GSOE9210 Engineering Decisions

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Solving games

1. Modelling player behaviour
    - Solutions of zero-sum games
    - Best response
    - Repeated play; equilibria
    - Beliefs; rational solutions
    - Non-strictly-competitive games
    - Cooperation in games

Solving games

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Two-player zero-sum games: dominance

Example: simplify/reduce this two-player zero-sum game:

	b_1	b_2	b_3	b_4
a_1	0	1	7	6
a_2	4	2	3	4
a_3	3	1	0	2
a_4	0	0	7	3

- Round 1
- Round 2
- Round 3
- Round 4

- Common knowledge: a player won't play a dominated strategy; other players know this
- Game reduced (iterated dominance) to single strategy for each player
- Unique solution: (a_2, b_2)

Rational behaviour and strategic uncertainty

- In games the uncertainty for each player includes the *behaviour* of other players; *i.e.*, which strategy they'll choose
- This uncertainty can be reduced if players have *common knowledge* about the preferences and rationality of other players
- Dominance reduces *strategic uncertainty* about rational behaviour of other players (*e.g.*, rational players will never play dominated strategies)
- General principle about rational behaviour: *best response* . . .

Best response

Re-visit previous zero-sum game:

	b_1	b_2	b_3	b_4
a_1	0	1	7	6
a_2	4	2	3	4
a_3	3	1	0	2
a_4	0	0	7	3

- Play (a_2, b_2) is maximal in its column and minimal in its row
- *i.e.*, if column player plays b_2, then a_2 gives best possible outcome for row player
- Conversely, if row player plays a_2, then b_2 gives best possible outcome for column player

Best response: zero-sum games

Definition (Best response)

A player's strategy s^* is a *best response* to another player's strategy s if s^* gives a preference maximal outcome against s.

	b_1	b_2
a_1	${}_{*}2$	0^{*}
a_2	1^{*}	${}_{*}3$

In a zero-sum game:

- for any strategy of the column player, a best response of the row player is a strategy which maximises the column value ($*$)
- for any strategy of the row player, a best response of the column player is a strategy which minimises the row value ($*$)

Modelling player behaviour | Best response

Best response

	b_1	b_2	b_3	min
a_1	1^{*}	2	${}_{*}6$	1
a_2	${}_{*}7$	${}_{*}3^{*}$	3^{*}	3
a_3	3	2^{*}	5	2
max	7	3	6	

- Strictly dominated strategies are never best responses to any other player's strategies
- Column player's best responses are minimal in their row
- Every strategy has at least one best response; a_2 has two
- Row player's are maximal in their column
- Multiple best responses must have same payoff

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Best response: *Maximin*

	b_1	b_2	b_3	min
a_1	1^*	2	$_*6$	1
a_2	$_*7$	$_*3^*$	3^*	3
a_3	3	2^*	5	2
max	7	3	6	

- Row player's *Maximin* strategy is best strategy against 'perfect play' by opponent
- Above, row player's *Maximin* strategy is a_2; column player's *Maximin* strategy (*i.e.*, *miniMax* strategy) is b_2
- *Maximin* is rational sometimes: *e.g.*, if opponent can see your move

Repeated play

	b_1	b_2	b_3
a_1	1^*	2	$_*6$
a_2	4	$_*3^*$	4
a_3	$_*7$	2^*	5

- Suppose initially row player plays a_3, hoping for best outcome; similarly column player plays b_1; play (a_3, b_1)
- Row player happy (best response)
- Column player unhappy, so switches to best response b_2; in response row player plays a_2; ...
- Play 'stabilises' at (a_2, b_2)

Equilibrium

In 'stable' play (a_2, b_2) each strategy is a best response to the others.

	$\cancel{b_1}$	b_2	$\cancel{b_3}$
$\cancel{a_1}$	1^{*}	2	${}_{*}6$
a_2	4	${}_{*}3^{*}$	4
$\cancel{a_3}$	${}_{*}7$	2^{*}	5

John F. Nash $(1928\text{–}2015^{\dagger})$

Definition (Nash equilibrium)

A play is in *equilibrium* if each of its strategies is a best response to the others.

Equilibrium: belief interpretation

	b_1	b_2	b_3
a_1	1^{*}	2	6
a_2	${}_{*}4$	${}_{*}3^{*}$	4
a_3	${}_{*}7$	2^{*}	5

- If row player believes column player will play b_2, then row player cannot improve outcome by switching, and vice versa
- More generally, if each player believes the other will play according to their equilibrium strategy, then neither can improve their outcome by deviating from their equilibrium strategy

Equilibrium: existence and uniqueness

- Not all games have an equilibrium ... in pure strategies

	b_1	b_2
a_1	${}_{*}2$	0^{*}
a_2	1^{*}	${}_{*}3$

- Some games have multiple equilibria:

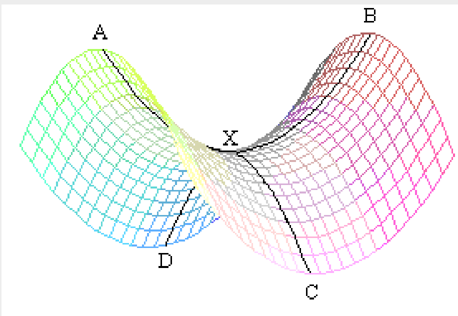
	b_1	b_2	b_3	b_4
a_1	4	${}_{*}2^{*}$	5	${}_{*}2^{*}$
a_2	2	1	-1	-2
a_3	3	${}_{*}2^{*}$	4	${}_{*}2^{*}$
a_4	-1	0	6	1

Zero-sum games: finding equilibria

Definition (Saddle point)

An entry in a matrix is called a *saddle point* iff it is minimal in its row and maximal in its column.

	b_1	b_2	b_3
a_1	1^{*}	3	4
a_2	${}_{*}7$	${}_{*}5^{*}$	6^{*}
a_3	3^{*}	4	${}_{*}8$



Theorem (Minimax)

In zero-sum games, saddle points represent equilibria.

Zero-sum games: solutions

Theorem

If a zero-sum game has an equilibrium, then it corresponds to the players playing Maximin strategies.

	b_1	b_2	b_3	min
a_1	1	3	4	1
a_2	7	$_*5^*$	6	5
a_3	3	4	8	3
max	7	5	8	

Because matrix entries are payoffs for row player, the column player's *Maximin* strategy translates to a *miniMax* strategy.

Zero-sum games: equilibrium

	b_1	b_2	b_3	min
a_1	1	3	4	1
a_2	7	$_*5^*$	6	5
a_3	3	4	8	3
max	7	5	8	

Theorem (Unique value)

All equilibria in a zero-sum game yield the same payoff. This payoff is said to be the value of the game.

- The value of the game above is 5
- Equilibria in zero-sum games are paired *Maximin* strategies (*miniMax* for column player)

Zero-sum games: finding equilibria

	b_1	b_2	b_3	min
a_1	1	3	4	1
a_2	7	5*	6	5
a_3	3	4	8	3
max	7	5	8	

- Saddle points are equilibria
- To find equilibria:
  - Use *Maximin* to evaluate each of the players' strategies (*i.e.*, *miniMax* for column player)
  - If the *Maximin* values agree (*e.g.*, 5 above), then that play is a saddle point of the game

Behaviour and beliefs

- A game matrix includes all possible strategies and outcomes, but says nothing about the players' preferences or *behaviour*; *i.e.*, which strategies the players should play
- Dominance and best response are principles about preference and rational *behaviour*
- An agent's behaviour should depend on its *beliefs* about the other players' behaviour (including likelihoods)
- In order to better explain behaviour we must formulate an agent's beliefs

Beliefs and behaviour

- Beliefs about the other players' play can be represented by a mixture of the other players' pure strategies
- Player A assigns to player B's strategy b_j a 'proportion' p_j if A's belief in the 'degree of likelihood' that B will play b_j is p_j
- Recall that utilities encode preferences in the presence of uncertainty (risk)

Modelling player behaviour | Beliefs; rational solutions

Best response to beliefs: zero-sum games

- Suppose player A believes that player B is twice as likely to play b_2 as b_1; *i.e.*, B will play b_1 with probability $\frac{1}{3}$ and b_2 with probability $\frac{2}{3}$
- Let $\beta \sim (\frac{1}{3}, \frac{2}{3})$ represent A's 'belief' about B's behaviour

B

A		b_1	b_2
	a_1	2	0
	a_2	1	3

- For belief β calculate the *Bayes* values of A's strategies:

$$V_B^{\beta}(a_1) = \tfrac{1}{3}(2) + \tfrac{2}{3}(0) = \tfrac{2}{3}$$
$$V_B^{\beta}(a_2) = \tfrac{1}{3}(1) + \tfrac{2}{3}(3) = \tfrac{7}{3}$$

- Therefore, A's best response given belief β about B is a_2.

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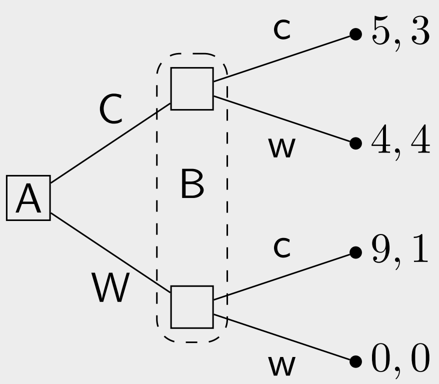
Rationalisation of behaviour and belief

- Any strategy by another player which will not be played should receive degree of belief (*i.e.*, probability) 0
- In general, a strategy which isn't *Bayes* for some belief β can be eliminated; compare with *admissibility*
- In a zero-sum game, rational strategies (strategies which aren't eliminated) must be on the player's 'admissibility frontier'

Non-zero-sum games: best response

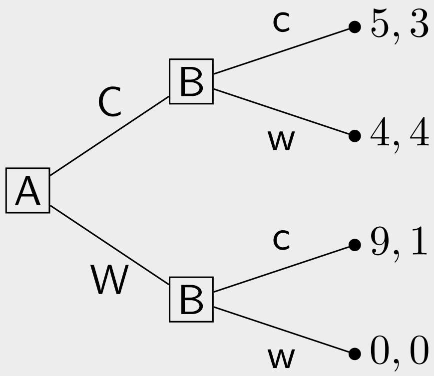


		B: c	B: w
A	C	$5, 3$	$4, 4$
	W	$9, 1$	$0, 0$

- If Alice were to wait, then Bob's best counter-move would be to climb
- Conversely, if Bob were to climb, then Alice's best counter-move would be to wait below

Solving games

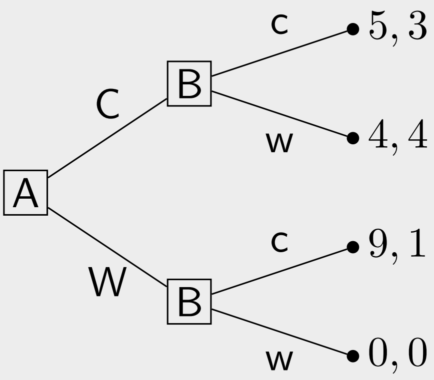
- What if Alice moves first?

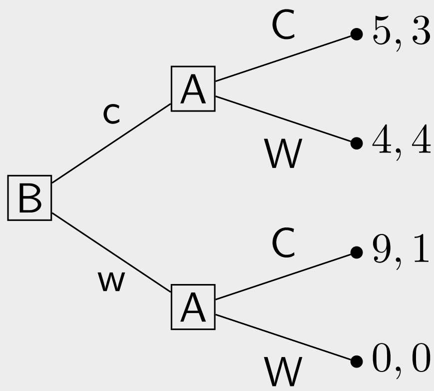


Exercises

- What is Bob's best response to Alice waiting? To Alice Climbing?
- Are there any equilibrium pairs/points? If so, which are they?

Equilibrium and solutions





Exercise

For the problems above, find all the equilibrium plays.

- In games that aren't strictly competitive, solution are less clear, because opportunities for co-operation arise
- Other considerations include: group benefit (Pareto optimality), initial tendencies (equilibrium), *etc.*

Non-strictly-competitive games

Example (The Prisoner's Dilemma)

Alice and Bob are suspects in a joint crime. The police doesn't yet have enough evidence to convict both/either, so it is trying to get either to implicate the other. The police inspector offers each separately a reduced sentence if they 'defect' (D) by implicating their accomplice.

If both suspects defect they will get a moderate sentence each (2 years). If only one defects he will get immunity, and the other will get the full sentence (3 years). If neither defects—*i.e.*, they both cooperate (C) with each other—both will be charged for only a minor offence (1 year).

	d	c
D	$1, 1$	$3, 0$
C	$0, 3$	$2, 2$

The payoff is the *reduction* in the player's sentence: $3 - s$, where $s \in \{0, 1, 2, 3\}$ is the length of the sentence.

Cooperation in non-zero-sum games

		B	
		d	c
A	D	$1, 1$	$3, 0$
	C	$0, 3$	$\boxed{2, 2}$

- Individual rationality (dominance) suggests both should defect (Dd); but mutual cooperation (Cc) better outcome for both
- In games which aren't strictly competitive cooperation may be possible
- What's best individually (individual rationality) may not best collectively, and vice versa
- Here Cc gives each player a better payoff than the individually rational play Dd

The Prisoner's Dilemma

Definition (Pareto optimality)

An outcome is *Pareto optimal* iff there is no other outcome which is at least as good or better for all the agents. not dominated

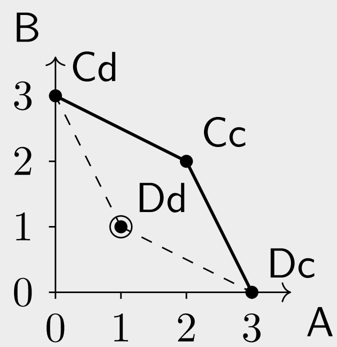
Pareto principle

Pareto optimal outcomes are optimal for a group.

Two-player *play diagram*:

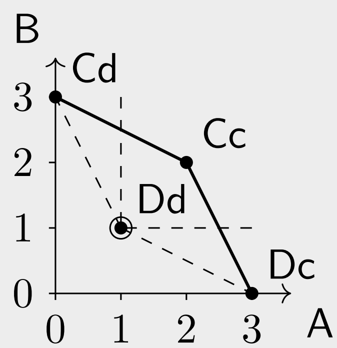
- x-value (abscissa) is A's payoff
- y-value (ordinate) is B's payoff

Pareto optimal outcomes represented by points on solid line



The Prisoner's Dilemma

		B	
		d	c
A	D	$1,1$	$3,0$
	C	$0,3$	$2,2$

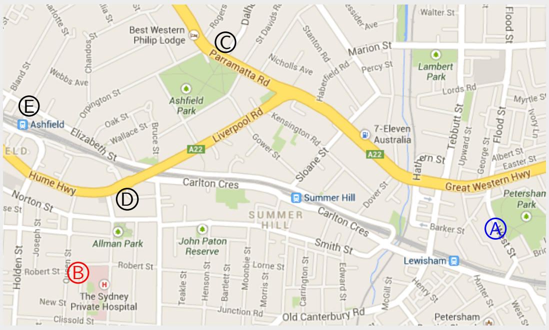


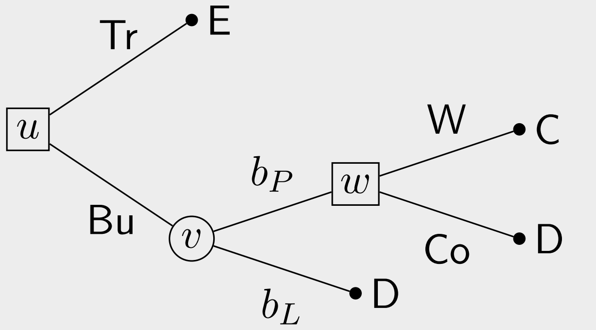
- The equilibrium is Dd (circled)
- The Pareto optimal outcomes are: Cc, Cd, Dc
- Play Cc, which is Pareto optimal, is better than Dd for both players

Conclusion

In two-player non-strictly-competitive games, what's best for the individual may not be best for the group; *i.e.*, *cooperation* desirable.

'Nature' as a player





- Can regarded single-agent decisions as games against a neutral player called 'Nature', or 'Chance', who has no preferences*
- Games in which some of the players' preferences are unknown are said to have *incomplete information*—as opposed to *imperfect information*, in which information sets may have multiple nodes
- In extensive form, Nature's moves take place at chance nodes, and they correspond to chance events

Summary

- Best response strategies
- Equilibrium in zero-sum games: saddle-points
- Solutions: rational reduction
- Non-zero-sum games: group preference and Pareto optimality; cooperation
- Single agent decisions are 'games against nature'