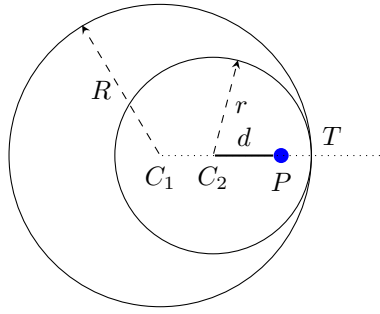


NOTES ON HYPOTROCHOIDS AND EPITROCHOIDS

ERIC MARTIN

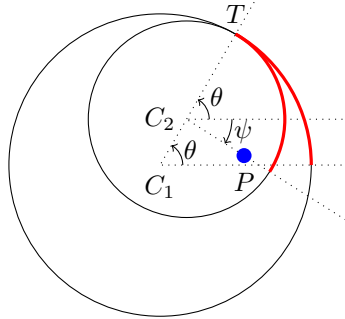
1. HYPOTROCHOIDS

A *hypotrochoid* is the curve obtained by tracing the positions taken by a point P rigidly attached to a circle \mathcal{C}_2 of centre C_2 and radius r , P being at a distance d from C_2 , with \mathcal{C}_2 rolling around the inside of another circle \mathcal{C}_1 of centre C_1 and radius R . To compute the equation of the curve, one assumes that C_1 is located at the origin of the plane, so has coordinates $(0, 0)$, and C_1 , C_2 and P are horizontally aligned, in that order from left to right, as shown in the following picture.



As \mathcal{C}_2 rotates clockwise and moves anticlockwise around the inside of \mathcal{C}_1 , when $\overrightarrow{C_1C_2}$ has gone from an angle of 0 to a positive angle of θ , and $\overrightarrow{C_2P}$ from an angle of 0 to a negative angle of ψ , the point of contact T between both circles has travelled the same distance along both circles—represented in red in the picture below—, namely, θR on \mathcal{C}_1 , and $(\theta - \psi)r$ on \mathcal{C}_2 . Hence:

$$\psi = -\frac{R-r}{r}\theta$$



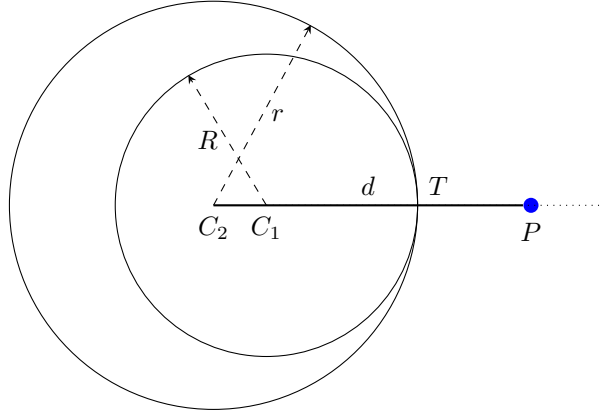
At this stage, since $\overrightarrow{C_1P} = \overrightarrow{C_1C_2} + \overrightarrow{C_2P}$, the point P has coordinates:

$$\begin{aligned} x &= (R-r)\cos(\theta) + d\cos(-\psi) \\ y &= (R-r)\sin(\theta) + d\sin(-\psi) \end{aligned}$$

that is:

$$\begin{aligned} x &= (R-r)\cos(\theta) + d\cos\left(\frac{R-r}{r}\theta\right) \\ y &= (R-r)\sin(\theta) - d\sin\left(\frac{R-r}{r}\theta\right) \end{aligned}$$

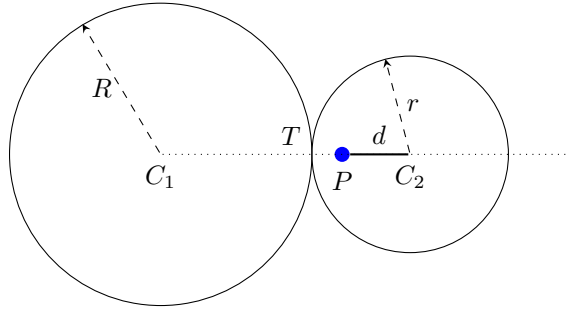
Note that P can “stick out” of C_2 , that is, d can be larger than r , as shown in the following picture, which also illustrates that C_2 can be larger than C_1 , that is, r can be greater than R ; that does not change the above reasoning and the equations still hold.



The *period* of a hypotrochoid is the number of times T gets back to its original position, as C_2 keeps rotating around the inside of C_1 , for P to get back to its original position. It is equal to the least strictly positive integer ρ such that $\rho \times 2\pi R$ is a multiple of $2\pi r$; hence it is equal to $\frac{r}{\gcd(r, R)}$.

2. EPITROCHOIDS

If we let C_2 roll around the outside rather than the inside of C_1 , then the curve obtained by tracing the positions taken by P is called an *epitrochoid*. To compute the equation of the curve, one assumes that C_1 , C_2 and P are horizontally aligned, with C_2 to the right of C_1 and with P to the left of C_2 , and also to the left of C_1 in case d is greater than $R + r$; the following picture illustrates the case where $r < R$ and $d < r$.



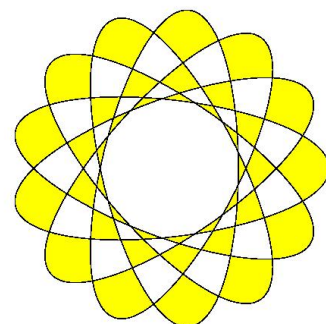
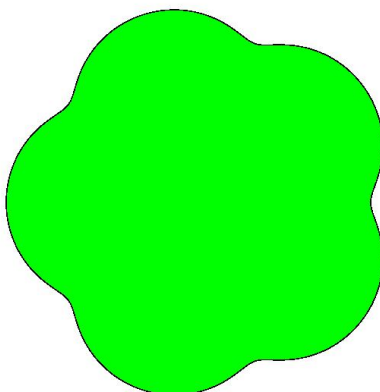
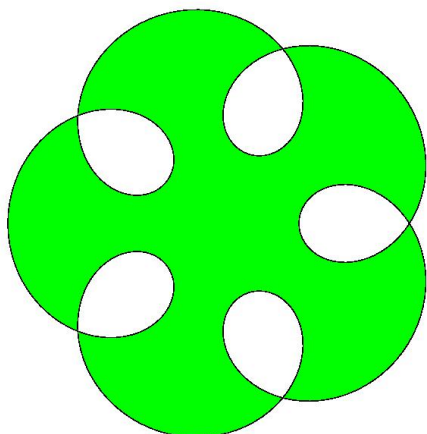
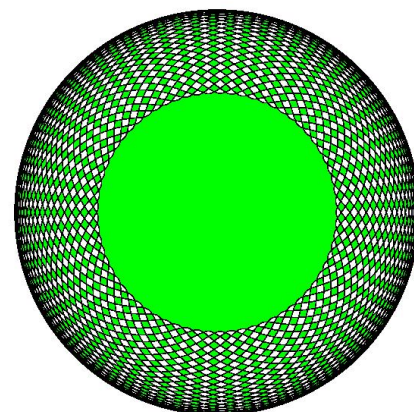
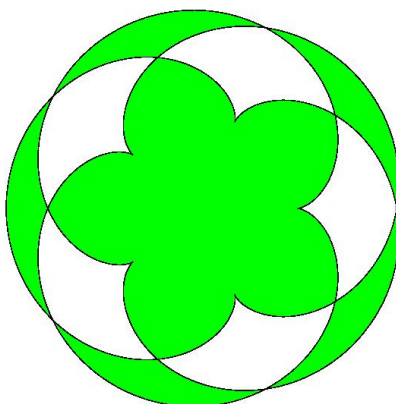
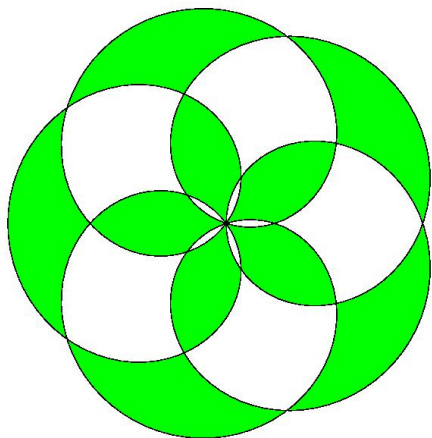
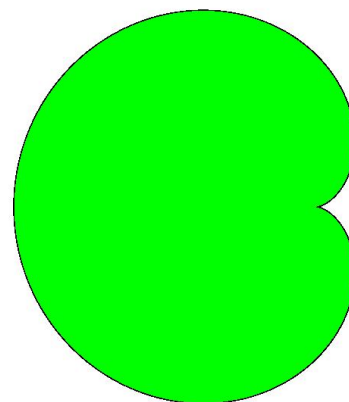
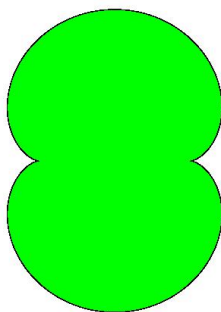
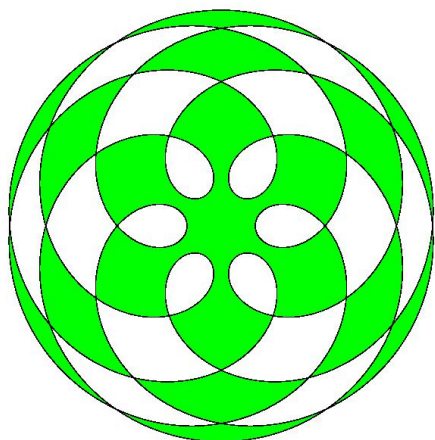
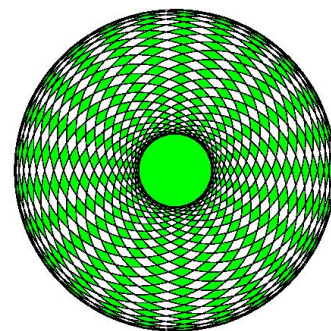
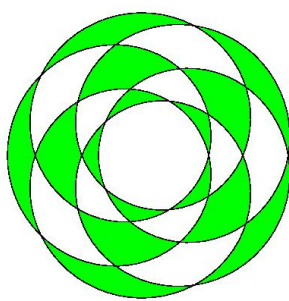
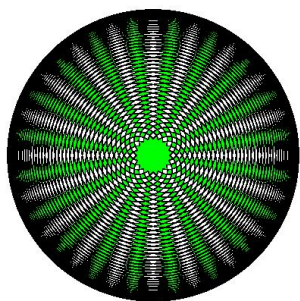
The reasoning that yields the equations for hypotrochoids can be immediately adapted to epitrochoids and result in the following equations:

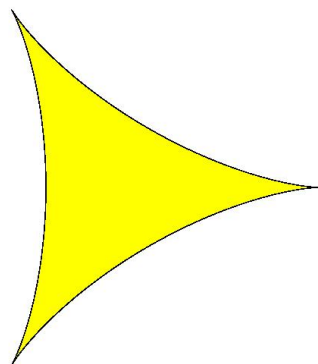
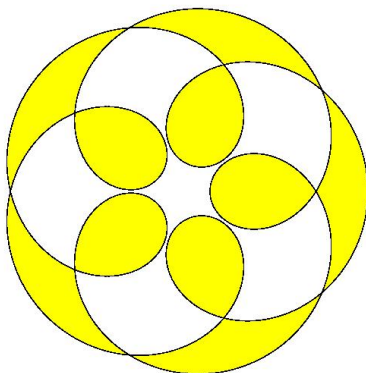
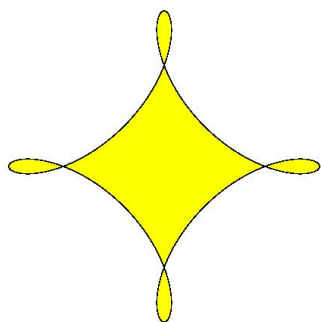
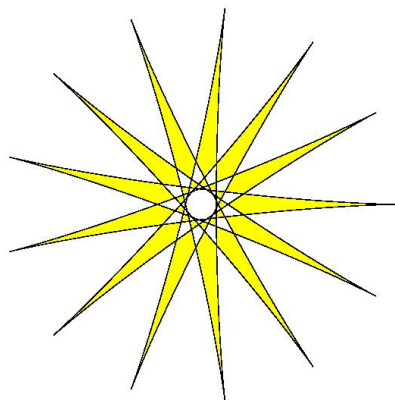
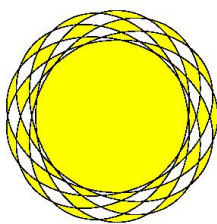
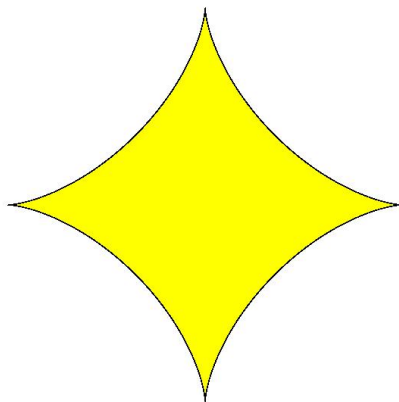
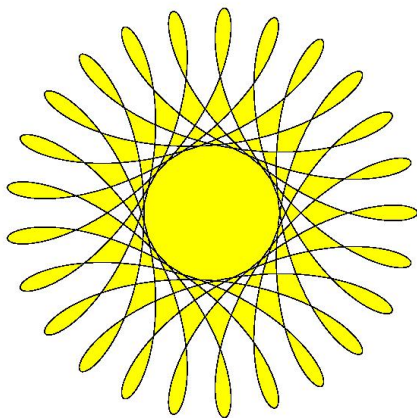
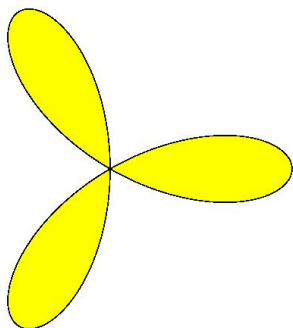
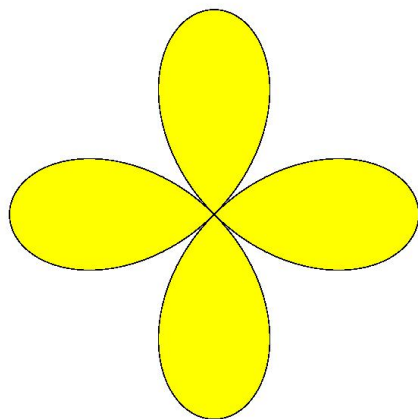
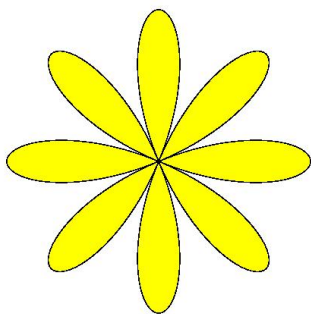
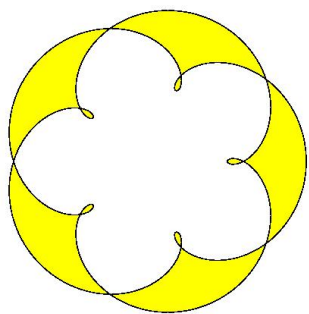
$$\begin{aligned} x &= (R + r) \cos(\theta) - d \cos\left(\frac{R + r}{r} \theta\right) \\ y &= (R + r) \sin(\theta) - d \sin\left(\frac{R + r}{r} \theta\right) \end{aligned}$$

The period of an epitrochoid is also equal to $\frac{r}{\gcd(r, R)}$.

3. PARTICULAR CASES

Ellipse, deltoid, astroid, nephroid, cardioid and *roses* are amongst the following pictures of epitrochoids (with a green filling) and hypotrochoids (with a yellow filling).





The following table shows how ellipse, deltoid, astroid, nephroid and a few other particular cases are obtained. When d is equal to r , hypotrochoids are also called *hypocycloids*, and epitrochoids are also called *epicycloids*.

	Hypotrochoids					Epitrochoids	
	$r = \frac{R}{2}$	$r \in \{\frac{R}{3}, \frac{2R}{3}\}$	$r \in \{\frac{R}{4}, \frac{3R}{4}\}$	$r = \frac{3R}{2}$	$r = 2R$	$r = \frac{R}{2}$	$r = R$
$d = r$	ellipse	deltoid	astroid	nephroid	cardioid	nephroid	cardioid
$d = 0$	segment	circle					
Any d							Pascal limaçon

To be complete, one should let R be ∞ ; then \mathcal{C}_1 is a line and the associated curves are called *trochoids*, with *cycloids* as a particular case when $d = r$...