

Answers to Analysis I Exercises, Third Edition

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Chapter 1

Introduction

1.1 What is analysis?

N/A

1.2 Why do analysis?

N/A

Chapter 2

Starting at the beginning: the natural numbers

2.1 The Peano axioms

N/A

2.2 Addition

Exercise 2.2.1

We shall use induction on a , fixing b and c . First, we shall prove the base case. If $a = 0$, $a + b = 0 + b = b$ by the definition of addition of natural numbers (Definition 2.2.1). Therefore, $(a + b) + c = b + c$. Also by Definition 2.2.1, $a + (b + c) = 0 + (b + c) = b + c$. We now have $(a + b) + c = b + c = a + (b + c)$.

Now suppose that $(a + b) + c = a + (b + c)$. We then have, by Definition 2.2.1, that $((a++) + b) + c = ((a + b)++) + c = ((a + b) + c)++ = (a + (b + c))++$. Because of Definition 2.2.1 again, $(a + (b + c))++ = (a++) + (b + c)$. Now we have $((a++) + b) + c = (a++) + (b + c)$, which closes the induction.

Exercise 2.2.2

We shall use induction on a . The base case is vacuously true since 0 is not positive. For the inductive step, we have to show that $a++$ being positive implies that $a++ = b++$ for exactly one natural number b if a being positive

implies that $a = c++$ for exactly one natural number c . But a is equal to b if $a++$ is equal to $b++$, by Axiom 2.4. This closes the induction.

Exercise 2.2.3

(a) By Lemma 2.2.2, $a = a + 0$, so $a \geq a$.

(b) Because $a \geq b$ and $b \geq c$, $a = b + n$ and $b = c + m$ for some n and m . Therefore, $a = b + n = (c + m) + n$. By Proposition 2.2.5, $(c + m) + n = c + (m + n)$, so $a = c + (m + n)$. $n + m$ is a natural number, so $a \geq c$.

(c) Because $a \geq b$ and $b \geq a$, $a = b + n$ and $b = a + m$. Therefore, $a = b + n = (a + m) + n$. By Proposition 2.2.5, $(a + m) + n = a + (m + n)$. Now we have $a = a + (m + n)$. By Lemma 2.2.2, $a = a + 0$, so we can use Proposition 2.2.6 to get $0 = m + n$. Using Corollary 2.2.9, we can deduce that n and m are both equal to 0. Therefore, $a = b$.

(d) If $a \geq b$, $a = b + d$ for some d . Because of this, $a + c = (b + d) + c$. By Propositions 2.2.4 and 2.2.5, $(b + d) + c = b + (d + c) = b + (c + d) = (b + c) + d$. Therefore, $a + c \geq b + c$.

Conversely, if $a + c \geq b + c$, $a + c = (b + c) + d$ for some d . By Propositions 2.2.4 and 2.2.5, $(b + c) + d = b + (c + d) = b + (d + c) = (b + d) + c$. Using Proposition 2.2.6, we get $a = b + d$, and therefore, $a \geq b$.

(f) If $a < b$, $b = a + c$ for some c , and $b \neq a$. Therefore, c is positive, because otherwise $b = a + 0 = a$. ($a + 0 = a$ because of Lemma 2.2.2.)

Conversely, if $b = a + c$ for some positive number c , we know that $a \leq b$. If $a = b$, $a = a + c$. By Lemma 2.2.2, $a = a + 0$, so $a + 0 = a + c$. We can use Proposition 2.2.6 to get $0 = c$, but c is positive. Therefore, $a \neq b$, and combined with $a \leq b$, we get $a < b$.

(e) We first prove that $a < b$ implies that $a++ \leq b$. By (f), $b = a + c$ for some positive c . By Lemma 2.2.10, exactly one d exists such that $d++ = c$. Therefore, $b = a + d++$. By Lemma 2.2.3 and Definition 2.2.1, $b = a + d++ = (a + d)++ = (a++) + d$. Therefore, $a++ \leq b$.

Conversely, if $a++ \leq b$, we need to prove that $a < b$. Since $a++ = a + 1 \geq a$, by (b), $a \leq b$. If $a = b$, then $a++ \leq a$, and we get, from (c), that $a++ = a$. This is a contradiction, from Axiom 2.3 if $a = 0$ and Axiom 2.4 otherwise. Therefore, $a < b$.

Exercise 2.2.6 (unfinished)

We use induction on n . When $n = 0$, $m \leq n$ implies that $0 = m + a$. But by Corollary 2.2.9, $m = 0 = n$, so $P(n)$ being true implies that $P(m)$ is true.

If $a \leq n$ and $P(n)$ together imply $P(a)$, we now need to prove that $b \leq n++$ and $P(n++)$ together imply $P(b)$. Since $b \leq n++$, $n++ = b + c$ for some c . If c is positive, by Lemma 2.2.10, $n++ = b + d++$ for some d , so by Lemma 2.2.3 and Axiom 2.4, $n = b + d$, and $b \leq n$. Therefore, $b \leq n++$ implies that $b \leq n$ or $n++ = b + 0 = b$. Because of Lemmas 2.2.2 and 2.2.3, $n++ = (n + 0)++ = n + 0++ = n + 1$. Since $P(n++)$ implies $P(n)$, $P(n)$ is true. But then ?

2.3 Multiplication

Exercise 2.3.1

First, we prove that $n \cdot 0 = 0$. We induct on n . When $n = 0$, $n \cdot 0 = 0 \cdot 0 = 0$, because of Definition 2.3.1. For the inductive hypothesis, we assume $n \cdot 0 = 0$. Then we need to prove that $n++ \cdot 0 = 0$. By Definition 2.3.1, $n++ \cdot 0 = n \cdot 0 + 0 = n \cdot 0$. But we already know that $n \cdot 0 = 0$, so $n++ \cdot 0 = 0$. This closes the induction.

Next, we prove that $n \cdot m++ = nm + n$ using induction on n . When $n = 0$, $0 \cdot m++ = 0 = 0 \cdot m + 0$. Now, if $n \cdot m++ = nm + n$, we need to prove that $n++ \cdot m++ = n++ \cdot m + n++$. We can deduce from $n \cdot m++ = nm + n$ that $n++ \cdot m++ = (n \cdot m++) + m++ = nm + n + m++ = nm + (n + m)++ = nm + n++ + m = (n++) \cdot m + n++$. This closes the induction.

Now we can prove that $nm = mn$. We induct on m . When $m = 0$, we have $n \cdot 0 = 0 = 0 \cdot n$. For the inductive hypothesis, we assume $nm = mn$. Now we need to prove $n \cdot m++ = m++ \cdot n$. We know that $n \cdot m++ = nm + n$ and $m++ \cdot n = mn + n$. But $nm = mn$, so $n \cdot m++ = m++ \cdot n$. This closes the induction.

Exercise 2.3.4

By Proposition 3.4, $(a + b)^2 = (a + b) \cdot (a + b) = (a + b) \cdot a + (a + b) \cdot b = a \cdot a + b \cdot a + a \cdot b + b \cdot b$. We can rewrite $b \cdot a + a \cdot b$ as $a \cdot b + a \cdot b$ because of Lemma 2.3.2. We can further rewrite this as $2ab$ because $2(ab) = 1(ab) + ab =$

$0(ab) + ab + ab = 0 + ab + ab = ab + ab$. We can also rewrite $a \cdot a$ and $b \cdot b$ as a^2 and b^2 respectively. Therefore, $(a + b)^2 = a^2 + 2ab + b^2$.

Chapter 3

Set theory

3.1 Fundamentals

Exercise 3.3.1

We know that at least one of the statements “ $a = c$ ” and “ $a = d$ ” is true since $\{a, b\} = \{c, d\}$. Similarly, at least one of the statements “ $b = c$ ” and “ $b = d$ ” is true. If we have that $a = c$ and $b = d$ at the same time, then we have proved what we want. (Same for $a = d$ and $b = c$ at the same time.) Otherwise, then we must have that $a = c = b$ or $a = d = b$. If both of these are true, then $a = b = c = d$, and therefore both $a = c$ and $b = d$. If only one of these is true, which we will assume to be the statement that $a = c = b$ as a similar argument holds assuming $a = d = b$, then $d \notin \{a, b\} = \{a, a\}$, while $d \in \{c, d\} = \{a, d\}$, so $\{a, b\} \neq \{c, d\}$, a contradiction. Therefore, if $\{a, b\} = \{c, d\}$, either both $a = c$ and $b = d$ or $a = d$ and $b = c$.

Exercise 3.3.2

First, we show that these four sets exist. Axiom 3.3 tells us that \emptyset exists. The singleton set axiom (part of Axiom 3.4) and Axiom 3.1 (so that we are allowed to construct the singleton set whose element is another set) tell us that $\{\emptyset\}$ exists. Applying them again gives that $\{\{\emptyset\}\}$ exists. Using the pair set axiom (also part of Axiom 3.4) and Axiom 3.1 allows us to create the set $\{\emptyset, \{\emptyset\}\}$.

Now we show that these four sets are distinct. Since the sets $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ obviously all contain at least one element, if one of them is equal

to \emptyset , then by Axiom 3.2, it contains no elements. But this contradicts that it has at least one element, and we have a contradiction. Now we need to prove that $\{\emptyset\}$ is not equal to $\{\{\emptyset\}\}$ or $\{\emptyset, \{\emptyset\}\}$. Both $\{\emptyset\} \in \{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are true, but $\{\emptyset\} \notin \{\emptyset\}$. Therefore, by Axiom 3.2, $\{\emptyset\}$ is not equal to any of the other sets. (We have already proved that $\emptyset \neq \{\emptyset\}$, so $\{\emptyset\} \neq \emptyset$). The last thing that we need to prove now is that $\{\{\emptyset\}\} \neq \{\emptyset, \{\emptyset\}\}$. Since $\emptyset \in \{\emptyset, \{\emptyset\}\}$ and $\emptyset \notin \{\{\emptyset\}\}$, by Axiom 3.2, these two sets are not equal. Therefore, all four sets are distinct.

Exercise 3.1.3

Proving that $\{a, b\} = \{a\} \cup \{b\}$ We shall prove that $\{a, b\} = \{a\} \cup \{b\}$. The following are equivalent:

- $c \in \{a, b\}$
- c is either equal to a or b
- $c \in \{a\}$ or $c \in \{b\}$
- $c \in \{a\} \cup \{b\}$

Therefore, $\{a, b\} = \{a\} \cup \{b\}$.

Proving that $A \cup B = B \cup A$ We now shall prove that $A \cup B = B \cup A$. If $c \in A \cup B$, then at least one of the statements $c \in A$ and $c \in B$ are true. By Axiom 3.5, this means that $c \in B \cup A$. We can replace A with B and B with A to show the other direction. Therefore, $A \cup B = B \cup A$.

Proving that $A \cup A = A \cup \emptyset = \emptyset \cup A = A$ Because it is impossible for c to be in \emptyset , the following are equivalent:

- $c \in A$ or $c \in A$
- $c \in A$ or $c \in \emptyset$
- $c \in \emptyset$ or $c \in A$
- $c \in A$

These four statements are equivalent to $c \in A \cup A$, $c \in A \cup \emptyset$, $c \in \emptyset \cup A$, and $c \in A$, respectively. Therefore, $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.

Exercise 3.1.11

The next paragraph will be written like a proof of Axiom 3.6 from Axiom 3.7: we will not define again A , $P(x)$, and so on.

If $Q(x, y)$ is the statement that $x = y$ and $P(x)$ are both true, then by Axiom 3.7 and the fact that for each x there exists at most one y such that $Q(x, y)$, there exists a set $B = \{y : x = y \text{ and } P(x) \text{ for some } x \in A\}$. For any $z \in B$, we know that $z \in A$ and that $P(z)$ is true. For any $x \in A$ such that $P(x)$ is true, we similarly know that $x \in B$. Therefore, $A = B$.

3.2 Russell's paradox (Optional)**Exercise 3.2.1**

Proving that Axiom 3.9 implies Axiom 3.3 We can create a property $P(x)$ that is always false regardless to the choice of x . Then the set $\{x : P(x) \text{ is true}\}$ has no elements, as $P(x)$ is false all the time.

Proving that Axiom 3.9 implies Axiom 3.4 If $P(x)$ is the property that $x = a$, then $\{x : P(x) \text{ is true}\}$ has only the element a . Similarly, if $Q(x)$ is the property that $x = a$ or $x = b$, then $\{x : Q(x) \text{ is true}\}$ has only the elements a and b .

Proving that Axiom 3.9 implies Axiom 3.5 If $P(x)$ is the property that $x \in A$ or $x \in B$, then $\{x : P(x) \text{ is true}\}$ exists, so $A \cup B$ exists.

Proving that Axiom 3.9 implies Axiom 3.6 If $Q(x)$ is the property that $P(x)$ is true and $x \in A$, then $\{x : Q(x) \text{ is true}\} = \{x \in A : P(x)\}$ exists.

Proving that Axiom 3.9 implies Axiom 3.7 If $Q(y)$ is the property that there exists some x such that $P(x, y)$ is true, then $\{y : Q(y) \text{ is true}\} = \{y : P(x, y) \text{ is true for some } x \in A\}$ exists.

Proving that Axiom 3.9 implies Axiom 3.8, assuming that all natural numbers are objects If $P(x)$ is the property that x is a natural number, then $\mathbf{N} = \{x : P(x) \text{ is true}\}$ exists, by Axiom 3.9.

3.3 Functions

Exercise 3.3.2

Proving that f and g being injective implies that $g \circ f$ is too We will show that if $x \neq x'$, then $(g \circ f)(x) \neq (g \circ f)(x')$. First, since f is injective, $f(x) \neq f(x')$. Since g is also injective, $g(f(x)) \neq g(f(x'))$. Therefore, $(g \circ f)(x) \neq (g \circ f)(x')$.

Proving that f and g being surjective implies that $g \circ f$ is too We will show that $g \circ f$ is surjective by showing that for any $c \in Z$, there exists some $a \in X$ such that $(g \circ f)(a) = c$. Since g is surjective, there exists some $b \in Y$ such that $g(b) = c$. Because f is also surjective, there exists some $a \in X$ such that $f(a) = b$. Therefore, $(g \circ f)(a) = c$, and $g \circ f$ is surjective.

Exercise 3.3.3

Finding when the empty function is injective We will prove that the empty function $f: \emptyset \rightarrow X$ is injective for any X . Since we cannot find any $x, x' \in \emptyset$ that are unequal, the empty function is (vacuously) injective.

Finding when the empty function is surjective We will prove that the empty function $f: \emptyset \rightarrow X$ is surjective only when $X = \emptyset$. The statement

“For every $y \in X$, there exists $x \in \emptyset$ such that $f(x) = y$ ”

can only be true when it is impossible that $y \in X$, because $x \in \emptyset$ is impossible. But then we have that $X = \emptyset$, and we have proved our claim.

Finding when the empty function is bijective We will prove that the empty function $f: \emptyset \rightarrow X$ is bijective only when $X = \emptyset$. Since f is always injective regardless of the choice of X , the empty function being bijective is equivalent to it being surjective. Therefore, since f is surjective precisely when $X = \emptyset$, the empty function is bijective only when $X = \emptyset$.

Exercise 3.3.4

Showing that if $g \circ f = g \circ \tilde{f}$ and g is injective, then $f = \tilde{f}$ We will use proof by contradiction. If $f \neq \tilde{f}$, then for some $x \in X$, we have $f(x) \neq \tilde{f}(x)$.

Therefore, since g is injective, $(g \circ f)(x) \neq (g \circ \tilde{f})(x)$, and $g \circ f \neq g \circ \tilde{f}$. This is a contradiction, and therefore, $f = \tilde{f}$. This statement is not necessarily true if g is not injective. If $X = Y = Z = \mathbf{N}$, $f(x) = 0$, $\tilde{f}(x) = 1$, and $g(x) = 0$, then $f \neq \tilde{f}$, while $(g \circ f)(x) = 0 = (g \circ \tilde{f})(x)$.

Showing that if $g \circ f = \tilde{g} \circ f$ and f is surjective, then $g = \tilde{g}$ We will use proof by contradiction again. If $g \neq \tilde{g}$, then for some $y \in Y$, we have $g(y) \neq \tilde{g}(y)$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. But then $(g \circ f)(x) \neq (\tilde{g} \circ f)(x)$. This is a contradiction, and therefore, $g = \tilde{g}$. This statement is not necessarily true if f is not surjective. If $X = Y = Z = \mathbf{N}$, $f(x) = 0$, $g(x) = x$, and $\tilde{g}(x) = 0$, then $g \neq \tilde{g}$, even though $(g \circ f)(x) = 0 = (\tilde{g} \circ f)(x)$.

Exercise 3.3.6

If $f(x) = a$, then by the definition of f^{-1} , $f^{-1}(f(x)) = f^{-1}(a) = x$. If $f^{-1}(y) = b$, then $f(b) = y$, so $f(f^{-1}(y)) = f(b) = y$.

We can deduce that f^{-1} is bijective from Exercise 3.3.5 and the fact that the identity map $\iota_{X \rightarrow X}$ defined as $\iota_{X \rightarrow X}(x) = x$ for all $x \in X$ is obviously bijective.

Exercise 3.3.7

By Exercise 3.3.2, $g \circ f$ is both injective and surjective, and is therefore bijective. The only thing left we have to prove is that $(f^{-1} \circ g^{-1} \circ g \circ f)(x) = x$. But $(f^{-1} \circ g^{-1} \circ g \circ f)(x) = (f^{-1} \circ f)(x) = x$, and we have proved what we want.

3.4 Images and inverse images

Challenge to define $f(S)$ using the axiom of specification

We can define $f(S) := \{y \in Y : \text{there exists } x \in S \text{ such that } f(x) = y\}$.

Exercise 3.4.1

First, we will show that any element of the forward image of V under f^{-1} is contained in the inverse image of V under f . If x is in the forward image of V

under f^{-1} , then there exists v such that $f^{-1}(v) = x$ and therefore $f(x) = v$. But then x is in the inverse image of V under f .

Next, we will show that any element v of the inverse image of V under f is contained in the forward image of V under f^{-1} . Since there exists x such that $f(x) = v$ and therefore $f^{-1}(v) = x$, x is in the forward image of V under f .

Finally, we have that the forward image of V under f^{-1} is equal to the inverse image of V under f , and that it is valid to use the notation $f^{-1}(V)$.

Exercise 3.4.6

By Axiom 3.11, $\{0, 1\}^X$ exists. We can use Axiom 3.7 (the axiom of replacement) to create a set

$$Y = \{f^{-1}(\{1\}) : f \in \{0, 1\}^X\}.$$

Every S which is a subset of X is in Y , because defining $f(x)$ to be 1 when x is in S and to be 0 otherwise means that $f^{-1}(\{1\}) = S$. Every element of Y is a subset of X , because every element of $f^{-1}(\{1\})$ has to be in X .

Exercise 3.4.9

We will show that

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

We know that $y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$ if and only if $y \in A_\beta$ and $y \in A_\alpha$ for all $\alpha \in I$. But the latter statement implies the former, so $y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$ if and only if $y \in A_\alpha$ for all $\alpha \in I$.

We can do something similar for $A_{\beta'}$. Therefore,

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

3.5 Cartesian products

Exercise 3.5.1

First, we will show that $(x, y) = (x', y')$ implies that both $x = x'$ and $y = y'$. Since $(x, y) := \{\{x\}, \{x, y\}\}$, we have to show that $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$.

$\{\{x'\}, \{x', y'\}\}$ implies that both $x = x'$ and $y = y'$. If $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$, then $\{x\} \in \{\{x'\}, \{x', y'\}\}$. In order for this to be true, we must have $\{x\} = \{x'\}$, which implies $x = x'$, or $\{x\} = \{x', y'\}$, which implies $x' = y' = x$. But the latter case implies the former case, so we can ignore the latter one. Doing the same with $\{x, y\}$, $\{x'\}$, and $\{x', y'\}$ yields that $x = x'$ and $y = y'$. The converse is true by Axiom 3.2.

We can show that $X \times Y$ is a set if X and Y are both sets using Axiom 3.1 (sets are objects), Axiom 3.7 (axiom of replacement), and Axiom 3.12 (axiom of union). By Axioms 3.1 and 3.7, we can create the set $Z = \{\{(x, y)\} : y \in Y\} : x \in X\}$. By Axiom 3.12, we can also create the set $\bigcup Z$, which we can define to be $X \times Y$.

I won't do the additional challenge.

Exercise 3.5.6

We will show that if $A \times B \subseteq C \times D$ and $A, B, C, D \neq \emptyset$, then $A \subseteq C$ and $B \subseteq D$. Since A and B are nonempty, by the single choice lemma (Lemma 3.1.5), there exist some $a \in A$ and $b \in B$. Now, for any $x \in A$, since $(x, b) \in A \times B \subseteq C \times D$, we have $x \in C$. Similarly, for any $y \in B$, since $(a, y) \in A \times B \subseteq C \times D$, we have $y \in D$. Therefore, $A \subseteq C$ and $B \subseteq D$.

However, we can show that if we remove the assumption that A, B, C , and D are nonempty, then this statement is false. If $A = \{0\}$, $C = \{1\}$, and $B = D = \emptyset$, then $A \times B$ is empty, as there are no elements of B . Similarly, $C \times D = \emptyset$, and therefore $A \times B \subseteq C \times D$. However, $A \not\subseteq C$.

If $A \subseteq C$ and $B \subseteq D$, all ordered pairs (a, b) where $a \in A$ and $b \in B$ are in $C \times D$ because $a \in C$ and $b \in D$. Hence, $A \times B \subseteq C \times D$. This statement holds even when A, B, C , and D are not all nonempty.

Assuming that $A \times B = C \times D$ and $A, B, C, D \neq \emptyset$, $A = C$ and $B = D$. This is because $A \times B \subseteq C \times D$ and $C \times D \subseteq A \times B$, and therefore $A \subseteq C$, $A \supseteq C$, $B \subseteq D$, and $B \supseteq D$.

If we remove the assumption that $A, B, C, D \neq \emptyset$, the statement is false. The counterexample from before ($A = \{0\}$, $C = \{1\}$, $B = D = \emptyset$) also works here. We have that $A \times B = C \times D = \emptyset$, but $A \neq C$.

We have that $A = C$ and $B = D$ imply that $A \times B = C \times D$ because $A \subseteq C$ and $B \subseteq D$ imply, together, that $A \times B \subseteq C \times D$ and the same with \subseteq replaced by \supseteq .

Exercise 3.5.13 (unfinished)

?

3.6 Cardinality of sets**Exercise 3.6.2**

If X has cardinality 0, there exists a bijection $f: X \rightarrow \{i \in \mathbf{N} : 1 \leq i \leq 0\}$. The set $\{i \in \mathbf{N} : 1 \leq i \leq 0\}$ is obviously empty. If $X \neq \emptyset$, then there exists some $x \in X$. But then $f(x)$ is in \emptyset , and therefore X must be empty.

If X is empty, the function $f: X \rightarrow \emptyset$ exists (we don't need to give how to derive $f(x)$ given x , since $x \in X$ is impossible) and is a bijection (injectivity is because there are no elements in the domain at all, and surjectivity is from that the range is empty). Since $\emptyset = \{i \in \mathbf{N} : 1 \leq i \leq 0\}$, X has cardinality 0.

Exercise 3.6.3

We will induct on n . When $n = 0$, $\{i \in N : 1 \leq i \leq n\}$ is empty, so we can make M equal to any natural number, say, 0. Now, assuming that all $f: \{i \in N : 1 \leq i \leq n\} \rightarrow \mathbf{N}$ satisfy the property that there exists some natural number M such that $f(i) \leq M$ for all $1 \leq i \leq n$, we will now show that for every $g: \{i \in N : 1 \leq i \leq n++\} \rightarrow \mathbf{N}$, there exists some natural number a such that $g(i) \leq a$ for all $1 \leq i \leq n++$. If $h: \{i \in N : 1 \leq i \leq n++\} \rightarrow \mathbf{N}$ is defined so that $h(x) := g(x)$ for all $x \in \{i \in N : 1 \leq i \leq n++\}$, there exists some b such that $h(i) \leq b$ for all $1 \leq i \leq n$. Defining

$a := \begin{cases} b & \text{if } g(n++) \leq b \\ g(n++) & \text{if } g(n++) > b \end{cases}$ ensures that a will always be greater than or equal to all elements of $g(\{i \in N : 1 \leq i \leq n++\})$.

Exercise 3.6.5

We will define $f: A \times B \rightarrow B \times A$ by $f(a, b) := (b, a)$, where $a \in A$ and $b \in B$. Injectivity of f comes from $(b, a) = (d, c)$ implying that $b = d$ and $a = c$, further implying that $(a, b) = (c, d)$. Surjectivity comes from the fact that

for every $(b, a) \in B \times A$ has a corresponding ordered pair (namely, (a, b)) in $A \times B$ where $f(a, b) = (b, a)$.

Now, we will prove that $ab = ba$. We will define A as $\{i \in \mathbf{N} : 1 \leq i \leq a\}$ and B as $\{j \in \mathbf{N} : 1 \leq j \leq b\}$. Since A has equal cardinality with itself and the same for B (Proposition 3.6.4), A and B are both finite and have the cardinalities a and b , respectively. By Proposition 3.6.14(e), $\#(A) \cdot \#(B) = \#(A \times B)$ and $\#(B) \cdot \#(A) = \#(B \times A)$. We also know that $\#(A \times B) = \#(B \times A)$, and therefore $ab = ba$.

Exercise 3.6.7 (unfinished)

?

Exercise 3.6.10 (unfinished)

Attempt 1 We induct on n . (*I don't think we can induct, actually, since $\#(\bigcup_{i \in \{1, \dots, n+1\}} A_i) > n+1$ does not necessarily imply $\#(\bigcup_{i \in \{1, \dots, n\}} A_i) > n$.*) When $n = 0$, $\#(\bigcup_{i \in \{1, \dots, n\}} A_i) = \#(\emptyset) = 0$. It is obvious that $0 \not> 0$, so the pigeonhole principle holds vacuously when $n = 0$. When the pigeonhole principle is true for n finite sets, now we will show that it also is true for A_1, \dots, A_{n++} . By Proposition 3.6.14(b),

$$\#(\bigcup_{i \in \{1, \dots, n++\}} A_i) \leq \#(\bigcup_{i \in \{1, \dots, n\}} A_i) + \#(A_{n++}).$$

Chapter 4

Integers and rationals

4.1 The integers

Exercise 4.1.1

Reflexive If $a, b \in \mathbf{N}$, $a + b = b + a$, so by Definition 4.1.1, $a - b = a - b$.

Symmetric If $a - b = c - d$, $a + d = c + b$. Now, since $c + b = a + d$, $c - d = a - b$.

Exercise 4.1.3

We know that $-1 = -(1 - 0) = 0 - 1$. If $a = a_1 - a_2$, $(-1) \cdot a = (0 \cdot a_1 + 1 \cdot a_2) - (0 \cdot a_2 + 1 \cdot a_1) = a_2 - a_1 = -a$.

Exercise 4.1.4

I will write $x = a - b$, $y = c - d$, and $z = e - f$, where $a, b, c, d, e, f \in \mathbf{N}$.

First, we will prove $x + y = y + x$:

$$\begin{aligned}x + y &= (a - b) + (c - d) \\&= (a + c) - (b + d) \\y + x &= (c - d) + (a - b) \\&= (c + a) - (d + b)\end{aligned}$$

and therefore, because of natural number addition's commutativity, $x + y = y + x$.

Next, we will prove that $(x + y) + z = x + (y + z)$:

$$\begin{aligned}
 (x + y) + z &= ((a - b) + (c - d)) + (e - f) \\
 &= ((a + c) - (b + d)) + (e - f) \\
 &= ((a + c) + e) - ((b + d) + f) \\
 x + (y + z) &= (a - b) + ((c - d) + (e - f)) \\
 &= (a - b) + ((c + e) - (d + f)) \\
 &= (a + (c + e)) - (b + (d + f))
 \end{aligned}$$

and therefore, because of natural number addition's associativity, $(x + y) + z = x + (y + z)$.

Now, we will prove that $x + 0 = 0 + x = x$. We have proved above that $x + y = y + x$ and therefore $x + 0 = 0 + x$, so we only need to show $x + 0 = x$:

$$\begin{aligned}
 x + 0 &= (a - b) + (0 - 0) \\
 &= (a + 0) - (b + 0).
 \end{aligned}$$

We know that $a + 0 = a$ and $b + 0 = b$, so $x + 0 = x$, and hence $x + 0 = 0 + x = x$.

Next, we will prove that $x + (-x) = (-x) + x = 0$. We have already proved that $x + y = y + x$ and therefore $x + (-x) = (-x) + x$, so the only thing left is to show $x + (-x) = 0$.

$$\begin{aligned}
 x + (-x) &= (a - b) + (b - a) \\
 &= (a + b) - (b + a) \\
 &= (a + b) - (a + b) \\
 &= 0 - 0 \\
 &= 0.
 \end{aligned}$$

Therefore, $x + (-x) = (-x) + x = 0$.

To show that $xy = yx$, we will use the definition of integer multiplication:

$$\begin{aligned}
 xy &= (a - b)(c - d) \\
 &= (ac + bd) - (ad + bc) \\
 yx &= (c - d)(a - b) \\
 &= (ca + db) - (cb + da) \\
 &= (ac + bd) - (ad + bc) \\
 &= xy.
 \end{aligned}$$

Note that $(xy)z = x(yz)$ was proved in in the section about proving Proposition 4.1.6 on page 4.1.

For the statement $x1 = 1x = x$, we only have to prove that $1x = x$, because we have already proved that $xy = yx$ earlier:

$$\begin{aligned} 1x &= (1 - 0) \cdot (a - b) \\ &= (1a + 0b) - (1b + 0a) \\ &= a - b \\ &= x. \end{aligned}$$

Now we will show that $x(y + z) = xy + xz$:

$$\begin{aligned} x(y + z) &= (a - b)((c - d) + (e - f)) \\ &= (a - b)((c + e) - (d + f)) \\ &= (a(c + e) + b(d + f)) - (a(d + f) + b(c + e)) \\ &= (ac + ae + bd + bf) - (ad + af + bc + be) \\ &= ((ac + bd) - (ad + bc)) + ((ae + bf) - (af + be)) \\ &= xy + xz. \end{aligned}$$

We can show that $(y + z)x = yx + zx$ using $xy = yx$ and $x(y + z) = xy + xz$ by simply swapping $x(y + z)$ with $(y + z)x$, xy with yx , and xz with zx .

Exercise 4.1.7

(a) If $a > b$, we will show that $a - b$ is a positive natural number. Since $a \geq b$, there exists $d \in \mathbf{N}$ such that $a = b + d$. Subtracting b from both sides gives $a - b = (b + d) - b$. We know that

$$(b + d) - b = (b + d) + (-b) = (d + b) + (-b) = d + (b + (-b)) = d + 0 = d.$$

Therefore, $a - b = d$. Since $d = 0$ implies $a = b + 0 = b$, which contradicts $a > b$, $a - b$ must be a nonzero, or positive, natural number.

Now we will show that if $a - b$ is a positive natural number, $a > b$. We know that $a \geq b$, since $a = a + (-b) + b = (a - b) + b$. We also know that $a \neq b$ because otherwise, $a - b$ would be 0. Therefore, $a > b$.

(b) When $a > b$, by (a), $a - b \neq 0$. Since $c - c = c + (-c) = 0$ (by Proposition 4.1.6 and the definition of subtraction on page 79), $a - b + c - c \neq 0 + c - c = 0$.

By Proposition 4.1.6 and the fact that $-(-x) = x$ (if $x = a - b$, $-x = b - a$, and $-(-x)$ is just $a - b$ again),

$$\begin{aligned} a - b + c - c &= a + (-b) + c + (-c) \\ &= (a + (-c)) + (-b) + (-(-c)) = (a - c) - (b - c). \end{aligned}$$

Therefore, $a + c > b + c$.

(c) Since $a > b$, $a - b$ is a positive natural number. Since c is also a positive natural number, by Lemma 2.3.3, $(a - b)c$ is also a positive natural number. The distributive law for natural numbers (Proposition 2.3.4) tells us that $(a - b)c = ac - bc$, and therefore $ac > bc$.

(d) Since $a > b$, $a - b$ is a positive natural number. But this means that $-a < -b$, since $-b = (a - b) + (-a)$.

(e) Since $a > b$ and $b > c$, $a - b$ and $b - c$ are both positive natural numbers. By Proposition 2.2.8, $(a - b) + (b - c)$ is also a positive natural number. We know that

$$(a - b) + (b - c) = a + (-b) + b + (-c) = a + 0 + (-c) = a + (-c) = a.$$

Therefore, $a > c$.

(f) First, we will show that at least one of the statements $a > b$, $a < b$, and $a = b$ is true. By Lemma 4.1.5 (trichotomy of integers), $a - b$ is either zero, positive, or negative. If it is 0,

$$a = (a - b) + b = 0 + b = b.$$

If $a - b$ is positive, by part (a), $a > b$. When $a - b$ is negative, it is the negation of some positive natural number d . Since $-(-d)$ is equal to d itself and $-(a - b) = b - a$, $d = b - a$. Therefore, $b > a$. We have now proved that at least one of the statements $a > b$, $a < b$, and $a = b$ is true.

Now, we will show that no two of these statements can be true at the same time. It is obvious from the definition of ordering on the integers (Definition 4.1.10) that $a = b$ cannot be true at the same time as $a > b$ or $a < b$. Now the only thing left is to prove that $a > b$ and $a < b$ cannot both be true. We will prove this using proof by contradiction. The statement that $a > b$

is equivalent to $a - b$ being positive, and $a < b$ is equivalent to $b - a$ being positive. But by Lemma 4.1.5 (the trichotomy of integers) and the fact that $b - a = -(a - b)$ give us a contradiction, as $b - a$ cannot both be positive and negative. We have now proved that exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.

Exercise 4.1.8

If $P(n)$ is the property that $n > -1$, $P(0)$ is true, because $0 = (-1) + 1$ and $0 \neq -1$. Also, if $P(n)$ is true, since $n++ > n$, by Lemma 4.1.11(e), $P(n++)$ is true. However, $P(-1)$ is obviously false, since $-1 = -1$.

4.2 The rationals

Exercise 4.2.1

Proving that equality is reflexive (for rational numbers) We know that for any $a, b \in \mathbf{Z}$ where $b \neq 0$, $ab = ab$. Therefore, $a//b = a//b$.

Proving that equality is symmetric If for $a, b, c, d \in \mathbf{Z}$ that satisfy $b, d \neq 0$, we have $a//b = c//d$, then $ad = cb$. Hence, $cb = ad$, and $c//d = a//b$.

Proving that equality is transitive If $a, b, c, d, e, f \in \mathbf{Z}$ where $b, d, f \neq 0$, $a//b = c//d$, and $c//d = e//f$, then we must have $ad = cb$ and $cf = ed$. Hence, $ad \cdot cf = cb \cdot ed$. By Corollary 4.1.9, we can remove c and d from both sides:

$$a \cdot f = b \cdot e.$$

Therefore, by Definition 4.2.1, $a//b = e//f$, and we have proved that rational number equality is reflexive, symmetric, and transitive.

Exercise 4.2.3

I will assume that $x = a//b$, $y = c//d$, and $z = e//f$, where $a, b, c, d, e, f \in \mathbf{Z}$ and $b, d, f \neq 0$.

First, we will prove that $x + y = y + x$:

$$\begin{aligned}
 x + y &= (a//b) + (c//d) \\
 &= (ad + bc)//(bd) \\
 y + x &= (c//d) + (a//b) \\
 &= (cb + da)//(db) \\
 &= (ad + bc)//(bd) \\
 &= x + y.
 \end{aligned}$$

The identity $(x + y) + z$ was proved on page 84 already.

Next, we will prove that $x + 0 = 0 + x = x$. We have already proved that $x + y = y + x$, so we only need to show that $x + 0 = x$:

$$\begin{aligned}
 x + 0 &= (a//b) + (0//1) \\
 &= (a \cdot 1 + b \cdot 0)//(b \cdot 1) \\
 &= a//b \\
 &= x.
 \end{aligned}$$

For the next identity, $x + (-x) = (-x) + x = 0$, we have already shown that $x + y = y + x$, so we only need to prove $x + (-x) = 0$:

$$\begin{aligned}
 x + (-x) &= (a//b) + ((-a)//b) \\
 &= (ab + b \cdot (-a))//(b \cdot b) \\
 &= 0//(b \cdot b) \\
 &= 0.
 \end{aligned}$$

The last step is valid because $0 \cdot 1 = 0 \cdot (b \cdot b)$.

Now we prove that $xy = yx$:

$$\begin{aligned}
 xy &= (a//b) \cdot (c//d) \\
 &= (ac)//(bd) \\
 yx &= (c//d) \cdot (a//b) \\
 &= (ca)//(db) \\
 &= xy.
 \end{aligned}$$

We now prove that $(xy)z = x(yz)$:

$$\begin{aligned}
 (xy)z &= ((a/b) \cdot (c/d)) \cdot (e/f) \\
 &= ((ac)/(bd)) \cdot (e/f) \\
 &= (ace)/(bdf) \\
 x(yz) &= (a/b) \cdot ((c/d) \cdot (e/f)) \\
 &= (a/b) \cdot ((ce)/(df)) \\
 &= (ace)/(bdf) \\
 &= (xy)z.
 \end{aligned}$$

Next, we prove $x1 = 1x = x$. We have already proved that $xy = yx$, so we only need to prove $x \cdot 1 = x$:

$$\begin{aligned}
 x \cdot 1 &= (a/b) \cdot (1/1) \\
 &= (a \cdot 1)/(b \cdot 1) \\
 &= a/b \\
 &= x.
 \end{aligned}$$

The next identity to prove is $x(y + z) = xy + xz$:

$$\begin{aligned}
 x(y + z) &= (a/b)((c/d) + (e/f)) \\
 &= (a/b)((cf + de)/(df)) \\
 &= (a(cf + de))/(bdf) \\
 &= (acf + ade)/(bdf) \\
 xy + xz &= ((ac)/(bd)) + ((ae)/(bf)) \\
 &= (acbf + bdae)/(b^2df) \\
 &= (acf + dae)/(bdf) \\
 &= x(y + z).
 \end{aligned}$$

For the identity $(y + z)x = yx + zx$, we can just apply $xy = yx$ and $x(y + z) = xy + xz$ together.

Finally, we will prove that $xx^{-1} = x^{-1}x = 1$, when $x \neq 0$. We know that x^{-1} exists because otherwise, $a \neq 0$, and therefore x would be 0. Since $xy = yx$, we only need to show that $xx^{-1} = 1$:

$$\begin{aligned}
 xx^{-1} &= (a/b)(b/a) \\
 &= (ab)/(ab) \\
 &= 1.
 \end{aligned}$$

(The last step comes from Definition 4.2.1).

Exercise 4.2.5

(a) By Lemma 4.2.7, we know that exactly one of the statements “ $x - y = 0$ ”, “ $x - y$ is positive”, or “ $x - y$ is negative” is true. We now can use Definition 4.2.8 to deduce that exactly one of the statements $x = y$, $x > y$, or $x < y$ is true.

(b) If $x < y$, $x - y$ must be a negative rational number. Hence, $y - x$ is positive, and $y > x$. Similarly, we can show that $y > x$ implies $x < y$.

(c) By (b), we know that $y > x$ and $z > y$. Thus, $y - x$ and $z - y$ are positive. If $y - x = a/b$ and $z - y = c/d$ (where a , b , c , and d are positive; this is possible by Definition 4.2.6),

$$\begin{aligned} z - x &= z - y + y - x \\ &= (ad + bc)/(bd) \end{aligned}$$

which is positive, and therefore $z > x$. We now have proved that $x < z$.

(d) By (b), we know that $y > x$, and hence, $y - x$ is positive. If $y - x = a/b$, we know that

$$\begin{aligned} y + z - (x + z) &= y + z - x - z \\ &= y - x \end{aligned}$$

and therefore $y + z > x + z$. Hence, $x + z < y + z$.

(e) Because of part (b), we know that $y > x$, and thus, $y - x$ is positive. Assuming that $y - x = a/b$ and $z = c/d$ (where a , b , c , and d are positive; this is possible by Definition 4.2.6),

$$\begin{aligned} yz - xz &= (y - x)z \\ &= (a/b)(c/d) \\ &= (ac)/(bd) \end{aligned}$$

and hence, $yz > xz$ and $xz < yz$.

Exercise 4.2.6

We know that $x - y$ is positive. Assuming that $y - x = a//b$ and $z = c//d$ (where a, b, c , and d are positive; this is always possible by Definition 4.2.6), we have that

$$\begin{aligned} yz - xz &= (y - x)z \\ &= (ac)//(bd) \end{aligned}$$

and therefore, $xz > yz$.

4.3 Absolute value and exponentiation**Exercise 4.3.3(a)**

First, we prove that $x^n x^m = x^{n+m}$. We induct on m . When $m = 0$, $x^m = 1$, and hence, $x^n x^m = x^n$. But $x^{n+m} = x^{n+0} = x^n$, and therefore, $x^n x^m = x^{n+m}$ is true when $m = 0$. When $x^n x^m = x^{n+m}$,

$$x^n x^{m++} = x^n x^m \cdot x = x^{n+m} \cdot x = x^{n+(m++)},$$

and we have proved that $x^n x^m = x^{n+m}$ for all n and m .

Next, we prove that $(x^n)^m = x^{nm}$, again by induction on m . When $m = 0$, it is obvious that both $(x^n)^m$ and x^{nm} are both 1, since anything to the power of 0 is 1. Now, we prove that if $(x^n)^m = x^{nm}$ for some n and m , $(x^n)^{m++} = x^{n(m++)}$. We know that $(x^n)^{m++} = x^n \cdot (x^n)^m$. Using Definition 4.3.9, we get that

$$x^n \cdot (x^n)^m = (x^n)^{m++}.$$

We have finished proving that $(x^n)^m = x^{nm}$ for all n and m .

We will now prove that $(xy)^n = x^n y^n$, again using induction, except on n . When $n = 0$, it is obvious that both sides are equal to 1. If $(xy)^n = x^n y^n$, we can multiply both sides of the equation by xy to get that

$$(xy)^{n++} = x^{n++} y^{n++}.$$

We have completed showing that $(xy)^n = x^n y^n$.

Exercise 4.3.5

We use induction on N , with base case $N = 1$. For the base case, $2^1 = 2 > 1$. For the inductive step, we multiply both sides of $2^N \geq N$ by 2 to get $2^{N++} \geq 2N$. Since $2N = N + N \geq N + 1$, $2^{N++} \geq N++$. We have finished proving that $2^N \geq N$ for all positive integers N .

4.4 Gaps in the rational numbers**Exercise 4.4.1**

If $x = \frac{a}{b}$ and $a, b \in \mathbf{N}$, by Proposition 2.3.9 (the Euclidean algorithm), there exist m and r in \mathbf{N} such that $0 \leq r < b$ and $a = mb + r$. Dividing both sides by b yields that

$$x = m + \frac{r}{b}.$$

Because $0 \leq r < b$, $0 \leq \frac{r}{b} < 1$. Adding m to 0, $\frac{r}{b}$, and 1 gives that $m \leq m + \frac{r}{b} < m + 1$. But $m + \frac{r}{b} = x$, so we have proved that $m \leq x < m + 1$. To prove that there exists some natural number N greater than x , we set N to be $m + 1$ when $m + 1 \geq 0$ and 0 when $m + 1 < 0$. (This works because when $m + 1 \geq 0$, it is a natural number and greater than x , as we proved before. When $m + 1 < 0$, $N = 0$ works because $0 > m + 1 > x$.)

Exercise 4.4.3

Showing that every natural number is either even or odd, but not both Let n be a natural number. By the Euclidean algorithm (Proposition 2.3.9), every natural number is equal to $2m + r$, where $m, r \in \mathbf{N}$ and $0 \leq r < 2$. This implies that every natural number is either even or odd. Now, we show that n cannot be both even and odd, by showing that having both at the same time will cause a contradiction. A natural number n being both even and odd must be equal to $2a$ and $2b + 1$ at the same time, where $a, b \in \mathbf{N}$. If $a > b$, $2a > 2b + 1$, and we have a contradiction, because then $2a \neq 2b + 1$. When $b \geq a$, $2b + 1 > 2a$, and we also have a contradiction.

Showing that an odd natural number n has an odd square n^2 We know that $n = 2m + 1$ for some $m \in \mathbf{N}$. Therefore, $n^2 = 4m^2 + 2m + 1 = 2m(2m + 1) + 1$, and therefore, n^2 is also odd.

Chapter 5

The real numbers

5.1 Cauchy sequences

Exercise 5.1.1

Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, it is eventually 1-steady. Thus, for some m , $(a_n)_{n=m}^{\infty}$ is 1-steady. By Lemma 5.1.14, the finite sequence a_1, \dots, a_{m-1} is bounded. Also, $(a_n)_{n=m}^{\infty}$ is bounded by $|a_m| + 1$. Therefore, $(a_n)_{n=1}^{\infty}$ is bounded by the greater of the bounds for a_1, \dots, a_{m-1} and $(a_n)_{n=m}^{\infty}$.

5.2 Equivalent Cauchy sequences

Exercise 5.2.1

We will only prove that $(a_n)_{n=1}^{\infty}$ being a Cauchy sequence implies that $(b_n)_{n=1}^{\infty}$ is also a Cauchy sequence; we can replace a_n with b_n and b_n with a_n , anyways.

For any rational $\epsilon > 0$, there exist natural numbers m_1 and m_2 such that $(a_n)_{n=m_1}^{\infty}$ is $\frac{\epsilon}{3}$ -steady and such that $(a_n)_{n=m_2}^{\infty}$ and $(b_n)_{n=m_2}^{\infty}$ are $\frac{\epsilon}{3}$ -close. If m is defined to be m_1 when $m_1 \geq m_2$ and m_2 when $m_1 < m_2$, $(b_n)_{n=m}^{\infty}$ is ϵ -steady, because for any $x, y \in \mathbf{N}$ greater than or equal to m ,

$$d(a_x, a_y), d(a_x, b_x), d(a_y, b_y) \leq \frac{\epsilon}{3}.$$

5.3 The construction of the real numbers

Exercise 5.3.1

We will assume here that ϵ is a rational number greater than 0.

We know that $(a_n)_{n=1}^\infty$ is always ϵ -close to itself. Therefore, $x = x$.

It is obvious that if $(a_n)_{n=1}^\infty$ is eventually ϵ -close to $(b_n)_{n=1}^\infty$, $(b_n)_{n=1}^\infty$ is eventually ϵ -close to $(a_n)_{n=1}^\infty$. Hence, $x = y$ implies $y = x$.

We also know that when $(a_n)_{n=1}^\infty$ is eventually $\epsilon/2$ -close to $(b_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ is eventually $\epsilon/2$ -close to $(c_n)_{n=1}^\infty$, $(a_n)_{n=1}^\infty$ is eventually ϵ -close to $(c_n)_{n=1}^\infty$. (This is because the same thing holds if we replace sequences with rational numbers and “eventually $\epsilon/2$ -close” with “ ϵ -close”). Thus, $x = y$ and $y = z$ together imply $x = z$.

Exercise 5.3.3

If $\text{LIM}_{n \rightarrow \infty} a = \text{LIM}_{n \rightarrow \infty} b$, we will prove that $a = b$. Assume for the sake of contradiction that $a \neq b$. Then, $|a - b| \neq 0$. But then a, a, a, a, \dots and b, b, b, b, \dots are not eventually $\frac{|a-b|}{2}$ -close. But this contradicts our assumption that $\text{LIM}_{n \rightarrow \infty} a = \text{LIM}_{n \rightarrow \infty} b$, and thus $a = b$.

To prove that $a = b$ implies $\text{LIM}_{n \rightarrow \infty} a = \text{LIM}_{n \rightarrow \infty} b$, we can use the fact that a is ϵ -close to itself for any rational $\epsilon > 0$.

Exercise 5.3.5

For any rational $\epsilon > 0$, there is some m greater than $\frac{1}{\epsilon}$, by Proposition 4.4.1. Since any natural number m' greater than or equal to m satisfies

$$\frac{1}{m'} \leq \frac{1}{m} \leq \epsilon,$$

the sequence $1/1, 1/2, 1/3, \dots$ is eventually ϵ -close to the sequence $0, 0, 0, \dots$. Therefore, $\text{LIM}_{n \rightarrow \infty} 1/n = 0$.

5.4 Ordering the reals

Exercise 5.4.2

(a) By Proposition 5.4.4, exactly one of the statements “ $x - y$ is positive”, “ $x - y$ is equal to 0”, or “ $x - y$ is negative” is true. These statements

correspond to “ $x > y$ ”, “ $x = y$ ”, and “ $x < y$ ” respectively.

(b) By Proposition 5.4.4, $x - y$ is negative if and only if $y - x$ is positive. Hence, $x < y$ if and only if $y > x$.

(c) Since $x < y$ and $y < z$, $y > x$ and $z > y$ (by part (b)). Thus, $y - x$ and $z - y$ are positive. We know that $z - y + y - x = z - x$. Therefore, by Proposition 5.4.4, $z - x$ is positive and $z > x$. Using part (b) again, we get that $x < z$.

(d) By part (b), $y - x$ is positive. Since $y + z - (x + z)$ is equal to $y - x$, $x + z < y + z$.

Exercise 5.4.4

By Proposition 5.4.12, there exist some integers a and b such that $a, b \neq 0$ and $0 < \frac{a}{b} \leq x$. Since

$$0 < \frac{1}{2b|a|/a} \leq \frac{|a|}{2b|a|/a} = \frac{a}{2b} < \frac{a}{b}$$

and $2b|a|/a$ is a positive integer, we can define N as $2b|a|/a$.

Exercise 5.4.5

By Corollary 5.4.13 (the Archimedean property), there exists a positive integer b such that $b(y - x) > 1$. Thus, $a := \begin{cases} y - 1 & \text{if } y \in \mathbf{Z} \\ \lfloor y \rfloor & \text{if } y \notin \mathbf{Z} \end{cases}$ is greater than bx and less than by . Therefore, $x < \frac{a}{b} < y$.

5.5 The least upper bound property

Exercise 5.5.1

First, we show that $-M$ is a lower bound for $-E$. We assume for the sake of contradiction that some element of E , which we call $-a$, is less than $-M$. Then, we have that $a > M$. But a is in E , and therefore, $M \neq \sup(E)$.

Thus, our assumption that $-a < -M$ is wrong, and $-M$ actually is a lower bound for $-E$.

Next, we show that no other lower bound for $-E$, $-N$, is greater than $-M$. We assume for the sake of contradiction that $-N > -M$. Then, $N < M$ and N is an upper bound for E lower than M . But this is impossible from the definition of least upper bound. Hence, $-N$ is actually equal to or less than $-M$. This means that $-M = \inf(-E)$.

Exercise 5.5.3

Assume for the sake of contradiction that $m \neq m'$. Therefore, either $m > m'$ or $m' > m$. If $m' > m$, we can just switch the names of m and m' , so we can just assume $m > m'$. Hence, $m \geq m' + 1$. Thus, $\frac{m-1}{n} \geq \frac{m'}{n}$. We know that $\frac{m'}{n}$ is an upper bound for E , so $\frac{m-1}{n}$ is too. But we assumed that $\frac{m-1}{n}$ is not an upper bound for E , so m is actually equal to m' .

Exercise 5.5.5

For any two real numbers $x < y$, there is a rational number a such that $x < a < y$, by Proposition 5.4.14. Using Proposition 5.4.14 again, we can see that there also is a rational number b such that $a < b < y$. Therefore, $x < a < b < y$. We can choose some rational number c such that $a - \sqrt{2} < c < b - \sqrt{2}$. Thus, $a < c + \sqrt{2} < b$. If $c + \sqrt{2} \in \mathbf{Q}$, $c + \sqrt{2} - c = \sqrt{2}$ is also rational. Therefore, $c + \sqrt{2}$ is irrational. We have finished constructing an irrational number $c + \sqrt{2}$ such that $x < c + \sqrt{2} < y$.

5.6 Real exponentiation, part I

Exercise 5.6.2 (unfinished)

Assume that $q = a/b$ and $r = c/d$, where a, b, c , and d are integers. Also, we assume that b and d are nonzero.

(a) By Definition 5.6.7, $x^q = (x^{1/b})^a$. We know that x is positive, so $x^{1/b}$ is also positive. Hence, $x^q = (x^{1/b})^a$ is also positive.

(b) Since $q = \frac{ad}{bd}$ and $r = \frac{bc}{bd}$,

$$x^{q+r} = x^{(ad+bc)/bd} = (x^{1/bd})^{ad+bc}.$$

By Proposition 5.6.3, $(x^{1/bd})^{ad+bc} = (x^{1/bd})^{ad}(x^{1/bd})^{bc}$. Thus, $x^{q+r} = x^q x^r$.

Similarly, we can prove that $(x^q)^r = x^{qr}$. First, we manipulate $(x^q)^r$:

$$\begin{aligned} (x^q)^r &= (x^{ad/bd})^{c/d} \\ &= (((x^{1/bd})^{ad})^{1/d})^c \\ &= ((x^{1/bd})^a)^c \\ &= (x^{1/bd})^{ac} \\ &= x^{ac/bd}. \end{aligned}$$

By replacing ac/bd with qr now, we obtain the result.

(c) We know that

$$x^{-q} = (x^{1/b})^{-a}.$$

Therefore, $x^{-q} = \frac{1}{(x^{1/b})^a} = \frac{1}{x^q}$.

(d) Since q is positive, we can safely assume that a and b are also positive. (If they are not, they must be both negative, and we can use $-a$ and $-b$ instead of a and b). If $x > y$, by Proposition 5.6.3 and Lemma 5.6.6(d), $(x^{1/b})^a > (y^{1/b})^a$. The left side is equal to x^q and the right side is equal to y^q . Therefore, $x^q > y^q$.

If $x^q > y^q$, $(x^{1/b})^a > (y^{1/b})^a$; thus, $((x^{1/b})^a)^{1/a} > ((y^{1/b})^a)^{1/a}$. Therefore, $x^{1/b} > y^{1/b}$. Similarly, we can obtain that $x > y$.

(e) ?

Exercise 5.6.3

If x is positive, by Proposition 5.6.3 and Proposition 4.3.12(b), $y > 0$ implies $y^2 \leq x^2$ if and only if $y \leq x$. By Definition 5.6.4, the definition of \sqrt{x} , $\sqrt{x^2} = \sup\{y \in \mathbf{R} : y \geq 0 \text{ and } y \leq x\}$. Therefore, $\sqrt{x^2} = x$ when x is positive.

If x is 0, it is obvious that $\sqrt{x^2} = \sup\{y \in \mathbf{R} : y = 0\} = 0$.

If x is negative, we can use the case for when x is positive except for replacing x with $-x$.

Chapter 6

Limits of sequences

6.1 Convergence and limit laws

Exercise 6.1.1

We induct on m , starting from $m = n + 1$. We have assumed that $a_{n+1} > a_n$ already. Now, if $a_m > a_n$ and $m > n$, $a_{m+1} > a_m > a_n$. We have proved $a_m > a_n$ for all $m > n$, as desired.

Exercise 6.1.3

If $(a_n)_{n=m}^{\infty}$ converges to c , for any real number $\epsilon > 0$, there exists a natural number $k \geq m$ such that $(a_n)_{n=k}^{\infty}$ is ϵ -close to c . We can define k' to be k if $k \geq m'$ and m' if $k < m'$. We know that $k' \geq m'$ and that $(a_n)_{n=k'}^{\infty}$ is ϵ -close to c from this definition. Since we can do this for any $\epsilon > 0$, $(a_n)_{n=m'}^{\infty}$ converges to c .

If $(a_n)_{n=m'}^{\infty}$ converges to c , for any real number $\epsilon > 0$, there exists a natural number $k \geq m'$ such that $(a_n)_{n=k}^{\infty}$ is ϵ -close to c . Since $k \geq m' \geq m$, $(a_n)_{n=m}^{\infty}$ converges to c .

Exercise 6.1.5

Assume $(a_n)_{n=m}^{\infty}$ converges to c , a real number. Then, for any real number $\epsilon > 0$, there exists a natural number $k \geq m$ such that $(a_n)_{n=k}^{\infty}$ is $\frac{\epsilon}{2}$ -close to c . For any $A, B \geq k$, a_A and a_B must then both be $\frac{\epsilon}{2}$ -close to c , and therefore $d(a_A, a_B) \leq \epsilon$. Hence, $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence.

Exercise 6.1.7

A sequence $(a_n)_{n=m}^{\infty}$ that is bounded by a positive rational number M in \mathbf{Q} is obviously bounded in \mathbf{R} by M .

If $(a_n)_{n=m}^{\infty}$ is bounded by a positive real number M in \mathbf{R} , Proposition 5.4.12 gives us a positive integer N that is greater than M . We know $N \in \mathbf{Q}$, and therefore N is a bound for $(a_n)_{n=m}^{\infty}$ using Definition 5.1.12.

Exercise 6.1.9

If $a_n = 1$ and $b_n = \frac{1}{n}$ for all natural numbers $n \geq m$,

$$\lim_{n \rightarrow \infty} (b_n) = 0.$$

But then, $a_n/b_n = n$. Therefore, $(a_n/b_n)_{n=m}^{\infty}$ is not bounded. By Corollary 6.1.17, $(a_n/b_n)_{n=m}^{\infty}$ is not convergent, and hence, Theorem 6.1.19(f) does not necessarily work when $\lim_{n \rightarrow \infty} (b_n) = 0$.

6.2 The Extended real number system**Exercise 6.2.2**

(a) If $E \subseteq \mathbf{R}$, we use Definition 5.5.10 to obtain this part of this theorem. If $+\infty \in E$, $\sup E = +\infty$ and therefore $x \leq \sup E$ for any x . If $+\infty \notin E$ and $-\infty \in E$, $\sup E = \sup(E \setminus \{-\infty\})$. All finite elements of E will then be less than or equal to $\sup E$, and $\sup E \geq -\infty$.

Since $-x \leq \sup(\{-y : y \in E\})$ for all $x \in E$, all elements of E are greater than or equal to $-\sup(\{-y : y \in E\}) = \inf(E)$.

(b) When $E \subseteq \mathbf{R}$, E is non-empty and has an upper bound in \mathbf{R} , E is non-empty but has no upper bound in \mathbf{R} , or E is empty. In the first case, $\sup E \in \mathbf{R}$, and $-\infty$ is not an upper bound for E , since E is not empty. Also, $+\infty$ would be greater than $\sup E$. In the second case, $-\infty$ is not an upper bound for E and $+\infty = \sup E$. In the last case, $-\infty = \sup E$ and $+\infty > \sup E$. Therefore, $\sup E \leq M$ for any M that is an upper bound for E , since it is true both when we restrict M to be $+\infty$ or $-\infty$ and when M is forced to be real (Definition 5.5.10).

If $+\infty \in E$, only $+\infty$ is an upper bound for E , so $\sup E = +\infty = M$.

If $-\infty \in E$ but $+\infty \notin E$, $-\infty$ is not both an upper bound for E and less than $\sup E$, from the case for when $E \subseteq \mathbf{R}$ applied to $E \setminus \{-\infty\}$.

(c) We know that $\inf(E) = -\sup(\{-x : x \in E\})$. Since $-M$ is an upper bound for $\{-x : x \in E\}$, $\sup(\{-x : x \in E\}) \leq -M$ from part (b). Therefore, $\inf(E) \geq M$.

6.3 Suprema and Infima of sequences

Exercise 6.3.2

Let S be the set $\{a_n : n \geq m\}$. Then, $x = \sup(S)$ and $a_n \in S$ for any natural number $n \geq m$. Therefore, Theorem 6.2.11(a) tells us that $a_n \leq x$ for all $n \geq m$. When $M \in \mathbf{R}^*$ is an upper bound for $(a_n)_{n=m}^\infty$, M is also an upper bound for S . Hence, Theorem 6.2.11(b) implies that $x \leq M$. For every extended real number y for which $y < x$, there exists at least one $n \geq m$ such that $y < a_n \leq x$ because otherwise y would be an upper bound for S lower than x , which is impossible.

Exercise 6.3.4

The sequence $(x^n)_{n=1}^\infty$ does not converge if $x > 1$. Therefore, we cannot use our trick with $xL = L$. In fact, by Theorem 6.1.19(b), Proposition 6.3.10, and $(\frac{1}{x})^n x^n = 1$, the sequence $(x^n)_{n=1}^\infty$ converging would imply

$$1 = \lim_{n \rightarrow \infty} 1 = \lim_{n \rightarrow \infty} \left(\frac{1}{x}\right)^n \lim_{n \rightarrow \infty} (x^n) = 0 \cdot \lim_{n \rightarrow \infty} (x^n) = 0.$$

We know that $1 \neq 0$, so $(x^n)_{n=1}^\infty$ diverges when $x > 1$.

In Example 1.2.3, we assumed implicitly that $L = \lim_{n \rightarrow \infty} x^n$ existed. Hence, when we made the absurd conclusion that $x = 1$ or $L = 0$, we were actually concluding that $x = 1$ or $L = 0$ when L is well-defined.