Complex Analysis

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Contents

Chapter 1

Introduction

1.1 References

References that will be used for this subject are:

1. Main References:

   (a) Brown, J. W., & Churchill, R. V. (2009). Complex variables and applications. (8th ed.). McGraw-Hill.

2. Additional References:

   (a) Ahlfors, L. V. (1979). Complex analysis: an introduction to the theory of analytic functions of one complex variable. (3rd ed). McGraw-Hill.

   (b) Rudin, W. (1987). Real and complex analysis. (3rd ed.). McGraw-Hill.

   (c) Saff, E. B., & A. D. Snider. (2003). Fundamentals of Complex Analysis with Applications to Engineering and Science. Prentice Hall.

1.2 Square root of negative numbers

Solving a quadratic equation can be treated as finding the number of intersection between two lines.

Example 1.2.1. Consider the equation

$$x^2 + 5 = 0.$$

Solving this equation is equivalent to finding intersection of the curve $f(x) = x^2 + 5$ and the line $g(x) = 0$. See figure below.

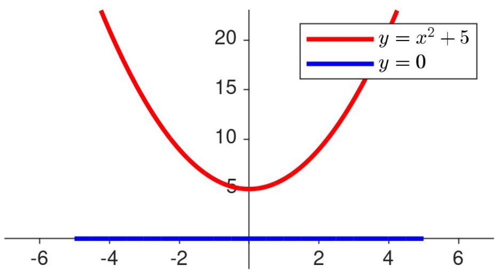


Figure 1.1: Intersection of $f(x) = x^2 + 5$ and $g(x) = 0$.

By writing this equation as $x^2 = -5$, we can also be viewed as intersection between $f(x) = x^2$, and $g(x) = -5$. There is no intersection between these two lines.

Similarly, $x^2+2x+1=5$ can be viewed as intersection of $f(x)=x^2+2x+1$ and $g(x)=5$. In this case, there are two intersection points. It is well known that the intersection points between $f(x)=ax^2+bx+c$ and $g(x)=0$ can be determined by the equation $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and

1. if $b^2-4ac>0$, then two distinct intersection points,
2. if $b^2-4ac=0$, then the quadratic curve touches x-axis at one point,
3. if $b^2-4ac<0$, then no intersection point.

Discriminant equals to zero implies that there is no real solution to the quadratic equation. However, complex numbers do not arise from solving quadratic equation, but from cubic equation.

The general cubic equation, $x^3+ax^2+bx+c=0$ can be reduced to *depressed cubic form*:

$$w^3+pw+q=0, \tag{1.1}$$

where $w=x+\frac{b}{3a}, p=\frac{3ac-b^2}{3a^2}$ and $q=\frac{2b^3-9acb+27a^2d}{27a^3}$. This implies that solving Equation (1.1), is equivalent to finding the intersection points between a cubic curve and a straight line.

Example 1.2.2. Consider the equation

$$x^3=2x+3.$$

The curve $f(x)=x^3$ and the line $g(x)=2x+3$ are shown below:

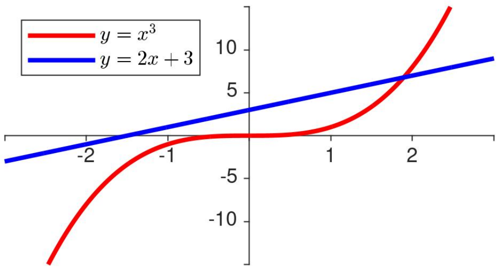


Figure 1.2: Intersection of $f(x)=x^3$ and $g(x)=2x+3$.

It can be easily seen that for any $p,q\in\mathbb{R}$, there is at least one intersection point.

Scipione del Ferro discovered the following formula for solving Equation (1.1):

$$w=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}. \tag{1.2}$$

In fact, it was also proven that if the discriminant, $\Delta=-(4p^3+27q^2)$,

1. $\Delta>0$, then Equation (1.1) has three distinct real roots,
2. $\Delta=0$, then Equation (1.1) has three real roots, where two of them are equal,
3. $\Delta<0$, then Equation (1.1) has one real and two complex roots.

Example 1.2.3. Consider the equation

$$x^3-2x+1=0.$$

This cubic intersects x-axis at three different points.

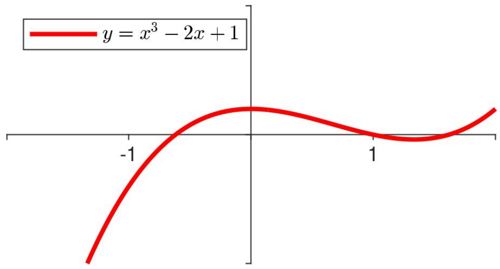


Figure 1.3: $f(x) = x^3 - 2 + 1$.

However, from Equation (1.2), we have that

$$x = \sqrt[3]{-\frac{1}{2} + \sqrt{-\frac{5}{108}}} + \sqrt[3]{-\frac{1}{2} - \sqrt{-\frac{5}{108}}},$$

which involves square root of negative number.

The existence of real roots implies the necessity of square root of negative numbers.

For any non-negative x, the square root of negative x can be written as $\sqrt{-x} = i\sqrt{x}$, where $i = \sqrt{-1}$ is a notation introduced by Euler. In fact, any complex number can be written as $x + iy$ where $x, y \in \mathbb{R}$ and can be visualized as point with rectangular coordinates. This will be discussed in more details in the next chapter.

1.3 Number system

The following figure appears frequently when one studies high school mathematics.

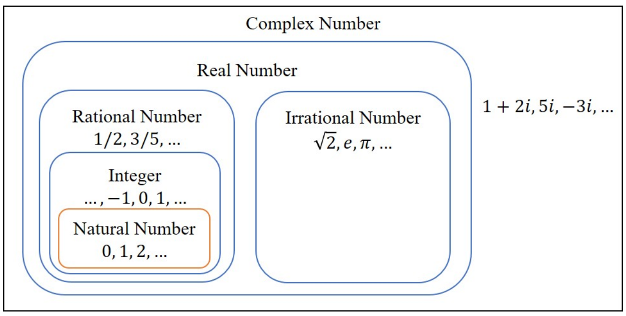


Figure 1.4: Number system

Note that there are two definitions of natural numbers, $\mathbb{N}$, which are positive integers ($\mathbb{N} = \{1, 2, 3, \ldots\}$) and non-negative integers ($\mathbb{N} = \{0, 1, 2, \ldots\}$).

1.3.1 Construction of natural numbers

Using set theory, one can construct the natural numbers as follows:

1. Set $0 = \emptyset$,
2. Define the successor function, $S(n) = n \cup \{n\}$,
3. $1 = 0 \cup \{0\} = \emptyset \cup \{0\} = \{0\} = \{\emptyset\}$,
4. $2 = 1 \cup \{1\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$.

Continuing in this fashion, one can defined all natural numbers. In fact, by axiom of infinity, there exist a set which contains 0 and is closed under the successor function. Such sets are said to be inductive. The intersection of all such inductive sets is defined to be the set of natural numbers.

1.3.2 Construction of integers

With p, q to be natural numbers and $p > q$, the equation $x + p = q$ has no solution in natural numbers. Thus, negative numbers are invented. The set of integers can be constructed as the equivalence classes of ordered pair of natural numbers, (a, b).

Definition 1.3.1. For $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$, a relation $\sim$ is defined as

$$(a, b) \sim (c, d) \quad \text{if and only if } a + d = b + c,$$

and $\sim$ is an equivalence relation.

It can be easily check that $(a, b) \sim (a+1, b+1) \sim \dots$, it can also be shown that if $a \leq b$, then $(a, b) \sim (0, k)$ where $a+k = b+0$. If $a > b$, then $(a, b) \sim (k, 0)$ where $a+0 = b+k$. For example, $(3, 5) \sim (0, 2) \sim (1, 3) \sim (2, 4) \sim \dots$.

We can then write all equivalent classes as:

- $[(0, 1)] = \{(0, 1), (1, 2), (2, 3), \dots\}$,
- $[(0, 0)] = \{(1, 1), (2, 2), (3, 3), \dots\}$,
- $[(1, 0)] = \{(1, 0), (2, 1), (3, 2), \dots\}$.

Hence the integers, $\mathbb{Z}$, are defined as all the equivalence classes:

$$\dots, [(0, 2)], [(0, 1)], [(0, 0)], [(1, 0)], [(2, 0)], \dots$$

If we plot these ordered pairs in rectangular coordinates, then each equivalent class is a straight line.

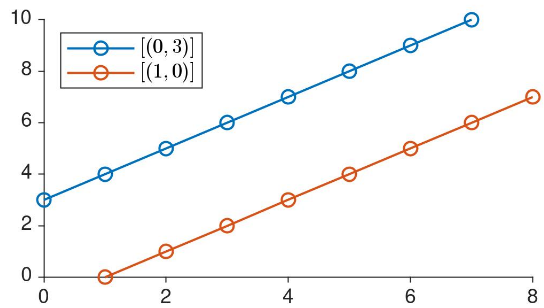


Figure 1.5: Equivalence classes.

Thus, the horizontal axis can be viewed as positive integer and vertical axis can be viewed as negative integer.

1.3.3 Construction of rational numbers

Similar to integers, rational numbers can be constructed as the equivalence classes of ordered pairs of integers.

Definition 1.3.2. For $(a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, an equivalence relation $\sim$ is defined as

$$(a, b) \sim (c, d) \quad \text{if and only if } ad = bc,$$

Addition and multiplication are defined as:

$$\begin{aligned}(a, b) + (c, d) &\equiv (ad + bc, bd),\\ (a, b) \times (c, d) &\equiv (ac, bd).\end{aligned}$$

The equivalence class of (a, b) can be denoted as $\frac{a}{b}$ and integer a can be defined as rational number $\frac{a}{1}$.

1.3.4 Construction of real numbers

There are several ways of constructing real numbers, we discuss only the method by using *Dedekind cuts*.

One can list rational numbers on a number line where numbers on the left are smaller than the numbers on the right of the number line. However, it can be easily checked that equation $x^2 = 2$ has no solution in rational number system. In fact, $x = \sqrt{2}$ is a *irrational number*.

Let A be the set of all positive rational number p such that $p^2 < 2$. Define

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2},$$

clearly, $q > p$ and $q^2 - 2 = \frac{4p^2+8p+4}{p^2+4p+4} - 2 = \frac{2(p^2-2)}{(p+2)^2} > 0$ implies that $q \in A$. Similarly, let B be the set of all positive rational number p such that $p^2 > 2$. Then $q = p - \frac{p^2-2}{p+2}$ implies that $q < p$ and $q \in B$. Thus, we showed that A has no largest number and B has no smallest. This implies that there are "gaps" in the rational number system.

In fact, this partition of rational numbers into set A and B is in fact a Dedekind cut. A **Dedekind cut** is a partition (α, β) such that α is nonempty and closed downwards, α is nonempty and closed upwards, and α has no greatest element.

Real numbers can be defined as all Dedekind cuts of $\mathbb{Q}$, that is, a real number α is subset of $\mathbb{Q}$ with the following properties:

1. α is not empty and $\alpha \neq \mathbb{Q}$,
2. α is closed downwards, that is, $\forall x, y \in \mathbb{Q}$, if $x < y$ and $y \in \alpha$, then $x \in \alpha$,
3. α has no greatest element.

Details are skipped here. Interested readers are encouraged to read Chapter 1 of "Principles of Mathematical Analysis" by Walter Rudin.

Theorem 1.3.1. There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. Moreover, $\mathbb{R}$ contains Q as a subfield.

Proof. Proof is omitted. □

Chapter 2

Complex Number

2.1 Algebraic Structure and Operations of $\mathbb{C}$

The set of all complex numbers is denoted as $\mathbb{C}$ and the set of all non-zero complex numbers is denoted as $\mathbb{C}^{\times}$.

Definition 2.1.1. A complex number z can be defined as an ordered pair (x, y) of real numbers.

Two complex numbers $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$ simultaneously.

Theorem 2.1.1. Let $z_1, z_2 \in \mathbb{C}$ and define the following operations:

1. $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$,
2. $z_1 z_2 = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2)$,

with $(0, 0)$ and $(1, 0)$ in the role of 0 and 1 is a *field*.

Proof. Proof is omitted. $\square$

Recall that a **field** is a set F with two operations, called *addition* and *multiplication*, which satisfy the field axioms:

1. Axioms for addition:
   (a) If $z_1, z_2 \in F$ then $z_1 + z_2 \in F$.
   (b) Commutative: $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in F$.
   (c) Associative: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for all $z_1, z_2, z_3 \in F$.
   (d) Additive identity: F contains an element 0 such that $0 + z = z$ for all $z \in F$.
   (e) Additive inverse: For all $z \in F$, there exists an element $-z \in F$ such that $z + (-z) = 0$.
2. Axioms for multiplication:
   (a) If $z_1, z_2 \in F$ then $z_1 z_2 \in F$.
   (b) Commutative: $z_1 z_2 = z_2 z_1$ for all $z_1, z_2 \in F$.
   (c) Associative: $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ for all $z_1, z_2, z_3 \in F$.
   (d) Multiplicative identity: F contains an element 1 such that $1z = z$ for all $z \in F$.
   (e) Multiplicative inverse: If $z \in F$ and $z \neq 0$ then there exists an element $1/z \in F$ such that $z(1/z) = 1$.
3. Distributive law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ for all $z_1, z_2, z_3 \in F$.

It can be easily checked that $\mathbb{C}$ satisfies all the field axioms.

Example 2.1.1. Show that for all $z \in \mathbb{C}$, there exist an identity element $0 \in \mathbb{C}$, such that $z + 0 = z$.

Solution:

Definition 2.1.2. Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be complex numbers. We define the arithmetic operations of complex numbers as follows:

1. Negation: $-z_1 = (-x, -y)$;
2. Addition: $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$;
3. Subtraction: $z_1 - z_2 = (x_1 - x_2, y_1 - y_2)$;
4. Multiplication: $z_1 z_2 = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2)$;
5. Division: $\frac{z_1}{z_2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right)$.

Example 2.1.2. Show that for all $z \in \mathbb{C}^\times$, there exist an inverse element $1/z \in \mathbb{C}^\times$, such that $z(1/z) = (1, 0) = 1$.

Solution:

Example 2.1.3. Show that $z_1(z_2 z_3) = (z_1 z_2) z_3$ for any $z_1, z_2, z_3 \in \mathbb{C}$.

Solution:

Definition 2.1.3. The n-th power of a complex number z, is defined as

$$z^0 = 1, z^n = z \cdot z^{n-1}, \quad \text{for } n \geq 1.$$

Theorem 2.1.2. For any real number x, y, we have

$$(x,0)+(y,0)=(x+y,0), \quad (x,0)(y,0)=(xy,0).$$

Proof. Proof is trivial □

From Theorem 2.1.2, we know that complex number $(x,0)$ has the same arithmetic properties as real number x. Therefore $(x,0)$ can be considered as a real number embedded into $\mathbb{C}$ through the map $\phi : \mathbb{R} \to \mathbb{C}, x \mapsto (x,0)$.

We have defined complex numbers without using "square root of -1". We now introduce the notation i and show that (x,y) is equivalent to $x+iy$.

Definition 2.1.4. $i=(0,1)$.

Theorem 2.1.3. $i^2=-1$.

Proof. **Solution**:

Theorem 2.1.4. For $x, y \in \mathbb{R}$, we have

$$(x,y)=x+iy$$

Proof. **Solution**:

Example 2.1.4. Let $z_1 = 3+2i$, $z_2 = 5-2i$ and $z_3 = 6-8i$. Compute the following:

1. $z_1 z_2$,
2. $\sqrt{z_3}$.

Solution:

Definition 2.1.5. Let $x, y \in \mathbb{R}$ and $z = x + iy$. The **conjugate** of z is defined as

$$\overline{z} = x - iy.$$

The number x and y are the **real part** and **imaginary part** of z, respectively. These are denoted as

$$x = \operatorname{Re}(z), \quad y = \operatorname{Im}(z).$$

To compute the division of complex numbers, we can multiply both numerator and denominator by conjugate of denominator.

Example 2.1.5. Simplify $\frac{2+3i}{4-i}$.

Solution:

Theorem 2.1.5. Let z and w be any complex numbers. We have

1. $\overline{\overline{z}} = z$,
2. $\overline{z + w} = \overline{z} + \overline{w}$,
3. $\overline{zw} = \overline{z} \cdot \overline{w}$,
4. $z + \overline{z} = 2\operatorname{Re}(z)$, $z - \overline{z} = 2i\operatorname{Im}(z)$,
5. $z\overline{z}$ is real and positive (except when $z = 0$).

Proof. Part 1 to 4 are trivial. For part 5, let $z = x + iy$.

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2.$$

Thus $z\overline{z}$ is always non-negative and zero if and only if $z = 0$. $\square$

Example 2.1.6. Let $a_i \in \mathbb{R}$ be some constants. If z_0 is a solution of the following equation

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

Show that $\overline{z_0}$ is also a solution of the equation.

Solution:

Definition 2.1.6. An **ordered field** is a field F which is also an *ordered set*, such that

1. $x + y < x + z$ if $x, y, z \in F$ and $y < z$,
2. $xy > 0$ if $x, y \in F, x > 0$ and $y > 0$.

If $x > 0$, we call x positive; if $x < 0$, x is negative.

Note that while $\mathbb{R}$ is an ordered field, $\mathbb{C}$ is not.

Definition 2.1.7. **Total ordering** is a property that implies only one of the following is true:

$$a < b, a = b, a > b.$$

Example 2.1.7. Show that it is impossible to have a total order for $\mathbb{C}$.

Solution:

2.2 Geometric interpretation

We start with one result related to real numbers.

Theorem 2.2.1. For all $x > 0$ and every integer $n > 0$, there is one and only one positive real y such that $y^n = x$. This number y is written as $\sqrt[n]{x}$ or $x^{1/n}$.

Proof. Suppose that there are more than one such y. Let $y_1 < y_2 < \cdots < y_k$, then $y_1^n < y_2^n < \cdots < y_k^n$. This implies that there is at most one such y that satisfy $y^n = x$.

Let E be the set consisting all positive real numbers t such that $t^n < x$.

To show that E is not empty, let $t = \frac{x}{1+x}$, then $0 < t < 1$ and $t^n < t < x$.

Clearly, $1 + x$ is an upper bound of E. From Theorem 1.3.1, we know that the least upper bound exists and take $y = \sup E$.

The identity $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ yields the inequality

$$b^n - a^n < (b - a)nb^{n-1}, \quad \text{for } 0 < a < b.$$

Assume $y^n < x$. We know that $(y+\epsilon)^n - y^n < n\epsilon(y+\epsilon)^{n-1} < x - y^n$ if $0 < \epsilon < 1$ and $\epsilon = \frac{x-y^n}{n(y+1)^{n-1}}$. This implies that $(y+\epsilon)^n < x$, or $y+\epsilon \in E$. This contradicts with the fact that y is an upper bound of E.

Assume $y^n > x$. We have $y^n - \epsilon^n < (y-\epsilon)ny^{n-1} = y^n - x$ when $\epsilon = y - \frac{y^n-x}{ny^{n-1}}$. This implies that $\epsilon^n > x$ and hence $\epsilon \notin E$. Thus, $\epsilon = y - \frac{y^n-x}{ny^{n-1}}$ is an upper bound of E, but this contradicts with the fact that y is the least upper bound of E.

Thus, $y^n = x$. □

Definition 2.2.1. If z is a complex number, then its **modulus** (also known as absolute value), $|z|$ is the non-negative square root of $z\overline{z}$, $(z\overline{z})^{1/2}$.

The existence of modulus is guaranteed by the fact that $z\overline{z}$ is real and positive ($z \neq 0$) and Theorem 2.2.1 ensures its uniqueness.

Since complex numbers are ordered pairs, a complex number $(x, y) \in \mathbb{C}$ can be represented by a point in the Euclidean space $\mathbb{R}^2$. This plane is also known as **complex plane** or **Argand diagram**.

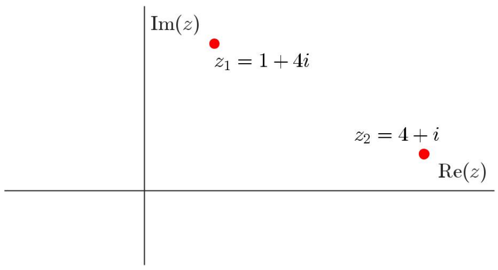


Figure 2.1: Complex plane

Any complex number $z = x + iy$ can also be viewed as the point (x, y) in rectangular coordinate. In fact, complex number z can be associated with a vector, from origin to the point (x, y). Figure below illustrates this.

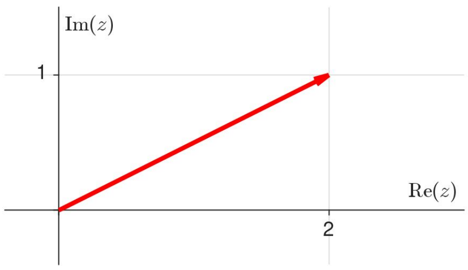


Figure 2.2: $z = 2 + i$.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the sum $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ corresponds to sum of two vectors, z_1 and z_2.

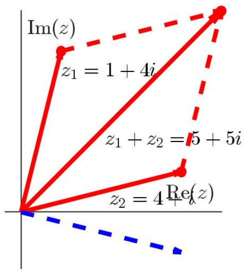


Figure 2.3: Sum of two complex numbers.

Conjugate is obtained by reflection along the real axis.

With this vector interpretation, one can also extend the concept of absolute value of real numbers to the complex plane. Geometrically, the **modulus** $|z|$ is the length of vector z and can be computed as

$$|z| = \sqrt{x^2 + y^2}$$

Definition 2.2.2. The argument of $z = x + iy \neq 0$, $\arg(z)$, is defined as the angle between vector z and positive x-axis (also known as real axis). Thus $\arg(z)$ is defined as that number θ for which

$$\cos\theta = \mathrm{Re}(z)/|z|; \quad \sin\theta = \mathrm{Im}(z)/|z|.$$

As cos and sin functions are 2π-periodic, we know that there are infinitely many θ such that $\cos\theta = \mathrm{Re}(z)/|z|$ and $\sin\theta = \mathrm{Im}(z)/|z|$ are satisfied:

$$\arg(z) = \Theta + 2n\pi, \quad (n = 0, \pm 1, \pm 2, \dots).$$

where Θ is known as the **principal argument** of $z = x + iy$:

$$\tan(\mathrm{Arg}(z)) = \frac{y}{x}, \quad -\pi < \mathrm{Arg}(z) \leq \pi.$$

For z is negative real number, $\mathrm{Arg}(z) = \pi$.

Example 2.2.1. Compute the modulus and principal argument of $z = 4 + 5i$.

Solution:

Example 2.2.2. Consider the complex number $z = -1 - i$. From definition, we know that

$$\arg(z) = -\frac{3\pi}{4} + 2n\pi, \quad (\ n = 0, \pm 1, \pm 2, \dots).$$

Since $\arg(z)$ is in fact a infinite set of values, we can also write

$$\arg(z) = \frac{5\pi}{4} + 2n\pi, \quad (\ n = 0, \pm 1, \pm 2, \dots).$$

Plotting it on Argand diagram shows that z falls in third quadrant and has principal argument $\mathrm{Arg}(z) = -\frac{3\pi}{4}$. Note that $\mathrm{Arg}(z) \neq \frac{5\pi}{4}$.

Theorem 2.2.2. Let $z, w \in \mathbb{C}$. Then

1. $|z| > 0$ unless $z = 0$, $|0| = 0$,
2. $|\overline{z}| = |z|$,
3. $|zw| = |z||w|$,
4. $\operatorname{Re}(z) \leq |z|, \operatorname{Im}(z) \leq |z|$,
5. $|z + w| \leq |z| + |w|$.

Proof. Part 1, 2 and 4 are trivial.

For part 3, let $z = x_1 + iy_1, w = x_2 + iy_2$.

$$|zw|^2 = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2 = (a^2 + b^2)(c^2 + d^2) = (|z||w|)^2 .$$

Taking square root on both sides of the equation above, Theorem 2.2.1 ensures the uniqueness the solution.

For part 5, note that $\overline{z + \overline{w}} = \overline{z} + w$. Thus $z\overline{w} + \overline{z}w = 2\operatorname{Re}(z\overline{w})$. Hence

$$\begin{aligned}
|z + w|^2 &= (z + w)\overline{(z + w)} = (z + w)(\overline{z} + \overline{w}) \\
&= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w} \\
&= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2 \\
&\leq |z|^2 + 2|z\overline{w}| + |w^2| \\
&= |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2 .
\end{aligned}$$

Taking square root on both sides, we have the desired result. □

Part 5 of Theorem 2.2.2 is widely known as **triangle inequality**.

Example 2.2.3. Show that $|z| - |w| \leq |z - w|$, for any $z, w \in \mathbb{C}$.

Solution:

Example 2.2.4. Prove that $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$ for any positive integer n and any $z_i \in \mathbb{C}$.

Solution:

2.2.1 Polar form

Suppose that we have $z_1 = 3 + i$ and $z_2 = -2 + 4i$. Plotting both z_1, z_2, z_1z_2 in complex plane:

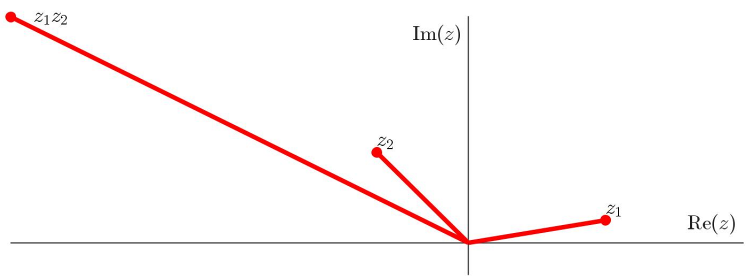


Figure 2.4: z_1z_2.

It can be seen that multiplying z_1 by z_2 rotates vector z_2 counterclockwise by the angle $\arg(z)$ and $|z_1z_2| = |z_1||z_2|$. Similarly, multiplication by i is equivalent to a counterclockwise rotation of $90°$.

Although this interpretation of multiplication of complex numbers can be verified geometrically, it can also be shown directly by using polar coordinates to represent all non-zero complex number.

Definition 2.2.3. For any complex number $z = x + iy \in \mathbb{C}^{\times}$, it can be written as polar form:

$$z = r\left(\cos\theta + i\sin\theta\right), \tag{2.1}$$

where $r = \sqrt{x^2 + y^2} = |z|$ and $\theta = \arg(z)$.

We abbreviate $\cos\theta + i\sin\theta$ as $\operatorname{cis}\theta$.

Example 2.2.5. Let $z_1 = r_2(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. We have

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta2)) \\ &= r_1r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right) \end{aligned}$$

Thus, z_1z_2 is a complex number with $\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$ and $|z_1z_2| = r_1r_2 = |z_1||z_2|$. This product is equivalent to a counterclockwise rotation.

Thus, complex numbers are multiplied by multiplying their *modulus* and adding their *arguments*.

Example 2.2.6. Using polar coordinates, compute $\frac{z_1}{z_2}$ where $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ and $z_1, z_2 \in \mathbb{C}^{\times}$.

Solution:

Similarly, z_1/z_2 can be obtained by dividing their *modulus* and substracting their *arguments*.

While $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$ is true, however, $\operatorname{Arg}(z_1z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ is **not** always true.

Example 2.2.7. Let $z_1 = -1$ and $z_2 = i$, $\operatorname{Arg}(z_1z_2) = -\pi/2$ but $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \pi + \pi/2 = 1.5\pi$.

Graphically,

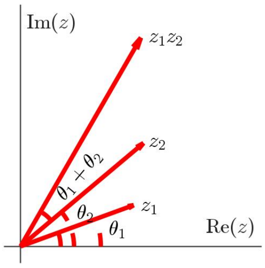


Figure 2.5: Multiplication of z_1 and z_2.

2.2.2 Exponential form

Theorem 2.2.3. (Euler's formula)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Proof. Substituting $x = i\theta$ in the Taylor series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ yields

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \dots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) \\
&= \cos\theta + i\sin\theta.
\end{aligned}$$

□

From Theorem 2.2.3, we have that $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. Using the properties of exponential function, we have

Theorem 2.2.4. Let $z = re^{i\theta}, w = r'e^{i\theta'}$.

1. $zw = rr'e^{i(\theta+\theta')}$,
2. $\frac{z}{w} = \frac{r}{r'}e^{i(\theta-\theta')}$,
3. $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$,

4. $z^n = r^n e^{in\theta}$ for $n = 0, \pm 1, \pm 2, \ldots$.

Proof. Part 1 to 3 is straight forward. For part 4, it can be proven by induction. For $n = 1$, $z = re^{i\theta}$. Assume that this equation is valid for $n = m$, we have

$$\begin{aligned} z^m &= r^m e^{im\theta} \\ z^{m+1} &= r^m e^{im\theta} \times re^{i\theta} \\ &= r^{m+1} e^{i(m+1)\theta}, \end{aligned}$$

implies that $z^n = r^n e^{in\theta}$ is true for all $n \geq 1$. For the case $n = 0$, it can also be easily verified.

For the case when $n = -t$ is negative, where $t = 1, 2, \ldots$

$$\begin{aligned} z^n &= \left(z^{-1}\right)^t \\ &= \left(\frac{1}{r} e^{i(-\theta)}\right)^t \\ &= \left(\frac{1}{r}\right)^t e^{it(-\theta)} \\ &= r^n e^{in\theta}. \end{aligned}$$

□

Example 2.2.8. Compute $(2 + 2i)^5$.

Solution:

Using part 4 of Theorem 2.2.4 with $r = 1$, we have

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Theorem 2.2.5. (de Moivre's formula) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in \mathbb{Z}$.

Example 2.2.9. Compute $(2 + 2i)^5$ using de Moivre's formula.

Solution:

Now consider finding roots of complex numbers.

Example 2.2.10. Find the cube root of 1.

Solution:

Now we consider raising a complex number to rational power.

Theorem 2.2.6. For $n, m \in \mathbb{Z}$, $m \neq 0$ and $\theta \in \mathbb{R}$, we have

$$z^{\frac{n}{m}} = r^{\frac{n}{m}} \left[\cos\left(\frac{n}{m}\theta\right) + i \sin\left(\frac{n}{m}\theta\right) \right],$$

for all $z = re^{i\theta}$.

Proof. Let $z^{\frac{n}{m}} = r_0 e^{i\theta_0}$. We have that

$$\begin{aligned} r_0^m e^{im\theta_0} &= z^n \\ r_0^m e^{im\theta_0} &= r^n e^{in\theta} \end{aligned}$$

Thus, $r_0 = r^{\frac{n}{m}}$ and $m\theta_0 = n\theta + 2k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$, or

$$c_k = z^{\frac{n}{m}} = r^{\frac{n}{m}} \left(\cos \frac{n\theta + 2k\pi}{m} + i \sin \frac{n\theta + 2k\pi}{m} \right), \quad \text{for } k = 0, 1, 2, \ldots, n-1$$

to denote the n distinct roots. □

Example 2.2.11. Find all five roots of the equation $z^5 = 1$.

Solution:

Generally, if $w = r(\cos\theta + i \sin\theta)$ and $w^n = z = \rho(\cos\psi + i\sin\psi)$ then

$$w = \sqrt[n]{|z|} \left(\cos \frac{\psi + 2k\pi}{n} + i \sin \frac{\psi + 2k\pi}{n} \right).$$

A complex number a is called n^{th} **root of unity**, if $a^n = 1$ and n is positive integer. Geometrically all the n^{th} roots of unity are lying on the unit circle with centre at the origin. Note that one root is at (1,0), i.e, $z = 1$ is one of the nth roots of unity. The other $n-1$ roots ω, ω^2, $\cdots$, ω^{n-1} are located equidistance on the unity circle and satisfies

$$\begin{aligned} \omega^n = 1 \Rightarrow \omega^n - 1 = 0 &\Rightarrow (\omega - 1)(\omega^{n-1} + \omega^{n-2} + \cdots + \omega + 1) = 0 \\ &\Rightarrow \omega^{n-1} + \omega^{n-2} + \cdots + \omega + 1 = 0. \end{aligned}$$

Example 2.2.12. Find all five roots of the equation $z^5 - 32 = 0$.

Solution:

Plotting all roots of $z^5 - 32 = 0$ in complex plane, we can observe that each of these points has distance 2 to origin, joining all these points gives us a pentagon and these points lie on a circle of radius 2 centered at origin.

In fact, all roots of $(z-a)^n = r$ form a n-sided polygon and lie on circle of radius $\sqrt[n]{r}$ centered at $z = a$. Example below illustrate this.

Example 2.2.13. All 5^{th} root of unity:

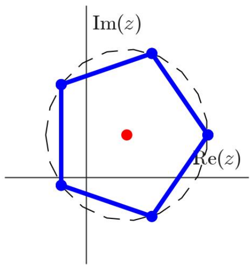


Figure 2.6: 5^{th} root of unity.

2.3 Topological aspects of the complex plane

In this section, some basic topological properties of $\mathbb{C}$ will be discussed.

2.3.1 Sequence and series

Definition 2.3.1. A sequence $z_1, z_2, z_3, \dots$ converges to z if the sequence $|z_n - z|$ converges to zero. The sequence z_n is **convergent** and any sequence that is not convergent is called **divergent**.

We write the convergence of sequence as $z_n \to z$ or $\lim_{n\to\infty} z_n = z$. Geometrically, $z_n \to z$ if for all $\epsilon > 0$, one can draw a circle with radius ϵ centered at z, which contains all but finitely many of the member of the sequence $\{z_n\}$.

Theorem 2.3.1. $z_n \to z$ if and only if $\mathrm{Re}(z_n) \to \mathrm{Re}(z)$ and $\mathrm{Im}(z_n) \to \mathrm{Im}(z)$.

Proof. Let $z_n = x_n + iy_n, z = x + iy$. If $\mathrm{Re}(z_n) \to \mathrm{Re}(z)$ and $\mathrm{Im}(z_n) \to \mathrm{Im}(z)$ then for all $\epsilon > 0$, we have

$$|x_n - x| < \frac{\epsilon}{2} \quad \text{for } n > n_1$$

and

$$|y_n - y| < \frac{\epsilon}{2} \quad \text{for } n > n_2.$$

Thus,

$$|x_n - x| + |y_n - y| < \epsilon \quad \text{for } n > \max(n_1, n_2) = n_0.$$

Since $|(x_n + iy_n) - (x + iy)| = |(x_n - x) + i(y_n - y)| \le |x_n - x| + |y_n - y|$, we have that

$$|z_n - z| < \epsilon \quad \text{for } n > n_0.$$

Conversely, if $z_n \to z$, we have that

$$|(x_n + iy_n) - (x + iy)| < \epsilon \quad \text{for all } n > n_0$$
$$|(x_n - x) + i(y_n - y)| < \epsilon \quad \text{for all } n > n_0,$$

which implies $|x_n - x| < \epsilon$ and $|y_n - y| < \epsilon$ for all $n > n_0$. $\square$

Theorem 2.3.1 implies that we can compute the limit of a complex sequence using knowledge from real numbers:

$$\lim_{n\to\infty}(x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

Example 2.3.1. Find the limit of $z_n = -2 + i\frac{(-1)^n}{n^2}$ when $n \to \infty$.

Solution:

Definition 2.3.2. $\{z_n\}$ is called a **Cauchy sequence** if for all $\epsilon > 0$, there is an integer N such that

$$n, m > N \implies |z_n - z_m| < \epsilon.$$

Example 2.3.2. Show that $z_n = -2 + i\frac{(-1)^n}{n^2}$ is a Cauchy sequence.

Solution:

Proposition 2.3.1. $\{z_n\}$ converges if and only if $\{z_n\}$ is a Cauchy sequence.

Proof. Let $z_n = x_n + iy_n$ and $z = x + iy$. Recall that from real analysis, we know that any convergent sequence is a Cauchy sequence. Thus, if $z_n \to z$, then $\operatorname{Re}(z_n) \to \operatorname{Re}(z), \operatorname{Im}(z_n) \to \operatorname{Im}(z)$ implies that $\{\operatorname{Re}(z_n)\}, \{\operatorname{Im}(z_n)\}$ are Cauchy sequence. Since

$$|z_n - z_m| \leq |\operatorname{Re}(z_n - z_m)| + |\operatorname{Im}(z_n - z_m)| = |\operatorname{Re}(z_n) - \operatorname{Re}(z_m)| + |\operatorname{Im}(z_n) - \operatorname{Im}(z_m)|,$$

$\{z_n\}$ is also a Cauchy sequence.

Conversely, $\{z_n\}$ is a Cauchy sequence implies that both $\operatorname{Re}(z_n)$ and $\operatorname{Im}(z_n)$ are Cauchy sequences and hence both real and imaginary parts converge. Thus $\{z_n\}$ converges. $\square$

Definition 2.3.3. An **infinite series** $\sum_{k=1}^{\infty} z_k$ is convergent if the sequence $\{s_n\}$ of partial sums, defined by $s_n = z_1 + z_2 + \cdots + z_n$, converges. The sum of series equals to

$$\sum_{k=1}^{\infty} z_k = \lim_{n\to\infty} s_n.$$

Theorem 2.3.2. Basic properties of infinite series:

1. The sum and difference of two convergent series are convergent,
2. A necessary condition for $\sum_{k=1}^{\infty} z_k$ to converge is that $z_n \to 0$ as $n \to \infty$,
3. A sufficient condition for $\sum_{k=1}^{\infty} z_k$ to converge is that $\sum_{k=1}^{\infty} z_k$ is **absolutely convergent**, that is, $\sum_{k=1}^{\infty} |z_k|$ converges.

Proof. Proof for part 1 and 2 will be omitted and for part 3, let $t_n = |z_1| + |z_2| + \cdots + |z_n|$ and $s_n = z_1 + z_2 + \cdots + z_n$. If $\sum_{k=1}^{\infty} |z_k|$ converges, then t_n is a Cauchy sequence, from Proposition 2.3.1. Thus,

$$\begin{aligned} |s_n - s_m| &= |z_{m+1} + z_{m+2} + \cdots + z_n| \\ &\leq |z_{m+1}| + |z_{m+2}| + \cdots + |z_n| = |t_n - t_m|, \end{aligned}$$

implies that $\sum_{k=1}^{\infty} z_k$ converges. $\square$

Example 2.3.3. Determine whether $\sum_{k=1}^{\infty} \frac{i^k}{k^2+i}$ is a convergent series.

Solution:

2.3.2 Classification of sets in the complex plane

Definition 2.3.4. The set $D(z_0; r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ is an open disc of radius r centered at z_0 and is known as **neighborhood** (or r-neighborhood) of z_0.

Definition 2.3.5. The set $C(z_0; r) = \{z \in \mathbb{C} : |z - z_0| = r\}$ is a circle radius r centered at z_0.

Definition 2.3.6. A set S is **open** if for any $z \in S$, there exists $\delta > 0$ such that $D(z; \delta) \subset S$.

A set S is **closed set** if its complement, $\tilde{S} = \mathbb{C} \setminus S$, if open set. Equivalently, S is closed if $\{z_n\} \in S$ and $z_n \to z$ imply $z \in S$.

Definition 2.3.7. The **boundary** of S, ∂S, is defined as the set of points whose δ-neighborhood have a nonempty intersection with both S and $\tilde{S}$, for all $\delta > 0$.

Definition 2.3.8. The **closure** of S, $\overline{S}$, is given by $\overline{S} = S \cup \partial S$.

Thus, we can defined a "closed disc" as $\overline{D(z_0; r)}$ as the set $\{z \in \mathbb{C} : |z - z_0| \leq r\}$.

Definition 2.3.9. S is **bounded** if it is contained in $D(0; M)$ for some $M > 0$.

Sets that are bounded and closed are called **compact**.

Definition 2.3.10. A set S is **disconnected** if there exist two disjoint open sets A and B such that

$$S \subseteq A \cup B \text{ and } S \not\subseteq A, S \not\subseteq B.$$

If S is not disconnected, then it is **connected**.

Definition 2.3.11. Let $[z_1, z_2]$ denotes the line segment with endpoints z_1 and z_2. Then a **polygonal line** is a finite union of line segments of the form $[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n]$.

If any two points of S can be connected by a polygonal line contained in S, S is said to be **polygonally connected**.

It is obvious that a polygonally connected set is connected.

Definition 2.3.12. The union of a nonempty open connected set with some, none or all its boundary points is called **region**. If none of the boundary points are included, it is named as **open region**; if all of the boundary points are included, it is a **closed region**.

Note that some authors use the term **domain** instead of open region.

Proposition 2.3.2. An open region S is polygonally connected.

Proof. Suppose $z_0 \in S$.

1. Let A be the set of points of S such that each point in A can be polygonally connected to z_0,
2. If $z \in A$, then z can be connected to z_0 via a polygonal line,
3. Consider $D(z; \delta)$, every point in $D(z; \delta)$ can be connected to z,
4. Thus, A is open,
5. Let B be the set of points of S such that each point in B cannot be polygonally connected to z_0,
6. Similar argument shows that B is open,
7. Since S is connected, $S = A \cup B$, and A is nonempty, we must have B is empty, that is, $S = A$.

□

The concepts of *open* and *polygonally connected* are important in complex analysis as we will see in other chapters.

2.4 Regions in the complex plane

Consider the equation $|z - z_0| = r$. This is an equation of a circle with radius r and centered at z_0. Similarly, we can obtain *closed disk*, *open disk* and *annulus* below:

$$|z - z_0| \leq r, \quad |z - z_0| < r, \quad r_1 \leq |z - z_0| \leq r_2.$$

Example 2.4.1. Sketch the region of the following:

1. The set of points given by the equation $\text{Re}(z) > 0$,
2. $\{z : z = \overline{z}\}$,
3. $\{z : -\theta < \text{Arg}(z) < \theta\}$,
4. $\{z : \left|\text{Arg}(z) - \frac{\pi}{2}\right| < \frac{\pi}{2}\}$,
5. $\{z : |z + 1| < 1\}$.

Solution:

Chapter 3

Complex Function

3.1 Introduction

In this chapter, we introduce functions of complex variables, with the aims of introducing *analytic functions*. Another different name that refers to the same class of functions is *holomorphic functions*.

Definition 3.1.1. A function f is said to be **holomorphic at a point** z_0 if and only if it is differentiable at z_0 and at every point in some neighbourhood of z_0.

Definition 3.1.2. A function f is **holomorphic in an open set** if it has a derivative everywhere in that set.

Although the term "analytic" and "holomorphic" are always interchangeable, it is in fact defined in a broader sense:

Definition 3.1.3. A function f is **analytic at** z_0 if and only if it can be written as a convergent power series.

However, it can be proven that all holomorphic functions are analytic and vice versa. Thus, in complex analysis, these two terms are referring to the same class of functions (similarly, regular functions are also used in some literature).

Definition 3.1.4. An **entire function** is a function that is analytic at each point in the whole complex plane.

Definition 3.1.5. If a function f fails to be analytic at a point z_0 but is analytic at some point in every neighbourhood of z_0, then z_0 is a **singular point**.

We know that any analytic function is differentiable and a differentiable function is continuous. Converse of this is not necessary true. Our other objective in this chapter is to develop the *sufficient conditions for complex differentiability*.

Discussion on the analyticity (or holomorphicity) of a function in other chapter. We now formally define the *domain of definition* of a function.

Definition 3.1.6. Let $S \subseteq \mathbb{C}$. f is a function if it is a rule that assigns each z in S to a complex number w, known as the value of f at z and denoted as

$$w = f(z).$$

The set S is the **domain of definition** of f.

Let $w = f(z)$ and $w = u + iv$, $z = x + iy$, then

$$u + iv = f(x + iy).$$

Note that by definition of complex number, u, v, x, y are all real numbers and by comparing real and imaginary parts, we have

$$f(z) = u(x, y) + v(x, y). \tag{3.1}$$

If z is written in polar coordinates, then

$$f(z) = f\left(re^{i\theta}\right) = u(r, \theta) + iv(r, \theta). \tag{3.2}$$

Example 3.1.1. Let f be defined on the set $z \neq 0$ and $w = 1/z$. By writing $z = x + iy$ and $w = f(z) = u(x, y) + iv(x, y)$, we have

$$\begin{aligned}
f(z) &= u(x, y) + iv(x, y)\\
\frac{1}{x + iy} &= u(x, y) + iv(x, y)\\
\frac{x - iy}{x^2 + y^2} &= u(x, y) + iv(x, y).
\end{aligned}$$

Thus,

$$u(x, y) = \frac{x}{x^2 + y^2}, \quad v(x, y) = -\frac{y}{x^2 + y^2}.$$

It is important that both *domain of definition* and *a rule* are needed in order for a function f to be well defined.

Example 3.1.2. Using polar coordinates, determine the functions $u(r, \theta)$ and $v(r, \theta)$ such that $w = u(r, \theta) + iv(r, \theta) = z^2$.

Solution:

Similar to real-valued function, we define the polynomial and rational function as

Definition 3.1.7. A **polynomial** function is a function P over $\mathbb{C}$:

$$P : \mathbb{C} \to \mathbb{C}, z \mapsto a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n.$$

A **rational** function is a function R over D:

$$R : D \to \mathbb{C}, z \mapsto \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n}.$$

where $D = \mathbb{C} \setminus \{z_0, z_1, \ldots, z_n\}$ and z_i (for $i = 1, \ldots, n$) are the zeroes of the polynomial function $b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n$.

Example 3.1.3. If $f(z) = z^{10} + z^5 - z^3$, determine a, b if

$$f\left(\frac{1+i}{\sqrt{2}}\right) = a + ib.$$

Solution:

3.2 Transformation $w = z^2$

A real valued function can be represented by a graph in xy-plane. For example, $f(x) = 2x + 3$ is a straight line with gradient 2 and y-intercept at $y = 3$. However, a complex valued function $f(z) = w$ can not be represented by a graph in a xy-plane as both z, w are located in a plane rather than a line.

To graphically present a complex function, it can be done by drawing two planes, i.e. z-plane that maps to w-plane, and this is also viewed as a **mapping**.

Example 3.2.1. Consider the mapping $w = z + 1$, where $z = x + iy$. Then

$$w = (x + 1) + iy.$$

This mapping can be thought of as a *translation* of each point z one unit to the right.

Example 3.2.2. Let $w = iz$. What is the geometric characteristics of this mapping?

Solution:

The mapping $w = z^2$ and $z = x + iy$ can be thought of as the transformation:

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$$

from the xy-plane into the uv-plane. Specifically, this maps hyperbola in xy-plane into straight line.

Example 3.2.3. Consider the mapping $f(z) = z^2$. The image of a hyperbola, $x^2 - y^2 = c_1$ where c_1 is a constant, in uv-plane is a vertical straight line $u = c_1$. The graph of $x^2 - y^2 = c_1$ is shown below:

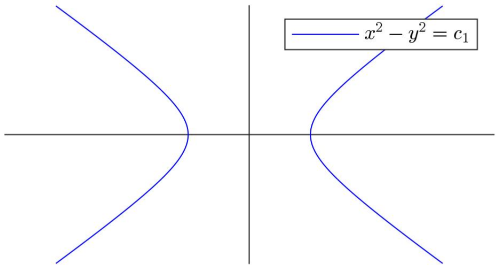


Figure 3.1: $x^2 - y^2 = c_1$

By substituting $v = 2xy$ into $u(x, y)$, we have that

$$u = c_1, \quad v = \pm 2y\sqrt{y^2 + c_1}, \quad (-\infty < y < \infty),$$

where the $\pm$ correspond to the right and left branches of the graph. This maps to a vertical line intercept at $u = c_1$.

Example 3.2.4. Find the curve in $\mathbb{C}$ such that the mapping $w = z^2$ is a horizontal line.

Solution:

Using polar coordinates, we can discover other mapping. Note that although not explicitly mention, the r in polar coordinate must be non-negative as $r = |z|$.

Example 3.2.5. Let $z = r_0 e^{i\theta}$ where $0 \le \theta \le \frac{\pi}{2}$. What is the image $w = z^2$?

Solution:

It can be easily verified that upper half plane ($0 \le \theta \le \pi$) is mapped onto the entire w-plane. However, this is not one-to-one mapping as both positive and negative real axis in the z-plane are mapped onto the positive real axis in the w-plane.

3.3 Limit and continuity

3.3.1 Limit at $\mathbb{C}$

Definition 3.3.1. The symbolism

$$\lim_{z \to z_0} f(z) = w_0$$

means that for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever $0 < |z - z_0| < \delta$, then $|f(z) - w_0| < \epsilon$.

This concept of limit can be expressed in the terminology of *neighborhood*. This is equivalent to saying that for each ϵ-neighborhood, $D(z_0; r)$ there is a deleted δ-neighborhood (also known as a punctured disc of radius r about $z = z_0$) such that every point in z in it has an image w lying in the ϵ-neighborhood. Graphically,

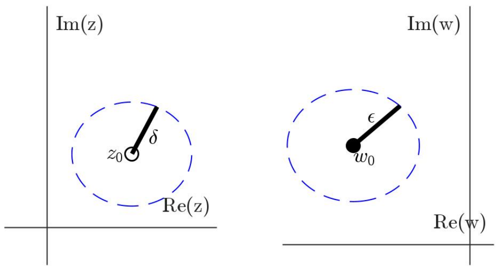


Figure 3.2: Definition of limit

Theorem 3.3.1. When a limit of a function f exists at a point z_0, it is unique.

Proof. Suppose that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} f(z) = w_1$. Then for each $\epsilon > 0$, we have $\delta_0 > 0, \delta_1 > 0$ such that

$$0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \epsilon,$$

and

$$0 < |z - z_0| < \delta_1 \implies |f(z) - w_1| < \epsilon.$$

Consider $|z - z_0| < \delta$ where $\delta = \min(\delta_0, \delta_1)$, we have

$$|w_1 - w_0| = |(f(z) - w_0) - (f(z) - w_1)| \le |f(z) - w_0| + |f(z) - w_1| < \epsilon + \epsilon = 2\epsilon.$$

Since $\epsilon > 0$ and can be chosen arbitrarily small, we must have $w_1 - w_0 = 0$. □

Example 3.3.1. Find the limit $\lim_{z\to 1} i\bar{z}$ for $|z| < 1$.

Solution:

By Definition 3.3.1, if limit of a function exist, the limit must be unique and it can be approached in an arbitrary manner. Thus a criterion for the nonexistence of a limit is as follows:

Theorem 3.3.2 (Limit Non-Existence Theorem)**.** If f approaches two complex numbers $w_1 \neq w_2$ for two different curves or paths through z_0, then $\lim_{z\to z_0} f(z)$ does not exist.

Example 3.3.2. Show that the limit $\lim_{z\to 0} \dfrac{z}{\bar{z}}$ does not exist.

Solution:

We now develop some theorems on limits that connect limits of complex valued-functions to real-valued functions.

Theorem 3.3.3. Suppose that $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. If $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$, then $\lim_{z \to z_0} f(z) = w_0$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

Theorem 3.3.4. Suppose $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$. Then

1. $\lim_{z \to z_0} (f(z) + F(z)) = w_0 + W_0$,
2. $\lim_{z \to z_0} f(z)F(z) = w_0 W_0$,
3. $\lim_{z \to z_0} \dfrac{f(z)}{F(z)} = \dfrac{w_0}{W_0}$ if $W_0 \neq 0$.

Proof. **Solution**:

Example 3.3.3. Evaluate $\lim_{z \to 1-i} [x + i(2x + y)]$ where $z = x + iy$.

Solution:

Example 3.3.4. To show that $\lim_{z \to z_0} z^n = z_0^n$ for $n = 1, 2, \ldots,$ use mathematical induction on n. It is clear that this relation is true when $n = 1$ and assume it is true for $n = m$.

$$\begin{aligned}\lim_{z\to z_0} z^{m+1} &= \lim_{z\to z_0} z^m \times \lim_{z\to z_0} z \\ &= z_0^m \times z_0 \\ &= z_0^{m+1}.\end{aligned}$$

Thus, this is true for $n = 1, 2, 3, \ldots$.

Example 3.3.5. Find the limit of $f(z) = \frac{1}{z^n}$ for $z \to z_0$ $(z_0 \neq 0)$.

Solution:

Example 3.3.6. Compute the limit of $f(z) = \frac{z^2-1}{z-1}$ when $z \to 1$.

Solution:

Definition 3.3.2. A complex function f is said to be *bounded* on region D if there exists a real constant M such that

$$|f(z)| \leq M \qquad \text{for all } z.$$

Example 3.3.7 (April 2009 Exam, Q2(a))**.** Suppose $f(z)$ is bounded in some neighborhood of z_0 and $\lim_{z\to z_0} g(z) = 0$. Show that $\lim_{z\to z_0} f(z)g(z) = 0$.

Solution:

3.3.2 Stereographic projection and point at infinity

Complex numbers can be represented by the points on the surface of a deleted (punctured) sphere, a sphere with point $(0,0,1)$ removed. Define a sphere with radius $\frac{1}{2}$ and centered at $\left(0,0,\frac{1}{2}\right)$:

$$\Sigma = \left\{(\xi,\eta,\zeta) : \xi^2+\eta^2+\left(\zeta-\frac{1}{2}\right)^2 = \frac{1}{4}\right\}.$$

This can viewed as a sphere touching the z-plane at $z=0$ and it can be shown that there is a one-to-one correspondence between points on the surface to z-plane. See figure below:

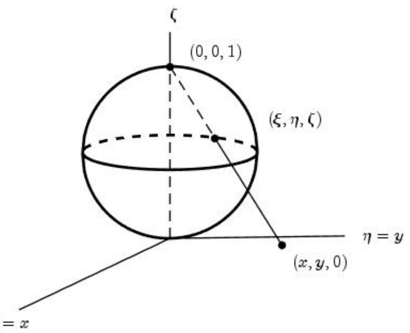


Figure 3.3: Stereographic projection

Since $(0,0,1),(\xi,\eta,\zeta)$ and $(x,y,0)$ lie on the same straight line, we have

$$\frac{x}{\xi} = \frac{y}{\eta} = \frac{1}{1-\zeta},$$

or,

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}.$$

Define $\{\sigma_k\} = \{(\xi_k,\eta_k,\zeta_k)\}$ be a sequence of points of Σ and $\{z_k\}$ be the corresponding sequence of points in $\mathbb{C}$. Thus, if $|z_k| \to \infty$ then $\Sigma_k \to (0,0,1)$.

It can also be shown that

$$\xi = \frac{x}{x^2+y^2+1}, \quad \eta = \frac{y}{x^2+y^2+1}, \quad \zeta = \frac{z}{x^2+y^2+1}.$$

Similarly, this implies that $|z_k| \to \infty$ when $\Sigma \to (0,0,1)$.

Thus, we can define the **point at infinity** and its neighborhoods is defined as the sets in $\mathbb{C}$ corresponding to the spherical neighborhoods of $(0,0,1)$. This sphere is also known as **Riemann sphere**.

Definition 3.3.3. The **extended complex plane** is defined as the complex plane altogether with the point at infinity.

With the point at infinity defined, we have the following operations:

$$\infty \pm a = a \pm \infty = \infty, \quad \frac{\infty}{a} = \infty, \quad \frac{a}{\infty} = 0, \quad \infty \cdot b = b \cdot \infty = \infty, \quad \frac{b}{0} = \infty,$$

where $a \neq \infty$ and $b \neq 0$ (note that b can be ∞).

Since point at infinity and its neighborhoods have been properly defined, the definition for limit also follows. We present the following theorem:

Theorem 3.3.5. If z_0 and w_0 are complex numbers in z-plane and w-plane, respectively, then

- $\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \dfrac{1}{f(z)} = 0$;
- $\lim_{z \to \infty} f(z) = w_0$ if and only if $\lim_{z \to 0} f\left(\dfrac{1}{z}\right) = w_0$;
- $\lim_{z \to \infty} f(z) = \infty$ if and only if $\lim_{z \to 0} \dfrac{1}{f\left(\frac{1}{z}\right)} = 0$.

Proof. Omitted. □

Example 3.3.8. Evaluate the limit $\lim_{z \to -2} \dfrac{iz-3}{z+2}$.

Solution:

3.3.3 Continuity

It is convenient to introduce the concept of *accumulation point*, which is also one of the fundamental concepts in point set theory.

Definition 3.3.4. Let S be a set in E and $x \in E$. x is called **accumulation point** if and only if every neighborhoods of x contains at least one point on S distinct from x.

Example 3.3.9. Suppose that $S = (0,1) \in \mathbb{R}$. It is clear that all elements in $(0,1)$ are accumulation points of S. Consider the number 0 and 1, for all $\epsilon > 0$, we know that $(0-\epsilon, 0+\epsilon)$ and $(1-\epsilon, 1+\epsilon)$ intersect with S. Thus, both $0,1$ are accumulation points of S too.

Example 3.3.10. Consider the singleton $\{1\}$. The set contains a sequence, $\{1,1,\dots\}$, that converges to 1. However, 1 is not a accumulation point as any neighborhoods of 1 does not contain element in $\{1\}$ that is distinct from 1.

Definition 3.3.5. A function f is said to be continuous at $z = z_0$ if and only if all the following conditions are satisfied:

- $f(z_0)$ exists,
- $\lim_{z \to z_0} f(z)$ exists,
- $\lim_{z \to z_0} f(z) = f(z_0)$.

Example 3.3.11. Let $f : D \to \mathbb{C}$ and $l \in \mathbb{C}$. The statement

$$f(z) \to l, \quad \text{if} \quad z \to a$$

means

- a is an accumulation point of D,
- The function,

$$\tilde{f} : D \cup \{a\} \to \mathbb{C}, z \mapsto \tilde{f}(z) = \begin{cases} f(z), & \text{if } z \neq a, z \in D, \\ l, & \text{if } z = a. \end{cases},$$

is continuous at a.

We have an equivalent definition of *continuity*:

Definition 3.3.6. f is continuous at z_0 if for any $\epsilon > 0$ there exist a $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

Let f, g be two functions that are continuous at z_0, then $f(z_0) + g(z_0)$ and $f(z_0)g(z_0)$ are continuous at that point. Furthermore, if $g(z_0) \neq 0$, then $f(z_0)/g(z_0)$ is also continuous.

Suppose that $f(z)$ is defined in a neighborhood of z_0, $|z - z_0| < \delta$. The domain of definition of $g(f(z))$ contains the image of the neighborhood of z_0. Thus, $g(f(z))$ is defined for all z in the neighborhood $|z - z_0| < \delta$. If f is continuous at z_0 and g is continuous at $f(z_0) = w_0$, then

$$\begin{aligned} |f(z) - f(z_0)| < \epsilon_1 \quad &\text{whenever} \quad |z - z_0| < \delta_1 \\ |g(w) - g(w_0)| < \epsilon_2 \quad &\text{whenever} \quad |w - w_0| < \delta_2 \end{aligned}$$

Since f is continuous at z_0, we can choose δ to be small enough such that $\epsilon_1 \leq \delta_2$. Thus,

$$|g(f(z)) - g(f(z_0))| < \epsilon_2 \quad \text{whenever} \quad |z - z_0| < \delta.$$

Theorem 3.3.6. Composition of two continuous functions is itself continuous.

Theorem 3.3.7. If $f(z)$ is continuous and $f(z_0) \neq 0$, then $f(z) \neq 0$ throughout some neighborhood of that point.

Proof. As $f(z)$ is continuous at z_0, for any $\epsilon > 0$, there exist a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

Choose $\epsilon = |f(z_0)/2|$ and there is a $\delta = \delta'$ such that

$$|f(z) - f(z_0)| < \frac{|f(z_0)|}{2} \quad \text{whenever} \quad |z - z_0| < \delta'.$$

If there is a point z in the neighborhood $|z - z_0| < \delta$, where $f(z) = 0$, we have the contradiction: $|f(z_0)| < |f(z_0)|/2$. ☐

Let $f(z) = u(x, y) + iv(x, y)$. Similar to *limit*, the function is continuous at z_0 if and only if $u(x, y)$ and $v(x, y)$ are continuous at z_0.

Theorem 3.3.8. If a complex function f is continuous on a closed and bounded region D, then f is bounded on D.

Proof. Suppose $f(z) = u(x,y) + iv(x,y)$. Then u and v are continuous for $(x,y) \in D$. It follows that $|f(z)| = \sqrt{u^2+v^2}$ is continuous in D. Hence f is bounded on D. □

Example 3.3.12 (April 2010 Exam, Q3(a)). Let $f(z) = \begin{cases} \dfrac{|\operatorname{Re}(z)|^4}{|z|}, & z \neq 0, \\ 0, & z = 0. \end{cases}$

Is $f(z)$ continuous at $z = 0$? (5 marks)

Solution:

3.4 Derivatives

Let f be a complex function whose domain D of definition contains a neighborhood of a point z_0.

Definition 3.4.1. The *derivative* of f at z_0, is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and f is **differentiable at** z_0 if $f'(z_0)$ exists.

Writing $\Delta z = z - z_0$, we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{3.3}$$

Example 3.4.1. Find the derivative of $f(z) = z^3$ at $z \in \mathbb{C}$.

Solution:

Theorem 3.4.1. For $n \in \mathbb{Z}$, the derivative of $f(z) = z^n$ where $z \in \mathbb{C}$ is

$$f'(z) = nz^{n-1}$$

Proof. As $z^{-n} = (z^{-1})^n$, thus without loss of generality, we consider cases where $n = 0, 1, 2, \ldots$.

Using binomial expansion, we have

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} = \lim_{\Delta z \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} z^k \Delta z^{n-k} - z^n}{\Delta z} = nz^{n-1}$$

$\square$

Example 3.4.2. Any polynomial is an entire function, as its derivative exists everywhere.

Theorem 3.4.2. If $f(z)$ is differentiable, then $f(z)$ is continuous.

Proof.

$$\lim_{\Delta z \to 0} [f(z + \Delta z) - f(z)] = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \lim_{\Delta z \to 0} \Delta z = f'(z) \times \lim_{\Delta z \to 0} \Delta z = 0.$$

Thus,

$$\lim_{\Delta z \to 0} f(z + \Delta z) = f(z),$$

implies that f is continuous. $\square$

A complex function that is differentiable at z_0 is continuous at z_0, however the converse is not necessary true. Example below illustrates this.

Example 3.4.3. Consider $f(z) = |z|^2$. Determine if f is differentiable.

Solution:

Note that $f(z) = |z|$ is a continuous but this continuity does not imply the existence of a derivative.

Example 3.4.4 (April 2008 Exam, Q1(c))**.** Prove that $\dfrac{d\overline{z}}{dz}$ does not exist anywhere. (7 marks)

Solution:

Now we present some differentiation formulas that can be derived in the same fashion as the ones we learned in functions of real variables.

Theorem 3.4.3. Let c be a complex constant and f, g are functions differentiable at z, then

- $\frac{d}{dz}(cf(z)) = cf'(z)$,
- $\frac{d}{dz}(f(z) + g(z)) = f'(z) + g'(z)$,
- $\frac{d}{dz}(f(z)g(z)) = f(z)g'(z) + f'(z)g(z)$,
- $\frac{d}{dz}(\frac{f(z)}{g(z)}) = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$, if $g(z) \neq 0$,
- $\frac{d}{dz}g[f(z)] = g'[f(z)]f'(z)$ (chain rule).

Proof. To derive the chain rule, we assume that f is differentiable at z_0 where $f(z_0) = w_0$ and g is differentiable at w_0. Define a function Φ:

$$\Phi(w) = \begin{cases} \frac{g(w)-g(w_0)}{w-w_0}, & \text{if} \quad w \neq w_0, \\ 0, \quad \text{if} \quad w = w_0. \end{cases}$$

It can be seen that $\lim_{\Delta w \to w_0} \Phi(w) = \Phi(w_0) = 0$. Thus, Φ is continuous at w_0. From definition of Φ, we have

$$g(w) - g(w_0) = [g'(w_0) + \Phi(w)](w - w_0), \quad \text{for} \quad |w - w_0| < \epsilon.$$

Also, f is differentiable at z_0 implies that f is continuous there.

Choose a $\delta > 0$ such that for all z in the δ-neighborhood, $|z - z_0| < \delta$, of z_0 we have $f(z)$ lies in the ϵ-neighborhood, $|w - w_0| < \epsilon$, of w_0. Thus, we can replace the w by $f(z)$ in the equation above,

$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \{g'[f(z_0)] + \Phi[f(z)]\} \times \frac{f(z) - f(z_0)}{z - z_0}, \quad \text{for} \quad 0 < |z - z_0| < \delta.$$

Taking limit on both sides, we have the chain rule proven. □

Example 3.4.5. Find the derivative of $(5z^2 + i)^3$.

Solution:

The L'Hôspital rule can also be generalized to the complex case below:

Theorem 3.4.4 (L'Hôspital Rule)**.** Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Then
$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Proof. Let $\Delta = z - z_0$.
$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \lim_{\Delta\to 0} \frac{\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}}{\frac{g(z_0+\Delta z)-g(z_0)}{\Delta z}} = \frac{\lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}}{\lim_{\Delta z\to 0} \frac{g(z_0+\Delta z)-g(z_0)}{\Delta z}} = \frac{f'(z_0)}{g'(z_0)}$$
□

Example 3.4.6. Compute $\displaystyle\lim_{z\to 1-i} \frac{z^2 - (1+2i)z + 3 + 3i}{z - 1 + i}$.

Solution:

3.5 Cauchy-Riemann equations

In this section, we study the Cauchy-Riemann equations and develop the sufficient conditions for complex differentiability.

Equation 3.4 are named Cauchy-Riemann equations in honour of the French mathematician Augustin Louis Cauchy (1789∼1857), who discovered and used them, and in honour of the German mathematician Georg Friedrich Riemann (1826∼1866), who made them fundamental in his development of the theory of functions of complex variable.

Theorem 3.5.1. Suppose that $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0) and satisfy the **Cauchy-Riemann equations**:
$$u_x = v_y, \quad u_y = -v_x. \tag{3.4}$$
Furthermore,
$$f'(z_0) = u_x + iv_x = v_y - iu_y, \tag{3.5}$$
where both partial derivatives are evaluated at (x_0, y_0).

Proof. Let $z_0 = x_0 + iy_0$ and $\Delta z = \Delta x + i\Delta y$. We have
$$f'(z_0) = \lim_{(\Delta x,\Delta y)\to(0,0)} \left\{ \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i\frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y} \right\}. \tag{3.6}$$
Since $f'(z_0)$ exist implies that the limit above exists and $f'(z_0)$ is the same when $(\Delta x, \Delta y)$ tends to $(0, 0)$ in any manner.

In particular, we let $(\Delta x, \Delta y) \to 0$ horizontally through the points $(\Delta x, 0)$, Equation 3.6 becomes
$$f'(z_0) = \lim_{(\Delta x,0)\to(0,0)} \left\{ \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right\} = u_x + iv_x.$$
Or, if $(\Delta x, \Delta y) \to (0, 0)$ vertically, then we have
$$f'(z_0) = \lim_{(0,\Delta y)\to(0,0)} \left\{ \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + i\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y} \right\} = v_y - iu_y.$$
□

Example 3.5.1. Consider $f(z) = |z|^2 = x^2 + y^2$. Show that f is not differentiable at all $z_0 \neq 0$.

Solution:

Example 3.5.2. The function $f(z) = |z|^2$ has no *singular point* as it is nowhere analytic.

Note that the Cauchy-Riemann equations can be combined into a single equation as:

$$f_x = -if_y. \tag{3.7}$$

Example 3.5.3. Consider the function $f(z) = z^2$. Determine if f satisfies Cauchy-Riemann equations.

Solution:

Theorem 3.5.1 states the necessary conditions for the existence of the derivative of a function f at a point z_0, but not the sufficient conditions.

Example 3.5.4. Consider the function

$$f(z) = \begin{cases} z^5/|z|^4, & \text{if } |z| \neq 0 \\ 0, & \text{if } z = 0 \end{cases},$$

is continuous everywhere, satisfies Cauchy-Riemann equations at $z = 0$, but is not differentiable there.

Theorem below present the **sufficient conditions for complex differentiability**.

Theorem 3.5.2. Let $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that

1. the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood,
2. those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations at (x_0, y_0).

Then $f'(z_0)$ exists and $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Proof. Assume that $0 < |\Delta z| = |\Delta x + i\Delta y| < \epsilon$. Let

$$\Delta w = f(z + \Delta z) - f(z) = u(x + \Delta x, y + \Delta y) - u(x, y) + i[v(x + \Delta x, y + \Delta y) - v(x, y)] = \Delta u + i\Delta v$$

The assumption that first-order derivatives of u and v are continuous at (x_0, y_0) implies that

$$\Delta u = u_x \Delta x + u_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$
$$\Delta v = v_x \Delta x + v_y \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. Thus

$$\Delta w = u_x \Delta x + u_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y + i[v_x \Delta x + v_y \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y]$$

Replacing u_y by $-v_x$ and v_y by u_x and then dividing Δw by Δz gives

$$\frac{\Delta w}{\Delta z} = u_x + iv_x + (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z}.$$

Since $\Delta z = \Delta x + i\Delta y$, we have that $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$. So,

$$\left|(\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z}\right| \le |\epsilon_1 + i\epsilon_3| \le |\epsilon_1| + |\epsilon_3|$$
$$\left|(\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z}\right| \le |\epsilon_2 + i\epsilon_4| \le |\epsilon_2| + |\epsilon_4|.$$

Hence $\lim\limits_{\Delta z \to 0} \dfrac{\Delta w}{\Delta z} = u_x + iv_x$. □

A polynomial function has derivatives everywhere, so every polynomial is analytic and is an *entire* function.

Example 3.5.5. Consider $f(z) = e^z$ where $z = x + iy$. Show that $f(z)$ is differentiable at all z and find $f'(z)$.

Solution:

Example 3.5.6. Let $f(z) = u + iv$ be analytic in a domain D and u is constant. Show that f is constant.

Solution:

3.5.1 Harmonic functions

If the derivatives in the Cauchy-Riemann equations are sufficiently smooth and if we differentiate in the following way

$$u_x = v_y \Rightarrow u_{xx} = v_{xy}, \quad u_y = -v_x \Rightarrow u_{yy} = -v_{yx}$$

we obtain $u_{xx} + u_{yy} = v_{xy} - v_{yx} \Rightarrow u_{xx} + v_{yy} = 0$. This is the famous Laplace equation.

Definition 3.5.1. A real-valued function of two variable $h(x, y)$ is said to be **harmonic** if it satisfies the *Laplace equation*

$$h_{xx} + h_{yy} = 0. \tag{3.8}$$

It is not difficult to derive the following theorem relating holomorphic functions and harmonic functions.

Theorem 3.5.3. If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v are harmonic.

Definition 3.5.2. If $u(x, y)$ and $v(x, y)$ are harmonic functions defined in domain D and satisfy the Cauchy-Riemann equations throughout D, then we say that v is a **harmonic conjugate** of u.

Theorem 3.5.4. A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.

Example 3.5.7 (April 2009 Exam, Q3(a))**.**

1. Show that $u(x, y) = 2x(1 - y)$ is harmonic.
2. Find a function $v(x, y)$ such that $f(x + iy) = u(x, y) + iv(x, y)$ is analytic.
3. Express function f in terms of z.

Solution:

3.5.2 Cauchy-Riemann Equations in Polar Coordinates

For $z = x + iy \neq 0$, we use the polar coordinate:

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{3.9}$$

Consider a function $f(z) = w$ where $w = u + iv$, u, v can be expressed in terms of either the variables x, y or r, θ, depending on whether we write

$$z = x + iy \quad \text{or} \quad z = re^{i\theta}.$$

From chain rule, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}.$$

Thus,

$$u_r = u_x\cos\theta + u_y\sin\theta, \quad u_\theta = -u_x r\sin\theta + u_y r\cos\theta,$$
$$v_r = v_x\cos\theta + v_y\sin\theta, \quad v_\theta = -v_x r\sin\theta + v_y r\cos\theta,$$

If the partial derivatives of u, v with respect to x, y satisfy the Cauchy-Riemann equations: $u_x = v_y, u_y = -v_x$ at z_0, then equation above becomes

$$\begin{aligned} u_r &= u_x \cos\theta + u_y \sin\theta, \quad u_\theta = -u_x r\sin\theta + u_y r\cos\theta, \\ v_r &= -u_y \cos\theta + u_x \sin\theta, \quad v_\theta = u_y r\sin\theta + u_x r\cos\theta, \end{aligned}$$

Comparing these two, we have that

$$r u_r = v_\theta, \quad u_\theta = -r v_r. \tag{3.10}$$

This result can be formulated in the theorem below.

Theorem 3.5.5. If $f(z) = u(r,\theta) + iv(r,\theta)$ is defined throughout some ϵ-neighborhood of a nonzero point $z_0 = r_0 e^{i\theta_0}$, and suppose that

1. the first-order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood,
2. those partial derivatives are continuous at (r_0, θ_0) and satisfy the Cauchy-Riemann equations:

   $$r u_r = v_\theta, \quad u_\theta = -r v_r$$

   at (r_0, θ_0).

Furthermore, $f'(z) = e^{-i\theta}(u_r + iv_r)$ at $z = z_0$.

Proof. The polar form of the Cauchy-Riemann equations have been proven previously. In order to show that $f'(z) = e^{-i\theta}(u_r + iv_r)$, we rewrite the expression

$$f'(z) = u_x + iv_x$$

in polar form. Since $r^2 = x^2 + y^2$ and $\tan\theta = y/x$, we have

$$\frac{\partial r}{\partial x} = x/r = \cos\theta, \quad \frac{\partial \theta}{\partial x} = \cos^2(\theta) \times -y/x^2 = -\frac{\sin\theta}{r}$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= u_r \cos\theta - u_\theta \frac{\sin\theta}{r}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= v_r \cos\theta - v_\theta \frac{\sin\theta}{r}. \end{aligned}$$

Using these results and Cauchy-Riemann equations, we have

$$\begin{aligned} f'(z) = u_x + iv_x &= u_r\cos\theta - u_\theta \frac{\sin\theta}{r} + i\left(v_r\cos\theta - v_\theta\frac{\sin\theta}{r}\right) \\ &= u_r\cos\theta + v_r\sin\theta + i\left(v_r\cos\theta - u_r\sin\theta\right) \\ &= u_r\cos(-\theta) - v_r\sin(-\theta) + i\left(v_r\cos(-\theta) + u_r\sin(-\theta)\right) \\ &= u_r\cos(-\theta) + iu_r\sin(-\theta) - v_r\sin(-\theta) + iv_r\cos(-\theta) \\ &= u_r[\cos(-\theta) + i\sin(-\theta)] + iv_r[\cos(-\theta) + i\sin(-\theta)] = e^{-i\theta}(u_r + iv_r) \end{aligned}$$

□

Example 3.5.8. Given $f(z) = \dfrac{1}{z}$, $z \neq 0$. Verify that the polar form of Cauchy-Riemann equations are fulfilled and find $f'(z)$.

Solution:

Example 3.5.9. $f(z) = \frac{1}{z}$ is analytic at each nonzero point in the complex plane. Thus, the only singular point of $f(z)$ is $z = 0$.

3.6 Elementary functions

In this section, we will define the exponential function for complex variable z and also some trigonometric functions.

3.6.1 Complex exponential function

Define an analytic function f such that

$$\begin{aligned} f(z_1 + z_2) &= f(z_1)f(z_2) \\ f(x) &= e^x \quad \text{for all } x \in \mathbb{R}. \end{aligned} \tag{3.11}$$

From the first equation of Equation 3.11, we know that $f(x + iy) = f(x)f(iy) = e^x f(iy)$. By writing $f(iy) = A(y) + iB(y)$, we have that

$$f(z) = e^x A(y) + ie^x B(y),$$

where $z = x + iy$.

Since f is analytic, Cauchy-Riemann equations implies that

$$A(y) = B'(y) \quad \text{and} \quad A'(y) = -B(y) \implies A''(y) = -A(y).$$

As y is real and from $A''(y) = -A(y)$, we take $A(y) = \alpha \cos y + \beta \sin y$ which leads to

$$B(y) = -A'(y) = \alpha \sin y - \beta \cos y.$$

$f(x) = e^x$ implies that $A(0) = \alpha = 1$ and $B(0) = -\beta = 0$. Thus,

$$f(z) = e^x \cos y + ie^x \sin y = e^{x+iy}.$$

Theorem 3.6.1. Suppose that $f(z) = e^z$ and $z = x + iy$. Then

1. $|e^z| = e^x$,
2. $e^z \neq 0$,
3. $e^{iy} = \cos y + i \sin y$,
4. $e^z = \alpha$ has infinitely many solutions for any $\alpha \neq 0$,

5. $(e^z)' = e^z$.

Proof. **Solution**:

Theorem 3.6.2 (Algebraic Properties of Exponential Function)**.** If z, z_1 and z_2 are complex numbers, then

1. $e^0 = 1$,
2. $e^{z_1}e^{z_2} = e^{z_1+z_2}$,
3. $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$,
4. $(e^z)^n = e^{nz}$, $n = 0, \pm1, \pm2, \cdots$,
5. $(e^z)^{-1} = (e^{-1})^z$.

Proof. These direct results from Equation 3.11. □

As the exponential function satisfies the Cauchy-Riemann equations, and the partial derivatives u_x, u_y, v_x and v_y are continuous for all $z \in \mathbb{C}$, we know that e^z is holomorphic for all $z \in \mathbb{C}$ and hence is entire.

Example 3.6.1. Solve the equation

$$e^z = 1.$$

Solution:

3.6.2 Complex trigonometric functions

Note that from Euler formula, we have

$$\begin{aligned} e^{iy} &= \cos y + i \sin y \\ e^{-iy} &= \cos y - i \sin y. \end{aligned}$$

Thus,

$$\begin{aligned} \sin y &= \frac{1}{2i}\left(e^{iy} - e^{-iy}\right) \\ \cos y &= \frac{1}{2}\left(e^{iy} + e^{-iy}\right) \end{aligned},$$

which hold for all real y. Similarly, we extend the definition to complex z:

$$\begin{aligned} \sin z &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\ \cos z &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \end{aligned}.$$

Some properties of these functions are described below:

Theorem 3.6.3. For $z \in \mathbb{C}$, we have

1. $\sin^2 z + \cos^2 z = 1$,
2. $(\sin z)' = \cos z$,
3. $(\cos z)' = -\sin z$,
4. $\sin(-z) = -\sin z$,
5. $\cos(-z) = \cos z$
6. $\sin(z + 2\pi) = \sin z$,
7. $\cos(z + 2\pi) = \cos z$.

As both sin and cos functions are written in terms of exponential function, we know that they are also entire.

Many other properties of the sin and cos functions remain valid in the larger setting of complex plane, such as

- $\tan z = \frac{\sin z}{\cos z}$,
- $\sec z = \frac{1}{\cos z}$,
- $\csc z = \frac{1}{\sin z}$,
- $\cot z = \frac{1}{\tan z}$.

or,

- $\frac{d}{dz} \sin z = \cos z$,
- $\frac{d}{dz} \cos z = -\sin z$,
- $\frac{d}{dz} \tan z = \sec^2 z$,
- $\frac{d}{dz} \sec z = \sec z \tan z$,
- $\frac{d}{dz} \csc z = \csc z \cot z$,
- $\frac{d}{dz} \cot z = -\csc^2 z$.

For $z_1, z_2 \in \mathbb{C}$, we have

- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$,
- $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$.

However, take note that $\sin z$ is not bounded in modulus by 1. For example, $|\sin 5i| = \frac{1}{2}\left(e^5 - e^{-5}\right) > 1$.

Example 3.6.2. Find $\sin^{-1} 2$.

Solution:

Example 3.6.3. What is the image of the vertical line $x = a$ in the z-plane under the mapping $w = \sin z$?

Solution:

3.6.3 Hyperbolic Functions

The complex hyperbolic functions are defined by replacing the variable x in the exponential function with variable z as follows.

Definition 3.6.1. The **(complex) hyperbolic cosine function** is defined as

$$\cosh : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \cosh z = \frac{e^z + e^{-z}}{2}. \tag{3.12}$$

The **(complex) hyperbolic sine function** is defined as

$$\sinh : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sinh z = \frac{e^z - e^{-z}}{2}. \tag{3.13}$$

The **(complex) hyperbolic tangent function** is defined as

$$\tanh : \mathbb{C} \setminus \{\frac{\pi i}{2} \pm k\pi i\} \to \mathbb{C}, \quad z \mapsto \tanh z = \frac{\sinh z}{\cosh z}. \tag{3.14}$$

The **hyperbolic secant**, **hyperbolic cosecant**, and **hyperbolic cotangent** functions are defined as the inverse of hyperbolic cosine, hyperbolic sine and hyperbolic tangent functions respectively as follows:

$$\operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z}, \quad \coth z = \frac{1}{\tanh z}.$$

Since the linear combination of entire functions is entire, it follows that $\cosh z$ and $\sinh z$ are entire.

By looking at the definitions of hyperbolic functions and trigonometric functions, we find that they are related by

$$\sin z = -i\sinh(iz), \quad \cos z = \cosh(iz), \quad \tan z = -i\tanh(iz).$$

It is also not difficult for us to obtain the properties of hyperbolic functions based on the properties of trigonometric functions:

- $\sinh(-z) = -\sinh z$, $\cosh(-z) = \cosh z$. In other words, hyperbolic sine function is odd and hyperbolic cosine function is even;
- $\cosh^2 z - \sinh^2 z = 1$;
- $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$;
- $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$;
- sinh and cosh are $2\pi i$-periodic entire functions.

It is also easy to obtain derivatives of hyperbolic functions by using definition.

Theorem 3.6.4. For appropriate z,

$$\frac{d}{dz}\sinh z = \cosh z, \quad \frac{d}{dz}\cosh z = \sinh z, \quad \frac{d}{dz}\tanh z = \operatorname{sech}^2 z,$$
$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z, \quad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \cot z, \quad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z.$$

3.7 Inverse functions

In this section, we construct the inverse functions of $f(z) = z^n$ and $f(z) = e^z$.

3.7.1 n-th root function

Example 3.7.1. Find all solutions of $z^3 = 2 + 2i$.

Solution:

Thus, we can define the n-th root function as a set-valued function below:

Definition 3.7.1. An **n-th root function** is a set-valued mapping

$$(\ )^{1/n} : \mathbb{C} \to 2^{\mathbb{C}}, \quad z \mapsto \{ \sqrt[n]{|z|} \exp \frac{i \arg z}{n} \}. \tag{3.15}$$

A **principal nth root function** is defined as

$$\sqrt[n]{\ } : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sqrt[n]{z} = \sqrt[n]{|z|} \exp \frac{i \operatorname{Arg}(z)}{n}.$$

Example 3.7.2. Find the images of the line $x = a$ and $y = b$ under the mapping $z \mapsto \sqrt{z}$.

Solution:

Example 3.7.3. Suppose that $z = i\pi$. Is $\sqrt{z^2e^z} = ze^{\frac{z}{2}}$?

Solution:

It is possible for us to calculate the derivative of principal n-th root function but not the n-th root function (because it is not a function!):

$$(\sqrt[n]{z})' = \lim_{\Delta z \to 0} \frac{\sqrt[n]{z+\Delta z} - \sqrt[n]{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{\frac{\Delta z}{\sqrt[n]{z+\Delta z} - \sqrt[n]{z}}} = \lim_{\Delta w \to 0} \frac{1}{\frac{(w+\Delta w)^n - w^n}{w+\Delta w - w}} = \frac{1}{nw^{n-1}} = \frac{w}{nw^n} = \frac{\sqrt[n]{z}}{nz}$$

where $w + \Delta w = \sqrt[n]{z+\Delta z} \Rightarrow (w+\Delta w)^n = z + \Delta z = w^n + \Delta z$ and $\Delta w \to 0$ as $\Delta z \to 0$.

We have to be very careful here because writing something below by copying the formula directly from Calculus does not work:

$$(\sqrt[n]{z})' \neq \frac{1}{n} z^{1/n-1}$$

This is because $z^{1/n-1} = z^{\frac{1-n}{n}}$ should have n values but the left hand side $(\sqrt[n]{z})'$ is single-valued.

3.7.2 Complex logarithm

We will use the notation $\ln x$ to denote the real valued natural logarithm, that is if $x = e^y$, then $y = \ln x$. Consider $z = e^w$ where $z, w \in \mathbb{C}$, we define *complex logarithm*, $\log z$, as the inverse function of $z = e^w$.

We wish to define the logarithm function in complex plane in order to solve the equation:

$$e^w = z,$$

where z is any nonzero complex number. Write $z = re^{i\Theta}$ $(-\pi < \Theta \le \pi)$ and $w = u + iv$, we have

$$\begin{aligned} e^u e^{iv} &= re^{i\Theta} \\ e^u &= r, \quad v \equiv \Theta \pmod{2\pi}. \end{aligned}$$

Hence, $w = \ln r + i(\Theta + 2n\pi)$ for $n = 0, \pm1, \pm2, \ldots$ and we define the logarithm function as:

$$\log z = \ln r + i(\Theta + 2n\pi), \quad \text{for } n = 0, \pm1, \pm2, \ldots.$$

From Equation 3.16, we know that complex logarithm function is a multi-valued function. When $n = 0$, we have the principal value of $\log z$.

Recall that $\arg z = \operatorname{Arg} z + 2n\pi$ for $n = 0, \pm1, \pm2, \ldots$. We can rewrite the complex logarithm in just z.

Definition 3.7.2. The **complex logarithm** is defined as

$$\log z = \ln|z| + i \arg z, \tag{3.16}$$

where $z \in \mathbb{C}$.

The **principal value** of $\log z$ (or, principal logarithm) is denoted as $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$.

Example 3.7.4. Let $z = 1 + i\sqrt{3}$. Find all solutions of $\log z$.

Solution:

Example 3.7.5. Let $z = re^{i\Theta} \in \mathbb{C}^{\times}$. To compute $e^{\log z}$, we use the definition of logarithm and for $n = 0, \pm 1, \pm 2, \dots$, we have

$$e^{\log z} = e^{\ln r + i(\Theta + 2n\pi)} = e^{\ln r}\left(\cos(\Theta + 2n\pi) + i\sin(\Theta + 2n\pi)\right) = r\left(\cos\Theta + i\sin\Theta\right) = re^{i\Theta}.$$

Thus, $e^{\log z} = z$.

Example 3.7.6. Suppose that $z = x + iy$. Compute $\log(e^z)$.

Solution:

Example 3.7.7. Evaluate $\log(-1)$.

Solution:

3.7.3 Branch points and branch cuts

We shall discuss the concepts of branch points and branch cuts using complex logarithm function only. However, discussion here are also applicable to other *multi-valued* functions, such as *square root function*.

Let $z = re^{i\theta}$ and consider the complex valued function $\log z = \ln r + i(\Theta + 2n\pi)$ where $n = 0, \pm 1, \pm 2, \dots$ and $\operatorname{Arg} z = \Theta$. As this is multi-valued function, and if one travels around the red closed path in figure below, starting counter-clockwise from point z and returning to z, argument of z increases by 2π and $\log z$ become $\log z + i(2\pi)$.

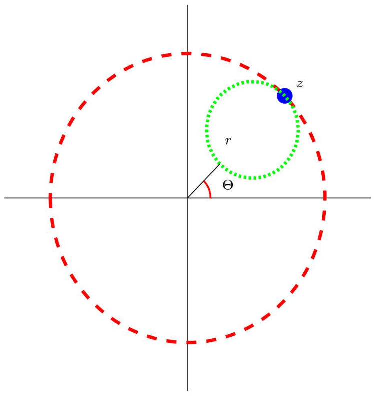


Figure 3.4: Closed curves in z-plane.

This issue (multiple values represented in a single point) arises at every point in the complex plane except at origin. Observe that, if one travels along green closed path, the argument and logarithm do not change in value.

For complex logarithm function, any closed path that encloses origin, will return at *different* value. Any such point is called *branch point* of a complex function.

Definition 3.7.3. If f is a multi-valued complex function, z_0 in the z-plane is called **branch point** if there exist a simple closed curve l surrounding z_0 such that $l(0) = l(1)$ and $f(l(0)) \neq f(l(1))$.

Since $\log z$ is a multi-valued function, $\log z$ is not differentiable. In order to avoid this issue, we let α be any real number and $\alpha < \theta < \alpha + 2\pi$. Thus,

$$\log z = \ln r + i\theta \quad (\alpha < \theta < \alpha + 2\pi),$$

By doing this, we define a "cut" on the plane and points on this cut is not defined.

In this new domain, one can check that $\log z = \ln r + i\theta$ where $\alpha < \theta < \alpha + 2\pi$ is analytic and we have that

$$\frac{d}{dz} \log z = \frac{1}{z} \quad (\alpha < \arg z < \alpha + 2\pi).$$

For each α, the single-valued function is a branch of the multi-valued function.

Definition 3.7.4. A **branch** of a multi-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f.

Definition 3.7.5. A **branch cut** is a portion of line or curve that is introduced in order to define a branch F of a multi-valued function f.

Definition 3.7.6. Any point that is common to all branch cuts of f is called a **branch point**.

Branch cut is defined so that there is no closed curve that encloses origin (branch point) and hence the issue of multiple values at a single point can be avoided.

By considering *Riemann sphere*, one can enclose the "point at infinity" by enclosing the north pole of Riemann sphere. Thus, it can be seen as a "large closed curve". Hence there are two branch points of $\log z$, that is, $z = 0$ and $z = \infty$. We choose a branch cut that connects all the branch points.

It is common (although not necessary) that the branch cut of $\log z$ is taken to be the negative real axis.

Definition 3.7.7. The **principal branch** of complex logarithm is defined as:

$$\operatorname{Log} z = \ln r + i\Theta \quad (-\pi < \Theta < \pi).$$

Example 3.7.8. Compute $\operatorname{Log}(i^3)$ and $3 \operatorname{Log} i$.

Solution:

Example 3.7.9. Find all the branch points of $f(z) = \log(z+1)$.

Solution:

Example 3.7.10. Find all the branch points of $f(z) = z^{1/2}$.

Solution:

3.7.4 Riemann surfaces

With branch cut, we are able to define complex logarithm as different branches that is single value. We could represent each of the branches in different complex planes. Figure below illustrate this.

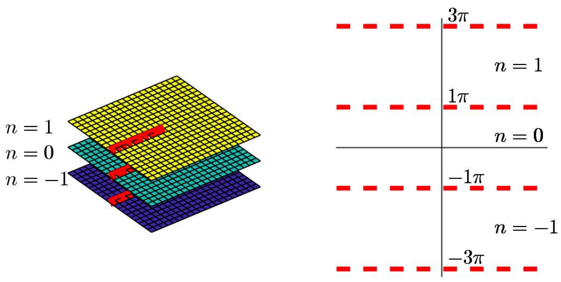


Figure 3.5: Different branches of complex logarithm.

If we join these planes together on the branch cuts, such that a counter-clockwise rotation will travel from branch $n = k$ to $n = k + 1$, we have the **Riemann surface** as below.

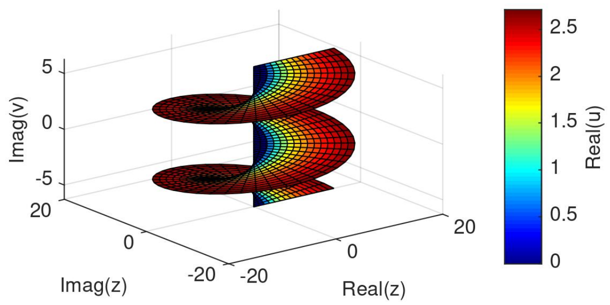


Figure 3.6: Riemann surface of $\log z$.

Similarly, for square root function, its Riemann surface is:

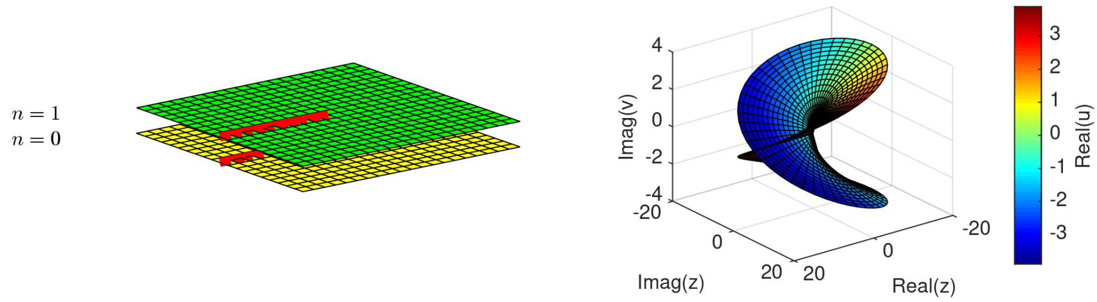


Figure 3.7: Riemann surface of $z^{1/2}$.

Chapter 4

Complex Integration

4.1 Derivatives and integral of $w(t)$

Let $w(t)$ to be a complex-valued function of a real variable t and is written as

$$w(t) = u(t) + iv(t),$$

where u and v are real-valued functions of t.

Definition 4.1.1. The derivative $w'(t)$ is defined as

$$w'(t) = u'(t) + iv'(t).$$

From the definition, it can be easily shown that

$$\frac{d}{dt}[z_0 w(t)] = z_0 w'(t).$$

Example 4.1.1. If $w(t) = e^{z_0 t}$, then $w'(t) = z_0 e^{z_0 t}$.

Solution:

Recall that the mean value theorem states that if a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

For complex functions, the mean value theorem for $w'(t)$ does not hold.

Example 4.1.2. Suppose that $w(t) = e^{it}$ for $0 \leq t \leq 2\pi$. We have that

$$w'(t) = ie^{it} \implies |w'(t)| = |ie^{it}| = 1.$$

Thus, the derivatives is nonzero. However, $w(2\pi) - w(0) = 0$.

The integration of function $w(t) = u(t) + iv(t)$ over interval $a \le t \le b$, can be computed by integrating its real and imaginary parts:

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt,$$

provided the individual integrals on the right exist (or, u, v are *piecewise continuous*).

Example 4.1.3.

$$\int_0^1 (1-it)^2\,dt = \int_0^1 (1-t^2)\,dt - i\int_0^1 2t\,dt = \frac{2}{3} - i.$$

Also, the real and imaginary parts are

$$\operatorname{Re}\int_a^b w(t)\,dt = \int_a^b \operatorname{Re}[w(t)]\,dt \quad \text{and} \quad \operatorname{Im}\int_a^b w(t)\,dt = \int_a^b \operatorname{Im}[w(t)]\,dt.$$

The integration on u, v are in fact integration of real functions. Thus, results from calculus can be extended to $w(t)$. For example,

1. $\int_a^b w(t)\,dt = \int_a^c w(t)\,dt + \int_c^b w(t)\,dt$,
2. $\int_a^b w(t)\,dt = U(t)\big|_a^b + iV(t)\big|_a^b = [U(b) + iV(b)] - [U(a) + iV(a)] = W(t)\big|_a^b$ where $U'(t) = u(t)$ and $V'(t) = v(t)$.

Example 4.1.4. Evaluate $\int_0^{\pi/4} e^{it}\,dt$.

Solution:

4.2 Contours

In the single-variable Calculus, there is only one "path" to integrate from point a to b. However, on a complex plane, when we want to "integrate" from a point z_0 to z_1, there are many "ways" to travel from z_0 to z_1, called arcs or paths.

Definition 4.2.1. An **arc**, or a **path**, or a **(parameterized) curve** C is a set of points $z(t) = x(t) + iy(t)$ where

$$x = x(t), \quad y = y(t) \quad (a \le t \le b),$$

and both $x(t), y(t)$ are continuous functions of the real parameter t.

The values $z(a)$ and $z(b)$ are called the **initial** and **end** or (**terminal**) points of C.

Definition 4.2.2. A curve C is said to be **smooth** (or **regular**) if $x'(t)$ and $y'(t)$ are continuous on $[a, b]$ and not simultaneously zero on the open interval (a, b). In other words, $|z'(t)| \ne 0$ for $t \in (a, b)$.

Definition 4.2.3. A curve C is said to be **piecewise smooth** if it consists of a finite number of smooth curves C_1, C_2, $\cdots$, C_n joined end to end, that is, the terminal point of curve C_k coinciding with the initial point of C_{k+1} for $k = 1, \cdots, n-1$.

Definition 4.2.4. A piecewise smooth curve on $\mathbb{C}$ is called a **contour**.

Definition 4.2.5. A curve C is called a **simple curve** if C does not cross itself except possibly at $t = a$ and $t = b$. It is called a **closed curve** if $z(a) = z(b)$ and if the curve C is simple and closed, then C is a **simple closed curve** or a **Jordan curve**.

Example 4.2.1. For $0 \leq \theta \leq 2\pi$, the following equations describe different type of circle (simple closed curve).

1. $z = e^{i\theta}$. Unit circle centered at origin and oriented counterclockwise (positively oriented),
2. $z = z_0 + Re^{i\theta}$. Circle with radius R and centered at $z = z_0$. Oriented counterclockwise,
3. $z = e^{-i\theta}$. Unit circle centered at origin and oriented clockwise (negatively oriented),
4. $z = e^{-i2\theta}$. Unit circle centered at origin and is traversed counterclockwise **twice**.

Recall that parameterization of a curve is not unique. For example, the smooth curve $y = x^2$ for $-\infty < x < \infty$, can be represented by $x(t) = t, y(t) = t^2$, or, $x(t) = -t, y(t) = t^2$ for $-\infty < t < \infty$.

Suppose that $z = z(t)$ is a parametric representation of a curve C, and

$$t = \phi(\tau) \quad (\alpha \leq \tau \leq \beta),$$

where ϕ is a real-valued function that maps $\alpha \leq \tau \leq \beta$ onto $a \leq t \leq b$. Assume that ϕ is continuous and $\phi'(\tau)$ is always positive and continuous. Thus, we have

$$z = Z(\tau) = z\left[\phi(\tau)\right]. \tag{4.1}$$

Now we wish to show that the *arc length* of C, L, is the same for both representation. Let $z'(t) = x'(t) + iy'(t)$ be continuous in $a \leq t \leq b$. The real-valued function

$$|z'(t)| = \sqrt{|x'(t)|^2 + |y'(t)|^2}$$

is integrable over the defined integral. Also, the length of this **differentiable arc** is

$$L = \int_a^b |z'(t)|\, dt.$$

Substituting $t = \phi(\tau)$ into the equation above, we have that

$$\begin{aligned} L &= \int_\alpha^\beta |z'[\phi(\tau)]\phi'(\tau)\, d\tau \\ &= \int_\alpha^\beta |Z'(\tau)|\, d\tau \quad \text{(From Equation 4.1)}. \end{aligned}$$

Thus, both representation give the same arc length.

If $z = z(t) = x(t) + iy(t)$ is a differentiable arc and if $z'(t) \neq 0$ anywhere in the interval $a < t < b$, then the unit tangent vector

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}$$

is well defined for all t, with angle of inclination $\arg z'(t)$. When $\mathbf{T}$ turns, it turs continuously as t changes over the interval $a < t < b$. Such arc is said to *smooth*.

Theorem 4.2.1 (Jordan Curve Theorem)**.** Any Jordan curve separates the plane into two domains, each having the curve as its boundary. One domain, called the **interior**, is bounded, the other domain, called **exterior**, is unbounded.

4.3 Contour integrals

In this section, we deal with the complex-valued functions f of the complex variable z. Such integral is defined on a contour C in the complex plane and denoted as

$$\int_C f(z)\,dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z)\,dz,$$

where z_1 and z_2 are the initial point and end point, respectively.

Suppose that $f(z)$ is piecewise continuous on C and the parameterization $z = z(t)$ for $a \le t \le b$ is also continuous on $[a, b]$. Then $f[z(t)]$ is also piecewise continuous on $a \le t \le b$. The **line integral** or **contour integral**, of f along C in terms of t, can be written as

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt.$$

C is a contour implies that $z'(t)$ is also piecewise continuous and hence the existence of integral above is ensured (Recall the similar concept in calculus that existence of an integral is provided by its continuity).

Hence we have the following relations.

Theorem 4.3.1. Suppose f and g be continuous in a region D, and C is a smooth curve lying entirely inside D. Then

1. $\displaystyle\int_C kf(z)dz = k\int_C f(z)dz$
2. $\displaystyle\int_C (\alpha f(z) + \beta g(z))dz = \alpha\int_C f(z)dz + \beta\int_C g(z)dz$
3. $\displaystyle\int_{-C} f(z)dz = -\int_C f(z)dz$, where $-C$ denotes the curve having the opposite orientation of C.
4. $\displaystyle\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

Example 4.3.1. Evaluate $\displaystyle\int_C z^2\,dz$ along the curve given by $C : z(t) = t + it^2$ where $0 \le t \le 1$.

Solution:

Example 4.3.2. Evaluate $\displaystyle\int_C z^2\,dz$ from 0 to $1 + i$ along the curve $C : y = x^2$.

Solution:

Example 4.3.3. The integral $\int_C z^2\,dz$ from 0 to $1+i$ along the curve $C : y = x^2$ can also be evaluated by using the fact that $dz = dx + idy$.

$$\begin{aligned}\int_C z^2\,dz &= \int_C (x+iy)^2(dx+idy) = \int_C (x^2-y^2+2ixy)(dx+2ixdx) = \int_C (x^2-y^2+2ixy)(1+2ix)\,dx \\ &= \int_0^1 x^2-y^2-4x^2y+i(2xy+2x^3-2xy^2)\,dx \\ &= \int_0^1 x^2-5x^4+i(4x^3-2x^5)\,dx \\ &= \left[\frac{x^3}{3}-x^5+i\left(x^4-\frac{x^6}{3}\right)\right]_0^1 = -\frac{2}{3}+i\frac{2}{3}.\end{aligned}$$

Example 4.3.4. Find the value of the integral

$$I = \int_C \overline{z}\,dz,$$

where C is the semi-circle

$$z = re^{i\theta} \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right).$$

Solution:

Example 4.3.5. Let C_1 be the polygonal line OAB, where $O = 0$, $A = i$ and $B = 1+i$ are the points on the complex plane. If C_2 is the line OB, evaluate $\int_{C_1} f(z)dz$ and $\int_{C_2} f(z)dz$. Are these two integrals the same if $f(z) = z^2$?

Solution:

Example 4.3.6. Repeat the previous example with $f(z) = \text{Re}(z)$.

Solution:

We will study the conditions for an integration to be *path independent* in the section of *Antiderivatives.*

It is common to use the notation $\oint_C f(z)dz$ to represent the complex integral along a closed curve C in the positive (counterclockwise) orientation.

The next two examples illustrate contour integral that contains point on a branch cut.

Example 4.3.7. Evaluate the integral

$$I = \oint_C (z - z_0)^{a-1}\, dz,$$

where $(z - z_0)^{a-1}$ is defined using the principal branch:

$$(z - z_0)^{a-1} = e^{(a-1)\,\text{Log}(z-z_0)} \quad (|z| > 0, -\pi < \text{Arg}\, z < \pi),$$

and C is a positively oriented circle with radius R and centered at z_0 and $z_0 \in \mathbb{C}$ is a constant.
Solution: Note that the branch cut of $\text{Log}\, z$ is taken to be the negative real axis and hence integrand is not defined at the point $\text{Arg}\, z = \pi$.

Let $z = z_0 + Re^{i\theta}$. We have that

$$I = \oint_C (z - z_0)^{a-1}\, dz = \int_{-a}^{a} \left(Re^{i\theta}\right)^{a-1} iRe^{i\theta}\, d\theta.$$

It can be checked that the integrand, $iR^a e^{ia\theta} = -R^a \sin a\theta + iR^a \cos a\theta$ is continuous on $-\pi < \theta < \pi$. Thus, the integral exists for the contour $z = Re^{i\theta}$ where $-\pi \leq \theta \leq \pi$. Thus,

$$\begin{aligned} I &= iR^a \int_{-\pi}^{\pi} e^{ia\theta}\, d\theta \\ &= iR^a \left[\frac{e^{ia\theta}}{ia}\right]_{-\pi}^{\pi} = \frac{i2R^a}{a} \times \frac{e^{ia\pi} - e^{-ia\pi}}{2i} = i\frac{2R^a}{a} \sin a\pi \end{aligned}$$

Thus, if a is a nonzero integer, then $I = 0$. If $a = 0$, we have

$$\oint_C \frac{1}{z - z_0}\, dz = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta}\, d\theta = 2\pi i$$

Example 4.3.8. Let C denote the semi-circle $z = 3e^{i\theta}$, $0 \le \theta \le \pi$. Compute the integral $I = \int_C z^{1/2}\, dz$, if $z^{1/2}$ is defined on the branch

$$f(z) = z^{1/2} = e^{\frac{1}{2}\log z} \quad (|z| > 0, 0 < \arg z < 2\pi).$$

Solution:

4.4 Upper bounds for moduli of contour integrals

We start with a lemma.

Lemma 4.4.1. If $w(t)$ is a piecewise continuous complex-valued function defined on an interval $a \le t \le b$, then

$$\left| \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)|\, dt. \tag{4.2}$$

Proof. If $\left| \int_a^b w(t) \right| = 0$, it can be easily verified that Equation 4.2 holds.

Suppose that the left-hand side is nonzero, then we can write

$$\int_a^b w(t)\, dt = r_0 e^{i\theta_0} \implies r_0 = \int_a^b e^{-i\theta_0} w(t)\, dt.$$

Since $\left| \int_a^b w(t)\, dt \right| = \left| r_0 e^{i\theta_0} \right| = r_0$ is a real number, we have

$$\begin{aligned} r_0 = \operatorname{Re} \int_a^b e^{-i\theta_0} w(t)\, dt &= \int_a^b \operatorname{Re} \left[e^{-i\theta_0} w(t) \right] dt \\ &\le \int_a^b \left| e^{-i\theta_0} w(t) \right| dt \\ &\le \int_a^b \left| e^{-i\theta_0} \right| |w(t)|\, dt \\ &\le \int_a^b |w(t)|\, dt \end{aligned}$$

□

Now we can prove a useful theorem, which is known as the **ML-inequality**.

Theorem 4.4.1. If f is continuous on a smooth curve C and $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then $\left|\int_C f(z)dz\right| \leq ML$, where L is the length of C.

Proof. Note that $\int_a^b |\gamma'(t)|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt = L$ is the length of the smooth curve $C : \gamma(t) = x(t) + iy(t)$. Then

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \leq \int_a^b |f(\gamma(t))\gamma'(t)|dt \leq \int_a^b M|\gamma'(t)|dt = ML.$$

□

Example 4.4.1. If C is the arc of circle $|z| = 2$ from $z = 2$ to $z = 2i$. Show that

$$\left|\int_C \frac{z}{z^3 - 1}dz\right| \leq \frac{2\pi}{7},$$

using ML-inequality.

Solution:

Example 4.4.2. Show that

$$\lim_{R\to\infty} \int_{C_R} \frac{z^{1/2}}{z^2 + 1}dz = 0,$$

where C_R is the semicircle $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$ and $z^{1/2} = \sqrt{r}e^{i\theta/2}$.
Solution:

$$\begin{aligned}
\left|\int_{C_R} \frac{z^{1/2}}{z^2 + 1}\right| &\leq \int_{C_R} \left|\frac{z^{1/2}}{z^2 + 1}\right| dz \quad \text{(from Lemma 4.4.1)} \\
&= \int_{C_R} \frac{|z^{1/2}|}{|z^2 + 1|} dz \quad \left(\frac{|z^{1/2}|}{|z^2 + 1|} = \frac{\sqrt{R}}{|z^2 - (-1)|} \leq \frac{\sqrt{R}}{|z^2| - |-1|} = \frac{\sqrt{R}}{R^2 - 1}\right) \\
&\leq \frac{\sqrt{R}}{R^2 - 1} \times \pi R \\
&= \frac{\pi/\sqrt{R}}{1 - (1/R^2)}.
\end{aligned}$$

Thus, the limit tends to zero when $R \to \infty$.

4.5 Antiderivatives

Recall that the **fundamental theorem of calculus**:

Theorem 4.5.1. If f is a real-valued function on $[a, b]$ and if there is a differentiable function F on

$[a, b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{4.3}$$

For the proof of this theorem, readers can refer to the Chapter 6 of "Principles of Mathematical Analysis" by Walter Rudin.

Similar theorem can be constructed for complex functions.

Theorem 4.5.2. Let $f(z)$ denote a continuous functions on a domain D. If any one of the following statements is true, then so are the others:

1. $f(z)$ has an antiderivative $F(z)$ throughout D,
2. the contour integrals of $f(z)$ with contours lying entirely in D are path independent, i.e.

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

   where $F(z)$ is the antiderivative of f,
3. the integrals

$$\oint_C f(z)\,dz = 0$$

   where C is any closed contours lying entirely in D.

Proof. Proof is omitted. □

Note that an antiderivative must be an analytic function and is unique expect for an additive constant.

Example 4.5.1. Evaluate $\displaystyle\oint_C \frac{dz}{z^3}$, where C denotes the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$.
Solution: It can be seen that $F(z) = -\frac{1}{2z^2}$ is an antiderivative for $f(z) = \frac{1}{z^3}$.

$f(z)$ is continuous in the domain $\{D : |z| > 0\}$ everywhere except for $z = 0$ and has antiderivative $F(z)$ throughout D. Since C is a closed contour, we have

$$\int_C \frac{dz}{z^3} = 0.$$

Example below illustrates that $f(z) = 1/z$ cannot be integrated in similar way. This is because $\log z$ is not defined (and hence not differentiable) along its branch cut.

Example 4.5.2. Use antiderivative to evaluate $\displaystyle\oint_C \frac{dz}{z}$, where C denotes the circle $z = 2e^{i\theta}$ $(-\pi \le \theta \le \pi)$.

Solution:

Note that example above can also be solved by parameterization of the $z = e^{it}$ for $-\pi \le t \le \pi$.

4.6 Cauchy-Goursat theorem

4.6.1 Cauchy-Goursat theorem

The following theorem on a simple closed contour integration was first proved by Cauchy with the condition that f' is continuous. Goursat was the first to prove that the condition of f' is continuous can be omitted.

Theorem 4.6.1 (Cauchy-Goursat theorem)**.** If a function f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0.$$

Example 4.6.1. Evaluate $\displaystyle\int_C \frac{dz}{z}$ where $C : |z-2| = 1$.

Solution:

4.6.2 Simply connected domain

Definition 4.6.1. A **simply connected** domain D is a domain such that every simple closed contour within it encloses only points of D.

A domain that is not simply connected is **multiply connected**.

Example 4.6.2. Figure below shows different types of domains:

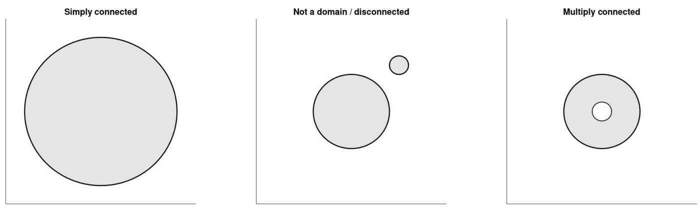


Figure 4.1: Types of domain

Note that in simply connected domain, one can replace the condition of *simple closed contour* in Theorem 4.6.1 with *closed contour*.

Theorem 4.6.2. If a function f is analytic throughout a simply connected domain D

$$\int_C f(z)\, dz = 0$$

for every closed contour C lying in D.

Example 4.6.3. Show that $\displaystyle\int_C \frac{dz}{z} = 0$ where C is any closed contour in the open disk $|z-3| < 2$.

Solution:

Example 4.6.4. A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.
Solution: Since f is analytic, f must be continuous in D and from Theorem 4.6.2,

$$\oint f(z)\, dz = 0.$$

Using Theorem 4.5.2, we know that f must have an antiderivative in D.

Example 4.6.5. Similarly, we know that all *entire functions* have antiderivatives.

4.6.3 Multiply connected domain

For multiply connected domain, we have the following theorem:

Theorem 4.6.3. Suppose that

1. C is a simple closed contour, described in the counterclockwise direction (positively oriented),
2. C_k for $k = 1, 2, \ldots n$ are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common.

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k , then

$$\int_C f(z)\,dz + \sum_{n=1}^{k} \int_{C_k} f(z)\,dz = 0.$$

Example 4.6.6. Let C_1 and C_2 denote positively oriented simple closed contours, where C_1 is interior to C_2. If a function f is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Solution:

4.7 Cauchy integral formula

The following result by Cauchy is a fundamental result in complex integration.

Theorem 4.7.1 (Cauchy integral formula)**.** Let f be analytic everywhere inside and on a simple closed contour C, oriented positively. Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\,dz \tag{4.4}$$

for all z_0 interior to C.

Proof. Let C_ρ to be the circle $|z - z_0| = \rho$, oriented counterclockwise. The value of ρ is chosen to be small enough such that C_ρ is interior to C.

$f(z)$ is analytic inside C, it follows that $f(z)/(z - z_0)$ is analytic in the region between C and C_ρ. Thus, by principle of deformation of paths, we know that

$$\begin{aligned}
\int_C \frac{f(z)}{z - z_0}\,dz &= \int_{C_\rho} \frac{f(z)}{z - z_0}\,dz \\
\int_C \frac{f(z)}{z - z_0}\,dz - f(z_0)\int_{C_\rho} \frac{dz}{z - z_0} &= \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,dz \\
\int_C \frac{f(z)}{z - z_0}\,dz - 2\pi i f(z_0) &= \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,dz,
\end{aligned}$$

where the last equality is using the fact that

$$\int_{C_\rho} \frac{dz}{z - z_0} = 2\pi i.$$

As f is analytic, we know that f continuous, at z_0 and by definition, for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

Let $\rho < \delta$, then $|z - z_0| = \rho < \delta$. Hence,

$$\left| \int_C \frac{f(z)}{z - z_0} \, dz - 2\pi i f(z_0) \right| = \left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| < \frac{\epsilon}{\rho} 2\pi\rho = 2\pi\epsilon$$

This implies that the left hand side of first equality equals to 0, or

$$\int_C \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0).$$

□

Example 4.7.1. Evaluate $\displaystyle\int_C \frac{z}{(z-3)(z-i)} \, dz$ where C is the positively oriented circle $|z| = 2$.

Solution:

Example 4.7.2. Evaluate $\displaystyle\int_C \frac{2z}{z^2 + 2z - 3} \, dz$, where C is the circle $|z - 2| = 2$ oriented positively.

Solution:

4.8 General Cauchy integral formula

Cauchy integral formula can be used to obtain the derivatives of an analytic function f.

Theorem 4.8.1. Let f be a function analytic inside and on a simple closed contour C that is oriented counterclockwise. The n-th derivative of f at z_0 (where z_0 is interior to C) is given by:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz. \tag{4.5}$$

Proof. This can be proved by using mathematical induction. □

Similarly, Equation 4.5 can be used to evaluate certain integrals.

Example 4.8.1. Let $f(z) = e^{3z}$ and C is the positively oriented unit circle $|z| = 2$. Evaluate

$$\int_C \frac{e^{3z}}{(z-1)^{11}}\, dz.$$

Solution:

Example 4.8.2. Evaluate $\displaystyle\int_C \frac{dz}{z}$ where $C : |z| = 1$.

Solution:

Example 4.8.3. Show that for $n \in \mathbb{Z}$ and $n \geq 2$,

$$\int_C \frac{dz}{(z-z_0)^n} = 0$$

where C is any simple closed contour enclosing z_0.
Solution: Similar to previous example, let $f(z) = 1$, then

$$\int_C \frac{dz}{(z-z_0)^n} = \int_C \frac{f(z)}{(z-z_0)^{(n-1)+1}}\, dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0) = 0.$$

We state some theorems before we end this section.

Theorem 4.8.2. If a function f is analytic at a given point, then its derivatives of all orders are analytic there too.

Proof. This can be proved by using results stated previously. Interested readers can refer to Section 52 of "Complex Variables and Applications, 8th edition" by J. W. Brown and R. V. Churchill. □

Theorem 4.8.3. Let f be continuous on a domain D. If $\int_C f(z)\, dz = 0$ for every closed contour C in D, then f is analytic throughout D.

Proof. Since f is continuous on a domain D, then from Theorem 4.5.2, we know that f has an antiderivative. Thus, there exist an analytic function F such that $F' = f$ and from previous theorem, we know that f is itself analytic. □

Theorem 4.8.4 (Cauchy's inequality)**.** Suppose that a function f is analytic inside and on a positively oriented circle C_R , centered at z_0 and with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R , then

$$|f^{(n)}(z_0)| \le \frac{n!M_R}{R^n} \tag{4.6}$$

for $n = 1, 2, \ldots$.

Proof. From Equation 4.5, we have

$$\begin{aligned}|f^{(n)}(z_0)| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| = \frac{n!}{2\pi}\left|\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \\ &\le \frac{n!}{2\pi}\frac{M_R}{R^{n+1}}2\pi R = \frac{n!M_R}{R^n}.\end{aligned}$$

The inequality comes from *ML-inequality.* □

4.9 Liouville's theorem and the fundamental theorem of algebra

Theorem 4.9.1 (Liouville's theorem)**.** If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Proof. Using Cauchy inequality,

$$|f'(z_0)| \le \frac{M_R}{R}$$

where z_0 can be any point on complex plane as f is entire. Since f is bounded, we know that $f \le M$ and $M_R \le M$, thus

$$|f'(z_0)| \le \frac{M}{R}.$$

Note that the value M is independent of R and hence if we take the contour to be arbitrary large (i.e. $R \to \infty$) then

$$|f'(z_0)| = 0.$$

Also, z_0 can be any point on complex plane, we conclude that f is a constant function. □

The **fundamental theorem of algebra** can also be proved by using Liouville's theorem.

Theorem 4.9.2 (Fundamental Theorem of Algebra)**.** Any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

where $a_n \ne 0$. Then $P(z)$ has at least one zero.

Proof. Note that $P(z)$ is a polynomial and entire in complex plane. If $P(z) \ne 0$ for all $z \in \mathbb{C}$, then

$$f(z) = \frac{1}{P(z)}$$

is also entire.

Let $w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}$, then $P(z) = (a_n + w)z^n$.

A sufficiently large positive number R can be found such that the modulus of each of the quotients in w is less than the number $|a_n|/(2n)$ when $|z| > R$. Hence

$$|w| < \frac{|a_n|}{2} \quad \text{whenever} \quad |z| > R.$$

Consequently,

$$|a_n + w| \ge ||a_n| - |w|| > \frac{|a_n|}{2} \quad \text{whenever} \quad |z| > R.$$

Or,

$$|P(z)| = |a_n + w||z^n| > \frac{|a_n|}{2}|z^n| > \frac{|a_n|}{2}R^n \quad \text{whenever} \quad |z| > R.$$

So

$$|f(z)| = 1/|P(z)| < \frac{2}{|a_n|R^n} \quad \text{whenever} \quad |z| > R.$$

Thus, f is bounded in the region exterior to the disk $|z| \leq R$. For interior to the disk, since f is continuous, f must be bounded also. Hence f is bounded in the entire plane.

By Theorem 4.9.1, f must be constant, and consequently $P(z)$. We have a contradiction and conclude that $P(z)$ must have at least one zero. □

4.10 Maximum modulus principle

In this section, we will state the results without proving them.

Lemma 4.10.1 (Local maximum modulus principle)**.** Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighborhood $|z - z_0| < \epsilon$ in which f is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.

Theorem 4.10.1 (Maximum modulus principle)**.** If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all points z in it.

With this theorem, for any f that is analytic and not constant in a **closed bounded region**, R, the maximum value of $|f(z)|$ in R is attained on the boundary of R.

Example 4.10.1. Let R denote the rectangular region $0 \leq x \leq \pi$, $0 \leq y \leq 1$. We know that $|\sin z|$ has a maximum value in R that occurs somewhere on the boundary of R and not in its interior (recall that $\sin z$ is entire function and hence analytic in R).

This can be verified directly by writing

$$|f(z)| = \sin 2x + \sinh 2y$$

and noting that the term $\sin 2x$ is greatest when $x = \pi/2$ and that the increasing function $\sinh 2y$ is greatest when $y = 1$.

Thus the maximum value of $|f(z)|$ in R occurs at the boundary point $z = (\pi/2, 1)$ and at no other point in R.

Chapter 5

Analyticity

5.1 Sequence and series

Definitions of sequence and series have been discussed in Chapter 2. In this chapter, we look at other sequence/series related topics.

Definition 5.1.1. An **infinite sequence**

$$z_1, z_2, \ldots, z_n, \ldots$$

of complex numbers has a limit z if, for each positive number ϵ, there exists a positive integer n_0 such that

$$|z_n - z| < \epsilon \quad \text{whenever} \quad n > n_0. \tag{5.1}$$

The sequence is convergence if the limit,

$$\lim_{n\to\infty} z_n = z, \tag{5.2}$$

exists.

Example 5.1.1. Figure below illustrates the Definition 5.1.1 geometrically.

Solution:

Similar to derivative, we have the following theorem:

Theorem 5.1.1. Suppose that $z_n = x_n + iy_n$, for $n = 1, 2, \ldots$ and $z = x + iy$. Then

$$\lim_{n\to\infty} z_n = z$$

if and only if

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

Proof. If $\lim_{n\to\infty} z_n = z$, then from Equation 5.1, there exist $n_0 > 0$ such that

$$|(x_n + iy_n) - (x + iy)| < \epsilon \quad \text{whenever} \quad n > n_0.$$

Since $|\alpha| \leq |\alpha + i\beta|$, we know that

$$|x_n - x| < \epsilon \quad \text{and} \quad |y_n - y| < \epsilon \quad \text{whenever} \quad n > n_0.$$

Conversely, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then for every $\epsilon > 0$, we have n_1 and n_2 such that

$$|x_n - x| < \epsilon/2 \quad \text{whenever} \quad n > n_1 \quad \text{and} \quad |y_n - y| < \epsilon/2 \quad \text{whenever} \quad n > n_2.$$

Since

$$|(x_n + iy_n) - (x + iy)| = |(x_n - x) + i(y_n - y)| \leq |x_n - x| + |y_n - y|,$$

and take $n_0 = \max(n_1, n_2)$ we have that

$$|z_n - z| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{whenever} \quad n > n_0.$$

□

5.1.1 Convergence of series

Theorem 5.1.2. Suppose that $z_n = x_n + iy_n$ for $n = 1, 2, \ldots$ and $S = X + iY$. Then

$$\sum_{n=1}^{\infty} z_n = S$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y.$$

Proof. Let $X_N = \sum_{n=1}^{N} x_n, Y_N = \sum_{n=1}^{N} y_n$ and $S_N = X_N + iY_N$.

From Theorem 5.1.1, we know that $\lim_{N\to\infty} S_N = S$ if and only if

$$\lim_{N\to\infty} X_N = X \quad \text{and} \quad \lim_{N\to\infty} Y_N = Y.$$

Thus, $\sum_{n=1}^{\infty} z_n = S$ if and only if

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y.$$

□

Recall that a sequence $\{z_n\}$ converges if and only if $\{z_n\}$ is a Cauchy sequence. Thus, a series converges if and only if S_N, n-th partial sum, is a Cauchy sequence. Following from Theorem 5.1.2, if a series of complex numbers converges, then

1. the n-th term converges to zero as n tends to infinity,
2. there exists a positive constant M such that $\sum_{n=1}^{\infty} |z_n| \leq M$ for all positive integer n,
3. the partial sum sequence, $\{S_N\}$, is a Cauchy sequence.

Example 5.1.2. Prove that if $\sum_{k=1}^{\infty} |z_k|$ converges, then $\sum_{k=1}^{\infty} z_k$ also converges.

Solution:

Example 5.1.3 (Cauchy sequence)**.** Show that the series

$$\sum_{n=1}^{\infty} \frac{i^n}{n^2}$$

converges as n tends to infinity.

Solution:

Definition 5.1.2. The **sequence of remainder** is defined as

$$\rho_N = S - S_N$$

where S_N is the partial sums.

It can be seen that $|S - S_N| = |S_N - S| = |\rho_N - 0| = |\rho_N|$, thus a series converges to S if and only if the sequence ρ_N converges to 0.

Example 5.1.4. Let $S = 1 + z + z^2 + \cdots + z^n$, then we have

$$S - zS = (1 + z + z^2 + \cdots + z^n) - z(1 + z + z^2 + \cdots + z^n) = 1 - z^{n+1} \implies S = \frac{1 - z^{n+1}}{1 - z},$$

for all $z \neq 1$.

Consider the partial sum of $\sum_{n=0}^{\infty} z^n$,

$$S_N = \sum_{n=0}^{N-1} z^n = \frac{1 - z^N}{1 - z}.$$

To show that the series converges to $1/(1-z)$ when n tends to infinity, we compute its remainder:

$$\rho_N = S - S_N = \frac{z^N}{1 - z}, \quad \text{for } z \neq 1.$$

It can be seen that $|\rho_N| = \frac{|z|^N}{|1-z|} \to 0$ if $|z| < 1$. Thus,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z} \quad \text{whenever} \quad |z| < 1.$$

Example 5.1.5. Show that for $0 < r < 1$,

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2} \quad \text{and}$$

Solution:

Two commonly used test for convergence of a series:

1. ratio test,
2. root test.

Theorem 5.1.3 ((Cauchy's) Ratio Test)**.** Suppose $\sum z_n$ is a series of nonzero complex numbers such that

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L.$$

- When $L < 1$, then $\sum z_n$ converges absolutely.
- When $L > 1$, then $\sum z_n$ diverges.

- When $L = 1$, inconclusive, other tests are required.

Theorem 5.1.4 ((Cauchy's) Root Test)**.** Suppose $\sum z_n$ is a series such that

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L.$$

- When $L < 1$, then $\sum z_n$ converges absolutely.
- When $L > 1$, then $\sum z_n$ diverges.
- When $L = 1$, inconclusive, other tests are required.

5.1.2 Power series

Definition 5.1.3. A **power series about** z_0 is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{5.3}$$

where the c_i's are the **coefficients**.

The radius of convergence of a power series can be obtained by using root test:

Theorem 5.1.5. Given a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. Let

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \quad \text{and} \quad R = 1/\alpha, \tag{5.4}$$

where $R = \infty$ if $\alpha = 0$ and $R = 0$ if $\alpha = \infty$.

The power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges for all $|z - z_0| < R$ and diverges if $|z - z_0| > R$. R is known as the **radius of convergence**.

A simple (and optional) example to illustrate the concept of $\limsup$ is given below:

Example 5.1.6. Recall two definitions below:

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right) \quad \text{and} \quad \limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right)$$

Consider the sequence $x_n = \frac{1}{n} + (-1)^n$. Then we have

- $\inf x_n = -1$,
- $\sup x_n = 3/2$,
- $\liminf x_n = -1$,
- $\limsup x_n = 1$.

Radius of convergence can also be computed by using *ratio test.* If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \gamma$, then the radius of convergence is $R = 1/\gamma$ where $R = \infty$ if $\gamma = 0$ and $R = 0$ if $\gamma = \infty$.

Example 5.1.7. Find the radius of convergence for $\sum_{n=1}^{\infty} \frac{\log n}{n!} z^n$.

Solution:

Example 5.1.8. Consider the power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Determine the radius of convergence and state whether the power series is convergent on the circle of convergence.

Solution:

5.2 Taylor Series

We now define the complex version of Taylor series:

Theorem 5.2.1 (Taylor's theorem)**.** Suppose that a function f is analytic throughout a disk $|z - z_0| < R_0$, Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \text{for } |z - z_0| < R_0, \tag{5.5}$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \text{where } n = 0, 1, 2, \ldots. \tag{5.6}$$

Proof. Let $|z| = r$ and C_0 be the positively orientated circle $|z| = r_0$ where $r < r_0 < R_0$. Since f is analytic throughout the disk $|z - z_0| < R_0$, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{1}{s - z} f(s)\, ds.$$

Note that for $|z| < 1$,

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = (1 + z + \ldots z^{N-1}) + z^N(1 + z + \ldots) = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z}.$$

Since $|z/s| < 1$, we can rewrite the integral above as

$$\begin{aligned} f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{1}{s} \frac{1}{1 - z/s} f(s)\, ds &= \frac{1}{2\pi i} \int_{C_0} \frac{1}{s} \left(\sum_{n=0}^{N-1} (z/s)^n + \frac{(z/s)^N}{1 - z/s} \right) f(s)\, ds \\ &= \sum_{n=0}^{N-1} \left(\int_{C_0} \frac{f(s)}{s^{n+1}}\, ds \right) z^n + z^N \int_{C_0} \frac{f(s)}{(s-z)s^N}\, ds \\ &= \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z), \end{aligned}$$

where

$$\rho_N(z) = \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)}{(s-z)s^N}\, ds.$$

Recall that $|z| = r$ and s is a point on C_0, thus,

$$|s - z| \geq ||s| - |z|| = r_0 - r.$$

Let M be the maximum value of $|f(s)|$ on C_0,

$$|\rho_N(z)| \leq \frac{r^N}{2\pi} \times \frac{M}{(r_0 - r) r_0^N} 2\pi r_0 = \frac{M r_0}{r_0 - r} \left(\frac{r}{r_0} \right)^N.$$

As $r/r_0 < 1$, we have that $|\rho_N(z)|$ tends to 0 and hence

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

To show that this is valid for all arbitrary point z_0, let $g(z + z_0) = f(z)$ and hence

$$g(z + z_0) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} z^n.$$

Replacing $z \to z - z_0$, we arrived at the Taylor series expansion. □

Recall that a function is analytic if and only if it has a derivative at each point in some neighborhood of z_0, thus, if f is analytic at z_0, then it must be analytic at each point in some neighborhood of $|z - z_0| < \epsilon$. So, the R_0 in Theorem 5.2.1 can be taken as $R_0 = \epsilon$.

If f is analytic everywhere inside a circle centered at z_0, its Taylor series about z_0 within this circle is convergent. When $z_0 = 0$, the series 5.5 becomes Maclaurin series.

Example 5.2.1. Find the Maclaurin series representation of $1/z$.

Solution:

Example 5.2.2. Consider the function

$$f(z) = \frac{1+2z^2}{z^3+z^5}.$$

Since $f(z)$ is not analytic at $z = 0$, we cannot find a Maclaurin series for it. This function, however, can be written as a series that includes *negative powers* of z. This will be discussed in the next section.

Example 5.2.3. Show that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \text{for } |z| < \infty.$$

Solution: Firstly, complex exponential is an entire function, thus it has a series representation that converges for all $|z| < \infty$.

As $f^{(n)} = e^z$, we have that

$$e^z = \sum_{n=0}^{\infty} \frac{e^0}{n!} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Example 5.2.4. Show that for all $|z| < \infty$,

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}.$$

Solution:

5.3 Laurent Series

If f is not analytic at $z = z_0$, we cannot apply Taylor's theorem to obtain a series representation. However it is possible to represent f as a series representation that includes negative powers, which is known as **Laurent series**.

Example 5.3.1. Find the Laurent series of

$$f(z) = \frac{1}{z^2 + z^3}.$$

Solution: Note that it is not possible to find the Maclaurin series of $f(z)$ as it is not analytic at $z = 0$. However

$$f(z) = \frac{1}{z^2 + z^3} = \frac{1}{z^2}\frac{1}{1+z}.$$

From Example 5.2.1, we know that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

and replacing $z \to -z$ gives

$$\frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n, \quad \text{for } |-z| = |z| < 1.$$

Thus,

$$f(z) = \frac{1}{z^2}(1 - z + z^2 - z^3 + z^4 - z^5 + \dots) = \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 - z^3 + \dots.$$

Theorem 5.3.1. Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{5.7}$$

where $R_1 < |z - z_0| < R_2$.

The coefficients are given by

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz \quad n = 0, 1, 2, \dots$$

and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{-n+1}}\,dz \quad n = 1, 2, 3, \dots$$

Proof. Omitted. □

The expression of Laurent series can also be written in an infinite series:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Example 5.3.2. The function $f(z) = 1/(z-2)^2$ is a Laurent series where $z_0 = 2$. That is,

$$\frac{1}{(z-2)^2} = \sum_{n=-\infty}^{\infty} c_n (z - i)^n$$

where $c_{-2} = 1$ and $c_j = 0$ for all $j \neq -2$.

Using Laurent's theorem, we know that

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz = \frac{1}{2\pi i} \int_C \frac{1}{(z-2)^{n+3}} \, dz$$

or,

$$\int_C \frac{1}{(z-2)} \, dz = 2\pi i.$$

Example 5.3.3. Find the series representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \frac{1}{1 + \frac{1}{z}}$$

that is valid when $1 < |z| < \infty$.

Solution:

Example 5.3.4. The function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

has two singular points $z = 1$ and $z = 2$. Find the series representation in the domain:

1. $D_1 : |z| < 1$,

2. $D_2 : 1 < |z| < 2$,
3. $D_3 : |z| > 2$.

Solution:

5.4 Other results on power series

This section collects other results related to power series without proving them (refer to "Section 63 - 67, Complex Variables and Applications, 8th edition" by J. W. Brown and R. V. Churchill).

Theorem 5.4.1 (Abosulutely convergence)**.** If a power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

converges when $z = z_1 (z_1 \neq z_0)$, then it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$.

Theorem 5.4.2 (Uniformly convergence)**.** If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

then that series must be uniformly convergent in the closed disk $|z - z_0| \leq R_1$, where $R_1 = |z_1 - z_0|$.

Theorem 5.4.3 (Continuity)**.** A power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

represents a continuous function $S(z)$ at each point inside its circle of convergence $|z - z_0| = R$.

Theorem 5.4.4 (Integrability)**.** Let C denote any contour interior to the circle of convergence of the power series

$$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

and let $g(z)$ be any function that is continuous on C. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over C; that is,

$$\int_C g(z)S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n\,dz.$$

Theorem 5.4.5 (Differentiability)**.** The power series

$$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

can be differentiated term by term. That is, at each point z interior to the circle of convergence of that series,

$$S'(z) = \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}.$$

Theorem 5.4.6 (Uniqueness for Taylor's series)**.** If a series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

converges to $f(z)$ at all points interior to some circle $|z - z_0| = R$, then it is the Taylor series expansion of for f in powers of $z - z_0$.

Theorem 5.4.7 (Uniqueness for Laurent's series)**.** If a series

$$\sum_{n=-\infty}^{\infty} c_n(z - z_0)^n = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

converges to $f(z)$ at all points interior to some annular domain about z_0, then it is the Laurent series expansion of for f in powers of $z - z_0$ for that domain.

The following sections are partial notes by Dr. Liew How Hui.

5.5 Holomorphicity and Analyticity

What makes complex analysis so special compare to Calculus is that in Calculus, a function many be smooth, i.e. infinite differentiable in a domain, but it may not be analytic, i.e. it may not be possible for us to represent the real-valued smooth function locally at a point with power series. However, in this section, we will see that holomorphicity (complex differentiability) implies analyticity!

Definition 5.5.1. We say that a function f is *analytic at* z_0 if f is **locally** given by a convergent power series, i.e. there exists a power series and some $\epsilon > 0$ such that the series converges absolutely for $|z - z_0| < \epsilon$ to $f(z)$.

Definition 5.5.2. f is called an *analytic function* on an open set D if f is analytic at every point of D.

The following two theorems are two basic properties of analytic functions.

Theorem 5.5.1. The sums, products, and compositions of analytic functions are analytic.

Theorem 5.5.2. The reciprocal of an analytic function that is nowhere zero is analytic, as is the inverse of an invertible analytic function whose derivative is nowhere zero.

Theorem 5.5.3. A power series can be integrated term-by-term within the circle of convergence: that is, if $R \neq 0$ and

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R,$$

then for every path C with initial point z_1 and end point z_2 lying inside the circle of convergence,

$$\int_C f(z)dz = \sum_{n=0}^{\infty} a_n \int_{z_1}^{z_2} (z - z_0)^n dz = \sum_{n=0}^{\infty} a_n \frac{(z - z_0)^{n+1}}{n+1}\Big|_{z=z_1}^{z=z_2}.$$

Proof: By 5.4.4, we conclude that term-by-term integration along C is permissible. Since each term $a_n(z - z_0)^n$ is holomorphic, the integrals are independent of path, so that $\int_{z_1}^{z_2} f(z)dz$ is independent of path. Hence

$$F(z) = \int_{z_0}^{z} f(z)dz$$

is well-defined for $|z - z_0| < R$. According to Morera's Theorem,

$$\frac{d}{dz}F(z) = f(z)$$

so that F is analytic.

One of the main results in complex analysis is that all analytic functions f defined on a region $D \subset \mathbb{C}$ can be represented as a convergent power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where z_0 is the centre of the largest disc $S \subset D$, and $z \in S$. Moreover, not only the convergent power series but all formal power series can be uniquely represented by equivalence classes of analytic functions.

Theorem 5.5.4 (Holomorphicity-Analyticity Theorem)**.** f is holomorphic in domain D if and only if f is analytic in domain D.

Proof by Alonso Delfín (CINVESTAV IPN mathematics department, http://www.math.cinvestav.mx/en/home).

Let $f : D \to \mathbb{C}$ be a holomorphic function, i.e. f is differentiable everywhere in D and satisfies Cauchy-Riemann equations. Also let $a \in D$ and $R > 0$ such that

$$\Delta_R(a) = \{z \in D : |z-a| < R\} \subset D,$$

then f is holomorphic in $\Delta_R(a)$. Now take any r such that $0 < r < R$ and define the path $\gamma(t) = a + re^{it}$ for $t \in [0, 2\pi]$. If $\zeta \in \partial\Delta_r(a) = \gamma([0, 2\pi])$ and $z \in \Delta_r(a)$ then clearly

$$\left|\frac{z-a}{\zeta-a}\right| < 1$$

which gives that

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\left(\frac{1}{1 - \frac{z-a}{\zeta-a}}\right) = \frac{1}{\zeta-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n = \sum_{n=0}^{\infty}\frac{(z-a)^n}{(\zeta-a)^{n+1}} \tag{5.8}$$

Furthermore, if $M = \max\{|f(\zeta)| \ : \ |\zeta - a| = r\}$, then since $|z-a|/r < 1$ and

$$|f(\zeta)| \times \frac{|z-a|^n}{|\zeta-a|^{n+1}} \le \frac{M}{r}(\frac{|z-a|}{r})^n$$

the Weierstrass M-test gives that the series (5.8) times $f(\zeta)$ converges uniformly to $f(\zeta)/(\zeta - z)$. That is

$$\frac{f(\zeta)}{\zeta - z} = \lim_{k\to\infty} f(\zeta)\sum_{n=0}^{k}\frac{(z-a)^n}{(\zeta-a)^{n+1}} \tag{5.9}$$

where the convergence is uniform.

Now, by the Cauchy Integral Formula on a Disk (Section 4.5.1 from Topic 4) we have that for all $z \in \Delta_r(a)$,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta \quad \text{(Section 4.5.1)}\\
&= \frac{1}{2\pi i}\int_\gamma \lim_{k\to\infty} f(\zeta)\sum_{n=0}^{k}\frac{(z-a)^n}{(\zeta-a)^{n+1}}d\zeta \quad \text{(by Equation (5.9))}\\
&= \lim_{k\to\infty}\sum_{n=0}^{k}\frac{1}{2\pi i}\int_\gamma f(\zeta)\frac{(z-a)^n}{(\zeta-a)^{n+1}}d\zeta \quad \text{(since the series converges uniformly)}\\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}d\zeta\right)(z-a)^n =: \sum_{n=0}^{\infty} a_n(\gamma)(z-a)^n
\end{aligned}$$

Moreover, since $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ then $a_n = f^{(n)}(a)/n!$, then $a_n(\gamma) = a_n = f^{(n)}(a)/n!$, the coefficients an don't depend on γ, thus they are independent of r. Then we now have that indeed f has a power expansion of the form $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, and this gives that f is analytic.

If f is analytic on D, then for each $z \in D$, there is a convergence power series at a ϵ-neighbourhood of z and by Theorem **??**, we know that the function is holomorphic on the ϵ-neighbourhood. Hence, f is holomorphic on D.

In order to understand the importance of the Holomorphicity-Analyticity Theorem, we will analyse the following example.

Example 5.5.1. Prove that the error function $\operatorname{erf}(z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z e^{-t^2} dt$ is entire.

Proof: By using the Maclaurin series of e^t, we obtain

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \int_0^z \frac{t^{2n}}{n!} dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)\cdot n!}, \quad z \in \mathbb{C}.$$

The radius of convergence remains ∞ according to

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+1)\cdot n!}{(2(n+1)+1)\cdot(n+1)!} = \lim_{n\to\infty} \frac{2n+1}{(2n+3)(n+1)} = 0.$$

So, erf is analytic on $\mathbb{C}$ and by the Holomorphicity-Analyticity Theorem, erf is entire.

5.6 Isolated Zeroes Principle

A very important nice property of analytic function is that its zeros are isolated.

Theorem 5.6.1 (Zeroes of Analytic Function are Isolated)**.** Let $D \subset \mathbb{C}$ be some open set and let $f : D \to \mathbb{C}$ be analytic. Then either f is a constant function, or the set $\{z \in D \ : \ f(z) = 0\}$ is totally disconnected.

ProofWiki:

Suppose f has no zeroes in D. Then the set described in the theorem is the empty set, and we're done.

So we suppose there is $z_0 \in D$ such that $f(z_0) = 0$.

Since f is analytic, there is a Taylor series for f at z_0 which converges for $|z - z_0| < R$. Since $f(z_0) = 0$, we know $a_0 = 0$. Other a_j may be 0 as well. So let k be the least number such that $a_j = 0$ for $0 \le j < k$, and $\boxed{a_k \neq 0}$. Then we can write the Taylor series for f about z_0 as:

$$\sum_{n=k}^{\infty} a_n (z - z_0)^n = (z - z_0)^k \underbrace{\sum_{n=0}^{\infty} a_{n+k} (z - z_0)^n}_{g(z)}.$$

Note that g is clearly analytic in $|z - z_0| < R$.

Since g is analytic, it is also holomorphic by Holomorphicity-Analyticity Theorem, it is also continuous. Since $g(z_0) = a_k \neq 0$, there is an $\delta > 0$ so that for every $z \in \mathbb{C}$,

$$|z - z_0| < \delta \Rightarrow |g(z) - a_k| < \frac{|a_k|}{2}.$$

But then $g(z)$ can't possibly be 0 in that disk. Hence the result.

5.7 Analytic Continuation

Analytic continuation is a technique to extend the domain of a function to a larger domain. For complex analysis, it is possible to have a beautiful result — the analytic continuation is unique.

The concept of a universal cover was first developed to define a natural domain for the analytic continuation of an analytic function. The idea of finding the maximal analytic continuation of a function in turn led to the development of the idea of *Riemann surfaces*.

The power series in the previous section is generalised by the idea of a *germ*. The general theory of analytic continuation and its generalisations are known as *sheaf theory*.

Complex analysis can be compared to Calculus of one-variable complex functions. Several complex variables, Calculus of multi-variable complex functions are far more complicated where carrying out analytic continuation since the singularities need not be isolated points, and its investigation was a major reason for the development of *sheaf cohomology*.

Definition 5.7.1. Suppose f is an analytic function defined on a non-empty open subset $D \subset \mathbb{C}$. If G is a larger open subset of $\mathbb{C}$ such that $D \subset G$, and F is an analytic function defined on G such that

$$F(z) = f(z) \qquad \forall z \in D,$$

then F is called an *analytic continuation* of f. In other words, the restriction of F to D is the function f we started with.

Example 5.7.1. Show that $F(z) = \frac{1}{1-z}$ is an analytic continuation of $f(z) = \sum_{n=0}^{\infty} z^n$.

Proof: Since f is defined only on $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$, while F is defined on $\mathbb{C} \setminus \{1\}$ and $F(z) = f(z)$ for $|z| < 1$, F is an analytic continuation of f.

A holomorphic function is completely determined by its values on a small neighbourhood in D. This is not true for differentiable functions in Calculus. In comparison, complex-differentiability, is a much more rigid notion.

Theorem 5.7.1 (Identity Theorem)**.** Given holomorphic functions f and g on a connected open set D, if $f = g$ on some open non-empty subset of D or along an arc interior to D, then $f = g$ on D.

Proof (Wikipedia):

The connectedness (Topic 1) assumption on the domain D is necessary and is in fact key to a short proof given here. Under this assumption, since we are given that the set is not empty, topologically the claim amounts to that f and g coincide on a set that is both open and closed.

The closedness is immediate from the continuity of f and g.

Therefore, the main issue is to show that the set on which $f = g$ is an open set.

Because a holomorphic function can be represented by its Taylor series everywhere on its domain since it is analytic, it is sufficient to consider the set

$$S = \{z \in D \mid f^{(k)}(z) = g^{(k)}(z) \quad \text{for all } k \geq 0\}.$$

Suppose w lies in S. Then, because the Taylor series of f and g at w have non-zero radius of convergence, the open disk $\Delta_r(w)$ also lies in S for some r. In fact, r can be anything less than the distance from w to the boundary of D. This shows S is open and proves the theorem.

With the identity theorem, is it possible for us to establish the uniqueness theorem for analytic continuations.

Theorem 5.7.2 (Analytic Continuation Theorem)**.** If G is the connected domain of two analytic functions F_1 and F_2 such that $D \subset G$ and for all z in D

$$F_1(z) = F_2(z) = f(z),$$

then

$$F_1 = F_2 \quad \text{on all of } G.$$

This is because $F_1 - F_2$ is an analytic function which vanishes on the open, connected domain D of f and hence must vanish on its entire domain according to the identity theorem.

A common way to define functions in complex analysis proceeds by first specifying the function on a "small domain" only, and then extending it by analytic continuation. There are various ways one might show that a holomorphic function can be extended to a larger domain. The *Schwarz Reflection Principle* is an important example: If an open set is symmetric about the real axis, then every function holomorphic in the upper half complex plane can be extended by mirror reflection. Sometimes, one can show that a function satisfies a "functional equation" and then use that to do the extending. Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

and *Riemann zeta function* are two famous examples of the analytic continuation of

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

By using the M-test, it is easy to see that $\zeta(z)$ converges for $\mathrm{Re}(z) > 1$. By the analytic continuation theorem, it is possible to extend the domain of $\Gamma(z)$ and $\zeta(z)$ to $\mathbb{C}\setminus\{0, -1, -2, \cdots\}$ and $\mathbb{C}\setminus\{1\}$ respectively.

So it is always possible for us to obtain an analytic continuation from a power series f? It is possible that f cannot be continued beyond the disk of definition if the boundary is full of non-isolated singularities. In this case, we say that the circumference is the natural boundary of the function.

Definition 5.7.2. For a power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$$

with

$$\liminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 1$$

the circle of convergence is a *natural boundary* or *natural barrier.* Such a power series is called *lacunary.*

A famous example is stated below.

Example 5.7.2. Show that $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ has a natural boundary $|z| = 1$.

Proof: By ratio test, it is easy to see that f converges for $|z| < 1$. Hence, $|z| = 1$ is a radius of convergence. To show that it is a natural boundary, we study

$$f(z^2) = \sum_{n=0}^{\infty} (z^2)^{2^n} = \sum_{n=0}^{\infty} z^{2^{n+1}} = f(z) - z.$$

By induction $f(z^{2^m}) = f(z) - \sum_{j=0}^{m-1} z^{2^j}$. Hence, the value of the function f at all points z_m on the unit circle satisfying the equation $z^{2^m} = 1$ is infinite:

$$f(z_m) = \sum_{n=0}^{\infty} z_m^{2^n} = \sum_{j=0}^{m-1} z_m^{2^j} + \sum_{n=0}^{\infty} z_m^{2^{m+n}} = \sum_{j=0}^{m-1} z_m^{2^j} + \sum_{n=0}^{\infty} 1 = \infty.$$

No matter how small an arc we take on the unit circle, there exist points z_m on the arc such that $f(z_m) = \infty$. Therefore, analytic continuation for f is impossible beyond $|z| = 1$.

The theoretical proof of analytic continuation theorem is complicated. We will just state an theorem which characterises how the largest domain of extension.

Theorem 5.7.3 (Monodromy Theorem)**.** Let D be a simply connected domain and f be analytic in some disk $D_0 \subset D$. If f can be analytically continued along any two distinct smooth contours C_1 and C_2 to a point in D, and if there are no singular points enclosed within C_1 and C_2 then the result of each analytic continuation is the same and the function is single-valued.

This theorem can be extended to the case where the region enclosed by contours C_1 and C_2 contains at most, isolated singular points.

Chapter 6

Residue Theorem and Its Applications

6.1 Isolated singular points and residues

In this section we will define isolated singular point and residues of a function at the isolated singular point.

Definition 6.1.1. A point z_0 is an **isolated singular** point if and only if there is a *deleted neighborhood* $0 < |z - z_0| < \epsilon$ throughout which f is analytic.

Example 6.1.1. The origin is an isolated singular point for $f(z) = 1/z$. However $z = 0$ is not an isolated singular point for $g(z) = \sqrt{z}$ because everywhere deleted neighborhood of it contains negative real axis, which the principal branch of square root function is not defined there.

Example 6.1.2. Determine all the singular points of $f(z) = \frac{1}{\sin(\pi/z)}$ and state whether they are isolated singular point.

Solution:

Note that if a function is analytic everywhere inside a simple closed contour C, except at a **finite number** of singular points, then all these points must be isolated singular points.

Definition 6.1.2. The **residue** of f at the isolated singular point z_0 is the complex coefficient, b_1, of its Laurent series at z_0. This is denoted as

$$b_1 = \operatorname*{Res}_{z=z_0} f(z).$$

Let z_0 to be an isolated singular point of $f(z)$ and R is positive number such that, $f(z)$ is analytic in $0 < |z - z_0| < R$. Then $f(z)$ has Laurent series representation in $0 < |z - z_0| < R$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots.$$

The complex constant b_1 here is the residue of $f(z)$ at z_0.

Recall that the complex coefficients of Laurent series is given by

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}}\, dz,$$

where C is any positively oriented simple simple closed contour that lies within $0 < |z - z_0| < R$. Thus, when $n = 1$, we have

$$b_1 = \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i} \int_C f(z)\, dz. \tag{6.1}$$

Example 6.1.3. Let C be the unit circle centered at origin. The integral

$$\int_C \frac{dz}{z}$$

can be evaluated using Equation 6.1. As the Laurent series of $1/z$ is $1/(z-0) = 1/z$ itself, we have that $b_1 = 1$ and hence

$$\int_C f(z)\, dz = 2\pi i b_1 = 2\pi i.$$

Example 6.1.4. Evaluate

$$\int_C z^4 \sin\left(\frac{1}{z}\right)\, dz,$$

where C is the positively oriented unit circle $|z| = 1$.

Solution:

Example 6.1.5. Evaluate the integral

$$\int_C \frac{dz}{z^2 - 4}$$

where C is the positively oriented circle $|z - 2| = 1$.

Solution:

The integration contour in previous example encloses only one singular point. If, a contour C' is defined to be the positively oriented circle $|z| = 3$, then C' encloses both singular points of $f(z) = 1/(z^2 - 4)$. The following theorem allows us to deal with contour that encloses *finite number* of singular points.

Theorem 6.1.1 (Cauchy's residue theorem)**.** Let C be a simple closed contour, oriented counterclockwise. If a function f is analytic inside and on C except for a finite number of singular points z_k, $k = 1, 2, \ldots, n$, inside C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Proof. Let C_k be the circle centered at z_k, oriented counterclockwise and interior to C. Let the radius of C_k to be small enough such that none of the circle C_k intersect each other.

Thus, C_k and simple closed contour C form a multiply domain where f is analytic inside and on the boundary.

By Cauchy-Goursat theorem, we have

$$\int_C f(z)\,dz - \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0,$$

or

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = \sum_{k=1}^{n} 2\pi i \operatorname*{Res}_{z=z_k} f(z),$$

for $k = 1, 2, \ldots, n$. $\square$

Example 6.1.6. Evaluate the integral

$$\int_{C'} \frac{dz}{z^2 - 4}$$

where C' is the positively oriented circle $|z| = 3$.

Solution:

Both singular points are interior to C'. Thus the integral can be evaluated by using Theorem 6.1.1.

Let $f(z) = \frac{1}{z^2-4}$ and rewriting it:

$$f(z) = \frac{1}{z-(-2)}\left(-\frac{1}{4}\frac{1}{1-\frac{z+2}{-4}}\right) = -\frac{1}{4}\frac{1}{z-(-2)}\sum_{n=0}^{\infty}\left(\frac{z+2}{-4}\right)^n = -\frac{1}{4}\frac{1}{z+2} + \sum_{n=0}^{\infty}\frac{1}{(-4)^{n+1}}(z+2)^n.$$

So, $\operatorname{Res}_{z=-2} f(z) = -1/4$. From Example 6.1.5, we know that $\operatorname{Res}_{z=2} f(z) = 1/4$. These give

$$\int_{C'} \frac{dz}{z^2-4} = 2\pi i(1/4 - 1/4) = 0.$$

Note that Example 6.1.5 and 6.1.5 can also be solved using Cauchy integral formula and Cauchy-Goursat theorem. Finding residues at $z = 2$ and $z = -2$ can also be simplified by using partial fractions:

$$\frac{1}{z^2-4} = \frac{1}{4}\frac{1}{z-2} + \frac{-1}{4}\frac{1}{z+2}.$$

Since $1/(z-2)$ is a Laurent series at $z = 2$ when $0 < |z-2| < 4$ and $1/(z+2)$ is a Laurent series when $0 < |z-1| < 4$, then

$$\operatorname*{Res}_{z=2} f(z) = \frac{1}{4}, \quad \operatorname*{Res}_{z=-2} f(z) = -\frac{1}{4}.$$

Example 6.1.7. Evaluate the integral

$$\int_C \frac{5z-2}{z(z-1)}\,dz,$$

where C is the circle $|z| = 2$, oriented counterclockwise.

Solution:

6.1.1 Point at infinity

It is convenient to consider the point at infinity as an isolated singular point if there exist a positive real number R such that the function f is analytic for $R < |z| < \infty$.

Theorem 6.1.2. If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$

Proof. Let R_1 be a positive number that is large enough such that C lies within $|z| = R_1$. So, f is analytic in $R_1 < |z| < \infty$ and hence the point at infinity is an isolated singular point of f.

Let C_0 be a circle of radius $R_0 > R_1$ and oriented **clockwise**, then

$$\int_{C_0} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\infty} f(z).$$

Since f is analytic at all points exterior to C, the principle of deformation of paths shows that

$$\int_C f(z)\,dz = \int_{-C_0} f(z)\,dz = -\int_{C_0} f(z)\,dz.$$

So,

$$\int_C f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z).$$

Expressing f using Laurent series at $z = 0$, for the range $R_1 < |z| < \infty$,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n\,dz,$$

where

$$c_n = \frac{1}{2\pi i}\int_{-C_0} \frac{f(z)}{z^{n+1}}\,dz, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Replace $z \to 1/z$, we obtain

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n}, \quad 0 < |z| < \frac{1}{R_1}$$

and

$$c_{-1} = \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = \frac{1}{2\pi i}\int_{-C_0} f(z)\,dz = -\operatorname*{Res}_{z=\infty} f(z),$$

or

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$

□

Example 6.1.8. Evaluate the integral of

$$f(z) = \frac{5z-2}{z(z-1)}$$

around the circle $|z| = 2$, oriented counterclockwise.

Solution:

6.2 Classification of isolated singularities

Definition 6.2.1. Negative power terms of $z - z_0$ in Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots$$

is known as the **principal part** of f at z_0.

We can now classify an isolated singular point, z_0, based on the number of nonzero term in the principal part of f at z_0:

1. **Pole of order m:** $b_m \neq 0$ and $b_{m+1} = b_{m+2} = \cdots = 0$ (a pole of order 1 is commonly known as simple pole),
2. **Removable singular point:** all coefficients in the principal part is zero,
3. **Essential singular point:** there are infinitely many $b_i \neq 0$.

Example 6.2.1. The function $\frac{z^2-2z+3}{z-2} = \frac{z(z-2)+3}{z-2} = z + \frac{3}{z-2} = 2 + (z-2) + \frac{3}{z-2}$ for $0 < |z-2| < \infty$ has a pole of order 1 (simple pole) at $z_0 = 2$ and $\operatorname{Res}_{z=2} f(z) = 3$.

Example 6.2.2. Determine the isolated singular point of $\frac{\sin z}{z}$ is a pole, removable singular point or an essential singular point.

Solution:

Suppose that $f(z)$ has a removable singular point at $z = z_0$. Then $f(z)$ can be written as a series representation in $0 < |z - z_0| < R$ and the singular point can be "removed" by defining

$$g(z) = \begin{cases} f(z) & \text{when } z \neq z_0 \\ a_0 & \text{when } z = z_0 \end{cases}.$$

It can be checked that $g(z)$ is analytic in $|z - z_0| < R$.

Example 6.2.3. Consider the function $e^{1/z}$, show that $z = 0$ is an essential singular point.

Solution:

Theorem 6.2.1 (Riemann's Theorem on Removable Singularities). If f has an isolated singularity at $z = z_0$ and if $\lim_{z\to z_0}(z - z_0)f(z) = 0$ then $z = z_0$ is a removable singular point.

Proof.

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots \\
(z - z_0)f(z) &= \sum_{n=0}^{\infty} a_n(z - z_0)^{n+1} + b_1 + \frac{b_2}{z - z_0} + \dots \\
\lim_{z\to z_0}(z - z_0)f(z) &= \lim_{z\to z_0}\sum_{n=0}^{\infty} a_n(z - z_0)^{n+1} + b_1 + \frac{b_2}{z - z_0} + \dots = 0 \implies b_1 = b_2 = b_3 = \dots = 0.
\end{aligned}$$

Thus, $z = z_0$ is a removable singular point. □

Theorem 6.2.2 (Riemann's Theorem on Poles). If f is analytic in a deleted neighborhood of z_0 and if there exists a positive integer k such that

$$\lim_{z\to z_0}(z - z_0)^k f(z) \neq 0 \quad \text{but} \quad \lim_{z\to z_0}(z - z_0)^{k+1} f(z) = 0$$

then f has a pole of order k at $z = z_0$.

Proof. Similar to previous theorem, we multiplying $(z - z_0)^{k+1}$ to the Laurent series of $f(z)$, we have

$$\begin{aligned}
(z - z_0)^{k+1}f(z) &= \sum_{n=0}^{\infty} a_n(z - z_0)^{n+k+1} + b_1(z - z_0)^k + b_2(z - z_0)^{k-1} + \dots + b_{k+1} + \frac{b_{k+2}}{z - z_0} + \dots \\
\lim_{z\to z_0}(z - z_0)^{k+1}f(z) &= \sum_{n=0}^{\infty} a_n(z - z_0)^{n+k+1} + b_1(z - z_0)^k + b_2(z - z_0)^{k-1} + \dots + b_{k+1} + \frac{b_{k+2}}{z - z_0} + \dots
\end{aligned}$$

$\implies b_1 = b_2 = b_3 = \dots = 0.$

If $\lim_{z\to z_0}(z - z_0)^{k+1}f(z) = 0$, then

$$b_{k+1} = b_{k+2} = \dots = 0,$$

and $\lim_{z\to z_0}(z - z_0)^k f(z) \neq 0$ ensures that $b_k \neq 0$.

Therefore the Laurent series expansion of f at z_0 reduces to

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \dots + \frac{b_k}{(z - z_0)^k}.$$

This shows that f has a pole of order n at z_0. □

6.3 Residue and its computation

The first part of the following theorem is equivalent to Theorem 6.2.2 and the second part provides an alternative way of finding residues at poles.

Theorem 6.3.1. An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic and nonzero at z_0. Moreover,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

Proof. Let $f(z) = \phi(z)/(z-z_0)^m$. Since $\phi(z)$ is analytic at z_0, for $0 < |z-z_0| < \epsilon$, we have

$$\begin{aligned}
\phi(z) &= \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n \\
f(z) &= \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \\
&= \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \\
&\quad + \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi'(z_0)/1!}{(z-z_0)^{m-1}} + \frac{\phi''(z_0)/2!}{(z-z_0)^{m-2}} + \cdots + \frac{\phi^{(m-1)}(z_0)/(m-1)!}{z-z_0}
\end{aligned}$$

$\phi(z_0) \neq 0$ implies that z_0 is a pole of order m and $\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$.

On the other hand, if $f(z)$ has a pole of order m at $z = z_0$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \cdots + \frac{b_m}{(z-z_0)^m}, \quad \text{for } 0 < |z-z_0| < R,$$

where $b_m \neq 0$.

Define the function, $\phi(z)$:

$$\phi(z) = \begin{cases} (z-z_0)^m f(z), & \text{when } z \neq z_0 \\ b_m, & \text{when } z = z_0 \end{cases}.$$

Writing this in power series, we have

$$\phi(z) = b_m + b_{m-1}(z-z_0) + \cdots + b_2(z-z_0)^{m-2} + b_1(z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n},$$

where is analytic in $|z-z_0| < R$ and $\phi(z_0) = b_m \neq 0$. □

Example 6.3.1. The function $f(z) = \frac{\cos z}{z^4}$ has Laurent series at $z = 0$:

$$f(z) = \frac{1}{z^4}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) = \frac{1}{z^4} - \frac{1}{4! z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \dots.$$

Thus, $f(z)$ has a pole of order 4 at $z = 0$ and its residue is zero.

Or, let $\phi(z) = \cos z$. We know that $\cos z$ is analytic at $z = 0$ and $\cos 0 \neq 0$, thus $f(z)$ has a pole of order 4 at $z = 0$ and its residue is

$$\operatorname*{Res}_{z=0} f(z) = \frac{\phi'''(z)|_{z=0}}{3!} = 0.$$

Example 6.3.2. Evaluate the following integral

$$\int_C \frac{z^3 + 2z}{(z-i)^3}\, dz$$

if C is $|z| = 2$, oriented counterclockwise.

Solution:

Example 6.3.3. Consider the function

$$f(z) = \frac{1}{z(e^z - 1)}.$$

Determine the order of pole at $z = 0$ and its residue.

Solution:

6.4 Zeros of analytic functions

In this section, we study properties of zeros of analytic functions and how they are related to poles.

Definition 6.4.1. An analytic function, f, has a **zero of order** m at z_0 if

$$f^{(n)}(z_0) = 0, \quad \text{for } n = 0, 1, 2, \ldots, m - 1$$

and

$$f^{(m)}(z_0) \neq 0.$$

Theorem 6.4.1. Let a function f be analytic at a point z_0. It has a zero of order m at z_0 if and only if there is a function g, which is analytic and nonzero at z_0, such that

$$f(z) = (z - z_0)^m g(z).$$

Proof. Suppose that g is analytic and nonzero and z_0. It has a Taylor series representation when $|z-z_0|<\epsilon$:

$$g(z)=\sum_{n=0}^{\infty}\frac{g^{(n)}(z_0)}{n!}(z-z_0)^n!.$$

Let $f(z)=(z-z_0)^m g(z)$, then

$$f(z)=\sum_{n=0}^{\infty}\frac{g^{(n)}(z_0)}{n!}(z-z_0)^{m+n},$$

implies that f is analytic at z_0 with $f(z_0)=f'(z_0)=\cdots=f^{(m-1)}(z_0)=0$ and $f^{(m)}(z_0)=m!g(z_0)\neq 0$. Thus, f has a zero of order m at z_0.

Conversely, if f is analytic at z_0 and has a zero of order m at z_0, then for some $\epsilon>0$, we have

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n=\sum_{n=m}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n=(z-z_0)^m\left[\sum_{n=0}^{\infty}\frac{f^{(m+n)}(z_0)}{(m+n)!}(z-z_0)^n\right]$$

when $|z-z_0|<\epsilon$. Take $g(z)=\frac{f^{(m+n)}(z_0)}{(m+n)!}(z-z_0)^n$, it can be seen that g is analytic in $|z-z_0|<\epsilon$ and $g(z_0)\neq 0$. $\square$

Example 6.4.1. The function $f(z)=z^3-27$ has zero of order 1 at $z=3$ because

$$f(z)=(z-3)(z^2+3z+9)=(z-3)g(z)$$

where $g(z)=z^2+3z+9$ and $g(3)=27\neq 0$.

Order of zeros can also be determined directly from definition.

Example 6.4.2. Determine the order of all zeros of function $f(z)=z^2(e^z-1)$.

Solution:

Theorem 6.4.2. Given a function f and a point z_0 , suppose that

1. f is analytic at z_0,
2. $f(z_0)=0$ but $f(z)$ is not identically equal to zero in any neighborhood of z_0.

Then $f(z)\neq 0$ throughout some deleted neighborhood $0<|z-z_0|<\epsilon$ of z_0.

Proof. As $f(z)$ is not identically zero in neighborhood of z_0, its derivatives must not be all zero at z_0. Since f is analytic and $f(z_0)=0$, the function can be written as

$$f(z)=(z-z_0)^m g(z),$$

where $g(z)$ is analytic and $g(z_0) \neq 0$, by Theorem 6.4.1.

$g(z)$ is analytic implies that $g(z)$ is continuous and by Theorem 3.3.7, we know that $g(z) \neq 0$ in $|z - z_0| < \epsilon$. Thus, $f(z) \neq 0$ in the deleted neighborhood $0 < |z - z_0| < \epsilon$. □

Theorem 6.4.3. If a function f is analytic throughout a neighborhood N_0 of z_0 and $f(z) = 0$ at each point z of a domain D or a line segment L containing z_0, then

$$f(z) \equiv 0 \quad \text{in } N_0.$$

Proof. This is a consequence from Theorem 6.4.2. □

6.4.1 Zeros and poles

Theorem 6.4.4. Suppose that p and q are analytic at z_0 and $p(z_0) \neq 0$. The function q has a zero of order m at z_0 if and only if the quotient $p(z)/q(z)$ has a pole of order m at z_0.

Proof. Since q has a zero of order m at z_0, from Theorem 6.4.2, we know that $q(z) \neq 0$ in $0 < |z - z_0| < \epsilon$ for some $\epsilon > 0$. So z_0 is a isolated singular point of the quotient $p(z)/q(z)$.

Write $q(z) = (z - z_0)^m g(z)$, we know that $g(z_0) \neq 0$ and g is analytic at z_0. Thus, $\phi(z) = p(z)/g(z)$ is also analytic and nonzero at z_0. Hence using the Taylor series representation of $\phi(z)$:

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m} = \sum_{n=0}^{\infty} a_n (z - z_0)^{n-m} = \sum_{n=-m}^{\infty} c_n (z - z_0)^n.$$

This implies that z_0 is a pole of order m.

If $p(z)/q(z)$ has a pole of order m at z_0, then

$$\frac{p(z)}{q(z)} = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} = \frac{\phi(z)}{(z - z_0)^m}, \quad b_m \neq 0,$$

where

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + b_1 (z - z_0)^{m-1} + \cdots + b_m,$$

and $\phi(z_0) \neq 0$. Rewriting this,

$$q(z) = (z - z_0)^m \frac{p(z)}{\phi(z)} = (z - z_0)^m g(z),$$

with $g(z)$ be analytic and nonzero at z_0. Thus, z_0 is a zero of order m. □

Example 6.4.3. Determine the order of pole at $z = 0$ of the function

$$f(z) = \frac{1}{z\,(e^z - 1)}.$$

Solution:

The following theorem shows another way of computing the residues at *simple poles* identified by previous theorem.

Theorem 6.4.5. Let two functions p and q be analytic at a point z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$ and $q'(z_0) \neq 0$, then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

Proof. Since $q(z_0) = 0$ and $q'(z_0) \neq 0$, z_0 is a zero of order 1 and by Theorem 6.4.4, z_0 is a simple pole of $p(z)/q(z)$.

Let $q(z) = (z - z_0)g(z)$ where $g(z)$ is analytic and nonzero at z_0. Then

$$\frac{p(z)}{q(z)} = \frac{p(z)}{(z - z_0)g(z)} = \frac{1}{z - z_0} \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The last equality follows from the fact that $p(z)/g(z)$ is analytic at z_0. Thus, residue of $p(z)/q(z)$ at z_0 is

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)}.$$

□

Theorem 6.4.6. If z_0 is a pole of a function f, then

$$\lim_{z \to z_0} f(z) = \infty$$

Proof. From Theorem 6.3.1, we know that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

and since

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z - z_0)^m}{\phi(z)} = 0 \implies \lim_{z \to z_0} f(z) = \infty.$$

□

To identify the residue at pole of order m where $m \geq 1$, we have the following:

Theorem 6.4.7. Let z_0 be a pole of order m of $f(z)$. The residue at z_0 can be computed by:

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right].$$

Proof. As $f(z)$ has a pole of order m at z_0, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}, \quad b_m \neq 0,$$

Multiplying both sides with $(z - z_0)^m$, we have

$$\begin{aligned}
(z - z_0)^m f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + b_1 (z - z_0)^{m-1} + \cdots + b_m \\
\frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)] &= \sum_{n=0}^{\infty} \frac{(n+m)!}{(n+1)!} a_n (z - z_0)^{n+1} + (m-1)! b_1 \\
\frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)] &= b_1.
\end{aligned}$$

□

Example 6.4.4. Find the residues at all singular points of the function

$$f(z) = \frac{e^{-1/z^4}}{z^2+1}.$$

Solution:

Example 6.4.5. Find the residue of the function

$$f(z) = \frac{\tanh z}{z^2}$$

at the singular point $z = \frac{\pi i}{2}$.

Solution:

6.5 Examples of applications

Residues concept is important in various applications such as evaluating contour integration by summing all residues at the isolated singular points of f that fall within the closed contour C.

In this section, we will demonstrate how residues concept can be applied to evaluate both proper and improper real integral. We start with an warm up example.

Example 6.5.1. Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)} dz$ where the contour C is given by:

1. a rectangle with vertices $z = i$, $z = 4 + i$, $z = -i$, $y = 4 - i$,
2. a circle $|z| = 2$.

Solution:

6.5.1 Definite integrals involving sines and cosines

Integrals of the type

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta,$$

can be evaluated by viewing θ as argument of a complex number z on a positively oriented circle C. Let

$$z = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi.$$

We can rewrite the integral as

$$\int_C F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{iz}.$$

This is obtained by observing that $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$.

Example 6.5.2. For $-1 < a < 1$, evaluate the integral

$$\int_0^{2\pi} \frac{1}{1 + a\sin\theta}\, d\theta.$$

Solution:

6.5.2 Improper integrals

The integration of $f(x)$ over the semi-infinite interval $0 \leq x < \infty$, is defined as

$$\int_0^\infty f(x)\,dx = \lim_{R\to\infty} \int_0^R f(x)\,dx.$$

Similarly, the **improper integral** over $\mathbb{R}$ is defined to be

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)dx \tag{6.2}$$

when both the limits exist.

Definition 6.5.1. The **Cauchy principal value** of the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is defined as

$$P.V. \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \tag{6.3}$$

If integral 6.2 converges, its Cauchy principal value 6.3 exists; and

$$\int_{-\infty}^{\infty} f(x)dx = P.V. \int_{-\infty}^{\infty} f(x)dx$$

However, it is not always true that integral 6.2 converges when its Cauchy principal value exists as demonstrated by the following example.

Example 6.5.3. Let $f(x) = x$. Determine whether $P.V. \int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx$ are the same.

Solution:

It can be shown that if $f(x)$ is even function, then

$$\int_{-\infty}^{\infty} f(x)\,dx = P.V. \int_{-\infty}^{\infty} f(x)\,dx.$$

Method involving sum of residues can be used to evaluate improper integral of rational functions $f(x) = p(x)/q(x)$. Let C_R be the positively oriented semi-circle centered at origin with radius R, then

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z),$$

and if $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$, then

$$P.V. \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Example 6.5.4. Evaluate the following integral

$$\int_{0}^{\infty} \frac{x^2}{x^6+1}\,dx$$

using sum of residues.

Solution:

6.5.3 Improper integrals from Fourier analysis

For the integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin ax\,dx, \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\cos ax\,dx,$$

we can use the technique of sum of residues by considering an integration over a contour from $-R$ to R along real axis, and C_R which is positively oriented semi-circle of radius R centered at origin.

The integration is modified by using the fact that

$$\int_{-R}^{R} f(x)\cos ax\,dx + i\int_{-R}^{R} f(x)\sin ax\,dx = \int_{-R}^{R} f(x)e^{iax}\,dx.$$

The example below illustrates this.

Example 6.5.5. Show that

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2}\,dx = \frac{2\pi}{e^3}.$$

Solution:

Chapter 7

Harmonic Function

7.1 Harmonic conjugate and relation with analytic function

In Section 3.5.1, we have defined **Laplace equation**, **harmonic function** (3.5.1) and **harmonic conjugate** (3.5.2) of a harmonic function. Recall that Theorem 3.5.3 show that if $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then u and v are harmonic in D. We have a slightly different version here:

Theorem 7.1.1. A function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.

Proof. This is direct result from the sufficient condition of differentiability of function. □

Now we show that harmonic function, $u(x, y)$, always has a harmonic conjugate $v(x, y)$ in D by deriving an expression for $v(x, y)$.

Theorem 7.1.2. Let $u(x, y)$ be any given harmonic function defined on a *simply connected domain* D. There is a harmonic conjugate $v(x, y)$ such that

$$u_x = v_y, \quad u_y = -v_x.$$

Proof. As u is harmonic function, its second-order partial derivatives are continuous in D, which implies the first-order partial derivatives of $-u_y$ and u_x are also continuous. Let (x_0, y_0) be a fixed point in D, define function

$$v(x, y) = \int_{(x_0,y_0)}^{(x,y)} -u_t(s, t)\, ds + u_s(s, t)\, dt.$$

As $u_{xx} + u_{yy} = 0$, we have $(-u_y)_y = (u_x)_x$ and hence the line integral is path-independent. Thus $v(x, y)$ is well defined for all (x, y) in D; and

$$v_x(x, y) = -u_y(x, y), \quad v_y(x, y) = u_x(x, y).$$

□

Following from the previous theorem, we have the following:

Theorem 7.1.3. Let u be a harmonic function in a simply connected domain D. Then there is a analytic function f such that $u = \mathrm{Re}(f)$ on D.

Example 7.1.1. Given $u(x, y) = y^3 - 3x^2y$. Show that $u(x, y)$ is harmonic and then find the corresponding analytic function, $f(z)$, such that

$$\mathrm{Re}\, f(z) = u(x, y).$$

Solution:

Note that not all harmonic function has harmonic conjugate. For example, the function $\ln|z|$ is harmonic in $D = \mathbb{C}^\times$ (not simply connected domain) has no harmonic conjugate. Theorem 7.1.3 indicates if u is harmonic in a simply connected domain then u has harmonic conjugate.

The following example shows that the harmonic conjugate is unique up to an additive constant.

Example 7.1.2. Show that if $u(x, y)$ is harmonic in a simply connected domain D, then its conjugate is unique up to an additive constant.

Solution: Since $u(x, y)$ is harmonic in simply connected domain, Theorem 7.1.3 guarantees the existence of harmonic conjugate. Let v and V to be harmonic conjugates of u. Thus,

$$u_x = v_y, \quad u_y = -v_x \quad \text{and} \quad u_x = V_y, \quad u_y = -V_x.$$

Let $w = v - V$, we have that

$$w_x = v_x - V_x = -u_y - (-u_y) = 0, \quad w_y = v_y - V_y = u_x - (u_x) = 0.$$

Thus, $w = C$ is a constant and hence $v(x, y) = V(x, y) + C$

Note that even if v is a harmonic conjugate of u, u is not necessary a harmonic conjugate of v (unlike the conjugate concept of complex numbers, $\overline{\overline{z}} = z$).

Example 7.1.3. Consider $f(z) = z^2 = (x^2 - y^2) + i(2xy) =: u + iv$. Show that v is a harmonic conjugate of u but u is not a harmonic conjugate of v.

Solution:

Thus, we see that if $f(z) = u(x, y) + iv(x, y)$ is analytic in D, $g(z) = v(x, y) + iu(x, y)$ is not necessary analytic in D.

Example 7.1.4 (April 2008 Exam, Q2(b)).

1. Prove that the function $\varphi(x, y) = e^{-x}(x \cos y + y \sin y)$ is harmonic. (4 marks)
2. Find an analytic (holomorphic) function f of $z = x + iy$ such that the real part of f is $\varphi(x, y)$. (6 marks)

7.2 Mean value property

Harmonic functions share many properties of analytic functions, especially the mean value property and the maximum value principle. The mean value property states that the average values of the function over a ball or sphere is equal to its value at the center.

Recall that a harmonic function, u, satisfies the Laplace equation:

$$\nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy} = 0.$$

Let $z = x + iy$ and $z_0 = x_0 + iy_0$. It can also be shown that $v = \ln |z - z_0|$ is a harmonic function ($z \neq z_0$):

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2} \Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{(x - x_0)^2 + (y - y_0)^2 - (x - x_0)[2(x - x_0)]}{[(x - x_0)^2 + (y - y_0)^2]^2} \\
&\qquad = \frac{(y - y_0)^2 - (x - x_0)^2}{[(x - x_0)^2 + (y - y_0)^2]^2} \quad \text{and} \\
\frac{\partial v}{\partial y} &= \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2} \Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{(x - x_0)^2 - (y - y_0)^2}{[(x - x_0)^2 + (y - y_0)^2]^2}.
\end{aligned}$$

Therefore, $v_{xx} + v_{yy} = 0$ for all $z \neq z_0$.

Theorem 7.2.1 (Mean value theorem). If $u(z)$ is a harmonic function in an open connected domain D, then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta. \tag{7.1}$$

The closed disc $\overline{D(z_0; r)}$ is within the domain D.

Solution: *Proof.* Let $\Omega : 0 < |z - z_0| \le r$. Since $\Delta u = 0$ and $\Delta \ln |z| = 0$ for $z \neq 0$, we have

$$\int_\Omega [u(z)\Delta \ln |z - z_0| - \ln |z - z_0| \Delta u(z)] \, dz = 0.$$

We rewrite the left hand side of this integral using divergence operator and recall that the divergence theorem (for 2D) states that

$$\int\int_\Omega \boldsymbol{\nabla} \cdot \vec{F} \, dA = \int_{\delta\Omega} \vec{F} \cdot \vec{n} \, ds,$$

thus,

$$\begin{aligned}\int_\Omega \boldsymbol{\nabla} \cdot [u(z)\nabla \ln |z - z_0| - \ln |z - z_0| \nabla u(z)] \, dz &= \int_{C_r} u(z)\nabla \ln |z - z_0| \cdot \vec{n} \, ds - \int_{C_\epsilon} u(z)\nabla \ln |z - z_0| \cdot \vec{n} \, ds \\ &+ \int_{C_r} \ln |z - z_0| \nabla u(z) \cdot \vec{n} \, ds - \int_{C_\epsilon} \ln |z - z_0| \nabla u(z) \cdot \vec{n} \, ds\end{aligned}$$

Let $\Omega' : |z - z_0| \le r$. By using divergence theorem,

$$\int_{C_r} \ln |z - z_0| \nabla u(z) \cdot \vec{n} \, ds = \ln r \int_{C_r} \nabla u(z) \cdot \vec{n} \, ds = \ln r \int_{\Omega'} \nabla \cdot \nabla u(z) \, dz = \ln r \int_{\Omega'} \Delta u(z) \, dz = 0,$$

Thus,

$$\int_{C_r} u(z)\nabla \ln |z - z_0| \cdot \vec{n} \, ds = \int_{C_\epsilon} u(z)\nabla \ln |z - z_0| \cdot \vec{n} \, ds.$$

Changing variable $z \to z_0 + re^{i\theta}$ and $\nabla \ln |z - z_0| \cdot \vec{n} = 1$, we obtain

$$\int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta = \int_0^{2\pi} u(z_0 + \epsilon e^{i\theta}) \, d\theta$$

or, taking $\epsilon \to 0$,

$$\int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta = 2\pi u(z_0).$$

□

Note that mean value theorem of harmonic function can also be written in the form of real function:

$$h(x, y) = \frac{1}{2\pi} \int_0^{2\pi} h(x + r\cos\theta, y + r\sin\theta), d\theta.$$

Example 7.2.1. Show that for any $a, b \in \mathbb{R}$,

$$\int_0^{2\pi} e^{b+\sin\theta} \cos(a + \cos\theta) d\theta = 2\pi e^b \cos a. \tag{7.2}$$

Solution:

The mean value property characterizes harmonic functions.

Theorem 7.2.2. If $u : D \subset \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying the mean value property, then it is harmonic.

A harmonic function satisfies the "maximum value principle": it does not assume its maximum on a region unless it is constant there. Indeed the conclusion is true for any continuous function with the mean-value property, or even just satisfying the inequality

$$h(x, y) \leq \frac{1}{2\pi} \int_0^{2\pi} h(x + r\cos\theta, y + r\sin\theta)d\theta.$$

The reverse inequality similarly shows that a harmonic function satisfies the "minimum value principle".

Theorem 7.2.3 (Maximum-minimum (value) principle)**.** Let D be a domain of a harmonic function h.

1. If h attains a maximum or a minimum inside D, then h is a constant on D.
2. If h extends continuously to $\overline{D}$ and $h \leq 0$ on ∂D, then $h \leq 0$ on D.

Proof. It is enough to prove the statement concerning the maximum of h. If we let $v = -h$, we have the v is harmonic and a minimum value of h corresponds to the maximum value of v.

Suppose that h attains a maximum value $(x_0, y_0) \in D$.

Step 1: Let C_r be any positively oriented circle in D with a centre at (x_0, y_0). By assumption, $M = h(x_0, y_0) \geq h(x, y)$ for all $(x, y) \in D$. By Mean Value Theorem, we have

$$M = h(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} h(x_0 + r\cos\theta, y_0 + r\sin\theta)d\theta.$$

Consider the continuous function $f(\theta) = h(x_0 + r\cos\theta, y_0 + r\sin\theta)$ for $\theta \in [0, 2\pi]$. Then

$$M \geq h(x, y) \Rightarrow f(\theta) \leq M \quad \text{for } 0 \leq \theta \leq 2\pi.$$

The average of $f(\theta)$ is equal to M implies that $f(\theta) = M$ (otherwise, since $f(\theta) \leq M$, the average should be less than M). Thus h is constant and equal to $h(x_0, y_0)$ on C_r. Since r is arbitrary, it follows that h is constant and equal to $h(x_0, y_0)$ on any circle that we can fit around (x_0, y_0) in D.

Step 2: Let (x, y) be any point in D. Join (x_0, y_0) to (x, y) using a polygonal line. Cover the polygonal line by finitely many overlapping disks S_0, S_1, $\cdots$, S_n with S_0 centred at (x_0, y_0) and S_j centred at (x_j, y_j), the pint of intersection of ∂S_{j-1} and the polygonal line.

Remark: It is possible to cover the polygonal line with finitely many disk because polygonal lines are closed and bounded and Heine-Borel Theorem (Topic 1) says that finite covers exists.

The function h is constant and equals the maximum value M on S_0. By continuity of h, it follows that $h(x_1, y_1) = M$. We learn from step 1 that h is constant in the largest disk we can fit around (x_1, y_1). Hence h is constant and equals M in S_1. Continue in this manner, we conclude that h is constant and equals M in S_2 to S_n, Thus $h(x, y) = M$. Since (x, y) is arbitrary it follows that h is constant in D. □

If the domain D is not bounded, the function h need not attain a maximum or minimum anywhere inside D or on its boundary. For example, $h(x, y) = xy$ is harmonic in $D = \mathbb{R} \times [0, \infty)$. The function h can be arbitrary large and arbitrary small in the interior of D and so there is not maximum or minimum value.

However, if D is bounded, we have the following corollary.

Corollary 7.2.1. Let D be a bounded domain and h a harmonic function on D such that h is continuous on ∂D, the boundary of D. Then h attains its maximum value M and its minimum value m on ∂D.

Example 7.2.2. Find the maximum and minimum values of $h(x, y) = x^2 - y^2$ for $(x, y) \in D = [0, 2] \times [0, 1]$.

Solution:

Harmonic function appears to be useful in solving Dirichlet problem. We shall delay the discussion of this application to the last chapter.

Chapter 8

Further Topic in Geometry

A complex function can be viewed as a mapping/transformation, geometrically. For example, the function $w = z + B$ where B is a complex constant, is a translation by B.

Consider the mapping $w = Az$, where $A \neq 0$ is a complex constant. For all $z \neq 0$, we rewrite the mapping as

$$w = Az = ae^{i\alpha} \times re^{i\theta} = (ar)e^{i(\alpha+\theta)}.$$

Thus, this mapping can be viewed as an expansion/contraction by a factor a and followed by a rotation of angle α. In this sense, we define the following.

Definition 8.0.1. The **linear transformation** $w = Az + B$ where $B \neq 0$, is a composition of the transformation

$$Z = Az \quad (A \neq 0) \quad \text{and} \quad w = Z + B.$$

Example 8.0.1. Find the image of R under the transformation $w = (1 + i)z + 2$ where R is the region bounded by $y = 0$, $y = 2$, $x = 0$ and $x = 1$.

Solution:

In this chapter, we will look at three different types of transformation, that is, the inverse function and linear fractional transformations (which is also known as *Möbius transformation*).

8.1 Transformation $w = \frac{1}{z}$

Now we introduce the transformation $w = \frac{1}{z}$. It can be checked that

$$w = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}.$$

Thus, this transformation can be viewed as a composition of

$$Z = \frac{z}{|z|^2} \quad (z \neq 0) \quad \text{and} \quad w = \overline{Z},$$

where the first transformation maps ($|Z| = \frac{1}{|z|}, \operatorname{Arg} Z = \operatorname{Arg} z$) all nonzero points inside the circle $|z| = 1$ to exterior to it, and conversely. Points on the circle are mapped to itself. If we include the *point at infinity* into the first transformation, then this transformation is continuous on the *extended complex plane.* While the second transformation, is merely a reflection about the real axis.

8.2 Mapping by $w = \frac{1}{z}$

Theorem 8.2.1. The mapping $w = 1/z$ transforms circles and lines onto circles and lines.

Proof. Let $z = x + iy$, then

$$\begin{aligned} w &= \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2} \\ u + iv &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}. \end{aligned}$$

Solving for x and y:

$$x = \frac{u}{u^2+v^2}, \quad y = -\frac{v}{u^2+v^2}.$$

Next, we shall use a general equation to describe any *circle or line*:

$$A(x^2+y^2) + Bx + Cy + D = 0. \tag{8.1}$$

It is clearly seen that $A = 0$ gives line equation when B and C are not both zero.

For $A \neq 0$:

$$\begin{aligned} (x^2+y^2) + \frac{B}{A}x + \frac{C}{A}y + \frac{D}{A} &= 0 \\ \left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 + \frac{D}{A} - \frac{B^2}{4A^2} - \frac{C^2}{4A^2} &= 0 \\ \left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 &= \left(\frac{B^2+C^2-4AD}{4A^2}\right) \end{aligned}$$

Since the left hand side must be greater than or equal to zero, we have that $B^2 + C^2 > 4AD$ when $A \neq 0$ and this is a circle of radius $\sqrt{B^2+C^2-4AD}/2A$.

Substituting $x = u/(u^2+v^2)$ and $y = -v/(u^2+v^2)$ into Equation 8.1, we have

$$D(u^2+v^2) + Bu - Cv + A = 0. \tag{8.2}$$

Thus, $w = 1/z$ maps circles and lines onto circles and lines. $\square$

From Equations 8.1 and 8.2, we observe the following:

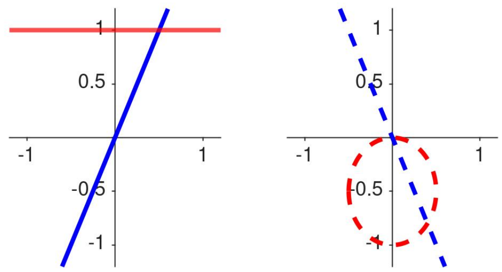
1. a circle ($A \neq 0$) not passing through the origin in the z-plane is transformed into a circle not passing through the origin in the w-plane,

2. a circle ($A \neq 0$) through the origin in the z-plane is transformed into a line that does not pass through the origin in the w-plane,

3. a line ($A = 0$) not passing through the origin in the z-plane is transformed into a circle through the origin in the w-plane,
4. a line ($A = 0$) through the origin in the z-plane is transformed into a line through the origin in the w-plane.

Example 8.2.1. Sketch the images of $y = 1$ and $y = 2x$ under the mapping $w = 1/z$.

Solution:



Example 8.2.2. Find the images of the line (a) $x = a$; (b) $y = b$, where $ab \neq 0$, under the mapping

$$w = f(z) = \frac{1}{z}. \tag{8.3}$$

Solution:

8.3 Linear fractional transformation

A more general transformation (as compared to $w = 1/z$) is linear fractional transformation (Möbius transformation).

Definition 8.3.1. A **linear fractional transformation** is defined as

$$w = \frac{az+b}{cz+d}, \tag{8.4}$$

where a, b, c, d are complex constant and $ad - bc \neq 0$.

When $c = 0$, the transformation can be rewritten as

$$w = \frac{a}{d}z + \frac{b}{d},$$

where $ad \neq 0$. This transformation reduces to a nonconstant linear function.

Similarly, for $c \neq 0$, we have

$$w = \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d}, \quad ad - bc \neq 0.$$

Thus, this is a composition of linear mappings and a reciprocal mapping.

To define this transformation in extended complex plane, we write

$$T(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

We then define $T(\infty) = \infty$ if $c = 0$, and $T(\infty) = \frac{a}{c}, T\left(-\frac{d}{c}\right) = \infty$ if $c \neq 0$. Thus, T is a one-to-one mapping from extended z-plane *onto* extended w-plane. This one-to-one and onto properties ensure the existence of inverse transformation T^{-1}. The inverse transformation can be obtained by solving Equation 8.4 for z:

$$T^{-1}(z) = \frac{-dw+b}{cw-a}, \quad ad - bc \neq 0.$$

Also, for the point at infinity,

$$\begin{aligned} &T^{-1}(\infty) = \infty, \quad \text{if } c = 0 \\ &T^{-1}\left(\frac{a}{c}\right) = \infty \quad \text{and} \quad T^{-1}(\infty) = -\frac{d}{c}, \quad \text{if } c \neq 0. \end{aligned} \tag{8.5}$$

Theorem 8.3.1. The composition of any two linear fractional transformations and inverse of linear fractional transformation is also a linear fractional transformation.

Proof. From Equation 8.5, we know that inverse of any linear fractional transformation is also a linear fractional transformation.

Suppose that we have

$$T(z) = \frac{az+b}{cz+d}, \quad T'(z) = \frac{a'z+b'}{c'z+d'}.$$

The composition $T(T'(z))$:

$$T(T'(z)) = \frac{a\frac{a'z+b'}{c'z+d'}+b}{c\frac{a'z+b'}{c'z+d'}+d} = \frac{a(a'z+b')+b(c'z+d')}{c(a'z+b')+d(c'z+d')} = \frac{(aa'+bc')z+(ab'+bd')}{(a'c+c'd)z+(b'c+dd')},$$

is just another linear fractional transformation. □

Example 8.3.1. Find the linear fractional transformation that maps the points

$$z_1 = -2, \quad z_2 = 0, \quad z_3 = 2$$

onto

$$w_1 = -2i, \quad w_2 = 1, \quad w_3 = 2i.$$

Solution:
Start with second point, we have

$$1 = \frac{b}{d} \implies b = d.$$

So,

$$\begin{aligned} -2i &= \frac{-2a+b}{-2c+b} \to 4ci - 2bi = -2a+b \\ 2i &= \frac{2a+b}{2c+b} \to 4ci + 2bi = 2a+b. \end{aligned}$$

with $b(a-c) \neq 0$. From these two equations, we have

$$b = 4ci \to c = -\frac{1}{4}bi, \quad a = bi.$$

Thus,

$$w = \frac{az+b}{cz+d} = \frac{bzi+b}{-\frac{1}{4}bzi+b} = \frac{zi+1}{-\frac{1}{4}zi+1} = \frac{4zi+4}{-zi+4}.$$

There is always a linear fractional transformation that maps three distinct points in z-plane onto three specified distinct points in w-plane.

8.3.1 Implicit form

Theorem 8.3.2. The *unique* linear fractional transformation that maps z_1, z_2, z_3 onto w_1, w_2, w_3 is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

Proof. Omitted. □

Theorem 8.3.3 (Fixed Points)**.** A linear fractional transformation that is not the identity map has one or two fixed points.

Proof. A fixed point has the property:

$$z = \frac{az+b}{cz+d}.$$

∞ is a fixed point iff $c = 0$.

The finite fixed points are the solutions in z of the equation $cz^2 + (d - a)z - b = 0$. If $c \neq 0$, this equation has either two distinct roots or one reported root. If $c = 0$ then there is one root for $d \neq a$ and zero roots for $d = a$. In both cases, the transformation has one or two fixed points. □

Theorem 8.3.4. A linear fractional transformation maps circles in extended z-plane onto circles and lines in extended w-plane. The image is a line if and only if $c \neq 0$ and the pole $z = -\frac{d}{c}$ is on the circle.

Proof. When $c = 0$,

$$w = \frac{a}{d}z + \frac{b}{d},$$

where $ad \neq 0$. This transformation is a nonconstant linear function. Thus, it maps circle onto circles.

When $c \neq 0$, we have

$$w = \frac{a}{c} + \frac{bc - ad}{c} \cdot \frac{1}{cz + d}, \quad ad - bc \neq 0.$$

This can be viewed as a composition of linear transformations, reciprocal mapping and linear transformations. Since $w = 1/z$ maps circles onto circles and lines, hence linear fractional transformation maps circles to circles and lines.

For $c \neq 0$, the type of image of the transformation is determined by the reciprocal mapping. From the discussion of $w = 1/z$, we know that $w = 1/z$ maps a circle that passes through the origin in the z-plane onto a line. Thus, $w = 1/(cz + d)$ maps a circle onto a line if the circle passes through $-d/c$. □

Example 8.3.2. Find the linear fractional transformation that maps $z_1 = -1, z_2 = 0, z_3 = 1$ onto $w_1 = -i, w_2 = 1, w_3 = i$.

Solution:

Example 8.3.3 (April 2010 Exam, Q5(a))**.** Find the image of the interior of the circle $C : |z - 2| = 2$ under the Möbius transformation $w = \dfrac{z}{2z - 8}$.

Solution:

Example 8.3.4. Find the image of the unit circle $|z| = 1$ under the linear fractional transformation

$$T(z) = \frac{z + 2}{z - 1}.$$

What is the image of the disk $|z| < 1$?

Solution:

Theorem 8.3.5. There exist infinitely many linear fractional transformation which transform a given circle into a second given circle.

Example 8.3.5. Find a linear transformation that maps the circle $|z - i| = 1$ to circle $|w| = 2$.
Solution: Observe that the Möbius transformation that we are looking for will translate the centre from i to 0 and magnify the radius from 1 to 2 Hence the linear transformation

$$T(z) = 2(z - i)$$

will do the job.

Example 8.3.6. Find a Möbius transformation that maps the circle $C : |z - 1| = 1$ to the circle $C' : |w - \frac{3i}{2}| = 2$.

Solution:

Example 8.3.7. Find a linear fractional transformation that maps $|z| < 2$ onto the right half-plane of w.

Solution:

Chapter 9

Conformal mapping and application

9.1 Conformal mapping

Definition 9.1.1. A mapping $w = f(z)$ is called **conformal mapping** at z_0 if and only if f preserves angles between curves at z_0 and preserves their angular orientation.

Example 9.1.1. Consider the linear mapping $w = az + b$ where $a, b \in \mathbb{C}$ and $a \neq 0$. Recall that this mapping can be viewed geometrically as a rotation of angle $\operatorname{Arg} a$, expansion/contraction by a factor $|a|$ ($Z = az$) and followed by a translated by b ($w = Z + b$).

Now take two smooth paths parametrized by γ_1 and γ_2. Suppose that these two paths intersect at z_0 with angle α. It can been seen that their images by f are two paths in the w-plane that intersect at $w_0 = f(z_0)$ with angle α. Furthermore, the angular orientation is also preserved.

Theorem 9.1.1. Suppose that f is analytic at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0.

Proof. Let γ be any smooth path through z_0, parametrized by $z(t) = x(t) + iy(t)$, with $z(t_0) = z_0$ and $z'(t) \neq 0$. Then the image of γ by f is a path parametrized by $f(z(t))$ that goes through $w_0 = \gamma(z_0)$ in the w-plane. Since $z'(t_0) \neq 0$, the direction of the tangent line to γ at z_0 is $\operatorname{Arg} z'(t_0)$.

Also,

$$\frac{d}{dt} f(z(t))\bigg|_{t_0} = f'(z(t_0))z'(t_0) \neq 0$$

and the direction of the tangent line to $f(z(t))$ at $w_0 = f(z_0)$ is

$$\operatorname{Arg}\left(\frac{d}{dt} f(z(t))\bigg|_{t_0}\right) = \operatorname{Arg}(f'(z(t_0))z'(t_0)) = \operatorname{Arg} f'(z(t_0)) + \operatorname{Arg} z'(t_0).$$

Thus, $f(z)$ rotates any tangent line by a fixed angle $\operatorname{Arg} f'(z_0)$. It preserves the angle of intersection and orientation. $\square$

9.2 Dirichlet problem

One of the examples of using *partial differential equations* as a modeling tool is time-dependent *Maxwell equation* in a free space:

$$\begin{aligned}
\nabla \times B &= \epsilon_0 \mu_0 \frac{\partial E}{\partial t}, \\
\nabla \times E &= -\frac{\partial B}{\partial t}, \\
\nabla \cdot B &= 0, \\
\nabla \cdot E &= 0
\end{aligned}$$

Definition 9.2.1. A problem consisting of a partial differential equation with specified *boundary conditions* are known as **boundary value problem**.

Boundary condition is the condition $u(x, y) = b(x, y)$ on the boundary of Ω (assuming that we are solving the partial differential equation inside Ω).

We start our first modeling with the modeling of traveling wave on a vibrating string.

Example 9.2.1. Suppose that the tension on both sides of a string is T and hence we know that both vertical and horizontal component of the tension are:

$$\begin{aligned} T_{Ax} &= T\cos\alpha, \quad T_{Ay} = -T\sin\alpha \\ T_{Bx} &= T\cos\beta, \quad T_{By} = T\sin\beta \end{aligned}$$

Since this is at equilibrium, we know that horizontal components must be added to (approximately) zero, so

$$T\cos\alpha \approx T\cos\beta,$$

and the vertical components:

$$-T\sin\alpha + T\sin\beta = ma = m\frac{\partial^2 u}{\partial t^2}.$$

If we divide both sides with $T\cos\alpha$ (remember that $T\cos\alpha \approx T\sin\beta$),

$$-\tan\alpha + \tan\beta = \frac{m}{\tau\cos\alpha}\frac{\partial^2 u}{\partial t^2} = \frac{\rho\Delta x}{\tau\cos\alpha}\frac{\partial^2 u}{\partial t^2}.$$

When α, β are small, tangents of the angles at the ends of the string piece are equal to the slopes at the ends.

$$\frac{1}{\Delta x}\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) = \frac{\rho}{\tau\cos\alpha}\frac{\partial^2 u}{\partial t^2},$$

or we rewrite it as

$$u_{tt} = c^2 u_{xx}, \quad c^2 = \frac{\tau\cos\alpha}{\rho\Delta x}$$

The above is a partial differential equation that models vibrating string. If we impose some conditions at its boundary, then we have a *boundary value problem* (BVP).

For boundary value problems, we usually specify the following three boundary conditions:

1. **Dirichlet Condition**: Condition that specifies the values of the solution u on the boundary,
2. **Neumann Condition**: Condition that specifies the normal derivative $\frac{\partial u}{\partial n}$ on the boundary,
3. **Robin Condition**: Condition that specifies $\frac{\partial u}{\partial n} + \alpha u$ on the boundary.

Example 9.2.2. Consider the one-dimensional heat equation and its boundary values:

$$\begin{aligned} \frac{\partial u}{\partial t} &= c^2\frac{\partial^2 u}{\partial t^2}, \ 0 < x < L, \ t > 0 \\ u(0, t) &= u(L, t) = 0, \ \forall t > 0 \\ u(x, 0) &= f(x), \ 0 < x < L \end{aligned}$$

We have the following general solution:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin \frac{n\pi x}{L}, \ \lambda_n = \frac{cn\pi}{L}$$
$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Definition 9.2.2. Dirichlet problem is a boundary value problem that involves Laplace's equations with specified boundary values.

It can be seen that if $f = u(x,y) + iv(x,y)$ is analytic then u, v satisfy the Cauchy-Riemann equations and it can be easily shown that both u and v satisfy the Laplace equation.

9.2.1 Boundary behaviour

Now we have defined what is *boundary conditions* and *Dirichlet problems*, we will use conformal mappings to transform boundary value problem consisting of Laplace equation along with boundary conditions.

The following theorem states that a Laplace equation is not changed under analytic mapping.

Theorem 9.2.1. If f is an analytic mapping of Ω into Ω' and U is a harmonic function on Ω'. Then $U \circ f$ is harmonic in Ω. Thus, if U satisfies $\Delta U = 0$ on Ω', then $u = U \circ f$ satisfies $\Delta u = 0$ on Ω.

The function U is a function of two variables, write $U \circ f(z) = U(\operatorname{Re} f, \operatorname{Im} f)$.

Proof. Let z_0 be a point in Ω and $w_0 = f(z_0)$. Theorem 7.1.2 states that U has a harmonic conjugate V in a disk around w_0. Then $U + iV$ is analytic in this disk, and by the composition of analytic function, $(U + iV) \circ f$ is analytic at z_0. Hence its real part, $\operatorname{Re}(U + iV) \circ f = \operatorname{Re}[U \circ f + i(V \circ f)] = U \circ f$ is harmonic. $\square$

Example 9.2.3. Consider $f(Z) = z^2$ and $\Omega = \{z = re^{i\theta} : \frac{1}{2} < r < 1, -\pi < \theta < \pi\}$. Then f is analytic and $f'(z) = 2z \neq 0$ for all $z \in \Omega$. Hence $f(z)$ is conformal at each $z \in \Omega$.

The image of Ω under $w = z^2$ is the annulus

$$f[\Omega] = \left\{ w = re^{i\theta} : \frac{1}{4} < r < 1, -2\pi < \theta < 2\pi \right\} = \left\{ w = re^{i\theta} : \frac{1}{4} < r < 1, 0 \leq \theta \leq 2\pi \right\}.$$

However, the boundary points are mapped to interior points. This can be easily checked by considering boundary points on negative real axis are mapped to positive real axis, which is interior to $f[\Omega]$.

Although mapping on boundary of Ω is not always *well-defined*, we say that f **maps the boundary of Ω onto the boundary of $f[\Omega]$** if for every point on w_0 on the boundary of $f[\Omega]$, we can find a sequence z_0 in Ω where $z_0 \to \alpha_0$ being the boundary of Ω and $f(z_0) \to w_0$.

Theorem 9.2.2. Suppose that f is analytic and one-to-one in a region Ω. Then f maps the boundary of Ω to the boundary of $f[\Omega]$ and this map is onto.

Proof. Omitted. $\square$

Example 9.2.4. Let $f(z) = \sin z$ and $\Omega = \left\{ z = x + iy : -\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0 \right\}$. Find the image $f[\Omega]$.

Solution:

9.3 Harmonic function and Dirichlet Problem

The Dirichlet problem can be viewed as a problem of finding a harmonic function on a given domain with prescribed boundary values. For "nice" domains, a solution is unique as shown below.

Theorem 9.3.1 (Uniquess Theorem)**.** Let D be a domain of $\mathbb{C}$ and $\phi : \partial D \to \mathbb{R}$ be a continuous function. The **Dirichlet problem** is to find a harmonic function h on D such that $\lim_{z\to z_0} h(z) = \phi(z_0)$ for all $z_0 \in \partial D$. There is at most one solution h to the Dirichlet problem.

Proof. If the Dirichlet problem has two solutions h_1 and h_2. Then $h_1 - h_2$ is harmonic on D and extends continuously to $\overline{D}$ and is zero on ∂D. By applying the Maximum Principle (Theorem 7.2.3, item 2.), since $h_1 - h_2$ is zero on the boundary, $h_1 - h_2 = 0$ in D. □

If the domain is not only nice, but also "simple" and related to the elementary functions we discussed in Topic 3 such as the logarithmic function, the we can obtain the solution.

Example 9.3.1. Find the solution to the Dirichlet problem in upper half-plane with boundary conditions:

$$u(x) = 0 \quad \text{if } x < 0,$$
$$u(x) = 100 \quad \text{if } x > 0.$$

Solution: Note that the boundary conditions in this example can be viewed as "do not depend on x, y" and "depend on $\operatorname{Arg} z$". Thus, we try harmonic function that is independent of r:

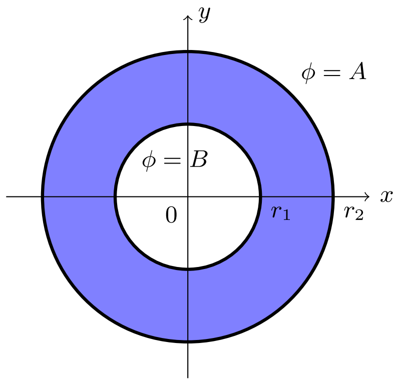
$$u(z) = \operatorname{Arg} z,$$

to ensure both boundary conditions, we choose $u(z) = \frac{100}{\pi}(\pi - \operatorname{Arg} z)$, which is harmonic in the upper half-plane and satisfies the boundary conditions.

Example 9.3.2. Find a harmonic function $h(x, y)$ in the upper half-plane and takes the following values on the x-axis: $h(x, 0) = 0$ for $x \in (-\infty, -1) \cup (1, \infty)$ and $h(x, 0) = 1$ for $x \in (-1, 1)$.

Solution:

Example 9.3.3. Find a general formula for a solution $\phi(x, y)$ to the Dirichlet problem inside the washer depicted in the following figure.



Solution:

9.4 Conformal mapping and Dirichlet problem

Now we look at an example of solving heat equation in unit disk. Note that heat equation in 2-dimensional space is given as

$$u_t = k\nabla u.$$

It is also common that we are looking for steady-state temperature, i.e. $u_t = 0$, and hence the heat equation reduces to Laplace equation.

Example 9.4.1. The boundary of a unit disk with insulated lateral surface is kept at a fixed temperature distribution equal to 100° on the upper semi-circle and 0° on the lower semi-circle. Find the steady-state temperature inside the plate.

Solution:

Example 9.4.2 (April 2008 Exam, Q4).

1. Prove that the transformation
$$f(z) = i(\frac{1-z}{1+z})$$
maps the interior of the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ conformally to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ and maps the upper and lower unit semicircles C_+ and C_- onto the positive and negative real axis $\mathbb{R}+$ and $\mathbb{R}-$ respectively. (12 marks)

2. Prove that the function
$$\varphi(u, v) = \frac{1}{2} + \frac{1}{\pi} \arctan\Big(\frac{u}{v}\Big)$$
solves the Dirichlet problem in the upper half-plane (**note**: there is a typo mistake in the original question):
$$\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2} = 0 \quad \text{on} \quad \mathbb{H}, \quad \varphi(u, v) = \begin{cases} 0, & \text{on } \mathbb{R}-, \\ 1, & \text{on } \mathbb{R}+. \end{cases}$$
(6 marks)

3. Determine the solution to the Dirichlet problem in the unit disk Δ:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{on} \quad \Delta, \quad \varphi(x, y) = \begin{cases} 0, & \text{on } C_-, \\ 1, & \text{on } C_+. \end{cases}$$

(6 marks)

Solution:

Example 9.4.3. Find the steady-state temperature $T(x, y)$ in the domain $D = \{z \in \mathbb{C} \,:\, 1 < |z| < 2\}$ and takes values $T(z) = 20^\circ$ when $|z| = 1$ and $T(z) = 30^\circ$ when $|z| = 2$.

Solution: