PMATH 450: Lesbegue Integration and Fourier Analysis

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Contents

1. Week 1

Goals of PMATH 450:

(1) Develop a theory of integration for functions $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ which is
   (a) more flexible (than Riemann) (applicable to more functions)
   (b) more rich (nicer results)
   (c) still extends Riemann integration

(2) Introduce Harmonic Analysis

General outline (first half):

(1) Which sets should we integrate over?
   - Measurable sets

(2) Which functions should we try to integrate?
   - Measurable functions

1.1 Borel Sets

Definition 1.1 — σ-algebra.
Consider a set X, we call $\mathcal{A} \subseteq \mathcal{P}(X)$ (which is the power set of X) a **σ-algebra** of subsets of X if

(1) $\varnothing \in \mathcal{A}$
(2) $A \in \mathcal{A} \implies X \backslash A \in \mathcal{A}$
(3) $A_1, A_2, A_3, \ldots \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

So a σ-algebra is a collection of subsets of X which contains the empty set, is closed under set difference and is closed under countable unions.

R Consider $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-algebra

(1) $X \in \mathcal{A}$
$X \backslash \varnothing = X \in \mathcal{A}$
(2) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$
$A \cup B = A \cup B \cup \varnothing \cup \varnothing \cup \cdots \in \mathcal{A}$

(3) $A_1, A_2, \ldots \in \mathcal{A} \implies \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$
$\bigcap_{i=1}^{\infty} A_i = X \setminus (\bigcup_{i=1}^{\infty} (X \setminus A_i))$
(4) $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$

■ Example 1.1

- $\{\varnothing, X\}$ is the smallest σ-algebra you could have given X
- $\mathcal{A} = \mathcal{P}(X)$ is a σ-algebra
- $\mathcal{A} = \{A \subseteq \mathbb{R} : A \text{ is open}\}$ is **NOT** a σ-algebra.
  it is not closed under set difference, consider $A = (0,1) \in \mathcal{A}$, $\mathbb{R} \setminus A = (-\infty, 0] \cup [1, \infty) \notin \mathcal{A}$ because it is not open.
- $\mathcal{A} = \{A \subseteq \mathbb{R} : A \text{ open or closed}\}$ is **NOT** a σ-algebra.
  Consider $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \notin \mathcal{A}$ since $\mathbb{Q}$ is neither open nor closed

Proposition 1.1
Given a set X and $\mathcal{C} \subseteq \mathcal{P}(X)$, then $\mathcal{A} = \cap\{\mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{-algebra }, \mathcal{C} \subseteq \mathcal{B}\}$ is a σ-algebra. It is the smallest σ-algebra containing $\mathcal{C}$.

Definition 1.2 — Borel Sets.
Consider $\mathcal{C} = \{A \subseteq \mathbb{R} : A \text{ is open}\}$ (this is s subset of power set of $\mathbb{R}$), then $\mathcal{A} = \bigcap\{\mathcal{B} : \mathcal{C} \subseteq \mathcal{B}, \mathcal{B} \text{ is } \sigma\text{-algebra}\}$ is called **Borel σ-algebra**. The elements of $\mathcal{A}$ is are called the **Borel sets**.

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(1) All the open sets are Borel. i.e. open $\implies$ Borel.
(2) All the closed sets are Borel. i.e. closed $\implies$ Borel.
since σ-algebra are closed under set difference, and $\mathbb{R}$ take away open is closed
(3) $\{X_1, X_2, \ldots\} = \bigcup_{i=1}^{\infty} \{X_i\}$ is Borel. i.e. countable $\implies$ Borel.
In particular, $\mathbb{Q}$ is a Borel set which is neither open nor closed.
(4) $[a, b) = [a, b] \setminus \{b\} = [a, b] \cap (\mathbb{R} \setminus \{b\})$ is Borel.

It is actually very hard to construct a set that is not Borel. The Borel sets are the propriate sets to integrate over.

1.2 Outer Measure

Idea

(1) Given $A \subseteq \mathbb{R}$, how should we "measure" the "size" of A
(2) Some sets have "sizes" which "measure" more nicely then others. Which ones? Borel sets?

Goal

Define a function $m : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{\infty\}$ (called a measure) such that

(1) $m((a, b)) = m([a, b]) = m((a, b]) = b - a$ (the measure of an interval I equals the length of I)
(2) $m(A \cup B) \leq m(A) + m(B)$
(3) If $A \cap B = \varnothing$, then $m(A \cup B) = m(A) + m(B)$

It will be shown later in the course that we may not use $\mathcal{P}(\mathbb{R})$

Idea

Given $A \subseteq \mathbb{R}$, there exists bounded, open intervals $I_i = (a_i, b_i)$ s.t. $A \subseteq \bigcup_{i=1}^{\infty} I_i$. We want:

$$m(A) \leq \sum_{i=1}^{\infty} m(I_i) = \sum_{i=1}^{\infty} \underbrace{L(I_i)}_{\text{the length of } I_i} = \sum_{i=1}^{\infty} (b_i - a_i)$$

Cover A by bounded, open, intervals as finely as possible.

Definition 1.3 — Outer Measure.
We define (Lebesgue) **outer measure** by $m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{\infty\}$ with

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} L(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i, I_i \text{ is a bounded, open interval} \right\}$$

■ **Example 1.2** Consider the $\varnothing$ (we would like to the size of it been zero). For any $\epsilon > 0$, $\varnothing \subseteq (0, \epsilon)$, by definition $m^*(\varnothing) \leq L((0, \epsilon)) = \epsilon$. Since $m^*(\varnothing) \geq 0$, $m^*(\varnothing) = 0$.

■ **Example 1.3** Consider a countable set $A = \{x_1, x_2, x_3, \ldots\}$, given any $\epsilon > 0$, then $A \subseteq \bigcup_{i=1}^{\infty}(x_i - \frac{\epsilon}{2^{i+1}}, x_i + \frac{\epsilon}{2^{i+1}})$. $m^*(A) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \frac{\epsilon}{2} \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = \frac{\epsilon}{2}(\frac{1}{1-\frac{1}{2}}) = \epsilon$. Since $\epsilon > 0$ was arbitrary, $m^*(A) = 0$
Follow a similar proof, we can show that the finite sets also have outer measure zero

Goal

Prove that if I is an interval, then $m^*(I) = L(I)$

Proposition 1.2
If $A \subseteq B$, then $m^*(A) \leq m^*(B)$

Proof. Sketch:
$X = \{\sum_{i=1}^{\infty} L(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i\}$
$Y = \{\sum_{i=1}^{\infty} L(I_i) : B \subseteq \bigcup_{i=1}^{\infty} I_i\}$
Clearly if $A \subseteq B$ then $Y \subseteq X$ (if you have intervals cover B then they must cover A), hence $\inf X \leq \inf Y$ using the ordering of the extended real numbers. i.e. $m^*(A) \leq m^*(B)$ ■

Lemma 1.3
If $a, b \in \mathbb{R}$ with $a \leq b$, then $m^*([a, b]) = b - a$

We start with the closed bounded intervals because they are compact, so as soon as you cover this closed interval with countable union of open intervals, then you only need to take finitely many of them because you are guaranteed to have a finite subcover to cover the interval.

Proof. Let $\epsilon > 0$ be given. Since $[a, b] \subseteq (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$, we see that $m^*([a, b]) \leq b - a + \epsilon$. Since $\epsilon > 0$ was arbitrary, $m^*([a, b]) \leq b - a$.
Let I_i ($i \in \mathbb{N}$) be bounded open intervals s.t. $[a, b] \subseteq \bigcup_{i=1}^{\infty} I_i$. Since $[a, b]$ is compact, there exists $n \in \mathbb{N}$ s.t. $[a, b] \subseteq \bigcup_{i=1}^{n} I_i$, hence $b - a \leq \sum_{i=1}^{n} L(I_i) \leq \sum_{i=1}^{\infty} L(I_i)$ (the first inequality can be proved by induction), and so $m^*([a, b]) \geq b - a$ (since $m^*([a, b])$ is the greatest lower bound) ■

Proposition 1.4
If I is an interval then $m^*(I) = L(I)$

Proof.
(1) Suppose I is bounded with endpoints $a \leq b$
Given $\epsilon > 0$, note $I \subseteq [a,b] \implies m^*(I) \leq m^*([a,b]) = b-a$. Also $[a+\frac{\epsilon}{2}, b-\frac{\epsilon}{2}] \subseteq I \implies b-a-\epsilon \leq m^*(I) \implies b-a \leq m^*(I)$.
(2) Suppose I is unbounded.
For all $n \in \mathbb{N}$, there exists I_n s.t. $I_n \subseteq I$, $L(I_n) = n$. Then $m^*(I) \geq m^*(I_n) = n$, hence $m^*(I) = \infty = L(I)$

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1.3 Basic Properties of Outer Measure

Outer measure is
(1) Translation Invariant
(2) Countably Subadditivity

Notation
Given $x \in \mathbb{R}, A \subseteq \mathbb{R}$, then $x + A = \{x + a : a \in A\}$

Proposition 1.5 — Translation Invariant.
$m^*(x + A) = m^*(A)$

Proof. Sketch:

$$\begin{aligned} m^*(x+A) &= \inf\left\{\sum L(I_i) : x + A \subseteq \bigcup I_i\right\} \\ &= \inf\left\{\sum L(I_i) : A \subseteq \bigcup(-x + I_i)\right\} \\ &= \inf\left\{\sum L(-x + I_i) : A \subseteq \bigcup(-x + I_i)\right\} \\ &= \inf\left\{\sum L(J_i) : A \subseteq \bigcup(J_i)\right\} \\ &= m^*(A) \end{aligned}$$

■

Proposition 1.6 — Countable Subadditivity.
If we take countably many subset $A_i \subseteq \mathbb{R} (i \in \mathbb{N})$ then $m^*(\bigcup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty m^*(A_i)$

Proof. We may assume each $m^*(A_i) < \infty$ (otherwise the result will be trivial). Let $\epsilon > 0$ be given and fix $i \in \mathbb{N}$. There exists open, bounded intervals $I_{i,j}$ s.t. $A_i \subseteq \bigcup_{i=1}^\infty I_{i,j}$ and $\sum_{j=1}^\infty L(I_{i,j}) \leq m^*(A_i) + \frac{\epsilon}{2^i}$ (Note that we add a little bit on the out measure which makes it no longer a lower bound hence we can find the $I_{i,j}$, this is a common technique when working with outer measure). We see that $\bigcup_{i=1}^\infty A_i \subseteq \bigcup_{i,j} I_{i,j}$ and so $m^*(A_i) \leq \sum_{i,j} L(I_{i,j}) = \sum_{i=1}^\infty \sum_{j=1}^\infty L(I_{i,j}) \leq \sum_{i=1}^\infty m^*(A_i) + \frac{\epsilon}{2^i} = \sum_{i=1}^\infty m^*(A_i) + \epsilon$. Since ϵ is arbitrary, the proposition follows. ■

Corollary 1.7 — Finite Subadditivity.
If $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{R})$, then $m^*(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n m^*(A_i)$

Proof. Sketch:
$A_1 \cup \cdots \cup A_n = A_1 \cup \cdots \cup A_n \cup \varnothing \cup \varnothing \cup \cdots$ ■

Problem

There exists $A, B \subseteq \mathbb{R}$ s.t. $A \cap B = \varnothing$ and $m^*(A \cup B) < m^*(A) + m^*(B)$. i.e. outer measure is not finitely additive. (We would like $m^*(A \cup B) = m^*(A) + m^*(B)$ for disjoint sets A, B)

Solution

Restrict the domain of m^* to only include sets which measure "nicely" (which are called measurable).

2. Week 2

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