PMATH 451: Measure and Integration

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Contents

1. Chapter 1

1.1 Lecture 1 A brief review of Riemann integrals

Limitations of Riemann Integration (R-int)

(1) Heavily rely on the structure of real line $\mathbb{R}$

(2) Not many functions are R-int

> **Theorem 1.1 — Lebesgue.**
> $f[a,b] \to \mathbb{R}$ is R-int if and only if the set of discontinuity of f is Lebesgue null set (has Lebesgue measure 0). (i.e. $\exists (a_n, b_n)$ s.t. the set of discontinuities $\subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$, $\sum(b_n - a_n) < \epsilon$)

■ **Example 1.1** $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$, $x \in [0,1]$. f is nowhere continuous. f is NOT R-int

(3) NOT well behaved under limits

■ **Example 1.2** Let $\{r_k\}_{k=1}^{\infty}$ be all $\mathbb{Q}$ in $[0,1]$, $f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \dots, r_n\} \\ 0 & \text{otherwise} \end{cases}$

- f_n is R-int
- $\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \notin \mathbb{Q} \cap [0,1] \end{cases} = f(x)$ (pointwise limit) is not R-int

Lebesgue's Idea

Ideally, define $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$

- $m([a,b]) = b - a$
- $m(A + x) = m(A)$
- $m(\bigcup_{n=1}^{\infty} A_n) = \sum m(A_n)$, A_n disjoint

Problem: m does not exists

Proof. Define $x \sim y$ if $x - y \in \mathbb{Q}$, consider $[0,1]$.

Let A = pick one x from each eq-class of $\sim$.
Let $\{r_k\}_{k=1}^{\infty}$ be all rationals in $[-1, 1]$.
Let $A_k = A + r_k$

(1) A_k are disjoint. If $x \in A_k \cap A_\ell$, then

$$x = \underbrace{a}_{\in A} + r_k = \underbrace{b}_{\in A} + r_\ell \implies a - b = r_\ell - r_k \in \mathbb{Q}$$

$$\implies a \sim b, a \neq b$$

not possible

(2) $[0, 1] \subseteq \bigcup_{n=1}^{\infty} A_n \subseteq [-1, 2]$
   (a) $A \subseteq [0, 1],\ -1 \leq r_k \leq 1,\ -1 \leq a + r_k \leq 2, A + r_k \subseteq [-1, 2]$
   (b) $\forall x \in [0, 1], \exists a \in A,\ a \sim x,\ x - a \in \mathbb{Q},\ -1 \leq x - a \leq 1, \implies x - a = r_k$ for some k, $x = a + r_k \in A + r_k \subseteq \bigcup_{n=1}^{\infty} A_n$
   (c)

$$1 = m([0, 1]) \leq m(\bigcup_{n=1}^{\infty} A_n) \leq m([-1, 2]) = 3$$

$$m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} m(A)$$

not possible

■

1.2 Lecture 2 Introduction to Sigma Algebra

Definition 1.1 — algebra.
Let X be a set. An **algebra** $\mathcal{A}$ is a collection of subsets of X ($\mathcal{A} \subseteq (x)$) s.t.
(1) $\varnothing \in \mathcal{A}$
(2) $X \backslash A \in \mathcal{A}$ for all $A \in \mathcal{A}$
(3) If $A_1, \ldots, A_n \in \mathcal{A}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$
We call $\mathcal{A}$ a $\sigma-$**algebra** if
(3') If $A_1, \ldots, A_n, \ldots \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

■ **Example 1.3**
(1) X be any set, $\{\varnothing, X\}$ is a $\sigma-$algebra

Note 1.1 If $\mathcal{A}$ is a $\sigma-$algebra, then it's an algebra

(2) $\mathcal{P}(X)$ is a $\sigma-$algebra
(3) Let X be an uncountable set (real line, Cantor set, etc.) Let $\mathcal{A} = \{E \subseteq X :$ either E is countable, or $X \backslash E$ is countable$\}$.
Claim: $\mathcal{A}$ is a $\sigma-$algebra

Proof. (a) $\varnothing \in \mathcal{A}$, $\varnothing$ is countable
(b) Let $E \in \mathcal{A}$,
- Case 1: E is countable, $X \backslash (X \backslash E) = E$ is countable, $\implies X \backslash E \in \mathcal{A}$
- Case 2: $X \backslash E$ is countable, $\implies X \backslash E \in \mathcal{A}$

(c) Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, consider $\bigcup_{n=1}^{\infty} E_n$
- Case 1: If all E_n are countable, $\bigcup_{n=1}^{\infty} E_n$ is countable $\implies \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$

- Case 2: $\exists E_N$, s.t. $X\backslash E_N$ is countable.

$$X\backslash\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcap_{n=1}^{\infty}(X\backslash E_n) \subseteq X\backslash E_N \implies \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$$

Hence $\mathcal{A}$ is a $\sigma-$algebra. ■

1.3 Lecture 3 Properties of Sigma Algebra

Basic properties of Algebra and $\sigma-$algebra

Let $\mathcal{A}$ be an algebra, $\mathcal{B}$ be a $\sigma-$algebra

- $E, F \in \mathcal{A} \implies E \cap F,\ E\backslash F \in \mathcal{A}$
- $\{E_n\} \subseteq \mathcal{B} \implies \bigcap_{n=1}^{\infty} E_n \in \mathcal{B}$

Proof.

$$X\backslash(E\cap F) = \underbrace{(X\backslash E)}_{\in\mathcal{A}} \cup \underbrace{(X\backslash F)}_{\in\mathcal{A}} \in \mathcal{A}$$
$$E \cap F = X\backslash(X\backslash(E\cup F)) \in \mathcal{A}$$
$$E\backslash F = E \cap (X\backslash F) \in \mathcal{A}$$

Exercise 1.1 $\bigcup_{n=1}^{k} E_n \in \mathcal{A}$, induction

$$X\backslash\left(\bigcap_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty}(X\backslash E_n) \in \mathcal{B}$$

■

Proposition 1.2
Let $\mathcal{A}$ be an algebra, $\forall\{A_n\}_{n=1}^{\infty}$ disjoint $\implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. Then $\mathcal{A}$ is a $\sigma-$algebra

Proof. Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, goal is to show $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. Consider

$$E_1 : A_1 = E_1$$
$$E_2 : A_2 = E_2\backslash E_1$$
$$E_3 : A_3 = E_3\backslash(E_1 \cup E_2)$$
$$\vdots$$
$$E_n : A_n = E_n\backslash\left(\bigcup_{i=1}^{n-1} E_i\right)$$
$$\vdots$$

Claim:

(1) $\bigcup_{n=1}^{k} A_n = \bigcup_{n=1}^{k} E_n$ (exercise using induction)

(2) A_n are disjoint.

$A_n \cap \left(\bigcup_{i=1}^{n-1} E_i\right) = \varnothing$

$A_n \cap \left(\bigcup_{i=1}^{n-1} A_i\right) = \varnothing$

(3) $A_n \in \mathcal{A}$.

By definition, $A_n = E_n\backslash\left(\bigcup_{i=1}^{n-1} E_i\right) \in \mathcal{A}$

$\implies \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ (by assumption)
$\implies \mathcal{A}$ is $\sigma-$algebra ■

Proposition 1.3
Suppose $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ are $\sigma-$algebras (on X). Then $\bigcap_{\lambda\in\Lambda} \mathcal{B}_\lambda$ is a $\sigma-$algebra

Proof.
(1) $\varnothing \in \mathcal{B}_\lambda,\ \forall\lambda \implies \varnothing \in \bigcap_{\lambda\in\Lambda} \mathcal{B}_\lambda$
(2) Take

$$\begin{aligned} A \in \bigcap_{\lambda\in\Lambda} \mathcal{B}_\lambda &\implies A \in \mathcal{B}_\lambda\ \forall\lambda \\ &\implies X\backslash A \in \mathcal{B}_\lambda\ \forall\lambda \\ &\implies X\backslash A \in \bigcap_{\lambda\in\Lambda} \mathcal{B}_\lambda\ \forall\lambda \end{aligned}$$

(3) Take

$$\{A_n\}_{n=1}^{\infty} \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{B}_\lambda \implies \bigcup_{n=1}^{\infty} A_n \in \bigcap_{\lambda\in\Lambda} \mathcal{B}_\lambda \text{ (exercise)}$$

■

Definition 1.2
Let $\mathcal{F} \subseteq \mathcal{P}(X)$. Define $\sigma-$algebra generated by $\mathcal{F}$ to be $\sigma(\mathcal{F}) = \bigcap_{\lambda\in\Lambda} \mathcal{B}_\lambda$, $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ are all σ-algebra containing $\mathcal{F}$

(1) $\mathcal{F} \subseteq \mathcal{P}(X)$, $\mathcal{P}(X)$ is a $\sigma-$algebra, so this intersection makes sense.
(2) Since $\mathcal{B}_\lambda$ is $\sigma-$algebra $\implies \bigcup_{\lambda\in\Lambda} \mathcal{B}_\lambda$ is a $\sigma-$algebra ($\sigma(\mathcal{F})$ is a $\sigma-$algebra)

Proposition 1.4
$\sigma(\mathcal{F})$ is the **smallest** $\sigma-$algebra containing $\mathcal{F}$.
By smallest: if $\exists\mathcal{B}$ $\sigma-$algebra, $\mathcal{B} \supseteq \mathcal{F}$, then $\sigma(\mathcal{F}) \subseteq B$

Proof. By defn, $\sigma(\mathcal{F}) = \bigcap_{\lambda\in\Lambda} \mathcal{B}_\lambda$, $\sigma-$alg $\mathcal{B} \supseteq \mathcal{F} \implies \mathcal{B} = \mathcal{B}_{\lambda_0}$ for some λ_0. $\sigma(\mathcal{F}) = \bigcap_{\lambda\in\Lambda} \mathcal{B}_\lambda \subseteq \mathcal{B}_{\lambda_0} = \mathcal{B}$ ■

1.3.1 Borel $\sigma-$algebra

Let X be a metric space, let $g = \{A \subseteq X : A \text{ open}\}$

Definition 1.3 — Borel $\sigma-$algebra.
the **Borel $\sigma-$alg** on X is $\sigma(\mathcal{G})$. Denote by $\mathcal{B}_X = \sigma(\mathcal{G})$

Notation 1.1.
- $\mathcal{G}$ = *set of open sets*
- $\mathcal{F}$ = *set of closed sets*
- $\mathcal{G}_\delta = \{\bigcap_{n=1}^{\infty} A_n : A_n \in \mathcal{G}\}$
- $\mathcal{F}_\sigma = \{\bigcup_{n=1}^{\infty} A_n : A_n \text{ closed}\}$
  $\mathcal{G}_\delta, \mathcal{F}_\sigma \subseteq \mathcal{B}_X$
- *a set* $A \subseteq X$ *is Borel if* $A \in \mathcal{B}_X$
  - *open set*

- *closed set*
- $\mathcal{G}_\delta,\ \mathcal{F}_\sigma$
- $X=\mathbb{R},\ (a,b]\in\mathcal{B}_X$

1.4 Lecture 4 Measure

Let $\mathcal{B}$ be a $\sigma-$algebra on X.

Definition 1.4 — measure, measurable space, measure space.
A **measure** on $(X,\mathcal{B})$ is a map $\mu:\mathcal{B}\to\mathbb{R}$, s.t.
(1) $\mu(\varnothing)=0$
(2) (Positivity) $\mu:\mathcal{B}\to[0,+\infty]$
(3) ($\sigma-$additivity) $\forall\{A_n\}\subseteq\mathcal{B}$ disjoint, $u\left(\bigcup_{n=1}^\infty A_n\right)=\sum_{n=1}^\infty\mu(A_n)$
We call $(X,\mathcal{B})$ a **measurable space**, $(X,\mathcal{B},\mu)$ a **measure space**

■ **Example 1.4**
(1) Let $\mu(B)=0,\ \forall B\in\mathcal{B}$
(2) Let $\mu(B)=\begin{cases}+\infty, & \forall B\in\mathcal{B}\\ 0, & B=\varnothing\end{cases}$
(3) (Dirac's measure) $x\in X$, $\delta_x(B)=\begin{cases}1, & \text{if } x\in B\\ 0, & \text{if } x\notin B\end{cases}$

Proof. δ_x is a measure:
(a) $\delta_x(\varnothing)=0, x\notin\varnothing$
(b) $\delta_x:\mathcal{B}\to\{0,1\}\subseteq[0,+\infty]$
(c) Let $\{A_n\}_{n=1}^\infty\subseteq\mathcal{B}$ disjoint. Consider $\delta_x\left(\bigcup_{n=1}^\infty A_n\right)$
case 1: $x\in\bigcup_{n=1}^\infty A_n$, $\delta_x\left(\bigcup_{n=1}^\infty A_n\right)=1$. $\exists k,\ x\in A_k,\ \delta_x(A_k)=1$. $\forall n\neq k$, by A_n disjoint, $x\notin A_n\cap A_k=\varnothing \implies x\notin A_n \implies \delta_x(A_n)=0$. $\sum_{n=1}^\infty\delta_x(A_n)=\delta(A_k)=1=\delta_x\left(\bigcup_{n=1}^\infty A_n\right)$
case 2: $x\notin\bigcup_{n=1}^\infty A_n \implies x\notin\bigcup_{n=1}^\infty A_n,\forall n \implies \delta_x(A_n)=0$
$\delta_x\left(\bigcup_{n=1}^\infty A_n\right)=0 \implies \sum_{n=1}^\infty\delta_x(A_n)=0$

■

(4) Assume μ_1,μ_2 are measures on $(X,\mathcal{B})$. μ_1,μ_2 are finite ($\mu_i(B)<+\infty$). Consider $t_1,t_2\geq0$, let $\mu=t_1\mu_1+t_2\mu_2$ (a linear combination). $\mu(B)=t_1\dot\mu_1(B)+t_2\dot\mu_2(B)$.
Claim: μ is a measure (exercise)

Definition 1.5 — finite, $\sigma-$finite, semi-finite, probability measure.
Let μ be a measure
(1) μ is **finite** is $\mu(B)<+\infty$ for all $B\in\mathcal{B}$. (equivalent to say $\mu(X)<+\infty$)
(2) μ is $\sigma-$**finite** if $X=\bigcup_{n=1}^\infty E_n,\ E_n\in\mathcal{B},\ \mu(E_n)<+\infty$. ($\mathbb{R}=\bigcup[-n,n]$)
(3) μ is **semi-finite** if $\forall F\in\mathcal{B}$, $\mu(F)\neq0$, then $\exists E\subseteq F$, $0<\mu(E)<\infty$
(4) μ is a **probability measure** if $\mu(X)=1$

Proposition 1.5 — Properties of Measures.

(1) Monotonicity: $E,F\in\mathcal{B}$, $E\subseteq F$, then $\mu(E)\leq\mu(F)$
(2) $\sigma-$subadditivity: If $\{E_n\}_{n=1}^\infty\subseteq\mathcal{B}$, $\mu\left(\bigcup_{n=1}^\infty E_n\right)\geq\sum_{n=1}^\infty\mu(E_n)$
(3) Continuity from below: If $E_1\subseteq E_2\subseteq\cdots$, then $\mu\left(\bigcup_{n=1}^\infty E_n\right)=\lim_{n\to\infty}\mu(E_n)$

(4) Continuity from above If $E_1 \supseteq E_2 \supseteq \cdots$ and $\mu(E_1) < \infty$, then $\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$

Proof.
(1) $\mu(F) = \mu(E \cup (F \backslash E)) = \mu(E) + \mu(F\backslash E) \geq \mu(E)$
(2) write a countable union as a countable disjoint union
$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n$, A_n disjoint. $(A_n = E_n \backslash \left(\bigcup_{i=1}^{n-1} E_i\right))$. $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$ ($\sigma-$additivity). (Each $\mu(A_n \leq \mu(E_n))$).

Note 1.2 $\mu\left(\bigcup_{n=1}^{N} E_n\right) \leq \sum_{n=1}^{N} \mu(E_n)$ this is called "subadditivity"

(3) Claim: Let $\{F_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$, $\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{k\to\infty} \mu\left(\bigcup_{n=1}^{k} F_n\right)$.

Proof. Write $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} A_n$, A_n disjoint. $A_n = F_n \backslash \left(\bigcup_{i=1}^{n-1} F_i\right)$, $\bigcup_{n=1}^{k} F_n = \bigcup_{n=1}^{k} A_n$. $\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \lim_{k\to\infty} \sum_{n=1}^{k} \mu(A_n) = \lim_{k\to\infty\mu(\bigcup_{n=1}^{k} A_n)} = \lim_{k\to\infty\mu(\bigcup_{n=1}^{\infty} F_n)}$ ∎

Now $E_1 \subseteq E_2 \subseteq \cdots$, $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{k\to\infty} \mu\left(\bigcup_{n=1}^{k} E_n\right) = \lim_{k\to\infty\, k}$

∎

Definition 1.6 — complete.
A measure μ on $(X, \mathcal{B})$ is **complete** if $\forall N \in \mathcal{B}$ with $\mu(N) = 0$, $E \subseteq N \implies E \in \mathcal{B}$

■ **Example 1.5** Take m, the Lebesgue measure on $\mathbb{R}$, $(\mathbb{R}, , m)$ is complete. (is the Lebesgue measurable sets). $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$ is not complete ($\mathcal{B}_{\mathbb{R}}$ is the Borel $\sigma-$algebra). In $\mathbb{R}$ you can find some measure zero set that is not Borel

Theorem 1.6
Let μ be a measure on $(X, \mathcal{B})$, let $\overline{B} = \sigma(\mathcal{B}, \{E : E \subseteq N, \mu(N) = 0\})$. Then there is a unique complete measure $\overline{\mu}$ on $(X, c\overline{B})$, $\overline{\mu}_{\mathcal{B}} = \mu$ (i.e. $\forall B \in \mathcal{B}, \overline{\mu}(B) = \mu(B)$)

Lemma 1.7
$\overline{B} = \{E \cup F : \ E \in \mathcal{B},\ F \subseteq N,\ \mu(N) = 0\}$

Proof. Suffices to show the RHS is a $\sigma-$alg.
- $\varnothing \in \overline{B}$
- $E \cup F \in \overline{B}$, $F \subseteq N$, $\mu(N) = 0$, $X\backslash(E \cup F) = \underbrace{X\backslash(E \cup N)}_{\in \mathcal{B}} \cup \underbrace{(N \backslash (E \cup F))}_{\subseteq N,\ \mu(N)=0} \in$ RHS.
- $E_i \cup F_i \in \overline{B}$, $F_i \subseteq N_i$, $\mu(N_i) = 0$. $\bigcup_{i=1}^{\infty}(E_i \cup F_i) = \underbrace{\left(\bigcup_{i=1}^{\infty} E_i\right)}_{\in \mathcal{B}} \cup \underbrace{\left(\bigcup_{i=1}^{\infty} F_i\right)}_{\subseteq \cup N_i,\ \mu(\cup N_i)=0}$ $\in$RHS,
$\bigcup_{i=1}^{\infty} F_i \subseteq \bigcup_{i=1}^{\infty} N_i$, $0 = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) \leq \sum_{i=1}^{\infty} \mu(N_i) = 0$

∎

Proof. thm 1.6:
On $\overline{\mathcal{B}}$, define $\overline{\mu}$ such that $\overline{\mu}(E \cup F) := \mu(E)$.
(1) We first need to show $\overline{\mu}$ is well-defined. Let $A = E_i \cup F_i = E_2 \cup F_2$, $F_i \subseteq N_i$. $\mu(N_i) = 0$. Need $\mu(E_1) = \mu(E_2)$. We claim $\mu(E_1) = \mu(E_2) = \mu(E_1 \cap E_2)$. Let $E = E_1 \cap E_2$, then

$\mu(E) \le \mu(E_i)$.
$E_1 \subseteq (N_1 \cup N_2) \cup (E_1 \cap E_2) = E \cup N_1 \cup N_2,\ \mu(E) \le \mu(E_1) \le \mu(E \cup N_1 \cup N_2) \le \mu(E) + \mu(N_1) + \mu(N_2) = \mu(E) \implies \mu(E) = \mu(E_1) = \mu(E_1)$

(2) We need to show $\overline{\mu}$ is a mesaure
- $\overline{\mu}(\varnothing) = 0$, $\overline{\mu} \ge 0$
- Let $E_n \cup F_n \in \overline{\mathcal{B}}$ disjoint.

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty}(E_n \cup F_n)\right) = \overline{\mu}\left(\underbrace{\left(\bigcup_{i=1}^{\infty} E_i\right)}_{\in \mathcal{B}} \cup \underbrace{\left(\bigcup_{i=1}^{\infty} F_i\right)}_{\subseteq \cup N_i,\ \mu(\cup N_i)=0}\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

$$E_n \text{ disjoint} = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \overline{\mu}(E_n \cup F_n)$$

- $\forall B \in \mathcal{B},\ B = B \cup \varnothing \in \overline{B}$, $\overline{\mu}(B \cup \varnothing) = \overline{\mu}(B) = \mu(B)$, $\overline{\mu}|_{\mathcal{B}} = \mu$
- $\overline{\mu}$ is complete. $\forall M \in \overline{\mathcal{B}}$, $\overline{\mu}(M) = 0$, $\forall G \subseteq M$. Need $G \in \overline{\mathcal{B}}$.
  $M \in \overline{\mathcal{B}} \implies M \in E \cup F,\ E \in \mathcal{B},\ F \subseteq N,\ \mu(N) = 0.\ \overline{\mu}(M) = 0 \implies \mu(E) = 0.$
  $G \subset M = E \cup F \subseteq E \cup N,\ \mu(E \cup N) = 0 \implies G \in \overline{\mathcal{B}}$

■

Definition 1.7 — completion.
$(X, \overline{\mathcal{B}}, \overline{\mu})$ is called the completion of $(X, \mathcal{B}, \mu)$