# term_paper_sagemath

November 28, 2024

[1]:
```
print(r'''Note: SageMath uses a different convention for quadratic forms. For␣
 ↪example, a quadratic form
[a b/2]
[b/2 c]
is instead shown as
[a b]
[* c].
Some cells here take a while to execute because my implementation of the truant␣
 ↪is inefficient.''')
```

```
Note: SageMath uses a different convention for quadratic forms. For example, a
quadratic form
[a b/2]
[b/2 c]
is instead shown as
[a b]
[* c].
Some cells here take a while to execute because my implementation of the truant
is inefficient.
```

[2]:
```
import itertools
```

[3]:
```
def truant(quad_form):
    max_for_variables = 6
    num_repr = []
    for num_pair in itertools.product([i for i in range(-max_for_variables,␣
 ↪max_for_variables)], repeat=quad_form.dim()):
        num_repr.append(quad_form(num_pair))
    for num_to_check in range(16):
        if num_to_check not in num_repr:
            return num_to_check
    return oo
# Test cases
# print(truant(QuadraticForm(ZZ, 4, [1,0,0,0,1,0,0,1,0,1])))
# print(truant(QuadraticForm(ZZ, 3, [1,0,0,1,0,1])))
```

[4]:
```
zero_dimensional_escalators = [QuadraticForm(ZZ, 0, [])]
```

[5]:
```
one_dimensional_escalators = [QuadraticForm(ZZ, 1, [1])]
```

[6]:
```
two_dimensional_escalators = [QuadraticForm(ZZ, 2, [1,0,1]), QuadraticForm(ZZ,␣
↪2, [1,0,2])]
```

[7]:
```
three_dimensional_escalators = [QuadraticForm(ZZ, 3, l) for l in␣
↪[[1,0,0,1,0,1], [1,0,0,1,0,2], [1,0,0,1,0,3], [1,0,0,2,0,2], [1,0,0,2,0,3],␣
↪[1,0,0,2,2,4], [1,0,0,2,0,4], [1,0,0,2,2,5], [1,0,0,2,0,5]]]
```

[8]:
```
print("Truants of three-dimensional integer-matrix escalators.
↪\n--------------------")
for form in three_dimensional_escalators:
    print(form)
    print(f'Truant: {truant(form)}')
    print('-'*10)
```

```
Truants of three-dimensional integer-matrix escalators.
--------------------
Quadratic form in 3 variables over Integer Ring with coefficients:
[ 1 0 0 ]
[ * 1 0 ]
[ * * 1 ]
Truant: 7
----------
Quadratic form in 3 variables over Integer Ring with coefficients:
[ 1 0 0 ]
[ * 1 0 ]
[ * * 2 ]
Truant: 14
----------
Quadratic form in 3 variables over Integer Ring with coefficients:
[ 1 0 0 ]
[ * 1 0 ]
[ * * 3 ]
Truant: 6
----------
Quadratic form in 3 variables over Integer Ring with coefficients:
[ 1 0 0 ]
[ * 2 0 ]
[ * * 2 ]
Truant: 7
----------
Quadratic form in 3 variables over Integer Ring with coefficients:
[ 1 0 0 ]
[ * 2 0 ]
[ * * 3 ]
Truant: 10
----------
```

```
Quadratic form in 3 variables over Integer Ring with coefficients:
[ 1 0 0 ]
[ * 2 2 ]
[ * * 4 ]
Truant: 7
----------
Quadratic form in 3 variables over Integer Ring with coefficients:
[ 1 0 0 ]
[ * 2 0 ]
[ * * 4 ]
Truant: 14
----------
Quadratic form in 3 variables over Integer Ring with coefficients:
[ 1 0 0 ]
[ * 2 2 ]
[ * * 5 ]
Truant: 7
----------
Quadratic form in 3 variables over Integer Ring with coefficients:
[ 1 0 0 ]
[ * 2 0 ]
[ * * 5 ]
Truant: 10
----------
```

[9]:
```
four_dimensional_escalators_raw_text = r"""1:,1,1,1,0,0,0
2:,1,1,2,0,0,0
3:,1,1,3,0,0,0
3:,1,2,2,2,0,0
4:,1,1,4,0,0,0
4:,1,2,2,0,0,0
4:,2,2,2,2,2,0
5:,1,1,5,0,0,0
5:,1,2,3,2,0,0
6:,1,1,6,0,0,0
6:,1,2,3,0,0,0
6:,2,2,2,2,0,0
7:,1,1,7,0,0,0
7:,1,2,4,2,0,0
7:,2,2,3,2,0,2
8:,1,2,4,0,0,0
8:,1,3,3,2,0,0
8:,2,2,2,0,0,0
8:,2,2,3,2,2,0
9:,1,2,5,2,0,0
9:,1,3,3,0,0,0
9:,2,2,3,0,0,2
```

```
10:,1,2,5,0,0,0
10:,2,2,3,2,0,0
10:,2,2,4,2,0,2
11:,1,2,6,2,0,0
11:,1,3,4,2,0,0
12:,1,2,6,0,0,0
12:,1,3,4,0,0,0
12:,2,2,3,0,0,0
12:,2,2,4,0,0,2
13:,2,2,5,2,0,2
13:,2,3,3,2,2,0
14:,1,2,7,0,0,0
14:,1,3,5,2,0,0
14:,2,2,4,2,0,0
15:,1,2,8,2,0,0
15:,1,3,5,0,0,0
15:,2,2,5,0,0,2
15:,2,3,3,0,2,0
16:,1,2,8,0,0,0
16:,2,2,4,0,0,0
16:,2,3,3,2,0,0
17:,1,2,9,2,0,0
17:,1,3,6,2,0,0
17:,2,3,4,0,2,2
18:,1,2,9,0,0,0
18:,1,3,6,0,0,0
18:,2,2,5,2,0,0
18:,2,3,3,0,0,0
18:,2,3,4,2,0,2
19:,1,2,10,2,0,0
19:,2,3,4,2,2,0
20:,1,2,10,0,0,0
20:,2,2,5,0,0,0
20:,2,2,6,2,2,0
20:,2,4,4,4,2,0
21:,2,3,4,0,2,0
22:,1,2,11,0,0,0
22:,2,2,6,2,0,0
22:,2,3,4,2,0,0
22:,2,3,5,0,2,2
23:,1,2,12,2,0,0
23:,2,3,5,2,0,2
24:,1,2,12,0,0,0
24:,2,2,6,0,0,0
24:,2,2,7,2,2,0
24:,2,3,4,0,0,0
24:,2,4,4,0,2,2
```

```
24:,2,4,4,4,0,0
25:,1,2,13,2,0,0
25:,2,3,5,2,2,0
26:,1,2,13,0,0,0
26:,2,2,7,2,0,0
26:,2,4,4,2,2,0
27:,1,2,14,2,0,0
27:,2,3,5,0,2,0
27:,2,4,5,4,0,2
28:,1,2,14,0,0,0
28:,2,2,7,0,0,0
28:,2,3,5,2,0,0
28:,2,4,4,0,2,0
28:,2,4,5,4,2,0
30:,2,3,5,0,0,0
30:,2,4,4,2,0,0
31:,2,3,6,2,2,0
31:,2,4,5,0,2,2
32:,2,4,4,0,0,0
32:,2,4,5,4,0,0
33:,2,3,6,0,2,0
33:,2,4,5,2,0,2
34:,2,3,6,2,0,0
34:,2,4,5,2,2,0
34:,2,4,6,4,0,2
35:,2,4,5,0,0,2
36:,2,3,6,0,0,0
36:,2,4,5,0,2,0
36:,2,4,6,4,2,0
36:,2,5,5,4,2,2
37:,2,5,5,4,2,0
38:,2,4,5,2,0,0
38:,2,4,6,0,2,2
39:,2,3,7,0,2,0
40:,2,3,7,2,0,0
40:,2,4,5,0,0,0
40:,2,4,6,2,0,2
40:,2,4,6,4,0,0
41:,2,4,7,4,0,2
42:,2,3,7,0,0,0
42:,2,4,6,0,0,2
42:,2,4,6,2,2,0
42:,2,5,5,4,0,0
43:,2,3,8,2,2,0
43:,2,5,5,2,0,2
44:,2,4,6,0,2,0
45:,2,4,7,0,2,2
```

```
45:,2,5,5,0,2,0
45:,2,5,6,4,2,2
46:,2,3,8,2,0,0
46:,2,4,6,2,0,0
46:,2,5,6,4,0,2
47:,2,4,7,2,0,2
47:,2,5,6,4,2,0
48:,2,3,8,0,0,0
48:,2,4,6,0,0,0
48:,2,5,5,2,0,0
49:,2,3,9,2,2,0
49:,2,4,7,0,0,2
49:,2,5,6,0,2,2
50:,2,4,7,2,2,0
50:,2,5,5,0,0,0
51:,2,3,9,0,2,0
52:,2,3,9,2,0,0
52:,2,5,6,2,0,2
52:,2,5,6,4,0,0
53:,2,5,6,2,2,0
54:,2,3,9,0,0,0
54:,2,4,7,2,0,0
54:,2,5,6,0,0,2
54:,2,5,7,4,2,2
55:,2,3,10,2,2,0
55:,2,5,6,0,2,0
55:,2,5,7,4,0,2
56:,2,4,7,0,0,0
56:,2,4,8,4,0,0
57:,2,3,10,0,2,0
58:,2,3,10,2,0,0
58:,2,4,8,2,2,0
58:,2,5,6,2,0,0
58:,2,5,7,0,2,2
60:,2,3,10,0,0,0
60:,2,4,9,4,2,0
60:,2,5,6,0,0,0
61:,2,5,7,2,0,2
62:,2,4,8,2,0,0
62:,2,5,7,4,0,0
63:,2,5,7,0,0,2
63:,2,5,7,2,2,0
64:,2,4,8,0,0,0
66:,2,4,9,2,2,0
67:,2,5,8,4,2,0
68:,2,4,9,0,2,0
68:,2,4,10,4,2,0
```

```
68:,2,5,7,2,0,0
70:,2,4,9,2,0,0
70:,2,5,7,0,0,0
72:,2,4,9,0,0,0
72:,2,4,10,4,0,0
72:,2,5,8,4,0,0
74:,2,4,10,2,2,0
76:,2,4,10,0,2,0
77:,2,5,9,4,2,0
78:,2,4,10,2,0,0
78:,2,5,8,2,0,0
80:,2,4,10,0,0,0
80:,2,4,11,4,0,0
80:,2,5,8,0,0,0
82:,2,4,11,2,2,0
82:,2,5,9,4,0,0
83:,2,5,9,2,2,0
85:,2,5,9,0,2,0
86:,2,4,11,2,0,0
87:,2,5,10,4,2,0
88:,2,4,11,0,0,0
88:,2,4,12,4,0,0
88:,2,5,9,2,0,0
90:,2,4,12,2,2,0
90:,2,5,9,0,0,0
92:,2,4,13,4,2,0
92:,2,5,10,4,0,0
93:,2,5,10,2,2,0
94:,2,4,12,2,0,0
95:,2,5,10,0,2,0
96:,2,4,12,0,0,0
96:,2,4,13,4,0,0
98:,2,4,13,2,2,0
98:,2,5,10,2,0,0
100:,2,4,13,0,2,0
100:,2,4,14,4,2,0
100:,2,5,10,0,0,0
102:,2,4,13,2,0,0
104:,2,4,13,0,0,0
104:,2,4,14,4,0,0
106:,2,4,14,2,2,0
108:,2,4,14,0,2,0
110:,2,4,14,2,0,0
112:,2,4,14,0,0,0"""
four_dimensional_escalators_processed_text = [[int(str_num) for str_num in␣
 ↪string.split(',')[1:]] for string in four_dimensional_escalators_raw_text.
 ↪splitlines()]
```

```
four_dimensional_escalators = []
for num_list in four_dimensional_escalators_processed_text:
    a,b,c,d,e,f = num_list
    four_dimensional_escalators.append(QuadraticForm(2*matrix([[1,0,0,0],␣
 ↪[0,a,f/2,e/2], [0,f/2,b,d/2], [0,e/2,d/2,c]])))
for quad_form in four_dimensional_escalators:
    print(quad_form.matrix()/2)
    print('\n')
print('Table of non-universal quaternary integer-matrix␣
 ↪escalators\n--------------------')
four_dimensional_non_universal_escalators = []
for quad_form in four_dimensional_escalators:
    a = truant(quad_form)
    if a != oo:
        four_dimensional_non_universal_escalators.append(quad_form)
        print(f'{quad_form}\nTruant: {a}')
        print('-'*10)

print(f'The number of nonuniversal quaternary escalators is␣
 ↪{len(four_dimensional_non_universal_escalators)}.')
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]


[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 2]


[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 3]


[1 0 0 0]
[0 1 0 0]
[0 0 2 1]
[0 0 1 2]


[1 0 0 0]
[0 1 0 0]
```

```
[0 0 1 0]
[0 0 0 4]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 0]
[0 0 0 2]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 2 1]
[0 1 1 2]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 5]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 1]
[0 0 1 3]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 6]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 0]
[0 0 0 3]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 1]
[0 0 1 2]
```

```
[1 0 0 0]
[0 1 0 0]
```

```
[0 0 1 0]
[0 0 0 7]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 1]
[0 0 1 4]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 2 1]
[0 0 1 3]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 0]
[0 0 0 4]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 3 1]
[0 0 1 3]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 0]
[0 0 0 2]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 2 1]
[0 1 1 3]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 1]
[0 0 1 5]
```

```
[1 0 0 0]
[0 1 0 0]
```

```
[0 0 3 0]
[0 0 0 3]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 2 0]
[0 0 0 3]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 0]
[0 0 0 5]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 1]
[0 0 1 3]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 2 1]
[0 0 1 4]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 1]
[0 0 1 6]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 3 1]
[0 0 1 4]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 0]
[0 0 0 6]
```

```
[1 0 0 0]
[0 1 0 0]
```

```
[0 0 3 0]
[0 0 0 4]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 0]
[0 0 0 3]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 2 0]
[0 0 0 4]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 2 1]
[0 0 1 5]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 3 1]
[0 1 1 3]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 0]
[0 0 0 7]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 3 1]
[0 0 1 5]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 1]
[0 0 1 4]
```

```
[1 0 0 0]
[0 1 0 0]
```

```
[0 0 2 1]
[0 0 1 8]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 3 0]
[0 0 0 5]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 2 0]
[0 0 0 5]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 3 0]
[0 1 0 3]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 0]
[0 0 0 8]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 0]
[0 0 0 4]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 3 1]
[0 0 1 3]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 1]
[0 0 1 9]
```

```
[1 0 0 0]
[0 1 0 0]
```

```
[0 0 3 1]
[0 0 1 6]
```

```
[1 0 0 0]
[0 2 1 1]
[0 1 3 0]
[0 1 0 4]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 2 0]
[0 0 0 9]
```

```
[1 0 0 0]
[0 1 0 0]
[0 0 3 0]
[0 0 0 6]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 1]
[0 0 1 5]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 3 0]
[0 0 0 3]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 3 1]
[0 0 1 4]
```

```
[ 1  0  0  0]
[ 0  1  0  0]
[ 0  0  2  1]
[ 0  0  1 10]
```

```
[1 0 0 0]
[0 2 0 1]
```

```
[0 0 3 1]
[0 1 1 4]
```

```
[ 1  0  0  0]
[ 0  1  0  0]
[ 0  0  2  0]
[ 0  0  0 10]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 0]
[0 0 0 5]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 2 1]
[0 1 1 6]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 2]
[0 1 2 4]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 3 0]
[0 1 0 4]
```

```
[ 1  0  0  0]
[ 0  1  0  0]
[ 0  0  2  0]
[ 0  0  0 11]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 1]
[0 0 1 6]
```

```
[1 0 0 0]
[0 2 0 0]
```

```
[0 0 3 1]
[0 0 1 4]
```

```
[1 0 0 0]
[0 2 1 1]
[0 1 3 0]
[0 1 0 5]
```

```
[ 1  0  0  0]
[ 0  1  0  0]
[ 0  0  2  1]
[ 0  0  1 12]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 3 1]
[0 0 1 5]
```

```
[ 1  0  0  0]
[ 0  1  0  0]
[ 0  0  2  0]
[ 0  0  0 12]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 0]
[0 0 0 6]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 2 1]
[0 1 1 7]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 3 0]
[0 0 0 4]
```

```
[1 0 0 0]
[0 2 1 1]
```

```
[0 1 4 0]
[0 1 0 4]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 2]
[0 0 2 4]
```

```
[ 1  0  0  0]
[ 0  1  0  0]
[ 0  0  2  1]
[ 0  0  1 13]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 3 1]
[0 1 1 5]
```

```
[ 1  0  0  0]
[ 0  1  0  0]
[ 0  0  2  0]
[ 0  0  0 13]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 1]
[0 0 1 7]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 1]
[0 1 1 4]
```

```
[ 1  0  0  0]
[ 0  1  0  0]
[ 0  0  2  1]
[ 0  0  1 14]
```

```
[1 0 0 0]
[0 2 0 1]
```

```
[0 0 3 0]
[0 1 0 5]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 4 2]
[0 0 2 5]
```

```
[ 1  0  0  0]
[ 0  1  0  0]
[ 0  0  2  0]
[ 0  0  0 14]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 2 0]
[0 0 0 7]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 3 1]
[0 0 1 5]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 0]
[0 1 0 4]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 2]
[0 1 2 5]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 3 0]
[0 0 0 5]
```

```
[1 0 0 0]
[0 2 0 0]
```

```
[0 0 4 1]
[0 0 1 4]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 3 1]
[0 1 1 6]
```

```
[1 0 0 0]
[0 2 1 1]
[0 1 4 0]
[0 1 0 5]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 0]
[0 0 0 4]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 2]
[0 0 2 5]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 3 0]
[0 1 0 6]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 4 1]
[0 0 1 5]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 3 1]
[0 0 1 6]
```

```
[1 0 0 0]
[0 2 0 1]
```

```
[0 0 4 1]
[0 1 1 5]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 4 2]
[0 0 2 6]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 4 0]
[0 0 0 5]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 3 0]
[0 0 0 6]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 0]
[0 1 0 5]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 2]
[0 1 2 6]
```

```
[1 0 0 0]
[0 2 1 1]
[0 1 5 2]
[0 1 2 5]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 5 2]
[0 1 2 5]
```

```
[1 0 0 0]
[0 2 0 0]
```

```
[0 0 4 1]
[0 0 1 5]
```

```
[1 0 0 0]
[0 2 1 1]
[0 1 4 0]
[0 1 0 6]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 3 0]
[0 1 0 7]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 3 1]
[0 0 1 7]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 0]
[0 0 0 5]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 4 1]
[0 0 1 6]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 2]
[0 0 2 6]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 4 2]
[0 0 2 7]
```

```
[1 0 0 0]
[0 2 0 0]
```

```
[0 0 3 0]
[0 0 0 7]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 4 0]
[0 0 0 6]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 1]
[0 1 1 6]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 2]
[0 0 2 5]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 3 1]
[0 1 1 8]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 5 1]
[0 0 1 5]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 0]
[0 1 0 6]
```

```
[1 0 0 0]
[0 2 1 1]
[0 1 4 0]
[0 1 0 7]
```

```
[1 0 0 0]
[0 2 0 1]
```

```
[0 0 5 0]
[0 1 0 5]
```

```
[1 0 0 0]
[0 2 1 1]
[0 1 5 2]
[0 1 2 6]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 3 1]
[0 0 1 8]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 1]
[0 0 1 6]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 5 2]
[0 0 2 6]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 4 1]
[0 0 1 7]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 5 2]
[0 1 2 6]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 3 0]
[0 0 0 8]
```

```
[1 0 0 0]
[0 2 0 0]
```

```
[0 0 4 0]
[0 0 0 6]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 1]
[0 0 1 5]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 3 1]
[0 1 1 9]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 4 0]
[0 0 0 7]
```

```
[1 0 0 0]
[0 2 1 1]
[0 1 5 0]
[0 1 0 6]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 1]
[0 1 1 7]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 0]
[0 0 0 5]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 3 0]
[0 1 0 9]
```

```
[1 0 0 0]
[0 2 0 0]
```

```
[0 0 3 1]
[0 0 1 9]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 5 1]
[0 0 1 6]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 2]
[0 0 2 6]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 5 1]
[0 1 1 6]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 3 0]
[0 0 0 9]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 1]
[0 0 1 7]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 5 0]
[0 0 0 6]
```

```
[1 0 0 0]
[0 2 1 1]
[0 1 5 2]
[0 1 2 7]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
```

```
[ 0  0  3  1]
[ 0  1  1 10]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 5 0]
[0 1 0 6]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 5 2]
[0 0 2 7]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 0]
[0 0 0 7]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 2]
[0 0 2 8]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  3  0]
[ 0  1  0 10]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  3  1]
[ 0  0  1 10]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 1]
[0 1 1 8]
```

```
[1 0 0 0]
[0 2 0 0]
```

```
[0 0 5 1]
[0 0 1 6]
```

```
[1 0 0 0]
[0 2 1 1]
[0 1 5 0]
[0 1 0 7]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  3  0]
[ 0  0  0 10]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 2]
[0 1 2 9]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 0]
[0 0 0 6]
```

```
[1 0 0 0]
[0 2 1 0]
[0 1 5 1]
[0 0 1 7]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 1]
[0 0 1 8]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 2]
[0 0 2 7]
```

```
[1 0 0 0]
[0 2 1 0]
```

```
[0 1 5 0]
[0 0 0 7]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 5 1]
[0 1 1 7]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 0]
[0 0 0 8]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 1]
[0 1 1 9]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 5 2]
[0 1 2 8]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 4 0]
[0 1 0 9]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  4  2]
[ 0  1  2 10]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 1]
[0 0 1 7]
```

```
[1 0 0 0]
[0 2 0 0]
```

```
[0 0 4 1]
[0 0 1 9]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 0]
[0 0 0 7]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 4 0]
[0 0 0 9]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  2]
[ 0  0  2 10]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 2]
[0 0 2 8]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  4  1]
[ 0  1  1 10]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  4  0]
[ 0  1  0 10]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 5 2]
[0 1 2 9]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
```

```
[ 0  0  4  1]
[ 0  0  1 10]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 1]
[0 0 1 8]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  0]
[ 0  0  0 10]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  2]
[ 0  0  2 11]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 0]
[0 0 0 8]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  4  1]
[ 0  1  1 11]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 2]
[0 0 2 9]
```

```
[1 0 0 0]
[0 2 0 1]
[0 0 5 1]
[0 1 1 9]
```

```
[1 0 0 0]
[0 2 0 1]
```

```
[0 0 5 0]
[0 1 0 9]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  1]
[ 0  0  1 11]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  5  2]
[ 0  1  2 10]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  0]
[ 0  0  0 11]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  2]
[ 0  0  2 12]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 1]
[0 0 1 9]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  4  1]
[ 0  1  1 12]
```

```
[1 0 0 0]
[0 2 0 0]
[0 0 5 0]
[0 0 0 9]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
```

```
[ 0  0  4  2]
[ 0  1  2 13]


[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  5  2]
[ 0  0  2 10]


[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  5  1]
[ 0  1  1 10]


[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  1]
[ 0  0  1 12]


[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  5  0]
[ 0  1  0 10]


[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  0]
[ 0  0  0 12]


[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  2]
[ 0  0  2 13]


[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  4  1]
[ 0  1  1 13]


[ 1  0  0  0]
[ 0  2  0  0]
```

```
[ 0  0  5  1]
[ 0  0  1 10]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  4  0]
[ 0  1  0 13]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  4  2]
[ 0  1  2 14]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  5  0]
[ 0  0  0 10]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  1]
[ 0  0  1 13]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  0]
[ 0  0  0 13]
```

```
[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  2]
[ 0  0  2 14]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
[ 0  0  4  1]
[ 0  1  1 14]
```

```
[ 1  0  0  0]
[ 0  2  0  1]
```

```
[ 0  0  4  0]
[ 0  1  0 14]


[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  1]
[ 0  0  1 14]


[ 1  0  0  0]
[ 0  2  0  0]
[ 0  0  4  0]
[ 0  0  0 14]


Table of non-universal quaternary integer-matrix escalators
--------------------
Quadratic form in 4 variables over Rational Field with coefficients:
[ 1 0 0 0 ]
[ * 2 0 2 ]
[ * * 3 0 ]
[ * * * 4 ]
Truant: 10
----------
Quadratic form in 4 variables over Rational Field with coefficients:
[ 1 0 0 0 ]
[ * 2 2 0 ]
[ * * 4 2 ]
[ * * * 5 ]
Truant: 10
----------
Quadratic form in 4 variables over Rational Field with coefficients:
[ 1 0 0 0 ]
[ * 2 2 0 ]
[ * * 5 2 ]
[ * * * 5 ]
Truant: 15
----------
Quadratic form in 4 variables over Rational Field with coefficients:
[ 1 0 0 0 ]
[ * 2 0 0 ]
[ * * 5 0 ]
[ * * * 5 ]
Truant: 15
----------
Quadratic form in 4 variables over Rational Field with coefficients:
[ 1 0 0 0 ]
```

```
[ * 2 0 2 ]
[ * * 5 4 ]
[ * * * 8 ]
Truant: 15
----------
Quadratic form in 4 variables over Rational Field with coefficients:
[ 1 0 0 0 ]
[ * 2 0 2 ]
[ * * 5 2 ]
[ * * * 9 ]
Truant: 15
----------
The number of nonuniversal quaternary escalators is 6.
```

[10]:
```
#not needed
three_dim_esc_lattice = Matrix([[1,0,0],[0,2,1],[0,1,4]])
three_dim_esc_quad_form = QuadraticForm(2*three_dim_esc_lattice)
Zmodsixteen = IntegerModRing(16)
nums_repr_by_lattice_mod_sixteen =␣
 ↪{Zmodsixteen(three_dim_esc_quad_form([a,b,c])) for a in range(16) for b in␣
 ↪range(16) for c in range(16)}
print(2,16)
print([num for num in Zmodsixteen if num not in␣
 ↪nums_repr_by_lattice_mod_sixteen])
print('----')
for prime_num in [3,5,7,11]:
    Zmodprimesq = IntegerModRing(prime_num^2)
    nums_repr_by_lattice_mod_primesq =␣
 ↪{Zmodprimesq(three_dim_esc_quad_form([a,b,c])) for a in range(prime_num^2)␣
 ↪for b in range(prime_num^2) for c in range(prime_num^2)}
    print(prime_num, prime_num^2)
    print(nums_repr_by_lattice_mod_primesq)
    print([num for num in Zmodprimesq if num not in␣
 ↪nums_repr_by_lattice_mod_primesq])
    print('----')
```

```
2 16
[]
----
3 9
{0, 1, 2, 3, 4, 5, 6, 7, 8}
[]
----
5 25
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,
22, 23, 24}
[]
----
```

```
7 49
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22,
23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 43, 44,
45, 46, 47, 48}
[21, 35, 42]
----
11 121
{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,
22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41,
42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61,
62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81,
82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100,
101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116,
117, 118, 119, 120}
[]
----
```

[ ]: