GSOE9210 Engineering Decisions

Problem Set 07

1. Show that an irreflexive and transitive relation is asymmetric.

   Solution
   Suppose R is irreflexive and transitive (*i.e.*, R is a strict preorder). Suppose xRy and yRx. Then by transitivity xRx, which would contradict irreflexivity. It follows that if xRy, then it cannot be that yRx. That is, R is asymmetric.

2. An *equivalence relation* on a set A is any binary relation which is: a) reflexive; b) symmetric; and c) transitive

   Show that for any fixed $m \in \mathbb{N}$, the relation $R_m \subseteq \mathbb{Z} \times \mathbb{Z}$ such that xR_my iff $x - y = km$ for some $k \in \mathbb{Z}$, is an equivalence relation.

   Define $[n]_m = [n]_{R_m}$. Describe the equivalence class $[3]_0$, $[3]_1$, and $[3]_2$, $[3]_3$. In general, describe the equivalence classes $[n]_m$? Show that $[m]_4 \subseteq [m]_2$, for any $m \in \mathbb{Z}$. More generally, show that if for some $k, n, p \in \mathbb{Z}$, $n = kp$, then $[m]_n \subseteq [m]_p$

   Solution
   Let $x \in [m]_4$; *i.e.*, mR_4x; *i.e.*, $m - x = 4k$, for some $k \in \mathbb{Z}$. Then $m - x = 2(2k)$. But then for some $j \in \mathbb{Z}$ (specifically $j = 2k$), $m - x = 2j$; *i.e.*, mR_2x. Therefore, $x \in [m]_2$, and hence, $[m]_4 \subseteq [m]_2$.

3. Verify that for any finite (or indeed infinite) sets A and B, the relation $A \simeq B$ iff $|A| = |B|$, where $|A|$ is the *cardinality* of A (*i.e.*, the number of elements in A) is an equivalence relation.

4. A *partial order* is any relation which is reflexive, antisymmetric, and transitive.

   Define the relation $| \subseteq \mathbb{N} \times \mathbb{N}$ by $x|y$ iff x divides y (or x is a factor of y, or y is a multiple of x). Show that $|$ is a partial order (*i.e.*, that it is reflexive, antisymmetric, and transitive).

   Solution
   $x|y$ iff there exists some $k \in \mathbb{Z}$ such that $y = kx$.

   (Reflexivity) Since $x = 1x$, it follows that if $y = x$ for $k = 1$, $y = 1x = x$ and hence $x|x$.

   (Antisymmetry) Suppose $x|y$ and $y|x$; *i.e.*, for some $k, j \in \mathbb{Z}$, $y = kx$ and $x = jy$. Substituting the second into the first: $y = k(jy) = (kj)y$. But this can only hold if $kj = 1$, making j the multiplicative inverse of k: $j = k^{-1}$. Since $k^{-1} \in \mathbb{Z}$ only for $k = 1$, in which case $j = 1$, it follows that $k = j = 1$, and $kj = 1$. Hence $y = 1x = x$, as required.

(Transitivity) Suppose $x|y$ and $y|z$; *i.e.*, $y = kx$ and $z = jy$, for some $k, j \in \mathbb{Z}$. But then $z = j(kx) = (jk)x$, hence there is some $n \in \mathbb{Z}$ (specifically $n = jk$) such that $z = nx$; *i.e.*, $x|z$.

Since $|$ is reflexive, antisymmetric, and transitive, it is a partial order. Note that it is not total; *e.g.*, neither $2|3$ nor $3|2$ (nor $2 = 3$).

5. For a weak preference relation $\succsim$, verify the following:

   (a) If an agent's preferences are consistent then $\sim$ is an equivalence relation
   (b) The corresponding strict preference relation $\succ$ is a strict total order
   (c) Strict preference satisfies an 'indifference version' of the trichotomy law; *i.e.*, exactly one of the following holds between any $x, y \in A$: $x \succ y$ or $x \sim y$ or $y \succ x$.

   Solution

   (a) If an agent's preferences are consistent then $\succsim$ is a weak total order; *i.e.*, it is reflexive, antisymmetric, transitive, and connected. Now $\sim$ is defined as $x \sim y$ iff $x \succsim y$ and $y \succsim x$.
   (Reflexivity) Since $\succsim$ is reflexive, for every $x \in A$, $x \succsim x$, hence $x \sim x$.
   (Symmetry) Suppose $x \sim y$; *i.e.*, $x \succsim y$ and $y \succsim x$. Since the order of these two facts is irrelevant, this implies $y \sim x$.
   (Transitivity) Suppose $x \sim y$ and $y \sim z$. Then $x \succsim y$ and $y \succsim x$, and $y \succsim z$ and $z \succsim y$. Since $\succsim$ is transitive, then $x \succsim z$ and $z \succsim x$; *i.e.*, $x \sim z$.
   (b) By definition $x \succ y$ iff it is not the case that $y \succsim x$. We need to show that $\succ$ is transitive and satisfies: either $x \succ y$ or $x \sim y$ or $y \succ x$. Suppose $x \succ y$ and $y \succ z$; *i.e.*, neither $y \succsim x$ nor $z \succsim y$ hold. Since $\succsim$ is connected, $x \succsim y$ and $y \succsim z$. But then $x \succsim z$, which, by ...

6. Verify that the following properties hold from the axiomatisation of $\succsim$ given in lectures.

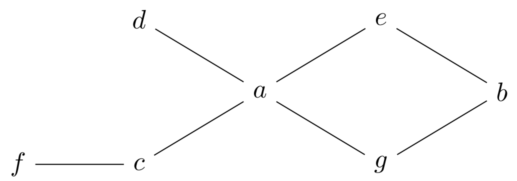
   - Strict preference properties:
     - if $x \succ y$, then it should be that $y \succ x$
     - if $x \succ y$ and $y \succ z$, then it should not be that $z \succ x$
   - Indifference properties:
     - if $x \sim y$, then $y \sim x$
     - if $x \sim y$ and $y \sim z$, then $x \sim z$
     - $x \sim x$ holds for any $x \in A$
   - Combined properties:
     - if $x \sim y$ and $z \succ x$, then $z \succ y$
     - if $x \sim y$ and $x \succ z$, then $y \succ z$
     - for any x, y *either* $x \succ y$ or $x \sim y$ or $y \succ x$

7. Let $[x]$ be an abbreviation for $[x]_\sim$, show that:

   (a) if $x \sim y$, then $[x] = [y]$
   (b) if $[x] \cap [y] \neq \varnothing$, then $[x] = [y]$

(c) if $x \succ y$, then if $a \in [x]$ and $b \in [y]$, then $a \succ b$

8. Left the left-to-right edges in the Hasse diagram below represent $\succ$.



In terms of $\succ$ what is the relationship between:

(a) d and a
(b) a and e
(c) a and b
(d) f and d

Solution

(a) $d \succ a$ as there is a left-to-right edge connecting d to a.
(b) $a \succ e$
(c) $a \succ b$, by transitivity, as $a \succ e$ and $e \succ b$. Alternatively, there is a left-to-right *path* from a to b.
(d) There is no left-to-right path from f to d or from d to f, so neither $f \succ d$ nor $d \succ f$.

9. Consider the following preferences on the set $A = \{a, b, c, d, e\}$:

$$c \succsim a \quad b \succsim d \quad e \succsim d \quad d \succsim a \quad d \succsim e \quad a \succsim c$$

(a) What additional instances of $\succsim$ can be inferred from the axioms given in lectures?
(b) Assume that any inference not present above, or inferred from them, is false. From the definition of $\succ$ in terms of $\succsim$, what are the instances of $\succ$?
(c) For an equivalence relation $(A, \sim)$, denote the set of all equivalence classes of A by $A/\sim$. (Sometimes $A/\sim$ is called the *quotient class* of A.) List the indifference classes in $A/\sim$?
(d) Draw the Hasse diagram for $\succ$.
(e) Draw the Hasse diagram for $\succ_I$: the preference relation on indifference classes.
(f) Define an ordinal function V on the members of $A/\sim$ (*i.e.*, $V : A/\sim \to \mathbb{R}$) and hence, one on A ($v : A \to \mathbb{R}$).

Solution

(a) All the reflexive instances:

$$a \succsim a, b \succsim b, c \succsim c, d \succsim d, e \succsim e$$

All the instances derived from transitivity:

$$b \succsim a, b \succsim e, b \succsim d, b \succsim c, e \succsim a, e \succsim c, d \succsim c, e \succsim a, e \succsim c$$

In summary (omitting reflexive instances):

$$
\begin{array}{l}
a \succsim c \\
b \succsim a, b \succsim c, b \succsim d, b \succsim e \\
c \succsim a \\
d \succsim a, d \succsim c, d \succsim e \\
e \succsim a, e \succsim c, e \succsim d
\end{array}
$$

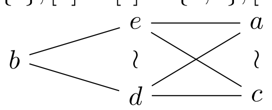
In tabular form, placing an $\times$ in row x and column y iff $x \succsim y$ is true:

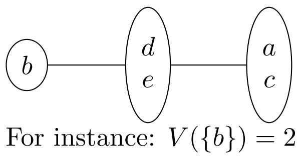
	a	b	c	d	e
a	$\times$		$\times$		
b	$\times$	$\times$	$\times$	$\times$	$\times$
c	$\times$		$\times$		
d	$\times$		$\times$	$\times$	$\times$
e	$\times$		$\times$	$\times$	$\times$

Note that the diagonal entries are all true, as $\succsim$ is reflexive.

(b) Assuming the table above is complete, we look for entries that are empty; *e.g.*, entry (a, b). This implies that it is not the case that $a \succsim b$, hence, by the assumption that any instance that can't be derived is false, we infer that $b \succ a$. In this way, we look down the columns for blank entries: for (column) a: there is nothing strictly less preferred than a (no blanks in column a); for b: $b \succ a, b \succ c, b \succ d, b \succ e$; for c: nothing; for d: $d \succ a, d \succ c$; for e: $e \succ a, e \succ c$.

(c) Since we have $a \succsim c$ and $c \succsim a$, then $a \sim c$. Similarly, as $e \succsim d$ and $d \succsim e$, then $d \sim e$. Therefore, the indifference classes are $[b] = \{b\}, [a] = [c] = \{a, c\}, [d] = [e] = \{d, e\}$.

(d) 

(e) 

(f) For instance: $V(\{b\}) = 2, V(\{d, e\}) = 1, V(\{a, c\}) = 0$.
That is $v(b) = 2, v(d) = v(e) = 1, v(a) = v(c) = 0$.

10. Show that the weak preference ordering $\succsim_I$ on indifference classes is anti-symmetric.

11. Show that for the weak preference relation $\succsim_I$ on indifference classes:

    (a) for any $X, Y \in A/\!\sim$, $X \succsim_I Y$ iff for *every* $x \in X$ and $y \in Y$, $x \succsim y$
    (b) $\succsim_I$ is a weak total order

12. Show that for any ordinal value function v:

    (a) $v(x) > v(y)$ iff $x \succ y$.
    (b) $v(x) = v(y)$ iff $x \sim y$.

Solution

(a) Let v be an ordinal value function; *i.e.*, $v(x) \geqslant v(y)$ iff $x \succsim y$.
Assume $v(x) > v(y)$. It follows that $v(y) \not\geqslant v(x)$. But then $y \not\succsim x$. By definition of $\succ$, then $x \succ y$, and conversely.

(b) Assume $v(x) = v(y)$. It follows that $v(x) \geqslant v(y)$ and $v(y) \geqslant v(x)$. Therefore, $x \succsim y$ and $y \succsim x$. By definition of $\sim$, $x \sim y$, and conversely.