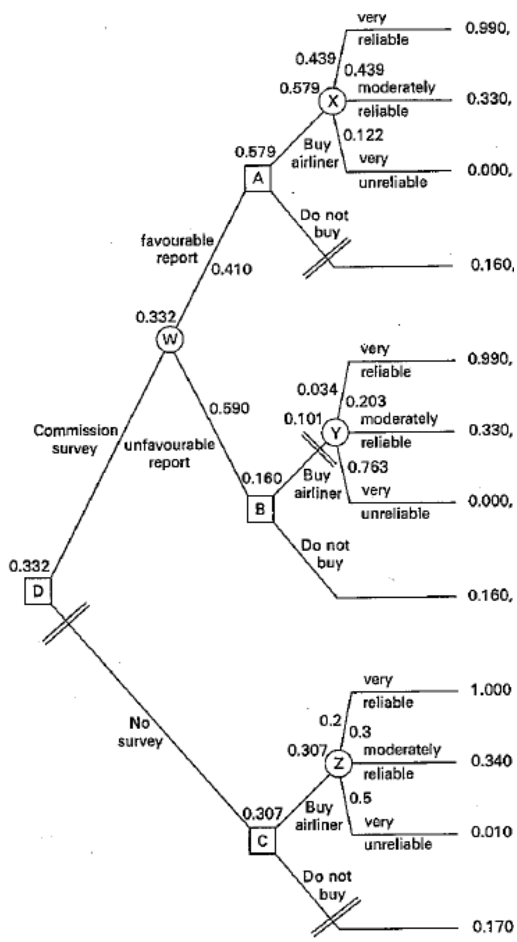
GSOE9210 Engineering Decisions

Problem Set 08

1. Consider the airliner problem discussed in lectures:



The decision-maker's initial epistemic state (*i.e.*, *prior* probabilities), based on general industry information, and preferences (utilities) for buying an aircraft, based on its operational reliability, are given below.

	Reliability		
	vR	mR	uR
Probability	0.2	0.3	0.5
Utility	1.0	0.34	0.01

The values at decision points A and B in the diagram above are based on the original consulting firm's (call it F_1) accuracy:

	... given:		
Probability of:	vR	mR	uR
f	0.9	0.6	0.1
u	0.1	0.4	0.9

Consider a second consulting firm (F_2) which charges the same \$10,000 fee, but for which:

	... given:		
Probability of:	vR	mR	uR
f	0.6	0.5	0.4
u	0.4	0.5	0.6

a) Verify that subtracting a fixed value from the utility of each outcome (*e.g.*, the cost of the report) of the sub-tree with root A results in the same reduction in the value of node A.
b) Would you expect the value of the information provided by F_2 to be better or worse than that of F_1?
c) Calculate the probabilities of the airliner's reliabilities (*i.e.*, $P(vR|f)$, *etc.*) based on a favourable assessment by F_2; *i.e.*, the updated probabilities for the branches at node X.
d) Repeat the above for node Y.
e) What are the utilities of buying the airliner if the report is favourable and unfavourable (nodes X and Y) respectively?
f) What would be the utility of commissioning the report if it turns out to be favourable (*i.e.*, the value at node A) and unfavourable (*i.e.*, the value at node B) respectively?
g) For firm F_2, determine the updated likelihoods of the report being favourable (f) and unfavourable (u) (*i.e.*, the probabilities associated with the branches at node W).
h) What is the value of commissioning F_2's report (*i.e.*, the utility at node W)?
i) Would it be worthwhile paying F_2's \$10,000 fee?
j) Suppose a third company, F_3, regularly gave 'incorrect' advice:

	... given:		
Probability of:	vR	mR	uR
f	0.1	0.5	0.8
u	0.9	0.5	0.2

How valuable would you expect their information to be?

Solution

For F_2, the distribution of probabilities over the three states is relatively uniform. This should suggest, at least intuitively, that F_2 is not a good discriminator of the reliability of aircraft. We might suspect, then, that their assessment may not provide much useful information, and that, hence, it may not be worth much.

a) Because *Bayes* utility is a linear operation, if we assume that the utility also varies approximately linearly with the cost of the fee, adding or subtracting a constant dollar amount from each outcome should have the same effect (reduction by that constant) on the *Bayes* utility. This can be verified by direct calculation.

b) The original consulting firm (call it F_1) would be expected to be a good predictor of reliability: *e.g.*, $P_{F_1}(f|vR)$ is 90%. Firm F_2 would be expected to be less so: *e.g.*, for F_2 we have that $P_{F_2}(f|vR)$ is only 60%; *i.e.*, the correlation between the reliability of the aircraft and the outcome of the report is lower. Consequently, we might expect the value of the information gained by commissioning F_2's report to be lower than that of F_1's.

c) At point C for F_2:[1]

$$\begin{aligned} P_{F_2}(vR|f) &= \tfrac{P_{F_2}(f|vR)P(vR)}{P_{F_2}(f|vR)P(vR)+P_{F_2}(f|mR)P(mR)+P_{F_2}(f|uR)P(uR)} \\ &= \frac{0.6(0.2)}{0.6(0.2)+0.5(0.3)+0.4(0.5)} \\ &= \frac{0.12}{0.47} \approx 0.26 \qquad (1) \end{aligned}$$

Similar calculations yield:

$$P_{F_2}(mR|f) \approx 0.32 \qquad P_{F_2}(uR|f) \approx 0.43$$

d) For an unfavourable report:

$$\begin{aligned} P_{F_2}(vR|u) &= \tfrac{P_{F_2}(u|vR)P(vR)}{P_{F_2}(u|vR)P(vR)+P_{F_2}(u|mR)P(mR)+P_{F_2}(u|uR)P(uR)} \\ &= \frac{0.4(0.2)}{0.4(0.2)+0.5(0.3)+0.6(0.5)} \\ &= \frac{0.08}{0.53} \approx 0.15 \qquad (2) \end{aligned}$$

Similarly:

$$\begin{aligned} P_{F_2}(mR|u) &\approx 0.28 \\ P_{F_2}(uR|u) &\approx 0.57 \end{aligned}$$

e) At chance nodes X (favourable) and Y (unfavourable) the *Bayes* value (expected utility) is:

$$\begin{aligned} U(X) &= 0.26(0.99) + 0.32(0.33) + 0.43(0) = 0.36 \\ U(Y) &= 0.15(0.99) + 0.28(0.33) + 0.57(0) = 0.25 \end{aligned}$$

f) The values above are both better than the status quo's (not buying) value of 0.16, which includes the cost of the consultation fee. These become the new utilities at decision nodes A and B.

[1] Note that the prior probabilities $P(vR)$, *etc.*, which appear at decision points A, B, and C are common to both F_1 and F_2.

g) The updated probabilities of favourable and unfavourable reports are just the denominators of equations (1) and (2):

$$P_{F_2}(f) = 0.47$$
$$P_{F_2}(u) = 0.53$$

h) The utility at node W is: $U(W) = 0.47(0.36) + 0.53(0.25) = 0.30$.

i) The value above is *less* than the utility of the decision at C in which the report is not commissioned. That is, commissioning F_2's report adds no value; in fact it has had a net reduction in utility, due to the consultation fee, compared to making a decision based on the original prior information from general industry reliability estimates. The best strategy for the company based on dealing with F_2 (instead of F_1) is now to: not commission the report, and buy the airliner. In other words, F_2's \$10,000 fee is not worth the value of the information it provides.

j) The table suggests that F_3's report has good predictive ability, albeit it happens that one would be better off taking the opposite recommendation. Or, alternatively, one should interpret their 'favourable' as 'unfavourable', and vice versa. One might expect the optimal strategy would then be:

> Commission the report; if the report is unfavourable, then buy; otherwise (report favourable), don't buy

Moreover, this strategy may well have greater utility than not commissioning the report.
Verifying this is left as an exercise.

2. Consider five possible prizes/outcomes, $x_1, \ldots, x_5$, listed by a rational agent in non-increasing order of preference (*i.e.*, $x_1 \succsim x_2 \succsim \cdots \succsim x_5$). Further, assume that when interviewed further the agent is unable to give precise preferences but specifies the following:

A $[0.9 : x_1 | 0.1 : x_5] \succ x_2 \succ [0.8 : x_1 | 0.2 : x_5]$
B $[0.42 : x_1 | 0.2 : x_4 | 0.38 : x_5] \succ [0.3 : x_1 | 0.6 : x_4 | 0.1 : x_5] \succ [0.38 : x_1 | 0.2 : x_4 | 0.42 : x_5]$
C $[0.7 : x_1 | 0.3 : x_5] \succ x_3 \succ [0.5 : x_2 | 0.5 : x_4]$

Given the uncertainty in the agent's utility estimates:

(a) Find the range of utility values for each of $x_1, \ldots, x_5$. You may assume utilities are in the range $[0, 1]$.
(b) Determine the agent's preference relation (*i.e.*, $\succ$, $\sim$, or indeterminate[2]) between the two lotteries: $[0.5 : x_3 | 0.5 : x_4]$ and $[0.5 : x_2 | 0.5 : x_3]$.
(c) Determine the agent's preference relation between the lottery $[0.3 : x_1 | 0.1 : x_2 | 0.5 : x_3 | 0.1 : x_4]$ and the outcome x_3 (You can think of the latter as a certain lottery; one which always results in x_3.)

[2] Because the utilities of the prizes are not precisely determined, it may be that for some utility values one lottery is preferred to another and for other values the opposite is the case. In this case the preference relation would be *indeterminate*.

(d) Determine the agent's preference relation between the two lotteries $[0.1 : x_2|0.6 : x_3|0.3 : x_4]$ and $[0.1 : x_2|0.7 : x_3|0.2 : x_4]$

(e) Determine the agent's preference relation between the two lotteries $[0.5 : x_1|0.5 : x_4]$ and $[0.2 : x_1|0.6 : x_3|0.2 : x_5]$

Solution

Note that from the condition $[0.9 : x_1|0.1 : x_5] \succ x_2 \succ [0.8 : x_1|0.2 : x_5]$, it follows that $x_1 \succ x_5$.

One thing to be careful of is that if two functions f and g both depend on the same variable x then the values of f and g are likely to be coupled to some extent. For example, if $x \in [0, 1]$ then $\mathrm{ran}(3x + 2) = [2, 5]$ and $\mathrm{ran}(2x + 3) = [3, 5]$. Now there are clearly some values of $u \in [3, 5] = \mathrm{ran}(2x+3)$ which are less than some $v \in [2, 5] = \mathrm{ran}(3x+2)$; *i.e.*, for which $u - v < 0$. But, there is no value of $x \in [0, 1]$ for which $2x + 3$ is less than $3x+2$ (for the same $x \in [0, 1]$). In particular, $2x+3-(3x+2) = 1-x \geqslant 0$ for all $x \in [0, 1]$.

This means that if we had two completely independent (uncoupled) functions f and g then we would have to compare their values for all values in their respective ranges. But if they are coupled on x, then we should compare only for variation in values other than in x.

To account for this coupling we compare differences in utilities: (*i.e.*, $U(\ell_1) - U(\ell_2)$) for each pair of lotteries. If the difference is positive for the entire range of possible utilities of the prizes, then the agent should choose ℓ_1; if it is always negative the agent should choose ℓ_2; if it is always zero then the agent is indifferent between the two lotteries. If for some combination of possible values the difference is positive, but negative for others, then the relation of preference between the two is *indeterminate*; *i.e.*, cannot be determined definitively.

In many cases, general reasoning may help avoid calculation altogether. Note these examples below.

Let $u_i = u(x_i)$ be the utility of prize i, and set $u_1 = 1$ and $u_5 = 0$.

(a) Then, from the agent's preferences above, we get:

$$\begin{aligned} u_1 &= 1 \\ 0.9 > u_2 &> 0.8 \\ 0.7 > u_3 &> 0.5u_2 + 0.5u_4 > 0.5 \\ 0.3 > u_4 &> 0.2 \\ u_5 &= 0 \end{aligned}$$

Because we can determine concrete (numerical) lower bounds for u_2 and u_4, from conditions A and B above, we can establish a concrete lower bound for u_3; *i.e.*, from C above, $u_3 > 0.5u_2+0.5u_4 > 0.5(0.8)+ 0.5(0.2) = 0.5(0.8 + 0.2) = 0.5$.

(b) Note that

$$\begin{aligned} U([0.5 : x_2|0.5 : x_3]) &= \tfrac{1}{2}u_2 + \tfrac{1}{2}u_3 \\ U([0.5 : x_3|0.5 : x_4]) &= \tfrac{1}{2}u_3 + \tfrac{1}{2}u_4 \end{aligned}$$

That is, the values of the two lotteries are coupled on u_3.
Because $x_2 \succ x_3$ and $x_3 \succ x_4$, it follows that

$$\begin{aligned} U([0.5 : x_2|0.5 : x_3]) - U([0.5 : x_3|0.5 : x_4]) &= \tfrac{1}{2}u_2 + \tfrac{1}{2}u_3 - (\tfrac{1}{2}u_3 + \tfrac{1}{2}u_4) \\ &= \tfrac{1}{2}(u_2 - u_4) \\ &> 0 \quad (\text{as } u_2 > u_3 > u_4) \end{aligned}$$

Note that the coupling term vanishes, which means the uncertainty in u_3 is eliminated.
Therefore, $[0.5 : x_2|0.5 : x_3] \succ [0.5 : x_3|0.5 : x_4]$.

(c) Subtracting expected utilities:

$$\begin{aligned} (0.3u_1 + 0.1u_2 + 0.5u_3 + 0.1u_4) - u_3 &= 0.3 + 0.1u_2 - 0.5u_3 + 0.1u_4 \\ &> 0.3 + 0.1(0.8) - 0.5(0.7) + 0.1(0.2) \\ &= 0.3 + 0.08 - 0.35 + 0.02 \\ &= 0.05 > 0 \end{aligned}$$

Note that because in this case the coefficient of u_3 is negative, the lower bound for the difference above is obtained by the *upper* bound for u_3 (0.7).
It follows that $[0.3 : x_1|0.1 : x_2|0.5 : x_3|0.1 : x_4] \succ x_3$.

(d) Subtracting expected utilities:

$$\begin{aligned} (0.1u_2 + 0.6u_3 + 0.3u_4) - (0.1u_2 + 0.7u_3 + 0.2u_4) &= -0.1u_3 + 0.1u_4 \\ &= 0.1(u_4 - u_3) < 0 \end{aligned}$$

So $[0.1 : x_2|0.7 : x_3|0.2 : x_4] \succ [0.1 : x_2|0.6 : x_3|0.3 : x_4]$.
Alternatively, observe that as $x_3 \succ x_4$, it follows by monotonicity that $[0.7 : x_3|0.2 : x_4] \succ [0.6 : x_3|0.3 : x_4]$, hence by substitution $[0.1 : x_2|0.7 : x_3|0.2 : x_4] \succ [0.1 : x_2|0.6 : x_3|0.3 : x_4]$.

(e) Subtracting expected utilities:

$$(0.5u_1 + 0.5u_4) - (0.2u_1 + 0.6u_3 + 0.2u_5) = 0.3 + 0.5u_4 - 0.6u_3$$

But:

$$\begin{aligned} 0.3 + 0.5(0.2) - 0.6(0.7) < 0.3 + 0.5u_4 - 0.6u_3 &< 0.3 + 0.5u_4 - 0.6(0.5u_2 + 0.5u_4) \\ 0.3 + 0.1 - 0.42 < 0.3 + 0.5u_4 - 0.6u_3 &< 0.3 - 0.3u_2 + (0.5 - 0.3)u_4 \\ -0.02 < 0.3 + 0.5u_4 - 0.6u_3 &< 0.3 - 0.3u_2 + 0.2u_4 \\ -0.02 < 0.3 + 0.5u_4 - 0.6u_3 &< 0.3 - 0.3(0.8) + 0.2(0.3) \\ -0.02 < 0.3 + 0.5u_4 - 0.6u_3 &< 0.3 - 0.24 + 0.06 = 0.06 + 0.06 \\ -0.02 < 0.3 + 0.5u_4 - 0.6u_3 &< 0.12 \end{aligned}$$

Since this range includes both positive and negative values, the preference between the lotteries is indeterminate (but not indifference).