CS 476: Numeric Computation for Financial Modeling

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Contents

1. Week 1

1.1 General Derivative Contracts

Outline

(1) General definition of a financial derivative contract
(2) Standard options
(3) Payoff function

> **Definition 1.1 — Financial options.**
> A **financial option/derivative** is a financial contract stipulated today at $t = 0$. The value of the contract at the future *expiry* T is determined exactly by the market price of an *underlying asset* at T.

We don't know the value of the underlying asset at T, but we know the contract's value in relation to the underlying asset price ar T.

- The underlying asset can be: stock, commodity, market index, interest rate/bond, exchange rate

Notation 1.1 (S_t). *Let S_t or $S(t)$ denote the underlying price at time t, a stochastic process.*

| Holder | | $S_t,\ T,\ \text{Payoff}(S_T)$ | | Writer |
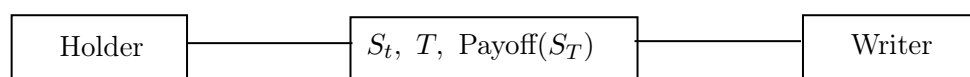
Knowing the future value of the contract in relation to the underlying allows it to be used as an insurance.

1.1.1 European calls and puts

> **Definition 1.2 — European call.**
> A **European call** option is the *right* to *buy* underlying asset at a preset strike price K. The right can only be exercised at the expiry T.

Asymmetry: holder has the option to exercise. Writer has the obligation

> **Definition 1.3 — European put.**
> A **European put** option is the *right* to sell underlying asset at a preset strike price K. The right can only be exercised at the expiry T.

> **Definition 1.4 — American put.**
> An American call option is the *right* to *sell* underlying asset at a preset strike price K. The right can be exercised any time from now to the expiry T.

- Holder: Buyer of the option, enters a **long** position
- Writer: Seller of the option, enters a *short* position

Notation 1.2 ($V(S(t), t)$ or V_t). *Let $V(S(t), t)$ or V_t denote the option value at time t.* $V_T = payoff(S_T)$

Central Question

- What is the fair value of V_0 of the option today?
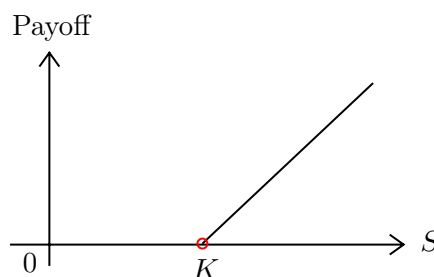- How should a writer hedge risk?

What are the payoff functions $V_T = \text{payoff}(S_T)$ for calls/puts?

1.1.2 Call value at the expiry T

If $S_T \leq K$, holder should not exercise the call. $V_T = 0$
If $S_T \geq K$, holder exercises the right $V_T = S_T - K$
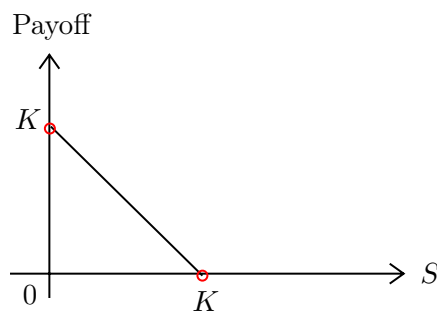$\implies V_T = \text{payoff}(S_T) = \max(S_t - K, 0)$

1.1.3 Put value at the expiry T

If $S_T \leq K$, holder should exercise the put. $V_T = K - S_T$
If $S_T \geq K$, holder should not exercises
$\implies V_T = \text{payoff}(S_T) = \max(K - S_t, 0)$

■ **Example 1.1**
- Bulb wholesaler can purchase a call to have the option of buying tulip \$0.5 a dozen at a fixed price in 3 months.
- Bulb growers can purchase a put to allow selling tulip \$1 a dozen at a fixed price in 3 months.

A bet on the underlying price can be done by trading either S_t or V_t.

Note 1.1 Option is more risky compared to the underlying (leverage effect):

$$\left| \frac{S_T - S_0}{S_O} \right| << \left| \frac{V_T - V_0}{V_0} \right|, \ v_t = 0 \implies \frac{V_T - V_0}{V_0} = -100\%$$

When option expires out of money, 100% loss for the option holder.

We focus on stock option with expiry $T \leq 1$ and interest rate randomness is reasonably ignored. (because we will only look into short term contracts in this course)
- Stock: a share in ownership of a company
- Dividend: payment to shareholder from the profits

Note 1.2 When stock pays dividend to the shareholder, holder of option on the stock receives nothing. Option is said to be *dividend protected*.
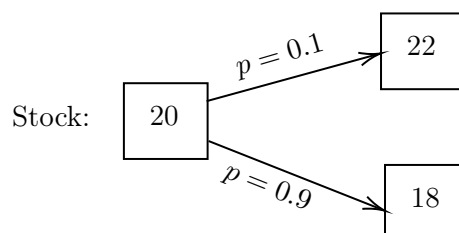
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2. Week 2

2.1 Option Pricing

Outline

(1) One period binomial
(2) Fair value of option
(3) Arbitrage
(4) Put - Call parity

2.1.1 One-period Binomial Case

Consider a one-period binomial case. Assume $T = 1$ and up probability $p = 0.1$

Stock: 20, with $p = 0.1$ going to 22 and $p = 0.9$ going to 18.

Consider a call with $K = 21$. What do we know of the option value at the expiry $T = 1$?

Call: ?, with $p = 0.1$ giving payoff $= 1$ and $p = 0.9$ giving payoff $= 0$.
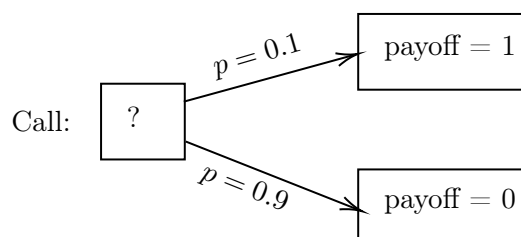
What is the option value today?
Is it $0.1 \cdot 1 + 0.9 \times 0 = 0.1$ using probability?
What about time value of the money, i.e. interest rate?

Riskless Asset (constant interest rate)

Cash account continuously compounds at risk free rate $r \geq 0$.

Lending (depositing) money to a bank $\implies$ buying a bond from a bank

Borrowing money from a bank $\implies$ selling a bond.

Let $\beta(t)$ denote the value of a riskless bond at time t

$$\frac{d\beta(\tau)}{\beta(\tau)} = rd\tau \implies \int_t^T \frac{d\beta(\tau)}{\beta(\tau)} = \int_t^T rd\tau$$

$$\log(\beta(T)) - \log(\beta(t)) = r(T - t)$$

- Discounting: $\beta(T) = 1 \implies \beta(t) = e^{-r(T-t)}$
- Compounding: $\beta(t) = 1 \implies \beta(T) = e^{r(T-t)}$

Back to the example 2.1.1, is the fair value $0.1 \times e^{-0.05}$ if $r = 0.05$?

No this is not the fair value.

Determine the fair value by trading

Determining the fair option value needs to consider trading in a financial market of bond, stock, option.

> **Definition 2.1 — Arbitrage.**
> An **arbitrage** is a trading opportunity to make a no-risk(guaranteed) profit which is greater than that of a bank deposit which earns the interest rate $r \geq 0$.

> **Definition 2.2 — Fair value.**
> The **fair value** of a financial instrument is the price which does not lead to arbitrage.

Why? Arbitrage can only occur momentarily.

How? Under no arbitrage: two instruments have the same values at a future time, they must be priced at the same price today.

■ **Example 2.1 — Constructing Arbitrage.** We represent a trading strategy as a portfolio. Buy one share of stock and borrow \$100 (sell bonds) today

$$\Pi_0 = \underbrace{1}_{\text{long}} \times S_0 \underbrace{-}_{\text{short}} 100 \quad \text{or} \quad \Pi_0 = \{S_0, -100\}$$

A long position benefits from increased prices and a short position benefits from decreased prices. The value of this portfolio at time t:

$$\Pi_t = S_t - 100e^{rt}$$

Mathematical Characterization of an Arbitrage Strategy

An arbitrage strategy can be described as
- A portfolio with an initial value $\Pi_0 = 0$ but $\Pi_T > 0$ for all $T > 0$
- A portfolio with an initial value $\Pi_0 < 0$ but $\Pi_T \geq 0$ for all $T > 0$

2.1.2 Put and Call Parity

> **Proposition 2.1 — Put-Call Parity.**
> Assume stock S_t does not pay dividend, interest rate $r \geq 0$, and no arbitrage. Then at any time $0 \geq t \geq T$, European call C_t and put P_t, with the same strike K and expiry T, on the same underlying, satisfy
>
> $$C_t = P_t + S_t - Ke^{-r(T-t)}$$

Proof. At time T we have
- $C_T = \max(S_T - K, 0)$
- $P_T = \max(K - S_T, 0)$
- $C_T - P_T = \max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K$

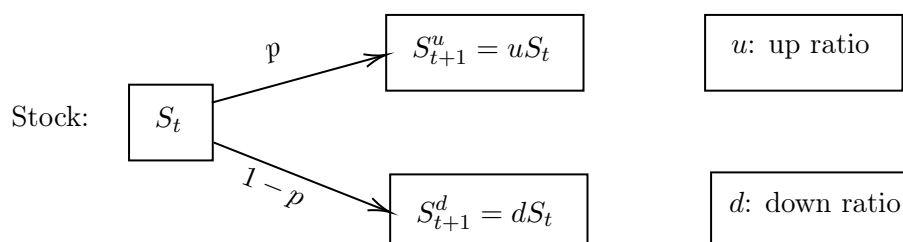Hence, since no arbitrage, $C_t - P_t = S_t - Ke^{-r(T-t)}$ ∎

Outline

Consider 1-period in a binomial model
- Option replication and hedging
- Computing option fair value
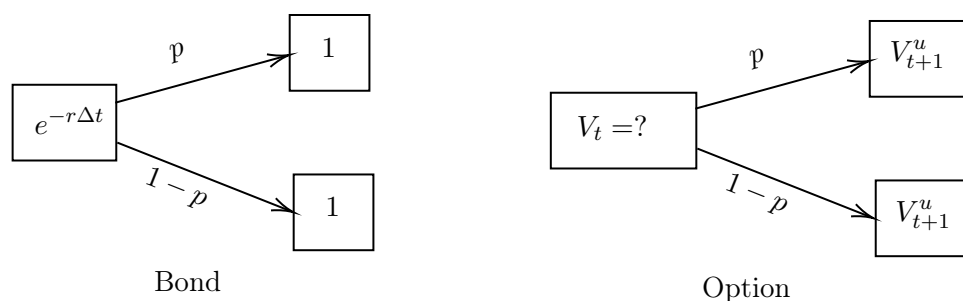- Risk neutral valuation

2.1.3 Pricing by Replication in a Binomial Model

Consider a binomial model with an up probability $p > 0$

Assume:
- $S_t > 0$
- the length of time interval $\Delta t > 0$
- $0 < d < u$, u, d be given
- **no arbitrage**

Let V_{t+1}^d and V_{t+1}^u be given, e.g. at T, they equal to corresponding payoffs. We would like to the the fair value of the option V_t.

At time t, construct portfolio $\{\delta_t S_t, \eta_t \beta_t\}$ such that replication equation is satisfied

$$\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{bond: } t+1} \eta_t + \underbrace{\begin{bmatrix} uS_t \\ dS_t \end{bmatrix}}_{\text{stock: } t+1} \delta_t = \underbrace{\begin{bmatrix} V^u_{t+1} \\ V^d_{t+1} \end{bmatrix}}_{\text{option at } t+1} \tag{2.1}$$

Note that the solution is unique (under the assumption $u > d$).

Hold δ_t unit of underlying stock, η_t unit of underlying bond.
For a simple binomial model, we can always construct a unique portfolio of underlying stock and bond to hold at the beginning of the period, such that this portfolio at the end of period, always equal to the value of the option

No arbitrage $\implies V_t = \delta_t S_t + \eta_t \beta_t = \delta_t S_t + \eta_t e^{-r\Delta t}$

The fact that **there exists a replicating portfolio of stock and bond for the option is important**: $\{V_t, -\delta_t S_t\}$ is risk-free (exactly the same as a bond).

Note that p is irrelevant. (This is a consequence of no arbitrage.)

■ **Example 2.2**
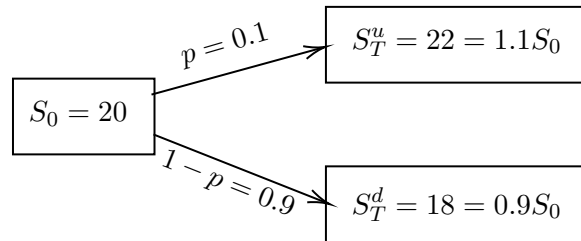
Assume $r = 0$. For a call with strike $K = 21$. To determine a replicating portfolio, solve

$$22\delta + 1 \cdot \eta = 1 \ (= C^u_T)$$
$$18\delta + 1 \cdot \eta = 0 \ (= C^d_T)$$

$$\implies \begin{cases} \delta &= \frac{C^u_T - C^d_T}{(u-d)S_0} = 0.25 \\ \eta &= -4.5 \text{ (sell bond (borrow cash))} \end{cases}$$

$$\implies C_0 = \delta S_0 + \eta \beta_0 = 0.25 \times 20 - 4.5 = 0.5$$

What if the market price of call is greater than 0.5, e.g., 0.95?

Position at $t = 0$: sell call for 0.95, buy $0.25 S_0$, cash $= -4.5$ (borrow): the portfolio value:

$$\Pi_0 = -0.95 + 0.025 \times 20 - 4.5 = -0.45$$

At time T, the total value of positions

$$-C_T + 0.25 S_T - 4.5 = 0$$

since $\delta = 0.25$ and $\eta = -4.5$ satisfies option replicating equations. Arbitrage!

If the market price is set to:10 (the expected payoff), find an arbitrage.

2.1.4 Risk Neutral Valuation

> **Note 2.1** No arbitrage assumption implies $d \leq e^{r\Delta t} \leq u$.
> (Otherwise, arbitrage can be found by trading S_t and risk-free bond)

Consider another 2-by-2 linear system ψ^u and ψ^d parameters for up and down state

$$\underbrace{\begin{bmatrix} 1 \\ uS_t \end{bmatrix}}_{\text{market up}} \psi^u + \underbrace{\begin{bmatrix} 1 \\ dS_t \end{bmatrix}}_{\text{down: } t+1} \psi^d = \underbrace{\begin{bmatrix} e^{-r\Delta t} \\ S_t \end{bmatrix}}_{\text{time } t}$$

The RHS will essentially have the value of the bond and the stock at the beginning of the period, the LHS, the first term consists the bond and stock value when the market goes up, the second term when the market goes down. We are putting a weight on up side and down side.

The unique solution is $\psi^u = e^{-r\Delta t} q^*$ and $\psi^d = e^{-r\Delta t}(1 - q^*)$

$$q^* = \frac{e^{r\Delta t} - d}{u - d}, \ 0 \leq q^* \leq 1$$

The second equation $\implies$

$$S_t = e^{-r\Delta t}(q^* u S_t + (1 - q^*) d S_t) = e^{-r\Delta t} \mathbf{E}^Q(S_{t+1}) \tag{2.2}$$

where $\mathbf{E}^Q(\cdot)$ is the expected value of S_{t+1} using q^* as the probability. Note: different notation of q^* is used to emphasize

The first equation $\implies$

$$\beta_t = e^{-r\Delta t}(q^* \cdot 1 + (1 - q^*) \cdot 1) = e^{-r\Delta t} \mathbf{E}^Q(\beta_{t+1}) \tag{2.3}$$

Let $\{\delta_t S_t, \eta_t \beta_t\}$ be the replicating portfolio. Using 2.2 and 2.3

$$\begin{aligned}
V_t &= \delta_t S_t + \eta_t \beta_t \quad \text{(Replicating portfolio at } t) \\
&= \delta_t \underbrace{e^{-r\Delta t}(q^* u S_t + (1 - q^*) d S_t)}_{\text{expected stock value}} + \eta_t \underbrace{e^{-r\Delta t}(q^* \cdot 1 + (1 - q^*) \cdot 1)}_{\text{expected bond value}} \\
&= \delta_t e^{-r\Delta t} q^* \underbrace{(\delta_t u S_t + \eta_t \cdot 1)}_{\text{portfolio value when market goes up}} + \eta_t e^{-r\Delta t}(1 - q^*) \underbrace{(\delta_t d S_t + \eta_t \cdot 1)}_{\text{portfolio value when market goes down}} \\
&= \delta_t e^{-r\Delta t} q^* V_{t+1}^u + \eta_t e^{-r\Delta t}(1 - q^*) V_{t+1}^d \quad \text{using 2.1} \\
&= e^{-r\Delta t} \mathbf{E}^Q(V_{t+1})
\end{aligned}$$

> **Proposition 2.2 — Risk Neutral Valuation.**
> If there exists no arbitrage, then there exists risk neutral probability $0 \leq q^* = \frac{e^{r\Delta t} - d}{u - d} \leq 1$ such that
>
> $$\beta_t = \underbrace{e^{r\Delta t}}_{\text{discounting using } r} \beta_{t+1} \quad \text{rate return} = r$$
>
> $$S_t = \underbrace{e^{r\Delta t}}_{\text{expected rate of return is } r \text{ under } Q} \mathbf{E}^Q(S_{t+1})$$
>
> $$V_t = \underbrace{e^{r\Delta t}}_{\text{expected rate of return is } r \text{ under } Q} \mathbf{E}^Q(V_{t+1})$$
>
> where V_t is derivative on S (converse is also true). This holds beyond the 1-period model.
>
> $$q^* = \frac{e^{r\Delta t} - d}{u - d}, \ 0 \leq q^* \leq 1$$
>
> Risk neutral pricing:
>
> $$V_t = e^{r\Delta t}\mathbf{E}^Q(V_{t+1})$$

No arbitrage assumption implies some relationship between traded assets, specifically in this case, no arbitrage implies constraint that there exists probability world q^* such that all the traded assets link its term value to future value in the simple expected future value.

■ **Example 2.3** Recall the example of pricing a call with $K = 21$ and
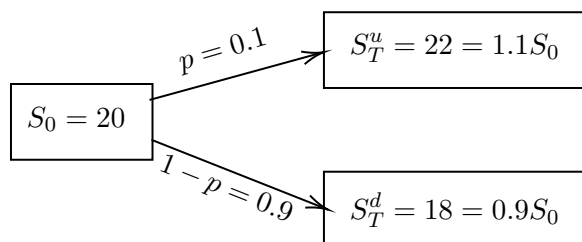
$$q^* = \frac{e^{r\Delta t - d}}{u - d} = \frac{1 - 0.9}{1.1 - 0.9} = 0.5$$
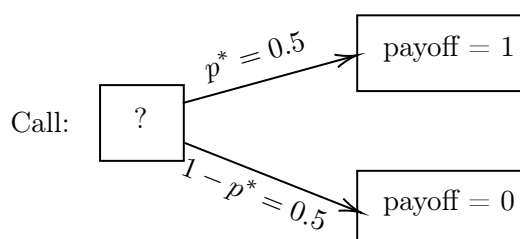
We get again $C_0 = 0.5$

3. Week 3

Outline
- Multi-period binomial lattice: multiplicative price model
- Multi-period binomial model : additive log price model
- Constructing a binomial lattice for underlying
- European option pricing on a binomial lattice

3.1 Multi-period binomial lattice

One-period binomial model is crude, to improve it, we use it in a small interval Δt only. Specifically, let $[0, T]$ be partitioned into

$$t_0 = 0 < t_1 < \cdots < t_n < \cdots < t_N = T$$

$$t_n = n\Delta t, \ \Delta t = \frac{T}{N}$$

In the binomial model, let $S_0^0 = S_0$ and S_j^n be the price at $S(t_n)$ with $j = 0, \ldots, n$, total of $n + 1$ prices.

Notation 3.1. *Instead of S_t (emphasize time), we will use*
- *$S_n = S(t_n)$ to denote value at t_n*
- *When both sub- and sup- scripts are used, the superscript denotes time*

Form S_j^n, in Δt, price goes up to $S_{j+1}^{n+1} = uS_j^n$ with probability p and down to $S_j^{n+1} = dS_j^n$ with probability $(1 - p)$
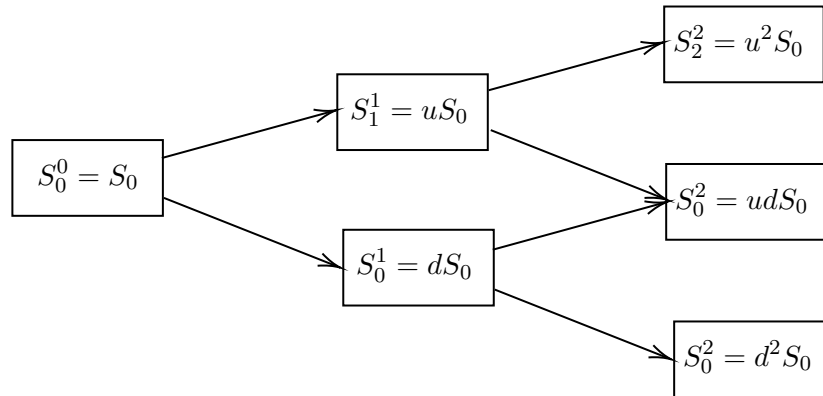
Assume further $d = \frac{1}{u}$, we have a non-drifting tree.

Given a binomial lattice which ensures that no arbitrage, i.e. $d < e^{r\delta t} < u$, any European option values can easily be done via backward iteration.

How to find the fair value?
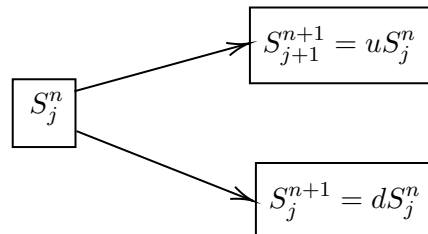
- At the final nodes:
    - $payoff(S_2^2)$
    - $payoff(S_1^2)$
    - $payoff(S_0^2)$
- At the middle node, each is a 1-period binomial movement: Discounting expected value (under risk neutral $q^* = \frac{e^{r\Delta t} - d}{u - d}$)

▍Exercise 3.1 Pricing a European option on the 2-period lattice

At t_n, the jth node is

How to construct the lattice (determine u, d, p) to mode a stock price?

What happens to the binomial latter when $\Delta t \to 0$?

3.1.1 Binomial Model: Logarithmic return

In finance, instead of price S_n, we often consider log return.

> **Definition 3.1 — logarithmic return.**
> The **logarithmic return** in the $(n+1)$ period is
> $$\Delta X_n = \log\left(\frac{S_{n+1}}{S_n}\right)$$
> where $X_n = \log(S_n)$ and $\Delta X_n = X_{n+1} - X_n$

> **Definition 3.2 — simple return.**
> $$\frac{S_{n+1} - S_n}{S_n} \approx \log\left(1 + \frac{S_{n+1} - S_n}{S_n}\right) = \log\left(\frac{S_{n+1}}{S_n}\right)$$

Then

$$X_{n+1} = \log(S_{n+1}) = \log(S_n) + (\log(S_{n+1}) - \log(S_n))$$
$$= X_n + \Delta X_n$$

where $\Delta X_n = \log\left(\frac{S_{n+1}}{S_n}\right)$.

Assume that the log return in the n-th period ΔX_n has the distribution

$$\Delta = \begin{cases} \Delta h, & \text{with probability } p \\ -\Delta h, & \text{with probability } 1 - p \end{cases}$$

with Δh is a given constant parameter, e.g. let $d = \frac{1}{u}$ and $\Delta h = \log(u)$.
In terms of the price, this corresponds to a binomial model

$$S_{n+1} = \begin{cases} S_n u, & \text{with probability } p \\ S_n d, & \text{with probability } 1 - p \end{cases}$$

where $u = e^{\Delta h}$, $d = e^{-\Delta h} = \frac{1}{u}$
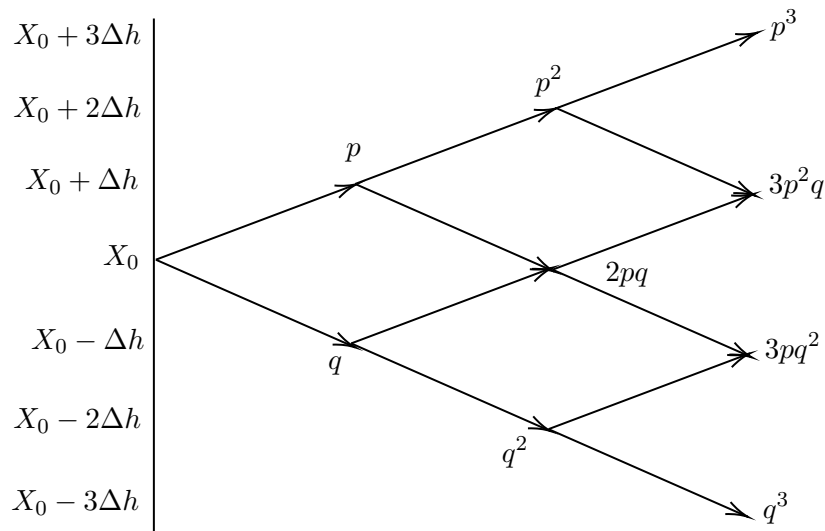Note that the cumulative log returns in the first n period is

$$\sum_{k=0}^{n+1} \Delta X_k$$

and the log price

$$X_n = X_0 + \sum_{k=0}^{n+1} \Delta X_k$$

(or a random walker's position at t_n, assuming $X_0 = 0$).
This allows us to analyze an additive model for the log price, instead of the multiplicative model for the price.

3.1.2   Discrete Random Walk and its Limit

Properties of discrete random walk

(1) ΔX_n has the identical distribution for any n

(2) ΔX_n and $\Delta X_{n'}$ are **independent** for $n \neq n'$. For any $n \neq n'$

$$\mathbf{E}(\Delta X_n \Delta X_{n'}) = \mathbf{E}(\Delta X_n)\mathbf{E}(\Delta X_{n'})$$

At t_n, the next price move is independent of its past moves, i.e. it is a random walk (Markov process)

(3) Statistical moments are determined by p and Δh (or p, u and d)

$$\mathbf{E}(\Delta X_n) = p\Delta h - (1-p)\Delta h = (2p-1)\Delta h = \alpha_0$$
$$\mathbf{E}(\Delta X_n^2) = p(\Delta h)^2 + (1-p)(\Delta h)2 = (\Delta h)^2$$
$$\mathbf{var}(\Delta X_n) = \mathbf{E}((\Delta X_n)^2) - (\mathbf{E}(\Delta X_n^2))^2$$
$$= (\Delta h)^2 - (2p-1)^2(\Delta h)^2 = \sigma_0^2$$

Following Central Limit Theorem, as $N \to \infty$, $X_N - X_0$ converges to a normal distribution.

$$P\left(\frac{\sum_{n=0}^{N-1}\Delta X_n - N\alpha_0}{\sigma_0\sqrt{N}} \leq x\right) \to F_{normal}(x)$$

$F_{normal}(x)$ is the cumulative distribution function of the standard normal, specifically,

$$F_{normal}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{s^2}{2}}ds$$

The log return has a normal distribution

How to choose p, u, d

Assume that underlying stock has the two statistics:

- the average log return per year is α
- the standard deviation of the annual return, volatility, is σ, or the variance σ^2

Then the expected log return in an interval of Δt is $\alpha\Delta t$ and variance $\sigma^2\Delta t$.

At t, in a period of Δt, we choose binomial parameters u, p, d to match underlying return in the mean and variance statistics.

(we can specifically choose u, d, p so that the stock's expected log return equals to α and volatility equals to $\sigma^2\Delta t$)

■ **Example 3.1** Consider $u = e^{\Delta h}$ and $d = \frac{1}{u}$. Given α and σ, choose p and Δh to satisfy

$$\mathbf{E}(\Delta X_n) = p\Delta h - (1-p)\Delta h = \mathbf{E}\left(\log\left(\frac{S_{t+\Delta t}}{S_t}\right)\right) = \alpha\Delta t$$

$$\mathbf{var}(\Delta X_n) = (\Delta h)^2 - (2p-1)^2(\Delta h)^2 = \mathbf{var}\left(\log\left(\frac{S_{t+\Delta t}}{S_t}\right)\right) = \sigma^2\Delta t$$

One frequent choice: CRR (79):

$$u = e^{\sigma\sqrt{\Delta t}}$$
$$d = \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}$$
$$p = \frac{1}{2} + \frac{1}{2}\frac{\alpha}{\sigma}\sqrt{\Delta t}$$

> **Note 3.1**
> - Under this construction, u and d depend on volatility only.
>   The expected log return α is irrelevant. Option value depends on volatility, not expected rate of return.
> - CRR leads to a recombining tree (non-drifting)
> - More generally, there are many different choices for (u, d, p) since we have two equations for the first two moments and three parameters. See course notes (and assignment 1) for alternative parameterizations
> - One can also directly set up the no-arbitrage lattice for which the expected rate of return is r (i.e. the probability is q^*)

CRR Model 2-period
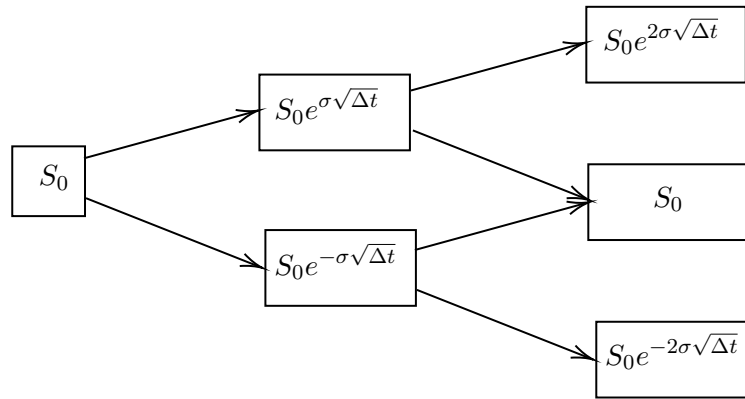
3.2 Pricing European option under no-arbitrage lattice

Pricing an European option with expiry T using an N-period lattice. Let $t_n = n\Delta t$ and $\Delta t = \frac{T}{N}$, and S_j^n denote the j-th price at t_n, $j = 0, 1, \ldots, n$

(1) Choose u, d and p to match log return statistics.

■ **Example 3.2** Let $\log(u) = \Delta h = \sigma\sqrt{\Delta t}$. Choose Δt sufficiently small so that the condition

$$e^{-\sigma\sqrt{\Delta t}} \le e^{r\Delta t} \le e^{\sigma\sqrt{\Delta t}}$$

is satisfied

(2) Construct lattice of prices: e.g. $u = e^{\sigma\sqrt{\Delta t}}, d = e^{-\sigma\sqrt{\Delta t}}$

$$S_{j+1}^{n+1} = uS_j^n \text{ and } S_j^{n+1} = dS_j^n, \ S_j^N = u^j d^{N-j} S_0$$

(3) At the expiry T, the value of the option is

$$V_j^N = \text{payoff}(S_j^N), \ 0 \le j \le N$$

e.g. call payoff$(S_j^N) = \max(S_j^N - K, 0)$

(4) Roll back through the lattice

$$
\begin{aligned}
&\text{for } \ n = N-1, N-2, \ldots, 0 \text{ do} \\
&\qquad \text{for } \ j = 0, 1, \ldots, n \text{ do} \\
&\qquad\quad V_j^n = e^{-r\Delta t}(q^* V_{j+1}^{n+1} + (1 - q^*)V_j^{n+1}) \\
&\qquad \text{end} \\
&\text{end}
\end{aligned}
$$

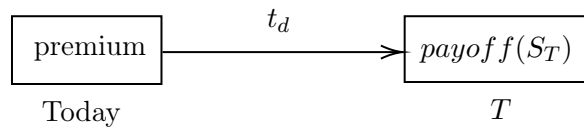where $q^* = \frac{e^{r\Delta t} - d}{u - d}$

outline

- European option pricing under a binomial lattice and dividend
- Pricing American option under a binomial lattice
- Convergence (and convergence rate) of option values under a binomial lattice
- Black-Scholes formula for European calls and puts

3.2.1 Dividend

Denote

- t^-: immediately before t
- t^+: immediately after t

Assume that, at $t = 0$, it is known that the underlying pays dividend D. e.g. $D = \rho S(t_d^-)$, at $t = t_d$, $0 \le t_d \le T$

$$\boxed{\text{premium}} \xrightarrow{\quad t_d \quad} \boxed{payoff(S_T)}$$

$$\text{Today} \qquad\qquad\qquad\qquad\qquad T$$

No arbitrage implies

$$S(t_d^+) = S(t_d^-) - D$$

However, this does not affect option holder/writer, i.e., option is dividend protected.
No arbitrage implies that $V(S(t), (t))$ ($\equiv \tilde{V}(t)$) is a **continuous** function of t, i.e.

$$V(S(t_d^-), t_d^-) = V(S(t_d^+), t_d^+) = V(S(t_d^-) - D, t_d^+) \tag{3.1}$$

Pricing under dividend payment: assuming t_d is dividend date

$$\text{for} \quad n = N-1, N-2, \ldots, 0 \text{ do}$$
$$\quad \text{for} \quad j = 0, 1, \ldots, n \text{ do}$$
$$\qquad V_j^n = e^{-r\Delta t}(q^* V_{j+1}^{n+1} + (1-q^*)V_j^{n+1})$$
$$\quad \text{end}$$
$$\quad \text{if } t_n \text{ is the closest time point to } t_d$$
$$\qquad \text{compute } V_j^n \text{ at } t_n^- \text{ by (3.1) and interpolation}$$
$$\quad \text{end}$$
$$\text{end}$$

> **Note 3.2** Interpolation is required since $S(t_d^-) - D$ may not equal to any node price S_j^n

3.3 Pricing American Options

Consider an American option which can be exercised at the discrete time t_n, $n = 0, \cdots, N$
Given a no-arbitrage lattice, the holder needs to optimally decide at any t_n, $n = 0, 1 \cdots, N$, whether to **exercise** or to continue to **hold**.
At the expiry $T = t_N$, the holder exercises if and only if payoff(S_j^N) > 0 and $V_j^N =$ payoff(S_j^N).
Dynamic Programming $\implies$ optimal strategy needs to be optimal at any node, *assuming optimal decision starting from that node forward.*
Hence a solution can be made one step a time, in backwards.

Backeard Iteration

- At the expiry $T = t_N$, $V_j^N = \text{payoff}(S_j^N)$
- At (t_n, S_j^n), the option has (continue-to-hold) **continuation value**

$$(V_j^n)^* = e^{-r\Delta t}\mathbf{E}(V^{n+1}|S_j^n) = e^{-r\Delta t}(q^* V_{j+1}^{n+1} + (1-q^*)V_j^{n+1})$$

and $\text{payoff}(S_j^n)$, *if exercised.* Hence the fair option value is

$$V_j^n = \max((V_j^n)^*, \text{payoff}(S_j^n)), \ j = 0, 1 \cdots, n$$

i.e., the optimal exercise strategy is to exercise if the continuation value is less than the payoff

Pricing American option

$$
\begin{aligned}
&\text{for} \ \ n = N-1, N-2, \ldots, 0 \ \text{do} \\
&\quad \text{for} \ \ j = 0, 1, \ldots, n \ \text{do} \\
&\quad\quad (V_j^n)^* = e^{-r\Delta t}(q^* V_{j+1}^{n+1} + (1-q^*)V_j^{n+1}) \ \ \% \ \text{continuation value} \\
&\quad\quad V_j^n = \max((V_j^n)^*, \text{payoff}(S_j^n)) \\
&\quad \text{end} \\
&\text{end}
\end{aligned}
$$

3.3.1 Computational Cost

- Floating point operations (FLOPS): $O(N^2)$ (for both European and American)
- Time efficiency: vectorization Matlab code for efficiency, see chapter 2.5 from the course notes
- Space cost: if only V_0 is needed, we can compute using $O(N)$ space

3.4 BS Formula

Central Limit Theorem implies that $\log(S_t) - \log(S_0)$ has a normal distribution with mean αt and variance $\sigma^2 t$.

It can be shown that the binomial lattice standard option value functions converges to the BS formulae.

Let $0 \le t \le T$. As $N \to \infty$, the call option value $C(S, t)$ is

$$C(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

- $T - t$: time to expiry
- $N(d)$: Cumulative distribution function for a standard normal

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}y^2} dy$$

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log\left(\frac{S}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

> **Note 3.3**
> - Option value only depends on σ, not the average rate of return
> - There is no analytic formula for American option pricing

3.4.1 Convergence Rate

Let V_0^{exact} be the BS value when $t = 0$ and $S = S_0$, i.e., $V_0^{\text{exact}} = V(S_0, 0)$.
Let $V_0^{\text{tree}}(\Delta t)$ be the binomial lattice value at $t = 0$ and $S = S_0$ when $\Delta t = \frac{T}{N}$.
PDE theory shows (later) that, if **strike price is at a binomial node**,

$$V_0^{\text{tree}}(\Delta t) = V_0^{\text{exact}} + \text{const} \cdot \Delta t + o(\Delta t)$$

Otherwise, smoothing of payoff is required (see chapter 5.4 from course notes).
If convergence rate is **linear**, then the ratio

$$\lim_{\Delta t \to 0} \frac{V_0^{\text{tree}}\left(\frac{\Delta t}{2}\right) - V_0^{\text{tree}}(\Delta t)}{V_0^{\text{tree}}\left(\frac{\Delta t}{4}\right) - V_0^{\text{tree}}\left(\frac{\Delta t}{2}\right)} \tag{3.2}$$

would approach 2.
If the convergence rate is **quadratic**, then

$$V_0^{\text{tree}}(\Delta t) = V_0^{\text{exact}} + \alpha(\Delta t)^2 + o((\Delta t)^2)$$

where α is some constant independent of Δt.
What would the ratio (3.2) approaches to if convergence rate is quadratic?
This allows us to computationally verify convergence rate of a method.