

PMATH333: Introduction to Real Analysis

Instructor Benjamin Passer
 \LaTeX er Iris Jiang

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Chapter 1

Real Number

1.1 Lecture 2, Sept. 6th

1.1.1 Dedekind Cut

Definition 1.1 (Dedekind cut). A **Dedekind cut** is a subset $A \subseteq \mathbb{Q}$, such that

1. $\emptyset \neq A \neq \mathbb{Q}$
2. If $x \in A$ and $q \in \mathbb{Q}$ with $q \leq x$, then $q \in A$
3. A has no largest element. That is, if $x \in A$, then there exists $y \in A$ with $x < y$

Example 1.1. If $q \in \mathbb{Q}$ is given, the $A_q := \{x \in \mathbb{Q} : x < q\}$ is a Dedekind cut

Proof. Let $q \in \mathbb{Q}$ and consider $A_q \subseteq \mathbb{Q}$

1. Since $q - 1$ is rational and $q - 1 < q$, we have $q - 1 \in A_q$ and $\emptyset \neq A_q$. Similarly, $q + 1$ is rational and $q + 1 \not< q$, So $q + 1 \notin A_q$, $A_q \neq \mathbb{Q}$. Hence $\emptyset \neq A_q \neq \mathbb{Q}$
2. Suppose $x \in A_q$ and $r \in \mathbb{Q}$ with $r \leq x$. By definition, $x < q$, so $r < q$ and $r \in A_q$.
3. Suppose $x \in A_q$, so by definition, $x < q$. Then $a < \frac{x+q}{2} < q$ and $\frac{x+q}{2}$ is rational, so $\frac{x+q}{2} \in A_q$. That is, x is not a largest element of A_q

□

Definition 1.2 (Rational Cut). A Dedekind cut is called a **rational cut** if $X = A_q$ for some $q \in \mathbb{Q}$. Otherwise we say the Dedekind cut X is an **irrational cut**

Rational Operations

For $q, r \in \mathbb{Q}$

$$q < r \iff r - q = \frac{a}{b} \text{ for some } a, b \in \mathbb{Z}^+$$
$$q \leq r \iff q < r \text{ or } q = r$$

For all $q, r, s \in \mathbb{Q}$

- Either $q < r$ or $r \leq q$. (any two of rational numbers are comparable)

- $q \leq r$ and $r \leq q \iff r = q$
- If $q \leq r$ and $r \leq s$, then $q \leq s$

For all $q, r, s, t \in \mathbb{Q}$

- $q \leq r$ and $s \leq t \implies q + s \leq r + t$
- If $0 \leq q$ and $0 \leq r$, then $qr \geq 0$
- If $q \leq r$, then $-r \leq -q$

Lemma. Let q be a rational number with $0 < q$ and $q^2 < 2$. Then there exists some $r \in \mathbb{Q}$ with $q < r$ and $r^2 < 2$

Proof. If $q \leq 1$, choose $r = 1.3$

Consider $r = q + \frac{1}{n}, n \in \mathbb{Z}^+$

$$\begin{aligned} (q + \frac{1}{n})^2 &= q^2 + 2q\frac{1}{n} + \frac{1}{n^2}, & \leftarrow \text{want it } < 2 \\ &< q^2 + 4\frac{1}{n} + \frac{1}{n}, & \text{since } q < q^2, q \leq 1 \\ &< q^2 + \frac{5}{n} \end{aligned}$$

By definition, since $q^2 < 2$, we can write $2 - q^2 = \frac{a}{b}$ for $a, b \in \mathbb{Z}^+$. Choose $n = 5b$, then $r_n^2 < q^2 + \frac{1}{b} \leq q^2 + \frac{a}{b} = 2$

We have found a rational number r_n with $q < r_n$ and $r_n^2 < 2$ □

Goal

We want to show that the collection of Dedekind cuts may be reasonably thought of as a set of numbers. We must be able to define an ordering and algebraic operations on cuts.

We also need the rational cuts A_q to behave just like the rational numbers under these new operations.

The ordering we will use on Dedekind cuts is \subseteq

Theorem 1.1. If X and Y are Dedekind cuts, then either $X \subseteq Y$ or $Y \subseteq X$

Proof Skeleton. Suppose $X \not\subseteq Y$. Then there exists some $x \in X$ such that $x \notin Y$. Let $y \in Y$

- I. If $x \leq y$, then because Y is a Dedekind cut, $x \in Y$, contradiction
- II. $y < x$. Since $x \in X$ and X is a Dedekind cut, we conclude $y \in X$.
We have proved $y \in Y \implies y \in X$, hence $Y \subseteq X$

□

1.2 Lecture 3, Sept. 9th

1.2.1 Real Number

Definition 1.3 (Real Number). A **real number** is a Dedekind cut.

We denote the set of real numbers by \mathbb{R} , and we use the following notation:

- If $q \in \mathbb{Q}$, we use " q " to mean A_q
- $\alpha \leq \beta$ means $\alpha \subseteq \beta$

The collection of rational cuts gives us a copy of \mathbb{Q} in \mathbb{R} , we write $\mathbb{Q} \subseteq \mathbb{R}$

Theorem 1.2 ((\mathbb{R}, \leq) is a **totally ordered** space). If $\alpha, \beta, \gamma \in \mathbb{R}$, then

1. Either $\alpha \leq \beta$ or $\beta \leq \alpha$
2. If $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$
3. If $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$

Proof. Translation into Dedekind Cuts, If W, X, Y are Dedekind cuts, then

1. Either $W \subseteq X$ or $X \subseteq W$
2. If $W \subseteq X$ and $X \subseteq W$, then $X = W$
3. If $W \subseteq X$ and $X \subseteq Y$, then $W \subseteq Y$

□

We also set

$$\alpha < \beta \iff \alpha \leq \beta \text{ and } \alpha \neq \beta$$

In the language of Dedekind cuts, this is \subsetneq ("proper subset").

Goal

The goal of \mathbb{R} is to fill the gaps in \mathbb{Q}

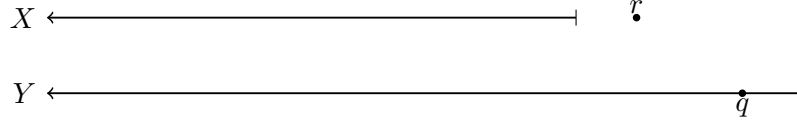
1. \mathbb{R} has no gap
2. Real number is not too large

We will prove that between any two real numbers, there are rational numbers

1.2.2 Density of \mathbb{Q} in \mathbb{R}

Theorem 1.3 (Density of \mathbb{Q} in \mathbb{R}). If $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, then there exists $q \in \mathbb{Q}$ with $\alpha < q < \beta$

Proof. We need to show that if X and Y are Dedekind cuts with $X \subsetneq Y$, then there exists a rational cut A_q with $X \subsetneq A_q \subsetneq Y$. By assumption, $X \subsetneq Y$ implies that there is some $r \in Y \subseteq \mathbb{Q}$ with $r \notin X$. Since $r \in Y$ and Dedekind cuts have no largest element, there exists some $q \in Y$ with $r < q$. We will prove $X \subsetneq A_q \subsetneq Y$



Let x be an arbitrary element of X . Suppose $q \leq x$, so $r < x$. Since X is a Dedekind cut, $r < x$ implies $r \in X$, a contradiction. We conclude $q \not\leq x$ and hence $x < q$. By definition, we have $x \in A_q$. Since $x \in X \implies x \in A_q$. We have $X \subsetneq A_q$. Further, $r \notin X$, but $r < q$, so $r \in A_q$, we therefore have $X \subsetneq A_q$. Suppose $s \in A_q$, so by definition $s \in \mathbb{Q}$ with $s < q$. Since $q \in Y$, and Y is a Dedekind cut, this implies $s \in Y$. Since $s \in A_q \implies s \in Y$, we have $A_q \in Y$. Since $q \in Y$ but $q \notin A_q$, we have $A_q \subsetneq Y$ \square

We just showed: for any open interval $(\alpha, \beta) = \{x \in \mathbb{R} : \alpha < x < \beta\}$ with $\alpha < \beta$ has $\mathbb{Q} \cap (\alpha, \beta) \neq \emptyset$.
(This will later tell us that every real number is a limit of rational numbers)

1.2.3 Bound

Definition 1.4 (Bound). If $\mathcal{S} \subseteq \mathbb{R}$, we say $\beta \in \mathbb{R}$ is an **upper bound** of \mathcal{S} if every $x \in \mathcal{S}$ has $x \leq \beta$. If an upper bound for \mathcal{S} exists, we say \mathcal{S} is **bounded above**

Definition 1.5 (Supremum). Let $\mathcal{S} \subseteq \mathbb{R}$ and $\beta \in \mathbb{R}$. Then β is a **supremum**, or **least upper bound** of \mathcal{S} if

1. β is an upper bound of \mathcal{S}
2. If γ is an upper bound for \mathcal{S} , then $\beta \leq \gamma$

Theorem 1.4. Let $\mathcal{S} \subseteq \mathbb{R}$. If a supremum of \mathcal{S} exists, it is unique.

Proof. If β_1 and β_2 are supremum of \mathcal{S} , then $\beta_1 \leq \beta_2$ because β_1 is a supremum and β_2 is an upper bound. Similarly, $\beta_2 \leq \beta_1$. Hence $\beta_1 = \beta_2$ \square

1.3 Lecture 4, Sept. 11th

1.3.1 The Completeness of \mathbb{R}

Theorem 1.5 (The Completeness of \mathbb{R}). Let $\mathcal{S} \subseteq \mathbb{R}$. If \mathcal{S} is nonempty and bounded above (\exists at least one upper bound), then the supremum of \mathcal{S} exists.

Proof. We seek a number $\beta \in \mathbb{R}$ which is an upper bound of s and has $\beta \leq \gamma$ for every upper bound γ .

Translation in to Dedekind cuts: If \mathcal{S} is a collection of Dedekind cuts which has

- 1). There exists at least one Dedekind cut $X \in \mathcal{S}$
- 2). There is some Dedekind cut \mathcal{D} with $X \in \mathcal{S} \implies X \subseteq \mathcal{D}$

We are looking for a Dedekind cut \mathcal{B} with

- a). $X \in \mathcal{S} \implies X \subseteq \mathcal{B}$
- b). Every \mathcal{D} as in property 2 has $\mathcal{B} \subseteq \mathcal{D}$

We define $\mathcal{B} = \bigcup_{X \in \mathcal{S}} X = \{x \in \mathbb{Q} : \exists X \in \mathcal{S}, x \in X\}$.

Suppose $X \in \mathcal{S}$. Consider an arbitrary element $x \in X \subseteq \mathbb{Q}$. By definition, $x \in \mathcal{B}$, hence $X \subseteq \mathcal{B}$. Suppose \mathcal{D} has the property that $X \in \mathcal{S} \implies X \subseteq \mathcal{D}$, we want $\mathcal{B} \subseteq \mathcal{D}$. Consider any $x \in \mathcal{B} \subseteq \mathbb{Q}$. By definition of \mathcal{B} , there is some $X \in \mathcal{S}$ such that $x \in X$. Since $X \subseteq \mathcal{D}$, we conclude $x \in \mathcal{D}$, hence $\mathcal{B} \subseteq \mathcal{D}$.

The only thing left to prove is that $\mathcal{B} \in \mathbb{R}$, i.e. \mathcal{B} is a Dedekind cut.

- 1). We want to show $\emptyset \neq \mathcal{B} \neq \mathbb{Q}$
 - Since $\mathcal{S} \neq \emptyset$, there is at least one Dedekind cut $X \in \mathcal{S}$. Since Dedekind cuts are nonempty, there exists $x \in X$. By definition, $x \in \mathcal{B}$. We proved $\mathcal{B} \neq \emptyset$
 - By assumption, there is a Dedekind cut \mathcal{D} such that if $X \in \mathcal{S}$ then $X \subseteq \mathcal{D}$. Since \mathcal{D} is a Dedekind cut, there is some rational number q with $q \notin \mathcal{D}$. Since $X \subseteq \mathcal{D}$, we must have $q \notin X$. That is $X \in \mathcal{S} \implies q \notin X$. Since \mathcal{B} contains precisely all of the elements of any $X \in \mathcal{S}$, we must have $q \notin \mathcal{B}$, hence $\mathcal{B} \neq \mathbb{Q}$
- 2). We want to show if $x \in \mathcal{B}, q \in \mathbb{Q}, q \leq x$, then $q \in \mathcal{B}$

If $x \in \mathcal{B}, q \in \mathbb{Q}, q \leq x$, by definition of \mathcal{B} , there exists some $X \in \mathcal{S}$ such that $x \in X, \mathcal{S} \subseteq \mathbb{R}$, so $X \in \mathbb{R}$ is a Dedekind cut. Since $q \in \mathbb{Q}$ and $q \leq x$, we have $q \in X$, but $X \in \mathcal{B}$, so $q \in \mathcal{B}$
- 3). We want to show that if $x \in \mathcal{B}$, then $\exists y \in \mathcal{B}$ with $x < y$

If $x \in \mathcal{B}$, by definition, there exists some $X \in \mathcal{S}$ such that $x \in X$ and X is a Dedekind cut, then \exists some $y \in X$ with $x < y$. But $X \subseteq \mathcal{B}$, so $y \in \mathcal{B}$

□

Dictionary: $\sup(\mathcal{S})$ exists in $\mathbb{R} \implies \bigcup \mathcal{S}$ is a Dedekind cut

Note. $\sup([0, 1)) = 1$, but $1 \notin [0, 1)$

1.4 Lecture 5, Sept. 13th

If we switch all inequalities (\leq to \geq)

Definition 1.6. A lower bound of a set $\mathcal{S} \subseteq \mathbb{R}$ is some $\gamma \in \mathbb{R}$ such that $x \in \mathcal{S} \implies \gamma \leq x$. If some lower bound exists, \mathcal{S} is bounded below.

An infimum of \mathcal{S} is a lower bound β with $\gamma \leq \beta$ for all lower bounds γ . If an infimum exists, it is unique and hence denoted by $\inf(\mathcal{S})$

Theorem 1.6. If $\mathcal{S} \subseteq \mathbb{R}$ is nonempty and bounded below, then $\inf(\mathcal{S})$ exists.

Note. *Proof goes roughly the same with $\mathcal{B} = \bigcap \mathcal{S}$. However, \mathcal{B} might not be a Dedekind cut, from existence of a largest element q . In this case, $\mathcal{B} \setminus \{q\}$ is the Dedekind cut you are looking for.*

Operations

If $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, then

- 1). $\alpha + \beta = \beta + \alpha$
- 2). $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$
- 3). If $\alpha \leq \beta$ and $\gamma \leq \delta$, then $\alpha + \gamma \leq \beta + \delta$

\mathbb{Q} also has an **additive identity**: $q \in \mathbb{Q} \implies q + 0 = q$

Theorem 1.7. If $\alpha \in \mathbb{R}$, then $\alpha + 0 = \alpha$. Moreover, if $\beta \in \mathbb{R}$ has $\alpha + \beta = \alpha$ for all $\alpha \in \mathbb{R}$, then $\beta = 0$

Proof. The official version of the first claim is $\alpha + A_0 = \alpha$, where $A_0 = \{q \in \mathbb{Q} : q < 0\}$, and α is any Dedekind cut. If $a \in \alpha$ and $q \in A_0$, then $a + q$ is rational with $a + q < a$ and $a + q \in \alpha$. That is $\alpha + A_0 \subseteq \alpha$.

If $a \in \alpha$, then since α is a Dedekind cut, there is some $b \in \alpha$ with $a < b$, then $a = b + (a - b) \in \alpha + A_0$, hence $\alpha \subseteq \alpha + A_0$ and $\alpha = \alpha + A_0$

For the second claim, suppose $\beta \in \mathbb{R}$ with $\alpha + \beta = \alpha$ for all $\alpha \in \mathbb{R}$. Then $\beta = \beta + 0 = 0 + \beta = 0$ \square