Tutorial 3 - Complex Integration

1. Evaluate $\int_C \overline{z}\, dz$ along the curve $C : z(t) = t^2 + it$ from $z = 0$ to $z = 4 + 2i$.

2. Let $f(z) = \frac{z+2}{z}$. Find the value of
$$\int_C f(z)\, dz$$
where C is the circle $z = 2e^{i\theta}, 0 \le \theta < 2\pi$, in the counterclockwise direction.

3. Let $f(z) = z^i = e^{i \operatorname{Log} z}$ where $|z| > 0, -\pi < \operatorname{Arg} z < \pi$ and C is the semicircle $z = e^{i\theta}$ for $0 \le \theta \le \pi$. Evaluate $\int_C z^i\, dz$.

4. Show that
$$\left| \int_C \frac{dz}{z^2 - 1} \right| \le \frac{\pi}{3}$$
where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$.

5. Let C be the line segment from $z = i$ to $z = -1$. Show that
$$\left| \int_C \frac{dz}{z^2} \right| \le 2\sqrt{2}.$$

6. Let C_R be the circle $|z| = R$ where $R > 1$ and oriented counterclockwise. Show that
$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right)$$
and then
$$\lim_{R \to \infty} \left| \int_{C_R} \frac{\operatorname{Log} z}{z^2} \right| = 0.$$

7. If C is any path between $z = 1$ to $z = i$, evaluate
$$\int_C e^{\pi z}\, dz.$$

8. Let C, C_0 and C_1 denote the following contours:

   $C : z = Re^{i\theta}$, $\quad C_0 : z = z_0 + Re^{i\theta}$, and C_1 : any closed contour does not pass through the point z_0,

   for $-\pi \le \theta \le \pi$.

   (a) Show that if f is piecewise continuous then $\int_{C_0} f(z - z_0)\, dz = \int_C f(z)\, dz$.

   (b) By using previous part, show that
$$\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \ (n = \pm 1, \pm 2, \dots) \quad \text{and} \quad \int_{C_0} \frac{1}{z - z_0}\, dz = 2\pi i.$$

(c) Show that

$$\int_{C_1} (z - z_0)^{n-1} \, dz = 0$$

for all $n \in \mathbb{Z}$.

(d) Can we use antiderivative to evaluate

$$\int_C \frac{dz}{z}$$

where C is a unit circle centered at origin and oriented counterclockwise.

9. Define the z^i to be the principal branch:

$$z^i = e^{i \operatorname{Log} z}, \quad |z| > 0, -\pi < \operatorname{Arg} z < \pi.$$

Show that

$$\int_{-1}^{1} z^i \, dz = \frac{1 + e^{-\pi}}{2}(1 - i).$$

The path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. [Hint: note that the point $z = -1$ is not defined for the principal branch.]

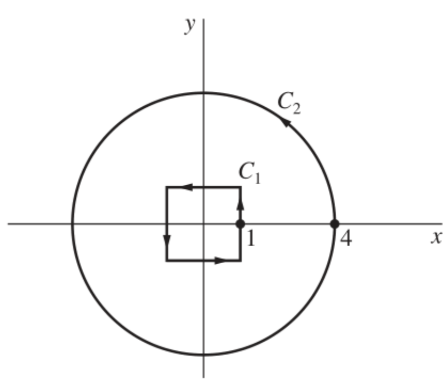
10. Evaluate $\int_C \frac{z^2+2z}{z-3} \, dz$ where C is the unit circle $|z| = 1$.

11. Evaluate $\int_C \frac{1}{z^2+2z+2} \, dz$ where C is the unit circle $|z| = 1$.

12. Let $f(z) = \frac{z+2}{\sin(z/2)}$. Show that

$$\int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz$$

where C_1, C_2 are described in figure below.



13. Use Cauchy integral formula to show that

$$\int_C \frac{dz}{z} = 2\pi i$$

for C is any positively oriented simple closed contour enclosing the origin.

14. Evaluate

$$\int_C \frac{1}{(z^2 + 9)^3} \, dz$$

where C is the circle $|z - 2i| = 2$.

15. Evaluate

$$\lim_{R\to\infty} \int_{C_1+C_R} \frac{1}{(z^2+1)^2}\,dz$$

where C_R is the upper half of the circle $|z| = R$, oriented counterclockwise and C_1 is the horizontal line from $z = -R$ to $z = R$.

16. By using result from previous question, show that

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{2}.$$

17. Let C be the circle $|z| = 3$, oriented counterclockwise. Let

$$g(w) = \int_C \frac{2s^2 - s - 2}{s - w}\,ds$$

where $|w| \neq 3$. Find $g(2)$ and $g(4)$.

18. Let $f(z) = (z+1)^2$. Find points in R where $|f(z)|$ has its maximum value, with R defined as the triangular region with vertices $z = 0, z = 2$ and $z = i$.

19. *Minimum Modulus Principle.* Let f be a continuous function on a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R. Prove that $|f(z)|$ has minimum value in R which occurs on the boundary of R.

20. *Gauss's mean value theorem.* Prove that when a function is analytic within and on a given circle, its value at the center equals the arithmetic mean of its values on the circle.