Chapter 1

Section 1.1

N/A

Section 1.2

N/A

Chapter 2

Section 2.1

N/A

Section 2.2

Exercise 2.2.1

We shall use induction on a, fixing b and c. First, we shall prove the base case. If $a = 0$, $a + b = 0 + b = b$ by the definition of addition of natural numbers (Definition 2.2.1). Therefore, $(a + b) + c = b + c$. Also by Definition 2.2.1, $a + (b + c) = 0 + (b + c) = b + c$. We now have $(a + b) + c = b + c = a + (b + c)$.

Now suppose that $(a + b) + c = a + (b + c)$. We then have, by Definition 2.2.1, that $((a{++}) + b) + c = ((a + b){++}) + c = ((a + b) + c){++} = (a + (b + c)){++}$. Because of Definition 2.2.1 again, $(a + (b + c)){++} = (a{++}) + (b + c)$. Now we have $((a{++}) + b) + c = (a{++}) + (b + c)$, which closes the induction.

Exercise 2.2.2

We shall use induction on a. The base case is vacuously true since 0 is not positive. For the inductive step, we have to show that $a{++}$ being positive implies that $a{++} = b{++}$ for exactly one natural number b if a being positive implies that $a = c{+}{+}$ for exactly one natural number c. But a is equal to b if $a{+}{+}$ is equal to $b{+}{+}$, by Axiom 2.4. This closes the induction.

Exercise 2.2.3

(a) By Lemma 2.2.2, $a = a + 0$, so $a \geq a$.
(b) Because $a \geq b$ and $b \geq c$, $a = b + n$ and $b = c + m$ for some n and m. Therefore, $a = b + n = (c + m) + n$. By Proposition 2.2.5, $(c + m) + n = c + (m + n)$, so $a = c + (m + n)$. $n + m$ is a natural number, so $a \geq c$.
(c) Because $a \geq b$ and $b \geq a$, $a = b + n$ and $b = a + m$. Therefore, $a = b + n = (a + m) + n$. By Proposition 2.2.5, $(a + m) + n = a + (m + n)$. Now we have $a = a + (m + n)$. By Lemma 2.2.2, $a = a + 0$, so we can use Proposition 2.2.6 to get $0 = m + n$. Using Corollary 2.2.9, we can deduce that n and m are both equal to 0. Therefore, $a = b$.
(d) If $a \geq b$, $a = b + d$ for some d. Because of this, $a + c = (b + d) + c$. By Propositions 2.2.4 and 2.2.5, $(b+d)+c = b+(d+c) = b+(c+d) = (b+c)+d$. Therefore, $a + c \geq b + c$.

Conversely, if $a+c \geq b+c$, $a+c = (b+c)+d$ for some d. By Propositions 2.2.4 and 2.2.5, $(b + c) + d = b + (c + d) + b + (d + c) = (b + d) + c$. Using Proposition 2.2.6, we get $a = b + d$, and therefore, $a \geq b$.
(f) If $a < b$, $b = a + c$ for some c, and $b \neq a$. Therefore, c is positive, because otherwise $b = a + 0 = a$. ($a + 0 = a$ because of Lemma 2.2.2.)

Conversely, if $b = a + c$ for some positive number c, we know that $a \leq b$. If $a = b$, $a = a + c$. By Lemma 2.2.2, $a = a + 0$, so $a + 0 = a + c$. We can use Proposition 2.2.6 to get $0 = c$, but c is positive. Therefore, $a \neq b$, and combined with $a \leq b$, we get $a < b$.
(e) We first prove that $a < b$ implies that $a{+}{+} \leq b$. By (f), $b = a + c$ for some positive c. By Lemma 2.2.10, exactly one d exists such that $d{+}{+} = c$. Therefore, $b = a+d{+}{+}$. By Lemma 2.2.3 and Definition 2.2.1, $b = a+d{+}{+} = (a + d){+}{+} = (a{+}{+}) + d$. Therefore, $a{+}{+} \leq b$.

Conversely, if $a{+}{+} \leq b$, we need to prove that $a < b$. Since $a{+}{+} = a + 1 \geq a$, by (b), $a \leq b$. If $a = b$, then $a{+}{+} \leq a$, and we get, from (c), that $a{+}{+} = a$. This is a contradiction, from Axiom 2.3 if $a = 0$ and Axiom 2.4 otherwise. Therefore, $a < b$.

Exercise 2.2.6 (unfinished)

We use induction on n. When $n = 0$, $m \le n$ implies that $0 = m + a$. But by Corollary 2.2.9, $m = 0 = n$, so $P(n)$ being true implies that $P(m)$ is true.

If $a \le n$ and $P(n)$ together imply $P(a)$, we now need to prove that $b \le n{+}{+}$ and $P(n{+}{+})$ together imply $P(b)$. Since $b \le n{+}{+}$, $n{+}{+} = b + c$ for some c. If c is positive, by Lemma 2.2.10, $n{+}{+} = b + d{+}{+}$ for some d, so by Lemma 2.2.3 and Axiom 2.4, $n = b + d$, and $b \le n$. Therefore, $b \le n{+}{+}$ implies that $b \le n$ or $n{+}{+} = b + 0 = b$. Because of Lemmas 2.2.2 and 2.2.3, $n{+}{+} = (n + 0){+}{+} = n + 0{+}{+} = n + 1$. Since $P(n{+}{+})$ implies $P(n)$, $P(n)$ is true. But then **?**

Section 2.3

Exercise 2.3.1

First, we prove that $n \cdot 0 = 0$. We induct on n. When $n = 0$, $n \cdot 0 = 0 \cdot 0 = 0$, because of Definition 2.3.1. For the inductive hypothesis, we assume $n \cdot 0 = 0$. Then we need to prove that $n{+}{+} \cdot 0 = 0$. By Definition 2.3.1, $n{+}{+} \cdot 0 = n \cdot 0 + 0 = n \cdot 0$. But we already know that $n \cdot 0 = 0$, so $n{+}{+} \cdot 0 = 0$. This closes the induction.

Next, we prove that $n \cdot m{+}{+} = nm + n$ using induction on n. When $n = 0$, $0 \cdot m{+}{+} = 0 = 0 \cdot m + 0$. Now, if $n \cdot m{+}{+} = nm + n$, we need to prove that $n{+}{+} \cdot m{+}{+} = n{+}{+} \cdot m + n{+}{+}$. We can deduce from $n \cdot m{+}{+} = nm + n$ that $n{+}{+} \cdot m{+}{+} = (n \cdot m{+}{+}) + m{+}{+} = nm + n + m{+}{+} = nm + (n + m){+}{+} = nm + n{+}{+} + m = (n{+}{+}) \cdot m + n{+}{+}$. This closes the induction.

Now we can prove that $nm = mn$. We induct on m. When $m = 0$, we have $n \cdot 0 = 0 = 0 \cdot n$. For the inductive hypothesis, we assume $nm = mn$. Now we need to prove $n \cdot m{+}{+} = m{+}{+} \cdot n$. We know that $n \cdot m{+}{+} = nm + n$ and $m{+}{+} \cdot n = mn + n$. But $nm = mn$, so $n \cdot m{+}{+} = m{+}{+} \cdot n$. This closes the induction.

Exercise 2.3.4

By Proposition 3.4, $(a + b)^2 = (a + b) \cdot (a + b) = (a + b) \cdot a + (a + b) \cdot b = a \cdot a + b \cdot a + a \cdot b + b \cdot b$. We can rewrite $b \cdot a + a \cdot b$ as $a \cdot b + a \cdot b$ because of Lemma 2.3.2. We can further rewrite this as $2ab$ because $2(ab) = 1(ab) + ab =$

$0(ab) + ab + ab = 0 + ab + ab = ab + ab$. We can also rewrite $a \cdot a$ and $b \cdot b$ as a^2 and b^2 respectively. Therefore, $(a + b)^2 = a^2 + 2ab + b^2$.

Chapter 3

Section 3.1

Exercise 3.3.1

We know that at least one of the statements "$a = c$" and "$a = d$" is true since $\{a, b\} = \{c, d\}$. Similarly, at least one of the statements "$b = c$" and "$b = d$" is true. If we have that $a = c$ and $b = d$ at the same time, then we have proved what we want. (Same for $a = d$ and $b = c$ at the same time.) Otherwise, then we must have that $a = c = b$ or $a = d = b$. If both of these are true, then $a = b = c = d$, and therefore both $a = c$ and $b = d$. If only one of these is true, which we will assume to be the statement that $a = c = b$ as a similar argument holds assuming $a = d = b$, then $d \notin \{a, b\} = \{a, a\}$, while $d \in \{c, d\} = \{a, d\}$, so $\{a, b\} \neq \{c, d\}$, a contradiction. Therefore, if $\{a, b\} = \{c, d\}$, either both $a = c$ and $b = d$ or $a = d$ and $b = c$.

Exercise 3.3.2

First, we show that these four sets exist. Axiom 3.3 tells us that $\emptyset$ exists. The singleton set axiom (part of Axiom 3.4) and Axiom 3.1 (so that we are allowed to construct the singleton set whose element is another set) tell us that $\{\emptyset\}$ exists. Applying them again gives that $\{\{\emptyset\}\}$ exists. Using the pair set axiom (also part of Axiom 3.4) and Axiom 3.1 allows us to create the set $\{\emptyset, \{\emptyset\}\}$.

Now we show that these four sets are distinct. Since the sets $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ obviously all contain at least one element, if one of them is equal to $\emptyset$, then by Axiom 3.2, it contains no elements. But this contradicts that it has at least one element, and we have a contradiction. Now we need to prove that $\{\emptyset\}$ is not equal to $\{\{\emptyset\}\}$ or $\{\emptyset, \{\emptyset\}\}$. Both $\{\emptyset\} \in \{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are true, but $\{\emptyset\} \notin \{\emptyset\}$. Therefore, by Axiom 3.2, $\{\emptyset\}$ is not equal to any of the other sets. (We have already proved that $\emptyset \neq \{\emptyset\}$, so $\{\emptyset\} \neq \emptyset$). The last thing that we need to prove now is that $\{\{\emptyset\}\} \neq \{\emptyset, \{\emptyset\}\}$. Since $\emptyset \in \{\emptyset, \{\emptyset\}\}$

and $\emptyset \notin \{\{\emptyset\}\}$, by Axiom 3.2, these two sets are not equal. Therefore, all four sets are distinct.

Exercise 3.1.3

Proving that $\{a, b\} = \{a\} \cup \{b\}$

We shall prove that $\{a, b\} = \{a\} \cup \{b\}$. The following are equivalent:

- $c \in \{a, b\}$
- c is either equal to a or b
- $c \in \{a\}$ or $c \in \{b\}$
- $c \in \{a\} \cup \{b\}$

Therefore, $\{a, b\} = \{a\} \cup \{b\}$.

Proving that $A \cup B = B \cup A$

We now shall prove that $A \cup B = B \cup A$. If $c \in A \cup B$, then at least one of the statements $c \in A$ and $c \in B$ are true. By Axiom 3.5, this means that $c \in B \cup A$. We can replace A with B and B with A to show the other direction. Therefore, $A \cup B = B \cup A$.

Proving that $A \cup A = A \cup \emptyset = \emptyset \cup A = A$

Because it is impossible for c to be in $\emptyset$, the following are equivalent:

- $c \in A$ or $c \in A$
- $c \in A$ or $c \in \emptyset$
- $c \in \emptyset$ or $c \in A$
- $c \in A$

These four statements are equivalent to $c \in A \cup A$, $c \in A \cup \emptyset$, $c \in \emptyset cup A$, and $c \in A$, respectively. Therefore, $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.

Exercise 3.1.11

The next paragraph will be written like a proof of Axiom 3.6 from Axiom 3.7: we will not define again A, $P(x)$, and so on.

If $Q(x, y)$ is the statement that $x = y$ and $P(x)$ are both true, then by Axiom 3.7 and the fact that for each x there exists at most one y such that $Q(x, y)$, there exists a set $B = \{y : x = y \text{ and } P(x) \text{ for some } x \in A\}$. For any $z \in B$, we know that $z \in A$ and that $P(z)$ is true. For any $x \in A$ such that $P(x)$ is true, we similarly know that $z \in B$. Therefore, $A = B$.

Section 3.2

Exercise 3.2.1

Proving that Axiom 3.9 implies Axiom 3.3

We can create a property $P(x)$ that is always false regardless to the choice of x. Then the set $\{x : P(x) \text{ is true}\}$ has no elements, as $P(x)$ is false all the time.

Proving that Axiom 3.9 implies Axiom 3.4

If $P(x)$ is the property that $x = a$, then $\{x : P(x) \text{ is true}\}$ has only the element a. Similarly, if $Q(x)$ is the property that $x = a$ or $x = b$, then $\{x : Q(x) \text{ is true}\}$ has only the elements a and b.

Proving that Axiom 3.9 implies Axiom 3.5

If $P(x)$ is the property that $x \in A$ or $x \in B$, then $\{x : P(x) \text{ is true}\}$ exists, so $A \cup B$ exists.

Proving that Axiom 3.9 implies Axiom 3.6

If $Q(x)$ is the property that $P(x)$ is true and $x \in A$, then $\{x : Q(x) \text{ is true}\} = \{x \in A : P(x)\}$ exists.

Proving that Axiom 3.9 implies Axiom 3.7

If $Q(y)$ is the property that there exists some x such that $P(x, y)$ is true, then $\{y : Q(y) \text{ is true}\} = \{y : P(x, y) \text{ is true for some x } \in A\}$ exists.

Proving that Axiom 3.9 implies Axiom 3.8, assuming that all natural numbers are objects

If $P(x)$ is the property that x is a natural number, then $\mathbf{N} = \{x : P(x) \text{ is true}\}$ exists, by Axiom 3.9.

Section 3.3

Exercise 3.3.2

Proving that f and g being injective implies that $g \circ f$ is too

We will show that if $x \neq x'$, then $(g \circ f)(x) \neq (g \circ f)(x')$. First, since f is injective, $f(x) \neq f(x')$. Since g is also injective, $g(f(x)) \neq g(f(x'))$. Therefore, $(g \circ f)(x) \neq (g \circ f)(x')$.

Proving that f and g being surjective implies that $g \circ f$ is too

We will show that $g \circ f$ is surjective by showing that for any $c \in Z$, there exists some $a \in X$ such that $(g \circ f)(a) = c$. Since g is surjective, there exists some $b \in Y$ such that $g(b) = c$. Because f is also surjective, there exists some $a \in X$ such that $f(a) = b$. Therefore, $(g \circ f)(a) = c$, and $g \circ f$ is surjective.

Exercise 3.3.3

Finding when the empty function is injective

We will prove that the empty function $f : \emptyset \to X$ is injective for any X. Since we cannot find any $x, x' \in \emptyset$ that are unequal, the empty function is (vacuously) injective.

Finding when the empty function is surjective

We will prove that the empty function $f : \emptyset \to X$ is surjective only when $X = \emptyset$. The statement

"For every $y \in X$, there exists $x \in \emptyset$ such that $f(x) = y$"

can only be true when it is impossible that $y \in X$, because $x \in \emptyset$ is impossible. But then we have that $X = \emptyset$, and we have proved our claim.

Finding when the empty function is bijective

We will prove that the empty function $f\colon \emptyset \to X$ is bijective only when $X = \emptyset$. Since f is always injective regardless of the choice of X, the empty function being bijective is equivalent to it being surjective. Therefore, since f is surjective precisely when $X = \emptyset$, the empty function is bijective only when $X = \emptyset$.

Exercise 3.3.4

Showing that if $g \circ f = g \circ \tilde{f}$ and g is injective, then $f = \tilde{f}$

We will use proof by contradiction. If $f \neq \tilde{f}$, then for some $x \in X$, we have $f(x) \neq \tilde{f}(x)$. Therefore, since g is injective, $(g \circ f)(x) \neq (g \circ \tilde{f})(x)$, and $g \circ f \neq g \circ \tilde{f}$. This is a contradiction, and therefore, $f = \tilde{f}$. This statement is not necessarily true if g is not injective. If $X = Y = Z = \mathbf{N}$, $f(x) = 0$, $\tilde{f}(x) = 1$, and $g(x) = 0$, then $f \neq \tilde{f}$, while $(g \circ f)(x) = 0 = (g \circ \tilde{f})(x)$.

Showing that if $g \circ f = \tilde{g} \circ f$ and f is surjective, then $g = \tilde{g}$

We will use proof by contradiction again. If $g \neq \tilde{g}$, then for some $y \in Y$, we have $g(y) \neq \tilde{g}(y)$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. But then $(g \circ f)(x) \neq (\tilde{g} \circ f)(x)$. This is a contradiction, and therefore, $g = \tilde{g}$. This statement is not necessarily true if f is not surjective. If $X = Y = Z = \mathbf{N}$, $f(x) = 0$, $g(x) = x$, and $\tilde{g}(x) = 0$, then $g \neq \tilde{g}$, even though $(g \circ f)(x) = 0 = (\tilde{g} \circ f)(x)$.

Exercise 3.3.6

If $f(x) = a$, then by the definition of f^{-1}, $f^{-1}(f(x)) = f^{-1}(a) = x$. If $f^{-1}(y) = b$, then $f(b) = y$, so $f(f^{-1}(y)) = f(b) = y$.

We can deduce that f^{-1} is bijective from Exercise 3.3.5 and the fact that the identity map $\imath_{X \to X}$ defined as $\imath_{X \to X}(x) = x$ for all $x \in X$ is obviously bijective.

Exercise 3.3.7

By Exercise 3.3.2, $g \circ f$ is both injective and surjective, and is therefore bijective. The only thing left we have to prove is that $(f^{-1} \circ g^{-1} \circ g \circ f)(x) = x$.

But $(f^{-1} \circ g^{-1} \circ g \circ f)(x) = (f^{-1} \circ f)(x) = x$, and we have proved what we want.

Section 3.4

Challenge to define $f(S)$ using the axiom of specification

We can define $f(S) \coloneqq \{y \in Y : \text{there exists } x \in S \text{ such that } f(x) = y\}$.

Exercise 3.4.1

First, we will show that any element of the forward image of V under f^{-1} is contained in the inverse image of V under f. If x is in the forward image of V under f^{-1}, then there exists v such that $f^{-1}(v) = x$ and therefore $f(x) = v$. But then x is in the inverse image of V under f.

Next, we will show that any element v of the inverse image of V under f is contained in the forward image of V under f^{-1}. Since there exists x such that $f(x) = v$ and therefore $f^{-1}(v) = x$, x is in the forward image of V under f.

Finally, we have that the forward image of V under f^{-1} is equal to the inverse image of V under f, and that it is valid to use the notation $f^{-1}(V)$.

Exercise 3.4.6

By Axiom 3.11, $\{0,1\}^X$ exists. We can use Axiom 3.7 (the axiom of replacement) to create a set

$$Y = \{f^{-1}(\{1\}) : f \in \{0,1\}^X\}.$$

Every S which is a subset of X is in Y, because defining $f(x)$ to be 1 when x is in S and to be 0 otherwise means that $f^{-1}(\{1\}) = S$. Every element of Y is a subset of X, because every element of $f^{-1}(\{1\})$ has to be in X.

Exercise 3.4.9

We will show that

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

We know that $y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$ if and only if $y \in A_\beta$ and $y \in A_\alpha$ for all $\alpha \in I$. But the latter statement implies the former, so $y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$ if and only if $y \in A_\alpha$ for all $\alpha \in I$.

We can do something similar for $A_{\beta'}$. Therefore,

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

Section 3.5

Exercise 3.5.1

First, we will show that $(x, y) = (x', y')$ implies that both $x = x'$ and $y = y'$. Since $(x, y) \coloneqq \{\{x\}, \{x, y\}\}$, we have to show that $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$ implies that both $x = x'$ and $y = y'$. If $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$, then $\{x\} \in \{\{x'\}, \{x', y'\}\}$. In order for this to be true, we must have $\{x\} = \{x'\}$, which implies $x = x'$, or $\{x\} = \{x', y'\}$, which implies $x' = y' = x$. But the latter case implies the former case, so we can ignore the latter one. Doing the same with $\{x, y\}$, $\{x'\}$, and $\{x', y'\}$ yields that $x = x'$ and $y = y'$. The converse is true by Axiom 3.2.

We can show that $X \times Y$ is a set if X and Y are both sets using Axiom 3.1 (sets are objects), Axiom 3.7 (axiom of replacement), and Axiom 3.12 (axiom of union). By Axioms 3.1 and 3.7, we can create the set $Z = \{\{(x, y)\} : y \in Y\} : x \in X\}$. By Axiom 3.12, we can also create the set $\bigcup Z$, which we can define to be $X \times Y$.

I won't do the additional challenge.

Exercise 3.5.6

We will show that if $A \times B \subseteq C \times D$ and $A, B, C, D \neq \emptyset$, then $A \subseteq C$ and $B \subseteq D$. Since A and B are nonempty, by the single choice lemma (Lemma 3.1.5), there exist some $a \in A$ and $b \in B$. Now, for any $x \in A$, since $(x, b) \in A \times B \subseteq C \times D$, we have $x \in C$. Similarly, for any $y \in B$, since $(a, y) \in A \times B \subseteq C \times D$, we have $y \in D$. Therefore, $A \subseteq C$ and $B \subseteq D$.

However, we can show that if we remove the assumption that A, B, C, and D are nonempty, then this statement is false. If $A = \{0\}$, $C = \{1\}$, and $B = D = \emptyset$, then $A \times B$ is empty, as there are no elements of B. Similarly, $C \times D = \emptyset$, and therefore $A \times B \subseteq C \times D$. However, $A \not\subseteq C$.

If $A \subseteq C$ and $B \subseteq D$, all ordered pairs (a, b) where $a \in A$ and $b \in B$ are in $C \times D$ because $a \in C$ and $b \in D$. Hence, $A \times B \subseteq C \times D$. This statement holds even when A, B, C, and D are not all nonempty.

Assuming that $A \times B = C \times D$ and $A, B, C, D \neq \emptyset$, $A = C$ and $B = D$. This is because $A \times B \subseteq C \times D$ and $C \times D \supseteq A \times B$, and therefore $A \subseteq C$, $A \supseteq C$, $B \subseteq D$, and $B \supseteq D$.

If we remove the assumption that $A, B, C, D \neq \emptyset$, the statement is false. The counterexample from before ($A = \{0\}$, $C = \{1\}$, $B = D = \emptyset$) also works here. We have that $A \times B = C \times D = \emptyset$, but $A \neq C$.

We have that $A = C$ and $B = D$ imply that $A \times B = C \times D$ because $A \subseteq C$ and $B \subseteq D$ imply, together, that $A \times B \subseteq C \times D$ and the same with $\subseteq$ replaced by $\supseteq$.

Exercise 3.5.13