GSOE9210 Engineering Decisions

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Risk attitudes and Utility

Outline

1. Risk preferences and bets
   - Bets and odds
   - Expected monetary value
2. Utility of money
3. Risk attitudes

Introduction to risk preference

- Class poll: You have $\$1000$. Would you risk it to play 'double or nothing' on the toss of a fair coin? *i.e.*, to win $\$2000$ on heads, and $\$0$ on tails?
- Measured in dollars, $v_{\$}(\$x) = x$, the two have equal *Bayes* value; *i.e.*, $v_{\$}(\$1000) = 1000 = V_B([\frac{1}{2} : \$2000|\frac{1}{2} : \$0])$
- Most people prefer a certain $\$1000$ over an even chance at $\$2000$ or $\$0$; *i.e.*, prefer $\$1000$ to $[\frac{1}{2} : \$2000|\frac{1}{2} : \$0]$
- Which value function, v, would satisfy:

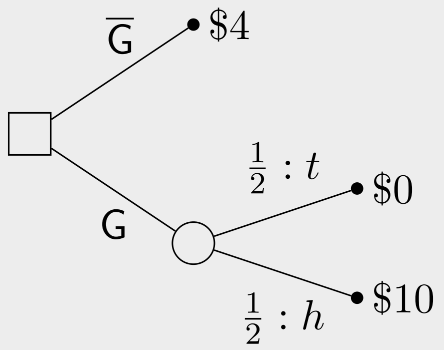
$$V_B([\$1000]) = v(\$1000) > V_B([\frac{1}{2} : \$2000|\frac{1}{2} : \$0])$$

Money bets and odds

Example (Betting)

Alice has $\$4$ to bet on the toss of a fair coin to win $\$10$ on heads.

Should Alice gamble?



$$\ell_{\overline{\mathsf{G}}} = [\$4]$$
$$\ell_{\mathsf{G}} = [\tfrac{1}{2} : \$10 | \tfrac{1}{2} : \$0]$$

Definition (Expected monetary value)

The *expected monetary value* (EMV) of a lottery, denoted $V_\$$, is the *Bayes* value of the lottery when outcomes are valued in $\$$ (*i.e.*, $v = v_\$$).

$$\begin{aligned} V_\$(\ell_{\overline{\mathsf{G}}}) &= 4 \\ V_\$(\ell_{\mathsf{G}}) &= \tfrac{1}{2} v_\$(h) + \tfrac{1}{2} v_\$(t) \\ &= \tfrac{1}{2}(10) + \tfrac{1}{2}(0) \\ &= 5 \end{aligned}$$

Expected monetary value

Definition (Fair bet)

A two-way gamble/bet is *fair* for an agent if the expected monetary value for the corresponding lottery is no less than the value of not gambling; *i.e.*,

$$V_\$(\ell_{\mathsf{G}}) = E(v_\$) \geqslant V_\$(\ell_{\overline{\mathsf{G}}})$$

- The bet Alice was offered was fair—indeed 'favourable'—for Alice; *i.e.*, $V_\$(\ell_{\mathsf{G}}) > V_\$(\ell_{\overline{\mathsf{G}}})$

Bets, stakes, and odds

Example (The races)

Alice is at the races and she's offered odds of '13 to 2' ($13 : 2$) on a horse by a bookmaker; *i.e.*, for every $\$2$ she puts in (her *stake*), the bookmaker puts in $\$13$, and the winner takes the entire *pool* ($\$15 = \$13 + \$2$).

Should Alice gamble? *i.e.*, is the bet favourable for Alice?

Bets, stakes, and odds

Definition (Favourable bet)

A bet is *favourable* to an agent if the value of the corresponding lottery for the agent is greater than that of not gambling. It is *unfavourable* if it is neither fair nor favourable.

Theorem (Fair bets)

Let a be agent A's stake and b be B's stake in a bet in which p is A's probability of winning. The bet is fair iff:

$$\frac{a}{b} = \frac{p}{1-p}$$

Bets: subjective belief

- Suppose Alice believes that her horse has a 20% chance of winning:

$$V_{\$}(\ell_{\mathsf{G}}) = \tfrac{1}{5}(15) + \tfrac{4}{5}(0) = \$3$$
$$V_{\$}(\ell_{\overline{\mathsf{G}}}) = \$2.$$

- Alice considers bet to be favourable based on her *beliefs* about her chances of winning

Exercises

- Prove the theorem on fair bets.
- For what probabilities of winning should Alice bet on her horse?

Betting example

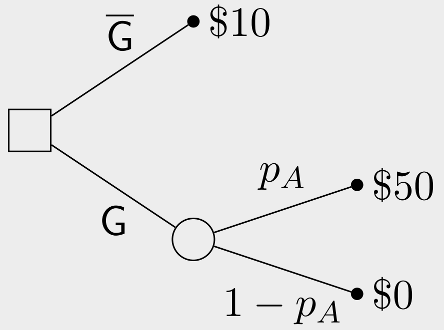
Example (Betting)

A bookmaker (B) offers Alice (A) odds '4 to 1' ($4:1$) on her team—a strong underdog—to win a football match. Alice has $\$10$ to bet on her team.

- The 'bookie' *puts up* $\$4$ for every $\$1$ Alice bets, so bookie has to put $\$40$ into pool to match Alice's $\$10$
- Alice's outcomes: balance of $\$50$ or $\$0$, depending on whether her team wins or loses
- a bet is *fair overall* if it is fair to both parties

Fair bets for odds

The decision tree for the two-way bet:



where G means Alice's agrees to gamble, and p_A is the probability that Alice wins $(p_A + p_B = 1)$

Fair odds (in \$):

$$p_A(50) + (1 - p_A)(0) \geqslant 10$$
$$\textit{i.e.} \quad p_A \geqslant \frac{10}{40+10} = \frac{1}{5}$$

In general, a bet is fair for A if:

$$p_A \geqslant \frac{x_A}{x_A + x_B}$$

where
x_A is A's stake (\$10)
x_B is B's stake (\$40).

Utility of bets

- At odds $4 : 1$ Alice should be if she *believes* chances of her team winning exceed 1 in 5 $\ldots$ Suppose Alice needs \$10 to buy dinner; should Alice gamble?
- Alice's risk preference: I'll gamble (risk going hungry) only if I believe my team's chances are at least even (*i.e.*, greater than 1 in 2)
- Alice indifferent between certain \$10 and $\ell = [\frac{1}{2} : \$50 | \frac{1}{2} : \$0]$:

$$\begin{aligned} u(\$10) &= U([\tfrac{1}{2} : \$50 | \tfrac{1}{2} : \$0]) = E_u(\ell) \\ &= V_B([\tfrac{1}{2} : \$50 | \tfrac{1}{2} : \$0]) \qquad \text{using } u \text{ rather than } v_{\$} \\ &= \tfrac{1}{2}u(\$50) + \tfrac{1}{2}u(\$0) \end{aligned}$$

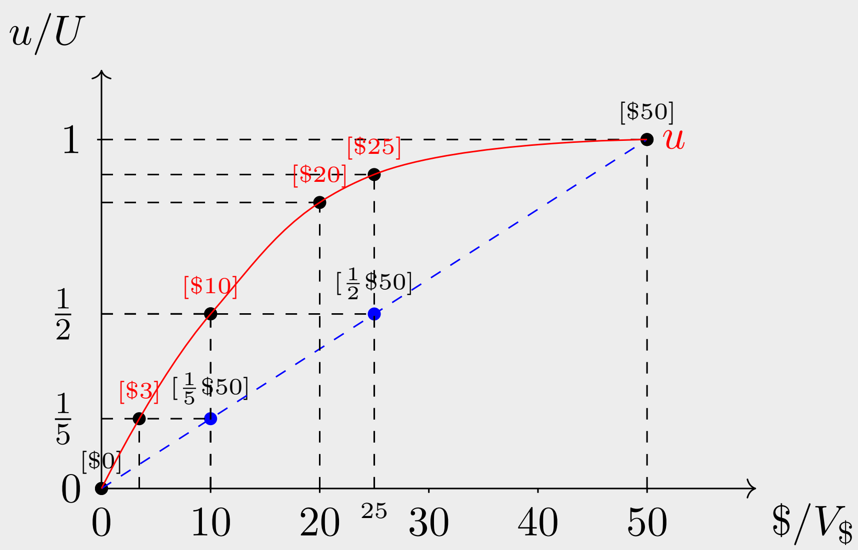
- What does u look like?

Outline

Utility of money

Fix u scale:

$$u(\$0) = 0$$
$$u(\$50) = 1$$

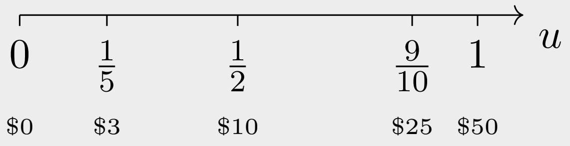


Possible gambles lie on diagonal:

$$U([\tfrac{1}{2} : \$50 | \tfrac{1}{2} : \$0]) = \tfrac{1}{2}u(\$50) + \tfrac{1}{2}u(\$0) = \tfrac{1}{2}$$
$$U([p : \$50 | (1-p) : \$0]) = p$$

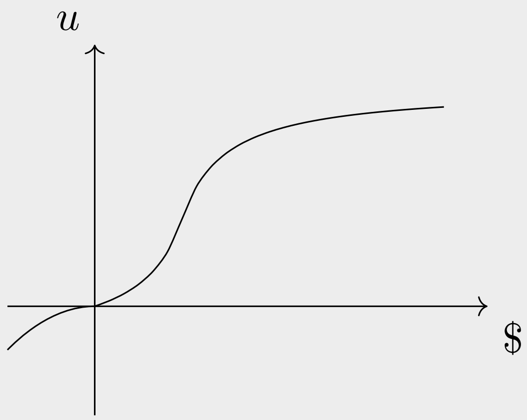
Utility for money

On Alice's utility scale the monetary outcomes are arranged as follows:



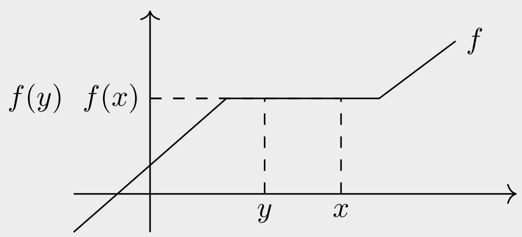
Question

What properties do typical utility functions for money have?



Utility values should increase with increasing money

Functions on ordered sets



Definition (Monotonic increasing function)

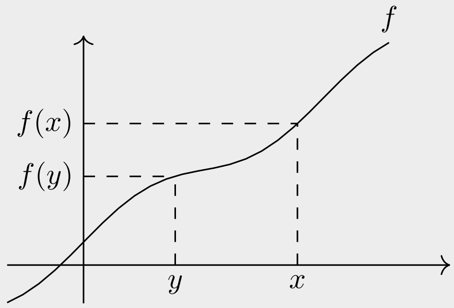
A real-valued function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *monotonically increasing*, or *non-decreasing*, iff for any $x, y \in \mathbb{R}$, if $x \geqslant y$, then $f(x) \geqslant f(y)$.

Examples: the following are non-decreasing functions on $\mathbb{R}$: $f(x) = \frac{1}{10}x$, $f(x) = x$, $f(x) = c$, for any fixed $c \in \mathbb{R}$

Exercise

Does this imply the converse; *i.e.*, if $f(x) \geqslant f(y)$, then $x \geqslant y$?

Strictly increasing functions

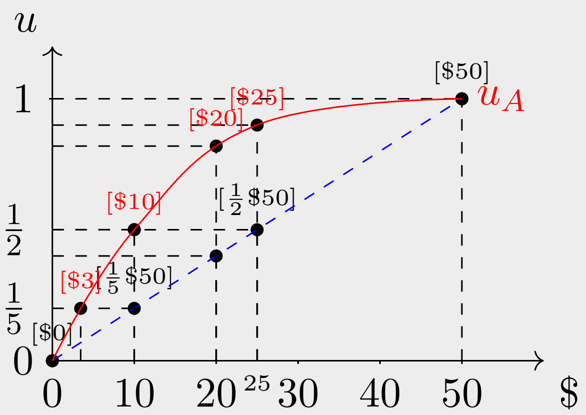


Definition (Strictly increasing function)

A real-valued function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* iff for any $x, y \in \mathbb{R}$, if $x > y$, then $f(x) > f(y)$.

Examples: $f(x) = \frac{1}{10}x$, $f(x) = x$, $f(x) = 3x + 2$, $f(x) = x^2$, $f(x) = \log_2 x$

Utility for money



How much money is $[\frac{1}{2}\$50]$ worth to Alice? $\$10$ xc

The EMV of $[\frac{1}{2}\$50]$ is $\$25$. How much of that amount is Alice willing to give up for a certain $\$10$? Up to $\$25 - \$10 = \$15$

Definition (Certainty equivalent)

An agent's *certainty equivalent* for a lottery is the value x_c for which the agent would be indifferent between it and the lottery; *i.e.*, $u(x_c) = U(\ell)$.

Definition (Risk premium)

The *risk premium* of an agent for lottery ℓ is the difference between the EMV of the lottery and the certainty equivalent: $V_\$(\ell) - x_c$.

Repeated trials

Example (Alice and Bob)

Alice and her twin, Bob, have $\$10$ each and they are offered, separately, 4 to 1 odds on a team in two different football matches (*e.g.*, home and away). They believe the team has a 2 in 5 chance of winning each match.

- Should Alice bet?

  In terms of the individual outcomes of Alice and Bob:

  $$\ell_{AB} = [\tfrac{9}{25} : (\$0, \$0) | \tfrac{6}{25} : (\$0, \$50) | \tfrac{6}{25} : (\$50, \$0) | \tfrac{4}{25} : (\$50, \$50)]$$

  If Alice and Bob share the risk/gain then:

  $$(\$x, \$y) \sim \$\left(\tfrac{x+y}{2}\right) \qquad \textit{i.e. } u_A(x, y) = u_A\left(\tfrac{x+y}{2}\right)$$

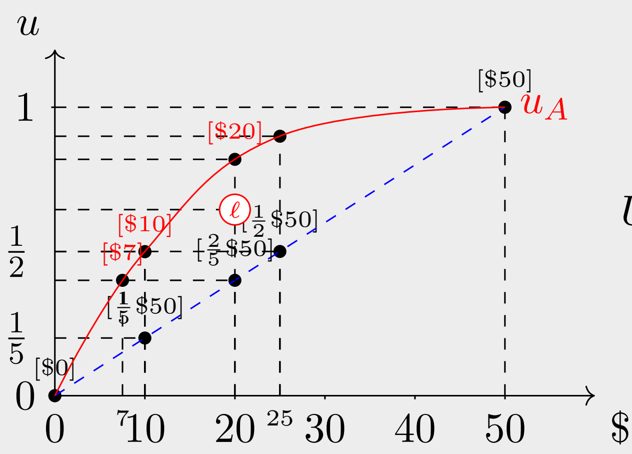
  So for Alice:

  $$\begin{aligned} \ell_A &= [\tfrac{9}{25} : \$0 | \tfrac{6}{25} : \$25 | \tfrac{6}{25} : \$25 | \tfrac{4}{25} : \$50] \\ &= [\tfrac{9}{25} : \$0 | \tfrac{12}{25} : \$25 | \tfrac{4}{25} : \$50] \end{aligned}$$

Repeated trials

Where does ℓ_A fit in in the scheme of things?

$$\ell_A = [\tfrac{9}{25} : \$0 | \tfrac{12}{25} : \$25 | \tfrac{4}{25} : \$50]$$

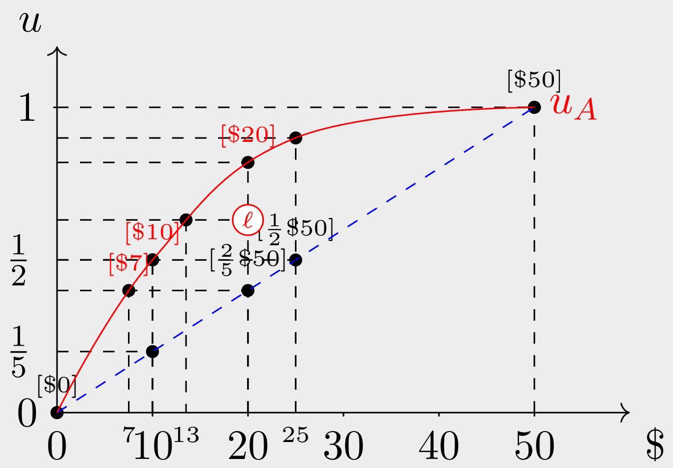


$$V_\$(\ell_A) = \tfrac{12}{25}(25) + \tfrac{4}{25}(50) = 20$$

$$\begin{aligned} U_A(\ell_A) &= \tfrac{9}{25}(0) + \tfrac{12}{25} u_A(\$25) + \tfrac{4}{25}(1) \\ &= 0 + \tfrac{12}{25}(\tfrac{9}{10}) + \tfrac{4}{25} = \tfrac{4}{25}(\tfrac{37}{10}) \\ &> \tfrac{4}{25}(\tfrac{35}{10}) = \tfrac{14}{25} > \tfrac{1}{2} = u_A(\$10) \end{aligned}$$

Alice should bet, sharing the risk and the winnings!

Repeated trials



- The individual bets are favourable for both Alice and Bob
- Despite this neither Alice nor Bob would take their respective individual bets
- However, they should bet together over multiple bets/trials

Outline

Risk attitudes

Definition (Risk attitudes)

An agent is:

- *risk averse* iff its certainty equivalent is less than the lottery's expected value; *i.e.*, it values the lottery to be worth less than the expected value.
- *risk seeking* (*risk prone*) iff its certainty equivalent is greater than the lottery's expected value.
- *risk-neutral* otherwise.

Exercises

- What is Alice's certainty equivalent for the lottery with Bob?
- The risk premium in what range if the agent is: risk averse? risk seeking? risk neutral?

Risk attitudes

More generally:

Definition (Risk averse)

An agent is *risk averse* if its utility function is concave down.

Definition (Risk seeking)

An agent is *risk seeking* if its utility function is concave up (convex).

Definition (Risk neutral)

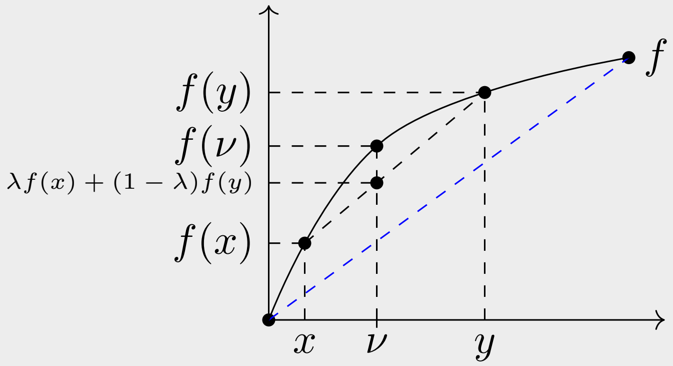
An agent is *risk neutral* if its utility function both concave down and up; *i.e.*, linear.

Concave and convex functions

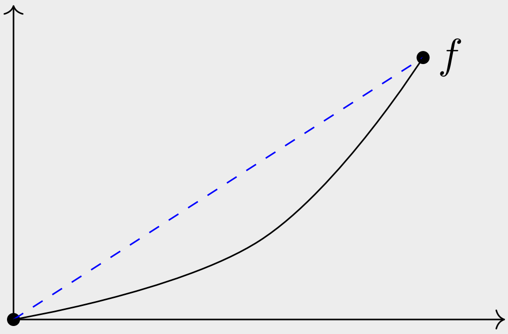
Definition (Concave and convex)

A function $f : \mathbb{R} \to \mathbb{R}$ is *concave down* in the interval $[a, b]$ if for all $x, y \in [a, b]$, and all $\lambda \in [0, 1]$, $f(\lambda x + (1 - \lambda)y) \geqslant \lambda f(x) + (1 - \lambda)f(y)$, and *concave up* (or *convex*) if $f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y)$.

concave



convex



where $\nu = \lambda x + (1 - \lambda)y$

Summary: risk attitudes and utility

- Not all quantities (*e.g.*, \$) accurately represent preference over outcomes
- Expected values on these quantities may not accurately represent preference
- Measure preference in terms of utility; agent must calibrate utilities against uncertain outcomes (lotteries)
- An agent's utility is personal/subjective; *i.e.*, particular to him. Different agents may have different utilities for the same 'outcome'
- Utility functions are non-decreasing; this means that over many trials *Bayes* utilities approach expected values
- The shape of an agent's utility curve/function determines its risk attitude