Answers to Analysis I Exercises, Third Edition

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N/A

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2.2 Addition

Exercise 2.2.1

We shall use induction on a, fixing b and c. First, we shall prove the base case. If $a = 0$, $a + b = 0 + b = b$ by the definition of addition of natural numbers (Definition 2.2.1). Therefore, $(a + b) + c = b + c$. Also by Definition 2.2.1, $a + (b + c) = 0 + (b + c) = b + c$. We now have $(a + b) + c = b + c = a + (b + c)$.

Now suppose that $(a + b) + c = a + (b + c)$. We then have, by Definition 2.2.1, that $((a{+}{+}) + b) + c = ((a + b){+}{+}) + c = ((a + b) + c){+}{+} = (a + (b + c)){+}{+}$. Because of Definition 2.2.1 again, $(a + (b + c)){+}{+} = (a{+}{+}) + (b + c)$. Now we have $((a{+}{+}) + b) + c = (a{+}{+}) + (b + c)$, which closes the induction.

Exercise 2.2.2

We shall use induction on a. The base case is vacuously true since 0 is not positive. For the inductive step, we have to show that $a{+}{+}$ being positive implies that $a{+}{+} = b{+}{+}$ for exactly one natural number b if a being positive implies that $a = c{+}{+}$ for exactly one natural number c. But a is equal to b if $a{+}{+}$ is equal to $b{+}{+}$, by Axiom 2.4. This closes the induction.

Exercise 2.2.3

(a) By Lemma 2.2.2, $a = a + 0$, so $a \geq a$.

(b) Because $a \geq b$ and $b \geq c$, $a = b + n$ and $b = c + m$ for some n and m. Therefore, $a = b + n = (c + m) + n$. By Proposition 2.2.5, $(c + m) + n = c + (m + n)$, so $a = c + (m + n)$. $n + m$ is a natural number, so $a \geq c$.

(c) Because $a \geq b$ and $b \geq a$, $a = b + n$ and $b = a + m$. Therefore, $a = b + n = (a + m) + n$. By Proposition 2.2.5, $(a + m) + n = a + (m + n)$. Now we have $a = a + (m + n)$. By Lemma 2.2.2, $a = a + 0$, so we can use Proposition 2.2.6 to get $0 = m + n$. Using Corollary 2.2.9, we can deduce that n and m are both equal to 0. Therefore, $a = b$.

(d) If $a \geq b$, $a = b + d$ for some d. Because of this, $a + c = (b + d) + c$. By Propositions 2.2.4 and 2.2.5, $(b+d)+c = b+(d+c) = b+(c+d) = (b+c)+d$. Therefore, $a + c \geq b + c$.

Conversely, if $a+c \geq b+c$, $a+c = (b+c)+d$ for some d. By Propositions 2.2.4 and 2.2.5, $(b + c) + d = b + (c + d) + b + (d + c) = (b + d) + c$. Using Proposition 2.2.6, we get $a = b + d$, and therefore, $a \geq b$.

(f) If $a < b$, $b = a + c$ for some c, and $b \neq a$. Therefore, c is positive, because otherwise $b = a + 0 = a$. ($a + 0 = a$ because of Lemma 2.2.2.)

Conversely, if $b = a + c$ for some positive number c, we know that $a \leq b$. If $a = b$, $a = a + c$. By Lemma 2.2.2, $a = a + 0$, so $a + 0 = a + c$. We can use Proposition 2.2.6 to get $0 = c$, but c is positive. Therefore, $a \neq b$, and combined with $a \leq b$, we get $a < b$.

(e) We first prove that $a < b$ implies that $a{+}{+} \leq b$. By (f), $b = a + c$ for some positive c. By Lemma 2.2.10, exactly one d exists such that $d{+}{+} = c$. Therefore, $b = a + d{+}{+}$. By Lemma 2.2.3 and Definition 2.2.1, $b = a + d{+}{+} = (a + d){+}{+} = (a{+}{+}) + d$. Therefore, $a{+}{+} \leq b$.

Conversely, if $a{+}{+} \leq b$, we need to prove that $a < b$. Since $a{+}{+} = a + 1 \geq a$, by (b), $a \leq b$. If $a = b$, then $a{+}{+} \leq a$, and we get, from (c), that $a{+}{+} = a$. This is a contradiction, from Axiom 2.3 if $a = 0$ and Axiom 2.4 otherwise. Therefore, $a < b$.

Exercise 2.2.6 (unfinished)

We use induction on n. When $n = 0$, $m \leq n$ implies that $0 = m + a$. But by Corollary 2.2.9, $m = 0 = n$, so $P(n)$ being true implies that $P(m)$ is true.

If $a \leq n$ and $P(n)$ together imply $P(a)$, we now need to prove that $b \leq n{++}$ and $P(n{++})$ together imply $P(b)$. Since $b \leq n{++}$, $n{++} = b + c$ for some c. If c is positive, by Lemma 2.2.10, $n{++} = b + d{++}$ for some d, so by Lemma 2.2.3 and Axiom 2.4, $n = b + d$, and $b \leq n$. Therefore, $b \leq n{++}$ implies that $b \leq n$ or $n{++} = b + 0 = b$. Because of Lemmas 2.2.2 and 2.2.3, $n{++} = (n + 0){++} = n + 0{++} = n + 1$. Since $P(n{++})$ implies $P(n)$, $P(n)$ is true. But then **?**

2.3 Multiplication

Exercise 2.3.1

First, we prove that $n \cdot 0 = 0$. We induct on n. When $n = 0$, $n \cdot 0 = 0 \cdot 0 = 0$, because of Definition 2.3.1. For the inductive hypothesis, we assume $n \cdot 0 = 0$. Then we need to prove that $n{++} \cdot 0 = 0$. By Definition 2.3.1, $n{++} \cdot 0 = n \cdot 0 + 0 = n \cdot 0$. But we already know that $n \cdot 0 = 0$, so $n{++} \cdot 0 = 0$. This closes the induction.

Next, we prove that $n \cdot m{++} = nm + n$ using induction on n. When $n = 0$, $0 \cdot m{++} = 0 = 0 \cdot m + 0$. Now, if $n \cdot m{++} = nm + n$, we need to prove that $n{++} \cdot m{++} = n{++} \cdot m + n{++}$. We can deduce from $n \cdot m{++} = nm + n$ that $n{++} \cdot m{++} = (n \cdot m{++}) + m{++} = nm + n + m{++} = nm + (n+m){++} = nm + n{++} + m = (n{++}) \cdot m + n{++}$. This closes the induction.

Now we can prove that $nm = mn$. We induct on m. When $m = 0$, we have $n \cdot 0 = 0 = 0 \cdot n$. For the inductive hypothesis, we assume $nm = mn$. Now we need to prove $n \cdot m{++} = m{++} \cdot n$. We know that $n \cdot m{++} = nm + n$ and $m{++} \cdot n = mn + n$. But $nm = mn$, so $n \cdot m{++} = m{++} \cdot n$. This closes the induction.

Exercise 2.3.4

By Proposition 3.4, $(a + b)^2 = (a + b) \cdot (a + b) = (a + b) \cdot a + (a + b) \cdot b = a \cdot a + b \cdot a + a \cdot b + b \cdot b$. We can rewrite $b \cdot a + a \cdot b$ as $a \cdot b + a \cdot b$ because of Lemma 2.3.2. We can further rewrite this as $2ab$ because $2(ab) = 1(ab) + ab =$

$0(ab) + ab + ab = 0 + ab + ab = ab + ab$. We can also rewrite $a \cdot a$ and $b \cdot b$ as a^2 and b^2 respectively. Therefore, $(a + b)^2 = a^2 + 2ab + b^2$.

Chapter 3

Set theory

3.1 Fundamentals

Exercise 3.3.1

We know that at least one of the statements "$a = c$" and "$a = d$" is true since $\{a, b\} = \{c, d\}$. Similarly, at least one of the statements "$b = c$" and "$b = d$" is true. If we have that $a = c$ and $b = d$ at the same time, then we have proved what we want. (Same for $a = d$ and $b = c$ at the same time.) Otherwise, then we must have that $a = c = b$ or $a = d = b$. If both of these are true, then $a = b = c = d$, and therefore both $a = c$ and $b = d$. If only one of these is true, which we will assume to be the statement that $a = c = b$ as a similar argument holds assuming $a = d = b$, then $d \notin \{a, b\} = \{a, a\}$, while $d \in \{c, d\} = \{a, d\}$, so $\{a, b\} \neq \{c, d\}$, a contradiction. Therefore, if $\{a, b\} = \{c, d\}$, either both $a = c$ and $b = d$ or $a = d$ and $b = c$.

Exercise 3.3.2

First, we show that these four sets exist. Axiom 3.3 tells us that $\emptyset$ exists. The singleton set axiom (part of Axiom 3.4) and Axiom 3.1 (so that we are allowed to construct the singleton set whose element is another set) tell us that $\{\emptyset\}$ exists. Applying them again gives that $\{\{\emptyset\}\}$ exists. Using the pair set axiom (also part of Axiom 3.4) and Axiom 3.1 allows us to create the set $\{\emptyset, \{\emptyset\}\}$.

Now we show that these four sets are distinct. Since the sets $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ obviously all contain at least one element, if one of them is equal to $\emptyset$, then by Axiom 3.2, it contains no elements. But this contradicts that it has at least one element, and we have a contradiction. Now we need to prove that $\{\emptyset\}$ is not equal to $\{\{\emptyset\}\}$ or $\{\emptyset, \{\emptyset\}\}$. Both $\{\emptyset\} \in \{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are true, but $\{\emptyset\} \notin \{\emptyset\}$. Therefore, by Axiom 3.2, $\{\emptyset\}$ is not equal to any of the other sets. (We have already proved that $\emptyset \neq \{\emptyset\}$, so $\{\emptyset\} \neq \emptyset$). The last thing that we need to prove now is that $\{\{\emptyset\}\} \neq \{\emptyset, \{\emptyset\}\}$. Since $\emptyset \in \{\emptyset, \{\emptyset\}\}$ and $\emptyset \notin \{\{\emptyset\}\}$, by Axiom 3.2, these two sets are not equal. Therefore, all four sets are distinct.

Exercise 3.1.3

Proving that $\{a, b\} = \{a\} \cup \{b\}$ We shall prove that $\{a, b\} = \{a\} \cup \{b\}$. The following are equivalent:

- $c \in \{a, b\}$
- c is either equal to a or b
- $c \in \{a\}$ or $c \in \{b\}$
- $c \in \{a\} \cup \{b\}$

Therefore, $\{a, b\} = \{a\} \cup \{b\}$.

Proving that $A \cup B = B \cup A$ We now shall prove that $A \cup B = B \cup A$. If $c \in A \cup B$, then at least one of the statements $c \in A$ and $c \in B$ are true. By Axiom 3.5, this means that $c \in B \cup A$. We can replace A with B and B with A to show the other direction. Therefore, $A \cup B = B \cup A$.

Proving that $A \cup A = A \cup \emptyset = \emptyset \cup A = A$ Because it is impossible for c to be in $\emptyset$, the following are equivalent:

- $c \in A$ or $c \in A$
- $c \in A$ or $c \in \emptyset$
- $c \in \emptyset$ or $c \in A$
- $c \in A$

These four statements are equivalent to $c \in A \cup A$, $c \in A \cup \emptyset$, $c \in \emptyset cup A$, and $c \in A$, respectively. Therefore, $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.

Exercise 3.1.11

The next paragraph will be written like a proof of Axiom 3.6 from Axiom 3.7: we will not define again A, $P(x)$, and so on.

If $Q(x, y)$ is the statement that $x = y$ and $P(x)$ are both true, then by Axiom 3.7 and the fact that for each x there exists at most one y such that $Q(x, y)$, there exists a set $B = \{y : x = y \text{ and } P(x) \text{ for some } x \in A\}$. For any $z \in B$, we know that $z \in A$ and that $P(z)$ is true. For any $x \in A$ such that $P(x)$ is true, we similarly know that $z \in B$. Therefore, $A = B$.

3.2 Russell's paradox (Optional)

Exercise 3.2.1

Proving that Axiom 3.9 implies Axiom 3.3 We can create a property $P(x)$ that is always false regardless to the choice of x. Then the set $\{x : P(x) \text{ is true}\}$ has no elements, as $P(x)$ is false all the time.

Proving that Axiom 3.9 implies Axiom 3.4 If $P(x)$ is the property that $x = a$, then $\{x : P(x) \text{ is true}\}$ has only the element a. Similarly, if $Q(x)$ is the property that $x = a$ or $x = b$, then $\{x : Q(x) \text{ is true}\}$ has only the elements a and b.

Proving that Axiom 3.9 implies Axiom 3.5 If $P(x)$ is the property that $x \in A$ or $x \in B$, then $\{x : P(x) \text{ is true}\}$ exists, so $A \cup B$ exists.

Proving that Axiom 3.9 implies Axiom 3.6 If $Q(x)$ is the property that $P(x)$ is true and $x \in A$, then $\{x : Q(x) \text{ is true}\} = \{x \in A : P(x)\}$ exists.

Proving that Axiom 3.9 implies Axiom 3.7 If $Q(y)$ is the property that there exists some x such that $P(x, y)$ is true, then $\{y : Q(y) \text{ is true}\} = \{y : P(x, y) \text{ is true for some } x \in A\}$ exists.

Proving that Axiom 3.9 implies Axiom 3.8, assuming that all natural numbers are objects If $P(x)$ is the property that x is a natural number, then $\mathbf{N} = \{x : P(x) \text{ is true}\}$ exists, by Axiom 3.9.

3.3 Functions

Exercise 3.3.2

Proving that f and g being injective implies that $g \circ f$ is too We will show that if $x \neq x'$, then $(g \circ f)(x) \neq (g \circ f)(x')$. First, since f is injective, $f(x) \neq f(x')$. Since g is also injective, $g(f(x)) \neq g(f(x'))$. Therefore, $(g \circ f)(x) \neq (g \circ f)(x')$.

Proving that f and g being surjective implies that $g \circ f$ is too We will show that $g \circ f$ is surjective by showing that for any $c \in Z$, there exists some $a \in X$ such that $(g \circ f)(a) = c$. Since g is surjective, there exists some $b \in Y$ such that $g(b) = c$. Because f is also surjective, there exists some $a \in X$ such that $f(a) = b$. Therefore, $(g \circ f)(a) = c$, and $g \circ f$ is surjective.

Exercise 3.3.3

Finding when the empty function is injective We will prove that the empty function $f \colon \emptyset \to X$ is injective for any X. Since we cannot find any $x, x' \in \emptyset$ that are unequal, the empty function is (vacuously) injective.

Finding when the empty function is surjective We will prove that the empty function $f \colon \emptyset \to X$ is surjective only when $X = \emptyset$. The statement

"For every $y \in X$, there exists $x \in \emptyset$ such that $f(x) = y$"

can only be true when it is impossible that $y \in X$, because $x \in \emptyset$ is impossible. But then we have that $X = \emptyset$, and we have proved our claim.

Finding when the empty function is bijective We will prove that the empty function $f \colon \emptyset \to X$ is bijective only when $X = \emptyset$. Since f is always injective regardless of the choice of X, the empty function being bijective is equivalent to it being surjective. Therefore, since f is surjective precisely when $X = \emptyset$, the empty function is bijective only when $X = \emptyset$.

Exercise 3.3.4

Showing that if $g \circ f = g \circ \tilde{f}$ and g is injective, then $f = \tilde{f}$ We will use proof by contradiction. If $f \neq \tilde{f}$, then for some $x \in X$, we have $f(x) \neq \tilde{f}(x)$.

Therefore, since g is injective, $(g \circ f)(x) \neq (g \circ \tilde{f})(x)$, and $g \circ f \neq g \circ \tilde{f}$. This is a contradiction, and therefore, $f = \tilde{f}$. This statement is not necessarily true if g is not injective. If $X = Y = Z = \mathbf{N}$, $f(x) = 0$, $\tilde{f}(x) = 1$, and $g(x) = 0$, then $f \neq \tilde{f}$, while $(g \circ f)(x) = 0 = (g \circ \tilde{f})(x)$.

Showing that if $g \circ f = \tilde{g} \circ f$ and f is surjective, then $g = \tilde{g}$ We will use proof by contradiction again. If $g \neq \tilde{g}$, then for some $y \in Y$, we have $g(y) \neq \tilde{g}(y)$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. But then $(g \circ f)(x) \neq (\tilde{g} \circ f)(x)$. This is a contradiction, and therefore, $g = \tilde{g}$. This statement is not necessarily true if f is not surjective. If $X = Y = Z = \mathbf{N}$, $f(x) = 0$, $g(x) = x$, and $\tilde{g}(x) = 0$, then $g \neq \tilde{g}$, even though $(g \circ f)(x) = 0 = (\tilde{g} \circ f)(x)$.

Exercise 3.3.6

If $f(x) = a$, then by the definition of f^{-1}, $f^{-1}(f(x)) = f^{-1}(a) = x$. If $f^{-1}(y) = b$, then $f(b) = y$, so $f(f^{-1}(y)) = f(b) = y$.

We can deduce that f^{-1} is bijective from Exercise 3.3.5 and the fact that the identity map $\imath_{X \to X}$ defined as $\imath_{X \to X}(x) = x$ for all $x \in X$ is obviously bijective.

Exercise 3.3.7

By Exercise 3.3.2, $g \circ f$ is both injective and surjective, and is therefore bijective. The only thing left we have to prove is that $(f^{-1} \circ g^{-1} \circ g \circ f)(x) = x$. But $(f^{-1} \circ g^{-1} \circ g \circ f)(x) = (f^{-1} \circ f)(x) = x$, and we have proved what we want.

3.4 Images and inverse images

Challenge to define $f(S)$ using the axiom of specification

We can define $f(S) := \{y \in Y : \text{there exists } x \in S \text{ such that } f(x) = y\}$.

Exercise 3.4.1

First, we will show that any element of the forward image of V under f^{-1} is contained in the inverse image of V under f. If x is in the forward image of V

under f^{-1}, then there exists v such that $f^{-1}(v) = x$ and therefore $f(x) = v$. But then x is in the inverse image of V under f.

Next, we will show that any element v of the inverse image of V under f is contained in the forward image of V under f^{-1}. Since there exists x such that $f(x) = v$ and therefore $f^{-1}(v) = x$, x is in the forward image of V under f.

Finally, we have that the forward image of V under f^{-1} is equal to the inverse image of V under f, and that it is valid to use the notation $f^{-1}(V)$.

Exercise 3.4.6

By Axiom 3.11, $\{0,1\}^X$ exists. We can use Axiom 3.7 (the axiom of replacement) to create a set

$$Y = \{f^{-1}(\{1\}) : f \in \{0,1\}^X\}.$$

Every S which is a subset of X is in Y, because defining $f(x)$ to be 1 when x is in S and to be 0 otherwise means that $f^{-1}(\{1\}) = S$. Every element of Y is a subset of X, because every element of $f^{-1}(\{1\})$ has to be in X.

Exercise 3.4.9

We will show that

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

We know that $y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$ if and only if $y \in A_\beta$ and $y \in A_\alpha$ for all $\alpha \in I$. But the latter statement implies the former, so $y \in \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$ if and only if $y \in A_\alpha$ for all $\alpha \in I$.

We can do something similar for $A_{\beta'}$. Therefore,

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

3.5 Cartesian products

Exercise 3.5.1

First, we will show that $(x, y) = (x', y')$ implies that both $x = x'$ and $y = y'$. Since $(x, y) := \{\{x\}, \{x, y\}\}$, we have to show that $\{\{x\}, \{x, y\}\} =$

$\{\{x'\},\{x',y'\}\}$ implies that both $x = x'$ and $y = y'$. If $\{\{x\},\{x,y\}\} = \{\{x'\},\{x',y'\}\}$, then $\{x\} \in \{\{x'\},\{x',y'\}\}$. In order for this to be true, we must have $\{x\} = \{x'\}$, which implies $x = x'$, or $\{x\} = \{x',y'\}$, which implies $x' = y' = x$. But the latter case implies the former case, so we can ignore the latter one. Doing the same with $\{x,y\}$, $\{x'\}$, and $\{x',y'\}$ yields that $x = x'$ and $y = y'$. The converse is true by Axiom 3.2.

We can show that $X \times Y$ is a set if X and Y are both sets using Axiom 3.1 (sets are objects), Axiom 3.7 (axiom of replacement), and Axiom 3.12 (axiom of union). By Axioms 3.1 and 3.7, we can create the set $Z = \{\{(x,y)\} : y \in Y\} : x \in X\}$. By Axiom 3.12, we can also create the set $\bigcup Z$, which we can define to be $X \times Y$.

I won't do the additional challenge.

Exercise 3.5.6

We will show that if $A \times B \subseteq C \times D$ and $A, B, C, D \neq \emptyset$, then $A \subseteq C$ and $B \subseteq\ D$. Since A and B are nonempty, by the single choice lemma (Lemma 3.1.5), there exist some $a \in A$ and $b \in B$. Now, for any $x \in A$, since $(x,b) \in A \times B \subseteq C \times D$, we have $x \in C$. Similarly, for any $y \in B$, since $(a,y) \in A \times B \subseteq C \times D$, we have $y \in D$. Therefore, $A \subseteq C$ and $B \subseteq D$.

However, we can show that if we remove the assumption that A, B, C, and D are nonempty, then this statement is false. If $A = \{0\}$, $C = \{1\}$, and $B = D = \emptyset$, then $A \times B$ is empty, as there are no elements of B. Similarly, $C \times D = \emptyset$, and therefore $A \times B \subseteq C \times D$. However, $A \not\subseteq C$.

If $A \subseteq C$ and $B \subseteq D$, all ordered pairs (a,b) where $a \in A$ and $b \in B$ are in $C \times D$ because $a \in C$ and $b \in D$. Hence, $A \times B \subseteq C \times D$. This statement holds even when A, B, C, and D are not all nonempty.

Assuming that $A \times B = C \times D$ and $A, B, C, D \neq \emptyset$, $A = C$ and $B = D$. This is because $A \times B \subseteq C \times D$ and $C \times D \supseteq A \times B$, and therefore $A \subseteq C$, $A \supseteq C$, $B \subseteq D$, and $B \supseteq D$.

If we remove the assumption that $A, B, C, D \neq \emptyset$, the statement is false. The counterexample from before ($A = \{0\}$, $C = \{1\}$, $B = D = \emptyset$) also works here. We have that $A \times B = C \times D = \emptyset$, but $A \neq C$.

We have that $A = C$ and $B = D$ imply that $A \times B = C \times D$ because $A \subseteq C$ and $B \subseteq D$ imply, together, that $A \times B \subseteq C \times D$ and the same with $\subseteq$ replaced by $\supseteq$.

Exercise 3.5.13 (unfinished)

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3.6 Cardinality of sets

Exercise 3.6.2

If X has cardinality 0, there exists a bijection $f\colon X \to \{i \in \mathbf{N} : 1 \leq i \leq 0\}$. The set $\{i \in \mathbf{N} : 1 \leq i \leq 0\}$ is obviously empty. If $X \neq \emptyset$, then there exists some $x \in X$. But then $f(x)$ is in $\emptyset$, and therefore X must be empty.

If X is empty, the function $f\colon X \to \emptyset$ exists (we don't need to give how to derive $f(x)$ given x, since $x \in X$ is impossible) and is a bijection (injectivity is because there are no elements in the domain at all, and surjectivity is from that the range is empty). Since $\emptyset = \{i \in \mathbf{N} : 1 \leq i \leq 0\}$, X has cardinality 0.

Exercise 3.6.3

We will induct on n. When $n = 0$, $\{i \in N : 1 \leq i \leq n\}$ is empty, so we can make M equal to any natural number, say, 0. Now, assuming that all $f\colon \{i \in N : 1 \leq i \leq n\} \to \mathbf{N}$ satisfy the property that there exists some natural number M such that $f(i) \leq M$ for all $1 \leq i \leq n$, we will now show that for every $g\colon \{i \in N : 1 \leq i \leq n{+}{+}\} \to \mathbf{N}$, there exists some natural number a such that $g(i) \leq a$ for all $1 \leq i \leq n{+}{+}$. If $h\colon \{i \in N : 1 \leq i \leq n{+}{+}\} \to \mathbf{N}$ is defined so that $h(x) \coloneqq g(x)$ for all $x \in \{i \in N : 1 \leq i \leq n{+}{+}\}$, there exists some b such that $h(i) \leq b$ for all $1 \leq i \leq n$. Defining $a \coloneqq \begin{cases} b & \text{if } g(n{+}{+}) \leq b \\ g(n{+}{+}) & \text{if } g(n{+}{+}) > b \end{cases}$ ensures that a will always be greater than or equal to all elements of $g(\{i \in N : 1 \leq i \leq n{+}{+}\})$.

Exercise 3.6.5

We will define $f\colon A \times B \to B \times A$ by $f(a, b) \coloneqq (b, a)$, where $a \in A$ and $b \in B$. Injectivity of f comes from $(b, a) = (d, c)$ implying that $b = d$ and $a = c$, further implying that $(a, b) = (c, d)$. Surjectivity comes from the fact that for every $(b,a) \in B \times A$ has a corresponding ordered pair (namely, (a,b)) in $A \times B$ where $f(a,b) = (b,a)$.

Now, we will prove that $ab = ba$. We will define A as $\{i \in \mathbf{N} : 1 \leq i \leq a\}$ and B as $\{j \in \mathbf{N} : 1 \leq j \leq b\}$. Since A has equal cardinality with itself and the same for B (Proposition 3.6.4), A and B are both finite and have the cardinalities a and b, respectively. By Proposition 3.6.14(e), $\#(A) \cdot \#(B) = \#(A \times B)$ and $\#(B) \cdot \#(A) = \#(B \times A)$. We also know that $\#(A \times B) = \#(B \times A)$, and therefore $ab = ba$.

Exercise 3.6.7 (unfinished)

?

Exercise 3.6.10 (unfinished)

Attempt 1 We induct on n. (*I don't think we can induct, actually, since* $\#(\bigcup_{i \in \{1,\ldots,n+1\}} A_i) > n+1$ *does not necessarily imply* $\#(\bigcup_{i \in \{1,\ldots,n\}} A_i) > n$). When $n = 0$, $\#(\bigcup_{i \in \{1,\ldots,n\}} A_i) = \#(\emptyset) = 0$. It is obvious that $0 \not> 0$, so the pigeonhole principle holds vacuously when $n = 0$. When the pigeonhole principle is true for n finite sets, now we will show that it also is true for $A_1, \ldots, A_{n++}$. By Proposition 3.6.14(b),

$$\#(\bigcup_{i \in \{1,\ldots,n++\}} A_i) \leq \#(\bigcup_{i \in \{1,\ldots,n\}} A_i) + \#(A_{n++}).$$

Chapter 4

Integers and rationals

4.1 The integers

Exercise 4.1.1

Reflexive If $a, b \in \mathbf{N}$, $a + b = b + a$, so by Definition 4.1.1, $a — b = a — b$.

Symmetric If $a — b = c — d$, $a + d = c + b$. Now, since $c + b = a + d$, $c — d = a — b$.

Exercise 4.1.3

We know that $-1 = -(1 — 0) = 0 — 1$. If $a = a_1 — a_2$, $(-1) \cdot a = (0 \cdot a_1 + 1 \cdot a_2) — (0 \cdot a_2 + 1 \cdot a_1) = a_2 — a_1 = -a$.

Exercise 4.1.4

I will write $x = a — b$, $y = c — d$, and $z = e — f$, where $a, b, c, d, e, f \in \mathbf{N}$.

First, we will prove $x + y = y + x$:

$$\begin{aligned}
x + y &= (a — b) + (c — d) \\
&= (a + c) — (b + d) \\
y + x &= (c — d) + (a — b) \\
&= (c + a) — (d + b)
\end{aligned}$$

and therefore, because of natural number addition's commutativity, $x + y = y + x$.

Next, we will prove that $(x + y) + z = x + (y + z)$:

$$\begin{aligned}
(x + y) + z &= ((a—b) + (c—d)) + (e—f) \\
&= ((a + c)—(b + d)) + (e—f) \\
&= ((a + c) + e)—((b + d) + f) \\
x + (y + z) &= (a—b) + ((c—d) + (e—f)) \\
&= (a—b) + ((c + e)—(d + f)) \\
&= (a + (c + e))—(b + (d + f))
\end{aligned}$$

and therefore, because of natural number addition's associativity, $(x+y)+z = x + (y + z)$.

Now, we will prove that $x + 0 = 0 + x = x$. We have proved above that $x + y = y + x$ and therefore $x + 0 = 0 + x$, so we only need to show $x + 0 = x$:

$$\begin{aligned}
x + 0 &= (a—b) + (0—0) \\
&= (a + 0)—(b + 0).
\end{aligned}$$

We know that $a+0 = a$ and $b+0 = b$, so $x+0 = x$, and hence $x+0 = 0+x = x$.

Next, we will prove that $x + (-x) = (-x) + x = 0$. We have already proved that $x + y = y + x$ and therefore $x + (-x) = (-x) + x$, so the only thing left is to show $x + (-x) = 0$.

$$\begin{aligned}
x + (-x) &= (a—b) + (b—a) \\
&= (a + b)—(b + a) \\
&= (a + b)—(a + b) \\
&= 0—0 \\
&= 0.
\end{aligned}$$

Therefore, $x + (-x) = (-x) + x = 0$.

To show that $xy = yx$, we will use the definition of integer multiplication:

$$\begin{aligned}
xy &= (a—b)(c—d) \\
&= (ac + bd)—(ad + bc) \\
yx &= (c—d)(a—b) \\
&= (ca + db)—(cb + da) \\
&= (ac + bd)—(ad + bc) \\
&= xy.
\end{aligned}$$

Note that $(xy)z = x(yz)$ was proved in in the section about proving Proposition 4.1.6 on page 4.1.

For the statement $x1 = 1x = x$, we only have to prove that $1x = x$, because we have already proved that $xy = yx$ earlier:

$$\begin{aligned}
1x &= (1 \text{—} 0) \cdot (a \text{—} b) \\
&= (1a + 0b) \text{—} (1b + 0a) \\
&= a \text{—} b \\
&= x.
\end{aligned}$$

Now we will show that $x(y + z) = xy + xz$:

$$\begin{aligned}
x(y + z) &= (a \text{—} b)((c \text{—} d) + (e \text{—} f)) \\
&= (a \text{—} b)((c + e) \text{—} (d + f)) \\
&= (a(c + e) + b(d + f)) \text{—} (a(d + f) + b(c + e)) \\
&= (ac + ae + bd + bf) \text{—} (ad + af + bc + be) \\
&= ((ac + bd) \text{—} (ad + bc)) + ((ae + bf) \text{—} (af + be)) \\
&= xy + xz.
\end{aligned}$$

We can show that $(y+z)x = yx+zx$ using $xy = yx$ and $x(y+z) = xy+xz$ by simply swapping $x(y + z)$ with $(y + z)x$, xy with yx, and xz with zx.

Exercise 4.1.7

(a) If $a > b$, we will show that $a - b$ is a positive natural number. Since $a \geq b$, there exists $d \in \mathbf{N}$ such that $a = b + d$. Subtracting b from both sides gives $a - b = (b + d) - b$. We know that

$$(b + d) - b = (b + d) + (-b) = (d + b) + (-b) = d + (b + (-b)) = d + 0 = d.$$

Therefore, $a - b = d$. Since $d = 0$ implies $a = b + 0 = b$, which contradicts $a > b$, $a - b$ must be a nonzero, or positive, natural number.

Now we will show that if $a - b$ is a positive natural number, $a > b$. We know that $a \geq b$, since $a = a + (-b) + b = (a - b) + b$. We also know that $a \neq b$ because otherwise, $a - b$ would be 0. Therefore, $a > b$.

(b) When $a > b$, by (a), $a - b \neq 0$. Since $c - c = c + (-c) = 0$ (by Proposition 4.1.6 and the definition of subtraction on page 79), $a - b + c - c \neq 0 + c - c = 0$.

By Proposition 4.1.6 and the fact that $-(-x) = x$ (if $x = a—b$, $-x = b—a$, and $-(-x)$ is just $a—b$ again),

$$\begin{aligned} a - b + c - c &= a + (-b) + c + (-c) \\ &= (a + (-c)) + (-b) + (-(-c)) = (a - c) - (b - c). \end{aligned}$$

Therefore, $a + c > b + c$.

(c) Since $a > b$, $a - b$ is a positive natural number. Since c is also a positive natural number, by Lemma 2.3.3, $(a - b)c$ is also a positive natural number. The distributive law for natural numbers (Proposition 2.3.4) tells us that $(a - b)c = ac - bc$, and therefore $ac > bc$.

(d) Since $a > b$, $a - b$ is a positive natural number. But this means that $-a < -b$, since $-b = (a - b) + (-a)$.

(e) Since $a > b$ and $b > c$, $a - b$ and $b - c$ are both positive natural numbers. By Proposition 2.2.8, $(a - b) + (b - c)$ is also a positive natural number. We know that

$$(a - b) + (b - c) = a + (-b) + b + (-c) = a + 0 + (-c) = a + (-c) = a.$$

Therefore, $a > c$.

(f) First, we will show that at least one of the statements $a > b$, $a < b$, and $a = b$ is true. By Lemma 4.1.5 (trichotomy of integers), $a - b$ is either zero, positive, or negative. If it is 0,

$$a = (a - b) + b = 0 + b = b.$$

If $a - b$ is positive, by part (a), $a > b$. When $a - b$ is negative, it is the negation of some positive natural number d. Since $-(-d)$ is equal to d itself and $-(a - b) = b - a$, $d = b - a$. Therefore, $b > a$. We have now proved that at least one of the statements $a > b$, $a < b$, and $a = b$ is true.

Now, we will show that no two of these statements can be true at the same time. It is obvious from the definition of ordering on the integers (Definition 4.1.10) that $a = b$ cannot be true at the same time as $a > b$ or $a < b$. Now the only thing left is to prove that $a > b$ and $a < b$ cannot both be true. We will prove this using proof by contradiction. The statement that $a > b$

is equivalent to $a - b$ being positive, and $a < b$ is equivalent to $b - a$ being positive. But by Lemma 4.1.5 (the trichotomy of integers) and the fact that $b - a = -(a - b)$ give us a contradiction, as $b - a$ cannot both be positive and negative. We have now proved that exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.

Exercise 4.1.8

If $P(n)$ is the property that $n > -1$, $P(0)$ is true, because $0 = (-1) + 1$ and $0 \neq -1$. Also, if $P(n)$ is true, since $n{+}{+} > n$, by Lemma 4.1.11(e), $P(n{+}{+})$ is true. However, $P(-1)$ is obviously false, since $-1 = -1$.

4.2 The rationals

Exercise 4.2.1

Proving that equality is reflexive (for rational numbers) We know that for any $a, b \in \mathbf{Z}$ where $b \neq 0$, $ab = ab$. Therefore, $a//b = a//b$.

Proving that equality is symmetric If for $a, b, c, d \in \mathbf{Z}$ that satisfy $b, d \neq 0$, we have $a//b = c//d$, then $ad = cb$. Hence, $cb = ad$, and $c//d = a//b$.

Proving that equality is transitive If $a, b, c, d, e, f \in \mathbf{Z}$ where $b, d, f \neq 0$, $a//b = c//d$, and $c//d = e//f$, then we must have $ad = cb$ and $cf = ed$. Hence, $ad \cdot cf = cb \cdot ed$. By Corollary 4.1.9, we can remove c and d from both sides:

$$a \cdot f = b \cdot e.$$

Therefore, by Definition 4.2.1, $a//b = e//f$, and we have proved that rational number equality is reflexive, symmetric, and transitive.

Exercise 4.2.3

I will assume that $x = a//b$, $y = c//d$, and $z = e//f$, where $a, b, c, d, e, f \in \mathbf{Z}$ and $b, d, f \neq 0$.

First, we will prove that $x + y = y + x$:

$$\begin{aligned} x + y &= (a//b) + (c//d) \\ &= (ad + bc)//(bd) \\ y + x &= (c//d) + (a//b) \\ &= (cb + da)//(db) \\ &= (ad + bc)//(bd) \\ &= x + y. \end{aligned}$$

The identity $(x + y) + z$ was proved on page 84 already.

Next, we will prove that $x + 0 = 0 + x = x$. We have already proved that $x + y = y + x$, so we only need to show that $x + 0 = x$:

$$\begin{aligned} x + 0 &= (a//b) + (0//1) \\ &= (a \cdot 1 + b \cdot 0)//(b \cdot 1) \\ &= a//b \\ &= x. \end{aligned}$$

For the next identity, $x + (-x) = (-x) + x = 0$, we have already shown that $x + y = y + x$, so we only need to prove $x + (-x) + 0$:

$$\begin{aligned} x + (-x) &= (a//b) + ((-a)//b) \\ &= (ab + b \cdot (-a))//(b \cdot b) \\ &= 0//(b \cdot b) \\ &= 0. \end{aligned}$$

The last step is valid because $0 \cdot 1 = 0 \cdot (b \cdot b)$.

Now we prove that $xy = yx$:

$$\begin{aligned} xy &= (a//b) \cdot (c//d) \\ &= (ac)//(bd) \\ yx &= (c//d) \cdot (a//b) \\ &= (ca)//(db) \\ &= xy. \end{aligned}$$

We now prove that $(xy)z = x(yz)$:

$$\begin{aligned}
(xy)z &= ((a//b) \cdot (c//d)) \cdot (e//f) \\
&= ((ac)//(bd)) \cdot (e//f) \\
&= (ace)//(bdf) \\
x(yz) &= (a//b) \cdot ((c//d) \cdot (e//f)) \\
&= (a//b) \cdot ((ce)//(df)) \\
&= (ace)//(bdf) \\
&= (xy)z.
\end{aligned}$$

Next, we prove $x1 = 1x = x$. We have already proved that $xy = yx$, so we only need to prove $x \cdot 1 = x$:

$$\begin{aligned}
x \cdot 1 &= (a//b) \cdot (1//1) \\
&= (a \cdot 1)//(b \cdot 1) \\
&= a//b \\
&= x.
\end{aligned}$$

The next identity to prove is $x(y + z) = xy + xz$:

$$\begin{aligned}
x(y+z) &= (a//b)((c//d) + (e//f)) \\
&= (a//b)((cf + de)//(df)) \\
&= (a(cf + de))//(bdf) \\
&= (acf + ade)//(bdf) \\
xy + xz &= ((ac)//(bd)) + ((ae)//(bf)) \\
&= (acbf + bdae)//(b^2df) \\
&= (acf + dae)//(bdf) \\
&= x(y+z).
\end{aligned}$$

For the identity $(y + z)x = yx + zx$, we can just apply $xy = yx$ and $x(y + z) = xy + xz$ together.

Finally, we will prove that $xx^{-1} = x^{-1}x = 1$, when $x \neq 0$. We know that x^{-1} exists because otherwise, $a \neq 0$, and therefore x would be 0. Since $xy = yx$, we only need to show that $xx^{-1} = 1$:

$$\begin{aligned}
xx^{-1} &= (a//b)(b//a) \\
&= (ab)//(ab) \\
&= 1.
\end{aligned}$$

(The last step comes from Definition 4.2.1).

Exercise 4.2.5

(a) By Lemma 4.2.7, we know that exactly one of the statements "$x-y=0$", "$x-y$ is positive", or "$x-y$ is negative" is true. We now can use Definition 4.2.8 to deduce that exactly one of the statements $x=y$, $x>y$, or $x<y$ is true.

(b) If $x<y$, $x-y$ must be a negative rational number. Hence, $y-x$ is positive, and $y>x$. Similarly, we can show that $y>x$ implies $x<y$.

(c) By (b), we know that $y>x$ and $z>y$. Thus, $y-x$ and $z-y$ are positive. If $y-x=a//b$ and $z-y=c//d$ (where a, b, c, and d are positive; this is possible by Definition 4.2.6),

$$\begin{aligned} z-x &= z-y+y-x \\ &= (ad+bc)//(bd) \end{aligned}$$

which is positive, and therefore $z>x$. We now have proved that $x<z$.

(d) By (b), we know that $y>x$, and hence, $y-x$ is positive. If $y-x=a//b$, we know that

$$\begin{aligned} y+z-(x+z) &= y+z-x-z \\ &= y-x \end{aligned}$$

and therefore $y+z>x+z$. Hence, $x+z<y+z$.

(e) Because of part (b), we know that $y>x$, and thus, $y-x$ is positive. Assuming that $y-x=a//b$ and $z=c//d$ (where a, b, c, and d are positive; this is possible by Definition 4.2.6),

$$\begin{aligned} yz-xz &= (y-x)z \\ &= (a//b)(c//d) \\ &= (ac)//(bd) \end{aligned}$$

and hence, $yz>xz$ and $xz<yz$.

Exercise 4.2.6

We know that $x - y$ is positive. Assuming that $y - x = a//b$ and $z = c//d$ (where a, b, c, and d are positive; this is always possible by Definition 4.2.6), we have that

$$\begin{aligned} yz - xz &= (y - x)z \\ &= (ac)//(bd) \end{aligned}$$

and therefore, $xz > yz$.

4.3 Absolute value and exponentiation

Exercise 4.3.3(a)

First, we prove that $x^n x^m = x^{n+m}$. We induct on m. When $m = 0$, $x^m = 1$, and hence, $x^n x^m = x^n$. But $x^{n+m} = x^{n+0} = x^n$, and therefore, $x^n x^m = x^{n+m}$ is true when $m = 0$. When $x^n x^m = x^{n+m}$,

$$x^n x^{m++} = x^n x^m \cdot x = x^{n+m} \cdot x = x^{n+(m++)},$$

and we have proved that $x^n x^m = x^{n+m}$ for all n and m.

Next, we prove that $(x^n)^m = x^{nm}$, again by induction on m. When $m = 0$, it is obvious that both $(x^n)^m$ and x^{nm} are both 1, since anything to the power of 0 is 1. Now, we prove that if $(x^n)^m = x^{nm}$ for some n and m, $(x^n)^{m++} = x^{n(m++)}$. We know that $(x^n)^{m++} = x^n \cdot (x^n)^m$. Using Definition 4.3.9, we get that

$$x^n \cdot (x^n)^m = (x^n)^{m++}.$$

We have finished proving that $(x^n)^m = x^{nm}$ for all n and m.

We will now prove that $(xy)^n = x^n y^n$, again using induction, except on n. When $n = 0$, it is obvious that both sides are equal to 1. If $(xy)^n = x^n y^n$, we can multiply both sides of the equation by xy to get that

$$(xy)^{n++} = x^{n++} y^{n++}.$$

We have completed showing that $(xy)^n = x^n y^n$.

Exercise 4.3.5

We use induction on N, with base case $N = 1$. For the base case, $2^1 = 2 > 1$. For the inductive step, we multiply both sides of $2^N \geq N$ by 2 to get $2^{N++} \geq 2N$. Since $2N = N + N \geq N + 1$, $2^{N++} \geq N{++}$. We have finished proving that $2^N \geq N$ for all positive integers N.

4.4 Gaps in the rational numbers

Exercise 4.4.1

If $x = \frac{a}{b}$ and $a, b \in \mathbf{N}$, by Proposition 2.3.9 (the Euclidean algorithm), there exist m and r in $\mathbf{N}$ such that $0 \leq r < b$ and $a = mb + r$. Dividing both sides by b yields that

$$x = m + \frac{r}{b}.$$

Because $0 \leq r < b$, $0 \leq \frac{r}{b} < 1$. Adding m to 0, $\frac{r}{b}$, and 1 gives that $m \leq m + \frac{r}{b} < m + 1$. But $m + \frac{r}{b} = x$, so we have proved that $m \leq x < m + 1$. To prove that there exists some natural number N greater than x, we set N to be $m + 1$ when $m + 1 \geq 0$ and 0 when $m + 1 < 0$. (This works because when $m + 1 \geq 0$, it is a natural number and greater than x, as we proved before. When $m + 1 < 0$, $N = 0$ works because $0 > m + 1 > x$.)

Exercise 4.4.3

Showing that every natural number is either even or odd, but not both Let n be a natural number. By the Euclidean algorithm (Proposition 2.3.9), every natural number is equal to $2m + r$, where $m, r \in \mathbf{N}$ and $0 \leq r < 2$. This implies that every natural number is either even or odd. Now, we show that n cannot be both even and odd, by showing that having both at the same time will cause a contradiction. A natural number n being both even and odd must be equal to $2a$ and $2b+1$ at the same time, where $a, b \in \mathbf{N}$. If $a > b$, $2a > 2b + 1$, and we have a contradiction, because then $2a \neq 2b + 1$. When $b \geq a$, $2b + 1 > 2a$, and we also have a contradiction.

Showing that an odd natural number n has an odd square n^2 We know that $n = 2m + 1$ for some $m \in \mathbf{N}$. Therefore, $n^2 = 4m^2 + 2m + 1 = 2m(2m + 1) + 1$, and therefore, n^2 is also odd.

Chapter 5

The real numbers

5.1 Cauchy sequences

Exercise 5.1.1

Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, it is eventually 1-steady. Thus, for some m, $(a_n)_{n=m}^{\infty}$ is 1-steady. By Lemma 5.1.14, the finite sequence $a_1, \ldots, a_{m-1}$ is bounded. Also, $(a_n)_{n=m}^{\infty}$ is bounded by $|a_m| + 1$. Therefore, $(a_n)_{n=1}^{\infty}$ is bounded by the greater of the bounds for $a_1, \ldots, a_{m-1}$ and $(a_n)_{n=m}^{\infty}$.

5.2 Equivalent Cauchy sequences

Exercise 5.2.1

We will only prove that $(a_n)_{n=1}^{\infty}$ being a Cauchy sequence implies that $(b_n)_{n=1}^{\infty}$ is also a Cauchy sequence; we can replace a_n with b_n and b_n with a_n, anyways.

For any rational $\epsilon > 0$, there exist natural numbers m_1 and m_2 such that $(a_n)_{n=m_1}^{\infty}$ is $\frac{\epsilon}{3}$-steady and such that $(a_n)_{n=m_2}^{\infty}$ and $(b_n)_{n=m_2}^{\infty}$ are $\frac{\epsilon}{3}$-close. If m is defined to be m_1 when $m_1 \geq m_2$ and m_2 when $m_1 < m_2$, $(b_n)_{n=m}^{\infty}$ is ϵ-steady, because for any $x, y \in \mathbf{N}$ greater than or equal to m,

$$d(a_x, a_y), d(a_x, b_x), d(a_y, b_y) \leq \frac{\epsilon}{3}.$$

5.3 The construction of the real numbers

Exercise 5.3.1

We will assume here that ϵ is a rational number greater than 0.

We know that $(a_n)_{n=1}^{\infty}$ is always ϵ-close to itself. Therefore, $x = x$.

It is obvious that if $(a_n)_{n=1}^{\infty}$ is eventually ϵ-close to $(b_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ is eventually ϵ-close to $(a_n)_{n=1}^{\infty}$. Hence, $x = y$ implies $y = x$.

We also know that when $(a_n)_{n=1}^{\infty}$ is eventually $\epsilon/2$-close to $(b_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ is eventually $\epsilon/2$-close to $(c_n)_{n=1}^{\infty}$, $(a_n)_{n=1}^{\infty}$ is eventually ϵ-close to $(c_n)_{n=1}^{\infty}$. (This is because the same thing holds if we replace sequences with rational numbers and "eventually $\epsilon/2$-close" with "ϵ-close"). Thus, $x = y$ and $y = z$ together imply $x = z$.

Exercise 5.3.3

If $\mathrm{LIM}_{n\to\infty}\, a = \mathrm{LIM}_{n\to\infty}\, b$, we will prove that $a = b$. Assume for the sake of contradiction that $a \neq b$. Then, $|a - b| \neq 0$. But then $a, a, a, a, \ldots$ and $b, b, b, b, \ldots$ are not eventually $\frac{|a-b|}{2}$-close. But this contradicts our assumption that $\mathrm{LIM}_{n\to\infty}\, a = \mathrm{LIM}_{n\to\infty}\, b$, and thus $a = b$.

To prove that $a = b$ implies $\mathrm{LIM}_{n\to\infty}\, a = \mathrm{LIM}_{n\to\infty}\, b$, we can use the fact that a is ϵ-close to itself for any rational $\epsilon > 0$.

Exercise 5.3.5

For any rational $\epsilon > 0$, there is some m greater than $\frac{1}{\epsilon}$, by Proposition 4.4.1. Since any natural number m' greater than or equal to m satisfies

$$\frac{1}{m'} \leq \frac{1}{m} \leq \epsilon,$$

the sequence $1/1, 1/2, 1/3, \ldots$ is eventually ϵ-close to the sequence $0, 0, 0, \ldots$. Therefore, $\mathrm{LIM}_{n\to\infty}\, 1/n = 0$.

5.4 Ordering the reals

Exercise 5.4.2

(a) By Proposition 5.4.4, exactly one of the statements "$x - y$ is positive", "$x - y$ is equal to 0", or "$x - y$ is negative" is true. These statements correspond to "$x > y$", "$x = y$", and "$x < y$" respectively.

(b) By Proposition 5.4.4, $x - y$ is negative if and only if $y - x$ is positive. Hence, $x < y$ if and only if $y > x$.

(c) Since $x < y$ and $y < z$, $y > x$ and $z > y$ (by part (b)). Thus, $y - x$ and $z - y$ are positive. We know that $z - y + y - x = z - x$. Therefore, by Proposition 5.4.4, $z - x$ is positive and $z > x$. Using part (b) again, we get that $x < z$.

(d) By part (b), $y - x$ is positive. Since $y + z - (x + z)$ is equal to $y - x$, $x + z < y + z$.

Exercise 5.4.4

By Proposition 5.4.12, there exist some integers a and b such that $a, b \neq 0$ and $0 < \frac{a}{b} \leq x$. Since

$$0 < \frac{1}{2b|a|/a} \leq \frac{|a|}{2b|a|/a} = \frac{a}{2b} < \frac{a}{b}$$

and $2b|a|/a$ is a positive integer, we can define N as $2b|a|/a$.

Exercise 5.4.5

By Corollary 5.4.13 (the Archimedean property), there exists a positive integer b such that $b(y - x) > 1$. Thus, $a := \begin{cases} y - 1 & \text{if } y \in \mathbf{Z} \\ \lfloor y \rfloor & \text{if } y \notin \mathbf{Z} \end{cases}$ is greater than bx and less than by. Therefore, $x < \frac{a}{b} < y$.

5.5 The least upper bound property

Exercise 5.5.1

First, we show that $-M$ is a lower bound for $-E$. We assume for the sake of contradiction that some element of E, which we call $-a$, is less than $-M$. Then, we have that $a > M$. But a is in E, and therefore, $M \neq \sup(E)$.

Thus, our assumption that $-a < -M$ is wrong, and $-M$ actually is a lower bound for $-E$.

Next, we show that no other lower bound for $-E$, $-N$, is greater than $-M$. We assume for the sake of contradiction that $-N > -M$. Then, $N < M$ and N is an upper bound for E lower than M. But this is impossible from the definition of least upper bound. Hence, $-N$ is actually equal to or less than $-M$. This means that $-M = \inf(-E)$.

Exercise 5.5.3

Assume for the sake of contradiction that $m \neq m'$. Therefore, either $m > m'$ or $m' > m$. If $m' > m$, we can just switch the names of m and m', so we can just assume $m > m'$. Hence, $m \geq m' + 1$. Thus, $\frac{m-1}{n} \geq \frac{m'}{n}$. We know that $\frac{m'}{n}$ is an upper bound for E, so $\frac{m-1}{n}$ is too. But we assumed that $\frac{m-1}{n}$ is not an upper bound for E, so m is actually equal to m'.

Exercise 5.5.5

For any two real numbers $x < y$, there is a rational number a such that $x < a < y$, by Proposition 5.4.14. Using Proposition 5.4.14 again, we can see that there also is a rational number b such that $a < b < y$. Therefore, $x < a < b < y$. We can choose some rational number c such that $a - \sqrt{2} < c < b - \sqrt{2}$. Thus, $a < c + \sqrt{2} < b$. If $c + \sqrt{2} \in \mathbf{Q}$, $c + \sqrt{2} - c = \sqrt{2}$ is also rational. Therefore, $c + \sqrt{2}$ is irrational. We have finished constructing an irrational number $c + \sqrt{2}$ such that $x < c + \sqrt{2} < y$.

5.6 Real exponentiation, part I

Exercise 5.6.2 (unfinished)

Assume that $q = a/b$ and $r = c/d$, where a, b, c, and d are integers. Also, we assume that b and d are nonzero.

(a) By Definition 5.6.7, $x^q = (x^{1/b})^a$. We know that x is positive, so $x^{1/b}$ is also positive. Hence, $x^q = (x^{1/b})^a$ is also positive.

(b) Since $q = \frac{ad}{bd}$ and $r = \frac{bc}{bd}$,

$$x^{q+r} = x^{(ad+bc)/bd} = (x^{1/bd})^{ad+bc}.$$

By Proposition 5.6.3, $(x^{1/bd})^{ad+bc} = (x^{1/bd})^{ad}(x^{1/bd})^{bc}$. Thus, $x^{q+r} = x^q x^r$.

Similarly, we can prove that $(x^q)^r = x^{qr}$. First, we manipulate $(x^q)^r$:

$$\begin{aligned}(x^q)^r &= (x^{ad/bd})^{c/d} \\ &= (((x^{1/bd})^{ad})^{1/d})^c \\ &= ((x^{1/bd})^a)^c \\ &= (x^{1/bd})^{ac} \\ &= x^{ac/bd}.\end{aligned}$$

By replacing ac/bd with qr now, we obtain the result.

(c) We know that

$$x^{-q} = (x^{1/b})^{-a}.$$

Therefore, $x^{-q} = \frac{1}{(x^{1/b})^a} = \frac{1}{x^q}$.

(d) Since q is positive, we can safely assume that a and b are also positive. (If they are not, they must be both negative, and we can use $-a$ and $-b$ instead of a and b). If $x > y$, by Proposition 5.6.3 and Lemma 5.6.6(d), $(x^{1/b})^a > (y^{1/b})^a$. The left side is equal to x^q and the right side is equal to y^q. Therefore, $x^q > y^q$.

If $x^q > y^q$, $(x^{1/b})^a > (y^{1/b})^a$; thus, $((x^{1/b})^a)^{1/a} > ((y^{1/b})^a)^{1/a}$. Therefore, $x^{1/b} > y^{1/b}$. Similarly, we can obtain that $x > y$.

(e) ?

Exercise 5.6.3

If x is positive, by Proposition 5.6.3 and Proposition 4.3.12(b), $y > 0$ implies $y^2 \leq x^2$ if and only if $y \leq x$. By Definition 5.6.4, the definition of $\sqrt{x}$, $\sqrt{x^2} = \sup\{y \in \mathbf{R} : y \geq 0 \text{ and } y \leq x\}$. Therefore, $\sqrt{x^2} = x$ when x is positive.

If x is 0, it is obvious that $\sqrt{x^2} = \sup\{y \in \mathbf{R} : y = 0\} = 0$.

If x is negative, we can use the case for when x is positive except for replacing x with $-x$.

Chapter 6

Limits of sequences

6.1 Convergence and limit laws

Exercise 6.1.1

We induct on m, starting from $m = n + 1$. We have assumed that $a_{n+1} > a_n$ already. Now, if $a_m > a_n$ and $m > n$, $a_{m+1} > a_m > a_n$. We have proved $a_m > a_n$ for all $m > n$, as desired.

Exercise 6.1.3

If $(a_n)_{n=m}^{\infty}$ converges to c, for any real number $\epsilon > 0$, there exists a natural number $k \geq m$ such that $(a_n)_{n=k}^{\infty}$ is ϵ-close to c. We can define k' to be k if $k \geq m'$ and m' if $k < m'$. We know that $k' \geq m'$ and that $(a_n)_{n=k'}^{\infty}$ is ϵ-close to c from this definition. Since we can do this for any $\epsilon > 0$, $(a_n)_{n=m'}^{\infty}$ converges to c.

If $(a_n)_{n=m'}^{\infty}$ converges to c, for any real number $\epsilon > 0$, there exists a natural number $k \geq m'$ such that $(a_n)_{n=k}^{\infty}$ is ϵ-close to c. Since $k \geq m' \geq m$, $(a_n)_{n=m}^{\infty}$ converges to c.

Exercise 6.1.5

Assume $(a_n)_{n=m}^{\infty}$ converges to c, a real number. Then, for any real number $\epsilon > 0$, there exists a natural number $k \geq m$ such that $(a_n)_{n=k}^{\infty}$ is $\frac{\epsilon}{2}$-close to c. For any $A, B \geq k$, a_A and a_B must then both be $\frac{\epsilon}{2}$-close to c, and therefore $d(a_A, a_B) \leq \epsilon$. Hence, $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence.

Exercise 6.1.7

A sequence $(a_n)_{n=m}^{\infty}$ that is bounded by a positive rational number M in $\mathbf{Q}$ is obviously bounded in $\mathbf{R}$ by M.

If $(a_n)_{n=m}^{\infty}$ is bounded by a positive real number M in $\mathbf{R}$, Proposition 5.4.12 gives us a positive integer N that is greater than M. We know $N \in \mathbf{Q}$, and therefore N is a bound for $(a_n)_{n=m}^{\infty}$ using Definition 5.1.12.

Exercise 6.1.9

If $a_n = 1$ and $b_n = \frac{1}{n}$ for all natural numbers $n \geq m$,

$$\lim_{n\to\infty}(b_n) = 0.$$

But then, $a_n/b_n = n$. Therefore, $(a_n/b_n)_{n=m}^{\infty}$ is not bounded. By Corollary 6.1.17, $(a_n/b_n)_{n=m}^{\infty}$ is not convergent, and hence, Theorem 6.1.19(f) does not necessarily work when $\lim_{n\to\infty}(b_n) = 0$.

6.2 The Extended real number system

Exercise 6.2.2

(a) If $E \subseteq \mathbf{R}$, we use Definition 5.5.10 to obtain this part of this theorem. If $+\infty \in E$, $\sup E = +\infty$ and therefore $x \leq \sup E$ for any x. If $+\infty \notin E$ and $-\infty \in E$, $\sup E = \sup(E \backslash \{-\infty\})$. All finite elements of E will then be less than or equal to $\sup E$, and $\sup E \geq -\infty$.

Since $-x \leq \sup(\{-y : y \in E\})$ for all $x \in E$, all elements of E are greater than or equal to $-\sup(\{-y : y \in E\}) = \inf(E)$.

(b) When $E \subseteq \mathbf{R}$, E is non-empty and has an upper bound in $\mathbf{R}$, E is non-empty but has no upper bound in $\mathbf{R}$, or E is empty. In the first case, $\sup E \in \mathbf{R}$, and $-\infty$ is not an upper bound for E, since E is not empty. Also, $+\infty$ would be greater than $\sup E$. In the second case, $-\infty$ is not an upper bound for E and $+\infty = \sup E$. In the last case, $-\infty = \sup E$ and $+\infty > \sup E$. Therefore, $\sup E \leq M$ for any M that is an upper bound for E, since it is true both when we restrict M to be $+\infty$ or $-\infty$ and when M is forced to be real (Definition 5.5.10).

If $+\infty \in E$, only $+\infty$ is an upper bound for E, so $\sup E = +\infty = M$.

If $-\infty \in E$ but $+\infty \notin E$, $-\infty$ is not both an upper bound for E and less than $\sup E$, from the case for when $E \subseteq \mathbf{R}$ applied to $E \backslash \{-\infty\}$.

(c) We know that $\inf(E) = -\sup(\{-x : x \in E\})$. Since $-M$ is an upper bound for $\{-x : x \in E\}$, $\sup(\{-x : x \in E\}) \leq -M$ from part (b). Therefore, $\inf(E) \geq M$.

6.3 Suprema and Infima of sequences

Exercise 6.3.2

Let S be the set $\{a_n : n \geq m\}$. Then, $x = \sup(S)$ and $a_n \in S$ for any natural number $n \geq m$. Therefore, Theorem 6.2.11(a) tells us that $a_n \leq x$ for all $n \geq m$. When $M \in \mathbf{R}^*$ is an upper bound for $(a_n)_{n=m}^{\infty}$, M is also an upper bound for S. Hence, Theorem 6.2.11(b) implies that $x \leq M$. For every extended real number y for which $y < x$, there exists at least one $n \geq m$ such that $y < a_n \leq x$ because otherwise y would be an upper bound for S lower than x, which is impossible.