CS 476: Numeric Computation for Financial Modeling

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Contents

1. Week 1

1.1 General Derivative Contracts

Outline

(1) General definition of a financial derivative contract
(2) Standard options
(3) Payoff function

Definition 1.1 — Financial options.
A **financial option/derivative** is a financial contract stipulated today at $t = 0$. The value of the contract at the future *expiry* T is determined exactly by the market price of an *underlying asset* at T.

We don't know the value of the underlying asset at T, but we know the contract's value in relation to the underlying asset price ar T.

- The underlying asset can be: stock, commodity, market index, interest rate/bond, exchange rate

Notation 1.1 (S_t)**.** *Let S_t or $S(t)$ denote the underlying price at time t, a stochastic process.*

$$\boxed{\text{Holder}} \;\text{———}\; \boxed{S_t,\ T,\ \text{Payoff}(S_T)} \;\text{———}\; \boxed{\text{Writer}}$$

Knowing the future value of the contract in relation to the underlying allows it to be used as an insurance.

1.1.1 European calls and puts

Definition 1.2 — European call.
A **European call** option is the *right* to *buy* underlying asset at a preset strike price K. The right can only be exercised at the expiry T.

Asymmetry: holder has the option to exercise. Writer has the obligation

Definition 1.3 — European put.
A **European put** option is the *right* to sell underlying asset at a preset strike price K. The right can only be exercised at the expiry T.

Definition 1.4 — American put.
An American call option is the *right* to *sell* underlying asset at a preset strike price K. The right can be exercised any time from now to the expiry T.

- Holder: Buyer of the option, enters a **long** position
- Writer: Seller of the option, enters a *short* position

Notation 1.2 ($V(S(t), t)$ or V_t). *Let* $V(S(t), t)$ *or* V_t *denote the option value at time* t.
$V_T = payoff(S_T)$

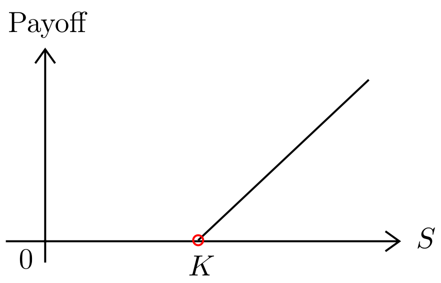
Central Question

- What is the fair value of V_0 of the option today?
- How should a writer hedge risk?

What are the payoff functions $V_T = \text{payoff}(S_T)$ for calls/puts?

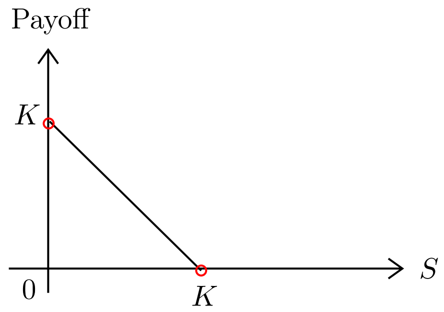
1.1.2 Call value at the expiry T

If $S_T \leq K$, holder should not exercise the call. $V_T = 0$
If $S_T \geq K$, holder exercises the right $V_T = S_T - K$
$\implies V_T = \text{payoff}(S_T) = \max(S_t - K, 0)$



1.1.3 Put value at the expiry T

If $S_T \leq K$, holder should exercise the put. $V_T = K - S_T$
If $S_T \geq K$, holder should not exercises
$\implies V_T = \text{payoff}(S_T) = \max(K - S_t, 0)$



■ **Example 1.1**

- Bulb wholesaler can purchase a call to have the option of buying tulip \$0.5 a dozen at a fixed price in 3 months.
- Bulb growers can purchase a put to allow selling tulip \$1 a dozen at a fixed price in 3 months.

A bet on the underlying price can be done by trading either S_t or V_t.

Note 1.1 Option is more risky compared to the underlying (leverage effect):

$$\left|\frac{S_T - S_0}{S_O}\right| << \left|\frac{V_T - V_0}{V_0}\right|, \ v_t = 0 \implies \frac{V_T - V_0}{V_0} = -100\%$$

When option expires out of money, 100% loss for the option holder.

We focus on stock option with expiry $T \leq 1$ and interest rate randomness is reasonably ignored. (because we will only look into short term contracts in this course)

- Stock: a share in ownership of a company
- Dividend: payment to shareholder from the profits

Note 1.2 When stock pays dividend to the shareholder, holder of option on the stock receives nothing. Option is said to be *dividend protected*.

2. Week 2

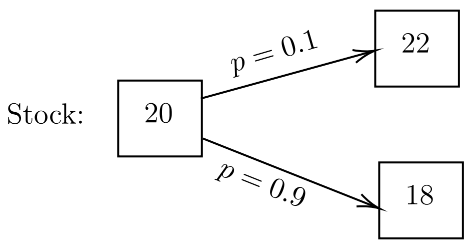
2.1 Option Pricing

Outline

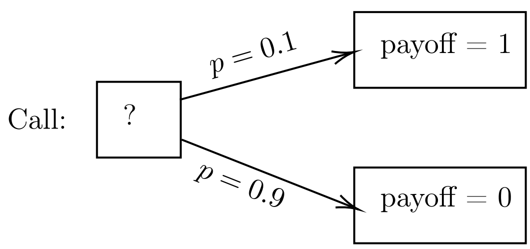
(1) One period binomial
(2) Fair value of option
(3) Arbitrage
(4) Put - Call parity

2.1.1 One-period Binomial Case

Consider a one-period binomial case. Assume $T = 1$ and up probability $p = 0.1$



Consider a call with $K = 21$. What do we know of the option value at the expiry $T = 1$?



What is the option value today?
Is it $0.11 + 0.9 \times 0 = 0.1$ using probability?
What about time value of the money, i.e. interest rate?

Riskless Asset (constant interest rate)

Cash account continuously compounds at risk free rate $r \geq 0$.
Lending (depositing) money to a bank $\implies$ buying a bond from a bank
Borrowing money from a bank $\implies$ selling a bond.
Let $\beta(t)$ denote the value of a riskless bond at time t

$$\frac{d\beta(\tau)}{\beta(\tau)} = r d\tau \implies \int_t^T \frac{d\beta(\tau)}{\beta(\tau)} = \int_t^T r d\tau$$

$$\log(\beta(T)) - \log(\beta(t)) = r(T-t)$$

- Discounting: $\beta(T) = 1 \implies \beta(t) = e^{-r(T-t)}$
- Compounding: $\beta(t) = 1 \implies \beta(T) = e^{r(T-t)}$

Back to the example 2.1.1, is the fair value $0.1 \times e^{-0.05}$ if $r = 0.05$?
No this is not the fair value.

Determine the fair value by trading

Determining the fair option value needs to consider trading in a financial market of bond, stock, option.

Definition 2.1 — Arbitrage.
An **arbitrage** is a trading opportunity to make a no-risk(guaranteed) profit which is greater than that of a bank deposit which earns the interest rate $r \geq 0$.

Definition 2.2 — Fair value.
The **fair value** of a financial instrument is the price which does not lead to arbitrage.

Why? Arbitrage can only occur momentarily.
How? Under no arbitrage: two instruments have the same values at a future time, they must be priced at the same price today.

■ **Example 2.1 — Constructing Arbitrage.** We represent a trading strategy as a portfolio. Buy one share of stock and borrow \$100 (sell bonds) today

$$\Pi_0 = \underbrace{1}_{\text{long}} \times S_0 \underbrace{-}_{\text{short}} 100 \quad \text{or} \quad \Pi_0 = \{S_0, -100\}$$

A long position benefits from increased prices and a short position benefits from decreased prices. The value of this portfolio at time t:

$$\Pi_t = S_t - 100e^{rt}$$

Mathematical Characterization of an Arbitrage Strategy

An arbitrage strategy can be described as

- A portfolio with an initial value $\Pi_0 = 0$ but $\Pi_T > 0$ for all $T > 0$
- A portfolio with an initial value $\Pi_0 < 0$ but $\Pi_T \geq 0$ for all $T > 0$

2.1.2 Put and Call Parity

Proposition 2.1 — Put-Call Parity.
Assume stock S_t does not pay dividend, interest rate $r \geq 0$, and no arbitrage. Then at any time $0 \geq t \geq T$, European call C_t and put P_t, with the same strike K and expiry T, on the same underlying, satisfy

$$C_t = P_t + S_t - Ke^{-r(T-t)}$$

Proof. At time T we have

- $C_T = \max(S_T - K, 0)$
- $P_T = \max(K - S_T, 0)$
- $C_T - P_T = \max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K$

Hence, since no arbitrage, $C_t - P_t = S_t - Ke^{-r(T-t)}$ ■

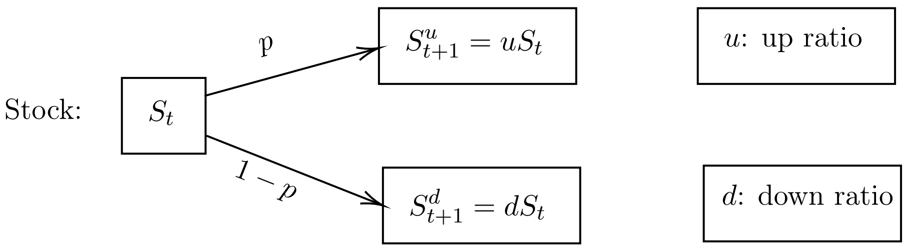
Outline

Consider 1-period in a binomial model

- Option replication and hedging
- Computing option fair value
- Risk neutral valuation

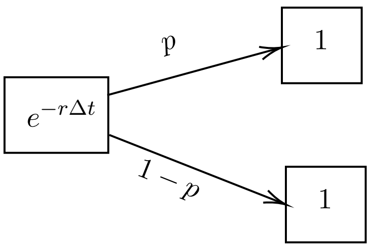
2.1.3 Pricing by Replication in a Binomial Model

Consider a binomial model with an up probability $p > 0$

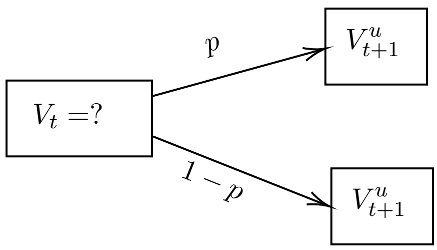


Assume:

- $S_t > 0$
- the length of time interval $\Delta t > 0$
- $0 < d < u$, u, d be given
- **no arbitrage**



Bond



Option

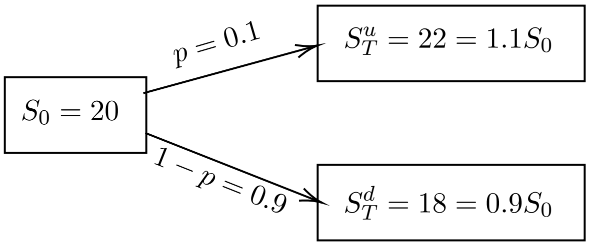
Let V^d_{t+1} and V^u_{t+1} be given, e.g. at T, they equal to corresponding payoffs. We would like to the the fair value of the option V_t.

At time t, construct portfolio $\{\delta_t S_t, \eta_t \beta_t\}$ such that replication equation is satisfied

$$\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\text{bond: } t+1} \eta_t + \underbrace{\begin{bmatrix}uS_t\\dS_t\end{bmatrix}}_{\text{stock: } t+1} \delta_t = \underbrace{\begin{bmatrix}V_{t+1}^u\\V_{t+1}^d\end{bmatrix}}_{\text{option at } t+1} \tag{2.1}$$

Note that the solution is unique (under the assumption $u > d$).
Hold δ_t unit of underlying stock, η_t unit of underlying bond.
For a simple binomial model, we can always construct a unique portfolio of underlying stock and bond to hold at the beginning of the period, such that this portfolio at the end of period, always equal to the value of the option
No arbitrage $\implies V_t = \delta_t S_t + \eta_t \beta_t = \delta_t S_t + \eta_t e^{-r\Delta t}$
The fact that **there exists a replicating portfolio of stock and bond for the option is important**: $\{V_t, -\delta_t S_t\}$ is risk-free (exactly the same as a bond).
Note that p is irrelevant. (This is a consequence of no arbitrage.)

■ Example 2.2



Assume $r = 0$. For a call with strike $K = 21$. To determine a replicating portfolio, solve

$$22\delta + 1 \cdot \eta = 1 \ (= C_T^u)$$
$$18\delta + 1 \cdot \eta = 0 \ (= C_T^d)$$

$$\implies \begin{cases} \delta & = \frac{C_T^u - C_T^d}{(u-d)S_0} = 0.25 \\ \eta & = -4.5 \text{ (sell bond (borrow cash))} \end{cases}$$

$$\implies C_0 = \delta S_0 + \eta \beta_0 = 0.25 \times 20 - 4.5 = 0.5$$

What if the market price of call is greater than 0.5, e.g., 0.95?
Position at $t = 0$: sell call for 0.95, buy $0.25S_0$, cash $= -4.5$ (borrow): the portfolio value:

$$\Pi_0 = -0.95 + 0.025 \times 20 - 4.5 = -0.45$$

At time T, the total value of positions

$$-C_T + 0.25S_T - 4.5 = 0$$

since $\delta = 0.25$ and $\eta = -4.5$ satisfies option replicating equations. Arbitrage!
If the market price is set to:10 (the expected payoff), find an arbitrage.

2.1.4 Risk Neutral Valuation

Note 2.1 No arbitrage assumption implies $d \leq e^{r\Delta t} \leq u$.
(Otherwise, arbitrage can be found by trading S_t and risk-free bond)

Consider another 2-by-2 linear system ψ^u and ψ^d parameters for up and down state

$$\underbrace{\begin{bmatrix} 1 \\ uS_t \end{bmatrix}}_{\text{market up}} \psi^u + \underbrace{\begin{bmatrix} 1 \\ dS_t \end{bmatrix}}_{\text{down: } t+1} \psi^d = \underbrace{\begin{bmatrix} e^{-r\Delta t} \\ S_t \end{bmatrix}}_{\text{time } t}$$

The RHS will essentially have the value of the bond and the stock at the beginning of the period, the LHS, the first term consists the bond and stock value when the market goes up, the second term when the market goes down. We are putting a weight on up side and down side.
The unique solution is $\psi^u = e^{-r\Delta t} q^*$ and $\psi^d = e^{-r\Delta t}(1 - q^*)$

$$q^* = \frac{e^{r\Delta t} - d}{u - d}, \ 0 \leq q^* \leq 1$$

The second equation $\implies$

$$S_t = e^{-r\Delta t}(q^* uS_t + (1 - q^*)dS_t) = e^{-r\Delta t}\mathbf{E}^Q(S_{t+1}) \tag{2.2}$$

where $\mathbf{E}^Q(\cdot)$ is the expected value of S_{t+1} using q^* as the probability. Note: different notation of q^* is used to emphasize
The first equation $\implies$

$$\beta_t = e^{-r\Delta t}(q^* \cdot 1 + (1 - q^*) \cdot 1) = e^{-r\Delta t}\mathbf{E}^Q(\beta_{t+1}) \tag{2.3}$$

Let $\{\delta_t S_t, \eta_t \beta_t\}$ be the replicating portfolio. Using 2.2 and 2.3

$$\begin{aligned}
V_t &= \delta_t S_t + \eta_t \beta_t \quad \text{(Replicating portfolio at } t\text{)} \\
&= \delta_t \underbrace{e^{-r\Delta t}(q^* uS_t + (1 - q^*)dS_t)}_{\text{expected stock value}} + \eta_t \underbrace{e^{-r\Delta t}(q^* \cdot 1 + (1 - q^*) \cdot 1)}_{\text{expected bond value}} \\
&= \delta_t e^{-r\Delta t} q^* \underbrace{(\delta_t uS_t + \eta_t \cdot 1)}_{\text{portfolio value when market goes up}} + \eta_t e^{-r\Delta t}(1 - q^*) \underbrace{(\delta_t dS_t + \eta_t \cdot 1)}_{\text{portfolio value when market goes down}} \\
&= \delta_t e^{-r\Delta t} q^* V^u_{t+1} + \eta_t e^{-r\Delta t}(1 - q^*)V^d_{t+1} \quad \text{using 2.1} \\
&= e^{-r\Delta t}\mathbf{E}^Q(V_{t+1})
\end{aligned}$$

Proposition 2.2 — Risk Neutral Valuation.
If there exists no arbitrage, then there exists risk neutral probability $0 \le q^* = \frac{e^{r\Delta t}-d}{u-d} \le 1$ such that

$$\beta_t = \underbrace{e^{r\Delta t}}_{\text{discounting using } r} \beta_{t+1} \quad \text{rate return} = r$$
$$S_t = \underbrace{e^{r\Delta t}}_{\text{expected rate of return is } r \text{ under } Q} \mathbf{E}^Q(S_{t+1})$$
$$V_t = \underbrace{e^{r\Delta t}}_{\text{expected rate of return is } r \text{ under } Q} \mathbf{E}^Q(V_{t+1})$$

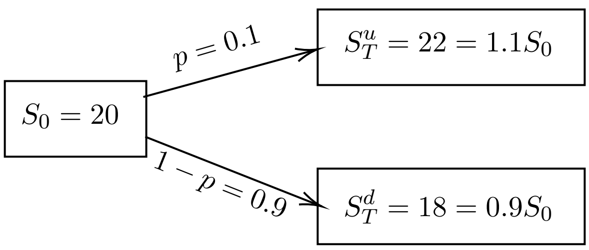
where V_t is derivative on S (converse is also true). This holds beyond the 1-period model.

$$q^* = \frac{e^{r\Delta t} - d}{u - d}, \ 0 \le q^* \le 1$$

Risk neutral pricing:
$$V_t = e^{r\Delta t}\mathbf{E}^Q(V_{t+1})$$

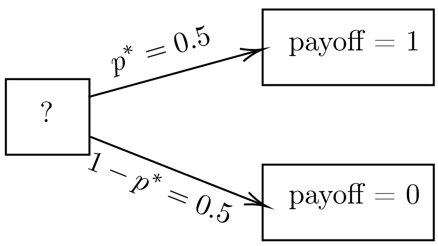
No arbitrage assumption implies some relationship between traded assets, specifically in this case, no arbitrage implies constraint that there exists probability world q^* such that all the traded assets link its term value to future value in the simple expected future value.

■ **Example 2.3** Recall the example of pricing a call with $K = 21$ and



$q^* = \frac{e^{r\Delta t-d}}{u-d} = \frac{1-0.9}{1.1-0.9} = 0.5$

Call:



We get again $C_0 = 0.5$

3. Week 3

Outline

- Multi-period binomial lattice: multiplicative price model
- Multi-period binomial model : additive log price model
- Constructing a binomial lattice for underlying
- European option pricing on a binomial lattice

3.1 Multi-period binomial lattice

One-period binomial model is crude, to improve it, we use it in a small interval Δt only. Specifically, let $[0, T]$ be partitioned into

$$t_0 = 0 < t_1 < \cdots < t_n < \cdots < t_N = T$$

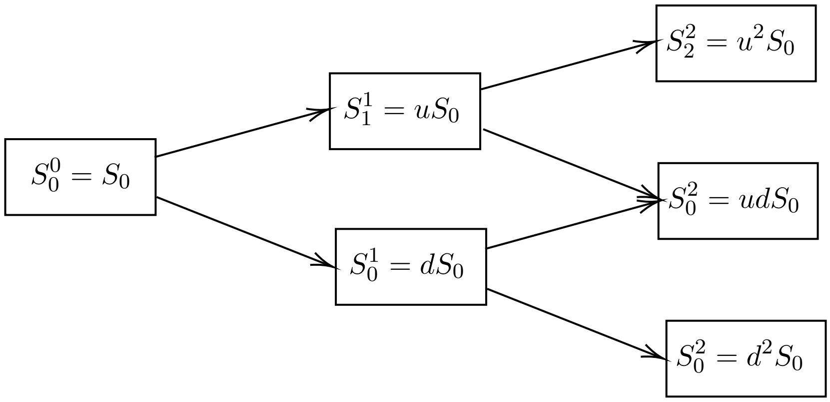
$$t_n = n\Delta t, \ \ \Delta t = \frac{T}{N}$$

In the binomial model, let $S_0^0 = S_0$ and S_j^n be the price at $S(t_n)$ with $j = 0, \ldots, n$, total of $n + 1$ prices.

Notation 3.1. *Instead of S_t (emphasize time), we will use*

- *$S_n = S(t_n)$ to denote value at t_n*
- *When both sub- and sup- scripts are used, the superscript denotes time*

Form S_j^n, in Δt, price goes up to $S_{j+1}^{n+1} = uS_j^n$ with probability p and down to $S_j^{n+1} = dS_j^n$ with probability $(1 - p)$

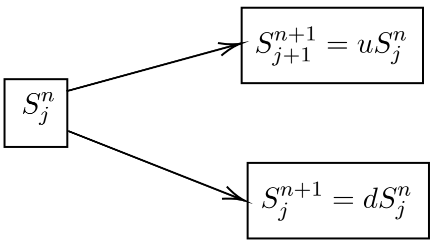


Assume further $d = \frac{1}{u}$, we have a non-drifting tree.
Given a binomial lattice which ensures that no arbitrage, i.e. $d < e^{r\delta t} < u$, any European option values can easily be done via backward iteration.
How to find the fair value?

- At the final nodes:
  - $payoff(S_2^2)$
  - $payoff(S_1^2)$
  - $payoff(S_0^2)$
- At the middle node, each is a 1-period binomial movement: Discounting expected value (under risk neutral $q^* = \frac{e^{r\Delta t} - d}{u-d}$)

Exercise 3.1 Pricing a European option on the 2-period lattice

At t_n, the j^{th} node is



How to construct the lattice (determine u, d, p) to mode a stock price?
What happens to the binomial latter when $\Delta t \to 0$?

3.1.1 Binomial Model: Logarithmic return

In finance, instead of price S_n, we often consider log return.

Definition 3.1 — logarithmic return.
The **logarithmic return** in the $(n+1)$ period is

$$\Delta X_n = \log\left(\frac{S_{n+1}}{S_n}\right)$$

where $X_n = \log(S_n)$ and $\Delta X_n = X_{n+1} - X_n$

Definition 3.2 — simple return.

$$\frac{S_{n+1}-S_n}{S_n} \approx \log\left(1+\frac{S_{n+1}-S_n}{S_n}\right) = \log\left(\frac{S_{n+1}}{S_n}\right)$$

Then

$$\begin{aligned} X_{n+1} = \log(S_{n+1}) &= \log(S_n) + (\log(S_{n+1}) - \log(S_n)) \\ &= X_n + \Delta X_n \end{aligned}$$

where $\Delta X_n = \log\left(\frac{S_{n+1}}{S_n}\right)$.
Assume that the log return in the n-th period ΔX_n has the distribution

$$\Delta = \begin{cases} \Delta h, & \text{with probability } p \\ -\Delta h, & \text{with probability } 1-p \end{cases}$$

with Δh is a given constant parameter, e.g. let $d = \frac{1}{u}$ and $\Delta h = \log(u)$.
In terms of the price, this corresponds to a binomial model

$$S_{n+1} = \begin{cases} S_n u, & \text{with probability } p \\ S_n d, & \text{with probability } 1-p \end{cases}$$

where $u = e^{\Delta h},\ d = e^{-\Delta h} = \frac{1}{u}$
Note that the cumulative log returns in the first n period is

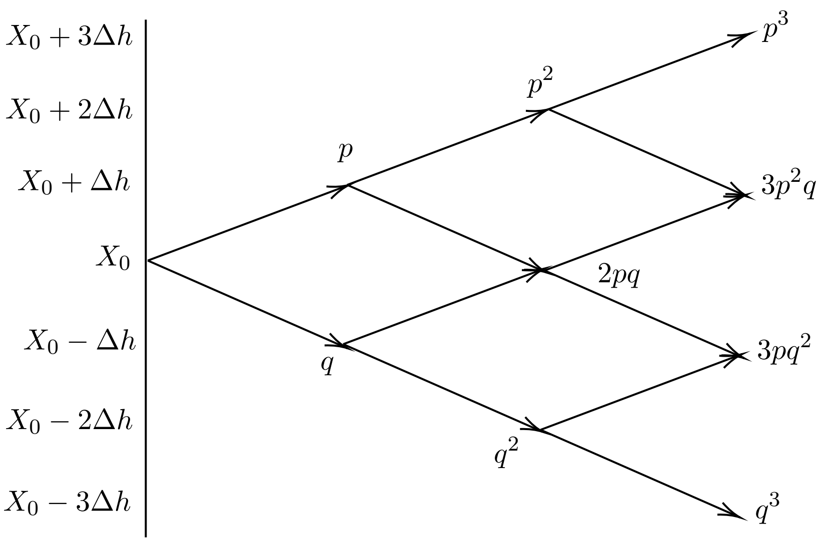
$$\sum_{k=0}^{n+1} \Delta X_k$$

and the log price

$$X_n = X_0 + \sum_{k=0}^{n+1} \Delta X_k$$

(or a random walker's position at t_n, assuming $X_0 = 0$).
This allows us to analyze an additive model for the log price, instead of the multiplicative model for the price.

3.1.2 Discrete Random Walk and its Limit



Properties of discrete random walk

(1) ΔX_n has the identical distribution for any n
(2) ΔX_n and $\Delta X_{n'}$ are **independent** for $n \neq n'$. For any $n \neq n'$

$$\mathbf{E}(\Delta X_n \Delta X_{n'}) = \mathbf{E}(\Delta X_n)\mathbf{E}(\Delta X_{n'})$$

At t_n, the next price move is independent of its past moves, i.e. it is a random walk (Markov process)
(3) Statistical moments are determined by p and Δh (or p, u and d)

$$\begin{aligned}
\mathbf{E}(\Delta X_n) &= p\Delta h - (1-p)\Delta h = (2p-1)\Delta h = \alpha_0 \\
\mathbf{E}(\Delta X_n^2) &= p(\Delta h)^2 + (1-p)(\Delta h)2 = (\Delta h)^2 \\
\mathbf{var}(\Delta X_n) &= \mathbf{E}((\Delta X_n)^2) - (\mathbf{E}(\Delta X_n^2))^2 \\
&= (\Delta h)^2 - (2p-1)^2(\Delta h)^2 = \sigma_0^2
\end{aligned}$$

Following Central Limit Theorem, as $N \to \infty$, $X_N - X_0$ converges to a normal distribution.

$$P\left(\frac{\sum_{n=0}^{N-1} \Delta X_n - N\alpha_0}{\sigma_0\sqrt{N}} \leq x\right) \to F_{normal}(x)$$

$F_{normal}(x)$ is the cumulative distribution function of the standard normal, specifically,

$$F_{normal}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$$

The log return has a normal distribution

How to choose p, u, d

Assume that underlying stock has the two statistics:

- the average log return per year is α
- the standard deviation of the annual return, volatility, is σ, or the variance σ^2

Then the expected log return in an interval of Δt is $\alpha\Delta t$ and variance $\sigma^2\Delta t$.
At t, in a period of Δt, we choose binomial parameters u, p, d to match underlying return in the mean and variance statistics.
(we can specifically choose u, d, p so that the stock's expected log return equals to α and volatility equals to $\sigma^2\Delta t$)

■ **Example 3.1** Consider $u = e^{\Delta h}$ and $d = \frac{1}{u}$. Given α and σ, choose p and Δh to satisfy

$$\mathbf{E}(\Delta X_n) = p\Delta h - (1-p)\Delta h = \mathbf{E}\left(\log\left(\frac{S_{t+\Delta t}}{S_t}\right)\right) = \alpha\Delta t$$

$$\mathbf{var}(\Delta X_n) = (\Delta h)^2 - (2p-1)^2(\Delta h)^2 = \mathbf{var}\left(\log\left(\frac{S_{t+\Delta t}}{S_t}\right)\right) = \sigma^2\Delta t$$

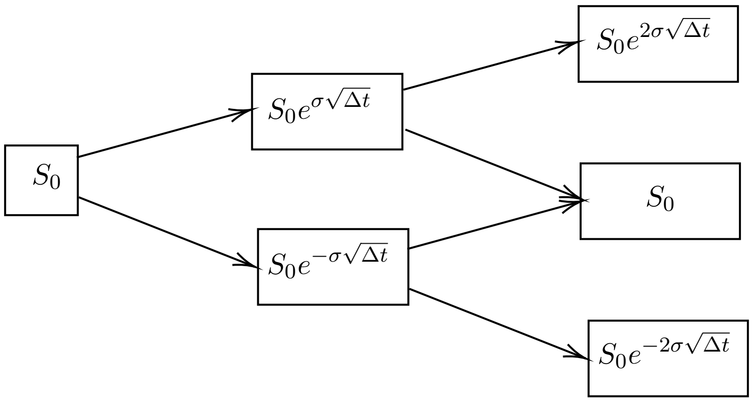
One frequent choice: CRR (79):

$$\begin{aligned}
u &= e^{\sigma\sqrt{\Delta t}} \\
d &= \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}} \\
p &= \frac{1}{2} + \frac{1}{2}\frac{\alpha}{\sigma}\sqrt{\Delta t}
\end{aligned}$$

Note 3.1

- Under this construction, u and d depend on volatility only.
  The expected log return α is irrelevant. Option value depends on volatility, not expected rate of return.
- CRR leads to a recombining tree (non-drifting)
- More generally, there are many different choices for (u, d, p) since we have two equations for the first two moments and three parameters. See course notes (and assignment 1) for alternative parameterizations
- One can also directly set up the no-arbitrage lattice for which the expected rate of return is r (i.e. the probability is q^*)

CRR Model 2-period



3.2 Pricing European option under no-arbitrage lattice

Pricing an European option with expiry T using an N-period lattice. Let $t_n = n\Delta t$ and $\Delta t = \frac{T}{N}$, and S_j^n denote the j-th price at t_n, $j = 0, 1, \ldots, n$

(1) Choose u, d and p to match log return statistics.

Example 3.2 Let $\log(u) = \Delta h = \sigma\sqrt{\Delta t}$. Choose Δt sufficiently small so that the condition

$$e^{-\sigma\sqrt{\Delta t}} \le e^{r\Delta t} \le e^{\sigma\sqrt{\Delta t}}$$

is satisfied

(2) Construct lattice of prices: e.g. $u = e^{\sigma\sqrt{\Delta t}}, d = e^{-\sigma\sqrt{\Delta t}}$

$$S_{j+1}^{n+1} = uS_j^n \text{ and } S_j^{n+1} = dS_j^n,\ S_j^N = u^j d^{N-j} S_0$$

(3) At the expiry T, the value of the option is

$$V_j^N = \text{payoff}(S_j^N),\ 0 \le j \le N$$

e.g. call payoff$(S_j^N) = \max(S_j^N - K, 0)$

(4) Roll back through the lattice

$$\begin{aligned}
&\text{for } \ n = N-1, N-2, \ldots, 0 \text{ do} \\
&\quad \text{for } \ j = 0, 1, \ldots, n \text{ do} \\
&\quad\quad V_j^n = e^{-r\Delta t}(q^* V_{j+1}^{n+1} + (1-q^*)V_j^{n+1}) \\
&\quad \text{end} \\
&\text{end}
\end{aligned}$$

where $q^* = \frac{e^{r\Delta t}-d}{u-d}$

outline

- European option pricing under a binomial lattice and dividend
- Pricing American option under a binomial lattice
- Convergence (and convergence rate) of option values under a binomial lattice
- Black-Scholes formula for European calls and puts

3.2.1 Dividend

Denote

- t^-: immediately before t
- t^+: immediately after t

Assume that, at $t = 0$, it is known that the underlying pays dividend D. e.g. $D = \rho S(t_d^-)$, at $t = t_d$, $0 \le t_d \le T$

$$\underset{\text{Today}}{\boxed{\text{premium}}} \xrightarrow{\quad t_d \quad} \underset{T}{\boxed{payoff(S_T)}}$$

No arbitrage implies

$$S(t_d^+) = S(t_d^-) - D$$

However, this does not affect option holder/writer, i.e., option is dividend protected.
No arbitrage implies that $V(S(t),(t))$ $(\equiv \tilde{V}(t))$ is a **continuous** function of t, i.e.

$$V(S(t_d^-), t_d^-) = V(S(t_d^+), t_d^+) = V(S(t_d^-) - D, t_d^+) \tag{3.1}$$

Pricing under dividend payment: assuming t_d is dividend date

for $n = N-1, N-2, \ldots, 0$ do
  for $j = 0, 1, \ldots, n$ do
    $V_j^n = e^{-r\Delta t}(q^* V_{j+1}^{n+1} + (1-q^*)V_j^{n+1})$
  end
  if t_n is the closest time point to t_d
    compute V_j^n at t_n^- by (3.1) and interpolation
  end
end

Note 3.2 Interpolation is required since $S(t_d^-) - D$ may not equal to any node price S_j^n

3.3 Pricing American Options

Consider an American option which can be exercised at the discrete time t_n, $n = 0, \cdots, N$
Given a no-arbitrage lattice, the holder needs to optimally decide at any t_n, $n = 0, 1 \cdots, N$, whether to **exercise** or to continue to **hold**.
At the expiry $T = t_N$, the holder exercises if and only if payoff$(S_j^N) > 0$ and $V_j^N =$ payoff(S_j^N).
Dynamic Programming $\implies$ optimal strategy needs to be optimal at any node, *assuming optimal decision starting from that node forward.*
Hence a solution can be made one step a time, in backwards.

Backeard Iteration

- At the expiry $T = t_N$, $V_j^N = \text{payoff}(S_j^N)$
- At (t_n, S_j^n), the option has (continue-to-hold) **continuation value**

$$(V_j^n)^* = e^{-r\Delta t}\mathbf{E}(V^{n+1}|S_j^n) = e^{-r\Delta t}(q^* V_{j+1}^{n+1} + (1-q^*)V_j^{n+1})$$

and payoff(S_j^n), *if exercised.* Hence the fair option value is

$$V_j^n = \max((V_j^n)^*, \text{payoff}(S_j^n)),\ \ j = 0, 1 \cdots, n$$

i.e., the optimal exercise strategy is to exercise if the continuation value is less than the payoff

Pricing American option

for $n = N-1, N-2, \ldots, 0$ do
  for $j = 0, 1, \ldots, n$ do
    $(V_j^n)^* = e^{-r\Delta t}(q^* V_{j+1}^{n+1} + (1-q^*)V_j^{n+1})$ % continuation value
    $V_j^n = \max((V_j^n)^*, \text{payoff}(S_j^n))$
  end
end

3.3.1 Computational Cost

- Floating point operations (FLOPS): $O(N^2)$ (for both European and American)
- Time efficiency: vectorization Matlab code for efficiency, see chapter 2.5 from the course notes
- Space cost: if only V_0 is needed, we can compute using $O(N)$ space

3.4 BS Formula

Central Limit Theorem implies that $\log(S_t) - \log(S_0)$ has a normal distribution with mean αt and variance $\sigma^2 t$.
It can be shown that the binomial lattice standard option value functions converges to the BS formulae.
Let $0 \le t \le T$. As $N \to \infty$, the call option value $C(S,t)$ is

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

- $T - t$: time to expiry
- $N(d)$: Cumulative distribution function for a standard normal

$$\begin{aligned} N(d) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}y^2} dy \\ d_1 &= \frac{\log\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\log\left(\frac{S}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

Note 3.3

- Option value only depends on σ, not the average rate of return
- There is no analytic formula for American option pricing

3.4.1 Convergence Rate

Let V_0^{exact} be the BS value when $t = 0$ and $S = S_0$, i.e., $V_0^{\text{exact}} = V(S_0, 0)$.
Let $V_0^{\text{tree}}(\Delta t)$ be the binomial lattice value at $t = 0$ and $S = S_0$ when $\Delta t = \frac{T}{N}$.
PDE theory shows (later) that, if **strike price is at a binomial node**,

$$V_0^{\text{tree}}(\Delta t) = V_0^{\text{exact}} + \text{const} \cdot \Delta t + o(\Delta t)$$

Otherwise, smoothing of payoff is required (see chapter 5.4 from course notes).
If convergence rate is **linear**, then the ratio

$$\lim_{\Delta t \to 0} \frac{V_0^{\text{tree}}\left(\frac{\Delta t}{2}\right) - V_0^{\text{tree}}(\Delta t)}{V_0^{\text{tree}}\left(\frac{\Delta t}{4}\right) - V_0^{\text{tree}}\left(\frac{\Delta t}{2}\right)} \tag{3.2}$$

would approach 2.
If the convergence rate is **quadratic**, then

$$V_0^{\text{tree}}(\Delta t) = V_0^{\text{exact}} + \alpha(\Delta t)^2 + o((\Delta t)^2)$$

where α is some constant independent of Δt.
What would the ratio (3.2) approaches to if convergence rate is quadratic?
This allows us to computationally verify convergence rate of a method.

4. Week 4

Outline
- Standard Brownian motion
- Geometric Brownian motion
- Ito's Lemma
- Explicit solution of BS model

4.1 Black-Scholes Model

4.1.1 A Special Binomial Lattice

Definition 4.1 — standard Brownian motion.
Consider a discrete random walk With $X_0 = 0$

$$\Delta X_n = \begin{cases} \sqrt{\Delta t} & \text{with probability } p = 0.5 \\ -\sqrt{\Delta t} & \text{with probability } 0.5 \end{cases}$$

(The change of position is related to the change in time)
Let $\Delta t \to 0$,

$$\Delta t \to dt,\ \Delta X_t \to dX_t$$

(Δ means finite changes while d means infinite changes)
The limit of this random walk is a continuous process, which is called **standard Brownian motion**

A standard Brownian motion Z_t has the following properties:
(1) $Z(0) = 0$
(2) For all $t \geq 0$ and $\Delta t > 0$, $Z(t + \Delta t) - Z(t)$ is normally distributed with mean 0 and variance Δt
(3) For any $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$, $Z(t_2) - Z(t_1)$ and $Z(t_4) - Z(t_3)$ are independent

Note 4.1
- Z_t has a normal distribution with mean zero and variance t
- The Markovian property is consistent with the market efficiency assumption

Definition 4.2 — Ito's Process.
Let a, b be constants (or functions), an **Ito's process** X_t is a solution to the *stochastic differential equation* below

$$dX_t = a \cdot dt + b \cdot dZ_t$$

- $a \cdot dt$: deterministic trend
- $b \cdot dZ_t$: random fluctuation of standard Brownian

4.1.2 Geometric Brownian Model

Definition 4.3 — Black-Scholes Model.
Black-Scholes Model: a stochastic differential equation SDE

$$\frac{dS_t}{S_t} = \mu \cdot dt + \sigma \cdot dZ_t$$

- $\mu \cdot dt$: deterministic trend
- $\sigma \cdot dZ_t$: random fluctuation of standard Brownian
- μ, σ are constants

(the return of an asset is described a Ito's process)
or

$$dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dZ_t$$

How to solve for price S_t?
Recall bond return ODE (a classical DE):

$$d\beta(\tau) = r\beta d\tau, \ \frac{d\beta}{\beta d\tau} = rt, \ d\log(\beta(\tau)) = rd\tau$$

Integrate

$$\log(\beta(t)) - \log(\beta(0)) = \int_0^t d\log(\beta(\tau)) = rt$$
$$\beta(t) = \beta(0)e^{rt}$$

What is the solution to the SDE

$$dX_t = a\ dt + b\ dZ_t?$$

$$\begin{aligned} X(t) &= X(0) + \int_0^t (a\ d\tau + b\ dZ(\tau)) \\ &= X(0) + \int_0^t a\ d\tau + \int_0^t b\ dZ(\tau) \\ &= X(0) + a \cdot t + b \cdot Z_t \quad \text{if } a, b \text{ are constants} \end{aligned}$$

What is $\int_0^T b(t)\ dZ_t$ when b is a function involving t?

4.1.3 Ito's Integral

In the classical (deterministic) calculus,

$$\int_0^T f(t_n)dt = \lim_{N\to+\infty}\left(\sum_{n=1}^{N-1} b(t_n)(Z(t_{n+1} - Z(t_n)))\right)$$

(the RHS is called Ito's sum)
where the limit is in the mean square sense, i.e.

$$\lim_{N\to+\infty}\mathbf{E}\left(\left(\sum_{k=1}^{N-1} b(t_{n+1})(Z(t_{n+1}) - Z(t_n)) - \int_0^T b(t)dZ_t\right)^2\right) = 0$$

> **Note 4.2**
> - $\sum\limits_{k=1}^{N-1} b(t_n)(Z(t_{n+1}) - Z(t_n))$ is a random variable, so is $\int_0^T b(t)dZ_t$
> - $\sum\limits_{k=1}^{N-1} b(t_{n+1})(Z(t_{n+1}) - Z(t_n))$ does not make sense financially, as it uses information ahead

4.1.4 More about Standard Brownian

$\Delta Z_t = Z(t+\Delta t) - Z(t)$ has a normal distribution with mean 0 and variance Δt
Assume $\phi_t \sim \mathcal{N}(0,1)$ is a standard normal. Then

$$\Delta Z_t = \phi_t\sqrt{\Delta t} \text{ or } dZ_t = \phi_t\sqrt{dt} \text{ equivalently } \frac{dZ_t}{d_t} = \frac{\phi_t}{\sqrt{dt}}$$

The standard Brownian motion $Z_t = Z_0 + \int_0^t dZ_s$ is **continuous** but **not differentiable**!
Note $\phi_t \sim \mathcal{N}(0,1)$

$$\mathbf{E}(\phi_t) = 0,\ \mathbf{E}(\phi_t^2) = 1,\ \mathbf{E}(\phi_t^3) = 0,\ \mathbf{E}(\phi_t^4) = 3$$

and

$$\begin{aligned}
\mathbf{E}((\Delta Z_t)^2) &= \mathbf{E}(\Delta t(\phi_t)^2) = \Delta t\mathbf{E}((\phi_t)^2) = \Delta t\\
\mathbf{var}((\Delta Z_t)^2) &= \mathbf{E}((\Delta Z_t)^4) - (\mathbf{E}((\Delta Z_t)^2))^2\\
&= (\Delta t)^2\mathbf{E}((\phi_t)^4) - (\Delta t)^2
\end{aligned}$$

$$= O((\Delta t)^2) \text{ Note: } \mathbf{E}(\phi_t^4) = 3$$

> **Note 4.3** Variance of $(\Delta Z_t)^2$ goes to zero quadratically! In fact we can think of $(dZ_t)^2$ as deterministic!
> We can write $(dZ_t)^2 = dt$

Important Properties of Z_t

Assume $\phi_t \sim \mathcal{N}(0,1)$ is a standard normal. Then

$$\Delta Z_t = \phi_t\sqrt{\Delta r} \text{ or } dZ_t = \phi_t\sqrt{dt},\ (dZ_t)^2 = dt$$

How does the option value $V(S,t)$ change with time?

4.1.5 Ito's Lemma

Lemma 4.1 — Ito's Lemma.
Assume that

$$dS_t = a(S_t,t)dt + b(S_t,t)dZ_t$$

Let $G(S,t): \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Then $Y_t = G(S_t,t)$ is an Ito's process which satisfies

$$dY_t = \left[\frac{\partial G}{\partial t} + a(S_t,t)\frac{\partial G}{\partial S} + \frac{1}{2}b(S_t,t)^2\frac{\partial^2 G}{\partial S^2}\right]dt + b(S_t,t)\frac{\partial G}{\partial S}dZ_t$$

Note: $a(S_t,t) = \mu S_t$ and $b(S_t,t) = \sigma S_t$ for BS model.

Proof. For notational simplicity, denote $a(S_t,t) \equiv a$ and $b(S_t,t) \equiv b$. The Taylor expansion:

$$\begin{aligned}\Delta Y_t &= G(S_t + \Delta S_t, t + \Delta t) - G(S_t,t)\\ &= \underbrace{\frac{\partial G}{\partial S}\Delta S_t + \frac{\partial G}{\partial t}\Delta t}_{\text{first order expansion}} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}(\Delta S_t)^2 + \frac{\partial^2 G}{\partial S\partial t}(\Delta S_t)(\Delta t) + O((\Delta t)^2)\end{aligned}$$

Since $dS_t = a\cdot dt + b\cdot dZ_t,\ dZ_t = O(\sqrt{dt}),\ \Delta S_t\Delta t = O\left((\Delta t)^{\frac{3}{2}}\right)$

$$\begin{aligned}\implies \quad \Delta Y_t &= G(S_t + \Delta S_t, t+\Delta t) - G(S_t,t)\\ &= \frac{\partial G}{\partial S}\Delta S_t + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}(\Delta S_t)^2 + O\left((\Delta t)^{\frac{3}{2}}\right)\end{aligned}$$

Note

$$\begin{aligned}(dS_t)^2 &= (a\cdot dt + b\cdot dZ_t)^2\\ &= a^2\cdot(dt)^2 + b\cdot(dZ_t)^2 + (2ab)\cdot dtdZ_t\\ &= b^2\cdot dt \quad (\text{using } (dZ_t)^2 = dt)\end{aligned}$$

Let $\Delta t \to 0$, we have

$$\begin{aligned}dY_t &= \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}b^2 dt\\ &= \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}(adt + bdZ_t) + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}b^2 dt\\ &= \left[\frac{\partial G}{\partial t} + a(S_t,t)\frac{\partial G}{\partial S} + \frac{1}{2}b(S_t,t)^2\frac{\partial^2 G}{\partial S^2}\right]dt + b(S_t,t)\frac{\partial G}{\partial S}dZ_t\end{aligned}$$

■

Solving BS Model

Black-Scholes Model: return follows the process

$$\frac{dS_t}{S_t} = \mu\cdot dt + \sigma\cdot dZ_t$$

- $\mu\cdot dt$: deterministic trend
- $\sigma\cdot dZ_t$: random fluctuation of standard Brownian
- μ,σ are constants

$$dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dZ_t$$

an Ito's process with $a(S,t) = \mu S_t$ and $b(S,t) = \sigma S$.
Consider $G(S,t) = \log(S)$. We have

$$\frac{\partial G}{\partial t} = 0, \quad \frac{\partial G}{\partial S} = \frac{1}{S}, \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$$

$$\begin{aligned} d\log(S_t) &= \left[\frac{\partial G}{\partial t} + a(S_t,t)\frac{\partial G}{\partial S} + \frac{1}{2}b(S_t,t)^2\frac{\partial^2 G}{\partial S^2}\right]dt + b(S_t,t)\frac{\partial G}{\partial S}dZ_t \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dZ_t \end{aligned}$$

Thus

$$\begin{aligned} \log(S_t) - \log(S_0) &= \int_0^t (\mu - \frac{1}{2}\sigma^2)ds + \int_0^t \sigma dZ_s \\ &= (\mu - \frac{1}{2}\sigma^2)t + \sigma(Z_t - Z_0) \end{aligned}$$

Or

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma Z_t}$$

Assuming $S_0 > 0$, under BS, stock price S_t is always positive and has a lognormal distribution

Outline

- Risk neutral option pricing
- MC for pricing European option
- MC for pricing European path dependent option
- Path simulation for generalized BS model
- Confidence interval for MC estimate

4.2 Risk Neutral Valuation

Recall that, for a binommial lattice, no arbitrage implies that there exists the risk neutral probability q^* such that

$$S(t_n) = e^{-r\Delta t}\mathbf{E}^Q(S(t_n+1)) \tag{4.1}$$

$$V(S(t_n), t_n) = e^{-r\Delta t}\mathbf{E}^Q(V(S(t_{n+1}), t_{n+1})) \tag{4.2}$$

where $\mathbf{E}^Q(\cdot)$ denotes the expectation using q^* and $V(S,t)$ is the European option value function.
Equation (4.1) can be written as (using Taylor's expansion)

$$\mathbf{E}^Q\left(\frac{S(t_{n+1})}{S(t_n)}\right) = e^{r\Delta t} = 1 + \Delta t + O((\Delta t)^2)$$

Thus

$$\mathbf{E}^Q\left(\frac{S(t_{n+1}) - S(t_n)}{S(t_n)}\right) = \Delta t + O((\Delta t)^2)$$

Let $\Delta t \to 0$, this implies that the expected rate of return is r. In fact, under risk neutral probability (this exists by the arbitrage assumption)

$$\frac{dS_t}{S_t} = rdt + \sigma dZ_t^Q \tag{4.3}$$

where Z_t^Q is a standard Brownain.
Note that the actual stock price follows

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t \tag{4.4}$$

Under continuous model (4.4), risk neutral pricing becomes this is the European option pricing formula

$$V(S,t) = e^{-r(T-t)\mathbf{E}^Q(\text{payoff}(S_T))}$$

where the expectation $\mathbf{E}^Q(\cdot)$ assumes the underlying asset price follows the risk neutral model (4.3)

Note 4.4 superscript Q emphasizes that path under the risk neutral model is not related to path under (4.4) for real underlying price.

4.3 MC Option Pricing

Using risk neutral pricing

$$V(S_0,0) = e^{-rT\mathbf{E}^Q(\text{payoff}(S_T))}$$

option value can be computed by generating M samples of $(S_T)^j$, $j = 1, \cdots M$, o f S_T under risk neutral model (4.3)

$$V(S_0,0) \approx \frac{1}{M}\sum_{j=1}^{M}\left(e^{-rT}\text{payoff}((S_T)^j)\right) \tag{4.5}$$

Note that, under BS, i.i.d. samples $\{(S_T)^j,\ j = 1, \cdots, M\}$ can be easily generated from standard normals, e.g. randn in Matlab.
Assume a BS model for S_t. Under risk neutral probability,

$$\frac{dS_t}{S_t} = rdt + \sigma dZ_t^Q$$

Under BS, we have the explicit solution

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma Z_t}$$

In Matlab, M independent samples of S_T can be generated by

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\cdot\sqrt{T}\cdot\text{randn}(M,1)} \tag{4.6}$$

where $\sqrt{T}\cdot\text{randn}(M,1)$ is generate Z_t. Here S_T is an M-by-1 vector.

4.3.1 Path Dependent Options

For European call and put options, the payoff only depends on the underlying asset's price at the expiration time S_T. We can have the options that the value you get at the expiration time depends on how the stock price has moved depending on the path.
A path dependent option has either some feature or payoff itself depending on the price path $S(t)$, $0 \le t \le R$

Definition 4.4 — Barrier Option.
A **barrier option** comes/ceases to exist when a barrier has been crossed, e.g. up/down, in/out.

- **Up-Out**: If S_t crosses an up-barrier S_u, option pays nothing. Otherwise, standard payoff.

To price options with payoff depending on path, we simulate price paths: $S(t_n)$, $n = 0, 1, \cdots, N$, e.g., $t_n = n \cdot \frac{T}{N}$

$$S(t_{n+1}) = S(t_n) .* e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma \cdot \sqrt{\Delta t} \cdot \mathrm{randn}(M,1)}$$

■ **Example 4.1** Consider a European Asian option (the payoff depends on the average of stock price during the time period) with payoff a function of the average below

$$\frac{1}{T}\int_0^T S_t dt = \frac{1}{T}\int_0^T S(t)dt$$

For example, an Asian call with pay off equal to

$$\max\left(\frac{1}{T}\int_0^T S_t dt - K, 0\right)$$

For computation implementation, discretization is needed.
Let $t_n = t_{n-1} + \Delta t$ and $\Delta t = \frac{T}{N}$. Let $A_n = \frac{1}{n}\sum\limits_{k=1}^{n} S(t_k) \approx \int_0^{t_n} S(\tau)d\tau$. Note

$$A_n = \frac{1}{n}((n-1)A_{n-1} + S(t_n)) = \frac{n-1}{n}A_{n-1} + \frac{1}{n}S(t_n)$$

psuedo code: Let $\mathrm{randn}(M, 1)$ denote M samples of standard normals

$$\begin{aligned}
&S_0 = S(0) \cdot \mathrm{ones}(M, 1)\\
&A_0 = 0\\
&\text{for } n = 0, 1, \cdots, N-1\\
&\qquad A_{n+1} = \frac{n}{n+1}A_n + \frac{1}{n+1}S_n\\
&\qquad S_{n+1} = S_n * e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma \cdot \sqrt{\Delta t} \cdot \mathrm{randn}(M,1)}\\
&\text{end}\\
&\text{payoff } = \max(A_N - K, 0)\\
&V_0 = e^{-rT}\mathrm{mean}(\text{payoff})
\end{aligned}$$

4.3.2 Sampling Error in MC Pricing

Using BS (4.5) and explicit formula (4.6) to price option, no time disretization error in simulated path, the only error comes from (4.5) is the sampling error.
Using M simulations, the sampling error is $O(\frac{1}{\sqrt{M}})$. Why?
Consider average of finite samples from the same distribution to estimate the mean,

$$V^M = \frac{Y^1 + \cdots + Y^M}{M}, \ Y^j \sim Y, \text{ e.g., discounted payoff}$$

$$\Longrightarrow \mathbf{E}(V^M) = \frac{1}{M}\sum_{j=1}^{M}\mathbf{E}(Y^j) = \mathbf{E}(Y)$$

Hence V^M is an unbiased estimator of $V_0 = \mathbf{E}(Y)$

$$\mathbf{var}(V^M) = \mathbf{E}\left(\left(\frac{Y^1 + \cdots + Y^M}{M} - \mathbf{E}(V^M)\right)^2\right)$$

$$= \frac{1}{M^2}\sum_{j=1}^{M}\mathbf{E}((Y^j - \mathbf{E}(Y))^2) \quad \text{using independence}$$

$$= \frac{1}{M}\mathbf{var}(Y) \quad \text{using } Y^j \sim Y$$

Hence the standard error of the estimator V^M is

$$\text{std}(V^M) = \frac{1}{\sqrt{M}}\text{std}(Y)$$

Following Central Limit Theorem:

$$V^M - \mathbf{E}(Y) \implies \mathcal{N}(0, \frac{1}{\sqrt{M}}\sigma_Y), \quad \text{as } M \to \infty$$

Note 4.5

- Sampling error does not depend on the number of risky assets if the payoff function depends on multiple assets; hence good for high dimensional problems
- V^M is random since each (payoff) Y^j os a random sample

4.3.3 Confidence Interval

σ_Y (e.g., standard deviation of discounted payoff) is unknown. We can approximate it. Let sample standard deviation be

$$\hat{\sigma} = \left[\frac{\sum_{j=1}^{M}(Y_i = V^M)^2}{M-1}\right]^{\frac{1}{2}} \approx \sigma_Y$$

Approximating σ_Y by $\hat{\sigma}$

$$V^M - V_0 \approx \mathcal{N}(0, \frac{\hat{\sigma}}{\sqrt{M}})$$

For standard normal distribution, given any confidence β, e.e., $\beta = 95\%$, it is possible to determine the interval $[-x^*, x^*]$ the standard normal is in by solving x^* from

$$\int_{-x^*}^{x^*} e^{-\frac{x^2}{2}}\,dx = \beta$$

where $p(x) = e^{-\frac{x^2}{2}}$ is the standard normal density.

Facts

If $\beta = 0.95 \implies x^* = 1.96$
If $\beta = 0.99 \implies x^* = 2.58$
Hence we can assert, with 99% confidence,

$$V_0 \in \left[V^M - \frac{2.58\hat{\sigma}}{\sqrt{M}}, V^M + \frac{2.58\hat{\sigma}}{\sqrt{M}}\right]$$

5. Week 5

Outline

- MC for Pricing European Option under Generalized BS Model
- Euler Method and Time Stepping
- Error in Euler's Method
- Dynamic Trading Performance Analysis via MC Simulation

5.0.1 A Brief Review

Option Pricing Under Continuous Model

Assume Black-Schole Model:

$$\frac{dS_t}{S_t} = \mu \cdot dt + \sigma \cdot dZ_t \tag{5.1}$$

- $\mu \cdot dt$: deterministic trend
- $\sigma \cdot dZ_t$: random fluctuation of standard Brownian
- μ, σ are constants

Using Ito's Lemma, we have explicit solution for (5.1)

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma Z_t} \tag{5.2}$$

- Sampling is easy, by sampling from normal distribution, e.g. in Matlab, randn yields a sample from standard normal
- No error due to time sidcretization error under BS

Risk Neutral Valuation

Under no arbitrage, fair value of European option satisfies

$$V(S,t) = e^{-r(T-t)\mathbf{E}^Q(\text{payoff}(S_T))}$$

where the expectation $\mathbf{E}^Q(\cdot)$ is under

$$\frac{dS_t}{S_t} = r \cdot dt + \sigma \cdot dZ_t$$

where r is the risk free rate, and Z_t is a standard Brownian

- Can generate sample paths $\{S(t_n), n = 0, 1, \ldots, N\}$
- Can price European path dependent exotic options
- Only sampling error if explicit solution (5.2) is used

5.1 Euler Method

Local Volatility Function (LVF) Model

$$\frac{dS_t}{S_t} = r \cdot dt + \sigma(S_t, t) \cdot dZ_t \tag{5.3}$$

Assume $t_n = n\Delta t,\ n = 0, 1, \ldots, N,\ \Delta t = \frac{T}{N}$. Integrate $[t_n, t_{n+1}]$, we have

$$\int_{t_n}^{t_{n+1}} dS_t = \int_{t_n}^{t_{n+1}} rS_t dt + \int_{t_n}^{t_{n+1}} \sigma(S_t, t) S_t dZ_t$$

Since $\Delta Z(t_n) = \sqrt{\Delta t}\phi_n, \phi_n \sim \mathcal{N}(0, 1)$

$$S_{n+1} = S_n + S_n(r\Delta t + \sigma(S_n, t_n)\sqrt{\Delta t}\phi_n),\ \ \phi_n \sim \mathcal{N}(0, 1) \tag{5.4}$$

This is called Euler's formula.
We can price any European options using MC under a generalized BS model.

Note 5.1
- Additional time discretization error when the Euler's method (5.4) is used for a generalized BS model

5.1.1 Error in Euler's Method

Recall that Euler's method

$$S_{n+1} = S_n + S_n(r\Delta t + \sigma(S_n, t_n)\sqrt{\Delta t}\phi_n),\ \ \phi_n \sim \mathcal{N}(0, 1)$$

comes from approximating Ito's Integral

$$\int_{t_n}^{t_{n+1}} dS_t = \int_{t_n}^{t_{n+1}} rS_t dt + \int_{t_n}^{t_{n+1}} \sigma(S_t, t) S_t dZ_t$$

For Euler's method with a time stepsize Δt, accuracy is Δt in the following sense (weak convergence)

$$\left|\mathbf{E}(S(T)) - \mathbf{E}(S_T^{\Delta t})\right| \leq C\Delta t$$

where
- $S(T)$: a price realization to SDE (5.3)
- $S_T^{\Delta t}$: approximation with time step Δt

Weak convergence with order 1 (linear convergence). See chapter 4.8.2 in course notes for more details.

Balance Time Stepping Error and Sampling Error

Assume that a numerical approximation to SDE has a time stepping accuracy of Δt, i.e.

$$\text{time discretization error} = O(\Delta t)$$

Sample error (approximationg using M simulation) is $O(\frac{1}{\sqrt{M}})$. Thus

$$\text{total error} = O(\Delta t) + O(\frac{1}{\sqrt{M}})$$

If we want
$$\text{total error} = O(\Delta t)$$
we need to choose
$$M \approx \frac{\text{constant}}{(\Delta t)^2}$$

How efficient is MC option pricing?
How much work is needed to achieve required accuracy when pricing using MC?
Investigate how computational cost (complexity) depends on Δt and M.
For European option pricing, how does error relate to the amount of work?
From
$$\begin{aligned}\text{complexity} &= O((\#\text{time step}) \times (\#\text{samples})) \\ &= O\left(\left(\frac{T}{\Delta t}\right) M\right) = O\left(\frac{M}{\Delta t}\right) \\ &= O\left(\frac{1}{(\Delta t)^3}\right), \quad M = O\left(\frac{1}{(\Delta t)^2}\right)\end{aligned}$$
We have
$$\Delta t = O\left(\frac{1}{\text{complexity}^{\frac{1}{3}}}\right)$$
For Euler's method, error $= O(\Delta t)$. Hence
$$\implies \text{error} = O(\Delta t) = O\left(\frac{1}{\text{complexity}^{\frac{1}{3}}}\right)$$
$\implies$ To reduce error by a factor of 10, increase computation by 10^3

5.2 (Dynamic Trading) Hedging Analysis

In addition to pricing, MC simulation allows us to analyze performance of complex trading strategies.
Consider option dynamic hedging:
- Hedging is a dynamic trading process and trading position rebalancing can only be implemented at discrete times
- Analyzing performance of dynamic hedging/trading process is an important part of risk management for financial institutions

In addition to back testing, we can perform hedging analysis based on a stochastic model.
Goal of analysis:
- How good is your hedging/trading strategy?
- How risky is it? (how much risk remains ?)
- How do we measure the risk?

We conduct model based hedging analysis
- assuming model is correct
- selling an option
- dynamically trading underlying, and cash account financing.

This corresponds to the dynamic trading strategy
$$\{-V_t, \delta_t S_t, B_t\}$$
where the bond account always ensures balancing of the account (self-financed)

Outline

- Dynamic hedging analysis
- Computing delta under binomial lattice and MC pricing
- Delta neutral, gamma neutral, and vegaa neutral hedging

Assume BS model for the underlying price

$$\frac{dS}{S} = \mu dt + \sigma dZ_t$$

Consider hedging a **short** option position, i.e., hedging $-V(S,t)$. Assume that trading time are $t_n = n\Delta t,\ \Delta t = \frac{T}{N}$ where

$$0 = t_0 < t_1 < \cdots < t_N = T$$

Assume that δ_n units of S are held in $(t_n, t_{n+1}]$

Definition 5.1 — Delta Hedging.
At time t_n, underlying price S_n, is a binomial lattice is used, the number of units in underlying is given by

$$\delta_j^n = \frac{V_{j+1}^{n+1} - V_j^{n+1}}{(u-d)S_j} \approx \frac{\partial V}{\partial S}(S_j^n, t_n)$$

More generally, hold $\delta_n = \frac{\partial V}{\partial S}(S_n, t_n)$ units of underlying.

We can generate underlying price paths using MC based on the model for the real market price and compute profit and loss (PL) of the dynamic trading strategy along each path.

Note 5.2 we do not use risk neutral model to generate underlying price paths when analyze hedging performance.

Initially $n = 0$, option: $-V_0 = -V(S_0, 0)$, underlying δ_0, balancing with cash account by setting

$$B_0 = V_0 - \delta_0 S_0$$

Portfolio $\Pi = -V + \delta S + B$ has initial value $\Pi_0 = 0$.
At t_n, $\Pi_n = -V(S_n, t_n) + \delta_n S_n + B_n$.
Consider $[t_n, t_{n+1})$. At t_{n+1}, rebalancing position in share and updating cash account so that $\Pi_{t_{n+1}^-} = \Pi_{t_{n+1}^+}$

$$-V(S_{n+1}, t_{n+1}) + \delta_n S_{n+1} + B_n e^{r\Delta t} = -V(S_{n+1}, t_{n+1}) + \delta_{n+1} S_{n+1} + B_{n+1}$$

$$\implies B_{n+1} + B_n e^{r\Delta t} + (\delta_n - \delta_{n+1}) S_{n+1}$$

If $\delta_{n+1} > \delta_n$, buy $\delta_{n+1} - \delta_n$ units.
If $\delta_{n+1} < \delta_n$, sell $\delta_{n+1} - \delta_n$ units.
At expiry $t_N = T$, liquid the portfolio formed at t_{N-1}, which has value

$$\Pi_N = -V(S_N, t_N) + \delta_{N-1} S_N + B_{N-1} e^{r\Delta t}$$