Tutorial 6 - Harmonic functions

1. Let $z = x + iy$. It can be seen that $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2}$. Thus, we can treat z and $\overline{z}$ as two independent variables. [Hint: Obtain the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.]

   (a) Show that for any function f, we have

   $$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right], \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right].$$

   (b) If f is analytic, show that

   $$\frac{\partial f}{\partial \overline{z}} = 0.$$

   This implies that f is independent of $\overline{z}$ whenever it is analytic.

   (c) Recall that $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Show that

   $$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta$$

   This shows that if $\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}u = 0$ then u is harmonic.

2. Determine the set on which the functions below is harmonic:

   (a) $u = x^2 - y^2 + 2x - y$,

   (b) $u = \frac{1}{x+y}$.

3. (Harmonic function independent of r) Let $z = x + iy$. Show that $u(x, y) = \operatorname{Arg} z$ on the region $D = \mathbb{C} \setminus (-\infty, 0]$.

4. (Harmonic function independent of y) Suppose that $u(x, y)$ is a harmonic function whose values depend only on x but not y. Determine $u(x, y)$.

5. (Harmonic function independent of θ) Suppose that $u(x, y)$ is a harmonic function and the values of u does not depend on θ.

   (a) Show that $u_{rr} + \frac{1}{r}u_r = 0$ by transforming the Laplace equation into polar form.

   (b) Show that $u(r, \theta) = u(r) = a \ln r + b$. [Hint: This is an Euler equation (ODE)]

6. Let $u(x, y) = x^2 - y^2 + 2x - y$, where x, y are real variables. Find a real-valued function v such that $f(x, y) = u(x, y) + iv(x, y)$ is analytic.

7. Let $u(x, y) = e^{-y} \sin x$ and $z = x + iy$ where $x, y \in \mathbb{R}$.

   (a) Show that u is harmonic function,

   (b) Find a harmonic conjugate of u,

   (c) Find an analytic function $f(z)$ such that $\operatorname{Re} f = u$. Express f in terms of z only.

8. Evaluate the integral

$$\int_0^{2\pi} \sin(1+\cos t)\sinh(2+\sin t)\,dt.$$

9. Find the maximum and minimum values of $u = x^2 - y^2 + xy$ in $D = [0,1] \times [0,1]$.