

Chapter 1

Section 1.1

N/A

Section 1.2

N/A

Chapter 2

Section 2.1

N/A

Section 2.2

Exercise 2.2.1

We shall use induction on a , fixing b and c . First, we shall prove the base case. If $a = 0$, $a + b = 0 + b = b$ by the definition of addition of natural numbers (Definition 2.2.1). Therefore, $(a + b) + c = b + c$. Also by Definition 2.2.1, $a + (b + c) = 0 + (b + c) = b + c$. We now have $(a + b) + c = b + c = a + (b + c)$.

Now suppose that $(a + b) + c = a + (b + c)$. We then have, by Definition 2.2.1, that $((a++) + b) + c = ((a + b)++) + c = ((a + b) + c)++ = (a + (b + c))++$. Because of Definition 2.2.1 again, $(a + (b + c))++ = (a++) + (b + c)$. Now we have $((a++) + b) + c = (a++) + (b + c)$, which closes the induction.

Exercise 2.2.2

We shall use induction on a . The base case is vacuously true since 0 is not positive. For the inductive step, we have to show that $a++$ being positive implies that $a++ = b++$ for exactly one natural number b if a being positive

implies that $a = c++$ for exactly one natural number c . But a is equal to b if $a++$ is equal to $b++$, by Axiom 2.4. This closes the induction.

Exercise 2.2.3

- (a) By Lemma 2.2.2, $a = a + 0$, so $a \geq a$.
- (b) Because $a \geq b$ and $b \geq c$, $a = b + n$ and $b = c + m$ for some n and m . Therefore, $a = b + n = (c + m) + n$. By Proposition 2.2.5, $(c + m) + n = c + (m + n)$, so $a = c + (m + n)$. $n + m$ is a natural number, so $a \geq c$.
- (c) Because $a \geq b$ and $b \geq a$, $a = b + n$ and $b = a + m$. Therefore, $a = b + n = (a + m) + n$. By Proposition 2.2.5, $(a + m) + n = a + (m + n)$. Now we have $a = a + (m + n)$. By Lemma 2.2.2, $a = a + 0$, so we can use Proposition 2.2.6 to get $0 = m + n$. Using Corollary 2.2.9, we can deduce that n and m are both equal to 0. Therefore, $a = b$.
- (d) If $a \geq b$, $a = b + d$ for some d . Because of this, $a + c = (b + d) + c$. By Propositions 2.2.4 and 2.2.5, $(b + d) + c = b + (d + c) = b + (c + d) = (b + c) + d$. Therefore, $a + c \geq b + c$.

Conversely, if $a + c \geq b + c$, $a + c = (b + c) + d$ for some d . By Propositions 2.2.4 and 2.2.5, $(b + c) + d = b + (c + d) + b + (d + c) = (b + d) + c$. Using Proposition 2.2.6, we get $a = b + d$, and therefore, $a \geq b$.

- (f) If $a < b$, $b = a + c$ for some c , and $b \neq a$. Therefore, c is positive, because otherwise $b = a + 0 = a$. ($a + 0 = a$ because of Lemma 2.2.2.)

Conversely, if $b = a + c$ for some positive number c , we know that $a \leq b$. If $a = b$, $a = a + c$. By Lemma 2.2.2, $a = a + 0$, so $a + 0 = a + c$. We can use Proposition 2.2.6 to get $0 = c$, but c is positive. Therefore, $a \neq b$, and combined with $a \leq b$, we get $a < b$.

- (e) We first prove that $a < b$ implies that $a++ \leq b$. By (f), $b = a + c$ for some positive c . By Lemma 2.2.10, exactly one d exists such that $d++ = c$. Therefore, $b = a + d++$. By Lemma 2.2.3 and Definition 2.2.1, $b = a + d++ = (a + d)++ = (a++) + d$. Therefore, $a++ \leq b$.

Conversely, if $a++ \leq b$, we need to prove that $a < b$. Since $a++ = a + 1 \geq a$, by (b), $a \leq b$. If $a = b$, then $a++ \leq a$, and we get, from (c), that $a++ = a$. This is a contradiction, from Axiom 2.3 if $a = 0$ and Axiom 2.4 otherwise. Therefore, $a < b$.

Exercise 2.2.6 (unfinished)

We use induction on n . When $n = 0$, $m \leq n$ implies that $0 = m + a$. But by Corollary 2.2.9, $m = 0 = n$, so $P(n)$ being true implies that $P(m)$ is true.

If $a \leq n$ and $P(n)$ together imply $P(a)$, we now need to prove that $b \leq n++$ and $P(n++)$ together imply $P(b)$. Since $b \leq n++$, $n++ = b + c$ for some c . If c is positive, by Lemma 2.2.10, $n++ = b + d++$ for some d , so by Lemma 2.2.3 and Axiom 2.4, $n = b + d$, and $b \leq n$. Therefore, $b \leq n++$ implies that $b \leq n$ or $n++ = b + 0 = b$. Because of Lemmas 2.2.2 and 2.2.3, $n++ = (n + 0)++ = n + 0++ = n + 1$. Since $P(n++)$ implies $P(n)$, $P(n)$ is true. But then ?

Section 2.3

Exercise 2.3.1

First, we prove that $n \cdot 0 = 0$. We induct on n . When $n = 0$, $n \cdot 0 = 0 \cdot 0 = 0$, because of Definition 2.3.1. For the inductive hypothesis, we assume $n \cdot 0 = 0$. Then we need to prove that $n++ \cdot 0 = 0$. By Definition 2.3.1, $n++ \cdot 0 = n \cdot 0 + 0 = n \cdot 0$. But we already know that $n \cdot 0 = 0$, so $n++ \cdot 0 = 0$. This closes the induction.

Next, we prove that $n \cdot m++ = nm + n$ using induction on n . When $n = 0$, $0 \cdot m++ = 0 = 0 \cdot m + 0$. Now, if $n \cdot m++ = nm + n$, we need to prove that $n++ \cdot m++ = n++ \cdot m + n++$. We can deduce from $n \cdot m++ = nm + n$ that $n++ \cdot m++ = (n \cdot m++) + m++ = nm + n + m++ = nm + (n + m)++ = nm + n++ + m = (n++) \cdot m + n++$. This closes the induction.

Now we can prove that $nm = mn$. We induct on m . When $m = 0$, we have $n \cdot 0 = 0 = 0 \cdot n$. For the inductive hypothesis, we assume $nm = mn$. Now we need to prove $n \cdot m++ = m++ \cdot n$. We know that $n \cdot m++ = nm + n$ and $m++ \cdot n = mn + n$. But $nm = mn$, so $n \cdot m++ = m++ \cdot n$. This closes the induction.

Exercise 2.3.4

By Proposition 3.4, $(a + b)^2 = (a + b) \cdot (a + b) = (a + b) \cdot a + (a + b) \cdot b = a \cdot a + b \cdot a + a \cdot b + b \cdot b$. We can rewrite $b \cdot a + a \cdot b$ as $a \cdot b + a \cdot b$ because of Lemma 2.3.2. We can further rewrite this as $2ab$ because $2(ab) = 1(ab) + ab =$

$0(ab) + ab + ab = 0 + ab + ab = ab + ab$. We can also rewrite $a \cdot a$ and $b \cdot b$ as a^2 and b^2 respectively. Therefore, $(a + b)^2 = a^2 + 2ab + b^2$.

Chapter 3

Section 3.1

Exercise 3.3.1

We know that at least one of the statements “ $a = c$ ” and “ $a = d$ ” is true since $\{a, b\} = \{c, d\}$. Similarly, at least one of the statements “ $b = c$ ” and “ $b = d$ ” is true. If we have that $a = c$ and $b = d$ at the same time, then we have proved what we want. (Same for $a = d$ and $b = c$ at the same time.) Otherwise, then we must have that $a = c = b$ or $a = d = b$. If both of these are true, then $a = b = c = d$, and therefore both $a = c$ and $b = d$. If only one of these is true, which we will assume to be the statement that $a = c = b$ as a similar argument holds assuming $a = d = b$, then $d \notin \{a, b\} = \{a, a\}$, while $d \in \{c, d\} = \{a, d\}$, so $\{a, b\} \neq \{c, d\}$, a contradiction. Therefore, if $\{a, b\} = \{c, d\}$, either both $a = c$ and $b = d$ or $a = d$ and $b = c$.

Exercise 3.3.2

First, we show that these four sets exist. Axiom 3.3 tells us that \emptyset exists. The singleton set axiom (part of Axiom 3.4) and Axiom 3.1 (so that we are allowed to construct the singleton set whose element is another set) tell us that $\{\emptyset\}$ exists. Applying them again gives that $\{\{\emptyset\}\}$ exists. Using the pair set axiom (also part of Axiom 3.4) and Axiom 3.1 allows us to create the set $\{\emptyset, \{\emptyset\}\}$.

Now we show that these four sets are distinct. Since the sets $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ obviously all contain at least one element, if one of them is equal to \emptyset , then by Axiom 3.2, it contains no elements. But this contradicts that it has at least one element, and we have a contradiction. Now we need to prove that $\{\emptyset\}$ is not equal to $\{\{\emptyset\}\}$ or $\{\emptyset, \{\emptyset\}\}$. Both $\{\emptyset\} \in \{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are true, but $\{\emptyset\} \notin \{\emptyset\}$. Therefore, by Axiom 3.2, $\{\emptyset\}$ is not equal to any of the other sets. (We have already proved that $\emptyset \neq \{\emptyset\}$, so $\{\emptyset\} \neq \emptyset$). The last thing that we need to prove now is that $\{\{\emptyset\}\} \neq \{\emptyset, \{\emptyset\}\}$. Since $\emptyset \in \{\emptyset, \{\emptyset\}\}$

and $\emptyset \notin \{\{\emptyset\}\}$, by Axiom 3.2, these two sets are not equal. Therefore, all four sets are distinct.

Exercise 3.1.3

Proving that $\{a, b\} = \{a\} \cup \{b\}$

We shall prove that $\{a, b\} = \{a\} \cup \{b\}$. The following are equivalent:

- $c \in \{a, b\}$
- c is either equal to a or b
- $c \in \{a\}$ or $c \in \{b\}$
- $c \in \{a\} \cup \{b\}$

Therefore, $\{a, b\} = \{a\} \cup \{b\}$.

Proving that $A \cup B = B \cup A$

We now shall prove that $A \cup B = B \cup A$. If $c \in A \cup B$, then at least one of the statements $c \in A$ and $c \in B$ are true. By Axiom 3.5, this means that $c \in B \cup A$. We can replace A with B and B with A to show the other direction. Therefore, $A \cup B = B \cup A$.

Proving that $A \cup A = A \cup \emptyset = \emptyset \cup A = A$

Because it is impossible for c to be in \emptyset , the following are equivalent:

- $c \in A$ or $c \in A$
- $c \in A$ or $c \in \emptyset$
- $c \in \emptyset$ or $c \in A$
- $c \in A$

These four statements are equivalent to $c \in A \cup A$, $c \in A \cup \emptyset$, $c \in \emptyset \cup A$, and $c \in A$, respectively. Therefore, $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.

Exercise 3.1.11

The next paragraph will be written like a proof of Axiom 3.6 from Axiom 3.7: we will not define again A , $P(x)$, and so on.

If $Q(x, y)$ is the statement that $x = y$ and $P(x)$ are both true, then by Axiom 3.7 and the fact that for each x there exists at most one y such that $Q(x, y)$, there exists a set $B = \{y : x = y \text{ and } P(x) \text{ for some } x \in A\}$. For any $z \in B$, we know that $z \in A$ and that $P(z)$ is true. For any $x \in A$ such that $P(x)$ is true, we similarly know that $x \in B$. Therefore, $A = B$.

Section 3.2

Exercise 3.2.1

Proving that Axiom 3.9 implies Axiom 3.3

We can create a property $P(x)$ that is always false regardless to the choice of x . Then the set $\{x : P(x) \text{ is true}\}$ has no elements, as $P(x)$ is false all the time.

Proving that Axiom 3.9 implies Axiom 3.4

If $P(x)$ is the property that $x = a$, then $\{x : P(x) \text{ is true}\}$ has only the element a . Similarly, if $Q(x)$ is the property that $x = a$ or $x = b$, then $\{x : Q(x) \text{ is true}\}$ has only the elements a and b .

Proving that Axiom 3.9 implies Axiom 3.5

If $P(x)$ is the property that $x \in A$ or $x \in B$, then $\{x : P(x) \text{ is true}\}$ exists, so $A \cup B$ exists.

Proving that Axiom 3.9 implies Axiom 3.6

If $Q(x)$ is the property that $P(x)$ is true and $x \in A$, then $\{x : Q(x) \text{ is true}\} = \{x \in A : P(x)\}$ exists.

Proving that Axiom 3.9 implies Axiom 3.7

If $Q(y)$ is the property that there exists some x such that $P(x, y)$ is true, then $\{y : Q(y) \text{ is true}\} = \{y : P(x, y) \text{ is true for some } x \in A\}$ exists.

Proving that Axiom 3.9 implies Axiom 3.8, assuming that all natural numbers are objects

If $P(x)$ is the property that x is a natural number, then $\mathbf{N} = \{x : P(x) \text{ is true}\}$ exists, by Axiom 3.9.

Section 3.3

Exercise 3.3.2

Proving that f and g being injective implies that $g \circ f$ is too

We will show that if $x \neq x'$, then $(g \circ f)(x) \neq (g \circ f)(x')$. First, since f is injective, $f(x) \neq f(x')$. Since g is also injective, $g(f(x)) \neq g(f(x'))$. Therefore, $(g \circ f)(x) \neq (g \circ f)(x')$.

Proving that f and g being surjective implies that $g \circ f$ is too

We will show that $g \circ f$ is surjective by showing that for any $c \in Z$, there exists some $a \in X$ such that $(g \circ f)(a) = c$. Since g is surjective, there exists some $b \in Y$ such that $g(b) = c$. Because f is also surjective, there exists some $a \in X$ such that $f(a) = b$. Therefore, $(g \circ f)(a) = c$, and $g \circ f$ is surjective.

Exercise 3.3.3

Finding when the empty function is injective

We will prove that the empty function $f: \emptyset \rightarrow X$ is injective for any X . Since we cannot find any $x, x' \in \emptyset$ that are unequal, the empty function is (vacuously) injective.

Finding when the empty function is surjective

We will prove that the empty function $f: \emptyset \rightarrow X$ is surjective only when $X = \emptyset$. The statement

“For every $y \in X$, there exists $x \in \emptyset$ such that $f(x) = y$ ”

can only be true when it is impossible that $y \in X$, because $x \in \emptyset$ is impossible. But then we have that $X = \emptyset$, and we have proved our claim.

Finding when the empty function is bijective

We will prove that the empty function $f: \emptyset \rightarrow X$ is bijective only when $X = \emptyset$. Since f is always injective regardless of the choice of X , the empty function being bijective is equivalent to it being surjective. Therefore, since f is surjective precisely when $X = \emptyset$, the empty function is bijective only when $X = \emptyset$.

Exercise 3.3.4

Showing that if $g \circ f = g \circ \tilde{f}$ and g is injective, then $f = \tilde{f}$

We will use proof by contradiction. If $f \neq \tilde{f}$, then for some $x \in X$, we have $f(x) \neq \tilde{f}(x)$. Therefore, since g is injective, $(g \circ f)(x) \neq (g \circ \tilde{f})(x)$, and $g \circ f \neq g \circ \tilde{f}$. This is a contradiction, and therefore, $f = \tilde{f}$. This statement is not necessarily true if g is not injective. If $X = Y = Z = \mathbf{N}$, $f(x) = 0$, $\tilde{f}(x) = 1$, and $g(x) = 0$, then $f \neq \tilde{f}$, while $(g \circ f)(x) = 0 = (g \circ \tilde{f})(x)$.

Showing that if $g \circ f = \tilde{g} \circ f$ and f is surjective, then $g = \tilde{g}$

We will use proof by contradiction again. If $g \neq \tilde{g}$, then for some $y \in Y$, we have $g(y) \neq \tilde{g}(y)$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. But then $(g \circ f)(x) \neq (\tilde{g} \circ f)(x)$. This is a contradiction, and therefore, $g = \tilde{g}$. This statement is not necessarily true if f is not surjective. If $X = Y = Z = \mathbf{N}$, $f(x) = 0$, $g(x) = x$, and $\tilde{g}(x) = 0$, then $g \neq \tilde{g}$, even though $(g \circ f)(x) = 0 = (\tilde{g} \circ f)(x)$.

Exercise 3.3.6

If $f(x) = a$, then by the definition of f^{-1} , $f^{-1}(f(x)) = f^{-1}(a) = x$. If $f^{-1}(y) = b$, then $f(b) = y$, so $f(f^{-1}(y)) = f(b) = y$.

We can deduce that f^{-1} is bijective from Exercise 3.3.5 and the fact that the identity map $\iota_{X \rightarrow X}$ defined as $\iota_{X \rightarrow X}(x) = x$ for all $x \in X$ is obviously bijective.

Exercise 3.3.7

By Exercise 3.3.2, $g \circ f$ is both injective and surjective, and is therefore bijective. The only thing left we have to prove is that $(f^{-1} \circ g^{-1} \circ g \circ f)(x) = x$.

But $(f^{-1} \circ g^{-1} \circ g \circ f)(x) = (f^{-1} \circ f)(x) = x$, and we have proved what we want.

Section 3.4

Challenge to define $f(S)$ using the axiom of specification

We can define $f(S) := \{y \in Y : \text{there exists } x \in S \text{ such that } f(x) = y\}$.

Exercise 3.4.1

First, we will show that any element of the forward image of V under f^{-1} is contained in the inverse image of V under f . If x is in the forward image of V under f^{-1} , then there exists v such that $f^{-1}(v) = x$ and therefore $f(x) = v$. But then x is in the inverse image of V under f .

Next, we will show that any element v of the inverse image of V under f is contained in the forward image of V under f^{-1} . Since there exists x such that $f(x) = v$ and therefore $f^{-1}(v) = x$, x is in the forward image of V under f .

Finally, we have that the forward image of V under f^{-1} is equal to the inverse image of V under f , and that it is valid to use the notation $f^{-1}(V)$.

Exercise 3.4.6

By Axiom 3.11, $\{0, 1\}^X$ exists. We can use Axiom 3.7 (the axiom of replacement) to create a set

$$Y = \{f^{-1}(\{1\}) : f \in \{0, 1\}^X\}.$$

Every S which is a subset of X is in Y , because defining $f(x)$ to be 1 when x is in S and to be 0 otherwise means that $f^{-1}(\{1\}) = S$. Every element of Y is a subset of X , because every element of $f^{-1}(\{1\})$ has to be in X .

Exercise 3.4.9

We will show that

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

We know that $y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$ if and only if $y \in A_\beta$ and $y \in A_\alpha$ for all $\alpha \in I$. But the latter statement implies the former, so $y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$ if and only if $y \in A_\alpha$ for all $\alpha \in I$.

We can do something similar for $A_{\beta'}$. Therefore,

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

Section 3.5

Exercise 3.5.1

First, we will show that $(x, y) = (x', y')$ implies that both $x = x'$ and $y = y'$. Since $(x, y) := \{\{x\}, \{x, y\}\}$, we have to show that $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$ implies that both $x = x'$ and $y = y'$. If $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$, then $\{x\} \in \{\{x'\}, \{x', y'\}\}$. In order for this to be true, we must have $\{x\} = \{x'\}$, which implies $x = x'$, or $\{x\} = \{x', y'\}$, which implies $x' = y' = x$. But the latter case implies the former case, so we can ignore the latter one. Doing the same with $\{x, y\}$, $\{x'\}$, and $\{x', y'\}$ yields that $x = x'$ and $y = y'$.

The converse is true by Axiom 3.2.

I won't do the additional challenge.

Exercise 3.5.6

Exercise 3.5.10

Exercise 3.5.11

Exercise 3.5.13