Tutorial 4 - Analyticity

1. Expand $f(z) = (z-1)e^{2z}$ about $z = 1$.
2. Find the Taylor series representation of
$$f(z) = \frac{1}{1-z}$$
at $z = i$ and state its circle of convergence.
3. Find the Maclaurin series representation and its circle of convergence if
$$f(z) = \sinh z.$$
4. Find the Taylor series representation of $f(z) = \sinh z$ about $z = \pi i$ and its circle of convergence.
5. Show that
$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots \quad \text{for } z \neq 0.$$
6. Find the Laurent series representation of
$$f(z) = z^2 \cos\left(\frac{1}{z^2}\right).$$
7. Find the Laurent series representation of
$$f(z) = \frac{1}{z^2(1-z)},$$
about $z = 0$, in the region:
   (a) $0 < |z| < 1$,
   (b) $1 < |z| < \infty$.
8. For the function
$$f(z) = \frac{z+1}{z-1},$$
   (a) find its Maclaurin series, and state where the representation is valid,
   (b) find its Laurent series in the domain $1 < |z| < \infty$.
9. Show that for $0 < |w| < \infty$
$$e^{\frac{z}{2}(w - \frac{1}{w})} = \sum_{n=-\infty}^{\infty} J_n(z) w^n,$$
where
$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\phi - z\sin\phi)}\, d\phi = \frac{1}{\pi} \int_0^{\pi} \cos(n\phi - z\sin\phi)\, d\phi,$$
where $n = 0, \pm 1, \pm 2, \dots$. [Hint: use the contour integral where C denote the unit circle $w = e^{i\phi}$, where $-\pi \leq \phi \leq \pi$ to evaluate the coefficients of Laurent series of $f(w) = \exp(\frac{z}{2}(w - \frac{1}{w}))$ about $w = 0$.]

10. The function $\frac{\sin z}{z}$ is not defined for $z = 0$. However, show that the function f defined by

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{when } z \neq 0 \\ 1 & \text{when } z = 0 \end{cases},$$

is entire.

11. Using the result

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1,$$

to find the Maclaurin series representation of

$$\frac{1}{(1-z)^2}.$$

12. Replace $z \to 1/(1-z)$ in previous exercise, show that

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n (n-1)}{(z-1)^n}, \quad 1 < |z-1| < \infty.$$

13. By integrating

$$\frac{1}{w} = \sum_{n=0}^{\infty} (-1)^n (w-1)^n, \quad |w-1| < 1$$

along a contour interior to the circle of convergence from $w = 1$ to $w = z$, show that

$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n, \quad |z-1| < 1.$$

14. Find the Maclaurin series representation of

$$\frac{e^z}{1-z}, \quad |z| < 1$$

using multiplication.

15. Using division, show that the Maclaurin series of $\csc z = 1/\sin z$ is given by

$$\csc z = \frac{1}{z} + \frac{1}{3!} z + \left[\frac{1}{(3!)^2} - \frac{1}{5!} \right] z^3 + \ldots, \quad 0 < |z| < \pi.$$