

# CS540 Fall 2023 Homework 5

## 1 Linear Regression on Lake Mendota Ice

The Wisconsin State Climatology Office keeps a record on the number of days Lake Mendota was covered by ice at [link](#). Click the "Winter" column so the year is in **ascending** order.

## 2 Question 1: Data Curation

As with any real problems, the data is not as clean or as organized as one would like for machine learning. Curate a clean dataset starting from year **1855-56** and ending in year **2021-22**(Not 2022-23). We care about the following aspects of the data:

- $x$ , the starting year: for 1855-56,  $x = 1855$ ; for 2008-09,  $x = 2008$ ; and so on.
- $y$ , the number of ice days in that year. For Mendota in 1855-56,  $y = 118$ ; for 2008-09,  $y = 97$ ; and so on.

Some years have multiple freeze thaw cycles such as 2001-02; you should use the aggregated number of days. In the table, this appears as a year followed by a "-"(dash). You'll notice exactly one of the two lines has a number for ice days. That number is the one you'll use as the  $y$  for that year. For example, in year 2001-02 your feature will be:  $x = 2001$ ,  $y = 21$ . Save your dataset as "hw5.csv". We gave you an example toy.csv with the correct format (but the numbers are fake). Your file should follow this standard format for ".csv" files. For example, the first 4 lines of your "hw5.csv" would be the following:

```
year,days
1855,118
1856,151
1857,121
```

Output: Create (possibly manually) a csv file and name it to be "hw5.csv", as described above.

### 3 Question 2: Visualize Data

For this and the following questions, you need to write a python program hw5.py. Your code should take a argument the name of the csv file that you want to read (eg. toy.csv or hw5.csv), as we will run your program by:

```
$ python3 hw5.py filename.csv
```

Your program should work for any filename.

To get the first argument you can use the following code:

```
import sys
```

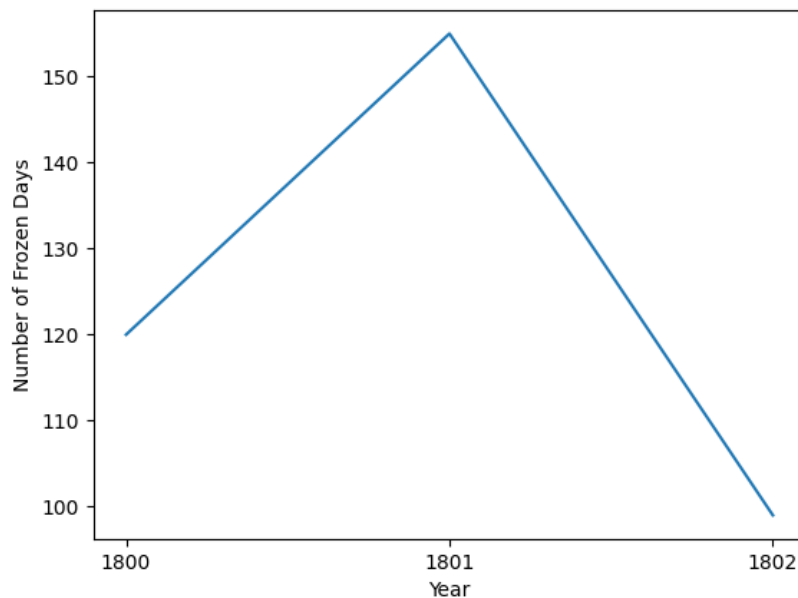
```
sys.argv[1] # the first argument as string
```

Your hw5.py then needs to produce the outputs in the order described in Question 2 - Question 6.

Plot Year vs. Number of Frozen Days from your dataset. You should save the plot as a plot.jpg. You can save the plot by using the following code:

```
plt.savefig("plot.jpg") # must be plot.jpg
```

For reference, we gave you the output plot.jpg for toy.csv:



Note: You do not need to fully match the plot style, but should have x-axis labels, y-axis labels, and the line.

Output: Your "hw5.py" would need to produce a "plot.jpg" file using plt.savefig.

## 4 Question 3: Linear Regression

Using the whole dataset as the training set, train a linear regression model:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

Recall, this means finding the closed-form MLE solution for  $\beta = (\beta_0, \beta_1)^\top$  :

$$\hat{\beta} = \arg \min_{\beta} \frac{1}{n} \sum_{i=1}^n (x_i^\top \beta - y_i)^2.$$

To write the solution in matrix form, we first augment the feature vector:

$$\mathbf{x}_i = \begin{pmatrix} 1 \\ x_i \end{pmatrix}$$

Note boldface  $\mathbf{x}_i$  is a vector while  $x_i$  is a scalar. Now organize the features in a  $n \times 2$  array, where  $n$  is the number of data points (roughly the number of rows in your computed csv).

$$X = \begin{pmatrix} \mathbf{x}_1^\top \\ \vdots \\ \mathbf{x}_n^\top \end{pmatrix}$$

And the  $y$  values into a vector:

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The MLE solution can be written as

$$\hat{\beta} = \arg \min_{\beta} \|X\beta - Y\|^2.$$

By setting the gradient to zero, we arrive at the closed-form MLE:

$$\hat{\beta} = (X^\top X)^{-1} X^\top Y.$$

This involves the inverse of  $X^\top X$ , which for our problem is invertible. Your program should compute  $\hat{\beta}$  as specified here. We will break down this process into several steps.

### 4.1 Q3a:

Represent the data as a matrix  $X$ , which will have dimension  $n \times 2$ . Recall that for each individual data point  $x_i$ , you should transform the point into a full feature vector

$$\mathbf{x}_i = \begin{pmatrix} 1 \\ x_i \end{pmatrix}$$

Then, you make each feature vector become a row of the overall data matrix  $X$  :

$$X = \begin{pmatrix} \mathbf{x}_1^\top \\ \vdots \\ \mathbf{x}_n^\top \end{pmatrix}$$

Your output for this section is formed by printing out  $X$  as follows:

```
print("Q3a:")
print(X) # Your X.dtype needs to be int64
Your output's dtype needs to be int64.
```

#### 4.2 Q3b:

Next, you need to place all the corresponding  $y_i$  values into a vector

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Your output for this section is formed by printing out  $Y$  as follows:

```
print("Q3b:")
print(Y) # Your Y.dtype needs to be int64
Your output's dtype needs to be int64.
```

#### 4.3 Q3c:

Next, compute the matrix product  $Z = X^T X$ .

Your output for this section is formed by printing out  $Z$  as follows:

```
print("Q3c:")
print(Z) # Your Z.dtype needs to be int64
Your output's dtype needs to be int64.
```

#### 4.4 Q3d:

Next, compute the inverse of  $X^T X$ , which we call  $I$ .

Your output for this section is formed by printing out  $I$  as follows:

```
print("Q3d:")
print(I)
```

#### 4.5 Q3e:

Next, compute what we call the pseudo-inverse of  $X$ , which we call  $PI$ . Mathematically,  $PI = (X^T X)^{-1} X^T$ . Your output for this section is formed by printing out  $PI$  as follows:

```
print("Q3e:")
print(PI)
```

#### 4.6 Q3f:

Lastly, compute  $\hat{\beta}$  using the results from the previous parts. Recall,

$$\hat{\beta} = (X^T X)^{-1} X^T Y.$$

Your output for this section is formed by printing out  $\hat{\beta}$  as follows:

```
print("Q3f:")
print( $\hat{\beta}$ )
```

### 5 Q3 Summary

Output: Your program should output the matrices  $X, Y, X^T X, (X^T X)^{-1}, (X^T X)^{-1} X^T$  and  $\hat{\beta}$ . If you do each part correctly, your code will print out the answers in the following format, where each variable will be replaced with the actual computed value:

```
Q3a:
X
Q3b:
Y
Q3c:
XTX
Q3d:
(XTX)-1
Q3e:
(XTX)-1XT
Q3f:
 $\hat{\beta}$ 
```

We gave you the complete sample output for toy.csv at the end of this file. You should test your code on this file.

### 6 Question 4: Prediction

Using your  $\hat{\beta}$ , predict the number of ice days for winter 2022-23(previous winter). Equivalently, we have a test item  $x_{\text{test}} = 2022$  and you should predict

$$\hat{y}_{\text{test}} = \hat{\beta}_0 + \hat{\beta}_1 x_{\text{test}} .$$

The Wisconsin State Climatology Office has released the 2022-23 official numbers, you may want to check to see how close your prediction was.

Output: Print the following similar to in Q3 but combine the question number and the answer:

Q4:  $\hat{y}_{\text{test}}$  You can do this with the following code:

```
print("Q4: " + str( $\hat{y}_{\text{test}}$ ))
```

You will use the same formatting to print the answers to the remaining questions. There should be a space after ":".

## 7 Question 5: Model Interpretation

(a) What is the sign of your  $\hat{\beta}_1$  ? Print a symbol where the symbol should be either  $>$ ,  $<$ ,  $=$  depending if the sign is positive, negative or zero.

(b) Interpret, in English, the meaning of the sign for Mendota ice. You will explain the meaning of all the three signs, individually. Print a short answer explanation. You can separate your sentences by space but not a new line.

Output: Your program should print the following lines:

Q5a: Symbol

Q5b: Short Answer

There should be a space after ":".

## 8 Question 6: Model Limitation

(a) Given your MLE  $\hat{\beta}$ , predict the year  $x^*$  by which Lake Mendota will no longer freeze. That is,

$$0 = \hat{\beta}_0 + \hat{\beta}_1 x^*.$$

Note  $x^*$  will in general be a real number instead of an integer.

(b) Discuss whether  $x^*$  is a compelling prediction based on the trends in the data, and why. Note that your answer should base on hw5.csv rather than toy.csv. You can separate your sentences by space but not a new line.

Output: Your program should print the following lines:

Q6a:  $x^*$

Q6b: Answer

Where Answer should be replaced with your answer for part b. There should be a space after ":".

## 9 Summary

You will create two files named hw5.csv and hw5.py.

hw5.csv: This will be a csv file containing two column headers: "year" and "days" in that order. Then, each row will contain the corresponding  $x$  and  $y$  defined in Question 1 in the **ascending** year order as they appear in the original Mendota dataset starting with year 1855-56.

hw5.py: This will be a python file that produces a plot for Q2 (**save it instead of plt.show() it!**) and then that prints out the answers to the following questions:

Q3a: Compute  $X$

Q3b: Compute  $Y$

Q3c: Compute  $X^T X$

Q3d: Compute  $(X^T X)^{-1}$

Q3e: Compute  $(X^T X)^{-1} X^T$

Q3f: Compute  $\hat{\beta}$

Q4: Compute a prediction for year  $x_{\text{test}} = 2022$  using your linear regression model.

Q5a: Compute the sign of  $\hat{\beta}_1$

Q5b: Explain what this sign could mean.

Q6a: Solve the equation  $0 = \hat{\beta}_0 + \hat{\beta}_1 x^*$  for  $x^*$ .

Q6b: Discuss whether this  $x^*$  makes sense given what we see in the data trends.

## 10 Submission Details

- There is no starter code for this assignment.
- All code, except importing modules, should be contained in functions or under `if __name__ == "__main__":`
- Please submit your hw5.csv and hw5.py to **Gradescope**. No late submission will be accepted.
- Only python built-in library, numpy, matplotlib and pandas are allowed.
- Remove all debugging output before submission.
- Do not include `plt.show()` in your submission.
- Q6b should address hw5.csv

## 11 Example Input/Output

**toy.csv:**

```
year,days
1800,120
1801,155
1802,99
```

**Run parameters:**

```
$ python3 hw5.py toy.csv
```

**Output:** (Note plot.jpg should be produced but is not shown here as it is in the section above)

Q3a:

```
[[ 1 1800]
 [ 1 1801]
 [ 1 1802]]
```

Q3b:

```
[120 155 99]
```

Q3c:

```
[[      3      5403]
 [ 5403 9730805]]
```

Q3d:

```
[[ 1.62180083e+06 -9.00500000e+02]
 [-9.00500000e+02  5.00000000e-01]]
```

Q3e:

```
[[ 9.00833334e+02  3.33333333e-01 -9.00166667e+02]
 [-5.00000000e-01  0.00000000e+00  5.00000000e-01]]
```

Q3f:

```
[ 1.90351667e+04 -1.05000000e+01]
```

Q4: -2195.833334037743

Q5a: <

Q5b: Answer

Q6a: 1812.873015872499

Q6b: Answer

We also gave you toy\_output.txt. You can use vimdiff on CSL Machines to compare your result with the example output.

```
$ python3 hw5.py toy.csv | tee your_output
$ vimdiff your_output toy_output.txt
```

Note that your result might be different from the example output because the code was running on different environments. We will grade your work on Gradescope which will run submissions on the same environment. In addition, a tolerance has been built into the grader.