

# Answers to Analysis I Exercises, Third Edition

my-cen

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# Chapter 1

## Introduction

### 1.1 What is analysis?

N/A

### 1.2 Why do analysis?

N/A

## Chapter 2

# Starting at the beginning: the natural numbers

### 2.1 The Peano axioms

N/A

### 2.2 Addition

#### Exercise 2.2.1

We shall use induction on  $a$ , fixing  $b$  and  $c$ . First, we shall prove the base case. If  $a = 0$ ,  $a + b = 0 + b = b$  by the definition of addition of natural numbers (Definition 2.2.1). Therefore,  $(a + b) + c = b + c$ . Also by Definition 2.2.1,  $a + (b + c) = 0 + (b + c) = b + c$ . We now have  $(a + b) + c = b + c = a + (b + c)$ .

Now suppose that  $(a + b) + c = a + (b + c)$ . We then have, by Definition 2.2.1, that  $((a++) + b) + c = ((a + b)++) + c = ((a + b) + c)++ = (a + (b + c))++$ . Because of Definition 2.2.1 again,  $(a + (b + c))++ = (a++) + (b + c)$ . Now we have  $((a++) + b) + c = (a++) + (b + c)$ , which closes the induction.

#### Exercise 2.2.2

We shall use induction on  $a$ . The base case is vacuously true since 0 is not positive. For the inductive step, we have to show that  $a++$  being positive implies that  $a++ = b++$  for exactly one natural number  $b$  if  $a$  being positive

implies that  $a = c++$  for exactly one natural number  $c$ . But  $a$  is equal to  $b$  if  $a++$  is equal to  $b++$ , by Axiom 2.4. This closes the induction.

### Exercise 2.2.3

(a) By Lemma 2.2.2,  $a = a + 0$ , so  $a \geq a$ .

(b) Because  $a \geq b$  and  $b \geq c$ ,  $a = b + n$  and  $b = c + m$  for some  $n$  and  $m$ . Therefore,  $a = b + n = (c + m) + n$ . By Proposition 2.2.5,  $(c + m) + n = c + (m + n)$ , so  $a = c + (m + n)$ .  $n + m$  is a natural number, so  $a \geq c$ .

(c) Because  $a \geq b$  and  $b \geq a$ ,  $a = b + n$  and  $b = a + m$ . Therefore,  $a = b + n = (a + m) + n$ . By Proposition 2.2.5,  $(a + m) + n = a + (m + n)$ . Now we have  $a = a + (m + n)$ . By Lemma 2.2.2,  $a = a + 0$ , so we can use Proposition 2.2.6 to get  $0 = m + n$ . Using Corollary 2.2.9, we can deduce that  $n$  and  $m$  are both equal to 0. Therefore,  $a = b$ .

(d) If  $a \geq b$ ,  $a = b + d$  for some  $d$ . Because of this,  $a + c = (b + d) + c$ . By Propositions 2.2.4 and 2.2.5,  $(b + d) + c = b + (d + c) = b + (c + d) = (b + c) + d$ . Therefore,  $a + c \geq b + c$ .

Conversely, if  $a + c \geq b + c$ ,  $a + c = (b + c) + d$  for some  $d$ . By Propositions 2.2.4 and 2.2.5,  $(b + c) + d = b + (c + d) = b + (d + c) = (b + d) + c$ . Using Proposition 2.2.6, we get  $a = b + d$ , and therefore,  $a \geq b$ .

(f) If  $a < b$ ,  $b = a + c$  for some  $c$ , and  $b \neq a$ . Therefore,  $c$  is positive, because otherwise  $b = a + 0 = a$ . ( $a + 0 = a$  because of Lemma 2.2.2.)

Conversely, if  $b = a + c$  for some positive number  $c$ , we know that  $a \leq b$ . If  $a = b$ ,  $a = a + c$ . By Lemma 2.2.2,  $a = a + 0$ , so  $a + 0 = a + c$ . We can use Proposition 2.2.6 to get  $0 = c$ , but  $c$  is positive. Therefore,  $a \neq b$ , and combined with  $a \leq b$ , we get  $a < b$ .

(e) We first prove that  $a < b$  implies that  $a++ \leq b$ . By (f),  $b = a + c$  for some positive  $c$ . By Lemma 2.2.10, exactly one  $d$  exists such that  $d++ = c$ . Therefore,  $b = a + d++$ . By Lemma 2.2.3 and Definition 2.2.1,  $b = a + d++ = (a + d)++ = (a++) + d$ . Therefore,  $a++ \leq b$ .

Conversely, if  $a++ \leq b$ , we need to prove that  $a < b$ . Since  $a++ = a + 1 \geq a$ , by (b),  $a \leq b$ . If  $a = b$ , then  $a++ \leq a$ , and we get, from (c), that  $a++ = a$ . This is a contradiction, from Axiom 2.3 if  $a = 0$  and Axiom 2.4 otherwise. Therefore,  $a < b$ .

### Exercise 2.2.6 (unfinished)

We use induction on  $n$ . When  $n = 0$ ,  $m \leq n$  implies that  $0 = m + a$ . But by Corollary 2.2.9,  $m = 0 = n$ , so  $P(n)$  being true implies that  $P(m)$  is true.

If  $a \leq n$  and  $P(n)$  together imply  $P(a)$ , we now need to prove that  $b \leq n++$  and  $P(n++)$  together imply  $P(b)$ . Since  $b \leq n++$ ,  $n++ = b + c$  for some  $c$ . If  $c$  is positive, by Lemma 2.2.10,  $n++ = b + d++$  for some  $d$ , so by Lemma 2.2.3 and Axiom 2.4,  $n = b + d$ , and  $b \leq n$ . Therefore,  $b \leq n++$  implies that  $b \leq n$  or  $n++ = b + 0 = b$ . Because of Lemmas 2.2.2 and 2.2.3,  $n++ = (n + 0)++ = n + 0++ = n + 1$ . Since  $P(n++)$  implies  $P(n)$ ,  $P(n)$  is true. But then ?

## 2.3 Multiplication

### Exercise 2.3.1

First, we prove that  $n \cdot 0 = 0$ . We induct on  $n$ . When  $n = 0$ ,  $n \cdot 0 = 0 \cdot 0 = 0$ , because of Definition 2.3.1. For the inductive hypothesis, we assume  $n \cdot 0 = 0$ . Then we need to prove that  $n++ \cdot 0 = 0$ . By Definition 2.3.1,  $n++ \cdot 0 = n \cdot 0 + 0 = n \cdot 0$ . But we already know that  $n \cdot 0 = 0$ , so  $n++ \cdot 0 = 0$ . This closes the induction.

Next, we prove that  $n \cdot m++ = nm + n$  using induction on  $n$ . When  $n = 0$ ,  $0 \cdot m++ = 0 = 0 \cdot m + 0$ . Now, if  $n \cdot m++ = nm + n$ , we need to prove that  $n++ \cdot m++ = n++ \cdot m + n++$ . We can deduce from  $n \cdot m++ = nm + n$  that  $n++ \cdot m++ = (n \cdot m++) + m++ = nm + n + m++ = nm + (n + m)++ = nm + n++ + m = (n++) \cdot m + n++$ . This closes the induction.

Now we can prove that  $nm = mn$ . We induct on  $m$ . When  $m = 0$ , we have  $n \cdot 0 = 0 = 0 \cdot n$ . For the inductive hypothesis, we assume  $nm = mn$ . Now we need to prove  $n \cdot m++ = m++ \cdot n$ . We know that  $n \cdot m++ = nm + n$  and  $m++ \cdot n = mn + n$ . But  $nm = mn$ , so  $n \cdot m++ = m++ \cdot n$ . This closes the induction.

### Exercise 2.3.4

By Proposition 3.4,  $(a + b)^2 = (a + b) \cdot (a + b) = (a + b) \cdot a + (a + b) \cdot b = a \cdot a + b \cdot a + a \cdot b + b \cdot b$ . We can rewrite  $b \cdot a + a \cdot b$  as  $a \cdot b + a \cdot b$  because of Lemma 2.3.2. We can further rewrite this as  $2ab$  because  $2(ab) = 1(ab) + ab =$

$0(ab) + ab + ab = 0 + ab + ab = ab + ab$ . We can also rewrite  $a \cdot a$  and  $b \cdot b$  as  $a^2$  and  $b^2$  respectively. Therefore,  $(a + b)^2 = a^2 + 2ab + b^2$ .



# Chapter 3

## Set theory

### 3.1 Fundamentals

#### Exercise 3.3.1

We know that at least one of the statements “ $a = c$ ” and “ $a = d$ ” is true since  $\{a, b\} = \{c, d\}$ . Similarly, at least one of the statements “ $b = c$ ” and “ $b = d$ ” is true. If we have that  $a = c$  and  $b = d$  at the same time, then we have proved what we want. (Same for  $a = d$  and  $b = c$  at the same time.) Otherwise, then we must have that  $a = c = b$  or  $a = d = b$ . If both of these are true, then  $a = b = c = d$ , and therefore both  $a = c$  and  $b = d$ . If only one of these is true, which we will assume to be the statement that  $a = c = b$  as a similar argument holds assuming  $a = d = b$ , then  $d \notin \{a, b\} = \{a, a\}$ , while  $d \in \{c, d\} = \{a, d\}$ , so  $\{a, b\} \neq \{c, d\}$ , a contradiction. Therefore, if  $\{a, b\} = \{c, d\}$ , either both  $a = c$  and  $b = d$  or  $a = d$  and  $b = c$ .

#### Exercise 3.3.2

First, we show that these four sets exist. Axiom 3.3 tells us that  $\emptyset$  exists. The singleton set axiom (part of Axiom 3.4) and Axiom 3.1 (so that we are allowed to construct the singleton set whose element is another set) tell us that  $\{\emptyset\}$  exists. Applying them again gives that  $\{\{\emptyset\}\}$  exists. Using the pair set axiom (also part of Axiom 3.4) and Axiom 3.1 allows us to create the set  $\{\emptyset, \{\emptyset\}\}$ .

Now we show that these four sets are distinct. Since the sets  $\{\emptyset\}$ ,  $\{\{\emptyset\}\}$ , and  $\{\emptyset, \{\emptyset\}\}$  obviously all contain at least one element, if one of them is equal

to  $\emptyset$ , then by Axiom 3.2, it contains no elements. But this contradicts that it has at least one element, and we have a contradiction. Now we need to prove that  $\{\emptyset\}$  is not equal to  $\{\{\emptyset\}\}$  or  $\{\emptyset, \{\emptyset\}\}$ . Both  $\{\emptyset\} \in \{\{\emptyset\}\}$  and  $\{\emptyset, \{\emptyset\}\}$  are true, but  $\{\emptyset\} \notin \{\emptyset\}$ . Therefore, by Axiom 3.2,  $\{\emptyset\}$  is not equal to any of the other sets. (We have already proved that  $\emptyset \neq \{\emptyset\}$ , so  $\{\emptyset\} \neq \emptyset$ ). The last thing that we need to prove now is that  $\{\{\emptyset\}\} \neq \{\emptyset, \{\emptyset\}\}$ . Since  $\emptyset \in \{\emptyset, \{\emptyset\}\}$  and  $\emptyset \notin \{\{\emptyset\}\}$ , by Axiom 3.2, these two sets are not equal. Therefore, all four sets are distinct.

### Exercise 3.1.3

**Proving that  $\{a, b\} = \{a\} \cup \{b\}$**  We shall prove that  $\{a, b\} = \{a\} \cup \{b\}$ . The following are equivalent:

- $c \in \{a, b\}$
- $c$  is either equal to  $a$  or  $b$
- $c \in \{a\}$  or  $c \in \{b\}$
- $c \in \{a\} \cup \{b\}$

Therefore,  $\{a, b\} = \{a\} \cup \{b\}$ .

**Proving that  $A \cup B = B \cup A$**  We now shall prove that  $A \cup B = B \cup A$ . If  $c \in A \cup B$ , then at least one of the statements  $c \in A$  and  $c \in B$  are true. By Axiom 3.5, this means that  $c \in B \cup A$ . We can replace  $A$  with  $B$  and  $B$  with  $A$  to show the other direction. Therefore,  $A \cup B = B \cup A$ .

**Proving that  $A \cup A = A \cup \emptyset = \emptyset \cup A = A$**  Because it is impossible for  $c$  to be in  $\emptyset$ , the following are equivalent:

- $c \in A$  or  $c \in A$
- $c \in A$  or  $c \in \emptyset$
- $c \in \emptyset$  or  $c \in A$
- $c \in A$

These four statements are equivalent to  $c \in A \cup A$ ,  $c \in A \cup \emptyset$ ,  $c \in \emptyset \cup A$ , and  $c \in A$ , respectively. Therefore,  $A \cup A = A \cup \emptyset = \emptyset \cup A = A$ .

**Exercise 3.1.11**

The next paragraph will be written like a proof of Axiom 3.6 from Axiom 3.7: we will not define again  $A$ ,  $P(x)$ , and so on.

If  $Q(x, y)$  is the statement that  $x = y$  and  $P(x)$  are both true, then by Axiom 3.7 and the fact that for each  $x$  there exists at most one  $y$  such that  $Q(x, y)$ , there exists a set  $B = \{y : x = y \text{ and } P(x) \text{ for some } x \in A\}$ . For any  $z \in B$ , we know that  $z \in A$  and that  $P(z)$  is true. For any  $x \in A$  such that  $P(x)$  is true, we similarly know that  $x \in B$ . Therefore,  $A = B$ .

**3.2 Russell's paradox (Optional)****Exercise 3.2.1**

**Proving that Axiom 3.9 implies Axiom 3.3** We can create a property  $P(x)$  that is always false regardless to the choice of  $x$ . Then the set  $\{x : P(x) \text{ is true}\}$  has no elements, as  $P(x)$  is false all the time.

**Proving that Axiom 3.9 implies Axiom 3.4** If  $P(x)$  is the property that  $x = a$ , then  $\{x : P(x) \text{ is true}\}$  has only the element  $a$ . Similarly, if  $Q(x)$  is the property that  $x = a$  or  $x = b$ , then  $\{x : Q(x) \text{ is true}\}$  has only the elements  $a$  and  $b$ .

**Proving that Axiom 3.9 implies Axiom 3.5** If  $P(x)$  is the property that  $x \in A$  or  $x \in B$ , then  $\{x : P(x) \text{ is true}\}$  exists, so  $A \cup B$  exists.

**Proving that Axiom 3.9 implies Axiom 3.6** If  $Q(x)$  is the property that  $P(x)$  is true and  $x \in A$ , then  $\{x : Q(x) \text{ is true}\} = \{x \in A : P(x)\}$  exists.

**Proving that Axiom 3.9 implies Axiom 3.7** If  $Q(y)$  is the property that there exists some  $x$  such that  $P(x, y)$  is true, then  $\{y : Q(y) \text{ is true}\} = \{y : P(x, y) \text{ is true for some } x \in A\}$  exists.

**Proving that Axiom 3.9 implies Axiom 3.8, assuming that all natural numbers are objects** If  $P(x)$  is the property that  $x$  is a natural number, then  $\mathbf{N} = \{x : P(x) \text{ is true}\}$  exists, by Axiom 3.9.

### 3.3 Functions

#### Exercise 3.3.2

**Proving that  $f$  and  $g$  being injective implies that  $g \circ f$  is too** We will show that if  $x \neq x'$ , then  $(g \circ f)(x) \neq (g \circ f)(x')$ . First, since  $f$  is injective,  $f(x) \neq f(x')$ . Since  $g$  is also injective,  $g(f(x)) \neq g(f(x'))$ . Therefore,  $(g \circ f)(x) \neq (g \circ f)(x')$ .

**Proving that  $f$  and  $g$  being surjective implies that  $g \circ f$  is too** We will show that  $g \circ f$  is surjective by showing that for any  $c \in Z$ , there exists some  $a \in X$  such that  $(g \circ f)(a) = c$ . Since  $g$  is surjective, there exists some  $b \in Y$  such that  $g(b) = c$ . Because  $f$  is also surjective, there exists some  $a \in X$  such that  $f(a) = b$ . Therefore,  $(g \circ f)(a) = c$ , and  $g \circ f$  is surjective.

#### Exercise 3.3.3

**Finding when the empty function is injective** We will prove that the empty function  $f: \emptyset \rightarrow X$  is injective for any  $X$ . Since we cannot find any  $x, x' \in \emptyset$  that are unequal, the empty function is (vacuously) injective.

**Finding when the empty function is surjective** We will prove that the empty function  $f: \emptyset \rightarrow X$  is surjective only when  $X = \emptyset$ . The statement

“For every  $y \in X$ , there exists  $x \in \emptyset$  such that  $f(x) = y$ ”

can only be true when it is impossible that  $y \in X$ , because  $x \in \emptyset$  is impossible. But then we have that  $X = \emptyset$ , and we have proved our claim.

**Finding when the empty function is bijective** We will prove that the empty function  $f: \emptyset \rightarrow X$  is bijective only when  $X = \emptyset$ . Since  $f$  is always injective regardless of the choice of  $X$ , the empty function being bijective is equivalent to it being surjective. Therefore, since  $f$  is surjective precisely when  $X = \emptyset$ , the empty function is bijective only when  $X = \emptyset$ .

#### Exercise 3.3.4

**Showing that if  $g \circ f = g \circ \tilde{f}$  and  $g$  is injective, then  $f = \tilde{f}$**  We will use proof by contradiction. If  $f \neq \tilde{f}$ , then for some  $x \in X$ , we have  $f(x) \neq \tilde{f}(x)$ .

Therefore, since  $g$  is injective,  $(g \circ f)(x) \neq (g \circ \tilde{f})(x)$ , and  $g \circ f \neq g \circ \tilde{f}$ . This is a contradiction, and therefore,  $f = \tilde{f}$ . This statement is not necessarily true if  $g$  is not injective. If  $X = Y = Z = \mathbf{N}$ ,  $f(x) = 0$ ,  $\tilde{f}(x) = 1$ , and  $g(x) = 0$ , then  $f \neq \tilde{f}$ , while  $(g \circ f)(x) = 0 = (g \circ \tilde{f})(x)$ .

**Showing that if  $g \circ f = \tilde{g} \circ f$  and  $f$  is surjective, then  $g = \tilde{g}$**  We will use proof by contradiction again. If  $g \neq \tilde{g}$ , then for some  $y \in Y$ , we have  $g(y) \neq \tilde{g}(y)$ . Since  $f$  is surjective, there exists  $x \in X$  such that  $f(x) = y$ . But then  $(g \circ f)(x) \neq (\tilde{g} \circ f)(x)$ . This is a contradiction, and therefore,  $g = \tilde{g}$ . This statement is not necessarily true if  $f$  is not surjective. If  $X = Y = Z = \mathbf{N}$ ,  $f(x) = 0$ ,  $g(x) = x$ , and  $\tilde{g}(x) = 0$ , then  $g \neq \tilde{g}$ , even though  $(g \circ f)(x) = 0 = (\tilde{g} \circ f)(x)$ .

### Exercise 3.3.6

If  $f(x) = a$ , then by the definition of  $f^{-1}$ ,  $f^{-1}(f(x)) = f^{-1}(a) = x$ . If  $f^{-1}(y) = b$ , then  $f(b) = y$ , so  $f(f^{-1}(y)) = f(b) = y$ .

We can deduce that  $f^{-1}$  is bijective from Exercise 3.3.5 and the fact that the identity map  $\iota_{X \rightarrow X}$  defined as  $\iota_{X \rightarrow X}(x) = x$  for all  $x \in X$  is obviously bijective.

### Exercise 3.3.7

By Exercise 3.3.2,  $g \circ f$  is both injective and surjective, and is therefore bijective. The only thing left we have to prove is that  $(f^{-1} \circ g^{-1} \circ g \circ f)(x) = x$ . But  $(f^{-1} \circ g^{-1} \circ g \circ f)(x) = (f^{-1} \circ f)(x) = x$ , and we have proved what we want.

## 3.4 Images and inverse images

### Challenge to define $f(S)$ using the axiom of specification

We can define  $f(S) := \{y \in Y : \text{there exists } x \in S \text{ such that } f(x) = y\}$ .

### Exercise 3.4.1

First, we will show that any element of the forward image of  $V$  under  $f^{-1}$  is contained in the inverse image of  $V$  under  $f$ . If  $x$  is in the forward image of  $V$

under  $f^{-1}$ , then there exists  $v$  such that  $f^{-1}(v) = x$  and therefore  $f(x) = v$ . But then  $x$  is in the inverse image of  $V$  under  $f$ .

Next, we will show that any element  $v$  of the inverse image of  $V$  under  $f$  is contained in the forward image of  $V$  under  $f^{-1}$ . Since there exists  $x$  such that  $f(x) = v$  and therefore  $f^{-1}(v) = x$ ,  $x$  is in the forward image of  $V$  under  $f$ .

Finally, we have that the forward image of  $V$  under  $f^{-1}$  is equal to the inverse image of  $V$  under  $f$ , and that it is valid to use the notation  $f^{-1}(V)$ .

### Exercise 3.4.6

By Axiom 3.11,  $\{0, 1\}^X$  exists. We can use Axiom 3.7 (the axiom of replacement) to create a set

$$Y = \{f^{-1}(\{1\}) : f \in \{0, 1\}^X\}.$$

Every  $S$  which is a subset of  $X$  is in  $Y$ , because defining  $f(x)$  to be 1 when  $x$  is in  $S$  and to be 0 otherwise means that  $f^{-1}(\{1\}) = S$ . Every element of  $Y$  is a subset of  $X$ , because every element of  $f^{-1}(\{1\})$  has to be in  $X$ .

### Exercise 3.4.9

We will show that

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

We know that  $y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$  if and only if  $y \in A_\beta$  and  $y \in A_\alpha$  for all  $\alpha \in I$ . But the latter statement implies the former, so  $y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$  if and only if  $y \in A_\alpha$  for all  $\alpha \in I$ .

We can do something similar for  $A_{\beta'}$ . Therefore,

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

## 3.5 Cartesian products

### Exercise 3.5.1

First, we will show that  $(x, y) = (x', y')$  implies that both  $x = x'$  and  $y = y'$ . Since  $(x, y) := \{\{x\}, \{x, y\}\}$ , we have to show that  $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$ .

$\{\{x'\}, \{x', y'\}\}$  implies that both  $x = x'$  and  $y = y'$ . If  $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$ , then  $\{x\} \in \{\{x'\}, \{x', y'\}\}$ . In order for this to be true, we must have  $\{x\} = \{x'\}$ , which implies  $x = x'$ , or  $\{x\} = \{x', y'\}$ , which implies  $x' = y' = x$ . But the latter case implies the former case, so we can ignore the latter one. Doing the same with  $\{x, y\}$ ,  $\{x'\}$ , and  $\{x', y'\}$  yields that  $x = x'$  and  $y = y'$ . The converse is true by Axiom 3.2.

We can show that  $X \times Y$  is a set if  $X$  and  $Y$  are both sets using Axiom 3.1 (sets are objects), Axiom 3.7 (axiom of replacement), and Axiom 3.12 (axiom of union). By Axioms 3.1 and 3.7, we can create the set  $Z = \{\{(x, y)\} : y \in Y\} : x \in X\}$ . By Axiom 3.12, we can also create the set  $\bigcup Z$ , which we can define to be  $X \times Y$ .

I won't do the additional challenge.

### Exercise 3.5.6

We will show that if  $A \times B \subseteq C \times D$  and  $A, B, C, D \neq \emptyset$ , then  $A \subseteq C$  and  $B \subseteq D$ . Since  $A$  and  $B$  are nonempty, by the single choice lemma (Lemma 3.1.5), there exist some  $a \in A$  and  $b \in B$ . Now, for any  $x \in A$ , since  $(x, b) \in A \times B \subseteq C \times D$ , we have  $x \in C$ . Similarly, for any  $y \in B$ , since  $(a, y) \in A \times B \subseteq C \times D$ , we have  $y \in D$ . Therefore,  $A \subseteq C$  and  $B \subseteq D$ .

However, we can show that if we remove the assumption that  $A, B, C$ , and  $D$  are nonempty, then this statement is false. If  $A = \{0\}$ ,  $C = \{1\}$ , and  $B = D = \emptyset$ , then  $A \times B$  is empty, as there are no elements of  $B$ . Similarly,  $C \times D = \emptyset$ , and therefore  $A \times B \subseteq C \times D$ . However,  $A \not\subseteq C$ .

If  $A \subseteq C$  and  $B \subseteq D$ , all ordered pairs  $(a, b)$  where  $a \in A$  and  $b \in B$  are in  $C \times D$  because  $a \in C$  and  $b \in D$ . Hence,  $A \times B \subseteq C \times D$ . This statement holds even when  $A, B, C$ , and  $D$  are not all nonempty.

Assuming that  $A \times B = C \times D$  and  $A, B, C, D \neq \emptyset$ ,  $A = C$  and  $B = D$ . This is because  $A \times B \subseteq C \times D$  and  $C \times D \subseteq A \times B$ , and therefore  $A \subseteq C$ ,  $A \supseteq C$ ,  $B \subseteq D$ , and  $B \supseteq D$ .

If we remove the assumption that  $A, B, C, D \neq \emptyset$ , the statement is false. The counterexample from before ( $A = \{0\}$ ,  $C = \{1\}$ ,  $B = D = \emptyset$ ) also works here. We have that  $A \times B = C \times D = \emptyset$ , but  $A \neq C$ .

We have that  $A = C$  and  $B = D$  imply that  $A \times B = C \times D$  because  $A \subseteq C$  and  $B \subseteq D$  imply, together, that  $A \times B \subseteq C \times D$  and the same with  $\subseteq$  replaced by  $\supseteq$ .

**Exercise 3.5.13 (unfinished)**

?

**3.6 Cardinality of sets****Exercise 3.6.2**

If  $X$  has cardinality 0, there exists a bijection  $f: X \rightarrow \{i \in \mathbf{N} : 1 \leq i \leq 0\}$ . The set  $\{i \in \mathbf{N} : 1 \leq i \leq 0\}$  is obviously empty. If  $X \neq \emptyset$ , then there exists some  $x \in X$ . But then  $f(x)$  is in  $\emptyset$ , and therefore  $X$  must be empty.

If  $X$  is empty, the function  $f: X \rightarrow \emptyset$  exists (we don't need to give how to derive  $f(x)$  given  $x$ , since  $x \in X$  is impossible) and is a bijection (injectivity is because there are no elements in the domain at all, and surjectivity is from that the range is empty). Since  $\emptyset = \{i \in \mathbf{N} : 1 \leq i \leq 0\}$ ,  $X$  has cardinality 0.

**Exercise 3.6.3**

We will induct on  $n$ . When  $n = 0$ ,  $\{i \in N : 1 \leq i \leq n\}$  is empty, so we can make  $M$  equal to any natural number, say, 0. Now, assuming that all  $f: \{i \in N : 1 \leq i \leq n\} \rightarrow \mathbf{N}$  satisfy the property that there exists some natural number  $M$  such that  $f(i) \leq M$  for all  $1 \leq i \leq n$ , we will now show that for every  $g: \{i \in N : 1 \leq i \leq n++\} \rightarrow \mathbf{N}$ , there exists some natural number  $a$  such that  $g(i) \leq a$  for all  $1 \leq i \leq n++$ . If  $h: \{i \in N : 1 \leq i \leq n++\} \rightarrow \mathbf{N}$  is defined so that  $h(x) := g(x)$  for all  $x \in \{i \in N : 1 \leq i \leq n++\}$ , there exists some  $b$  such that  $h(i) \leq b$  for all  $1 \leq i \leq n$ . Defining

$a := \begin{cases} b & \text{if } g(n++) \leq b \\ g(n++) & \text{if } g(n++) > b \end{cases}$  ensures that  $a$  will always be greater than or equal to all elements of  $g(\{i \in N : 1 \leq i \leq n++\})$ .

**Exercise 3.6.5**

We will define  $f: A \times B \rightarrow B \times A$  by  $f(a, b) := (b, a)$ , where  $a \in A$  and  $b \in B$ . Injectivity of  $f$  comes from  $(b, a) = (d, c)$  implying that  $b = d$  and  $a = c$ , further implying that  $(a, b) = (c, d)$ . Surjectivity comes from the fact that



for every  $(b, a) \in B \times A$  has a corresponding ordered pair (namely,  $(a, b)$ ) in  $A \times B$  where  $f(a, b) = (b, a)$ .

Now, we will prove that  $ab = ba$ . We will define  $A$  as  $\{i \in \mathbf{N} : 1 \leq i \leq a\}$  and  $B$  as  $\{j \in \mathbf{N} : 1 \leq j \leq b\}$ . Since  $A$  has equal cardinality with itself and the same for  $B$  (Proposition 3.6.4),  $A$  and  $B$  are both finite and have the cardinalities  $a$  and  $b$ , respectively. By Proposition 3.6.14(e),  $\#(A) \cdot \#(B) = \#(A \times B)$  and  $\#(B) \cdot \#(A) = \#(B \times A)$ . We also know that  $\#(A \times B) = \#(B \times A)$ , and therefore  $ab = ba$ .

### Exercise 3.6.7 (unfinished)

?

### Exercise 3.6.10 (unfinished)

**Attempt 1** We induct on  $n$ . (*I don't think we can induct, actually, since  $\#(\bigcup_{i \in \{1, \dots, n+1\}} A_i) > n+1$  does not necessarily imply  $\#(\bigcup_{i \in \{1, \dots, n\}} A_i) > n$ .*) When  $n = 0$ ,  $\#(\bigcup_{i \in \{1, \dots, n\}} A_i) = \#(\emptyset) = 0$ . It is obvious that  $0 \not> 0$ , so the pigeonhole principle holds vacuously when  $n = 0$ . When the pigeonhole principle is true for  $n$  finite sets, now we will show that it also is true for  $A_1, \dots, A_{n++}$ . By Proposition 3.6.14(b),

$$\#(\bigcup_{i \in \{1, \dots, n++\}} A_i) \leq \#(\bigcup_{i \in \{1, \dots, n\}} A_i) + \#(A_{n++}).$$

# Chapter 4

## Integers and rationals

### 4.1 The integers

#### Exercise 4.1.1

**Reflexive** If  $a, b \in \mathbf{N}$ ,  $a + b = b + a$ , so by Definition 4.1.1,  $a - b = a - b$ .

**Symmetric** If  $a - b = c - d$ ,  $a + d = c + b$ . Now, since  $c + b = a + d$ ,  $c - d = a - b$ .

#### Exercise 4.1.3

We know that  $-1 = -(1 - 0) = 0 - 1$ . If  $a = a_1 - a_2$ ,  $(-1) \cdot a = (0 \cdot a_1 + 1 \cdot a_2) - (0 \cdot a_2 + 1 \cdot a_1) = a_2 - a_1 = -a$ .

#### Exercise 4.1.4

I will write  $x = a - b$ ,  $y = c - d$ , and  $z = e - f$ , where  $a, b, c, d, e, f \in \mathbf{N}$ .

First, we will prove  $x + y = y + x$ :

$$\begin{aligned}x + y &= (a - b) + (c - d) \\&= (a + c) - (b + d) \\y + x &= (c - d) + (a - b) \\&= (c + a) - (d + b)\end{aligned}$$

and therefore, because of natural number addition's commutativity,  $x + y = y + x$ .

Next, we will prove that  $(x + y) + z = x + (y + z)$ :

$$\begin{aligned}
 (x + y) + z &= ((a - b) + (c - d)) + (e - f) \\
 &= ((a + c) - (b + d)) + (e - f) \\
 &= ((a + c) + e) - ((b + d) + f) \\
 x + (y + z) &= (a - b) + ((c - d) + (e - f)) \\
 &= (a - b) + ((c + e) - (d + f)) \\
 &= (a + (c + e)) - (b + (d + f))
 \end{aligned}$$

and therefore, because of natural number addition's associativity,  $(x + y) + z = x + (y + z)$ .

Now, we will prove that  $x + 0 = 0 + x = x$ . We have proved above that  $x + y = y + x$  and therefore  $x + 0 = 0 + x$ , so we only need to show  $x + 0 = x$ :

$$\begin{aligned}
 x + 0 &= (a - b) + (0 - 0) \\
 &= (a + 0) - (b + 0).
 \end{aligned}$$

We know that  $a + 0 = a$  and  $b + 0 = b$ , so  $x + 0 = x$ , and hence  $x + 0 = 0 + x = x$ .

Next, we will prove that  $x + (-x) = (-x) + x = 0$ . We have already proved that  $x + y = y + x$  and therefore  $x + (-x) = (-x) + x$ , so the only thing left is to show  $x + (-x) = 0$ .

$$\begin{aligned}
 x + (-x) &= (a - b) + (b - a) \\
 &= (a + b) - (b + a) \\
 &= (a + b) - (a + b) \\
 &= 0 - 0 \\
 &= 0.
 \end{aligned}$$

Therefore,  $x + (-x) = (-x) + x = 0$ .

To show that  $xy = yx$ , we will use the definition of integer multiplication:

$$\begin{aligned}
 xy &= (a - b)(c - d) \\
 &= (ac + bd) - (ad + bc) \\
 yx &= (c - d)(a - b) \\
 &= (ca + db) - (cb + da) \\
 &= (ac + bd) - (ad + bc) \\
 &= xy.
 \end{aligned}$$

Note that  $(xy)z = x(yz)$  was proved in the section about proving Proposition 4.1.6 on page 4.1.

For the statement  $x1 = 1x = x$ , we only have to prove that  $1x = x$ , because we have already proved that  $xy = yx$  earlier:

$$\begin{aligned} 1x &= (1 - 0) \cdot (a - b) \\ &= (1a + 0b) - (1b + 0a) \\ &= a - b \\ &= x. \end{aligned}$$

Now we will show that  $x(y + z) = xy + xz$ :

$$\begin{aligned} x(y + z) &= (a - b)((c - d) + (e - f)) \\ &= (a - b)((c + e) - (d + f)) \\ &= (a(c + e) + b(d + f)) - (a(d + f) + b(c + e)) \\ &= (ac + ae + bd + bf) - (ad + af + bc + be) \\ &= ((ac + bd) - (ad + bc)) + ((ae + bf) - (af + be)) \\ &= xy + xz. \end{aligned}$$

We can show that  $(y + z)x = yx + zx$  using  $xy = yx$  and  $x(y + z) = xy + xz$  by simply swapping  $x(y + z)$  with  $(y + z)x$ ,  $xy$  with  $yx$ , and  $xz$  with  $zx$ .

### Exercise 4.1.7

(a) If  $a > b$ , we will show that  $a - b$  is a positive natural number. Since  $a \geq b$ , there exists  $d \in \mathbf{N}$  such that  $a = b + d$ . Subtracting  $b$  from both sides gives  $a - b = (b + d) - b$ . We know that

$$(b + d) - b = (b + d) + (-b) = (d + b) + (-b) = d + (b + (-b)) = d + 0 = d.$$

Therefore,  $a - b = d$ . Since  $d = 0$  implies  $a = b + 0 = b$ , which contradicts  $a > b$ ,  $a - b$  must be a nonzero, or positive, natural number.

Now we will show that if  $a - b$  is a positive natural number,  $a > b$ . We know that  $a \geq b$ , since  $a = a + (-b) + b = (a - b) + b$ . We also know that  $a \neq b$  because otherwise,  $a - b$  would be 0. Therefore,  $a > b$ .

(b) When  $a > b$ , by (a),  $a - b \neq 0$ . Since  $c - c = c + (-c) = 0$  (by Proposition 4.1.6 and the definition of subtraction on page 79),  $a - b + c - c \neq 0 + c - c = 0$ .

By Proposition 4.1.6 and the fact that  $-(-x) = x$  (if  $x = a - b$ ,  $-x = b - a$ , and  $-(-x)$  is just  $a - b$  again),

$$\begin{aligned} a - b + c - c &= a + (-b) + c + (-c) \\ &= (a + (-c)) + (-b) + (-(-c)) = (a - c) - (b - c). \end{aligned}$$

Therefore,  $a + c > b + c$ .

(c) Since  $a > b$ ,  $a - b$  is a positive natural number. Since  $c$  is also a positive natural number, by Lemma 2.3.3,  $(a - b)c$  is also a positive natural number. The distributive law for natural numbers (Proposition 2.3.4) tells us that  $(a - b)c = ac - bc$ , and therefore  $ac > bc$ .

(d) Since  $a > b$ ,  $a - b$  is a positive natural number. But this means that  $-a < -b$ , since  $-b = (a - b) + (-a)$ .

(e) Since  $a > b$  and  $b > c$ ,  $a - b$  and  $b - c$  are both positive natural numbers. By Proposition 2.2.8,  $(a - b) + (b - c)$  is also a positive natural number. We know that

$$(a - b) + (b - c) = a + (-b) + b + (-c) = a + 0 + (-c) = a + (-c) = a.$$

Therefore,  $a > c$ .

(f) First, we will show that at least one of the statements  $a > b$ ,  $a < b$ , and  $a = b$  is true. By Lemma 4.1.5 (trichotomy of integers),  $a - b$  is either zero, positive, or negative. If it is 0,

$$a = (a - b) + b = 0 + b = b.$$

If  $a - b$  is positive, by part (a),  $a > b$ . When  $a - b$  is negative, it is the negation of some positive natural number  $d$ . Since  $-(-d)$  is equal to  $d$  itself and  $-(a - b) = b - a$ ,  $d = b - a$ . Therefore,  $b > a$ . We have now proved that at least one of the statements  $a > b$ ,  $a < b$ , and  $a = b$  is true.

Now, we will show that no two of these statements can be true at the same time. It is obvious from the definition of ordering on the integers (Definition 4.1.10) that  $a = b$  cannot be true at the same time as  $a > b$  or  $a < b$ . Now the only thing left is to prove that  $a > b$  and  $a < b$  cannot both be true. We will prove this using proof by contradiction. The statement that  $a > b$

is equivalent to  $a - b$  being positive, and  $a < b$  is equivalent to  $b - a$  being positive. But by Lemma 4.1.5 (the trichotomy of integers) and the fact that  $b - a = -(a - b)$  give us a contradiction, as  $b - a$  cannot both be positive and negative. We have now proved that exactly one of the statements  $a > b$ ,  $a < b$ , or  $a = b$  is true.

### Exercise 4.1.8

If  $P(n)$  is the property that  $n > -1$ ,  $P(0)$  is true, because  $0 = (-1) + 1$  and  $0 \neq -1$ . Also, if  $P(n)$  is true, since  $n++ > n$ , by Lemma 4.1.11(e),  $P(n++)$  is true. However,  $P(-1)$  is obviously false, since  $-1 = -1$ .

## 4.2 The rationals

### Exercise 4.2.1

**Proving that equality is reflexive (for rational numbers)** We know that for any  $a, b \in \mathbf{Z}$  where  $b \neq 0$ ,  $ab = ab$ . Therefore,  $a//b = a//b$ .

**Proving that equality is symmetric** If for  $a, b, c, d \in \mathbf{Z}$  that satisfy  $b, d \neq 0$ , we have  $a//b = c//d$ , then  $ad = cb$ . Hence,  $cb = ad$ , and  $c//d = a//b$ .

**Proving that equality is transitive** If  $a, b, c, d, e, f \in \mathbf{Z}$  where  $b, d, f \neq 0$ ,  $a//b = c//d$ , and  $c//d = e//f$ , then we must have  $ad = cb$  and  $cf = ed$ . Hence,  $ad \cdot cf = cb \cdot ed$ . By Corollary 4.1.9, we can remove  $c$  and  $d$  from both sides:

$$a \cdot f = b \cdot e.$$

Therefore, by Definition 4.2.1,  $a//b = e//f$ , and we have proved that rational number equality is reflexive, symmetric, and transitive.

### Exercise 4.2.3

I will assume that  $x = a//b$ ,  $y = c//d$ , and  $z = e//f$ , where  $a, b, c, d, e, f \in \mathbf{Z}$  and  $b, d, f \neq 0$ .

First, we will prove that  $x + y = y + x$ :

$$\begin{aligned}
 x + y &= (a/b) + (c/d) \\
 &= (ad + bc)/(bd) \\
 y + x &= (c/d) + (a/b) \\
 &= (cb + da)/(db) \\
 &= (ad + bc)/(bd) \\
 &= x + y.
 \end{aligned}$$

*The identity  $(x + y) + z$  was proved on page 84 already.*

Next, we will prove that  $x + 0 = 0 + x = x$ . We have already proved that  $x + y = y + x$ , so we only need to show that  $x + 0 = x$ :

$$\begin{aligned}
 x + 0 &= (a/b) + (0/1) \\
 &= (a \cdot 1 + b \cdot 0)/(b \cdot 1) \\
 &= a/b \\
 &= x.
 \end{aligned}$$

For the next identity,  $x + (-x) = (-x) + x = 0$ , we have already shown that  $x + y = y + x$ , so we only need to prove  $x + (-x) = 0$ :

$$\begin{aligned}
 x + (-x) &= (a/b) + ((-a)/b) \\
 &= (ab + b \cdot (-a))/(b \cdot b) \\
 &= 0/(b \cdot b) \\
 &= 0.
 \end{aligned}$$

The last step is valid because  $0 \cdot 1 = 0 \cdot (b \cdot b)$ .

Now we prove that  $xy = yx$ :

$$\begin{aligned}
 xy &= (a/b) \cdot (c/d) \\
 &= (ac)/(bd) \\
 yx &= (c/d) \cdot (a/b) \\
 &= (ca)/(db) \\
 &= xy.
 \end{aligned}$$

We now prove that  $(xy)z = x(yz)$ :

$$\begin{aligned}
 (xy)z &= ((a/b) \cdot (c/d)) \cdot (e/f) \\
 &= ((ac)/(bd)) \cdot (e/f) \\
 &= (ace)/(bdf) \\
 x(yz) &= (a/b) \cdot ((c/d) \cdot (e/f)) \\
 &= (a/b) \cdot ((ce)/(df)) \\
 &= (ace)/(bdf) \\
 &= (xy)z.
 \end{aligned}$$

Next, we prove  $x1 = 1x = x$ . We have already proved that  $xy = yx$ , so we only need to prove  $x \cdot 1 = x$ :

$$\begin{aligned}
 x \cdot 1 &= (a/b) \cdot (1/1) \\
 &= (a \cdot 1)/(b \cdot 1) \\
 &= a/b \\
 &= x.
 \end{aligned}$$

The next identity to prove is  $x(y + z) = xy + xz$ :

$$\begin{aligned}
 x(y + z) &= (a/b)((c/d) + (e/f)) \\
 &= (a/b)((cf + de)/(df)) \\
 &= (a(cf + de))/(bdf) \\
 &= (acf + ade)/(bdf) \\
 xy + xz &= ((ac)/(bd)) + ((ae)/(bf)) \\
 &= (acbf + bdae)/(b^2df) \\
 &= (acf + dae)/(bdf) \\
 &= x(y + z).
 \end{aligned}$$

For the identity  $(y + z)x = yx + zx$ , we can just apply  $xy = yx$  and  $x(y + z) = xy + xz$  together.

Finally, we will prove that  $xx^{-1} = x^{-1}x = 1$ , when  $x \neq 0$ . We know that  $x^{-1}$  exists because otherwise,  $a \neq 0$ , and therefore  $x$  would be 0. Since  $xy = yx$ , we only need to show that  $xx^{-1} = 1$ :

$$\begin{aligned}
 xx^{-1} &= (a/b)(b/a) \\
 &= (ab)/(ab) \\
 &= 1.
 \end{aligned}$$



(The last step comes from Definition 4.2.1).

### Exercise 4.2.5

(a) By Lemma 4.2.7, we know that exactly one of the statements “ $x - y = 0$ ”, “ $x - y$  is positive”, or “ $x - y$  is negative” is true. We now can use Definition 4.2.8 to deduce that exactly one of the statements  $x = y$ ,  $x > y$ , or  $x < y$  is true.

(b) If  $x < y$ ,  $x - y$  must be a negative rational number. Hence,  $y - x$  is positive, and  $y > x$ . Similarly, we can show that  $y > x$  implies  $x < y$ .

(c) By (b), we know that  $y > x$  and  $z > y$ . Thus,  $y - x$  and  $z - y$  are positive. If  $y - x = a/b$  and  $z - y = c/d$  (where  $a$ ,  $b$ ,  $c$ , and  $d$  are positive; this is possible by Definition 4.2.6),

$$\begin{aligned} z - x &= z - y + y - x \\ &= (ad + bc)/(bd) \end{aligned}$$

which is positive, and therefore  $z > x$ . We now have proved that  $x < z$ .

(d) By (b), we know that  $y > x$ , and hence,  $y - x$  is positive. If  $y - x = a/b$ , we know that

$$\begin{aligned} y + z - (x + z) &= y + z - x - z \\ &= y - x \end{aligned}$$

and therefore  $y + z > x + z$ . Hence,  $x + z < y + z$ .

(e) Because of part (b), we know that  $y > x$ , and thus,  $y - x$  is positive. Assuming that  $y - x = a/b$  and  $z = c/d$  (where  $a$ ,  $b$ ,  $c$ , and  $d$  are positive; this is possible by Definition 4.2.6),

$$\begin{aligned} yz - xz &= (y - x)z \\ &= (a/b)(c/d) \\ &= (ac)/(bd) \end{aligned}$$

and hence,  $yz > xz$  and  $xz < yz$ .

**Exercise 4.2.6**

We know that  $x - y$  is positive. Assuming that  $y - x = a//b$  and  $z = c//d$  (where  $a, b, c$ , and  $d$  are positive; this is always possible by Definition 4.2.6), we have that

$$\begin{aligned} yz - xz &= (y - x)z \\ &= (ac)//(bd) \end{aligned}$$

and therefore,  $xz > yz$ .

**4.3 Absolute value and exponentiation****Exercise 4.3.3(a)**

First, we prove that  $x^n x^m = x^{n+m}$ . We induct on  $m$ . When  $m = 0$ ,  $x^m = 1$ , and hence,  $x^n x^m = x^n$ . But  $x^{n+m} = x^{n+0} = x^n$ , and therefore,  $x^n x^m = x^{n+m}$  is true when  $m = 0$ . When  $x^n x^m = x^{n+m}$ ,

$$x^n x^{m++} = x^n x^m \cdot x = x^{n+m} \cdot x = x^{n+(m++)},$$

and we have proved that  $x^n x^m = x^{n+m}$  for all  $n$  and  $m$ .

Next, we prove that  $(x^n)^m = x^{nm}$ , again by induction on  $m$ . When  $m = 0$ , it is obvious that both  $(x^n)^m$  and  $x^{nm}$  are both 1, since anything to the power of 0 is 1. Now, we prove that if  $(x^n)^m = x^{nm}$  for some  $n$  and  $m$ ,  $(x^n)^{m++} = x^{n(m++)}$ . We know that  $(x^n)^{m++} = x^n \cdot (x^n)^m$ . Using Definition 4.3.9, we get that

$$x^n \cdot (x^n)^m = (x^n)^{m++}.$$

We have finished proving that  $(x^n)^m = x^{nm}$  for all  $n$  and  $m$ .

We will now prove that  $(xy)^n = x^n y^n$ , again using induction, except on  $n$ . When  $n = 0$ , it is obvious that both sides are equal to 1. If  $(xy)^n = x^n y^n$ , we can multiply both sides of the equation by  $xy$  to get that

$$(xy)^{n++} = x^{n++} y^{n++}.$$

We have completed showing that  $(xy)^n = x^n y^n$ .

**Exercise 4.3.5**

We use induction on  $N$ , with base case  $N = 1$ . For the base case,  $2^1 = 2 > 1$ . For the inductive step, we multiply both sides of  $2^N \geq N$  by 2 to get  $2^{N++} \geq 2N$ . Since  $2N = N + N \geq N + 1$ ,  $2^{N++} \geq N++$ . We have finished proving that  $2^N \geq N$  for all positive integers  $N$ .

**4.4 Gaps in the rational numbers****Exercise 4.4.1**

If  $x = \frac{a}{b}$  and  $a, b \in \mathbf{N}$ , by Proposition 2.3.9 (the Euclidean algorithm), there exist  $m$  and  $r$  in  $\mathbf{N}$  such that  $0 \leq r < b$  and  $a = mb + r$ . Dividing both sides by  $b$  yields that

$$x = m + \frac{r}{b}.$$

Because  $0 \leq r < b$ ,  $0 \leq \frac{r}{b} < 1$ . Adding  $m$  to 0,  $\frac{r}{b}$ , and 1 gives that  $m \leq m + \frac{r}{b} < m + 1$ . But  $m + \frac{r}{b} = x$ , so we have proved that  $m \leq x < m + 1$ . To prove that there exists some natural number  $N$  greater than  $x$ , we set  $N$  to be  $m + 1$  when  $m + 1 \geq 0$  and 0 when  $m + 1 < 0$ . (This works because when  $m + 1 \geq 0$ , it is a natural number and greater than  $x$ , as we proved before. When  $m + 1 < 0$ ,  $N = 0$  works because  $0 > m + 1 > x$ .)

**Exercise 4.4.3**

**Showing that every natural number is either even or odd, but not both** Let  $n$  be a natural number. By the Euclidean algorithm (Proposition 2.3.9), every natural number is equal to  $2m + r$ , where  $m, r \in \mathbf{N}$  and  $0 \leq r < 2$ . This implies that every natural number is either even or odd. Now, we show that  $n$  cannot be both even and odd, by showing that having both at the same time will cause a contradiction. A natural number  $n$  being both even and odd must be equal to  $2a$  and  $2b + 1$  at the same time, where  $a, b \in \mathbf{N}$ . If  $a > b$ ,  $2a > 2b + 1$ , and we have a contradiction, because then  $2a \neq 2b + 1$ . When  $b \geq a$ ,  $2b + 1 > 2a$ , and we also have a contradiction.

**Showing that an odd natural number  $n$  has an odd square  $n^2$**  We know that  $n = 2m + 1$  for some  $m \in \mathbf{N}$ . Therefore,  $n^2 = 4m^2 + 2m + 1 = 2m(2m + 1) + 1$ , and therefore,  $n^2$  is also odd.

# Chapter 5

## The real numbers

### 5.1 Cauchy sequences

#### Exercise 5.1.1

Since  $(a_n)_{n=1}^{\infty}$  is a Cauchy sequence, it is eventually 1-steady. Thus, for some  $m$ ,  $(a_n)_{n=m}^{\infty}$  is 1-steady. By Lemma 5.1.14, the finite sequence  $a_1, \dots, a_{m-1}$  is bounded. Also,  $(a_n)_{n=m}^{\infty}$  is bounded by  $|a_m| + 1$ . Therefore,  $(a_n)_{n=1}^{\infty}$  is bounded by the greater of the bounds for  $a_1, \dots, a_{m-1}$  and  $(a_n)_{n=m}^{\infty}$ .

### 5.2 Equivalent Cauchy sequences

#### Exercise 5.2.1

We will only prove that  $(a_n)_{n=1}^{\infty}$  being a Cauchy sequence implies that  $(b_n)_{n=1}^{\infty}$  is also a Cauchy sequence; we can replace  $a_n$  with  $b_n$  and  $b_n$  with  $a_n$ , anyways.

For any rational  $\epsilon > 0$ , there exist natural numbers  $m_1$  and  $m_2$  such that  $(a_n)_{n=m_1}^{\infty}$  is  $\frac{\epsilon}{3}$ -steady and such that  $(a_n)_{n=m_2}^{\infty}$  and  $(b_n)_{n=m_2}^{\infty}$  are  $\frac{\epsilon}{3}$ -close. If  $m$  is defined to be  $m_1$  when  $m_1 \geq m_2$  and  $m_2$  when  $m_1 < m_2$ ,  $(b_n)_{n=m}^{\infty}$  is  $\epsilon$ -steady, because for any  $x, y \in \mathbf{N}$  greater than or equal to  $m$ ,

$$d(a_x, a_y), d(a_x, b_x), d(a_y, b_y) \leq \frac{\epsilon}{3}.$$

## 5.3 The construction of the real numbers

### Exercise 5.3.1

We will assume here that  $\epsilon$  is a rational number greater than 0.

We know that  $(a_n)_{n=1}^\infty$  is always  $\epsilon$ -close to itself. Therefore,  $x = x$ .

It is obvious that if  $(a_n)_{n=1}^\infty$  is eventually  $\epsilon$ -close to  $(b_n)_{n=1}^\infty$ ,  $(b_n)_{n=1}^\infty$  is eventually  $\epsilon$ -close to  $(a_n)_{n=1}^\infty$ . Hence,  $x = y$  implies  $y = x$ .

We also know that when  $(a_n)_{n=1}^\infty$  is eventually  $\epsilon/2$ -close to  $(b_n)_{n=1}^\infty$  and  $(b_n)_{n=1}^\infty$  is eventually  $\epsilon/2$ -close to  $(c_n)_{n=1}^\infty$ ,  $(a_n)_{n=1}^\infty$  is eventually  $\epsilon$ -close to  $(c_n)_{n=1}^\infty$ . (This is because the same thing holds if we replace sequences with rational numbers and “eventually  $\epsilon/2$ -close” with “ $\epsilon$ -close”). Thus,  $x = y$  and  $y = z$  together imply  $x = z$ .

### Exercise 5.3.3

If  $\text{LIM}_{n \rightarrow \infty} a = \text{LIM}_{n \rightarrow \infty} b$ , we will prove that  $a = b$ . Assume for the sake of contradiction that  $a \neq b$ . Then,  $|a - b| \neq 0$ . But then  $a, a, a, a, \dots$  and  $b, b, b, b, \dots$  are not eventually  $\frac{|a-b|}{2}$ -close. But this contradicts our assumption that  $\text{LIM}_{n \rightarrow \infty} a = \text{LIM}_{n \rightarrow \infty} b$ , and thus  $a = b$ .

To prove that  $a = b$  implies  $\text{LIM}_{n \rightarrow \infty} a = \text{LIM}_{n \rightarrow \infty} b$ , we can use the fact that  $a$  is  $\epsilon$ -close to itself for any rational  $\epsilon > 0$ .

### Exercise 5.3.5

For any rational  $\epsilon > 0$ , there is some  $m$  greater than  $\frac{1}{\epsilon}$ , by Proposition 4.4.1. Since any natural number  $m'$  greater than or equal to  $m$  satisfies

$$\frac{1}{m'} \leq \frac{1}{m} \leq \epsilon,$$

the sequence  $1/1, 1/2, 1/3, \dots$  is eventually  $\epsilon$ -close to the sequence  $0, 0, 0, \dots$ . Therefore,  $\text{LIM}_{n \rightarrow \infty} 1/n = 0$ .

## 5.4 Ordering the reals

### Exercise 5.4.2

(a) By Proposition 5.4.4, exactly one of the statements “ $x - y$  is positive”, “ $x - y$  is equal to 0”, or “ $x - y$  is negative” is true. These statements

correspond to “ $x > y$ ”, “ $x = y$ ”, and “ $x < y$ ” respectively.

(b) By Proposition 5.4.4,  $x - y$  is negative if and only if  $y - x$  is positive. Hence,  $x < y$  if and only if  $y > x$ .

(c) Since  $x < y$  and  $y < z$ ,  $y > x$  and  $z > y$  (by part (b)). Thus,  $y - x$  and  $z - y$  are positive. We know that  $z - y + y - x = z - x$ . Therefore, by Proposition 5.4.4,  $z - x$  is positive and  $z > x$ . Using part (b) again, we get that  $x < z$ .

(d) By part (b),  $y - x$  is positive. Since  $y + z - (x + z)$  is equal to  $y - x$ ,  $x + z < y + z$ .

### Exercise 5.4.4

By Proposition 5.4.12, there exist some integers  $a$  and  $b$  such that  $a, b \neq 0$  and  $0 < \frac{a}{b} \leq x$ . Since

$$0 < \frac{1}{2b|a|/a} \leq \frac{|a|}{2b|a|/a} = \frac{a}{2b} < \frac{a}{b}$$

and  $2b|a|/a$  is a positive integer, we can define  $N$  as  $2b|a|/a$ .

### Exercise 5.4.5

By Corollary 5.4.13 (the Archimedean property), there exists a positive integer  $b$  such that  $b(y - x) > 1$ . Thus,  $a := \begin{cases} y - 1 & \text{if } y \in \mathbf{Z} \\ \lfloor y \rfloor & \text{if } y \notin \mathbf{Z} \end{cases}$  is greater than  $bx$  and less than  $by$ . Therefore,  $x < \frac{a}{b} < y$ .

## 5.5 The least upper bound property

### Exercise 5.5.1

First, we show that  $-M$  is a lower bound for  $-E$ . We assume for the sake of contradiction that some element of  $E$ , which we call  $-a$ , is less than  $-M$ . Then, we have that  $a > M$ . But  $a$  is in  $E$ , and therefore,  $M \neq \sup(E)$ .

Thus, our assumption that  $-a < -M$  is wrong, and  $-M$  actually is a lower bound for  $-E$ .

Next, we show that no other lower bound for  $-E$ ,  $-N$ , is greater than  $-M$ . We assume for the sake of contradiction that  $-N > -M$ . Then,  $N < M$  and  $N$  is an upper bound for  $E$  lower than  $M$ . But this is impossible from the definition of least upper bound. Hence,  $-N$  is actually equal to or less than  $-M$ . This means that  $-M = \inf(-E)$ .