Tutorial 2 - Complex Functions

1. Suppose that $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$. If $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$, show that $\lim_{z \to z_0} f(z) = w_0$ if and only if
$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0.$$

2. Find the limit of $\lim_{z\to -2i}(2z^2 + 7z + 8)$ using definition of limit.

3. Evaluate $\lim_{z \to -i} \dfrac{z^{11} - i}{z^7 - i}$.

4. By using the definition of limit, show that
$$\lim_{z \to z_0} f(z) = w_0 \implies \lim_{z \to z_0} |f(z)| = |w_0|.$$
Hint: Use the relation $|z_1 \pm z_2| \geq ||z_1| - |z_2||$.

5. Evaluate $\lim_{z \to \infty} \dfrac{2z^2}{(z-1)^2}$.

6. Evaluate $\lim_{z \to 1} \dfrac{1}{(z-1)^2}$.

7. Evaluate $\lim_{z \to \infty} \dfrac{z^2 + 1}{z - 1}$.

8. By using the definition of continuity, show that $f(z) = z^2$ is continuous at $z = i$.

9. Find $f'(z)$ if $f(z) = 3z^2 - 5z + 14$.

10. Find $f'(z)$ if $f(z) = \frac{z-1}{2z+1}$ with $z \neq -1/2$.

11. Use definition of derivative to proof that $\dfrac{d}{dz}\dfrac{1}{z} = -\dfrac{1}{z^2}$.

12. Show that $f(z) = \mathrm{Re}(z)$ is not differentiable at any point z.

13. Use Cauchy-Riemann equations to show that $f'(z)$ does not exist at any point if
    (a) $f(z) = \overline{z} - z$
    (b) $f(z) = 3x + ixy^2$

14. Determine where $f'(z)$ exists and find its value if $f(z) = 1/z$.

15. Show that $f(z) = |z|$ is not differentiable everywhere.

16. Show that $f(z) = \sqrt{r}e^{i\theta/2}$ is differentiable in the domain: $\{r > 0, \alpha < \theta < \alpha + 2\pi\}$.

17. Show that $f(z) = 4x + 3y + i(4y - 3x)$ is entire.

18. Is $f(z) = xy + iy$ analytic?

19. Show that $f(z) = x^2 + y^2 + 2ixy$ is differentiable along real axis but it is not analytic.

20. Evaluate $e^{2+3\pi i}$.

21. Show that $|e^{z^2}| \leq e^{|z|^2}$.

22. Evaluate $\operatorname{Log}(-ei)$.

23. Evaluate $\log(-ei)$.

24. Solve for all z such that $e^z = i$.

25. Choose a counter-example to show that

$$\operatorname{Log}(z_1 z_2) \neq \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

26. Show that for any two nonzero complex numbers z_1 and z_2,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2N\pi i$$

where N has one of the values $0, \pm 1$.