GSOE9210 Engineering Decisions

Problem Set 04

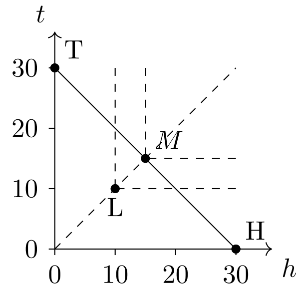
1. At a school fête, a suspicious-looking man is offering bets on the toss of a coin of questionable fairness. The man is offering \$2 for each dollar bet (plus your original dollar) if the contestant chooses the face on the coin correctly; *i.e.*, if you bet \$1 and win you will gain \$2, leaving you with \$3; if you lose you will lose the dollar you bet, leaving you with \$0.

   (a) Suppose Alice has a \$10 note in her pocket; represent the gamble as a decision table. Include the option of leaving (L) (*i.e.*, refusing to gamble).
   (b) If Alice were pessimistic (*i.e.*, she used *Maximin* as her decision rule), would Alice bet on heads or tails, or not bet?
   (c) Suppose Alice has a similarly pessimistic friend Bob, and both could bet together on the same toss, would that affect Alice's decision?
   (d) Suppose Alice was friendless but had—instead of one \$10 note—10 \$1 coins in her pocket. How might this affect her decision?

   Solution

   (a) Assuming that one always has the option to not gamble, this situation is represented below, showing Alice's cash balance.

	h	t
H	30	0
T	0	30
L	10	10
$\frac{1}{2}$H$\frac{1}{2}$T	$\frac{30}{2}$	$\frac{30}{2}$



   (b) Alice would prefer to leave (L), guaranteeing that she keeps her \$10, otherwise she risks losing it, despite the prospect of winning \$20 (*i.e.*, a total balance of \$30)—Alice is *risk averse.*
   (c) If they each bet on opposite outcomes one of the two will win, so they are *guaranteed* \$30 between them. If they divide the balance, they would both finish with \$15, each doing better than not betting. Mixed actions can be thought of as assuming the decision-maker(s) are making multiple independent decisions on a single trial.
   (d) If Alice put puts \$5 ($5 \times \$1$) on H and \$5 ($5 \times \$1$) on T at those odds she would similarly guarantee herself a return of \$15 in total. An average return of \$1.5 per dollar bet.

2. (a) Find the value $\frac{3}{4}$ of the way from 4 to 2 on the number line.

(b) Find a general expression for the value μ of the way from a to b on the number line.

Consider the decision problem below:

	s_1	s_2
A	2	3
B	4	0
C	3	3
D	5	2
E	3	5

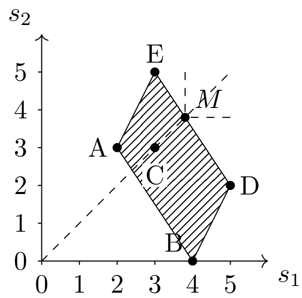
(c) Plot the actions as points on the Cartesian plane. Find the coordinates of the point μ of the way from B to A.

(d) Find the point, $P(x, y)$, on the segment AB that intersects with the diagonal $y = x$.

(e) Find the *Maximin* mixed action for the decision problem above.

Solution

	s_1	s_2
A	2	3
B	4	0
C	3	3
D	5	2
E	3	5



(a) The value is expressed as a convex combination with parameter μ: $x = 2\mu + 4(1 - \mu) = 4 + (2 - 4)\mu = 4 - 2\mu$

(b) In general: $x = b\mu + a(1 - \mu) = a + (b - a)\mu$

(c) The point P_μ, which is μ of the way from $B(4, 0)$ to $A(2, 3)$ is given by $P_\mu = (x, y)$, where x is μ of the way from 4 to 2 and y is μ of the way from 0 to 3 (by similar triangles). Hence:

$$\begin{aligned} x &= 2\mu + 4(1 - \mu) \\ y &= 3\mu + 0(1 - \mu) \end{aligned}$$

Alternatively, in vector form:

$$(x, y) = \mu(2, 3) + (1 - \mu)(4, 0)$$

(d) Setting $x = y$ gives:

$$\begin{aligned} 2\mu + 4(1 - \mu) &= 3\mu + 0(1 - \mu) \\ 4 - 2\mu &= 3\mu \\ 5\mu &= 4 \\ \therefore\ \mu &= \tfrac{4}{5} \end{aligned}$$

(e) From the diagram, by inspection, the *Maximin* mixed action must correspond to a point lying on the segment DE. The point which is μ of the way from E$(3,5)$ to D$(5,2)$ is given by $P_\mu = (x, y)$, where

$$x = 5\mu + 3(1-\mu)$$
$$y = 2\mu + 5(1-\mu)$$

Setting $x = y$ gives:

$$\begin{aligned} 5\mu + 3(1-\mu) &= 2\mu + 5(1-\mu) \\ 2\mu + 3 &= -3\mu + 5 \\ 5\mu &= 2 \\ \therefore\ \mu &= \tfrac{2}{5} \end{aligned}$$

Here $\mu_{\mathrm{D}} = \mu = \frac{2}{5}$, and, hence, $\mu_{\mathrm{E}} = (1-\mu) = \frac{3}{5}$. So the *Maximin* mixed action is $\frac{2}{5}\mathrm{D}\frac{3}{5}\mathrm{E}$. Setting $\mu = \frac{2}{5}$ in the expressions for x, y above, yields: $x = 5(\frac{2}{5}) + 3(\frac{3}{5}) = 2 + \frac{9}{5} = 3\frac{4}{5} = y$. Hence $P_\mu = (3\frac{4}{5}, 3\frac{4}{5}) = (3.8, 3.8)$. This is corroborated by the diagram.

3. Consider a scenario with four states, two pure actions, and their mixtures (M), where $\mu_A = \mu$:

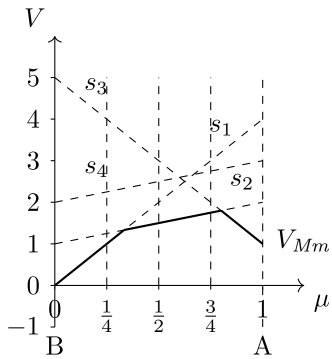
	s_1	s_2	s_3	s_4
A	4	2	1	3
B	0	1	5	2

(a) Draw the mixture plot for this problem.
(b) Do all states need to be considered when determining the *Maximin* mixed action?
(c) Find the *Maximin* mixed action, and the *Maximin* value—*i.e.*, the value of the *Maximin* mixed action—for this problem.

Solution
The decision table, including a generic mixture, is shown below:

	s_1	s_2	s_3	s_4
A	4	2	1	3
B	0	1	5	2
$M(\mu)$	4μ	$\mu+1$	$5-4\mu$	$\mu+2$

(a) The mixture plot is shown below.



(b) Consider an arbitrary mixed action $M(\mu)$. From the plot above, the value in s_4 is always going to be greater than that in s_2, so the former will never feature in *Maximin* considerations. It follows that s_4 can be ignored with respect to *Maximin*.

(c) The *Maximin* mixed strategy is obtained by finding the value of μ for which the values in s_2 and s_3 match—noting, as we did above, that s_2 and s_3 are the only two relevant states, as they are responsible for the minimum values of each mixed action.
Setting the values in s_2 and s_3 to be equal gives:

$$\begin{aligned} \mu + 1 &= 5 - 4\mu \\ 5\mu &= 4 \\ \therefore \quad \mu &= \tfrac{4}{5} \end{aligned}$$

The *Maximin* mixed action is $M(\frac{4}{5})$.
The *Maximin* value is the value of the *Maximin* mixed action:

$$\begin{aligned} V_{Mm}(M(\tfrac{4}{5})) &= v(M(\tfrac{4}{5}), s_2) \\ &= \tfrac{4}{5} + 1 = \tfrac{9}{5} \end{aligned}$$

4. Alice sells drinks at a local market once every month. She can order stock to sell several drink types: a) hot chocolate; b) iced tea; c) lemonade; d) orange juice.

   From past experience she knows that when she sells only one type of drink, on warm days her sales total for each type are: \$10 on hot chocolate, \$40 on iced tea, \$30 on lemonade, and \$40 on orange juice. On cool days, however, her sales totals are: \$30 on hot chocolate, \$0 on iced tea, \$20 on lemonade, and \$10 on orange juice.

   She has to order her stock weeks in advance, long before she can predict the temperature on the day of the market.

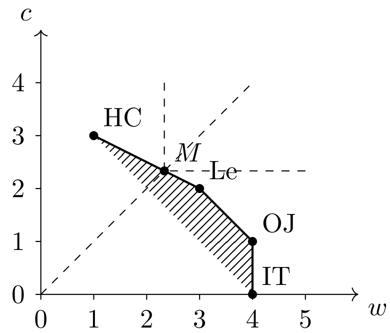
   (a) Produce a decision table for this problem.
   (b) If for the same total amount of stock she can stock different drinks, what proportion of drinks should she stock to maximise her guaranteed daily sales total regardless of the temperature?
   (c) Draw the admissibility frontier for this problem. Are any actions inadmissible?

Solution

(a) Consider the decision table below. Values are expressed in tens of dollars. The associated graph is also shown.

	w	c
HC	1	3
IT	4	0
Le	3	2
OJ	4	1

where: w warm day
c cold day



(b) She would maximise her guaranteed sales by having the mixture of stock which maximises the minimum sales irrespective of whether the day is warm or cold.
It is clear from the graph that the optimal mixture should comprise hot chocolate and lemonade only.
Let m_w be the average sales of the relevant mixture of drinks on a warm day and m_c the mixture's average sales on a cool day.
If μ is the desired proportion of hot chocolate in the mixture, then $M = (m_w, m_c) = (3, 2) + \mu[(1, 3) - (3, 2)]$; *i.e.*,

$$m_w = 3 + (1 - 3)\mu = 3 - 2\mu$$
$$m_c = 2 + (3 - 2)\mu = 2 + \mu$$

Setting $m_w = m_c$ to find the *Maximin* mixed strategy:

$$3 - 2\mu = 2 + \mu$$
$$1 = 3\mu$$
$$\therefore \quad \mu = \tfrac{1}{3}$$

That is, she should have a mixture consisting of one third of the units on sale being hot chocolate and the other two thirds lemonade.
Note that this is a ratio of two units of lemonade per unit of hot chocolate.

(c) The admissibility frontier is marked on the graph as the solid line connecting HC, Le, OJ, and IT. All pure strategies are admissible, provided we consider strict dominance. Under weak dominance IT would be inadmissible.

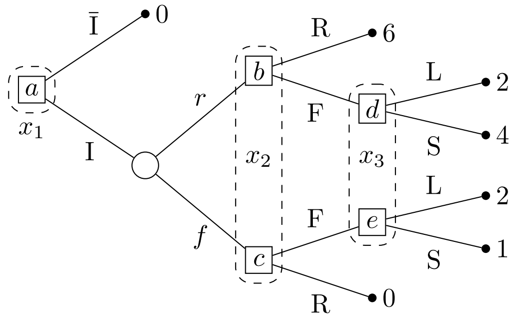
5. Alice is considering whether to invest \$1000, and if so on which type of investment and for how long (full term or half). She is looking at a risky stock (R) which will either rise (r) in value by 6% or flatten (f) to 0%. If she invests in the risky stock she must do so for the full term, before she knows how the stock's price will change. Another option is to invest in a fixed rate option (F), which gives a constant return of 2%. For the

fixed-term investment she has a further option to invest initially for a long term (L) (*i.e.*, for the full term), or a short term (S) (half term). At the end of the short term, she will know the movement of the risky stock, and will be able to re-invest for the remaining period by doing the following: (a) if the risky stock has risen, she will switch to it earning an average profit of 4%; (b) if the risky stock has fallen she'll reinvest in F but will receive a lower rate, which would reduce her gains to 1% per annum.

(a) Represent this situation as a decision tree (*i.e.*, in extensive form) and as a decision table.
(b) How many information sets are there in this problem?
(c) Assuming diversified stock portfolios (*i.e.*, mixtures of investments) are allowed, which mixed strategies are admissible?
(d) Which is the *Maximin* mixed strategy?
(e) Which is the *miniMax Regret* mixed strategy?

Solution

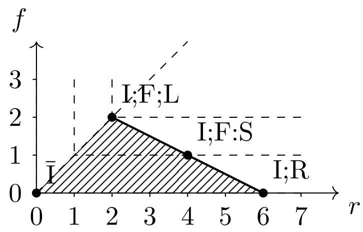
(a) One possible extensive form is shown below. The additional symbols are: I: invest; Ī: don't invest.



The decision table is shown below. Inadmissible (dominated) pure strategies have been struck out.

	r	f
~~Ī~~	~~0~~	~~0~~
I;F;L	2	2
I;F:S	4	1
I;R	6	0

(b) There are three information sets in total: $x_1 = \{a\}$, $x_2 = \{b, c\}$, and $x_3 = \{d, e\}$. Notice also that, because they are indistinguishable to the agent, the nodes in each information have the same number of identically labelled branches.
(c) Graphically:



Admissible mixed strategies are represented by points on the solid line. None of these strategies are better than any other in all (both) cases/states. All other mixed strategies are inadmissible.

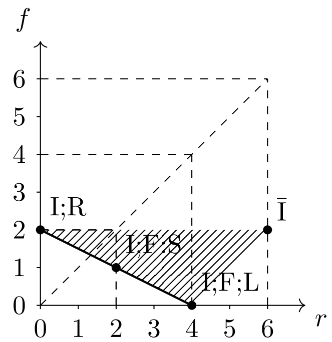
(d) It is clear from the graph than I;F;L is the *Maximin* mixed strategy. In this case, it happens to be a pure strategy.

(e) In terms of regret:

	r	f
~~$\bar{\mathrm{I}}$~~	~~6~~	~~2~~
I;F;L	4	0
I;F:S	2	1
I;R	0	2

Note that for regret we prefer smaller values, so a strategy is dominated if its values are greater than those of another action, as indicated above.

The regret graph is shown below:



Notice that the *miniMax Regret* pure strategies are I;F:S and I;R, whereas the *Maximin* pure strategy is I;F;L.

Would a person who wishes to guarantee the best possible return on her investment think in terms of value or regret?

Moreover, if the market was unpredictable, and the \$1000 can be divided among multiple investments (*e.g.*, as in an investment portfolio), then 'hedging', by investing in a mixture of I;R and I;F:S, would be best to minimise the maximum regret.

6. Consider the following decision table, in which mixed strategies are allowed.

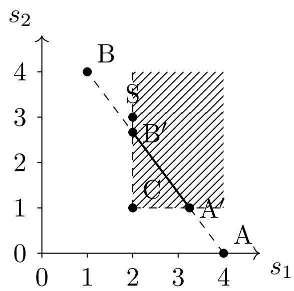
	s_1	s_2
A	4	0
B	1	4
C	2	1

(a) Which, if any, strategies are dominated?

(b) Prove that a possible strategy S, with payoffs 2 and 3 in states s_1 and s_1 respectively, would not be dominated.

Solution

(a) Strategy C is dominated by a mixture of A and B.



	s_1	s_2
A	4	0
B	1	4
~~C~~	~~2~~	~~1~~

The reduced matrix is shown above (right).

(b) Consider mixtures of A and B with parameter μ representing the proportion of A in the mixture; *i.e.*, $M(\mu) = \mu\text{B} + (1-\mu)\text{B}$.
It follows that $M(\mu) = (3\mu + 1, 4 - 4\mu)$. For any mixture $M(\mu)$ to strictly dominate S we require that $3\mu + 1 > 2$ and $4 - 4\mu > 3$; *i.e.*, $\mu > \frac{1}{3}$ and $\mu < \frac{1}{4}$. Since there is no μ that can simultaneously satisfy both conditions, it follows that there can be no mixture of A and B that dominates S.