Tutorial 8 - Conformal mapping and Dirichlet problem

1 Conformal mapping

1. Determine all the points in $\mathbb{C}$ for which the following mappings are not (locally) conformal.
   (a) $\sinh z$,
   (b) e^{z^4-32z}.

2. Show that the transformation $w = \sin z$ is conformal at all points except

$$z = \frac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \dots).$$

3. Joukowski's function has the form

$$J(z) = \frac{\alpha}{2}\left(z + \frac{1}{z}\right),$$

   where α is a positive constant. It maps a circle passing through -1 and containing 1 in its interior to an image called *Joukowski airfoil*. When a circle is appropriately chosen, the image looks like a cross-section of a wing. By adjusting the circle, one can adjust the exact shape of the airfoil. Using the explicit conformal map to the disk and explicit formulas for the solutions of Laplace equation on (the exterior of) a disk, one can calculate things like the drag and lift of the airfoil, and so this is a useful approach to airfoil design.

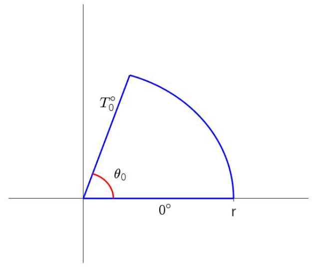
   (a) Show that Joukowski's function is formed from $f \circ g$ where

$$f(z) = \alpha\left[\frac{1+z}{1-z}\right], \quad g(z) = \left(\frac{z-1}{z+1}\right)^2.$$

   (b) Show that J is locally conformal in $\mathbb{C} \setminus \{\pm 1\}$.
   (c) Show that if $\alpha = 1$, J maps the upper semi-circle $\sigma = \left\{z : \ z = e^{i\theta}, 0 \leq \theta \leq \pi\right\}$ onto the real interval $J[\sigma] = [-1, 1]$, and the semi-infinite intervals $[1, \infty)$ and $(-\infty, -1]$ onto themselves.
   (d) Show that for $\alpha = 1$, J maps the set $\Omega = \{z : \ |z| \geq 1, 0 \leq \operatorname{Arg} z \leq \pi\}$ onto the upper half-plane $\{w = u + iv : \ v \geq 0\}$.

2 Harmonic function and Dirichlet problem

4. Show that the steady-state temperature in the shape below:

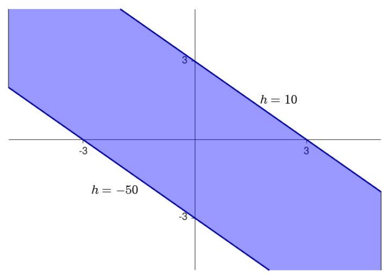


is

$$T(x,y) = \frac{T_0}{\theta_0} \tan^{-1}\left(\frac{y}{x}\right).$$

The arc $z = re^{i\theta}$ $(0 < \theta < \theta_0)$ is perfectly insulated.

5. Find a harmonic solution $h(x,y)$ to the infinite bar Dirichlet problem below and calculate $h(1,1)$.



6. Solve the following boundary value problem for the temperature function T:

$$\begin{aligned} T_{xx} + T_{yy} &= 0 \quad (x > 0, y > 0), \\ T(x,0) &= 0 \quad (x > 0), \\ T(0,y) &= 1 \quad (y > 0). \end{aligned}$$

3 Conformal mapping and Dirichlet problem

7. Use the function $\operatorname{Log} z$ to find an expression for the steady-state temperature (after the temperature reaches equilibrium distribution) in a plate $x > 0, y > 0$ if its faces are perfectly insulated and its edges have temperatures $T(x,0) = 0$ and $T(0,y) = 1$.

8. Solve the Dirichlet problem in the domain $D = \{z : |z| < 1\} \setminus \{z : |z - 0.3| \leq 0.3\}$ and takes that value 0 on the inner circle $C_1 \ : \ \{z : |z - 0.3| = 0.3\}$ and 1 on the outer circle $C_2 \ : \ \{z : |z| = 1\}$.

3.1 Interesting/challenging questions

9. $(w = \log \frac{z-1}{z+1})$ Solve the following Dirichlet problem:

$$T_{xx}(x,y) + T_{yy}(x,y) = 0 \quad (-\infty < x < \infty, y > 0),$$
$$\begin{aligned} T(x,0) &= 1 \quad \text{if } |x| < 1, \\ T(x,0) &= 0 \quad \text{if } |x| > 1. \end{aligned}$$

Also, $|T(x,y) < M|$ where M is some positive constant.

10. ($w = \sin z$) Solve the Dirichlet problem below:

$$
\begin{aligned}
\nabla T(x, y) &= 0 \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0\right), \\
T\left(-\frac{\pi}{2}, y\right) = T\left(\frac{\pi}{2}, y\right) &= 0 \quad (y > 0), \\
T(x, 0) &= 1 \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right).
\end{aligned}
$$