# Chapter 1

N/A

# Chapter 2

## 2.1

N/A

## 2.2

### Exercise 2.2.1

We shall use induction on $a$, fixing $b$ and $c$. First, we shall prove the base case. If $a = 0$, $a + b = 0 + b = b$ by the definition of addition of natural numbers (Definition 2.2.1). Therefore, $(a + b) + c = b + c$. Also by Definition 2.2.1, $a + (b + c) = 0 + (b + c) = b + c$. We now have $(a + b) + c = b + c = a + (b + c)$.

Now suppose that $(a+b)+c = a+(b+c)$. We then have, by Definition 2.2.1, that $((a{+}{+}) + b) + c = ((a + b){+}{+}) + c = ((a + b) + c){+}{+} = (a + (b + c)){+}{+}$. Because of Definition 2.2.1 again, $(a + (b + c)){+}{+} = (a{+}{+}) + (b + c)$. Now we have $((a{+}{+}) + b) + c = (a{+}{+}) + (b + c)$, which closes the induction.

### Exercise 2.2.2

We shall use induction on $a$. The base case is vacuously true since 0 is not positive. For the inductive step, we have to show that $a{+}{+}$ being positive implies that $a{+}{+} = b{+}{+}$ for exactly one natural number $b$ if $a$ being positive implies that $a = c{+}{+}$ for exactly one natural number $c$. But $a$ is equal to $b$ iff $a{+}{+}$ is equal to $b{+}{+}$, by Axiom 2.4. This closes the induction.

### Exercise 2.2.3

(a) By Lemma 2.2.2, $a = a + 0$, so $a \geq a$.
(b) Because $a \geq b$ and $b \geq c$, $a = b + n$ and $b = c + m$ for some $n$ and $m$. Therefore, $a = b+n = (c+m)+n$. By Proposition 2.2.5, $(c+m)+n = c+(m+n)$, so $a = c + (m + n)$. $n + m$ is a natural number, so $a \geq c$.
(c) Because $a \geq b$ and $b \geq a$, $a = b + n$ and $b = a + m$. Therefore, $a = b + n = (a + m) + n$. By Proposition 2.2.5, $(a + m) + n = a + (m + n)$. Now we have $a = a + (m + n)$. By Lemma 2.2.2, $a = a + 0$, so we can use Proposition 2.2.6 to get $0 = m + n$. Using Corollary 2.2.9, we can deduce that $n$ and $m$ are both equal to 0. Therefore, $a = b$.
(d) If $a \geq b$, $a = b + d$ for some $d$. Because of this, $a + c = (b + d) + c$. By Propositions 2.2.4 and 2.2.5, $(b + d) + c = b + (d + c) = b + (c + d) = (b + c) + d$. Therefore, $a + c \geq b + c$.

Conversely, if $a + c \geq b + c$, $a + c = (b + c) + d$ for some $d$. By Propositions 2.2.4 and 2.2.5, $(b + c) + d = b + (c + d) + b + (d + c) = (b + d) + c$. Using

Proposition 2.2.6, we get $a = b + d$, and therefore, $a \geq b$.
(f) If $a < b$, $b = a + c$ for some $c$, and $b \neq a$. Therefore, $c$ is positive, because otherwise $b = a + 0 = a$. ($a + 0 = a$ because of Lemma 2.2.2.)

Conversely, if $b = a + c$ for some positive number $c$, we know that $a \leq b$. If $a = b$, $a = a + c$. By Lemma 2.2.2, $a = a + 0$, so $a + 0 = a + c$. We can use Proposition 2.2.6 to get $0 = c$, but $c$ is positive. Therefore, $a \neq b$, and combined with $a \leq b$, we get $a < b$.
(e) We first prove that $a < b$ implies that $a{+}{+} \leq b$. By (f), $b = a + c$ for some positive $c$. By Lemma 2.2.10, exactly one $d$ exists such that $d{+}{+} = c$. Therefore, $b = a + d{+}{+}$. By Lemma 2.2.3 and Definition 2.2.1, $b = a + d{+}{+} = (a + d){+}{+} = (a{+}{+}) + d$. Therefore, $a{+}{+} \leq b$.

Conversely, if $a{+}{+} \leq b$, we need to prove that $a < b$. Since $a{+}{+} = a + 1 \geq a$, by (b), $a \leq b$. If $a = b$, then $a{+}{+} \leq a$, and we get, from (c), that $a{+}{+} = a$. This is a contradiction, from Axiom 2.3 if $a = 0$ and Axiom 2.4 otherwise. Therefore, $a < b$.

### Exercise 2.2.6 (unfinished)

We use induction on $n$. When $n = 0$, $m \leq n$ implies that $0 = m + a$. But by Corollary 2.2.9, $m = 0 = n$, so $P(n)$ being true implies that $P(m)$ is true.

If $a \leq n$ and $P(n)$ together imply $P(a)$, we now need to prove that $b \leq n{+}{+}$ and $P(n{+}{+})$ together imply $P(b)$. Since $b \leq n{+}{+}$, $n{+}{+} = b + c$ for some $c$. If $c$ is positive, by Lemma 2.2.10, $n{+}{+} = b + d{+}{+}$ for some $d$, so by Lemma 2.2.3 and Axiom 2.4, $n = b + d$, and $b \leq n$. Therefore, $b \leq n{+}{+}$ implies that $b \leq n$ or $n{+}{+} = b + 0 = b$. Because of Lemmas 2.2.2 and 2.2.3, $n{+}{+} = (n + 0){+}{+} = n + 0{+}{+} = n + 1$. Since $P(n{+}{+})$ implies $P(n)$, $P(n)$ is true. But then **?**

## 2.3

### Exercise 2.3.1

First, we prove that $n \cdot 0 = 0$. We induct on $n$. When $n = 0$, $n \cdot 0 = 0 \cdot 0 = 0$, because of Definition 2.3.1. For the inductive hypothesis, we assume $n \cdot 0 = 0$. Then we need to prove that $n{+}{+} \cdot 0 = 0$. By Definition 2.3.1, $n{+}{+} \cdot 0 = n \cdot 0 + 0 = n \cdot 0$. But we already know that $n \cdot 0 = 0$, so $n{+}{+} \cdot 0 = 0$. This closes the induction.

Next, we prove that $n \cdot m{+}{+} = nm + n$ using induction on $n$. When $n = 0$, $0 \cdot m{+}{+} = 0 = 0 \cdot m + 0$. Now, if $n \cdot m{+}{+} = nm + n$, we need to prove that $n{+}{+} \cdot m{+}{+} = n{+}{+} \cdot m + n{+}{+}$. We can deduce from $n \cdot m{+}{+} = nm + n$ that $n{+}{+} \cdot m{+}{+} = (n \cdot m{+}{+}) + m{+}{+} = nm + n + m{+}{+} = nm + (n + m){+}{+} = nm + n{+}{+} + m = (n{+}{+}) \cdot m + n{+}{+}$. This closes the induction.

Now we can prove that $nm = mn$. We induct on $m$. When $m = 0$, we have $n \cdot 0 = 0 = 0 \cdot n$. For the inductive hypothesis, we assume $nm = mn$. Now we need to prove $n \cdot m{+}{+} = m{+}{+} \cdot n$. We know that $n \cdot m{+}{+} = nm + n$ and $m{+}{+} \cdot n = mn + n$. But $nm = mn$, so $n \cdot m{+}{+} = m{+}{+} \cdot n$. This closes the induction.

### Exercise 2.3.4

By Proposition 3.4, $(a+b)^2 = (a+b)\cdot(a+b) = (a+b)\cdot a + (a+b)\cdot b = a\cdot a + b\cdot a + a\cdot b + b\cdot b$. We can rewrite $b\cdot a + a\cdot b$ as $a\cdot b + a\cdot b$ because of Lemma 2.3.2. We can further rewrite this as $2ab$ because $2(ab) = 1(ab) + ab = 0(ab) + ab + ab = 0 + ab + ab = ab + ab$. We can also rewrite $a\cdot a$ and $b\cdot b$ as $a^2$ and $b^2$ respectively. Therefore, $(a+b)^2 = a^2 + 2ab + b^2$.

# Chapter 3

## 3.1

### Exercise 3.3.1

We know that at least one of the statements "$a = c$" and "$a = d$" is true since $\{a, b\} = \{c, d\}$. Similarly, at least one of the statements "$b = c$" and "$b = d$" is true. If we have that $a = c$ and $b = d$ at the same time, then we have proved what we want. (Same for $a = d$ and $b = c$ at the same time.) Otherwise, then we must have that $a = c = b$ or $a = d = b$. If both of these are true, then $a = b = c = d$, and therefore both $a = c$ and $b = d$. If only one of these is true, which we will assume to be the statement that $a = c = b$ as a similar argument holds assuming $a = d = b$, then $d \notin \{a, b\} = \{a, a\}$, while $d \in \{c, d\} = \{a, d\}$, so $\{a, b\} \neq \{c, d\}$, a contradiction. Therefore, if $\{a, b\} = \{c, d\}$, either both $a = c$ and $b = d$ or $a = d$ and $b = c$.

### Exercise 3.3.2

First, we show that these four sets exist. Axiom 3.3 tells us that $\emptyset$ exists. The singleton set axiom (part of Axiom 3.4) and Axiom 3.1 (so that we are allowed to construct the singleton set whose element is another set) tell us that $\{\emptyset\}$ exists. Applying them again gives that $\{\{\emptyset\}\}$ exists. Using the pair set axiom (also part of Axiom 3.4) and Axiom 3.1 allow us to create the set $\{\emptyset, \{\emptyset\}\}$.