Tutorial 5 - Residues theorem and its applications

1. Find the residue at $z = 0$ of the function $f(z) = \frac{1}{z^2+2z}$.

2. Use Cauchy's residue theorem to evaluate

$$\int_C \frac{e^{-z}}{(z+1)^3}\, dz,$$

where C is the circle $|z| = 3$ oriented counterclockwise.

3. Let C be the circle $|z| = 2$ and oriented counterclockwise. Evaluate

$$\int_C \frac{dz}{1+z^3}.$$

4. By using principal part of Laurent series determine whether isolated singular point of functions below is a pole, removable singular point or an essential singular point. Determine its residue.

   (a) $ze^{1/z}$,

   (b) $\frac{z^2}{(z-1)^2}$,

   (c) $\frac{\cos z}{z}$.

5. Show that for $|z| > 0, -\pi < \arg z < \pi$,

$$\operatorname*{Res}_{z=1} \frac{z^{1/4}}{z-1} = 1$$

6. Evaluate the integral

$$\int_C \frac{3z^2+2}{z^3 - z^2 + 9z - 9}\, dz$$

where C is the positively oriented circle $|z-2| = 3$.

7. Show that

$$\operatorname*{Res}_{z=\pi i} \frac{z - \sinh z}{z^2 \sinh z} = \frac{i}{\pi}.$$

8. Evaluate

$$\int_C \frac{dz}{(z^2-1)^2+1},$$

where C is the circle $|z-2| = 2$

9. Determine the order of each pole and the corresponding residue.

   (a) $f(z) = \frac{z^2+2}{z-1}$,

   (b) $f(z) = \left(\frac{z}{2z+1}\right)^3$,

   (c) $f(z) = \frac{e^z}{z^2+\pi^2}$.

10. Evaluate
$$\int_0^\infty \frac{dx}{(x^2+1)}.$$

11. For $a > 0$, evaluate
$$\int_0^\infty \frac{\cos ax}{x^2+1}\,dx.$$

12. Evaluate
$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}.$$