PMATH 450: Lesbegue Integration and Fourier Analysis

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Contents

1. Week 1

Goals of PMATH 450:

(1) Develop a theory of integration for functions $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ which is
  (a) more flexible (than Riemann) (applicable to more functions)
  (b) more rich (nicer results)
  (c) still extends Riemann integration

(2) Introduce Harmonic Analysis

General outline (first half):

(1) Which sets should we integrate over?
  - Measurable sets

(2) Which functions should we try to integrate?
  - Measurable functions

1.1 Borel Sets

Definition 1.1 — σ-algebra.
Consider a set X, we call $\mathcal{A} \subseteq \mathcal{P}(X)$ (which is the power set of X) a **σ-algebra** of subsets of X if

(1) $\varnothing \in \mathcal{A}$
(2) $A \in \mathcal{A} \implies X \backslash A \in \mathcal{A}$
(3) $A_1, A_2, A_3, \ldots \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

So a σ-algebra is a collection of subsets of X which contains the empty set, is closed under set difference and is closed under countable unions.

R Consider $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-algebra

(1) $X \in \mathcal{A}$
$X \backslash \varnothing = X \in \mathcal{A}$
(2) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$
$A \cup B = A \cup B \cup \varnothing \cup \varnothing \cup \cdots \in \mathcal{A}$

(3) $A_1, A_2, \ldots \in \mathcal{A} \implies \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$

$\bigcap_{i=1}^{\infty} A_i = X \backslash \left(\bigcup_{i=1}^{\infty} (X \backslash A_i) \right)$

(4) $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$

■ **Example 1.1**

- $\{\varnothing, X\}$ is the smallest σ-algebra you could have given X
- $\mathcal{A} = \mathcal{P}(X)$ is a σ-algebra
- $\mathcal{A} = \{A \subseteq \mathbb{R} : A \text{ is open}\}$ is **NOT** a σ-algebra.
  it is not closed under set difference, consider $A = (0,1) \in \mathcal{A}$, $\mathbb{R} \backslash A = (-\infty, 0] \cup [1, \infty) \notin \mathcal{A}$ because it is not open.
- $\mathcal{A} = \{A \subseteq \mathbb{R} : A \text{ open or closed}\}$ is **NOT** a σ-algebra.
  Consider $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \notin \mathcal{A}$ since $\mathbb{Q}$ is neither open nor closed

Proposition 1.1
Given a set X and $\mathcal{C} \subseteq \mathcal{P}(X)$, then $\mathcal{A} = \cap\{\mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{-algebra }, \mathcal{C} \subseteq \mathcal{B}\}$ is a σ-algebra. It is the smallest σ-algebra containing $\mathcal{C}$.

Definition 1.2 — Borel Sets.
Consider $\mathcal{C} = \{A \subseteq \mathbb{R} : A \text{ is open}\}$ (this is s subset of power set of $\mathbb{R}$), then $\mathcal{A} = \bigcap\{\mathcal{B} : \mathcal{C} \subseteq \mathcal{B}, \mathcal{B} \text{ is } \sigma\text{-algebra}\}$ is called **Borel σ-algebra**. The elements of $\mathcal{A}$ is are called the **Borel sets**.

R

(1) All the open sets are Borel. i.e. open $\implies$ Borel.
(2) All the closed sets are Borel. i.e. closed $\implies$ Borel.
since σ-algebra are closed under set difference, and $\mathbb{R}$ take away open is closed
(3) $\{X_1, X_2, \ldots\} = \bigcup_{i=1}^{\infty} \{X_i\}$ is Borel. i.e. countable $\implies$ Borel.
In particular, $\mathbb{Q}$ is a Borel set which is neither open nor closed.
(4) $[a,b) = [a,b] \backslash \{b\} = [a,b] \cap (\mathbb{R} \backslash \{b\})$ is Borel.

It is actually very hard to construct a set that is not Borel. The Borel sets are the propriate sets to integrate over.

1.2 Outer Measure

Idea

(1) Given $A \subseteq \mathbb{R}$, how should we "measure" the "size" of A
(2) Some sets have "sizes" which "measure" more nicely then others. Which ones? Borel sets?

Goal

Define a function $m : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{\infty\}$ (called a measure) such that

(1) $m((a,b)) = m([a,b]) = m((a,b]) = b - a$ (the measure of an interval I equals the length of I)
(2) $m(A \cup B) \leq m(A) + m(B)$
(3) If $A \cap B = \varnothing$, then $m(A \cup B) = m(A) + m(B)$

It will be shown later in the course that we may not use $\mathcal{P}(\mathbb{R})$

Idea

Given $A \subseteq \mathbb{R}$, there exists bounded, open intervals $I_i = (a_i, b_i)$ s.t. $A \subseteq \bigcup_{i=1}^{\infty} I_i$. We want:

$$m(A) \leq \sum_{i=1}^{\infty} m(I_i) = \sum_{i=1}^{\infty} \underbrace{\mathscr{L}(I_i)}_{\text{the length of } I_i} = \sum_{i=1}^{\infty} (b_i - a_i)$$

Cover A by bounded, open, intervals as <u>finely</u> as possible.

Definition 1.3 — Outer Measure.
We define (Lebesgue) **outer measure** by $m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{\infty\}$ with

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \mathscr{L}(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i, I_i \text{ is a bounded, open interval} \right\}$$

■ **Example 1.2** Consider the $\varnothing$ (we would like to the size of it been zero). For any $\epsilon > 0$, $\varnothing \subseteq (0, \epsilon)$, by definition $m^*(\varnothing) \leq \mathscr{L}((0, \epsilon)) = \epsilon$. Since $m^*(\varnothing) \geq 0$, $m^*(\varnothing) = 0$.

■ **Example 1.3** Consider a countable set $A = \{x_1, x_2, x_3, \ldots\}$, given any $\epsilon > 0$, then $A \subseteq \bigcup_{i=1}^{\infty} (x_i - \frac{\epsilon}{2^{i+1}}, x_i + \frac{\epsilon}{2^{i+1}})$. $m^*(A) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \frac{\epsilon}{2} \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = \frac{\epsilon}{2}(\frac{1}{1-\frac{1}{2}}) = \epsilon$. Since $\epsilon > 0$ was arbitrary, $m^*(A) = 0$
Follow a similar proof, we can show that the finite sets also have outer measure zero

Goal

Prove that if I is an interval, then $m^*(I) = \mathscr{L}(I)$

Proposition 1.2
If $A \subseteq B$, then $m^*(A) \leq m^*(B)$

Proof. Sketch:
$X = \left\{ \sum_{i=1}^{\infty} \mathscr{L}(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i \right\}$
$Y = \left\{ \sum_{i=1}^{\infty} \mathscr{L}(I_i) : B \subseteq \bigcup_{i=1}^{\infty} I_i \right\}$
Clearly if $A \subseteq B$ then $Y \subseteq X$ (if you have intervals cover B then they must cover A), hence $\inf X \leq \inf Y$ using the ordering of the extended real numbers. i.e. $m^*(A) \leq m^*(B)$ ■

Lemma 1.3
If $a, b \in \mathbb{R}$ with $a \leq b$, then $m^*([a, b]) = b - a$

We start with the closed bounded intervals because they are compact, so as soon as you cover this closed interval with countable union of open intervals, then you only need to take finitely many of them because you are guaranteed to have a finite subcover to cover the interval.

Proof. Let $\epsilon > 0$ be given. Since $[a, b] \subseteq (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$, we see that $m^*([a, b]) \leq b - a + \epsilon$. Since $\epsilon > 0$ was arbitrary, $m^*([a, b]) \leq b - a$.

Let I_i $(i \in \mathbb{N})$ be bounded open intervals s.t. $[a, b] \subseteq \bigcup_{i=1}^{\infty} I_i$. Since $[a, b]$ is compact, there exists $n \in \mathbb{N}$ s.t. $[a, b] \subseteq \bigcup_{i=1}^{n} I_i$, hence $b - a \leq \sum_{i=1}^{n} \mathscr{L}(I_i) \leq \sum_{i=1}^{\infty} \mathscr{L}(I_i)$ (the first inequality can be proved by induction), and so $m^*([a,b]) \geq b-a$ (since $m^*([a,b])$ is the greatest lower bound) ■

Proposition 1.4
If I is an interval then $m^*(I) = \mathscr{L}(I)$

Proof.
(1) Suppose I is bounded with endpoints $a \leq b$
Given $\epsilon > 0$, note $I \subseteq [a,b] \implies m^*(I) \leq m^*([a,b]) = b-a$. Also $[a+\frac{\epsilon}{2}, b-\frac{\epsilon}{2}] \subseteq I \implies b-a-\epsilon \leq m^*(I) \implies b-a \leq m^*(I)$.
(2) Suppose I is unbounded.
For all $n \in \mathbb{N}$, there exists I_n s.t. $I_n \subseteq I$, $\mathscr{L}(I_n) = n$. Then $m^*(I) \geq m^*(I_n) = n$, hence $m^*(I) = \infty = \mathscr{L}(I)$

■

1.3 Basic Properties of Outer Measure

Outer measure is
(1) Translation Invariant
(2) Countably Subadditivity

Notation
Given $x \in \mathbb{R}, A \subseteq \mathbb{R}$, then $x + A = \{x + a : a \in A\}$

Proposition 1.5 — Translation Invariant.
$m^*(x+A) = m^*(A)$

Proof. Sketch:

$$\begin{aligned} m^*(x+A) &= \inf\left\{\sum \mathscr{L}(I_i) : x+A \subseteq \bigcup I_i\right\} \\ &= \inf\left\{\sum \mathscr{L}(I_i) : A \subseteq \bigcup(-x+I_i)\right\} \\ &= \inf\left\{\sum \mathscr{L}(-x+I_i) : A \subseteq \bigcup(-x+I_i)\right\} \\ &= \inf\left\{\sum \mathscr{L}(J_i) : A \subseteq \bigcup(J_i)\right\} \\ &= m^*(A) \end{aligned}$$

■

Proposition 1.6 — Countable Subadditivity.
If we take countably many subset $A_i \subseteq \mathbb{R} (i \in \mathbb{N})$ then $m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$

Proof. We may assume each $m^*(A_i) < \infty$ (otherwise the result will be trivial). Let $\epsilon > 0$ be given and fix $i \in \mathbb{N}$. There exists open, bounded intervals $I_{i,j}$ s.t. $A_i \subseteq \bigcup_{i=1}^{\infty} I_{i,j}$ and $\sum_{j=1}^{\infty} \mathscr{L}(I_{i,j}) \leq m^*(A_i) + \frac{\epsilon}{2^i}$ (Note that we add a little bit on the out measure which makes it no longer a lower bound hence we can find the $I_{i,j}$, this is a common technique when working with outer measure). We see that $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i,j} I_{i,j}$ and so $m^*(A_i) \leq \sum_{i,j} \mathscr{L}(I_{i,j}) =$

$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathscr{L}(I_{i,j}) \leq \sum_{i=1}^{\infty} m^*(A_i) + \frac{\epsilon}{2^i} = \sum_{i=1}^{\infty} m^*(A_i) + \epsilon$. Since ϵ is arbitrary, the proposition follows. ■

Corollary 1.7 — Finite Subadditivity.
If $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{R})$, then $m^*\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} m^*(A_i)$

Proof. Sketch:
$A_1 \cup \cdots \cup A_n = A_1 \cup \cdots \cup A_n \cup \varnothing \cup \varnothing \cup \cdots$ ■

Problem

There exists $A, B \subseteq \mathbb{R}$ s.t. $A \cap B = \varnothing$ and $m^*(A \cup B) < m^*(A) + m^*(B)$. i.e. outer measure is not finitely additive. (We would like $m^*(A \cup B) = m^*(A) + m^*(B)$ for disjoint sets A, B)

Solution

Restrict the domain of m^* to only include sets which measure "nicely" (which are called measurable).

2. Week 2

2.1 Measurable Sets

Goal

Restrict the domain of m^* to only include sets s.t. whenever $A \cap B = \varnothing$, $m^*(A \cup B) = m^*(A) + m^*(B)$

Definition 2.1 — Measurable.
We say $A \subseteq \mathbb{R}$ is **measurable** if $\forall X \subseteq \mathbb{R}$, $m^*(X) = m^*(X \cap A) + m^*(X \backslash A)$

R By finite subadditivity, we always have $m^*(X) \leq m^*(X \cap A) + m^*(X \backslash A)$ because $X = (X \cap A) \cup (X \backslash A)$

R If $\subseteq \mathbb{R}$ is measurable and $B \subseteq \mathbb{R}$ with $A \cap B = \varnothing$, then consider $X = A \cup B$, $m^*(A \cup B) = m^*(X \cap A) + m^*(X \backslash A) = m^*(A) + m^*(B)$.

Goal

Show a lot of sets are measurable.

Proposition 2.1
If $m^*(A) = 0$, then A is measurable.

Proof. Let $X \subseteq \mathbb{R}$. Since $X \cap A \subseteq A$, we have $0 \geq m^*(X \cap A) \leq m^*(A) = 0$, and so $m^*(X \cap A) = 0$. Then $m^*(X \cap A) + m^*(X \backslash A) = m^*(X \backslash A) \leq m^*(X)$. (The other inequality is trivial). ■

Proposition 2.2
If $A_1, A_2, \ldots, A_n$ are measurable, then $\bigcup\limits_{i=1}^{n} A_i$ is measurable.

Proof. It suffices to prove the result when $n = 2$ (then the rest is a trivial induction). Let $A, B \subseteq \mathbb{R}$ be measurable. Let $X \subseteq \mathbb{R}$. Then

$$\begin{aligned} m^*(X) &= m^*(X \cap A) + m^*(X \backslash A) \\ &= m^*(X \cap A) + m^*((X \backslash A) \cap B) + m^*((X \backslash A) \backslash B) \\ &= m^*(X \cap A) + m^*((X \backslash A) \cap B) + m^*(X \backslash (A \cup B)) \\ &\geq m^*((X \cap A) \cup ((X \backslash A) \cap B)) + m^*(X \backslash (A \cup B)) \\ &= m^*(X \cap (A \cup B)) + +m^*(X \backslash (A \cup B)) \end{aligned}$$

■

Proposition 2.3
Let $A_1, A_2, \ldots, A_n$ are measurable, $A_i \cap A_j = \varnothing$ for $i \neq j$. Let $A = A_1 \cup \cdots \cup A_n$. If $X \subseteq \mathbb{R}$, then $m^*(X \cap A) = \sum\limits_{i=1}^{n} m^*(X \cap A_i)$

Proof. It suffices to prove the result when $n = 2$ (then the rest is a trivial induction). Let $A, B \subseteq \mathbb{R}$ be measurable with $A \cap B = \varnothing$. Let $X \subseteq \mathbb{R}$. Then

$$\begin{aligned} m^*(X \cap (A \cup B)) &= m^*((X \cap (A \cup B)) \cap A) + m^*((X \cap (A \cup B)) \backslash A) \\ &= m^*(X \cap A) + m^*(X \cap B) \end{aligned}$$

We only used the measurability of A, so in some sense our assumption is stronger than it can be. When actually writing out the induction, you will need $A_1, \ldots, A_{n-1}$ to be measurable. It doesn't matter the last one A_n is measurable or not. ■

Corollary 2.4 — Finite additivity.
If $A_1, A_2, \ldots, A_n$ are measurable, $A_i \cap A_j = \varnothing$ for $i \neq j$. Then $m^*(A_1 \cup \cdots \cup A_n) = \sum\limits_{i=1}^{n} m^*(A_i)$

Proof. Take $X = \mathbb{R}$ in the previous prove. ■

2.2 Countable Additivity

Lemma 2.5
Consider $A_i \subseteq \mathbb{R}$ are measurable for $i \in \mathbb{N}$. If $A_i \cap A_j = \varnothing$ for $i \neq j$. Then $A := \bigcup\limits_{i=1}^{\infty} A_i$ is measurable.

Proof. Sketch: Consider $B_n = A_1 \cup A_2 \cup \cdots \cup A_n$ and $X \subseteq \mathbb{R}$. Then

$$\begin{aligned} m^*(X) &= m^*(X \cap B_n) + m^*(X \backslash B_n) \\ &\geq m^*(X \cap B_n) + m^*(X \backslash A) \\ &= \sum_{i=1}^{n} m^*(X \cap A_i) + m^*(X \backslash A) \text{ (Using the proposition 2.3)} \end{aligned}$$

Taking $n \to \infty$,

$$\begin{aligned} m^*(X) &\geq \sum_{i=1}^{\infty} m^*(X \cap A_i) + m^*(X \backslash A) \\ &\geq m^*\left(\bigcup_{i=1}^{\infty}(X \cap A_i)\right) + m^*(X \backslash A) \\ &= m^*(X \cap A) + m^*(X \backslash A) \end{aligned}$$

■

Proposition 2.6
If $A \subseteq \mathbb{R}$ is measurable, then $\mathbb{R} \backslash A$ is measurable.

Proof. Sketch: Take $X \subseteq \mathbb{R}$

$$\begin{aligned} m^*(X \cap (\mathbb{R} \backslash A)) + m^*(X \backslash (\mathbb{R} \backslash A)) &= m^*(X \backslash A) + m^*(X \cap A) \\ &= m^*(X) \text{ (Since } A \text{ is measurable)} \end{aligned}$$

■

Proposition 2.7
If $A_i \subseteq \mathbb{R}$ are measurable with $i \in \mathbb{N}$, then $A = \bigcup\limits_{i=1}^{\infty} A_i$ is measurable.

Proof. Sketch: Define $B_1 = A_1$ and $B_n = A_n \backslash (A_1 \cup A_2 \cup \cdots \cup A_{n-1})$ for $n \geq 2$. Then $B_n = A_n \cap (\mathbb{R} \backslash (A_1 \cup A_2 \cup \cdots \cup A_{n-1}))$ (these are two separate measurable sets). Therefore B_n is measurable and by construction, for $i \neq j$, $B_i \cap B_j = \varnothing$. Also $\bigcup\limits_{i=1}^{\infty} B_i = \bigcup\limits_{i=1}^{\infty} A_i$, which are measurable. ■

Corollary 2.8
The collection $\mathscr{L}$ of (Lebesgue) measurable sets is a σ-algebra of sets in $\mathbb{R}$

Corollary 2.9 — Countable additivity.
If $A_i \in \mathbb{R}$ are measurable with $i \in \mathbb{N}$, $A_i \cap A_j = \varnothing$ for $i \neq j$. Then $m^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \sum\limits_{i=1}^{\infty} m^*(A_i)$

Proof. Sketch: The countable subadditivity 1.6 tells us $m^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) \leq \sum\limits_{i=1}^{\infty} m^*(A_i)$.
Note that for any $n \in \mathbb{N}$

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geq m^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m^*(A_i)$$

Take $n \to \infty$, $m^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) \geq \sum\limits_{i=1}^{n} m^*(A_i)$ ■

2.3 Borel Implies Measurable

Goal

Show Borel sets are measurable.

Proposition 2.10
If $a \in \mathbb{R}$ then (a, ∞) is measurable.

Proof. Let $X \subseteq \mathbb{R}$. We want to show that $m^*(X \cap (a, \infty)) + m^*(X \backslash (a, \infty)) \leq m^*(X)$.

(1) Assume $a \notin X$. We want to show $m^*(X \cap (a, \infty)) + m^*(X \cap (-\infty, a)) \leq m^*(X)$. Let $X_1 = (X \cap (a, \infty))$ and $X_2 = (X \cap (-\infty, a))$. Let (I_i) be a sequence of bounded open intervals such that $X \subseteq \cup I_i$. Define for each i, $I_i' = I_i = \cap(a, \infty)$ and $I_i'' = I_i \cap (-\infty, a)$. Note that $X_1 \subseteq \cup I_i'$ and $X_2 \subseteq \cup I_i''$, ans so $m^*(X_1) \leq \sum \mathscr{L}(I_i')$ and $m^*(X_2) \leq \sum \mathscr{L}(I_i'')$. We then see that

$$
\begin{aligned}
m^*(X_1) + m^*(X_2) &\leq \sum \mathscr{L}(I_i') + \sum \mathscr{L}(I_i'') \\
&= \sum [\mathscr{L}(I_i') + \mathscr{L}(I_i'')] \\
&= \sum \mathscr{L}(I_i)
\end{aligned}
$$

Note that the $m^*(X)$ is the infimum of $\sum \mathscr{L}(I_i)$. By the definition of infimum, $m^*(X_1) + m^*(X_2) \leq m^*(X)$

(2) Assume $a \in X$. Let $X' = X \backslash \{a\}$. Then by the previous case, $m^*(X' \cap (a, \infty)) + m^*(X' \cap (-\infty, a)) \leq m^*(X')$. Also note $m^*(\{a\}) = 0$. Then

$$
\begin{aligned}
m^*(X \cap (a, \infty)) + m^*(X \backslash (a, \infty)) &= m^*(X' \cap (a, \infty)) + m^*((X' \backslash (a, \infty)) \cup \{a\}) \\
&= m^*(X' \cap (a, \infty)) + m^*((X' \cap (-\infty, a)) \cup \{a\}) \\
&\leq m^*(X' \cap (a, \infty)) + m^*(X' \cap (-\infty, a)) + m^*(\{a\}) \\
&= m^*(X' \cap (a, \infty)) + m^*(X' \cap (-\infty, a)) \\
&\leq m^*(X') \\
&\leq m^*(X)
\end{aligned}
$$

■

Theorem 2.11
Every Borel set is measurable.

Proof. Sketch: We will show that every open set is measurable, then the σ-algebra of measurable sets contains all the open sets, and so by smallness of Borel set it would be contained in the σ-algebra of Lebesgue measurable sets.

We have (a, ∞) is measurable, then $\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty)$ is measurable. Note $\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty) = [a, \infty$. Then $\mathbb{R} \backslash [a, \infty) = (-\infty, a]$ is also measurable. Then any open interval $(a, b) = (a, \infty) \cap (-\infty, b)$ is measurable. Hence, every open set in $\mathbb{R}$ is measurable. Recall the Borel σ-algebra is the σ-algebra generated by open sets, meaning it is the smallest σ-algebra of sets in $\mathbb{R}$ which contain open sets. Since the collection of measurable sets is a σ-algebra, now containing all open sets, we have the Borel σ-algebra has to be subset of the σ-algebra of the measurable sets. ■

Definition 2.2 — Lebesgue measurable.
We call $m : \mathscr{L} \to [0,\infty) \cup \{\infty\}$ where $\mathscr{L}$ is the σ-algebra of measurable sets given by $m(A) = m^*(A)$ Lebesgue measurable

Exercise 2.1 Prove that if $A \subseteq \mathbb{R}$ is measurable, then $x + A$ is measurable for any $x \in \mathbb{R}$

Solution. Let $X \subseteq \mathbb{R}$, we want to show $m^*(X) = m^*(X \setminus (x + A)) + m^*(X \cap (x + A)) = m^*(X \cap (x + A)^c)) + m^*(x \cap (x + A))$.

$$\begin{aligned} y \in X \cap (x + A) &\iff y \in X \text{ and } y \in x + A \\ &\iff y - x \in X - x \text{ and } y - x \in A \\ &\iff y - x \in (X - x) \cap A \\ &\iff y \in (X - x) \cap A + x \end{aligned}$$

Hence $X \cap (x+A) = (X-x) \cap A + x$. Similarly, one also has that $X \cap (x+A)^c = (X-x) \cap A^c + x$. Now, from the RHS, by the translation invariant property for outer measure, we have that

$$\begin{aligned} m^*(X \cap (x + A)^c)) + m^*(x \cap (x + A)) &= m^*((X - x) \cap A^c + x) + m^*((X - x) \cap A + x) \\ &= m^*((X - x) \cap A^c) + m^*((X - x) \cap A) \end{aligned}$$

Since A is measurable, $m^*((X - x) \cap A^c) + m^*((X - x) \cap A) = m^*(X - x) = m^*(X)$. It follows that $x + A$ is measurable. ■

2.4 Basic Properties of Lebesgue Measure

Proposition 2.12 — Excision Property.
If $A \subseteq B$ with A being measurable and $m(A) < \infty$, then $m^*(B \backslash A) = m^*(B) - m(A)$

Proof. Sketch: By the definition of measurable

$$\begin{aligned} m^*(B) &= m^*(B \cap A) + m^*(B \backslash A) \\ &= m(A) + m^*(B \backslash A) \end{aligned}$$

We want $m(A) > \infty$ to avoid the case $\infty - \infty$. ■

Theorem 2.13 — Continuity of Measure.

(1) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ are measurable, then $m\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} m(A_n)$

(2) If $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ are measurable, with $B_1 < \infty$, then $m\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} m(B_n)$

Proof. (1) Since $m(A_k) \leq m(\bigcup A_i)$ for all $k \in \mathbb{N}$, we have $\lim_{n\to\infty} m(A_n) \leq m(\bigcup A_i)$. If $\exists k \in \mathbb{N}$ s.t. $m(A_k) = \infty$, then $\lim_{n\to\infty} m(A_n) = \infty$ and we are done. Hence we may assume each $m(A_k) < \infty$ (so to use Excision Property 2.12).
(We would like to use Countable additivity 2.9, so we will replace A_i with a sequence of disjoint sets. For each $k \in \mathbb{N}$, let $D_k = A_k \backslash A_{k-1}$, $A_0 = \varnothing$.

Note 2.1
- The $D_k's$ are measurable
- The $D_k's$ are pariwise disjoint
- $\bigcup D_i = \bigcup A_i$

$$\begin{aligned} m\left(\bigcup A_i\right) = m\left(\bigcup D_i\right) &= \sum_{i=1}^{\infty} m(D_i) && \text{(by countable additivity)} \\ &= \sum_{i=1}^{\infty}(m(A_i) - m(A_{i-1})) && \text{(by excision property)} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n}(m(A_i) - m(A_{i-1})) \\ &= \lim_{n\to\infty} m(A_n) - m(A_0) \\ &= \lim_{n\to\infty} m(A_n) \end{aligned}$$

(2) For $k \in \mathbb{N}$, define $D_k = B_1 \backslash B_k$

Note 2.2
- $D_k's$ are measurable
- $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$

By (1), $m(\bigcup D_i) = \lim_{n\to\infty} m(D_n)$. We see that

$$\bigcup D_i = \bigcup_{i=1}^{\infty}(B_1 \backslash B_i) = B_1 \backslash \left(\bigcap_{i=1}^{\infty} B_i\right)$$

and so

$$\lim_{n\to\infty} m(D_n) = m\left(\bigcup D_i\right) = m\left(B_1 \backslash \left(\bigcap B_i\right)\right) = m(B_1) - m\left(\bigcap B_i\right)$$

Since $m(B_1) < \infty$, $B_i \subseteq B_1$ for all $i > 1$, hence $m(\bigcap B_i) < \infty$. However,

$$\lim_{n\to\infty} m(D_n) = \lim_{n\to\infty} m(B_1) - m(B_n) = m(B_1) - \lim_{n\to\infty} m(B_n)$$

■

■ **Example 2.1** $B_i = (i, \infty)$, $m\left(\bigcap B_i\right) = m(\varnothing) = 0$, however $\lim_{n\to\infty} m(B_n) = \infty$.

3. Week 3

Goals of the week:

(1) Construct an example of a non measurable set
(2) Construct an element in $\mathscr{L} \backslash \mathcal{B}$

3.1 A Non-measurable Set

Lemma 3.1
Assume we have a bounded, measurable set $A \subseteq \mathbb{R}$, and a bounded, countably infinite set $\Lambda \subseteq \mathbb{R}$. If $\lambda + A$, $\lambda \in \Lambda$, are pairwise disjoint, then $m(A) = 0$.

Proof. Sketch: Consider $\bigcup\limits_{\lambda}(\lambda + A)$ which is bounded, measurable. $m\left(\bigcup_{\lambda}(\lambda + A)\right) < \infty$ (since it is bounded).

$$m\left(\bigcup_{\lambda}(\lambda + A)\right) = \sum_{\lambda} m(\lambda + A) = \sum_{\lambda}(A) < \infty \implies m(A) = 0$$

■

Construction

Start with $\varnothing \neq A \subseteq \mathbb{R}$. Consider $a \sim b \iff a - b \in \mathbb{Q}$. Then N is an equivalence relation. Let C_A denote a single choice of equivalence class representatives for A relative to $\sim$.

R The sets $\lambda + C_A$, $\lambda \in \mathbb{Q}$, are pariwise disjoint.

Proof.

$$\begin{aligned} x \in (\lambda_1 + C_A) \cap (\lambda_2 + C_A) &\implies x = \lambda_1 + a = \lambda_2 + b,\ \ a, b \in C_A \\ &\implies a - b = \lambda_2 - \lambda_1 \in \mathbb{Q} \\ &\implies a \sim b \\ &\implies a = b \\ &\implies \lambda_1 = \lambda_2 \end{aligned}$$

■

Theorem 3.2 — Vitali Theorem.
Every set $A \subseteq \mathbb{R}$ with $m^*(A) > 0$ contains a non-measurable subset.

Proof. By Quiz 1, we may assume A is bounded, say $A \subseteq [-N, N]$, for some $N \in \mathbb{N}$. Claim: C_A is non-measurable. For contradiction, assume C_A is measurable. Let $\Lambda \subseteq \mathbb{Q}$ be bounded and infinite. By the lemma and the remark, $m(C_A) = 0$. Let $a \in A$. Then $a \sim b$ for some $b \in C_A$. In particular, $a - b = \lambda \in \mathbb{Q}$. Moreover, $\lambda \in [-2N, 2N]$. Taking $\Lambda_0 = \mathbb{Q} \cap [-2N, 2N]$, we have that $A \subseteq \bigcup\limits_{\lambda \in \Lambda_0} (\lambda + C_A)$. Contradiction to $m^*(A) > 0$. ■

Corollary 3.3
There exists $A, B \subseteq \mathbb{R}$ s.t.
(1) $A \cap B = \varnothing$
(2) $m^*(A \cup B) < m^*(A) + m^*(B)$

Proof. Sketch: Let C be a non-measurable set. By the definition, $\exists X \subseteq \mathbb{R}$, $m^*(X) < m^*(X \cap C) + m^*(X \backslash C)$ ■

3.2 Cantor-Lebesgue Function

Recall: Cantor Set

$$\begin{aligned} &I = [0, 1] \\ &C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \\ &C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \\ &\vdots \\ &C = \bigcap_{k=1}^{\infty} C_k \end{aligned}$$

The Cantor set is:
- uncountable
- closed

Proposition 3.4
The Cantor set if Borel and has measure zero.

Proof. Sketch: The Cantor set is closed so Borel. $C = \bigcap\limits_{k=1}^{\infty} C_k$, C_k's are measruable and

$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$, $m(C_1) < \infty$. by the continuity of measure,

$$m(C) = \lim_{k\to\infty} m(C_k) = \lim_{k\to\infty} \frac{2^k}{3^k} = 0$$

■

Construction

Uncountable set which has measure zero: Cantor-Lebesgue function

(1) For $k \in \mathbb{N}$, U_k = union of open intervals deleted in the process of constructing $C_1, C_2, \dots, C_k$, i.e. $U_k = [0,1] \backslash C_k$

(2) $U = \bigcup_{k=1}^{\infty} U_k$, i.e. $U = [0,1] \backslash C$

(3) Say $U_k = I_{k,1} \cup I_{k,2} \cup \cdots_{k,2^k-1}$ (in order) it can be checked that each U_k can be written as the union of $2^k - 1$ disjoint open intervals.

Define $\varphi : U_k \to [0,1]$ by $\varphi|_{I_{k,i}} = \frac{i}{2^k}$

■ Example 3.1

$U_1 = (\frac{1}{3}, \frac{2}{3}) \mapsto \frac{1}{2}$

$U_2 = \underbrace{(\frac{1}{9}, \frac{2}{9})}_{\mapsto \frac{1}{4}} \cup \underbrace{(\frac{1}{3}, \frac{2}{3})}_{\mapsto \frac{2}{4} = \frac{1}{2}} \cup \underbrace{(\frac{7}{9}, \frac{8}{9})}_{\mapsto \frac{3}{4}}$

etc.

(4) Define $\varphi : [0,1] \to [0,1]$ by: for $0 \neq x \in C$, $\varphi(x) = \sup\{\varphi(t) : t \in U \cap [0,x)\}$ and $\varphi(0) = 0$. This is the **Cantor-Lebesgue Function**

Things to know about φ:

(1) φ is increasing

(2) φ is continuous

(3) $\varphi : U \to [0,1]$ is differentiable and $\varphi' = 0$

(4) φ is onto

Proof. (2):

- φ is continuous on U. (φ is constant on each little intervals.)
- $x \in C$, $x \neq 0, 1$. For large k, $\exists a_k \in I_{k,i}$, $\exists b_k \in I_{k,i+1}$ s.t. $a_k < x < b_k$. But, $\varphi(b_k) - \varphi(a_k) = \frac{i+1}{2^k} - \frac{i}{2^k} = \frac{1}{2^k} \to 0$. No jump between , $\varphi(a_k), \varphi(b_k)$
- $x \in \{0,1\}$ can be proved similarly.

(4): $\varphi(0) = 0$, $\varphi(1) = 1$, by the IVT, φ hits all values in between ■

3.3 A non Borel Set

Let φ be the C-L function. Consider $\psi : [0,1] \to [0,2]$ defined by

$$\psi(x) = x + \varphi(x)$$

(1) ψ is **strictly** increasing
(the sum of a strictly increasing function and an increasing function)

(2) ψ is continuous

(3) ψ is onto (IVT)

$\implies$ ψ is invertible.

Proposition 3.5 (1) $\psi(C)$ is measurable and has **positive** measure.
It is turning a measure zero set to positive measure
(2) ψ maps a particular (measurable) subset of C to a non-measurable set

Proof.

(1) By Assignment 1, ψ^{-1} is continuous, then we have $\psi(C) = (\psi^{-1})^{-1}(C)$ is closed. $\implies$ $\psi(C)$ is measurable. Note that $[0,1] = C\dot{\cup}U$ for disjoint C and U, $\implies$ $[0,2] = \psi(C)\dot{\cup}\psi(U)$ $\implies$ $2 = m(\psi(C)) + m(\psi(U))$. It suffices to show that $m(\psi(U)) = 1$. (U is nicer to work with because by construction we can write it as a union of disjoint open intervals, as we know Lebesgue measure works quite well with disjoint union of measurable sets.)
Say $U = \dot{\bigcup}_{i=1}^{\infty} I_i$, where I_i's are disjoint open intervals. (remember that U is defined to be the countable union of U_k's, for each U_k is a finite union of disjoint open intervals).
Then, $\psi(U) = \dot{\bigcup}_{i=1}^{\infty}\psi(I_i)$ $\implies$ $m(\psi(U)) = \sum m(\psi(I_i))$. Note that for all $i \in \mathbb{N}$, $\exists r \in \mathbb{R}$, s.t. $\psi(x) = r$ for all $x \in I_i$. In particular, $\psi(x) = x + r$ for all $x \in I_i$, and so $\psi(I_i) = r + I_i$. Hence, $m(\psi(U)) = \sum m(I_i) = m(\dot{\bigcup} I_i) = m(U)$. Since $[0,1] = U\dot{\cup}C$, we have that $1 = m(U) + m(C) = m(U)$ (the measure of Cantor set is 0). Hence, $m(\psi(U)) = m(U) = 1 > 0$.

(2) By Vitali theorem 3.2, $\psi(C)$ contains a subset $A \subset \psi(C)$ which is non measurable. Let $B = \psi^{-1}(A) \subseteq C$. Then, $\psi(B) = A$ is non measurable as required.

■

Theorem 3.6
The Cantor set contains an element of $\mathscr{L}\backslash\mathcal{B}$

Proof. Sketch: $B \subseteq C$ $\implies$ B measurable. But we know $\psi(B)$ is non measurable. By Assignment 1, if B is Borel, then $\psi(B)$ is Borel. Therefore, B is NOT Borel.

■

4. Week 4

4.1 Measurable Functions

Question

Which functions are suitable for integration?

Definition 4.1 — measurable function.
Let $A \subseteq \mathbb{R}$ be measurable, we say $f : A \to \mathbb{R}$ is **measurable** if and only if for all open $\mathcal{U} \subseteq \mathbb{R}$, $f^{-1}(\mathcal{U})$ is **measurable**.

Proposition 4.1
If $A \subseteq \mathbb{R}$ is measurable and $f : A \to \mathbb{R}$ is continuous, then f is measurable.

Proof. Sketch: If $\mathcal{U} \subseteq \mathbb{R}$ is open, then $f^{-1}(\mathcal{U})$ is open, which is Borel, which is measurable ■

Proposition 4.2
If $A \subseteq \mathbb{R}$ is measurable and $\mathcal{X}_A : \mathbb{R} \to \mathbb{R}$, $\mathcal{X}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$, then $\mathcal{X}_A$ is measurable.

(This is called the characteristic function of A. In particular, if $A \neq \mathbb{R}$, then $\mathcal{X}_A$ is an example of measurable function which is not continuous)

Proof. Sketch: Take open set $\mathcal{U} \subseteq \mathbb{R}$, for $\mathcal{X}_A^{-1}(\mathcal{U}) = \mathbb{R}, A, \mathbb{R} \backslash A, \varnothing$ which are all measurable.
■

Proposition 4.3
Let $A \subseteq \mathbb{R}$ be measurable, $f : A \to \mathbb{R}$, the following are equivalent
(1) f is measurable
(2) $\forall a \in \mathbb{R}$, $f^{-1}((a, \infty))$ is measurable (This is always the given definition for measur-

able function)

(3) $\forall a < b$, $f^{-1}((a, b))$ is measurable

Proof.

(1) $\implies$ (2) Trivial.

(2) $\implies$ (3) Let $b \in \mathbb{R}$ so that $f^{-1}(b, \infty)$ is measurable. Then

$$\mathbb{R} \backslash f^{-1}((b, \infty)) = f^{-1}(\mathbb{R} \backslash (b, \infty)) = f^{-1}((-\infty, b])$$

is measurable as well. We see that $(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - \frac{1}{n}]$ and so $f^{-1}(-\infty, b) = \bigcup_{n=1}^{\infty} f^{-1}((-\infty, b - \frac{1}{n}])$ is measurable. Finally, for $a < b$, $(a, b) = (a, \infty) \cap (-\infty, b) \implies f^{-1}((a, b)) = f^{-1}((a, \infty)) + f^{-1}((-\infty, b))$ is measurable

(3) $\implies$ (1) Trivial.

■

4.2 Properties of Measurable Functions

Proposition 4.4

If $A \subseteq \mathbb{R}$ is measurable, $f, g : A \to \mathbb{R}$ are measurable

(1) For all $a, b \in \mathbb{R}$, $af + bg$ is measurable

(2) The function fg is measurable (fg denotes $f(x)g(x)$, not the composite function)

Proof.

(1) Let $a \in \mathbb{R}$.For $\alpha \in \mathbb{R}$, $(af)^{-1}((\alpha, \infty)) = \{x \in A : af(x) > \alpha\}$

(a) $a > 0$,

$$(af)^{-1}((a, \infty)) = \{x \in A : f(x) > \frac{\alpha}{a}\} = f^{-1}((\frac{\alpha}{a}, \infty))$$

which is measurable

(b) $a < 0$,

$$(af)^{-1}((a, \infty))f^{-1}((-\infty, \frac{\alpha}{a}))$$

(c) $a = 0$, af is continuous $\implies$ measurable

We now show that $f + g$ is measurable. For $\alpha \in \mathbb{R}$,

$$\begin{aligned}
(f + g)^{-1}((a, \infty)) &= \{x \in A : f(x) + g(x) > \alpha\} \\
&= \{x \in A : f(x) > \alpha - g(x)\} \\
&= \{x \in A : \exists 1 \in \mathbb{Q}, f(x) > q > \alpha - g(x)\} \\
&= \bigcup_{q \in \mathbb{Q}} (\{x \in A : f(x) > q\} \cap \{x \in A : g(x) > \alpha - q\}) \\
&= \bigcup_{q \in \mathbb{Q}} (f^{-1}((q, \infty)) \cap g^{-1}((\alpha - 1, \infty)))
\end{aligned}$$

which is measurable. Hence $f + g$ is measurable

(2) By the quiz, $|f|$ is measurable. For $\alpha \in \mathbb{R}$,

$$(f^2)^{-1}((\alpha,\infty)) = \{x \in A : f^2(x) > \alpha\}$$
$$= \begin{cases} A, & \alpha < 0 \\ \{x : A : |f|(x) > \sqrt{\alpha}\}, & \alpha \geq 0 \end{cases}$$
$$= \begin{cases} A, & \alpha < 0 \\ |f|^{-1}((\sqrt{\alpha},\infty)), & \alpha \geq 0 \end{cases}$$

which is measurable. Therefor f^2 is measurable.
Since $(f+g)^2 = f^2 + 2fg + g^2$ is measurable, we have that $2fg$ is measurable. By (1), fg is measurable as well.

■

■ **Example 4.1 — Not all compositions of measurable functions are measurable.** $\psi : [0,1] \to \mathbb{R}$, $\psi(x) = x + \varphi(x)$ (where φ is the Cantor-Lebesgue function) is not measurable. $\exists A \subseteq [0,1]$ s.t. A is measurable but $\psi(A)$ is not measurable. Extend $\psi : \mathbb{R} \to \mathbb{R}$ continuously to a strictly increasing surjective function s.t. ψ^{-1} is continuous.

Consider $\mathcal{X} \circ \psi^{-1}$ (we know $\mathcal{X}$ is measurable, be the continuity, ψ^{-1} is also measurable). Then

$$(\mathcal{X} \circ \psi^{-1})^{-1}(\frac{1}{2},\frac{2}{3}) = \psi(\mathcal{X}_A^{-1}(\frac{1}{2},\frac{2}{3})) = (A)$$

which is not measurable. Hence $\mathcal{X} \circ \psi^{-1}$ is not measurable.

Proposition 4.5
Given $A \subseteq \mathbb{R}$ is measurable. If $g : A \to \mathbb{R}$ is measurable and $f : \mathbb{R} \to \mathbb{R}$ is **continuous** then $f \circ g$ is measurable.

Proof. Sketch: Take $\mathcal{U} \subseteq \mathbb{R}$ be open. $(f \circ g)^{-1}(\mathcal{U}) = g^{-1}(f^{-1}(\mathcal{U}))$ is measurable since continuity gives $f^{-1}(\mathcal{U})$ is open. ■

Definition 4.2 — almost everywhere.
We say a property $P(x)$ $(x \in A)$ is true **almost everywhere** (ae) if $m(\{x \in A : P(x) \text{ is false}\}) = 0$

Proposition 4.6
$f : A \to \mathbb{R}$ is a measurable function. If $g : A \to \mathbb{R}$ is a function and $f = g$ almost everywhere, then g is measurable.
when we say a function is measurable, we implicitly say the domain is measurable

Proof. Sketch: Let $B = \{x \in A : f(x) \neq g(x)\}$, then $m(B) = 0$. Let $\alpha \in \mathbb{R}$,

$$g^{-1}((\alpha,\infty)) = \{x : A : g(x) > \alpha\}$$
$$= \{x \in A \backslash B : g(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\}$$
$$= \{x \in A \backslash B : f(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\}$$
$$= (f^{-1}(\alpha,\infty) \cap (A \backslash B)) \cup \{x \in B : g(x) > \alpha\}$$

$(f^{-1}(\alpha,\infty) \cap (A \backslash B))$ is clearly measurable, $\{x \in B : g(x) > \alpha\}$ is a subset of B which also has measure zero.
Hence the $g^{-1}((\alpha,\infty))$ is measurable ■

Proposition 4.7
Given A is measurable, $B \subseteq A$ is measurable. A function $f : A \to B$ is measurable if and only if $f|_B$ and $f|_{A\backslash B}$ are measurable.

Proof.
$\Longrightarrow$ Suppose $f : A \to \mathbb{R}$ is measurable. Let $\alpha \in \mathbb{R}$, then

$$\begin{aligned}(f|_B)^{-1}(\alpha, \infty) &= \{x \in B : f(X) > \alpha\}\\ &= f^{-1}(\alpha, \infty) \cap B\end{aligned}$$

which is measurable. Therefore $f|_B$ is measurable.
The proof for $f|_{A\to B}$ is identical.
$\Leftarrow$ Suppose $f|_B$ and $f|_{A\backslash B}$ are measurable. For $\alpha \in \mathbb{R}$,

$$\begin{aligned}f(\alpha, \infty)^{-1} &= \{x \in A : f(x) > \alpha\}\\ &= \{x \in B : f(x) > \alpha\} \cup \{x \in A\backslash B : f(x) > \alpha\}\\ &= (f|_B)^{-1}(\alpha, \infty) \cup (f|_{A\backslash B})^{-1}(\alpha, \infty)\end{aligned}$$

is measurable, and so f is a measurable function.

■

Proposition 4.8
A sequence of measurable functions (f_n), $f_n : A \to \mathbb{R}$. If $f_n \to f$ pointwise almost everywhere, then f is measurable.

Proof. Let $B = \{x \in A : f_n(x) \not\to f(x)\}$, so that $m(B) = 0$. For $\alpha \in \mathbb{R}$,

$$(f|_B)^{-1}(\alpha, \infty) = f^{-1}(\alpha, \infty) \cap B$$

which is measurable.
A function whose domain has measure zero is always measurable.
If suffices to show that $f|_{A\backslash B}$ is measurable. By replacing f be $f|_{A\backslash B}$, we may assume $f_n \to f$ pointwise. Let $\alpha \in \mathbb{R}$. Since $f_n \to f$ pointwise, we see that for $x \in A$

$$f(x) > \alpha \iff \exists n, N \in \mathbb{N}, \forall i \geq N, f_i(x) > \alpha + \frac{1}{n}$$

We then see that

$$f^{-1}(\alpha, \infty) = \bigcup_{n\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{i=N}^{\infty} f_i^{-1}(\alpha + \frac{1}{n}, \infty)$$

is measurable. Hence f is measurable.

■

4.3 Simple Approximation

Definition 4.3 — simple.
A function $\varphi : A \to \mathbb{R}$ is called **simple** if
(1) φ is measurable (the domain is measurable)
(2) $\varphi(A)$ is finite

Canonical Representation
$\varphi : A \to \mathbb{R}$ is a simple function. $\varphi(A)$ is finite, say $\varphi(A) = \{c_1, c_3, \dots, c_k\}$ for distinct c_i's. For every i, let $A_i = \varphi^{-1}(\{c_i\})$ which is measurable (by Quiz 4, the preimage of a Borel set under a measurable function is measurable).

Then $A = \dot{\bigcup}_{i=1}^{K} A_i$ and $\varphi = \sum\limits_{i=1}^{K} c_i \mathcal{X}_{A_i}$

(we can write any simple function as a finite linear combination of characteristic functions, note $\mathcal{X}_{A_i}$'s are restricted to A)

Goal

Show measurable functions can be approximated by simple functions

Lemma 4.9
Let $f : A \to \mathbb{R}$ be measurable and **bounded**. For all $\epsilon > 0$, there exists simple $\varphi_\epsilon, \psi_\epsilon : A \to \mathbb{R}$ such that
(1) $\varphi_\epsilon \le f \le \psi_\epsilon$
(2) $0 \le \psi_\epsilon - \varphi_\epsilon < \epsilon$

Proof. Sketch: $f(A) \subseteq [a, b]$ since f is bounded. Given $\epsilon > 0$, $a = y_0 < y_1 < y_2 < \cdots < y_n = b$, where every $y_{i+1} - y_i < \epsilon$. Let $I_k = [y_{k-1}, y_k)$ for $0 \le k \le n$. Let $A_k = f^{-1}(I_k)$ which is measurable since I_k is Borel. Define $\varphi_\epsilon, \psi_\epsilon : A \to \mathbb{R}$ as

$$\varphi_\epsilon = \sum_{k=1}^{n} y_{k-1} \mathcal{X}_{A_k}$$

$$\psi_\epsilon = \sum_{k=1}^{n} y_k \mathcal{X}_{A_k}$$

Each of these are measurable since each A_k is measurable and they are finite linear combination of measurable functions. And they are simple by design.
Let $x \in A$. Since $f(x) \in [a, b]$, there exists $k \in \{0, \dots, n\}$ such that $f(x) \in I_k$. i.e. $y_{k-1} \le f(x) < y_k$, $x \in A_k$. Moreover,

$$\varphi_\epsilon(x) = y_{k-1} \le f(x) < y_k = \psi_\epsilon(x)$$

and so $\varphi_\epsilon \le f < \psi_\epsilon$.
For the same x,

$$0 \le \psi_\epsilon(x) - \varphi_\epsilon(x) = y_k - y_{k-1} < \epsilon$$

■

Theorem 4.10 — Simple Approximation Theorem.
Let $A \subseteq \mathbb{R}$ be measurable. A function $f : A \to \mathbb{R}$ is measurable if and only if there is a sequence (φ_n) of simple functions on A such that
(1) $\varphi_n \to f$ pointwise
(2) $\forall n$, $|\varphi_n| \le |f|$

Proof.
$\Leftarrow$ Simple functions are measurable, pointwise limit of measurable functions is measurable
$\Longrightarrow$ Suppose $f : A \to \mathbb{R}$ is measurable.
Case 1. $f \ge 0$. For each $n \in \mathbb{N}$, define

$$A_n = \{x \in A : f(x) \le n\}$$

so that A_n is measurable and $f|_{A_n}$ is measurable and bounded. By the lemma, there exist simple functions (φ_n), (ψ_n) such that $\varphi_n \leq f \leq \psi_n$ on A_n and $0 \leq \psi_n - \varphi_n < \frac{1}{n}$ for every n.
Fix $n \in \mathbb{N}$. Extend $\varphi_n : A \to \mathbb{R}$ be setting $\varphi_n(x) = n$ if $x \notin A_n$. Hence $0 \leq \varphi_n \leq f$. For each $n \in \mathbb{N}$, $\varphi_n : A \to \mathbb{R}$ is simple.
Claim: $\varphi_n \to f$ pointwise. Let $x \in A$ and let $N \in \mathbb{N}$ such that $f(x) \leq N$ (i.e. $x \in A_n$). For $n \geq N$, $x \in A_n$ and so

$$0 \leq f(x) - \varphi_n(x) \leq \psi_n(x) - \varphi_n(x) < \frac{1}{n}$$

Case 2. $f : A \to \mathbb{R}$ is measurable. Let $B = \{x \in A : f(x) \geq 0\}$ and $C = \{x \in A : f(x) \leq 0\}$ wich are both measurable. Define $g, h : A \to \mathbb{R}$

$$g = \mathcal{X}_B f,\ h = -\mathcal{X}_C f$$

both are products of measurable functions hence are measurable and nonnegative. By Case 1, there exist sequences (φ_n), (ψ_n) of simple functions such that $\varphi_n \to f$ pointwise, $\psi_n \to h$ pointwise, $\leq \varphi_n \leq g$ and $0 \leq \psi_n \leq h$. Then

$$_{\varphi_n - \psi_n} \text{simple} \to g - h = f \quad \text{pointwise}$$

and

$$|\varphi_n - \psi_n| \leq |\varphi_n| + |\psi_n| = \varphi_n + \psi_n \leq g + h = |f|$$

■

5. Week 5: Littlewood's Principles

Up to certain finiteness conditions:

(1) Measurable sets are "almost" finite, disjoint, unions of bounded intervals
(2) Measurable functions are "almost" continuous
(3) Pointwise limits of measurable functions are "almost" uniform limits.

5.1 Littlewood 1

Theorem 5.1 — Littlewood 1.
Let A be a measurable set with $m(A) < \infty$. For all $\epsilon > 0$, there exists finitely many open, bounded, disjoint intervals $I_1, I_2, \ldots, I_n$ such that

$$m(A \Delta \mathcal{U}) < \epsilon$$

where $\mathcal{U} = I_1 \cup I_2 \cup \cdots \cup I_n$.
Note: $m(A\Delta\mathcal{U}) = m(A \backslash \mathcal{U}) + m(\mathcal{U} \backslash A)$

Proof. Let $\epsilon > 0$ be given. We may find an open set $\mathcal{U}$ such that $A \subseteq \mathcal{U}$ and

$$m(\mathcal{U} \backslash A) < \frac{\epsilon}{2}$$

(by Assignment 1). By PMATH351, there exist bounded, open, disjoint intervals I_i ($i \in \mathbb{N}$), such that

$$\mathcal{U} = \dot{\bigcup}_{i=1}^{\infty} I_i$$

Note that

$$\sum_{i=1}^{\infty} \ell(I_i) = m(\mathcal{U}) < \infty$$

In particular, there exists $N \in \mathbb{N}$ such that

$$\sum_{N+1}^{\infty} \ell(I_i) < \frac{\epsilon}{2}$$

Take $V = I_1 \cup \cdots \cup I_N$. We see that

$$m(A \backslash V) \leq m(\mathcal{U} \backslash V) = m(\bigcup_{i=N+1}^{\infty} I_i) = \sum_{i=N+1}^{\infty} \ell(I_i) < \frac{\epsilon}{2}$$

and

$$m(V \backslash A) \leq m(\mathcal{U} \backslash A) < \frac{\epsilon}{2}$$

Hence $m(A \Delta V) < \epsilon$ ■

5.2 Egoroff's Theorem (Littlewood 3)

Lemma 5.2
Let A be measurable, $m(A) < \infty$, (f_n) are measurable functions, $f_n : A \to \mathbb{R}$. Assume $f : A \to \mathbb{R}$ such that $f_n \to f$ pointwise. For all $\alpha, \beta > 0$, there exists a measurable subset $B \subseteq A$ and $N \in \mathbb{N}$ such that

(1) $|f_n(x) - f(x)| < \alpha$ for all $x \in B$, $n \geq N$
(2) $m(A \backslash B) < \beta$

Proof. Let $\alpha, \beta > 0$ be given. For $n \in \mathbb{N}$, define

$$\begin{aligned} A_n &= \{x \in A : |f_k(x) - f(x)| < \alpha\} \text{ for all } k \geq n \\ &= \bigcap_{k=n}^{\infty} |f_k - f|^{-1}(-\infty, \alpha) \end{aligned}$$

Therefore, every A_n is measurable. Since $f_n \to f$ pointwise,

$$A = \bigcup_{n=1}^{\infty} A_n$$

Since (A_n) is ascending ($A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$), by the continuity of measure,

$$m(A) = \lim_{n \to \infty} m(A_n) < \infty$$

We may find $N \in \mathbb{N}$ such that for all $n \geq N$,

$$m(A) - m(A_n) < \beta$$

Pick $B = A_N$. ■

Theorem 5.3 — Egoroff's Theorem (Littlewood 3).
Let A be measurable, $m(A) < \infty$, (f_n) are measurable functions, $f_n : A \to \mathbb{R}$. Assume $f : A \to \mathbb{R}$ such that $f_n \to f$ pointwise. For all $\epsilon > 0$, there exists a closed set $C \subseteq A$ such that

(1) $f_n \to f$ uniformly on C
(2) $m(A \backslash C) < \epsilon$

Proof. Let $\epsilon > 0$ be given. By the lemma, for every $n \in \mathbb{N}$, there exists measurable $A_n \subseteq A$ and $N(n) \in \mathbb{N}$ such that

(1) For all $x \in A_n$ and $k \geq N(n)$, $|f_k(x) - f|$
(2) $m(A \backslash A_n) < \frac{\epsilon}{2^{n+1}}$

Take $B = \bigcap_{n=1}^{\infty} A_n$ which is measurable. For $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$, $k \geq N(n)$ and $x \in B$, we have

$$|f_k(x) - f(x)| < \frac{1}{n} < \epsilon$$

Therefore $f_n \to f$ uniformly on B. Moreover,

$$m(A \backslash B) = m(A \backslash \cap A_n) = m(\cup(A \backslash A_n)) \leq \sum m(A \backslash A_n) < \sum \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2}$$

By A1, there exists a closed set C such that $C \subseteq B$ and $m(B \backslash C) < \frac{\epsilon}{2}$. Therefore

(1) Since $C \subseteq B$, $f_k \to f$ uniformly on C

(2) $m(A \backslash C) = m(A \backslash B) + m(B \backslash C) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

■

Exercise 5.1 — Warning. Consider $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{n}$. We see $f_n = 0$ pointwise, show $f_n \not\to 0$ uniformly on any measurable $B \subseteq \mathbb{R}$ such that $m(\mathbb{R} \backslash B) < 1$.

R To use Egoroff's Theorem, we need $m(A) < \infty$

5.3 Lusin's Theorem (Littlewood 2)

Lemma 5.4

Let $f : A \to \mathbb{R}$ be a simple function. For all $\epsilon > 0$ there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed $C \subseteq A$ such that

(1) $f = g$ on C

(2) $m(A \backslash C) < \epsilon$

Proof. Sketch: We can always write a simple function $f = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$ which is the canonical representation where $A_i = \{x \in A : f(x) = a_i\}$ is measurable. By A1, since A_i is measurable, we can find closed $C_i \subseteq A_i$ such that

$$m(A_i \backslash C_i) < \frac{\epsilon}{n}$$

Note $A = \dot{\bigcup}_{i=1}^{n} A_i$, we define $C := \dot{\bigcup}_{i=1}^{n} C_i$ is closed.

(1) For all $x \in C_i$, $f(x) = a_i$. By A_1, f is continuous on C, we then extend $f|_C$ to a continuous function $g : \mathbb{R} \to \mathbb{R}$

(2)

$$m(A \backslash C) = m\left(\dot{\bigcup}_{i=1}^{n} (A_i \backslash C_i)\right) = \sum_{i=1}^{n} m(A_i \backslash C_i) < \epsilon$$

■

Theorem 5.5 — Lusin's Theorem (Littlewood 2).

Let $f : A \to \mathbb{R}$ be a measurable function. For all $\epsilon > 0$ there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed $C \subseteq A$ such that

(1) $f = g$ on C

(2) $m(A \backslash C) < \epsilon$

Proof. Let $\epsilon > 0$ be given.

Case 1. $m(A) < \infty$. Let $f : A \to \mathbb{R}$ be measurable. By the Simple Approximation Theorem (4.10), there exists a sequence of simple functions (f_n) such that $f_n \to f$ pointwise. By the lemmar, there exists continuous $g_n : \mathbb{R} \to \mathbb{R}$ and claose $C_n \subseteq A$ such that
(1) $f_n = g_n$ on C_n and
(2) $m(A \backslash C_n) < \frac{\epsilon}{2^{n+1}}$
By Egoroff's Theorem (5.3), there exists closed set $C_0 \subseteq A$ such that $f_n \to f$ uniformly on C_0 and

$$m(A \backslash C_0) < \frac{\epsilon}{2}$$

Let $C = \bigcap_{i=0}^{\infty} C_i$. Note that
(1) $g_n = f_n \to f$ uniformly on $C \subseteq C_0$. Each g_n is continuous, therefore f is continuous on C. By A1, we may extend $f|_C$ to a continuous $g : \mathbb{R} \to \mathbb{R}$.
(2)

$$\begin{aligned} m(A \backslash C) &= m(A \backslash \bigcap_{i=0}^{\infty} C_i) = m(\bigcup_{i=1}^{\infty}(A \backslash C_i)) \\ &\leq \sum_{i=0}^{\infty} m(A \backslash C_i) = m(A \backslash C_0) + \sum_{i=1}^{\infty} m(A \backslash C_i) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Case 2. $m(A) = \infty$. For $n \in \mathbb{N}$, define $A_n := \{a \in A : |a| \in [n-1, n)\}$ so that $A = \dot{\bigcup}_{n=1}^{\infty} A_n$. By Case 1, there exists continuous $g_n : \mathbb{R} \to \mathbb{R}$ and closed $C_n \subseteq A_n$ such that
(1) $f = g_n$ on C_n
(2) $m(A_n \backslash C_n) < \frac{\epsilon}{2^n}$
Consider $C = \dot{\bigcup}_{n=1}^{\infty} C_n$. (exercise: show C is closed.)
(1)

$$m(A \backslash C) = m\left(\dot{\bigcup}(A_n \backslash C_n)\right) = \sum m(A_n \backslash C_n) < \epsilon$$

(2) $g : C \to \mathbb{R}$. Let $x \in C$ so that $c \in C_n$ for exactly one $n \in \mathbb{N}$. Define $g(x) = g_n(x) = f(x)$, remember each g_n is continuous. (exercise) Then g is continuous. By A1, extend g continuously to all $\mathbb{R}$.

■

6. Week 6

Outline

(1) Simple functions $\varphi : A \to \mathbb{R}$, $m(A) < \infty$
(2) $f : A \to \mathbb{R}$, bounded measurable functions with $m(A) < \infty$, $\varphi_\epsilon \le f \le \psi_\epsilon$
(3) $f : A \to \mathbb{R}$ measurable, $f \ge 0$, $\sup\{\int_A h : h \in (2) 0 \le h \le f\}$
(4) $f : A \to \mathbb{R}$ measurable, $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$

6.1 Integral 1

Step 1:

Simple functions $\varphi : A \to \mathbb{R}$, $m(A) < \infty$.

Definition 6.1 — Lebesgue integral.

$m(A) < \infty$, $\varphi : A \to \mathbb{R}$ is a simple function. Consider the Canonical Rep: $\varphi = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$.

The **(Lebesgue) integral of φ over A** is

$$\int_A \varphi = \sum_{i=1}^{n} a_i m(A_i)$$

Lemma 6.1

$m(A) < \infty$, A is measurable. If $B_1, B_2, \cdots, B_n \subseteq A$ are measurable and pairwise disjoint, and $\varphi : A \to \mathbb{R}$ is defined by

$$\varphi = \sum_{i=1}^{n} b_i \mathcal{X}_{B_i}$$

then $\int_A \varphi = \sum_{i=1}^{n} b_i m(B_i)$

Proof. Sketch: For $n = 2$: if $b_1 \ne b_2$, then $\varphi = b_1 \mathcal{X}_{B_1} + b_2 \mathcal{X}_{B_2}$ is the canonical representation.

If $b_1 = b_2$, then

$$\begin{aligned} b_1 \mathcal{X}_{B_1} + b_1 \mathcal{X}_{B_2} &= b_1(\mathcal{X}_{B_1} + \mathcal{X}_{B_2}) \\ &= b_1 \mathcal{X}_{B_1 \cup B_2} \end{aligned}$$

which is a canonical representation. Hence

$$\begin{aligned} \int_A \varphi &= b_1 m(B_1 \dot{\cup} B_2) \\ &= b_1(m(B_1) + m(B_2)) \\ &= b_1 m(B_1) + b_2 m(B_2) \end{aligned}$$

■

Proposition 6.2
$\varphi, \psi : A \to \mathbb{R}$ are simple functions with $m(A) < \infty$. For all $\alpha, \beta \in \mathbb{R}$,

$$\int_A (\alpha\varphi + \beta\psi) = \alpha \int_A \varphi + \beta \int_A \psi$$

Proof. Sketch: Since φ, ψ are simple, let's say

$$\varphi(A) = \{a_1, a_2, \cdots, a_n\}, a_i \text{ pairwise disjoint}$$

$$\psi(A) = \{b_1, b_2, \cdots, b_n\}, b_i \text{ pairwise disjoint}$$

Define

$$C_{ij} = \{x \in A : \varphi(x) = a_i, \psi(x) = b_j\} = \varphi^{-1}(\{a_i\}) \cap \psi^{-1}(\{b_j\})$$

which is measurable and pairwise disjoint.

$$\alpha\varphi + \beta\psi = \sum_{i,j} (\alpha a_i + \beta b_j) \mathcal{X}_{C_{ij}}$$

By the Lemma, we have

$$\begin{aligned} \int_A \alpha\varphi + \beta\psi &= \sum_{i,j} (\alpha a_i + \beta b_j) m(C_{ij}) \\ &= \sum_{i,j} \alpha a_i m(C_{i,j}) + \sum_{i,j} \beta b_j m(C_{ij}) \\ &= \sum_i \alpha a_i (\sum_j m(C_{ij})) + \sum_j \beta b_j (m(C_{ij})) \\ &= \alpha \sum_i a_i (m(\{x \in A : \varphi(x) = a_i\})) + \beta \sum_j b_j (m(\{x \in A : \psi(x) = b_j\})) \\ &= \alpha_a \varphi + \beta \int_A \psi \end{aligned}$$

■

Proposition 6.3
$\varphi, \psi : A \to \mathbb{R}$ are simple functions with $m(A) < \infty$. If $\varphi \le \psi$, then

$$\int_A \varphi \le \int_A \psi$$

Proof. Sketch:

$$\int_A \psi - \int_A \varphi = \int_A (\psi - \varphi) \ge 0$$

■

6.2 Integral 2

Step 2

$f : A \to \mathbb{R}$ bounded and measurable with $m(A) < \infty$

Recall

For all $\epsilon > 0$, there exist simple functions $\varphi_\epsilon \le f \le \psi_\epsilon$ such that $\psi_\epsilon - \varphi_\epsilon < \epsilon$

Definition 6.2 — Upper/Lower Lebesgue Integral.

$$\underline{\int_A} f = \sup\{\int_A \varphi : \varphi \le f \quad \textbf{is simple}\}$$
$$\overline{\int_A} f = \inf\{\int_A \psi : f \le \psi \quad \textbf{is simple}\}$$

Proposition 6.4
Let $m(A) < \infty$ and $f : A \to \mathbb{R}$ be bounded and measurable, then

$$\underline{\int_A} f = \overline{\int_A} f$$

Proof. For all $n \in \mathbb{N}$, $\exists$ simple function $\varphi_n, \psi_n : A \to \mathbb{R}$ such that

$$\varphi_n \le f \le \psi_n \textbf{and} \psi_n - \varphi_n < \frac{1}{n}$$

We see that

$$0 \le \overline{\int_A} f - \underline{\int_A} f \le \int_A \psi_n - \int_A \varphi_n = \int_A (\psi_n - \varphi_n) \le \int_A \frac{1}{n} = \frac{1}{n} \cdot m(A) \to 0$$

■

Definition 6.3 — Lebesgue Integral.
Let $m(A) < \infty$ and $f : A \to \mathbb{R}$ be bounded measurable functions, we define the **(Lebesgue Integral) of f over A** by

$$\int_A f = \underline{\int_A} f = \overline{\int_A} f$$

Proposition 6.5
Let $f.g : A \to \mathbb{R}$ be bounded measurable and $m(A) < \infty$. For any $\alpha, \beta \in \mathbb{R}$

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

Proof. Let $\varphi_1, \varphi_2, \psi_1, \psi_2$ be simple function where $\varphi_1 \leq f \leq \psi_1$ and $\varphi_2 \leq g \leq \psi_2$, so

$$\begin{aligned}
\int_A +f + g = \overline{\int}_A f + g &\leq \int_A (\psi_1 + \psi_2) \\
&= \int_A \psi_1 + \int_A \psi_2 \\
&\leq \inf\{\int_A \psi_1 + \int_A \psi_2 : f \leq \psi_1, g \leq \psi_2\} \\
&= \inf\{\int_A \psi_1 : f \leq \psi_1 \;\; \textbf{simple}\} + \inf\{\int_A \psi_2 : g \leq \psi_2 \;\; \textbf{simple}\} \\
&= \int_A f + \int_A g
\end{aligned}$$

$$\begin{aligned}
\int_A f + g = \underline{\int_A} f + g &\geq \int_A \varphi_1 + \varphi_2 \\
&= \int_A \varphi_1 + \int_A \varphi_2
\end{aligned}$$

Similarly, by taking sup we have $\int_A f + g \geq \int_A f + \int_A g$, so we have the addition

$$\int_A f + g = \int_A f + \int_A g$$

Scalar multiple is similar, then the results follows. ∎

Proposition 6.6
Let $f, g : A \to \mathbb{R}$ be bounded and measurable, $m(A) < \infty$. If $f \leq g$ then

$$\int_A f \leq \int_A g$$

Proof.

$$\int_A (g - f) \geq \int_A 0 = 0 \implies \int_A g - \int_A f \geq 0 \implies \int_A g \geq \int_A f$$

∎

6.3 Bounded Convergence Theorem

Proposition 6.7
Let $f : A \to \mathbb{R}$ be bounded and measurable, let $B \subseteq A$ be measurable and $m(A) < \infty$, then

$$\int_B f = \int_A (f \cdot \mathcal{X}_B)$$

Proof. If $f = \mathcal{X}_C$ and $C \subseteq A$ be measurable, then

$$\int_A \mathcal{X}_C \mathcal{X}_B = \int_A \mathcal{X}_{B\cap C} = B \cap C = \int_B \mathcal{X}_{C|B}$$

If f is simple, let $f = \sum_{i=1}^n a_i \mathcal{X}_{A_i}$, then

$$\int_A f\mathcal{X}_B = \sum a_i \int_A \mathcal{X}_{A_i}\mathcal{X}_B = \sum a_i \int_B \mathcal{X}_{A_i} = \int_B (\sum a_i \mathcal{X}_{A_i}) = \int_B f$$

Now $f : A \to \mathbb{R}$ bounded and measurable, let $f \le \psi$ be simple, so

$$\int_A f\mathcal{X}_B \le \int_A \psi\mathcal{X}_B = \int_B \psi$$

By taking the inf over all such ψ, we have that

$$\int_A f\mathcal{X}_B \le \overline{\int_B} f = \int_B f$$

Taking $\varphi \le f$, φ is simple, we obtain

$$\underline{\int_B} f = \int_B f \le \int_A f\mathcal{X}_b$$

as desired. ■

Proposition 6.8
Let $f : A \to \mathbb{R}$ be bounded measurable and $m(A) < \infty$. If $B, C \subseteq A$ are measurable and disjoint, then

$$\int_{B\cup C} f = \int_B f + \int_C f$$

Proof.

$$\int_{B\cup C} f = \int_A f\mathcal{X}_{B\cup C} = \int_A f \cdot (\mathcal{X}_B + \mathcal{X}_C) = \int_A f\mathcal{X}_B + \int_A f\mathcal{X}_C = \int_B f + \int_C f$$

■

Proposition 6.9
Let $f : A \to \mathbb{R}$ be bounded and measurable with $m(A) < \infty$, then

$$\int_A f \le \int_A f$$

Proof.

$$-f \le f \le f \implies -\int_A f \le \int_A f \le \int_A f$$

Take the absolve value we have

$$\int_A f \le \int_A f$$

as desired. ■

Proposition 6.10
Let (f_n) be bounded measurable sequence and $f_n : A \to \mathbb{R}$ with $m(A) < \infty$. If $f_n \to f$ uniformly then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof. Let $\varepsilon > 0$ be given and $N \in \mathbb{N}$ such that

$$f_n - f < \frac{\varepsilon}{m(A)+1}$$

for $n \geq N$, then for $n \geq N$ we have

$$\int_A f_n - \int_A f = \int_A (f_n - f) \leq \int_A f_n - f \leq m(A) \cdot \frac{\varepsilon}{m(A)+1} < \varepsilon$$

■

Exercise 6.1 Let $f_n : [0,1] \to \mathbb{R}$,

$$f_n(x) = \begin{cases} 0 & 0 \leq n \leq \frac{1}{n} \\ n & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \frac{2}{n} \leq x \end{cases}$$

We can see $f_n \to 0$ and

$$\int_{[0,1]} f_n = 1 \textbf{and} \int_{[0,1]} 0 = 1$$

Theorem 6.11 — Bounded Convergence Theorem.
Let (f_n) be a sequence of measurable functions and $f_n : A \to \mathbb{R}$ with $m(A) < \infty$. If $\exists M > 0$ such that $f_n \leq M$ for all n and $f_n \to f$ pointwise, then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof. Let $\varepsilon > 0$ be given, by **Egoroff's Theorem**, there exists measurable set $B \subseteq A$ and $N \in \mathbb{N}$ s.t. for $n \geq N$

$$f_n - f < \frac{\varepsilon}{2 \cdot (m(B)+1)} \textbf{and} m(A \backslash B) < \frac{\varepsilon}{4M}$$

For $n \geq N$ we have

$$\begin{aligned} \int_A f_n - \int_A f \leq \int_A f_n - f &= \int_B f_n - f + \int_{A\backslash B} f_n - f \\ &\leq \int_B f_n - f + \int_{A\backslash B} (f_n + f) \\ &\leq \int_B f_n - f + 2 \cdot M \cdot m(A\backslash B) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

■

6.4 Integral 3

Definition 6.4 — support, BF function.

(1) We say f has finite **support** if

$$A_0 = \{x \in A : f(x) \neq 0\}$$

has finite measure.

(2) We say f is **BF function** if f is bounded and has finite support.

(3) If $f : A \to \mathbb{R}$ is **BF** then

$$\int_A f = \int_{A_0} f$$

Definition 6.5

Let $f : A \to \mathbb{R}$ be measurable and $f \geq 0$, we define

$$\int_A f = \sup\{\int_A h : 0 \leq h \leq f \quad \mathbf{BF}\}$$

Proposition 6.12

Let $f, g : A \to \mathbb{R}$ be measurable function and $f, g \geq 0$, then

(1) $\forall \alpha, \beta \in \mathbb{R}$

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

(2) If $f \leq g$, then $\int_A f \leq \int_A g$

(3) If $B, C \subseteq A$ are measurable and $B \cap C = \varnothing$, then

$$\int_{B \cup C} f = \int_B f + \int_C f$$

Proposition 6.13 — Chebychev's Inequality.

$f : A \to \mathbb{R}$ be non-negative measurable function, then for all $\varepsilon > 0$

$$\{x \in A : f(x) \geq \varepsilon\} \leq \frac{1}{\varepsilon} \int_A f$$

Proof. Let $\varepsilon > 0$ be given and let

$$A_\varepsilon = \{x \in A : f(x) \geq \varepsilon\}$$

such that $A_\varepsilon < \varepsilon$ and $\underbrace{\varphi}_{\mathbf{BF}} = \varepsilon \cdot \mathcal{X}_{A_\varepsilon} \leq f$, so $\varepsilon A_\varepsilon = \int_A \varphi \leq \int_A f$

If $A_\varepsilon = \infty$, for $n \in \mathbb{N}$ define $A_{\varepsilon,n} = A_\varepsilon \cap [-n, n]$. By the continuity of measure

$$\infty = A_\varepsilon = \lim_{n\to\infty} A_{\varepsilon,n}$$

For $n \in \mathbb{N}$, $\varphi_n = \varepsilon \mathcal{X}_{A_{\varepsilon,n}}$ (BF) we see that $\varphi_n \leq f$. Therefore, we have

$$\infty = A_\varepsilon = \lim_{n\to\infty} A_{\varepsilon,n} = \lim_{n\to\infty} \frac{1}{\varepsilon} \int_A \varphi_n \leq \int_A f$$

■

Proposition 6.14
Let $f : A \to \mathbb{R}$ with $f \geq 0$, then

$$\int_A f = 0 \iff f = 0 \ \textbf{ae}$$

Proof. $\implies \int_A f = 0$.

$$\{x \in A : f(x) \neq 0\} \leq \sum \{x \in A : f(x) \geq \frac{1}{n}\} \leq \sum n \cdot \underbrace{\int_A f}_{=0} = 0$$

$\impliedby$ Suppose $B = \{x \in A : f(x) \neq 0\}$ has measure 0, so

$$\int_A f = \int_B f + \underbrace{\int_{A \setminus B} f}_{=0} = \int_B f = 0$$

■

6.5 Fatou's Lemma and MCT

Theorem 6.15 — Fatou's Lemma.
Let (f_n) be a measurable, non-negative sequence of functions and $f_n : A \to \mathbb{R}$. If $f_n \to f$ pointwise then

$$\int_A f \leq \liminf \int_A f_n$$

Proof. Let $0 \leq h \leq f$ be a **BF** function, we say $A_0 = \{x \in A : h(x) \neq 0\}$. It's suffices to show

$$\int_A h \leq \liminf \int_A f_n$$

Since for each $n \in \mathbb{N}$ we let

$$h_n = \min\{h, f_n\} \ \textbf{measurable}$$

Note 6.1
(1) $0 \leq h_n \leq h \leq M$ for some $M > 0$ for all $n \in \mathbb{N}$.
(2) For $x \in A_0$ and $n \in \mathbb{N}$, (a) $h_n(x) = h(x)$ or (b) $h_n(x) = f_n(x) \leq h(x)$ and

$$0 \leq h(x) - h_n(x) = h(x) - f_n(x) \leq f(x) - f_n(x) \to 0$$

Then $h_n \to h$ pointwise on A_0. By BCT 6.11

$$\lim_{n\to\infty} \int_{A_0} h_n = \int_{A_0} h \implies \lim_{n\to\infty} \int_A h_n = \int_A h$$

Since $h_n \leq f_n$ on A, so

$$\int_A h = \lim_{n\to\infty} \int_A h_n = \liminf_{n\to\infty} \int_A h_n \leq \liminf_{n\to\infty} \int_A f_n$$

■

Exercise 6.2 Let $A = (0,1]$ and $f_n = n \cdot \mathcal{X}_{(0,\frac{1}{n})}$, so $f_n \to 0$ pointwise. We also have

$$\int_A 0 = 0 \int_A f_n = n \cdot 0, \frac{1}{n} = 1 \lim_{n\to\infty} \inf \int_A f_n = 1$$

Theorem 6.16 — Monotone Convergence Theorem.
Let (f_n) be a non-negative measurable function and $f_n : A \to \mathbb{R}$. If (f_n) is increasing and $f_n \to f$ pointwise then

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof.

$$\int_A f \underbrace{\le}_{\textbf{FL}} \lim_{n\to\infty} \inf \int_A f_n \le \lim_{n\to\infty} \sup \int_A f_n \le \int_A f$$

■

R

(1) If $\varphi : A \to \mathbb{R}$ is simple and $m(A) < \infty$ then

$$\int_A \varphi < \infty$$

(2) If $f : A \to \mathbb{R}$ is bounded and measurable, also $m(A) < \infty$, then

$$\int_A f < \infty$$

Definition 6.6 — integrable.
If $f : A \to \mathbb{R}$ is measurable and $f \ge 0$, then we say f is **integrable** iff

$$\int_A f < \infty$$

7. Week 7

7.1 Integral 4

The general integral

Definition 7.1
$f : A \to \mathbb{R}$ be measurable:

$$f^+(x) = \max\{f(x), 0\} \quad \text{(positive part)}$$
$$f^-(x) = \max\{-f(x), 0\} \quad \text{(negative part)}$$

Note 7.1
(1) $f^+ + f^- = |f|$
(2) $f^+ - f^- = |f|$
(3) f^+, f^- are measurable

Proposition 7.1
If $f : A \to \mathbb{R}$ is measurable, then f^+, f^- are integrable if and only if $|f|$ is integrable

Proof. Sketch:
$\Longrightarrow$

$$|f| = f^+ + f^-, \int_A |f| = \underbrace{\int_A f^+}_{<\infty} + \underbrace{\int_A f^-}_{<\infty}$$

$\Longleftarrow$

$$\int_A f^+ \le \int_A |f| < \infty, \int_A f^- \le \int_A |f| < \infty$$

■

Definition 7.2 — integrable.
$f : A \to \mathbb{R}$ be measurable. We say f is **integrable**

- iff $|f|$ is integrable,
- iff f^+, f^- are integrable

and define

$$\int_A f = \int_A f^+ - \int_A f^-$$

Proposition 7.2 — Comparison Test.
$f : A \to \mathbb{R}$ measurable, $g : A \to \mathbb{R}$ non-negative and integrable. If $|f| \le g$, then f is integrable and $|\int_A f| \le \int_A |f|$

Proof. Sketch:

(1)

$$\int_A |f| \le \int_A g < \infty$$

(2)

$$\left|\int_A f\right| = \left|\int_A f^+ - \int_A f^-\right| \le \int_A f^+ + \int_A f^- = \int_A (f^+ + f^-) = \int_A |f|$$

■

Proposition 7.3
$f, g : A \to \mathbb{R}$ are integrable.
(1) $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is integrable and $\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$
(2) if $f \le g$, then $\int_A f \le \int_A g$
(3) if $B, C \subseteq A$ are measurable with $B \cap C = \varnothing$, then $\int_{B \cup C} = \int_B f + \int_C f$

Theorem 7.4 — Lebesgue Dominated Convergence Theorem.
Let (f_n) be a sequence of measurable function with $f_n : A \to \mathbb{R}$ and $f_n \to f$ pointwise. If there exists an integrable $g : A \to \mathbb{R}$ such that $f_n \le g$ for all $n \in \mathbb{N}$, then f is integrable and

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof. Since we can see that $f_n \to f$ pointwise and $f_n \le g$, and so $f \le g$. By comparision, f is integrable. Next, observe that $g - f \ge 0$, by Fatou's Lemma 6.15

$$\int_A g - \int_A f = \int_A (g - f) \le \lim_{n\to\infty} \inf \int_A (g - f_n) = \int_A g - \lim_{n\to\infty} \sup \int_A f_n$$

Then, cancel the g we have

$$\lim_{n\to\infty} \sup \int_A f_n \le \int_A f$$

Also

$$\int_A g + \int_A f = \int_A (g + f) \le \lim_{n\to\infty} \inf \int_A (g + f_n) = \int_A f + \lim_{n\to\infty} \inf \int_A f_n$$

Then, cancel the g again we have

$$\int_A f \le \lim_{n\to\infty} \inf \int_A f_n$$

so we have
$$\int_A f = \lim_{n\to\infty} \inf \int_A f_n = \lim_{n\to\infty} \sup \int_A f_n = \lim_{n\to\infty} \int_A f_n$$

■

7.2 Riemann Integration

Definition 7.3 — Riemann Sum.
Let $f : [a,b] \to \mathbb{R}$ be bounded function

(1) A **partition** of $[a,b]$ is a finite set $P = \{x_0, x_1, ..., x_n\} \subseteq \mathbb{R}$ such that
$$a = x_0 < x_1 < x_2 < .... < x_n = b$$

(2) Relative to P, we define the **lower Darboux sum:**
$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \textbf{ where } m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

(3) Similarly, we define the **upper Darboux sum:**
$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \textbf{ where } M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

Definition 7.4
Let $f : [a,b] \to \mathbb{R}$ be bounded function

(1) **Lower Riemann Integral:**
$$R\underline{\int_a^b} f = \sup\{L(f,P) : \textbf{P is a partition}\}$$

(2) **Upper Riemann Integral:**
$$R\overline{\int_a^b} f = \inf\{U(f,P) : \textbf{P is a partition}\}$$

(3) We say f is **Riemann Integrable if and only if**
$$R\overline{\int_a^b} f = R\underline{\int_a^b} f$$

Definition 7.5 — Step Function.
Let $I_i, ..., I_n$ be pointwise disjoint intervals such that
$$[a,b] = \bigcup_{i=1}^{n} I_i$$

A **Step function** is a function of the form

$$f = \sum_{i=1}^{n} a_i \mathcal{X}_{I_i}$$

for some $a_i \in \mathbb{R}$

R Let $f : [a, b] \to \mathbb{R}$ be a bounded fuction and

$$a = x_0 < x_1 < ...... < x_n = b$$

and $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, ..., n-1$ and $I_n = [x_{n-1}, x_n]$. Then

$$L(f, P) = \sum_{i=1}^{n} m_i \ell(I_i) = R\int_a^b \varphi$$

where $\varphi(x) = m_i$ on I_i $(\varphi \leq f)$ and

$$U(f, P) = \sum_{i=1}^{n} M_i \ell(I_i) = R\int_a^b \psi$$

where $\psi(x) = M_i$ on I_i $(f \leq \psi)$ and

R Let $f : [a, b] \to \mathbb{R}$ be a bounded fuction, then

$$R\underline{\int_a^b} f = \sup\left\{L(f, P) : \textbf{P is a partition}\right\} = \sup\left\{R\int_a^b \varphi : \varphi \leq f \textbf{is a step function}\right\}$$

$$R\overline{\int_a^b} f = \inf\left\{U(f, P) : \textbf{P is a partition}\right\} = \inf\left\{R\int_a^b \psi : f \leq \psi \textbf{is a step function}\right\}$$

7.3 Riemann Integral VS Lebesgue Integral

Definition 7.6
Let $f : [a, b] \to \mathbb{R}$ be bounded function and let $x \in [a, b]$ and $\delta > 0$

(1)
$$m_\delta(x) = \inf\left\{f(x) : x \in (x - \delta, x + \delta) \cap [a, b]\right\}$$

(2)
$$M_\delta(x) = \sup\left\{f(x) : x \in (x - \delta, x + \delta) \cap [a, b]\right\}$$

(3) **Lower boundary of** f:
$$m(x) = \lim_{\delta \to 0} m_\delta(x)$$

(4) **Upper boundary of** f:
$$M(x) = \lim_{\delta \to 0} M_\delta(x)$$

(5) **Oscillation of f:**
$$\omega(x) = M(x) - m(x)$$

R Let $f : [a, b] \to \mathbb{R}$ be bounded function, the following are equivalent:

(1) f is continuous at $x \in [a, b]$
(2) $M(x) = m(x)$
(3) $\omega(x) = 0$

Lemma 7.5
Let $f : [a, b] \to \mathbb{R}$ be bounded function, then
(1) m is measurable
(2) If $\varphi : [a, b] \to \mathbb{R}$ is a step function with $\varphi \leq f$, then
$$\varphi(x) \leq m(x)$$
at all points of continuity of φ
(3) $R\underline{\int_a^b} f = \int_{[a,b]} m$

Proof.
(1) Let $\alpha \in \mathbb{R}$ and $c \in [a, b]$ s.t. $m(c) > \alpha$. Choose any $m(c) > \beta > \alpha$, by the definition of m, there exists $\varepsilon > 0$ such that $m_\varepsilon > \beta$. However, this means that $f(x) > \beta$ for any $x \in (c - \varepsilon, c + \varepsilon) \cap [a, b]$. Take $x \in (c - \varepsilon, c + \varepsilon) \cap [a, b]$ so that there exists $\delta > 0$ such that $(x - \delta, x + \delta) \cap [a, b] \subseteq (c - \varepsilon, c + \varepsilon) \cap [a, b]$. It follows that $m_\delta(x) \geq \beta$ and so $m(x) \geq m_\delta(x) \geq \beta > \alpha$ as well. Therefore, $\{c \in [a, b] : m(c) > \alpha\}$ is relatively open in $[a, b]$ (i.e. is the intersection of an open set and $[a, b]$) and so is measurable.
(2) Suppose $\varphi \leq f$ is a step function and let x be a point of continuity of φ. Since x is not an endpoint of a middle step, we see that there exists $\delta > 0$ and $z \in \mathbb{R}$ such that $\varphi(y) = z$ for all $y \in (x - \delta, x + \delta) \cap [a, b]$. Therefore, for all $y \in (x - \delta, x + \delta) \cap [a, b]$, we have $f(y) \geq \varphi(y) = z$. Hence, $m(x) \geq m_\delta(x) \geq z = \varphi(x)$ as required.
(3) We begin by observing that if $\varphi \leq f$ is a step function then, by (2) $\varphi \leq m$ **a.e.** Therefore
$$R\underline{\int_a^b} f = \sup\left\{R\int_a^b \varphi : \varphi \leq f \quad \textbf{step}\right\} = \sup\left\{\int_{[a,b]} \varphi : \varphi \leq f \quad \textbf{step}\right\} \leq \int_{[a,b]} m$$
by monotonicity **a.e.**
Now for each $n \in \mathbb{N}$, let $P_n = \{a = x_0 < x_1 < .... < x_{2^n} = b\}$, where each $x_i - x_{i-1} = \frac{b-a}{2^n}$. Then let $I_{n,1} = [a, x_1]$ and $I_{n,k} = (x_{k-1}, x_k]$ for $2 \leq k \leq n$. Define a step function $\varphi_n \leq f$ by setting $\varphi_n(x) = \inf\{f(x) : x \in I_{n,k}\}$ for all $x \in I_{n,k}$. Let $P = P_i$ and note that P has measure 0 (countable)

Fix $x \in [a, b] \setminus P$. For all $n \in \mathbb{N}$, let $I_n(x)$ denote the interval $I_{n,k}$ (as above) which contains x. Let $\delta > 0$ be given and let $N \in \mathbb{N}$ be such that $I_n(x) \subseteq (x - \delta, x + \delta)$ for all $n \geq N$. By (2), for $n \geq N$ we have
$$m(x) \geq \varphi_n(x) \geq m_\delta(x)$$
as $\delta \to 0$ (and so $N \to \infty$) we see that
$$\lim_{n\to\infty} \varphi_n(x) = m(x)$$
In particular, we have that $\varphi_n \to m$ pointwise **a.e.**. Let $\alpha \in \mathbb{R}$ such that $f \leq \alpha$. Then $\varphi_n \leq \alpha$ for every n, where constant function α is integrable over $[a, b]$ and so we have by LDCT 7.4 that
$$\lim_{n\to\infty} \int_{[a,b]} \varphi_n = \int_{[a,b]} m$$
Since the Riemann and Lebesgue integrals clearly agree for step functions:
$$\lim_{n\to\infty} R\int_a^b]\varphi_n = \int_{[a,b]} m$$

Therefore,

$$\int_{[a,b]} m = \lim_{n\to\infty} R\int_a^b \varphi_n \le \sup\left\{R\int_a^b \varphi : \varphi \le f\textbf{step}\right\} = R\underline{\int_a^b} f$$

■

Lemma 7.6
Let $f : [a, b] \to \mathbb{R}$ be bounded function, then

(1) M is measurable

(2) If $\psi : [a, b] \to \mathbb{R}$ is a step function with $f \le \psi$, then

$$M(x) \le \psi(x)$$

at all points of continuity of ψ

(3) $R\overline{\int_a^b} f = \int_{[a,b]} M$

Proof. Similar as the last lemma. ■

Theorem 7.7 — Lebesgue.
Let $f : [a, b] \to \mathbb{R}$ be bounded function, then f is **Riemann Integrable** if and only if f is continuous **a.e.**. In that case:

$$R\int_a^b f = \int_{[a,b]} f$$

Proof. Note that

$$R\underline{\int_a^b} f = \int_{[a,b]} m \le \int_{[a,b]} M = R\overline{\int_a^b} f$$

so f is **Riemann integrable**. Then

$$\begin{aligned}\int_{[a,b]} m = \int_{[a,b]} M \iff \int_{[a,b]} (M - m) = 0 &\iff M = m \ \ \textbf{a.e.}\\ &\iff \omega = 0 \ \ \textbf{a.e.}\\ &\iff f \ \ \textbf{is continuous a.e.}\end{aligned}$$

If f is continuous **a.e.**, then f is measurable and

$$R\underline{\int_a^b} f \le \int_{[a,b]} m \le \int_{[a,b]} f \le \int_{[a,b]} M = R\overline{\int_a^b} f$$

Then we have

$$R\int_a^b f = \int_{[a,b]} f$$

as desired. ■

Exercise 7.1 Let $f : [0, 1] \to \mathbb{R}$ where

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

so f is discontinuous on $[0,1]$. Then f is **not Riemann Integrable**
However, $f = 0$ **a.e.** on $[0,1]$ and so

$$\int_{[0,1]} f = \int_{[0,1]} 0 = 0$$

so f is **Lebesgue Integrable**

Exercise 7.2 Let $\mathbb{Q} \cap [0,1] = \{q_1, q_2, ...\}$ and $f_n = \mathcal{X}_{\{q_1, q_2, \ldots, q_n\}}$ and $f_n \to f$ pointwise. Then f_n is increasing but $f_n \leq 1$, so

$$\underbrace{R\int_{[0,1]} f_n}_{=0} \not\to \underbrace{R\int_{[0,1]} f}_{\textbf{DNE}}$$

8. Week 8

8.1 L^p Spaces

Goal

Create Banach Spaces whose norm is given by Lebesgue Integration.
Recall

(1) For $1 \leq p < \infty$, $(C([a.b]), \|\cdot\|_p)$ is a normed vector space, where

$$\|f\|_p^p = \int_a^b f^p$$

(2) For $p = \infty$, $(C([a.b]), \|\cdot\|_\infty)$:

$$\|f\|_\infty = \sup\{f(x) : x \in [a, b]\}$$

is a Banach space.

Problem

Let $A \subseteq \mathbb{R}$ be measurable and $1 \leq p < \infty$, then

$$\|f\|_p = \left(\int_A f^p\right)^{\frac{1}{p}}$$

is not a norm on the vector space of integrable function $f : A \to \mathbb{R}$. Because $\int_A f^p = 0 \iff f = 0$ **a.e.**

Definition 8.1
Let $A \subseteq \mathbb{R}$ be measurable.

(1) $M(A) = \{f : A \to \mathbb{R}$ **measurable**$\}$ (vector space). $f \sim g$ if and only if $f = g$ **a.e.**. The $[f]$ is the equivalence class

(2) $M(A)/\sim = [f] : f \in M(A)$ (vector space) and

$$\alpha[f] + \beta[g] = [\alpha f + \beta g]$$

R If $f \sim g$ and f is integrable, then g is integrable and $\int_A f = \int_A g$

Definition 8.2 — L^p Space.
Let $A \subseteq \mathbb{R}$ be measurable set and $1 \le p < \infty$, the L^p space is defined by

$$L^p(A) = \{[f] \in M(A)/\sim : \int_A f^p < \infty\}$$

R Suppose $[f], [g] \in L^p(A)$, then $\int_A f^p, \int_A g^p < \infty$

(1)
$$f + g^p \le (f+g)^p \le 2\max\{f,g\})^p \le 2^p(f^p + g^p)$$
Then $f + g^p$ is integrable by comparison.

(2) $L^p(A)$ is a subspace of $M(A)/\sim$

Definition 8.3 — L^∞ Space.
Let $A \subseteq \mathbb{R}$ be measurable set, then $L^\infty(A)$ is defined by

$$L^\infty(A) = \{[f] \in M(A)/\sim : f \textbf{ is bounded a.e.}\}$$

R

(1) $[f], [g] \in L^\infty(A)$, we have $f \le M$ and $g \le N$, so we can find $B, C \subseteq A$ s.t. $B = C = 0$. For $x \notin B \cup C$, we have
$$f(x) + g(x) \le f(x) + g(x) \le M + N$$

(2) $L^\infty(A)$ is a subspace of $M(A)/\sim$

Proposition 8.1
Let $A \subseteq \mathbb{R}$ be measurable set, then

$$\|[f]\|_\infty = \inf\{M \ge 0 : f \le M \textbf{ a.e.}\}$$

is a norm on $L^\infty(A)$

R For all $n \in \mathbb{N}$,
$$f \le \|[f]\|_\infty + \frac{1}{n} \text{ off } A_n = 0$$
and
$$B = \bigcup A_n \to \textbf{measure 0}$$
so $f \le \|[f]\|_\infty$ off B.

Proof. Sketch:
(1) $\|[f]\|_\infty = 0 \implies f \le \|[f]\|_\infty$ **a.e.** so $[f] = [0]$ in $L^\infty(A)$

(2) $f \leq \|[f]\|_\infty$ off B and $g \leq \|[g]\|_\infty$ off C, off $B \cup C \to$ measure 0, then

$$f + g \leq f + g \leq \|[f]\|_\infty + \|[g]\|_\infty$$

By the definition of inf, we have

$$\|[f+g]\|_\infty = \|[f] + [g]\|_\infty \leq \|[f]\|_\infty + \|[g]\|_\infty$$

■

8.2 L^p Norm

Abusive Notation

$$f \equiv [f] \in L^p(A)$$

and $f = g$ in $L^p(A)$ means $f = g$ **a.e.**

Definition 8.4 — Holder Conjugates.
For $p \in (1, \infty)$ we define $q = \frac{p}{p-1}$ to be the **Holder conjugates** of p

Note 8.1
(1) $q = \frac{p}{p-1} \iff p = \frac{q}{q-1}$
(2) $\frac{1}{p} + \frac{1}{q} = 1$
(3) We also define 1 and ∞ to be **Holder conjugates**

Proposition 8.2 — Young's Inequality.
Let $p, q \in (1, \infty)$ be **Holder conjugates** , for all $a, b > 0$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof: We define $f(x) = \frac{1}{p}x^p + \frac{1}{q} - x$ where $x \in (0, \infty)$. Then we have $f'(x) = x^{p-1} - 1$ and $f''(x) = (p-1)x^{p-2}$. When $f'(x) = 0$, we can get the critical point of $f(x)$ at $x = 1$. Since the Holder conjugates $p, q \in (1, \infty)$, then $f''(x) = (p-1)x^{p-2} > 0$ for all $x \in (0, \infty)$. Therefore, we can know $f(x)$ has global minimum at $x = 1$. Since We have $\frac{1}{p} + \frac{1}{q} = 1$, so $f(1) = \frac{1}{p} + \frac{1}{q} - 1 = 0$, then $f(x) \geq 0$ on $x \in (0, \infty)$. Now we take $x = \frac{a}{b^{\frac{q}{p}}}$, then

$$f(\frac{a}{b^{\frac{q}{p}}}) = \frac{1}{p} \cdot (\frac{a}{b^{\frac{q}{p}}})^p + \frac{1}{q} - \frac{a}{b^{\frac{q}{p}}} \geq 0 \Longrightarrow \frac{1}{p} \cdot \frac{a^p}{b^q} + \frac{1}{q} - \frac{a}{b^{\frac{q}{p}}} \geq 0$$
$$\Longrightarrow \frac{a^p}{p} + \frac{b^q}{q} \geq ab^{q - \frac{q}{p}}$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, then we have $q - \frac{q}{p} = q \cdot (1 - \frac{1}{p}) = q \cdot \frac{1}{q} = 1$ Therefore, by $\frac{a^p}{p} + \frac{b^q}{q} \geq ab^{q-\frac{q}{p}}$ and $q - \frac{q}{p} = 1$, we have $\frac{a^p}{p} + \frac{b^q}{q} \geq ab$ as desired.

Proposition 8.3
Let $A \subseteq \mathbb{R}$ be measurable set and $1 \leq p < \infty$ and q is the **Holder conjugate** of p. If

$f \in L^p(A)$ and $g \in L^q(A)$, then $fg \in L^1(A)$ and

$$\int_A fg \leq \|f\|_p \|g\|_q$$

Proof. If $p = 1$ and $q = \infty$,

$$fg \leq fg \leq f \|g\|_\infty \quad \textbf{a.e.}$$

then $fg \in L^1(A)$.
If $1 < p < \infty$ and q is the conjugate of p, so

$$fg = fg \leq \frac{f^p}{p} + \frac{g^q}{q} \quad \text{by Young's Inequality 8.2}$$

so fg is integrable by comparison, then $fg \in L^1(A)$. Also we have

$$\int_A fg \leq \frac{1}{p}\int_A f^p + \frac{1}{q}\int_A g^q = \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q$$

Now we have two cases, Case 1: $\|f\|_p = \|g\|_q = 1$, so

$$\int_A fg \leq \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \|g\|_q$$

Case 2: $\frac{f}{\|f\|_p}, \frac{g}{\|g\|_q}$ by case 1 we have

$$\frac{1}{\|f\|_p \|g\|_q}\int_A fg \leq 1$$

■

Lemma 8.4
Let p, q be **Holder conjugate** and $f \in L^p(A)$, if $f \neq 0$

$$f^* = \|f\|_p^{1-p} \operatorname{sign}(f) f^{p-1}$$

is in $L^q(A)$ and

$$\int_A ff^* = \|f\|_p, \quad \|f^*\|_q = 1$$

Proof. If $p = 1$ and $q = \infty$, we have

$$f^* = f \in L^\infty(A)$$

and

$$\int_A ff^* = \int_A f = \|f\|_1$$

If $1 < p < \infty$ and q is the Holder conjugate of p,

$$\int_A ff^* = \|f\|_p^{1-p}\int_A f^p = \|f\|_p^{1-p}\|f\|_p^p = \|f\|_p$$

and

$$\|f^*\|_q^q = \|f\|_p^{(1-p)q}\int_A f^{(p-1)q} = \|f\|_p^{-p}\int_A f^p = \|f\|_p^{-p}\|f\|^p = 1$$

■

Theorem 8.5 — Minkowski's Inequality.
Let $A \subseteq \mathbb{R}$ be measurable and $1 \leq p < \infty$. If $f, g \in L^p(A)$, then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. If $p = 1$, the result is trivial. Now we look at $1 < p < \infty$, we can see that

$$\begin{aligned}\|f+g\|_p = \int_A (f+g)(f+g)^* &= \int_A f(f+g)^* + \int_A g(f+g)^* \\ &\leq \|f\|_p \|(f+g)^*\|_q + \|g\|_p \|(f+g)^*\|_q \\ &= \|f\|_p + \|g\|_p\end{aligned}$$

■

8.3 Completeness

Theorem 8.6 — Riesz-Fisher.
For every measurable set $A \subseteq \mathbb{R}$ and $1 \leq p \leq \infty$, $L^p(A)$ is a **Banach Space**

Proof. If $p = \infty$, it's trivial. Now we look at $1 \leq p < \infty$. Let $(f_n) \subseteq L^p(A)$ be strongly Cauchy. Then there exists $(\varepsilon_n) \subseteq \mathbb{R}$ such that

$$\|f_{n+1} - f_n\|_p \leq \varepsilon_n^2 \quad \textbf{and} \quad \sum \varepsilon_n < \infty$$

Since $\mathbb{R}$ is complete, if $(f_n(x))$ is strongly Cauchy, then it converges. Now for each $n \in \mathbb{N}$, we define

$$A_n := \{x \in A : f_{n+1}(x) - f_n(x) \geq \varepsilon\} = \{x \in A : f_{n+1}(x) - f_n(x)^p \geq \varepsilon^p\}$$

By Chebychev's Inequality 6.13

$$A_n \leq \frac{1}{\varepsilon_n^p} \int_A f_{n+1} - f_n{}^p \leq \frac{1}{\varepsilon_n^p} \cdot \varepsilon_n^{2p} = \varepsilon_n^p$$

Then we have

$$\sum A_n \leq \sum \varepsilon_n^p \leq (\sum \varepsilon_n^p) < \infty$$

so $\limsup_{n\to\infty} A_n = 0$. Now we fix $x \notin \limsup_{n\to\infty} A_n$, let

$$N = \max\{n : x \in A_n\}$$

and for $n > N$,

$$f_{n+1}(x) - f_n(x) < \varepsilon_n^2 \quad \textbf{and} \quad \sum \varepsilon_i < \infty$$

so $(f_n(x))$ is Cauchy. Then $f_n \to f$ pointwise **a.e.**. For $k \in \mathbb{N}$, we have

$$\|f_{n+k} - f_n\|_p \leq \sum_{i=n}^{\infty} \varepsilon_i^2$$

so $f_{n+k} - f_n{}^p \to f_n - f^p$ pointwise **a.e.** as $k \to \infty$. By Fatou's Lemma 6.15 we have

$$\int_A f_n - f^p \leq \liminf_{k\to\infty} \int_A f_{n+k} - f_n{}^p = \liminf_{k\to\infty} \|f_{n+k} - f_n\|_p^p \leq \left[\sum_{i=n}^{\infty} \varepsilon_i^2\right]^p \to 0$$

■

8.4 Separability

■ **Example 8.1** Let $p = \infty$, suppose $\{f_n : n \in \mathbb{N}\}$ is dense in $L^\infty[0,1]$. For every $x \in [0,1]$ we may find

$$\left\|\mathcal{X}_{0,x} - f_{\theta(x)}\right\|_\infty < \frac{1}{2}$$

For $x \neq y$ in $[0,1]$,

$$\left\|\mathcal{X}_{[0,x]-\mathcal{X}_{0,y}}\right\|_\infty = 1$$

so $\theta : [0,1] \to \mathbb{N}$ is injective, which is a contradiction

Notation 8.1.
(1) *Simp(A)=simple functions on measurable set A*
(2) *Step$[a,b]$= Step functions on $[a,b]$*
(3) *Step$_{\mathbb{Q}}[a,b]$=step functions on $[a,b]$, with rational partition function values.*

Proposition 8.7
Let $A \subseteq \mathbb{R}$ be measurable and $1 \leq p < \infty$, then Simp(A) is dense in $L^p(A)$

Proof. Let $f \in L^p(A)$ so f is measurable. Then $\exists(\varphi_n)$ simple function so that $\varphi_n \to f$ pointwise and $\varphi_n \leq f$, then ${\varphi_n}^p \leq f^p$. By comparison we have $(\varphi_n) \subseteq L^p(A)$. Note that

$$\|\varphi_n - f\|_p^p = \int_A \varphi_n - f^p \quad \textbf{and} \quad \varphi_n - f^p \leq 2^p({\varphi_n}^p + f^p) \leq 2^{p+1} f^p$$

which is integrable. By LDCT 7.4 we have

$$\lim_{n\to\infty} \int_A \varphi_n - f^p \int_A 0 = 0$$

as desired. (This is also true for $p = \infty$) ■

Proposition 8.8
Step$[a,b]$ is dense in $L^p[a,b]$

Proof. Let $A \subseteq [a,b]$ be measurable, so $\mathcal{X}_A : [a,b] \to \mathbb{R}$. By Littlewood 1 5.1, so for any $\varepsilon > 0$, there exists a collection of bounded open interval such that the disjoint union $\bigcup_{i=1}^n I_i = U$ and $U \Delta A < \varepsilon^p$. Since $\mathcal{X}_U$ is a step function so

$$\|\mathcal{X}_U - \mathcal{X}_A\|_p^p = \int_A \mathcal{X}_U - \mathcal{X}_A = A\Delta U$$

so we have $\|\mathcal{X}_U - \mathcal{X}_A\| < \varepsilon$ as desired. ■

Corollary 8.9
Let $1 \leq p < \infty$, Step$_{\mathbb{Q}}[a,b]$ is dense in $L^p[a,b]$, then $L^p[a,b]$ is separable.

Proposition 8.10
Let $1 \leq p < \infty$, $L^p(\mathbb{R})$ is separable

Proof. Consider to define $F_n = f \in L^p(\mathbb{R})$ where

$$F_n = \begin{cases} \text{Step}_{\mathbb{Q}}[-n,n] & \textbf{if } x \in [-n,n] \\ 0 & \textbf{if } x \notin [-n,n] \end{cases}$$

So we have $F = \bigcup F_i$ is countable. Take $f \in L^p(\mathbb{R})$, fix $n \in \mathbb{N}$ so $f\mid_{[-n,n]} \in L^p[-n,n]$, we show

$$f\mathcal{X}_{[-n,n]} \to f \quad \text{in} \quad L^p(\mathbb{R})$$

Note that

$$\left\|f\mathcal{X}_{[-n,n]} - f\right\|_p^p = \int_{\mathbb{R}} f\mathcal{X}_{[-n,n]} - f^p = \int_{\mathbb{R}\setminus[-n,n]} f^p = \int_{\mathbb{R}} f^p \mathcal{X}_{\mathbb{R}\setminus[-n,n]}$$

and

$$f^p \mathcal{X}_{\mathbb{R}\setminus[-n,n]} \leq f^p \quad \text{integrable}$$

By LDCT 7.4 we have

$$\lim_{n\to\infty} \left\|f\mathcal{X}_{[-n,n]} - f\right\|_p^p = \lim_{n\to\infty} \int_{\mathbb{R}} f\mathcal{X}_{[-n,n]} - f^p = \int_{\mathbb{R}} 0 = 0$$

so $\left\|f\mathcal{X}_{[-n,n]} - f\right\|_p \to 0$. Then for each $n \in \mathbb{N}$, $\exists \varphi_n \in F$ such that $\left\|f\mathcal{X}_{[-n,n]} - f\right\|_p < \frac{1}{n}$ so $\|\varphi_n - f\|_p \to 0$ as desired. ■

Theorem 8.11
Let $A \subseteq \mathbb{R}$ be measurable set and $1 \leq p < \infty$, then $L^p(A)$ is separable.

Proof. Similar as above. ■