Quantathon — February 22, 2025

Let N be a positive integer, and consider the following game played over N rounds. In each round $n = 1, \ldots, N$, a number U_n is drawn independently and uniformly at random from the unit interval $[0, 1]$. The value of U_n is not revealed to you, but only its *rank* R_n among the numbers drawn so far. The rank R_n is defined as the number $k \in \{1, \ldots, n\}$ such that U_n is the kth largest among the numbers $U_1, \ldots, U_n$ drawn so far. Thus $R_n = 1$ if U_n is the largest so far, $R_n = 2$ is U_n is the second largest so far, and so on. Because the numbers $U_1, \ldots, U_N$ are drawn independently and uniformly at random from the unit interval, they are all distinct with probability one, and there is no need to define the rank in case of ties.

Just before the nth draw, you must decide whether to Play or to Pass. Your decision can depend on the ranks $R_1, \ldots, R_{n-1}$ that you have observed in previous rounds, but not on the values $U_1, \ldots, U_{n-1}$ themselves (which were not revealed to you), nor on any outcomes from the current or future rounds. If you choose to Play, you pay $\$\frac{1}{n}$, and once the rank R_n is revealed, you receive \$1 if $R_n = 1$ and \$0 otherwise. If you Pass, you do not have to pay anything, but you also do not receive anything. Your overall gain/loss in round n is:

- If you choose to Play: $$\begin{cases} 1 - \dfrac{1}{n} & \text{if } R_n = 1, \\ -\dfrac{1}{n} & \text{otherwise} \end{cases} \tag{1}$$

- If you choose to Pass: $\quad 0$

We let W_N denote the total winnings at the end of the game. Thus W_N is the sum of all gains/losses over the N rounds. This may be positive, negative, or zero.

Example. Here is an example with $N = 4$ rounds:

Round n	1	2	3	4
U_n	.734	.8317	.7906	.023
R_n	1	1	2	4
Action	Pass	Play	Play	Pass
Gain/Loss	0	$1 - \frac{1}{2}$	$-\frac{1}{3}$	0

As a player, you do not get to see the values of U_n, only the sequence of ranks $(1, 1, 2, 4,$ revealed one round at a time.) Your action in each round has to be determined just before the draw in that round. In this example the following strategy was used: if a rank of one was just observed, then Play in the current round; otherwise Pass. The first action was Pass because nothing had been observed yet; the second action was Play because $R_1 = 1$ had just been observed; the third action was also Play because $R_2 = 1$ had just been observed; and the fourth action was Pass because $R_3 = 2 \neq 1$ had just been observed. The total winnings at the end of the game is $W_N = (1 - \frac{1}{2}) - \frac{1}{3} = \frac{1}{6}$ in this example. Note that W_N depends both on the random draws and on the strategy employed.

Problem 1. This problem considers the game described above.

(a) Let $N = 4$ and consider the strategy that chooses to Play in every round. Compute your expected winnings $\mathbb{E}[W_N]$ at the end of the game.

(b) Let $N = 4$. Can you find a strategy such that your expected winnings $\mathbb{E}[W_N]$ is higher than the value obtained in (a)?

(c) Let $N = 2025$. Find the largest possible value of the expected winnings $\mathbb{E}[W_N]$, and a strategy such that this value is achieved.

Problem 2. We now consider a variation of the game, where the player has more information. Not only are the ranks observed, but also the random draws U_n themselves. In this version of the game, your decision to Play or Pass in round n may depend on the values $U_1, \ldots, U_{n-1}$ drawn so far. (In round $n = 1$, no draws have yet been observed.)

(a) Let $N = 4$. Can you find a strategy such that the expected winnings are higher than the value obtained in Problem 1(a)?

(b) Let $N = 2025$. Find the largest possible expected winnings and a strategy such that this value is achieved.

Problem 3. As in Problem 2, you observe the draws U_n, not just the ranks. However, before each round, you must now decide whether to Play or to Stop. As soon as you Stop, the game is over, and you collect the winnings so far. More precisely:

Suppose you have chosen to Play in rounds $1, \ldots, n-1$. Going into round n, you use the values $U_1, \ldots, U_{n-1}$ observed so far to decide whether to Play or to Stop. If you choose to Stop, you collect the winnings W_{n-1} accumulated so far, and the game is over. If you choose to Play, the value of U_n is drawn, W_n is computed as W_{n-1} plus the amount in Eq. (1), and the game continues: you will now again be faced with the decision to Play or to Stop.

Prior to the first round $n = 1$, no draws have yet been observed, and the winnings so far are $W_0 = 0$. Prior to the final round $n = N$, if the game has not already ended and you decide to Play, you collect W_N and the game ends.

(a) Let $N = 3$. Find the largest possible expected winnings.

(b) Let $N = 2025$. Find the largest possible expected winnings.

(c) Present any other interesting observations or results that you have found. For instance: can you find the optimal strategy? What happens to the largest possible expected winnings as N tends to infinity?

Instructions. You may use mathematical arguments, computer simulations, numerical computations, or a combination of the three to obtain your answers. Regardless of your approach, please explain your solutions in detail. Partial solutions are often rewarded, so even if you are not able to fully solve some problems, be sure to show any progress you have made. Do not use generative AI tools such as ChatGPT.

Remarks.

(1) As stated above, $U_1, \ldots, U_N$ are independent and uniformly distributed in the unit interval. In particular, the cumulative distribution function (c.d.f.) of each U_n is $F(x) = P(U_n \leq x) = x$ for $x \in [0,1]$, $F(x) = 0$ for $x < 0$, and $F(x) = 1$ for $x > 1$. The probability density function (p.d.f.) is $f(x) = F'(x) = 1$ for $x \in (0,1)$ and $f(x) = 0$ for $x \notin (0,1)$.

(2) It is worth pointing out that $R_1 = 1$ always, regardless of the outcome of U_1.

(3) Your decision to Play or Pass (Stop in Problem 3) in the first round ($n = 1$) is made before you have observed anything at all. However, regardless of your decision, your gain/loss in the first round is always zero because $R_1 = 1$.