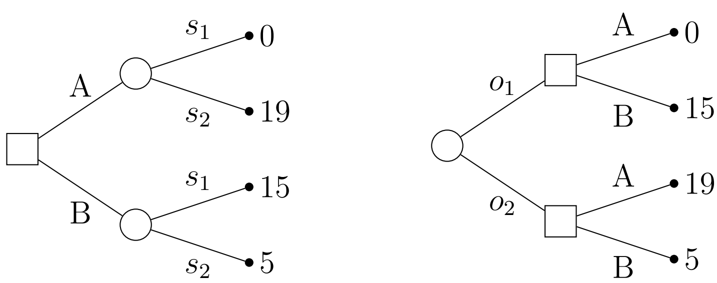



Figure 1: Decisions without and with the oracle for no fee.

situation after the oracle has been hired (assuming no fee). I have replaced $s_1$ and $s_2$ with $o_1$ and $o_2$ respectively, to represent the oracle's *prediction*. Note that in the tree on the right, at each of Alice's decision points she knows which of $o_1$ or $o_2$ has occurred.

Because the oracle is a perfect predictor, in this case the $o$ and $s$ events are equivalent, but if the oracle were not a perfect predictor a distinction would have to be made between the oracle's predictions ($o_i$) and the actual states ($s_i$): *e.g.*, the oracle might predict $o_1$ but $s_2$ occurs. One way to represent this on the tree on the right would be to replace the leaf nodes with chance nodes with branches for each of $s_1$ and $s_2$.

Let $\alpha$ be the strategy associated with hiring the oracle (excluding the oracle's fee).

$$\begin{aligned} V_B(\alpha) &= p(\max\{0, 15, 10\}) + (1-p)(\max\{19, 5, 12\}) \\ &= \tfrac{7}{10}(15) + \tfrac{3}{10}(19) \\ &= \tfrac{105}{10} + \tfrac{57}{10} = \tfrac{162}{10} = 16.2 \end{aligned}$$

That is, the oracle would average 16.2 sales per 100 leaflets sent. At \$10 per sale, this amounts to a profit of \$162. The *Bayes* value without the oracle would be that of B (12), resulting in a profit of $12 \times \$10 = \$120$ per 100 leaflets. Therefore, Alice should be willing to offer the oracle a rate of up to $\$162 - \$120 = \$42$ per 100 leaflets (\$0.42 per leaflet). For a fee of \$42 per 100 leaflets she'd make no gains by hiring the oracle.

---

*Questions 37 to 43 refer to zero-sum game matrix below.*

| | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
|---|---|---|---|---|
| $a_1$ | 4 | 2 | 5 | 2 |
| $a_2$ | 2 | 1 | $-1$ | $-2$ |
| $a_3$ | 3 | 2 | 4 | 2 |
| $a_4$ | $-6$ | 0 | 6 | 1 |

37. (*1 mark*) Which plays by the row player are best responses to column player's $b_3$? [41]

   a) $a_1$ only
   b) $a_2$ only
   c) $a_3$ only
   d) $a_4$ only
   e) there are multiple best responses

Solution

d)—$a_4$ is the unique best response to $b_3$.

Best responses are shown below:

| | $b_1$ | $b_2$ | $b_3$ | $b_4$ | min |
|---|---|---|---|---|---|
| $a_1$ | $4^{*}$ | $2^{**}$ | $5$ | $2^{**}$ | 2 |
| $a_2$ | $2$ | $1$ | $-1$ | $-2^{*}$ | $-2$ |
| $a_3$ | $3$ | $2^{**}$ | $4$ | $2^{**}$ | 2 |
| $a_4$ | $-6^{*}$ | $0$ | $6^{*}$ | $1$ | $-6$ |
| max | 4 | 2 | 6 | 2 | |

38. (*1 mark*) Which plays by the row player are best responses to column player's $b_2$? [42]

    a) $a_1$ only
    b) $a_2$ only
    c) $a_3$ only
    d) $a_4$ only
    e) there are multiple best responses

Solution

e)—there are multiple best responses: $a_1$ and $a_3$.

39. (*1 mark*) Which plays by the row player are best responses to column player's $b_1$? [43]

    a) $a_1$ only
    b) $a_2$ only
    c) $a_3$ only
    d) $a_4$ only
    e) there are multiple best responses

Solution

a)—$a_1$ is the unique best response to $b_1$.

40. (*1 mark*) Which plays by the column player are best responses to row player's $a_2$? [44]

a) $b_1$ only
b) $b_2$ only
c) $b_3$ only
d) $b_4$ only
e) there are multiple best responses

Solution
d)—$b_4$ is the unique best response to $a_2$.

41. (*1 mark*) How many saddle points does this game have? [45]

a) none
b) one
c) two
d) three
e) four

Solution
e)—there are four saddle points.

The *Maximin* and *miniMax* values are shown below:

| | $b_1$ | $b_2$ | $b_3$ | $b_4$ | min |
|---|---|---|---|---|---|
| $a_1$ | 4 | ${}_{*}2^{*}$ | 5 | ${}_{*}2^{*}$ | 2 |
| $a_2$ | 2 | 1 | $-1$ | $-2$ | $-2$ |
| $a_3$ | 3 | ${}_{*}2^{*}$ | 4 | ${}_{*}2^{*}$ | 2 |
| $a_4$ | $-6$ | 0 | 6 | 1 | $-6$ |
| max | 4 | 2 | 6 | 2 | |

Best responses coincide for $(a_1, b_2)$, $(a_1, b_4)$, $(a_3, b_2)$, and $(a_3, b_4)$. These are the four saddle points; *i.e.*, equilibrium plays in zero-sum games.

42. (*1 mark*) After simplification, how many strategies are left for the row player? [46]

a) none
b) one
c) two
d) three
e) four

Solution
b)—only one strategy is left for the row player: $a_1$ (see below).

| | ~~$b_1$~~ | $b_2$ | ~~$b_3$~~ | ~~$b_4$~~ |
|---|---|---|---|---|
| $a_1$ | 4 | 2 | 5 | 2 |
| ~~$a_2$~~ | ~~2~~ | ~~1~~ | ~~$-1$~~ | ~~$-2$~~ |
| ~~$a_3$~~ | ~~3~~ | ~~2~~ | ~~4~~ | ~~2~~ |
| ~~$a_4$~~ | ~~$-6$~~ | ~~0~~ | 6 | 1 |

43. (*1 mark*) After simplification, how many strategies are left for the column player? [47]

a) none
b) one
c) two
d) three
e) four

Solution
b)—only one pure strategy: $b_2$

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*Questions 44 to 47 refer to the game matrix below.*

| | $b_1$ | $b_2$ | $b_3$ |
|---|---|---|---|
| $a_1$ | $2, 6$ | $0, 4$ | $4, 4$ |
| $a_2$ | $3, 3$ | $0, 0$ | $1, 5$ |
| $a_3$ | $1, 1$ | $3, 5$ | $2, 3$ |

44. (*1 mark*) Which plays by the row player are best responses to the column player's $b_1$? [48]

a) $a_1$ only
b) $a_2$ only
c) $a_3$ only
d) there are two best responses
e) there are more than two best responses

Solution
b)—$a_2$.

The best responses are shown below:

| | $b_1$ | $b_2$ | $b_3$ |
|---|---|---|---|
| $a_1$ | $2,6^{*}$ | $0,4$ | $_{*}4,4$ |
| $a_2$ | $_{*}3,3$ | $0,0$ | $1,5^{*}$ |
| $a_3$ | $1,1$ | $_{*}3,5^{*}$ | $2,3$ |

The best response to $b_1$ is $a_2$.

45. (*1 mark*) Which plays by the column player are best responses to the row player's $a_3$? [49]

a) $b_1$ only
b) $b_2$ only
c) $b_3$ only
d) there are two best responses
e) there are more than two best responses

Solution
b)—the best response to $a_3$ is $b_2$.

46. (*2 marks*) Which plays by the row player are best responses to the column player's mixed action $\frac{1}{3}b_1\frac{1}{3}b_2\frac{1}{3}b_3$? [51]

a) $a_1$ only
b) $a_2$ only
c) $a_3$ only
d) there are two best responses
e) there are more than two best responses

Solution

d)—there are two best responses.

The *Bayes* values/utilities of $\frac{1}{3}b_1\frac{1}{3}b_2\frac{1}{3}b_3$ are:

$$\begin{aligned} U(a_1) &= \tfrac{1}{3}(2) + \tfrac{1}{3}(0) + \tfrac{1}{3}(4) \\ &= \tfrac{1}{3}(6) = 2 \\ U(a_2) &= \tfrac{1}{3}(3) + \tfrac{1}{3}(0) + \tfrac{1}{3}(1) \\ &= \tfrac{1}{3}(4) = \tfrac{4}{3} \\ U(a_3) &= \tfrac{1}{3}(1) + \tfrac{1}{3}(3) + \tfrac{1}{3}(2) \\ &= \tfrac{1}{3}(6) = 2 \end{aligned}$$

Therefore, $a_1$ and $a_3$ are the two best responses.

47. (*1 mark*) Which plays by the column player are best responses to the row player's mixed action $\frac{1}{2}a_1\frac{1}{4}a_2\frac{1}{4}a_3$? [52]

a) $b_1$ only
b) $b_2$ only
c) $b_3$ only
d) there are two best responses
e) there are more than two best responses

Solution

d)—there are two best responses.

The *Bayes* values/utilities of $\frac{1}{2}a_1\frac{1}{4}a_2\frac{1}{4}a_3$ are:

$$\begin{aligned} U(b_1) &= \tfrac{1}{2}(6) + \tfrac{1}{4}(3) + \tfrac{1}{4}(1) \\ &= \tfrac{1}{4}(16) = 4 \\ U(b_2) &= \tfrac{1}{2}(4) + \tfrac{1}{4}(0) + \tfrac{1}{4}(5) \\ &= \tfrac{1}{4}(13) = \tfrac{13}{4} \\ U(b_3) &= \tfrac{1}{2}(4) + \tfrac{1}{4}(5) + \tfrac{1}{4}(3) \\ &= \tfrac{1}{4}(16) = 4 \end{aligned}$$

Therefore, $b_1$ and $b_3$ are the two best responses.

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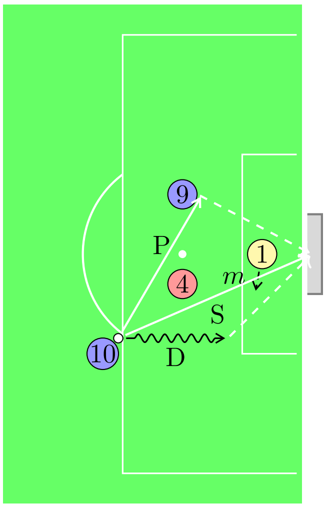
*Questions 48 to 53 refer to the problem below.*




Consider the football situation shown above, where Alice (yellow #10) has three options:

- P pass to her team-mate (yellow #9);
- D dribble closer to goal before shooting; or
- S shoot from where she is.

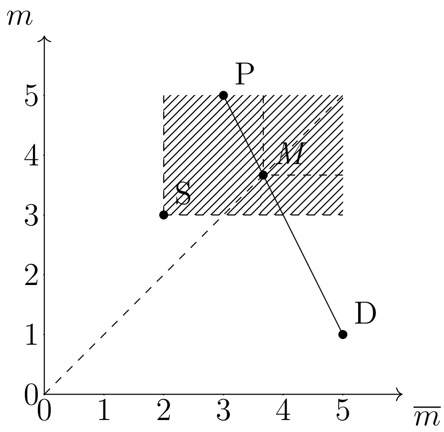
The chances of scoring if Alice passes (P) to her team-mate are 3 in 10. Her chances of scoring by first dribbling closer (D) to goal and then shooting are 5 in 10. Her chances of scoring by shooting from where she is (S) are 2 in 10.

Bob, the goal-keeper (blue #1), can choose to move ($m$) toward the ball as shown to reduce Alice's scoring chances if she dribbles to 1 in 10 at the expense of increasing her scoring chances by passing and shooting respectively to 5 and 3 in 10.

### Solution

The corresponding decision table is shown below:

| | $\overline{m}$ | $m$ | min |
|---|---|---|---|
| P | 3 | 5 | 3 |
| D | 5 | 1 | 1 |
| ~~S~~ | ~~2~~ | ~~3~~ | 2 |
| | 5 | 5 | |




48. (*1 mark*) Which is Alice's *Maximin* pure action? [53]

a) P
b) D
c) S
d) both P and D
e) none of the above

Solution
a)—P.

49. (*1 mark*) Which is Bob's *Maximin* pure action? [54]

a) $\overline{m}$
b) $m$
c) both $\overline{m}$ and $m$
d) neither $\overline{m}$ nor $m$
e) none of the above

Solution
c)—both $\overline{m}$ and $m$.

From the table above, both have the same *Maximin* (*i.e.*, *miniMax* for Bob) value of 5

50. (*2 marks*) How many pure strategy equilibria does this game have? [56]

a) 0
b) 1
c) 2
d) 3
e) none of the above

Solution
a)—0.

The game has no saddle points; *i.e.*, no equilibria.

51. (*2 marks*) Assuming that this situation were repeated many times (*i.e.*, mixed strategies are allowed), the lowest value to which Bob could restrict Alice's best response is: [58]

a) 7 in 10
b) 6 in 10
c) 5 in 10
d) 4 in 10
e) none of the above

Solution
d)—less than 4 in 10.

Let $\mu$ be the proportion of P in the mixture of P and D. Let the mixtures of P and D be represented by points $M = (x, y) = \mu(3, 5) + (1 - \mu)(5, 1) = (5 - 2\mu, 1 + 4\mu)$.

If Bob played optimally, Alice's best response would be her *Maximin* mixed strategy, $M^*$. To find this value set $x = y$ in $M$:

$$5 - 2\mu = 1 + 4\mu$$
$$4 = 6\mu$$
$$\mu = \tfrac{2}{3}$$

So Alice's *Maximin* strategy is $M^* = \frac{2}{3}\mathrm{P}\frac{1}{3}\mathrm{D}$. That is, Alice should pass twice as often as she dribbles. The *Bayes* value of $M^*$ is:

$$V_B(M^*) = 1 + 4(\tfrac{2}{3}) < 1 + 3 = 4$$

This value is consistent with the graph above. That is, Alice's chances of scoring are restricted to being no greater than 4 in 10.

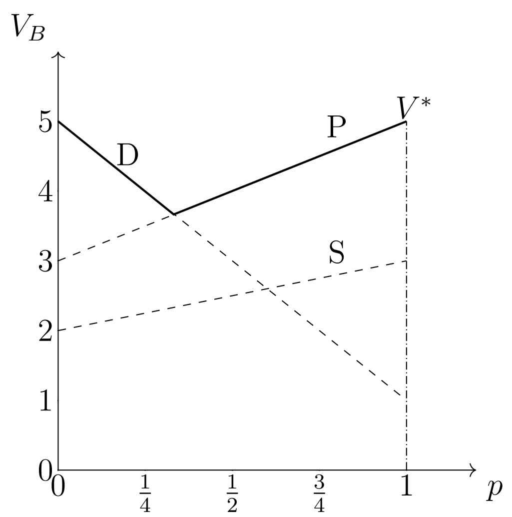
52. (*1 mark*) Let $p = P(m)$ be the probability that the goal-keeper will move. Which value of $p$ would restrict Alice's best response to the least chance of scoring? [59]

a) $p = \frac{1}{3}$
b) $p = \frac{3}{5}$
c) $p = \frac{2}{3}$
d) $p = \frac{2}{5}$
e) none of the above

Solution
a)—$p = \frac{1}{3}$.

The *Bayes* values of Alice's respective actions are given below:

$$\begin{aligned}
V_B(\mathrm{P}) &= (1-p)(3) + p(5) \\
&= 3 + 2p \\
V_B(\mathrm{D}) &= (1-p)(5) + p(1) \\
&= 5 - 4p \\
V_B(\mathrm{S}) &= (1-p)(2) + p(3) \\
&= 2 + p
\end{aligned}$$




The solid line represents the *Bayes* value of Alice's best response for each value of $\mu$.

Alice's best response is least for the value of $p$ which makes the *Bayes* values of P and D equal, as shown in the graph above. Any other value of $p$ would mean that Alice could play either P or D to get a higher value. This $p$ determines the strategy for Bob which results in the least favourable probability distribution for Alice.

Setting $V_B(\mathrm{P}) = V_B(\mathrm{D})$

$$\begin{aligned} 3 + 2p &= 5 - 4p \\ 6p &= 2 \\ p &= \tfrac{1}{3} \end{aligned}$$

Notice from the graph that if $p > \frac{1}{3}$ then Alice could respond with P to get a higher *Bayes* value. Similarly if $p < \frac{1}{3}$ the response D would also have a higher *Bayes* value.

Therefore, Bob—the goal-keeper—should play $\frac{2}{3}\overline{m}\frac{1}{3}m$; *i.e.*, move as shown one-third of the time.

53. (*1 mark*) If mixtures are allowed for both players, which of the following is an equilibrium? [60]

   a) $(\frac{1}{3}\mathrm{P}\frac{2}{3}\mathrm{D}, \frac{1}{3}\overline{m}\frac{2}{3}m)$
   b) $(\mathrm{P}, \overline{m})$
   c) $(\mathrm{D}, \frac{1}{3}\overline{m}\frac{2}{3}m)$
   d) $(\frac{1}{2}\mathrm{P}\frac{1}{2}\mathrm{D}, \frac{2}{3}\overline{m}\frac{1}{3}m)$
   e) none of the above

Solution

e)—none of the above.

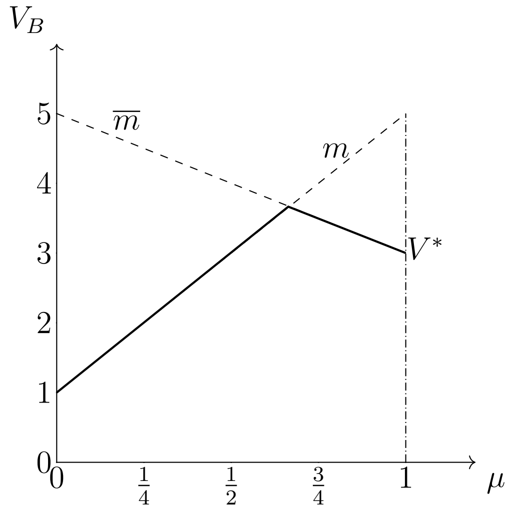
Equilibria in zero-sum games, with or without mixtures, are obtained by combining the *Maximin* strategies of both players; as a player's *Maximin* strategy is the best that the player can do when the other player plays a best response.

Alternatively, from Bob's perspective, the *Bayes* values of Bob's actions when Alice uses a mixed strategy (or given Bob's beliefs about Alice's strategy), excluding eliminated strategies (S in this case), are given below (where $\mu = P(\mathrm{P})$ is the proportion of P in the mixture played by Alice):

$$\begin{aligned} V_B(\overline{m}) &= \mu(3) + (1-\mu)(5) \qquad &\text{(first column)} \\ &= 5 - 2\mu \\ V_B(m) &= \mu(5) + (1-\mu)(1) \qquad &\text{(second column)} \\ &= 1 + 4\mu \end{aligned}$$

Note that these values coincide with the value of Alice's *Maximin* mixed strategy, derived in a previous problem.

The plots are shown, as functions of $\mu$, below, in which the solid line represents Alice's scoring chances given Bob's best response (*i.e.*, the lowest of the two lines) to Alice's mixed strategy:




The plot suggests that Alice can maximise her chances of scoring (*i.e.*, the payoffs) in spite of Bob's response by matching the *Bayes* values of Bob's actions. Therefore:

$$\begin{aligned} V_B(\overline{m}) &= V_B(m) \\ 5 - 2\mu &= 1 + 4\mu \\ 4 &= 6\mu \\ \therefore \quad \mu &= \tfrac{2}{3} \end{aligned}$$

Notice from the graph that if $\mu < \frac{2}{3}$, then Bob's $m$ would reduce Alice's chances of scoring. Similarly, if $\mu > \frac{2}{3}$ Bob could reduce Alice's chances of scoring by playing $\overline{m}$.

Therefore, Alice should play $\frac{2}{3}\text{P}\frac{1}{3}\text{D}$.

Combining this with Bob's best response, the equilibrium pair of strategies is $(\frac{2}{3}\text{P}\frac{1}{3}\text{D}, \frac{2}{3}\overline{m}\frac{1}{3}m)$, which is not in the list of answers provided.

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End of exam

Total questions: 53
Total marks: 60