CS 476: Numeric Computation for Financial Modeling

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Contents

1. Week 1

1.1 General Derivative Contracts

Outline

(1) General definition of a financial derivative contract
(2) Standard options
(3) Payoff function

Definition 1.1 — Financial options.
A **financial option/derivative** is a financial contract stipulated today at $t = 0$. The value of the contract at the future *expiry* T is determined exactly by the market price of an *underlying asset* at T.

We don't know the value of the underlying asset at T, but we know the contract's value in relation to the underlying asset price ar T.

- The underlying asset can be: stock, commodity, market index, interest rate/bond, exchange rate

Notation 1.1 (S_t)**.** *Let S_t or $S(t)$ denote the underlying price at time t, a stochastic process.*

$$\begin{tikzcd} \boxed{\text{Holder}} \arrow[r, dash] & \boxed{S_t,\ T,\ \text{Payoff}(S_T)} \arrow[r, dash] & \boxed{\text{Writer}} \end{tikzcd}$$

Knowing the future value of the contract in relation to the underlying allows it to be used as an insurance.

1.1.1 European calls and puts

Definition 1.2 — European call.
A **European call** option is the *right* to *buy* underlying asset at a preset strike price K. The right can only be exercised at the expiry T.

Asymmetry: holder has the option to exercise. Writer has the obligation

Definition 1.3 — European put.
A **European put** option is the *right* to sell underlying asset at a preset strike price K. The right can only be exercised at the expiry T.

Definition 1.4 — American put.
An American call option is the *right* to *sell* underlying asset at a preset strike price K. The right can be exercised any time from now to the expiry T.

- Holder: Buyer of the option, enters a **long** position
- Writer: Seller of the option, enters a *short* position

Notation 1.2 ($V(S(t),t)$ or V_t)**.** *Let* $V(S(t),t)$ *or* V_t *denote the option value at time* t.
$V_T = \textit{payoff}(S_T)$

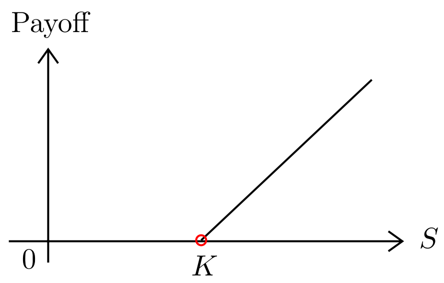
Central Question

- What is the fair value of V_0 of the option today?
- How should a writer hedge risk?

What are the payoff functions $V_T = \text{payoff}(S_T)$ for calls/puts?

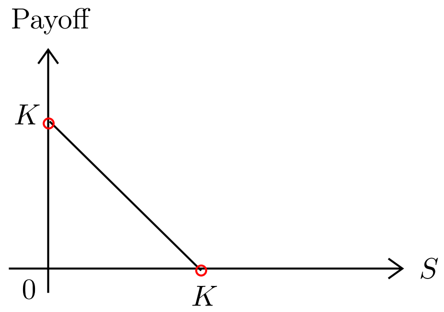
1.1.2 Call value at the expiry T

If $S_T \leq K$, holder should not exercise the call. $V_T = 0$
If $S_T \geq K$, holder exercises the right $V_T = S_T - K$
$\implies V_T = \text{payoff}(S_T) = \max(S_t - K, 0)$



1.1.3 Put value at the expiry T

If $S_T \leq K$, holder should exercise the put. $V_T = K - S_T$
If $S_T \geq K$, holder should not exercises
$\implies V_T = \text{payoff}(S_T) = \max(K - S_t, 0)$



■ **Example 1.1**
- Bulb wholesaler can purchase a call to have the option of buying tulip \$0.5 a dozen at a fixed price in 3 months.
- Bulb growers can purchase a put to allow selling tulip \$1 a dozen at a fixed price in 3 months.

A bet on the underlying price can be done by trading either S_t or V_t.

Note 1.1 Option is more risky compared to the underlying (leverage effect):

$$\left|\frac{S_T - S_0}{S_O}\right| << \left|\frac{V_T - V_0}{V_0}\right|, \; v_t = 0 \implies \frac{V_T - V_0}{V_0} = -100\%$$

When option expires out of money, 100% loss for the option holder.

We focus on stock option with expiry $T \leq 1$ and interest rate randomness is reasonably ignored. (because we will only look into short term contracts in this course)
- Stock: a share in ownership of a company
- Dividend: payment to shareholder from the profits

Note 1.2 When stock pays dividend to the shareholder, holder of option on the stock receives nothing. Option is said to be *dividend protected*.

2. Week 2

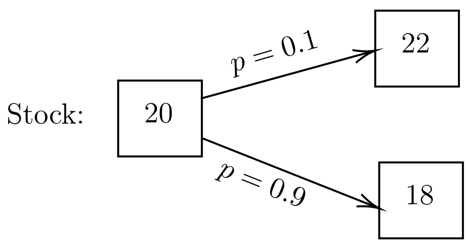
2.1 Option Pricing

Outline

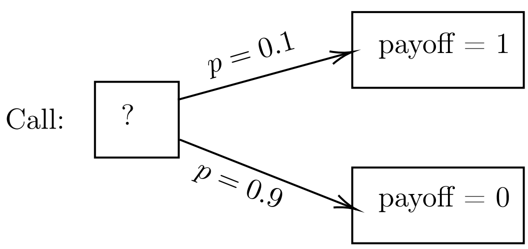
(1) One period binomial
(2) Fair value of option
(3) Arbitrage
(4) Put - Call parity

2.1.1 One-period Binomial Case

Consider a one-period binomial case. Assume $T = 1$ and up probability $p = 0.1$



Consider a call with $K = 21$. What do we know of the option value at the expiry $T = 1$?



What is the option value today?
Is it $0.11 + 0.9 \times 0 = 0.1$ using probability?
What about time value of the money, i.e. interest rate?

Riskless Asset (constant interest rate)

Cash account continuously compounds at risk free rate $r \geq 0$.
Lending (depositing) money to a bank $\implies$ buying a bond from a bank
Borrowing money from a bank $\implies$ selling a bond.
Let $\beta(t)$ denote the value of a riskless bond at time t

$$\frac{d\beta(\tau)}{\beta(\tau)} = r d\tau \implies \int_t^T \frac{d\beta(\tau)}{\beta(\tau)} = \int_t^T r d\tau$$

$$\log(\beta(T)) - \log(\beta(t)) = r(T - t)$$

- Discounting: $\beta(T) = 1 \implies \beta(t) = e^{-r(T-t)}$
- Compounding: $\beta(t) = 1 \implies \beta(T) = e^{r(T-t)}$

Back to the example 2.1.1, is the fair value $0.1 \times e^{-0.05}$ if $r = 0.05$?
No this is not the fair value.

Determine the fair value by trading

Determining the fair option value needs to consider trading in a financial market of bond, stock, option.

Definition 2.1 — Arbitrage.
An **arbitrage** is a trading opportunity to make a no-risk(guaranteed) profit which is greater than that of a bank deposit which earns the interest rate $r \geq 0$.

Definition 2.2 — Fair value.
The **fair value** of a financial instrument is the price which does not lead to arbitrage.

Why? Arbitrage can only occur momentarily.
How? Under no arbitrage: two instruments have the same values at a future time, they must be priced at the same price today.

■ **Example 2.1 — Constructing Arbitrage.** We represent a trading strategy as a portfolio. Buy one share of stock and borrow \$100 (sell bonds) today

$$\Pi_0 = \underbrace{1}_{\text{long}} \times S_0 \underbrace{-}_{short} 100 \quad \text{or} \quad \Pi_0 = \{S_0, -100\}$$

A long position benefits from increased prices and a short position benefits from decreased prices. The value of this portfolio at time t:

$$\Pi_t = S_t - 100e^{rt}$$

Mathematical Characterization of an Arbitrage Strategy

An arbitrage strategy can be described as

- A portfolio with an initial value $\Pi_0 = 0$ but $\Pi_T > 0$ for all $T > 0$
- A portfolio with an initial value $\Pi_0 < 0$ but $\Pi_T \geq 0$ for all $T > 0$

Put and Call Parity

Proposition 2.1 — Put-Call Parity.
Assume stock S_t does not pay dividend, interest rate $r \geq 0$, and no arbitrage. Then at any time $0 \geq t \geq T$, European call C_t and put P_t, with the same strike K and expiry T, on the same underlying, satisfy

$$C_t = P_t + S_t - Ke^{-r(T-t)}$$

Proof. At time T we have

- $C_T = \max(S_T - K, 0)$
- $P_T = \max(K - S_T, 0)$
- $C_T - P_T = \max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K$

Hence, since no arbitrage, $C_t - P_t = S_t - Ke^{-r(T-t)}$ ∎