PMATH 351: Real Analysis

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Contents

Chapter 1

Cardinality

Definition 1.1 (domain, range, image, inverse image)**.**
Let X and Y be sets and let $f : X \to Y$. Recall the **domain** of f and the **range** of f are the sets

$$\text{Domain}(f) = X, \ \text{Range}(f) = f(X) = \{f(x)|x \in X\}$$

for $A \subseteq X$, the **image** of A under f is the set

$$f(A) = \{f(x)|x \in A\}$$

For $B \subseteq Y$, the **inverse image** of B under f is the set

$$f^{-1}(B) = \{x \in X|f(x) \in B\}$$

Definition 1.2 (Composite)**.**
Let X, Y and Z be sets, let $f : X \to Y$ and let $g : Y \to Z$. We define the **composite** function $(g \circ f)(x) = g(f(x))$ for all $x \in X$

Definition 1.3 (injective, surjective, bijective)**.**
We say that f is **injective** (or **one-to-one**) when for every $y \in Y$ there exists at most one $x \in X$ such that $f(x) = y$. Equivalently, f is injective when for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
We say that f is **surjective** (or **onto**) when for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$. Equivalently, f is surjective when $\text{Range}(f) = Y$
We say that f is **bijective** (or **invertible**) when f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f(x) = y$. When f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f^{-1} : Y \to X$ such that for all $y \in Y$, $f^{-1}(y)$ is equal to the unique element $x \in X$ such that $f(x) = y$. Note that when f is bijective so is f^{-1}, and in this case we have $(f^{-1})^{-1} = f$

Theorem 1.1. Let $f : X \to Y$ and let $g : Y \to Z$. Then

1. If f and g are both injective then so is $g \circ f$
2. If f and g are both surjective then so is $g \circ f$
3. If f and g are both invertible then so is $g \circ f$, and in this case $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof.

1. Suppose that f and g are both injective. Let $x_1, x_2 \in X$. If $g(f(x_1)) = g(f(x_2))$ then since g is injective we have $f(x_1) = f(x_2)$, and then since f is injective we have $x_1 = x_2$. Thus $g \circ f$ is injective.

2. Suppose that f and g are both injective. Given $z \in Z$, since g is surjective we can choose $y \in Y$ so that $g(y) = z$, then since f is surjective we can choose $x \in X$ so that $f(x) = y$, and then we have $g(f(x)) = g(y) = z$. Thus $g \circ f$ is surjective.

3. Follows (1) and (2).

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Definition 1.4 (identity function)**.**
For a set X, we define the **identity function** on X to be the function $I_X : X \to X$ given by $I_X(x) = x$ for all $x \in X$. Note that for $f : X \to Y$ we have $f \circ I_X = f$ and $I_Y \circ f = f$.

Definition 1.5 (inverse)**.**
Let X and Y be sets and let $f : X \to Y$. A **left inverse** of f is a function $g : Y \to X$ given by $g \circ f = I_X$. Equivalently, a function $g : Y \to X$ is a left inverse of f when $g(f(x)) = x$ for all $x \in X$.
A **right inverse** of f is a function $h : Y \to X$ such that $f \circ h = I_Y$. Equivalently, a function $h : Y \to X$ is a right inverse of f when $f(h(y)) = y$ for all $y \in Y$.

Theorem 1.2. Let X and Y be nonempty sets and let $f : X \to Y$. Then

1. f is injective $\iff$ f has a left inverse.

2. f is surjective $\iff$ f has a right inverse.

3. f is bijective $\iff$ f has a left inverse g and a right inverse h, and in this case we have $g = h = f^{-1}$.

Proof.

1. Suppose first that f is injective. Since $X \neq \varnothing$ we can choose $a \in X$ and then define $g : Y \to X$ as follows: if $y \in \text{Range}(f)$ then (using the fact the f is injective) we define $g(y)$ to be the unique element $x_y \in X$ with $f(x_y) = y$, and if $y \notin \text{Range}(f)$, then we define $g(y) = a$. Then for every $x \in X$ we have $y = f(x) \in \text{Range}(f)$, so $g(y) = x_y = x$, that is $g(f(x)) = x$. Conversely, if f has a left inverse, say g, then f is injective since for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x = g(f(x_1)) = g(f(x_2)) = x_2$.

2. Suppose first that f is onto. For each $y \in Y$, choose $x_y \in X$ with $f(x_y) = y$, then define $g : X \to Y$ by $g(y) = x_y$ (We need the Axiom of Choice for this). Then g is a right inverse of f since for every $y \in Y$ we have $f(g(y)) = f(x_y) = y$. Conversely, if f has a right inverse, say g, then f is onto since given any $y \in Y$ we can choose $x = g(y)$ and then we have $f(x) = f(g(y)) = y$.

3. Suppose first that f is bijective. The inverse function $f^{-1} : Y \to X$ is a left inverse for f because given $x \in X$ we can let $y = f(x)$ and then $f^{-1}(y) = x$ so that $f^{-1}(f(x)) = f^{-1}(y) = x$. Similarly, f^{-1} is a right inverse for f because given $y \in Y$ we can let x be the unique element in X with $y = f(x)$ and then we have $x = f^{-1}(y)$ so that $f(f^{-1}(y)) = f(x) = y$.

Conversely, suppose that g is a left inverse for f and h is a right inverse for f. Since f has a left inverse, it is injective by (1). Since f has a right inverse, it is surjective by (2). Since f is injective and surjective, it is bijective. As shown above, the inverse function f^{-1} is both a left inverse and a right inverse. Finally, note that $g = f^{-1} = h$ because for all $y \in Y$ we have

$$g(y{=}g(f(f^{-1}(y))) = f^{-1}(y) = f^{-1}(f(h(y))) = h(y)$$

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Definition 1.6 (domain, range, image, inverse image).