CS 476: Numeric Computation for Financial Modeling

Professor Yuying Li
LaTeXer Iris Jiang

Winter 2021

Contents

1 Week 1 2

1.1 General Derivative Contracts 2

1.1.1 European calls and puts 3
1.1.2 Call value at the expiry T 3
1.1.3 Put value at the expiry T 3

2 Week 2 5

2.1 Option Pricing 5

2.1.1 One-period Binomial Case 5
2.1.2 Put and Call Parity 7
2.1.3 Pricing by Replication in a Binomial Model 7
2.1.4 Risk Neutral Valuation 8

1. Week 1

1.1 General Derivative Contracts

Outline

(1) General definition of a financial derivative contract
(2) Standard options
(3) Payoff function

Definition 1.1 — Financial options.
A **financial option/derivative** is a financial contract stipulated today at $t = 0$. The value of the contract at the future *expiry* T is determined exactly by the market price of an *underlying asset* at T.

We don't know the value of the underlying asset at T, but we know the contract's value in relation to the underlying asset price ar T.

- The underlying asset can be: stock, commodity, market index, interest rate/bond, exchange rate

Notation 1.1 (S_t)**.** *Let S_t or $S(t)$ denote the underlying price at time t, a stochastic process.*

$$\boxed{\text{Holder}} \;\text{———}\; \boxed{S_t,\ T,\ \text{Payoff}(S_T)} \;\text{———}\; \boxed{\text{Writer}}$$

Knowing the future value of the contract in relation to the underlying allows it to be used as an insurance.

1.1.1 European calls and puts

Definition 1.2 — European call.
A **European call** option is the *right* to *buy* underlying asset at a preset strike price K. The right can only be exercised at the expiry T.

Asymmetry: holder has the option to exercise. Writer has the obligation

Definition 1.3 — European put.
A **European put** option is the *right* to sell underlying asset at a preset strike price K. The right can only be exercised at the expiry T.

Definition 1.4 — American put.
An American call option is the *right* to *sell* underlying asset at a preset strike price K. The right can be exercised any time from now to the expiry T.

- Holder: Buyer of the option, enters a **long** position
- Writer: Seller of the option, enters a *short* position

Notation 1.2 ($V(S(t), t)$ or V_t)**.** *Let $V(S(t), t)$ or V_t denote the option value at time t.*
$V_T = \textit{payoff}(S_T)$

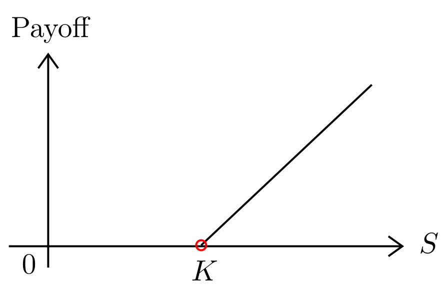
Central Question

- What is the fair value of V_0 of the option today?
- How should a writer hedge risk?

What are the payoff functions $V_T = \text{payoff}(S_T)$ for calls/puts?

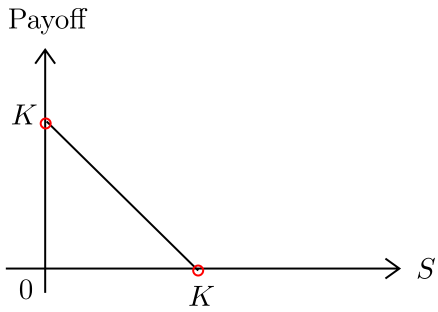
1.1.2 Call value at the expiry T

If $S_T \leq K$, holder should not exercise the call. $V_T = 0$
If $S_T \geq K$, holder exercises the right $V_T = S_T - K$
$\implies V_T = \text{payoff}(S_T) = \max(S_t - K, 0)$



1.1.3 Put value at the expiry T

If $S_T \leq K$, holder should exercise the put. $V_T = K - S_T$
If $S_T \geq K$, holder should not exercises
$\implies V_T = \text{payoff}(S_T) = \max(K - S_t, 0)$



■ **Example 1.1**

- Bulb wholesaler can purchase a call to have the option of buying tulip \$0.5 a dozen at a fixed price in 3 months.
- Bulb growers can purchase a put to allow selling tulip \$1 a dozen at a fixed price in 3 months.

A bet on the underlying price can be done by trading either S_t or V_t.

> **Note 1.1** Option is more risky compared to the underlying (leverage effect):
>
> $$\left|\frac{S_T - S_0}{S_O}\right| << \left|\frac{V_T - V_0}{V_0}\right|, \ v_t = 0 \implies \frac{V_T - V_0}{V_0} = -100\%$$
>
> When option expires out of money, 100% loss for the option holder.

We focus on stock option with expiry $T \leq 1$ and interest rate randomness is reasonably ignored. (because we will only look into short term contracts in this course)

- Stock: a share in ownership of a company
- Dividend: payment to shareholder from the profits

> **Note 1.2** When stock pays dividend to the shareholder, holder of option on the stock receives nothing. Option is said to be *dividend protected*.

2. Week 2

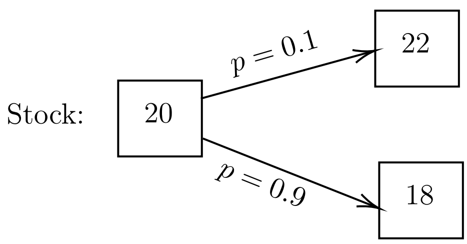
2.1 Option Pricing

Outline

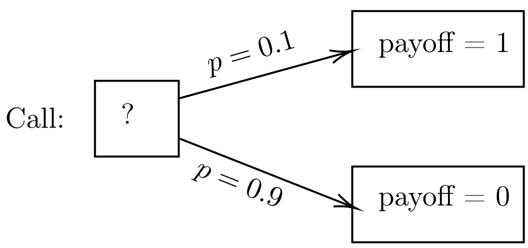
(1) One period binomial
(2) Fair value of option
(3) Arbitrage
(4) Put - Call parity

2.1.1 One-period Binomial Case

Consider a one-period binomial case. Assume $T = 1$ and up probability $p = 0.1$



Consider a call with $K = 21$. What do we know of the option value at the expiry $T = 1$?



What is the option value today?
Is it $0.11 + 0.9 \times 0 = 0.1$ using probability?
What about time value of the money, i.e. interest rate?

Riskless Asset (constant interest rate)

Cash account continuously compounds at risk free rate $r \geq 0$.
Lending (depositing) money to a bank $\Longrightarrow$ buying a bond from a bank
Borrowing money from a bank $\Longrightarrow$ selling a bond.
Let $\beta(t)$ denote the value of a riskless bond at time t

$$\frac{d\beta(\tau)}{\beta(\tau)} = rd\tau \implies \int_t^T \frac{d\beta(\tau)}{\beta(\tau)} = \int_t^T rd\tau$$

$$\log(\beta(T)) - \log(\beta(t)) = r(T-t)$$

- Discounting: $\beta(T) = 1 \implies \beta(t) = e^{-r(T-t)}$
- Compounding: $\beta(t) = 1 \implies \beta(T) = e^{r(T-t)}$

Back to the example 2.1.1, is the fair value $0.1 \times e^{-0.05}$ if $r = 0.05$?
No this is not the fair value.

Determine the fair value by trading

Determining the fair option value needs to consider trading in a financial market of bond, stock, option.

Definition 2.1 — Arbitrage.
An **arbitrage** is a trading opportunity to make a no-risk(guaranteed) profit which is greater than that of a bank deposit which earns the interest rate $r \geq 0$.

Definition 2.2 — Fair value.
The **fair value** of a financial instrument is the price which does not lead to arbitrage.

Why? Arbitrage can only occur momentarily.
How? Under no arbitrage: two instruments have the same values at a future time, they must be priced at the same price today.

■ **Example 2.1 — Constructing Arbitrage.** We represent a trading strategy as a portfolio. Buy one share of stock and borrow \$100 (sell bonds) today

$$\Pi_0 = \underbrace{1}_{\text{long}} \times S_0 \underbrace{-}_{\text{short}} 100 \quad \text{or} \quad \Pi_0 = \{S_0, -100\}$$

A long position benefits from increased prices and a short position benefits from decreased prices. The value of this portfolio at time t:

$$\Pi_t = S_t - 100e^{rt}$$

Mathematical Characterization of an Arbitrage Strategy

An arbitrage strategy can be described as

- A portfolio with an initial value $\Pi_0 = 0$ but $\Pi_T > 0$ for all $T > 0$
- A portfolio with an initial value $\Pi_0 < 0$ but $\Pi_T \geq 0$ for all $T > 0$

2.1.2 Put and Call Parity

Proposition 2.1 — Put-Call Parity.
Assume stock S_t does not pay dividend, interest rate $r \geq 0$, and no arbitrage. Then at any time $0 \geq t \geq T$, European call C_t and put P_t, with the same strike K and expiry T, on the same underlying, satisfy

$$C_t = P_t + S_t - Ke^{-r(T-t)}$$

Proof. At time T we have
- $C_T = \max(S_T - K, 0)$
- $P_T = \max(K - S_T, 0)$
- $C_T - P_T = \max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K$

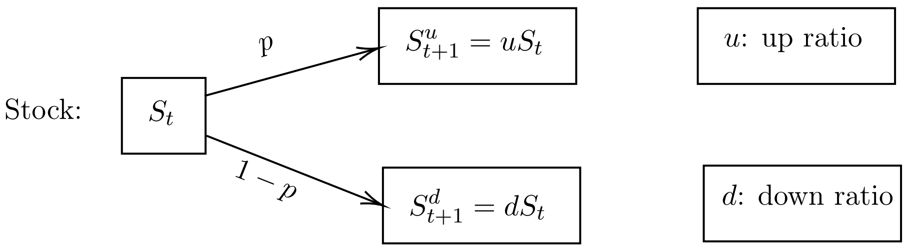
Hence, since no arbitrage, $C_t - P_t = S_t - Ke^{-r(T-t)}$ ■

Outline

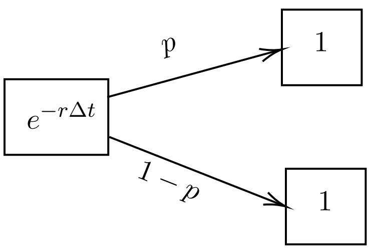
Consider 1-period in a binomial model
- Option replication and hedging
- Computing option fair value
- Risk neutral valuation

2.1.3 Pricing by Replication in a Binomial Model

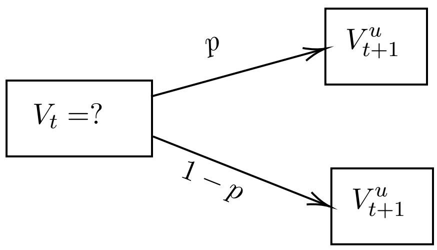
Consider a binomial model with an up probability $p > 0$



Assume:
- $S_t > 0$
- the length of time interval $\Delta t > 0$
- $0 < d < u$, u, d be given
- **no arbitrage**



Bond



Option

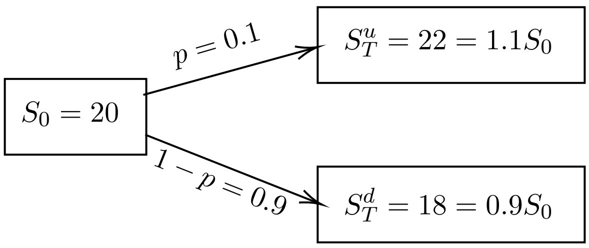
Let V^d_{t+1} and V^u_{t+1} be given, e.g. at T, they equal to corresponding payoffs. We would like to the the fair value of the option V_t.

At time t, construct portfolio $\{\delta_t S_t, \eta_t \beta_t\}$ such that replication equation is satisfied

$$\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\text{bond: } t+1} \eta_t + \underbrace{\begin{bmatrix}uS_t\\dS_t\end{bmatrix}}_{\text{stock: } t+1} \delta_t = \underbrace{\begin{bmatrix}V_{t+1}^u\\V_{t+1}^d\end{bmatrix}}_{\text{option at } t+1} \tag{2.1}$$

Note that the solution is unique (under the assumption $u > d$).
Hold δ_t unit of underlying stock, η_t unit of underlying bond.
For a simple binomial model, we can always construct a unique portfolio of underlying stock and bond to hold at the beginning of the period, such that this portfolio at the end of period, always equal to the value of the option
No arbitrage $\implies V_t = \delta_t S_t + \eta_t \beta_t = \delta_t S_t + \eta_t e^{-r\Delta t}$
The fact that **there exists a replicating portfolio of stock and bond for the option is important**: $\{V_t, -\delta_t S_t\}$ is risk-free (exactly the same as a bond).
Note that p is irrelevant. (This is a consequence of no arbitrage.)

■ **Example 2.2**



Assume $r = 0$. For a call with strike $K = 21$. To determine a replicating portfolio, solve

$$22\delta + 1 \cdot \eta = 1 \ (= C_T^u)$$
$$18\delta + 1 \cdot \eta = 0 \ (= C_T^d)$$

$$\implies \begin{cases} \delta & = \frac{C_T^u - C_T^d}{(u-d)S_0} = 0.25 \\ \eta & = -4.5 \text{ (sell bond (borrow cash))} \end{cases}$$

$$\implies C_0 = \delta S_0 + \eta \beta_0 = 0.25 \times 20 - 4.5 = 0.5$$

What if the market price of call is greater than 0.5, e.g., 0.95?
Position at $t = 0$: sell call for 0.95, buy $0.25S_0$, cash $= -4.5$ (borrow): the portfolio value:

$$\Pi_0 = -0.95 + 0.025 \times 20 - 4.5 = -0.45$$

At time T, the total value of positions

$$-C_T + 0.25S_T - 4.5 = 0$$

since $\delta = 0.25$ and $\eta = -4.5$ satisfies option replicating equations. Arbitrage!
If the market price is set to:10 (the expected payoff), find an arbitrage.

2.1.4 Risk Neutral Valuation

Note 2.1 No arbitrage assumption implies $d \leq e^{r\Delta t \leq u}$.
(Otherwise, arbitrage can be found by trading S_t and risk-free bond)

Consider another 2-by-2 linear system ψ^u and ψ^d parameters for up and down state

$$\underbrace{\begin{bmatrix} 1 \\ uS_t \end{bmatrix}}_{\text{market up}} \psi^u + \underbrace{\begin{bmatrix} 1 \\ dS_t \end{bmatrix}}_{\text{down: } t+1} \psi^d = \underbrace{\begin{bmatrix} e^{-r\Delta t} \\ S_t \end{bmatrix}}_{\text{time } t}$$

The RHS will essentially have the value of the bond and the stock at the beginning of the period, the LHS, the first term consists the bond and stock value when the market goes up, the second term when the market goes down. We are putting a weight on up side and down side.
The unique solution is $\psi^u = e^{-r\Delta t}q^*$ and $\psi^d = e^{-r\Delta t}(1-q^*)$

$$q^* = \frac{e^{r\Delta t} - d}{u - d},\ 0 \le q^* \le 1$$

The second equation $\implies$

$$S_t = e^{-r\Delta t}(q^* uS_t + (1-q^*)dS_t) = e^{-r\Delta t}\mathbf{E}^Q(S_{t+1}) \tag{2.2}$$

where $\mathbf{E}^Q(\cdot)$ is the expected value of S_{t+1} using q^* as the probability. Note: different notation of q^* is used to emphasize
The first equation $\implies$

$$\beta_t = e^{-r\Delta t}(q^* \cdot 1 + (1-q^*)\cdot 1) = e^{-r\Delta t}\mathbf{E}^Q(\beta_{t+1}) \tag{2.3}$$

Let $\{\delta_t S_t, \eta_t \beta_t\}$ be the replicating portfolio. Using 2.2 and 2.3

$$\begin{aligned}
V_t &= \delta_t S_t + \eta_t \beta_t \quad \text{(Replicating portfolio at } t\text{)} \\
&= \delta_t \underbrace{e^{-r\Delta t}(q^* uS_t + (1-q^*)dS_t)}_{\text{expected stock value}} + \eta_t \underbrace{e^{-r\Delta t}(q^* \cdot 1 + (1-q^*)\cdot 1)}_{\text{expected bond value}} \\
&= \delta_t e^{-r\Delta t} q^* \underbrace{(\delta_t uS_t + \eta_t \cdot 1)}_{\text{portfolio value when market goes up}} + \eta_t e^{-r\Delta t}(1-q^*) \underbrace{(\delta_t dS_t + \eta_t \cdot 1)}_{\text{portfolio value when market goes down}} \\
&= \delta_t e^{-r\Delta t} q^* V^u_{t+1} + \eta_t e^{-r\Delta t}(1-q^*)V^d_{t+1} \quad \text{using 2.1} \\
&= e^{-r\Delta t}\mathbf{E}^Q(V_{t+1})
\end{aligned}$$

Proposition 2.2 — Risk Neutral Valuation.
If there exists no arbitrage, then there exists risk neutral probability $0 \leq q^* = \frac{e^{r\Delta t}-d}{u-d} \leq 1$ such that

$$\beta_t = \underbrace{e^{r\Delta t}}_{\text{discounting using } r} \beta_{t+1} \quad \text{rate return} = r$$

$$S_t = \underbrace{e^{r\Delta t}}_{\text{expected rate of return is } r \text{ under } Q} \mathbf{E}^Q(S_{t+1})$$

$$V_t = \underbrace{e^{r\Delta t}}_{\text{expected rate of return is } r \text{ under } Q} \mathbf{E}^Q(V_{t+1})$$

where V_t is derivative on S (converse is also true). This holds beyond the 1-period model.

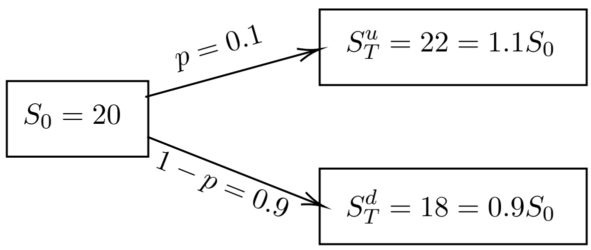
$$q^* = \frac{e^{r\Delta t}-d}{u-d}, \; 0 \leq q^* \leq 1$$

Risk neutral pricing:

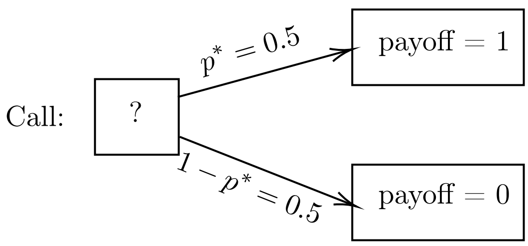
$$V_t = e^{r\Delta t}\mathbf{E}^Q(V_{t+1})$$

No arbitrage assumption implies some relationship between traded assets, specifically in this case, no arbitrage implies constraint that there exists probability world q^* such that all the traded assets link its term value to future value in the simple expected future value.

■ **Example 2.3** Recall the example of pricing a call with $K = 21$ and



$q^* = \frac{e^{r\Delta t-d}}{u-d} = \frac{1-0.9}{1.1-0.9} = 0.5$



We get again $C_0 = 0.5$