PMATH 351: Real Analysis

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Contents

1. Cardinality

Definition 1.1 — domain, range, image, inverse image.
Let X and Y be sets and let $f : X \to Y$. Recall the **domain** of f and the **range** of f are the sets

$$\text{Domain}(f) = X, \text{ Range}(f) = f(X) = \{f(x)|x \in X\}$$

for $A \subseteq X$, the **image** of A under f is the set

$$f(A) = \{f(x)|x \in A\}$$

For $B \subseteq Y$, the **inverse image** of B under f is the set

$$f^{-1}(B) = \{x \in X|f(x) \in B\}$$

Definition 1.2 — Composite.
Let X, Y and Z be sets, let $f : X \to Y$ and let $g : Y \to Z$. We define the **composite** function $(g \circ f)(x) = g(f(x))$ for all $x \in X$

Definition 1.3 — injective, surjective, bijective.
We say that f is **injective** (or **one-to-one**) when for every $y \in Y$ there exists at most one $x \in X$ such that $f(x) = y$. Equivalently, f is injective when for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
We say that f is **surjective** (or **onto**) when for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$. Equivalently, f is surjective when $\text{Range}(f) = Y$
We say that f is **bijective** (or **invertible**) when f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f(x) = y$. When f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f^{-1} : Y \to X$ such that for all $y \in Y$, $f^{-1}(y)$ is equal to the unique element $x \in X$ such that $f(x) = y$. Note that when f is bijective so is f^{-1}, and in this case we have $(f^{-1})^{-1} = f$

Theorem 1.1 Let $f : X \to Y$ and let $g : Y \to Z$. Then
(1) If f and g are both injective then so is $g \circ f$
(2) If f and g are both surjective then so is $g \circ f$
(3) If f and g are both invertible then so is $g \circ f$, and in this case $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof.
(1) Suppose that f and g are both injective. Let $x_1, x_2 \in X$. If $g(f(x_1)) = g(f(x_2))$ then since g is injective we have $f(x_1) = f(x_2)$, and then since f is injective we have $x_1 = x_2$. Thus $g \circ f$ is injective.
(2) Suppose that f and g are both injective. Given $z \in Z$, since g is surjective we can choose $y \in Y$ so that $g(y) = z$, then since f is surjective we can choose $x \in X$ so that $f(x) = y$, and then we have $g(f(x)) = g(y) = z$. Thus $g \circ f$ is surjective.
(3) Follows (1) and (2).

■

Definition 1.4 — identity function.
For a set X, we define the **identity function** on X to be the function $I_X : X \to X$ given by $I_X(x) = x$ for all $x \in X$. Note that for $f : X \to Y$ we have $f \circ I_X = f$ and $I_Y \circ f = f$.

Definition 1.5 — inverse.
Let X and Y be sets and let $f : X \to Y$. A **left inverse** of f is a function $g : Y \to X$ given by $g \circ f = I_X$. Equivalently, a function $g : Y \to X$ is a left inverse of f when $g(f(x)) = x$ for all $x \in X$.
A **right inverse** of f is a function $h : Y \to X$ such that $f \circ h = I_Y$. Equivalently, a function $h : Y \to X$ is a right inverse of f when $f(h(y)) = y$ for all $y \in Y$.

Theorem 1.2 Let X and Y be nonempty sets and let $f : X \to Y$. Then
(1) f is injective $\iff$ f has a left inverse.
(2) f is surjective $\iff$ f has a right inverse.
(3) f is bijective $\iff$ f has a left inverse g and a right inverse h, and in this case we have $g = h = f^{-1}$.

Proof.
(1) Suppose first that f is injective. Since $X \neq \emptyset$ we can choose $a \in X$ and then define $g : Y \to X$ as follows: if $y \in \text{Range}(f)$ then (using the fact the f is injective) we define $g(y)$ to be the unique element $x_y \in X$ with $f(x_y) = y$, and if $y \notin \text{Range}(f)$, then we define $g(y) = a$. Then for every $x \in X$ we have $y = f(x) \in \text{Range}(f)$, so $g(y) = x_y = x$, that is $g(f(x)) = x$. Conversely, if f has a left inverse, say g, then f is injective since for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x = g(f(x_1)) = g(f(x_2)) = x_2$.
(2) Suppose first that f is onto. For each $y \in Y$, choose $x_y \in X$ with $f(x_y) = y$, then define $g : X \to Y$ by $g(y) = x_y$ (We need the Axiom of Choice for this). Then g is a right inverse of f since for every $y \in Y$ we have $f(g(y)) = f(x_y) = y$. Conversely, if f has a right inverse, say g, then f is onto since given any $y \in Y$ we can choose $x = g(y)$ and then we have $f(x) = f(g(y)) = y$.
(3) Suppose first that f is bijective. The inverse function $f^{-1} : Y \to X$ is a left inverse for f because given $x \in X$ we can let $y = f(x)$ and then $f^{-1}(y) = x$ so that $f^{-1}(f(x)) = f^{-1}(y) = x$. Similarly, f^{-1} is a right inverse for f because given $y \in Y$ we can let x be the unique element in X with $y = f(x)$ and then we have $x = f^{-1}(y)$ so that $f(f^{-1}(y)) = f(x) = y$. Conversely, suppose that g is a left inverse for f and h

is a right inverse for f. Since f has a left inverse, it is injective by (1). Since f has a right inverse, it is surjective by (2). Since f is injective and surjective, it is bijective. As shown above, the inverse function f^{-1} is both a left inverse and a right inverse. Finally, note that $g = f^{-1} = h$ because for all $y \in Y$ we have

$$g(y{=}g(f(f^{-1}(y))) = f^{-1}(y) = f^{-1}(f(h(y))) = h(y)$$

■

Corollary 1.3
Let X and Y be sets. Then there exists an injective map $f : X \to Y$ if and only if there exists a surjective map $g : Y \to X$.

Proof. Suppose $f : X \to Y$ is an injective map. Then f has a left inverse. Let g be a left inverse of f. Since $g \circ f = I_X$, we see that f is a right inverse of g. Since g has a right inverse, g is surjective. Thus, there is a surjective map $g : Y \to X$. Similarly, if $g : Y \to X$ is surjective, then it has a right inverse $f : X \to Y$ which is injective. ■

Definition 1.6 — same cardinality, less than or equal to, less than.
Let A and B be sets. We say that A and B have the **same cardinality**, and write $|A| = |B|$, when there exists a bijective map: $f : A \to B$ (or equivalently when there exists a bijective map $g : B \to A$).
We say that the cardinality of A is **less than or equal to** the cardinality of B, and write $|A| \leq |B|$, when there exists an injective map $f : A \to B$ (or equivalently a surjective map $g : B \to A$).
We say that the cardinality of A is **less than** the cardinality of B, and write $|A| < |B|$, when $|A| \leq |B|$ and $|A| \neq |B|$, (that is when there exists an injective map $f : A \to B$ but there does not exist a bijective map $g : A \to B$).
We also write $|A| \geq |B|$ when $|B| \leq |A|$; and $|A| > |B|$ when $|B| < |A|$.

■ **Example 1.1** Let $\mathbb{N} = \{n \in \mathbb{Z} | n \geq 0\} = \{0, 1, 2, \cdots\}$.
(1) The map $f : \mathbb{N} \to 2\mathbb{N}$ given by $f(k) = 2k$ is bijective, so $|2\mathbb{N}| = |\mathbb{N}|$.
(2) The map $g : \mathbb{N} \to \mathbb{Z}$ given by $g(2k) = k$ and $g(2k+1) = -k-1$ for $k \in \mathbb{N}$ is bijective, so we have $|\mathbb{Z}| = |\mathbb{N}|$.
(3) The map $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $h(k, l) = 2^k(2l+1) - 1$ is bijective, so we have $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Theorem 1.4 For all sets A, B and C
(1) $|A| = |A|$
(2) If $|A| = |B|$ then $|B| = |A|$
(3) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$
(4) $|A| \leq |B| \iff (|A| = |B| \text{ or } |A| < |B|)$
(5) If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$

Proof.
(1) holds because the identity function $I_A : A \to A$ is bijective.
(2) holds because if $f : A \to B$ is bijective then so is $f^{-1} : B \to A$.
(3) holds because if $f : A \to B$ and $g : B \to C$ are bijective then so is the composite $g \circ f : A \to C$

■

Definition 1.7 — finite, infinite, countable.
Let A be a set. For each $n \in \mathbb{N}$, let $S_n = \{0, 1, 2, \cdots, n-1\}$. For $n \in \mathbb{N}$, we say that the cardinality of A is equal to n, or that A has n **elements**, and we write $|A| = n$, when $|A| = |S_n|$.
We say that A is **finite** when $|A| = n$ for some $n \in \mathbb{N}$. We say A is **infinite** when A is not finite. We say that A is **countable** when $|A| = |\mathbb{N}|$

Note 1.1 When a set A is finite with $|A| = n$, and when $f : A \to S_n$ is a bijection, if we let $a_k = f^{-1}(k)$ for each $k \in S_n$ then we have $A = \{a_0, a_1, \cdots, a_{k-1}\}$ with the elements a_k distinct. Conversely, if $A = \{a_0, a_1, \cdots, a_{k-1}\}$ with the elements a_k all distinct, then we define a bijection $f : A \to S_n$ by $f(a_k) = k$. Thus we see that A is finite with $|A| = n$ if and only if A is of the form $A = \{a_0, a_1, \cdots, a_{n-1}\}$ with the elements a_k all distinct. Similarly, a set A is countable if and only if A is of the form $A = \{a_0, a_1, a_2, \cdots\}$ with the elements a_k all distinct.

Note 1.2 For $n \in \mathbb{N}$, if A is a finite set with $|A| = n + 1$ and $a \in |A \backslash \{a\}| = n$. Indeed, if $A = \{a_0, a_1, \cdots, a_n\}$ with the elements a_i distinct, and if $a = a_k$ so that we have $A \backslash \{a\} = \{a_0, a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n\}$, then we can define a bijection $f : S_n \to A \backslash \{a\}$ by $f(i) = a_i$ for $0 \le i < k$ and $f(i) = a_{i+1}$ for $k \le i < n$.

Theorem 1.5 Let A be a set. Then the following are equivalent:
(1) A is infinite
(2) A contains a countable subset
(3) $|\mathbb{N}| \le |A|$
(4) There exists a map $f : A \to A$ which is injective but not surjective

Proof.
$(1) \implies (2)$ Suppose A is infinite. Since $A \neq \varnothing$ we can choose an element $a_0 \in A$. Since $A \neq \{a_0\}$ we can choose an element $a_1 \in A \backslash \{a_0\}$. Since $A \neq \{a_0, a_1\}$ we can choose $a_3 \in A \backslash \{a_0, a_1\}$. Continue this procedure: having chosen distinct elements $a_0, a_1, \cdots, a_{n-1} \in A$, since $A \neq \{a_0, a_1, \cdots, a_{n-1}\}$ we can choose $a_n \in A \backslash \{a_0, a_1, \cdots, a_{n-1}\}$. In this way we obtain $\{a_0, a_1, a_2, \cdots\} \subseteq A$.

$(2) \iff (3)$ Suppose that A contains a countable subset, say $\{a_0, a_1, a_2, \cdots\} \subseteq A$ with the element a_i distinct. Since a_i are distinct, the map $f : \mathbb{N} \to A$ given by $f(k) = a_k$ is injective, and so we have $|\mathbb{N}| \le |A|$. Conversely as a map from $\mathbb{N} \to f(\mathbb{N})$ where f is bijective, so we have $|\mathbb{N}| = |f(\mathbb{N})|$ hence $f(\mathbb{N})$ is a countable subset of A.

$(2) \implies (4)$ Suppose that A has a countable subset, say $\{a_0, a_1, a_2, \cdots\} \subseteq A$ with the element a_i distinct. Define $f : A \to A$ by $f(a_k) = a_{k+1}$ for all $k \in \mathbb{N}$ and by $f(b) = b$ for all $b \in A \backslash \{a_0, a_1, a_2, \cdots\}$. Then f is injective but not surjective (the element a_0 is not in the range of f).

$(4) \implies (1)$ To prove this we shall prove that if A is finite then every injective map $f : A \to A$ is surjective. We prove this by induction on the cardinality of A.
The only set A with $|A| = 0$ is the set $A \neq \varnothing$, and then the only function $f : A \to A$ is the empty function, which is surjective.
Since that base case may appear too trivial, let us consider the next case. Let $n = 1$ and let A be a set with $|A| = 1$, say $A = \{a\}$. The only function $f : A \to A$ is the function given by $f(a) = a$, which is surjective.
Let $n \ge 1$ and suppose, inductively, that for every set A with $|A| = n$, every injective

map $f : A \to A$ is surjective. Let B be a set with $|B| = n+1$ and let $g : B \to B$ be injective.
Suppose, for a contradiction, that g is not surjective. Choose an element $b \in B$ which is not in the range of g so that we have $g : B \to B \backslash \{b\}$. Let $A = B\backslash\{b\}$ and let $f : A \to A$ be given by $f(x) = g(x)$ for all $x \in A$. Since $g : B \to A$ is injective and $f(x) = g(x)$ for all $x \in A$, f is also injective. Again since g is injective, there is no element $x \in B\backslash\{b\}$ with $g(x) = g(b)$, so there is no element $x \in A$ with $f(x) = g(b)$, and so f is not surjective. Since $|A| = n$, this contradicts the induction hypothesis. Thus g must be surjective.
By the Principle of Induction, for every $n \in \mathbb{N}$ and for every set A with $|A| = n$, every injective function $f : A \to A$ is surjective.

■

Corollary 1.6
Let A and B be sets.
(1) If A is countable then A is infinite
(2) When $|A| \le |B|$, if B is finite so is A (equivalently if A is infinite then so is B)
(3) If $|A| = n$ and $|B| = m$ then $|A| = |B|$ if and only if $n = m$
(4) If $|A| = n$ and $|B| = m$ then $|A| \le |B|$ if and only if $n \le m$
(5) When one of the two sets A and B is finite, if $|A| \le |B|$ and $|B| \le |A|$ then $|A| = |B|$

Proof.
(1) If A is countable then A contains a countable subset (itself), so A is infinite by Theorem 1.5.
(2) Suppose that $|A| \le |B|$ and that $|A|$ is infinite. Since A is infinite, we have $|\mathbb{N}| \le |A|$ (by Theorem 1.5). Since $|\mathbb{N}| \le |A|$ and $|A| \le |B|$ we have $|\mathbb{N}| \le |B|$ (by Theorem 1.4). Since $|\mathbb{N}| \le |B|$, B is infinite (by Theorem 1.5).
(3) Suppose that $|A| = n$ and $|B| = m$. If $n = m$ then we have $S_n = S_m$ and so $|A| = |S_n| = |S_m| = |B|$. Conversely, suppose that $|A| = |B|$. Suppose, for a contradiction, that $n \ne m$, say $n > m$, and note that $S_m \subsetneq S_n$. Since $|A| = |B|$ we have $|S_n| = |A| = |B| = |S_m|$ so we must have $n = m$.
(4) Suppose $|A| = n$ and $|B| = m$. If $n \le m$ then $S_n \subseteq S_m$ so the inclusion map $I : S_n \to S_m$ is injective and we have $|A| = |S_n| \le |S_m| = |B|$. Conversely, suppose that $|A| \le |B|$ and suppose, for a contradiction, that $n > m$. Since $|A| \le |B|$ we have $|S_n| = |A| \le |B| = |S_m|$ so we can choose an injective map $f : S_n \to S_m$. Since $n > m$ we have $S_m \subsetneq S_n$ so we can consider f as a map $f : S_n \to S_m$, and this map is injective but not surjective. This contradicts Theorem 1.5, and so $n \le m$.
(5) Suppose that one of the two sets A and B is finite, and that $|A| \le |B|$ and $|B| \le |A|$. If $A$is finite then, since $|B| \le |A|$, (2) implies that B is finite. If B is finite then, since $|A| \le |B|$, (2) implies that A is finite. Thus, in either case, we see that A and B are both finite. Since A and B are both finite with $|A| \le |B|$ and $|B| \le |A|$, we must have $|A| = |B|$ by (3) and (4).

■

Theorem 1.7 Let A be a set. Then $|A| \le |\mathbb{N}| \iff A$ is finite or countable.

Proof. First we claim that every subset of $\mathbb{N}$ is either finite or countable. Let $A \subseteq \mathbb{N}$ and suppose that A is not finite.
Since $A \ne \varnothing$, we can set $a_0 = \min\{A\}$ (using the Well-Ordering Property of $\mathbb{N}$). Note that

$\{0, 10, \cdots, a_0\} \cap A = \{a_0\}$.
Since $A \neq \{a_0\}$ (so the set $A \backslash \{a_0\}$ is nonempty), we can set $a_1 = \min\{A \backslash \{a_0\}\}$. Then we have $a_0 < a_1$ and $\{0, 1, \cdots, a_1\} \cap A = \{a_0, a_1\}$.
Since $A \neq \{a_0, a_1\}$ we can set $a_2 = \min\{A \backslash \{a_0, a_1\}\}$. Then we have $a_0 < a_1 < a_2$ and $\{0, 1, 2, \cdots, a_3\} \cap A = \{a_0, a_1, a_2\}$
We continue the procedure: having chosen $a_0, a_1, \cdots, a_{n-1} \in A$ with $a_0 < a_1 < \cdots < a_{n-1}$ such that $\{0, 1, \cdots, a_{n-1}\} \cap A = \{a_0, a_1, \cdots, a_{n-1}\}$. Since $A \neq \{a_0, a_1, \cdots, a_{n-1}\}$, we can set $a_n = \min\{A \backslash \{a_0, a_1, \cdots, a_{n-1}\}\}$ and then we have $a_0 < a_1 < \cdots < a_{n-1} < a_n$ and $\{0, 1, \cdots, a_n\} \cap A = \{a_0, a_1, \cdots, a_n\}$.
In this way, we obtain a countable set $\{a_0, a_1, a_2, \cdots\} \subseteq A$ with $a_0 < a_1 < a_2 < \cdots$ with the property that for all $m \in \mathbb{N}$, $\{0, 1, 2, \cdots, a_m\} \cap A = \{a_0, a_1, \cdots, a_m\}$.
Since $0 \leq a_0 < a_1 < a_2 < \cdots$, it follows (by induction) that $a_k \geq k$ for all $k \in \mathbb{N}$. It follows in turn that $A \subseteq \{a_0, a_1, a_2, \cdots\}$ because given $m \in A$, since $m \leq a_m$ we have

$$m \in \{0, 1, 2, \cdots, m\} \cap A \subseteq \{0, 1, 2, \cdots, a_m\} \cap A = \{a_0, a_1, \cdots, a_m\}.$$

Thus $A = \{a_0, a_1, a - 2, \cdots\}$ and the elements a_i are distinct, so A is countable. This proves our claim that every subset of $\mathbb{N}$ is either finite or countable.
Suppose that $|A| \leq |\mathbb{N}|$ and choose an injective map $f : A \to \mathbb{N}$. Since f is injective, when we consider it as a map $f : A \to f(A)$, it is bijective, and so $|A| = |f(A)\}$. Since $f(A) \subseteq \mathbb{N}$, the previous paragraph shows that $f(A)$ is either finite or countable. If $f(A)$ is finite with $|f(A)| = n$ then $|A| = |f(A)| = |S_n|$, and if $f(A)$ is countable then we have $|A| = |f(A)| = |\mathbb{N}|$. Thus A is finite or countable.

■

Theorem 1.8 Let A be a set. Then
(1) $|A| < |\mathbb{N}| \iff A$ is finite
(2) $|\mathbb{N}| < |A| \iff A$ is neither finite nor countable
(3) if $|A| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |A|$ then $|A| = |\mathbb{N}|$

Proof.
(1) By Theorem 1.5

$$\begin{aligned} |A| < |\mathbb{N}| &\iff (|A| \leq |\mathbb{N}| \text{ and } |A| \neq |\mathbb{N}|) \\ &\iff (A \text{ if finite or countable and } A \text{ is not countable}) \\ &\iff A \text{ is finite} \end{aligned}$$

(2) By Theorem 1.7

$$\begin{aligned} |\mathbb{N}| < |A| &\iff (|\mathbb{N}| \leq |A| \text{ and } |\mathbb{N}| \neq |A|) \\ &\iff (A \text{ is not finite and } A \text{ is not countable}) \end{aligned}$$

(3) Suppose that $|A| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |A|$. Since $|A| \leq |\mathbb{N}|$, we know that A is finite or countable by Theorem 1.7. Since $|N| \leq |A|$, we know that A is infinite by Theorem 1.5. Since A is finite or countable and A is not finite, it follows that A is countable. Thus $|A| = |\mathbb{N}|$

■

Definition 1.8 — at most countable, uncountable.
Let A be a set. When A is countable we write $|A| = \aleph_0$. When A is finite we write $|A| < \aleph_0$. When A is infinite we write $|A| \geq \aleph_0$. When A is either finite or countable we write $|A| \leq \aleph_0$ and we say that A is **at most countable**. When A is neither finite nor

countable we write $|A| > \aleph_0$ and we say that A is **uncountable**.

Theorem 1.9
(1) If A and B are countable sets, then so is $A \times B$
(2) If A and B are countable sets, then so is $A \cup B$
(3) If $A_0, A_1, A_2, \cdots$ are countable sets, then so is $\bigcap_{k=0}^{\infty} A_k$
(4) $\mathbb{Q}$ is countable

Proof.
(1) Let $A = \{a_0, a_1, a_2, \cdots\}$ with the a_i distinct and let $B = \{b_0, b_1, b_2, \cdots\}$ with b_i distinct. Since every positive integer can be written uniquely in the form $2^k(2l+1)$ with $k, l \in \mathbb{N}$, the map $f : A \times B \to \mathbb{N}$ given by $f(a_k, b_l) = 2^k(2l+1) - 1$ is bijective, and so $|A \times B| = |\mathbb{N}|$
(2) Similar to (1), since the map $g : \mathbb{N} \to A \cup B$ given by $g(k) = a_k$ is injective, we have $|\mathbb{N}| \leq |A \cup B|$. Since the map $h : \mathbb{N} \to A \cup B$ given by $h(2k) = a_k$ and $h(2k+1) = b_k$ is surjective, we have $|A \cup B| \leq |\mathbb{N}|$. Since $|\mathbb{N}| \leq |A \cup B|$ and $|A \cup B| \leq |\mathbb{N}|$, we have $|A \cup B| = |\mathbb{N}|$ by Theorem 1.8
(3) For each $k \in \mathbb{N}$, let $A_k = \{a_{k0}, a_{k1}, a_{k2}, \cdots\}$ with the a_{ki} distinct. Since the map $f : \mathbb{N} \to \bigcap_{k=0}^{\infty} A_k$ given by $f(k) = a_{0,k}$ is injective, $|\mathbb{N}| \leq \left|\bigcap_{k=0}^{\infty} A_k\right|$. Since $\mathbb{N} \times \mathbb{N}$ is countable by (1), and since the map $g : \mathbb{N} \times \mathbb{N} \to \bigcap_{k=0}^{\infty} A_k$ given by $g(k, l) = a_{k,l}$ is surjective, we have $\left|\bigcap_{k=0}^{\infty} A_k\right| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. By Theorem 1.8, we have $\left|\bigcap_{k=0}^{\infty} A_k\right| = |\mathbb{N}|$.
(4) Since the map $f : \mathbb{N} \to \mathbb{Q}$ given by $f(k) = k$ is injective, we have $|\mathbb{N}| \leq |\mathbb{Q}|$. Since the map $g : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ given by $g(\frac{a}{b}) = (a, b)$ for all $a, b \in \mathbb{Z}$ with $b > 0$ and $\gcd(a, b) = 1$, is injective, and since $\mathbb{Z} \times \mathbb{Z}$ is countable, we have $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$. Since $|\mathbb{N}| \leq |\mathbb{Q}|$ and $|\mathbb{Q}| \leq |\mathbb{N}|$, we have $|\mathbb{Q}| = |\mathbb{N}|$

■

Exercise 1.1 Let A be a countable set. Show that the set of finite sequences with terms in A is countable. Show that the set of all finite subsets of A is countable.

Definition 1.9 — power set.
For a set A, let $\mathcal{P}(A)$ denote the **power set** of A, that is the set of all subsets of A, and let 2^A denote the set of all functions from A to $S_2 = \{0, 1\}$

Theorem 1.10
(1) For every set A, $\mathcal{P}(A) = |2^A|$
(2) For every set A, $|A| < \mathcal{P}(A)$
(3) $\mathbb{R}$ is uncountable

Proof.
(1) Let A be any set. Define a map $g : \mathcal{P}(A) \to 2^A$ as follows: given $S \in \mathcal{P}(A)$, that is given $S \subseteq A$, we define $g(S) \in 2^A$ to be the map $g(S) : A \to \{0, 1\}$ given by

$$g(S)(a) = \begin{cases} 1 & \text{if } a \in S, \\ 0 & \text{if } a \notin S. \end{cases}$$

Define map $h : 2^A \to \mathcal{P}(A)$ as follows: given $f \in 2^A$, that is given a map: $f : A \to \{0,1\}$, we define $h(f) \in (A)$ to be the subset

$$h(f) = \{a \in A | f(a) = 1\} \subseteq A$$

This maps g and h are the inverses of each other because for every $S \subseteq A$ and every $f : A \to \{0,1\}$ we have

$$\begin{aligned} f = g(S) &\iff \forall a \in A, f(a) = g(S)(a) \iff \forall a \in A, f(a) = \begin{cases} 1 & \text{if } a \in S, \\ 0 & \text{if } a \notin S. \end{cases} \\ &\iff \forall a \in A, (f(a) = 1 \iff a \in S) \iff \{a \in A | f(a) = 1\} = S \\ &\iff h(f) = S \end{aligned}$$

(2) Let A be any set. Since the map $f : A \to \mathcal{P}(A)$ given by $f(a) = \{a\}$ is injective, we have $|A| \leq |\mathcal{P}(A)|$. We need to show that $|A| \neq |\mathcal{P}(A)|$. Let $g : A \to \mathcal{P}(A)$ be any map. Let $S = \{a \in A | a \notin g(a)\}$. Note that S cannot be in the range of g because we could choose $a \in A$ so that $g(a) = S$ then, by the definition of S, we would have

$$a \in S \iff a \notin g(a) \iff a \notin S$$

which is impossible. Since S is not in the range of g, the map g is not surjective. Since g was an arbitrary map from A to $\mathcal{P}(A)$, it follows that there is no surjective map from A to $\mathcal{P}(A)$. Thus there is no bijective map from A to $\mathcal{P}(A)$ and so we have $|A| \neq |\mathcal{P}(A)|$.

(3) We prove $\mathbb{R}$ is uncountable using the fact that every real number has a unique decimal expansion which does not end with an infinite string of 9's. Define a map $g : 2^{\mathbb{N}} \to \mathbb{R}$ as follows: given $f \in 2^{\mathbb{N}}$, that is given a map $f : \mathbb{N} \to \{0,1\}$, we define $g(f)$ to be the real number of $g(f) \in [0,1)$ with the decimal expansion $g(f) = 0.f(1)f(2)f(3)\cdots$, that is $g(f) = \sum_{k=0}^{\infty} f(k)10^{-k-1}$. By the uniqueness of decimal expansions, the map g is injective, so we have $|2^{\mathbb{N}}| \leq |\mathbb{R}|$. Thus $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}| \leq |\mathbb{R}|$, and so $\mathbb{R}$ is uncountable by Theorem 1.8.

■

Theorem 1.11 — Cantor-Schroeder-Bernstein.
Let A and B be sets. Suppose that $|A| \leq |B|$ and $|B| \leq |A|$. Then $|A| = |B|$

Proof. We sketch a proof. Choose injective functions $f : A \to B$ and $g : B \to A$. Since the functions $f : A \to f(A)$, $g : B \to g(B)$ and $f : g(B) \to f(g(B))$ are bijective, we have $|A| = |f(A)|$ and $|B| = |g(B)| = |f(g(B))|$. Also note that $f(g(B)) \subseteq f(A) \subseteq B$. Let $X = f(g(B))$, $Y = f(A)$ and $Z = B$. Then we have $X \subseteq Y \subseteq Z$ and we have $|x| = |z|$ and we need to show that $|Y| = |Z|$. The composite $h = f \circ g : Z \to X$ is a bijective.
Define sets Z_n and Y_n for $n \in \mathbb{N}$ recursively by

$$Z_0 = Z, Z_n = h(Z_{n-1}) \text{ and } Y_0 = Y, Y_n = h(Y_{n-1})$$

Since $Y_0 = Y$, $Z_0 = Z$, $Z_1 = h(Z_0) = h(Z) = X$ and $X \subseteq Y \subseteq Z$, we have

$$Z_1 \subseteq Y \subseteq Z_0$$

Also note that for $1 \leq n \in \mathbb{N}$,

$$Z_n \subseteq Y_{n-1} \subseteq Z_{n-1} \implies h(Z_n) \subseteq h(Y_{n-1}) \subseteq h(Z_{n-1}) \implies Z_{n+1} \subseteq Y_n \subseteq Z_n$$

By the Induction Principle, it follows that $Z_n \subseteq Y_{n-1} \subseteq Z_{n-1}$ for all $n \geq 1$, so we have

$$Z_0 \supseteq Y_0 \supseteq Z_1 \supseteq Y_1 \supseteq Z_2 \supseteq Y_2 \supseteq \cdots$$

Let $U_n = \frac{Z_n}{Y_n}$, $U = \bigcup_{n=0}^{\infty} U_n$ and $V = \frac{Z}{U}$. Define $H : Z \to Y$ by

$$H(x) = \begin{cases} h(x) & \text{if } x \in U \\ x & \text{if } x \in V \end{cases}$$

Verify that H is bijective. ∎

Exercise 1.2 Show that $|\mathbb{R}| = |2^{\mathbb{N}}|$

Solution. $g : 2^{\mathbb{N}} \to \mathbb{R}$ as follows: for $f \in 2^{\mathbb{N}}$ we let $g(f)$ be the real number $g(f) \in [0,1)$ with decimal expansion $g(f) = 0.f(1)f(2)\cdots$. Then g is injective so $|2^{\mathbb{N}}| \leq \mathbb{R}$. Define $h : 2^{\mathbb{N}} \to [0,1)$ as follows: for $f \in 2^{\mathbb{N}}$ let $h(f)$ be the real number $h(f) \in [0,1]$ with binary expansion $h(f) = 0.f(0)f(1)f(2)\cdots$. Then h is surjective so we have $|[0,1]| \leq |2^{\mathbb{N}}|$. The map $k : \mathbb{R} \to [0,1]$ given by $k(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$ is injective, so we have $|\mathbb{R}| \leq |[0,1]|$. Since $|\mathbb{R}| \leq |[0,1]| \leq |2^{\mathbb{N}}|$ and $|2^{\mathbb{N}}| \leq \mathbb{R}$, we have $|\mathbb{R}| = |2^{\mathbb{N}}|$ by the Cantor-Schroeder-Bernstein Theorem (1.11) ∎

Notation 1.1 For sets A and B, we write A^B to denote the set of functions $f : B \to A$

Theorem 1.12 Let A and B be finite sets and let $\mathcal{P}(A)$ is the power set of A (that is the set of all subsets of A). Then

(1) if A and N are disjoint then $|A \cup B| = |A| + |B|$
(2) $|A \times B| = |A| \cdot |B|$
(3) $|A^B| = |A|^{|B|}$
(4) $|\mathcal{P}| = 2^{|A}$

Proof. The proof is left as an exercise ∎

Theorem 1.13 Let A, B, C and D be sets with $|A| = |C|$ and $|B| = |D|$. Then

(1) if $A \cap B = \varnothing$ and $C \cap D = \varnothing$ then $|A \cup B| = |C \cup D|$
(2) $|A \times B| = |c \times D|$
(3) $|A^B| = |C^D|$

Proof. The proof is left as an exercise ∎

R It is possible to define certain specific sets called **cardinals** such that for every set A there exists a unique cardinal κ with $|A| = |\kappa|$. We can then define the **cardinality** of a set A to be equal to the unique cardinal κ such that $|A| = ||$ and, in this case, we define the **cardinality** of the set A to be $|A| = \kappa$. In foundational set theory, the natural numbers are defined, formally, to be equal to the sets $0 = \varnothing$, $1 = \{0\} = \{\varnothing\}$, $2 = \{0,1\} = \{\varnothing, \{\varnothing\}\}$ and, in general,$n + 1 = n \cup \{n\}$ so that the natural number n is equal to the set that we previously denoted by S_n, that is $n = S_n = \{0, 1, \cdots, n-1\}$. The finite cardinals are equal to the natural numbers and the countable cardinal $\aleph_0$ is equal to the set of natural numbers. The previous theorem allows us to define **arithmetic operations** on cardinals which extend the usual arithmetic operations on the natural numbers. Given cardinals κ and λ we define $\kappa + \lambda$, $\kappa \cdot \lambda$ and κ^λ to be the cardinals such that

- $\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$

- $\kappa \cdot \lambda = |\kappa \times \lambda|$
- $\kappa^\lambda = |\kappa^\lambda|$

Theorem 1.14 Let κ, λ and μ be cardinals. Then

(1) $\kappa + \lambda = \lambda + \kappa$
(2) $(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu)$
(3) $\kappa + 0 = \kappa$
(4) $\lambda \leq \mu \implies \kappa + \lambda \leq \kappa + \mu$
(5) $\kappa \cdot \lambda = \lambda \cdot \kappa$
(6) $(\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu)$
(7) $\kappa \cdot 1 = \kappa$
(8) $\kappa \cdot (\lambda + \mu) = (\kappa \cdot \lambda) + (\kappa \cdot \lambda)$
(9) $\lambda \leq \mu \implies \kappa \cdot \lambda \leq \kappa \cdot \mu$
(10) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$
(11) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$
(12) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$
(13) $\lambda \leq \mu \implies \kappa^\lambda \leq \kappa^\mu$
(14) $\kappa \leq \lambda \implies \kappa^\mu \leq \lambda^\mu$

Proof. We sketch a proof for (9) and (11) and leave the rest as an exercise.

(9) Let A, B and C be sets with $|A| = \kappa$, $|B| = \lambda$ and $|C| = \mu$ and suppose that $|B| \leq |C|$. We need to show that $|A \times B| \leq |A \times C|$. Let $f : B \to C$ be an injective map. Define $F : A \times B \to A \times C$ by $F(a, b) = (a, f(b))$ then verify that F is injective.

(11) Let A, B and C be sets with $|A| = \kappa$, $|B| = \lambda$ and $|C| = \mu$. We need to show $|(A^B)^C| = |A^{B \times C}|$. Define $F : (A^B)^C \to A^{B \times C}$ by $F(f)(b, c) = f(c)(b)$. Verify that F is bijective with inverse $G : A^{B \times C} \to (A^B)^C$ given by $G(g)(c)(b) = g(b, c)$

■

Exercise 1.3 Show that $\left| \bigcup_{n=0}^{\infty} \mathbb{R}^n \right| = 2^{\aleph_0}$

Exercise 1.4 Find $\left|\mathbb{R}^{[0,1]}\right|$

2. Metric Spaces

Definition 2.1 — inner product, orthogonal, homomorphism, isomorphism.
isomorphism Let $F = \mathbb{R}$ or $\mathbb{C}$. Let U be a vector space over F. An **inner product** on U (over F) is function $\langle \ , \ \rangle : U \times U \to F$ (meaning that if $u, v \in U$ then $\langle u, v \rangle \in F$) such that for all $u, v, w \in U$ and all $t \in F$ we have

(Sesquilinearity)

(1) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle, \langle tu, v \rangle = t \langle u, v \rangle$
$\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle, \langle u, tv \rangle = \bar{t} \langle u, v \rangle$

(2) (Conjugate Symmetry)
$\langle u, v \rangle = \overline{\langle v, u \rangle}$

(3) (Positive Definition)
$\langle u, u \rangle \geq 0$ with $\langle u, u \rangle = 0 \iff u = 0$

For $u, v \in U$, $\langle u, v \rangle$ is called the **inner product** of u with v. We say that u and v are **orthogonal** when $\langle u, v \rangle = 0$. An **inner product space** (over F) is a vector space over F equipped with an inner product. Given two inner product spaces U and V over F, a linear map $L : U \to V$ is called a **homomorphism** of inner product spaces (or we say that L preserves inner product) when $\langle L(x), L(y) \rangle = \langle x, y \rangle$ for all $x, y \in U$. A bijection homomorphism is called an **isomorphism**.

Definition 2.2 — norm (length).
Let U be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$. For $u \in U$, we define the **norm** (or **length**) of u to be

$$\|u\| = \sqrt{\langle u, u \rangle}$$

Theorem 2.1 Let U be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$. For $u, v \in U$ and $t \in F$ we have

(1) (Scaling) $\|tu\| = |t| \|u\|$
(2) (Positive Definiteness) $\|u\| \geq 0$ with $\|u\| = 0 \iff u = 0$
(3) $\|u + v\|^2 = \|u\|^2 + 2\text{Re} \langle u, v \rangle + \|v\|^2$

(4) (Polarization Identity) if $F = \mathbb{R}$ then $\langle u, v\rangle = \frac{1}{4}(\|u+v\| - \|u-v\|)$ and if $F = \mathbb{C}$ then $\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 + i\|u+iv\|^2 - \|u-v\| - i\|u-iv\|^2)$
(5) (The Cauchy-Schwarz Inequality) $|\langle u, v\rangle| \leq \|u\|\|v\|$ with $|\langle u, v\rangle| = \|u\|\|v\|$ if and only if $\{u, v\}$ is linearly dependent
(6) (The Triangle Inequality) $|\|u\| - \|v\|| \leq \|u\| + \|v\|$

Proof. The first 4 parts are easy to prove.

(5) Suppose that $\{u, v\}$ is linearly dependent. Then one of x and y is a multiple of the other, say $v = tu$ with $t \in F$. Then we have $|\langle u, v\rangle| = |\langle u, tu\rangle| = |\bar{t}\langle u, u\rangle| = |t|\|u\|^2 = \|u\|\|tu\| = \|u\|\|v\|$. Next suppose that $\{u, v\}$ is linearly independent. Then $1 \cdot v + t \cdot u \neq 0$ for all $t \in F$, so in particular $v - \frac{\langle v,u\rangle}{\|u\|^2}u \neq 0$. Thus we have

$$\begin{aligned} 0 &< \|v - \frac{\langle v, u\rangle}{\|u\|^2}u\|^2 = \left\langle v - \frac{\langle v, u\rangle}{\|u\|^2}u, v - \frac{\langle v, u\rangle}{\|u\|^2}u \right\rangle \\ &= \langle v, v\rangle - \overline{\frac{\langle v, u\rangle}{\|u\|^2}}\langle v, u\rangle - \frac{\langle v, u\rangle}{\|u\|^2}\langle u, v\rangle + \frac{\langle v, u\rangle}{\|u\|^2}\overline{\frac{\langle v, u\rangle}{\|u\|^2}}\langle u, u\rangle \\ &= \|v\|^2 - \frac{|\langle v, u\rangle|^2}{\|u\|^2} \end{aligned}$$

So that $\frac{|\langle u,v\rangle|^2}{|u|^2} < |v|^2$ and hence $|\langle u, v\rangle| \leq |u||v|$

(6) Using (3) and (5), and the inequality $|\mathrm{Re}(z)| \leq |z|$ for $z \in \mathbb{C}$ (which follows Pythagoras' Theorem in $\mathbb{R}^2$), we have

$$\begin{aligned} \|u+v\|^2 &= \|u\|^2 + \mathrm{Re}\langle u, v\rangle + \|v\|^2 \leq \|u\|^2 + 2|\langle u, v\rangle| + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\| = (\|u\| + \|v\|)^2 \end{aligned}$$

Taking the square root on both sides gives $\|u+v\| \leq \|u\| + \|v\|$. Finally note that $\|u\| = \|(u+v) - v\| \leq \|u+v\| + \|-v\| = \|u+v\| + \|v\|$ so that we have $\|u\| - \|v\| \leq \|u+v\|$, and similarly $\|v\| - \|u\| \leq \|u+v\|$, hence $|\|u\| - \|v\|| \leq \|u+v\|$

■

Definition 2.3 — norm, unit vector, normed linear space, homomorphism, isomorphism. Let $F = \mathbb{R}$ or $\mathbb{C}$. Let U be a vector space over F. A **norm** on U is a function $\| \, \| : U \to \mathbb{R}$ (meaning that if $u \in U$ then $\|u\| \in \mathbb{R}$) such that for all $u, v \in U$ and all $t \in F$ we have
(1) (Scaling) $\|tu\| = |t|\|u\|$
(2) (Positive Definitenes) $\|u\| \geq 0$ with $\|u\| = 0 \iff u = 0$
(3) (Triangle Inequality) $\|u+v\| \leq \|u\| + \|v\|$

For $u \in U$ the real number $\|u\|$ is called the **norm** (or **length**) of u, and we say that u is a **unit vector** when $\|u\| = 1$. A **normed linear space** (over F) is a vector space equipped with a norm. Given two normed linear spaces U and V over F, a linear map $L : U \to V$ is called a homomorphism of normed linear spaces (or we say that L preserves norm) when $\|L(x)\| = \|x\|$ for all $x \in U$. A bijection homomorphism is called an **isomorphism**.

Definition 2.4 — distance. Let $F = \mathbb{R}$ or $\mathbb{C}$ and let U be a normed linear space over F. For $u, v \in U$, we define the **distance** between u and v to be

$$d(u, v) = \|v - u\|$$

Theorem 2.2 Let U be a normed linear space over $F = \mathbb{R}$ or $\mathbb{C}$. For all $u, v, w \in U$
(1) (Symmetry) $d(u,v) = d(v,u)$
(2) (Positive Definiteness) $d(u,v) \geq 0$ with $d(u,v) = 0 \iff u = v$
(3) (Triangle Inequality) $d(u,w) \leq d(u,v) + d(v,w)f$

Proof. The proof is left as exercise ■

Definition 2.5 — metric, distance, metric space, homomorphism, isomorphism.
Let X be a non-empty set. A **metric** on X is a map $d : X \times X \to \mathbb{R}$ such that for all $a, b, c \in X$ we have
(1) (Symmetry) $d(a,b) = d(b,a)$
(2) (Positive Definiteness) $d(a,b) \geq 0$ with $d(a,b) = 0 \iff a = b$
(3) (Triangle Inequality) $d(a,c) \leq d(a,b) + d(b,c)$
For $a, b \in X$, $d(a,b)$ is called the **distance** between a and b. A **metric space** is a set X which is equipped with a metric d, and we sometimes denote the metric space by X and sometimes by the pair (X,d). Given two metric spaces (X, d_X) and (Y, d_Y),, a map $f : X \to Y$ is called a homomorphism of metric spaces (or we say that f is **distance preserving**) when $d_Y(f(a), f(b)) = d_X(a,b)$ for all $a, b \in X$. A bijective homomorphism is called an **isomorphism** or an **isometry**.

Note 2.1 Every inner product space is also a normed linear space, using the induced norm given by $\|u\| = \sqrt{\langle u, u\rangle}$. Every normed linear space is also a metric space, using the induced metric given by $d(u,v) = \|u - v\|$. If U is an inner product space over $\mathbf{F} = \mathbb{R}$ or $\mathbb{C}$ then every subspace of U is also an inner product space using (the restriction of) the same inner product used in U. If U is a normed linear space over $\mathbf{F} = \mathbb{R}$ or $\mathbb{C}$ then every subspace of U is also a normed linear space using the same norm. If X is a metric space then so is every subset of X using the same metric.

■ **Example 2.1** Let $\mathbf{F} = \mathbb{R}$ or $\mathbb{C}$. The **standard inner product** on $\mathbf{F}^n$ is given by

$$\langle u, v\rangle = v * u = \sum_{i=1}^{n} u_i \overline{v_i}$$

The standard inner product induces the **standard norm** on $\mathbf{F}^n$, which is also called the **2-norm** on $\mathbf{F}^n$, given by

$$\|u\|_2 = \|u\| = \sqrt{\langle u, u\rangle} = \left(\sum_{i=1}^{n} |u_i|^2\right)^{\frac{1}{2}}$$

The standard norm on $\mathbf{F}^n$ induces the **standard metric** on $\mathbf{F}^n$, given by

$$d_2(u,v) = d(u,v) = \|v - u\| = \left(\sum_{i=1}^{n} |v_i - u_i|^2\right)^{\frac{1}{2}}$$

The **1-norm** on $\mathbf{F}^n$ is given by

$$\|u\|_1 = \sum_{i=1}^{n} |u_i|$$

and it induces the **1-metric** on $\mathbf{F}^n$ given by $d_1(u,v) = \|v - u\|_1$. The **supremum norm** also called **infinity norm**, on $\mathbf{F}^n$ is given by

$$\|u\|_\infty = \max\{|u_1|, |u_2|, \cdots, |u_n|\}$$

and it induces the **supremum metric** on $\mathbf{F}^n$ given by $d_\infty(u,v) = \|v-u\|_\infty$

Example 2.2 For $\mathbf{F} = \mathbb{R}$ or $\mathbb{C}$. We write

$$\begin{aligned}\mathbf{F}^\omega &= \{u = (u_1, u_2, u_3, \cdots) | \text{ each } u_i \in \mathbf{F}\}\\ \mathbf{F}^\infty &= \{u \in \mathbf{F}^\omega | \text{ there exists } n \in Z^+ \text{ such that } u_k = 0 \text{ for all } k \geq n\}\end{aligned}$$

Recall that $\mathbf{F}^\infty$ is a countable-dimensional vector space with standard basis $\{e_1, e_2, e_3, \cdots\}$ where $e_1 = (1,0,0,\cdots), e_2 = (0,1,0,\cdots)$ and so on. The **standard inner product** on $\mathbf{F}^\infty$ is given by

$$\langle u, v\rangle = \sum_{i=1}^{\infty} u_i \overline{v_i}$$

and it induces the **standard norm**, also called the **2-norm**, on $\mathbf{F}^\infty$ given by

$$\|u\|_2 = \sqrt{\langle u, v\rangle} = \left(\sum_{i=1}^{n} |u_i|^2\right)^{\frac{1}{2}}$$

The **1-norm** on $\mathbf{F}^\infty$ is given by

$$\|u\|_1 = \sum_{i=1}^{\infty} |u_i|$$

and it induces the **1-metric** on $\mathbf{F}^\infty$ given by $d_1(u,v) = \|v-u\|_1$. The **supremum norm** also called the **infinity norm**, on $\mathbf{F}^n$ is given by

$$\|u\|_\infty = \max\{|u_1|, |u_2|, \cdots, |u_n|\}$$

and it induces the **supremum metric** on $\mathbf{F}^n$ given by $d_\infty(u,v) = \|v-u\|_\infty$

Example 2.3 For $\mathbf{F} = \mathbb{R}$ or $\mathbb{C}$, the standard inner product, the 1-norm, the 2-norm and the ∞-norm, which are well defined on the vector space $\mathbf{F}^\infty$, do not extend naturally to give a well defined inner product or well-defined norms on the vector space $\mathbf{F}^\omega$ (because the relevant sums do not necessarily converge). But we can, and do, extend there definitions to various subspaces of $\mathbf{F}^\omega$. We define

$$\begin{aligned}\ell_1(\mathbf{F}) &= \{u \in \mathbf{F}^\omega \mid \sum_{i=1}^{\infty} |u_i| < \infty\}\\ \ell_2(\mathbf{F}) &= \{u \in \mathbf{F}^\omega \mid \sum_{i=1}^{\infty} |u_i|^2 < \infty\}\\ \ell_\infty(\mathbf{F}) &= \{u \in \mathbf{F}^\omega \mid \sup\{|u_1|, |u_2|, \cdots\} < \infty\}\end{aligned}$$

Verify that $\ell_1(\mathbf{F})$ is a normed linear space using **1-norm** given by $\|u\|_1 = \sum_{i=1}^{\infty} |u_i|$, hence $\ell_1(\mathbf{F})$ is also a metric space using the **1-metric** $d_1(u,v) = \|v-u\|_1$. Verify that $\ell_\infty(\mathbf{F})$ is a normed linear space using the **supremum norm**, also called the **infinity norm**, given by $\|u\|_\infty = \sup\{|u_1|, |u_2|, \cdots\}$, hence $\ell_\infty(\mathbf{F})$ is also a metric space using the **supremum metric** $d_\infty = \|v-u\|_\infty$. Verify that $\ell_2(\mathbf{F})$ is an inner product space using the **standard inner product** given by $\langle u, v\rangle = \sum_{i=1}^{\infty} u_i\overline{v_i}$. The standard inner product on $\ell_2(\mathbf{F})$ induces the **standard norm**, also called the **2-norm**, on $\ell_2(\mathbf{F})$ given by $\|u\|_2 = \left(\sum_{i=1}^{\infty} |u_i|^2\right)^{\frac{1}{2}}$ and the **standard metric**, or the **2-metric**, $d_2(u,v) = \|v-u\|_2$.
Since we shall usually work with the field $\mathbf{F} = \mathbb{R}$, for $p = 1, 2$ or ∞ we shall write

$$\ell_p = \ell_p(\mathbb{R})$$

■ **Example 2.4** For $\mathbf{F} = \mathbb{R}$ or $\mathbb{C}$ and for $a.b \in \mathbb{R}$ with $a \le b$, we write

$$\begin{aligned}\mathcal{F}([a,b],\mathbf{F}) &= \mathbf{F}^{[a,b]} = \{f : [a,b] \to \mathbf{F}\}\\ \mathcal{B}([a,b],\mathbf{F}) &= \{f : [a,b] \to \mathbf{F} \mid f \text{ is bounded}\}\\ \mathcal{C}([a,b],\mathbf{F}) &= \{f : [a,b] \to \mathbf{F} \mid f \text{ is continuous}\}\end{aligned}$$

Recall that for $f : [a,b] \to \mathbb{C}$ given by $f = u + iv$ where $u, v : [a,b] \to \mathbb{R}$, the function f is continuous if and only if both u and v are continuous and, in this case, $\int_a^b f = \int_a^b + i \int_a^b v$. In the space $\mathcal{C}([a,b],\mathbf{F})$ we have the **1-norm**, the **2-norm**, and the **supremum norm**

$$\begin{aligned}\|f\|_1 &= \int_a^b |f|\\ \|f\|_2 &= \left(\int_a^b |f|^2\right)^{\frac{1}{2}}\\ \|f\|_\infty &= \sup_{a \le x \le b} |f(x)|\end{aligned}$$

The supremum norm also gives a well-defined norm on the space $\mathcal{B}([a,b],\mathbf{F})$. The 2-norm on $\mathcal{C}([a,b],\mathbf{F})$ is induced by the inner product $\mathcal{C}([a,b],\mathbf{F})$ given by

$$\langle f, g\rangle = \int_a^b f\overline{g}$$

Since we shall usually work with the field $\mathbf{F} = \mathbb{R}$m we shall write

$$\mathcal{F}[a,b] = \mathcal{F}([a,b],\mathbb{R})\ ,\ \mathcal{B}[a,b] = \mathcal{B}([a,b],\mathbb{R}) \text{ and } \mathcal{C} = \mathcal{C}([a,b],\mathbb{R})$$

R For $\mathbf{F} = \mathbb{R}$ or $\mathbb{C}$ and for $1 \le p < \infty$, one can show that we can define a norm on $\mathbf{F}^n$ by

$$\|u\|_p = \left(\sum_{i=1}^n |u_i|^p\right)^{\frac{1}{p}}$$

and we can define a norm on $\mathbf{F}^\infty$ or on the space $\ell_\infty(\mathbf{F}) = \{u \in \mathbf{F}^\omega | \sum_{i=1}^\infty |u_i|^p < \infty\}$ by

$$\|u\|_p = \left(\sum_{i=1}^\infty |u_i|^p\right)^{\frac{1}{p}}$$

Also, we can define a norm on the space $\mathcal{C}([a,b],\mathbf{F})$ by

$$\|f\|_p = \left(\int_{i=a}^b |f|^p\right)^{\frac{1}{p}}$$

■ **Example 2.5** For any set $X \ne \varnothing$, the **discrete metric** on X is given by $d(x,y) = 1$ for all $x, y \in X$ with $x \ne y$ and $d(x,x) = 0$ for all $x \in X$.

Definition 2.6 — open ball, closed ball, punctured ball, bounded.
Let X be a metric space. For $a \in X$ and $0 < r \in \mathbb{R}$, the **open ball**, the **closed ball** and the (open) **punctured ball** in X centered at a of radius r are defined to be the sets

$$\begin{aligned}B(a,r) = B_X(a,r) &= \{x \in X \mid d(x,a) < r\}\\ \overline{B}(a,r) = \overline{B}_X(a,r) &= \{x \in X \mid d(x,a) \le r\}\\ B^*(a,r) = B^*_X(a,r) &= \{x \in X \mid 0 < d(x,a) < r\}\end{aligned}$$

When the metric on X denoted by d_p with $1 \le p \le \infty$, we often write $B(a,r)$, $\overline{B}(a,r)$ and $B^*(a,r)$ as $B_p(a,r)$, $\overline{B}_p(a,r)$ and $B_p^*(a,r)$. For $A \subseteq X$, we say that A is **bounded** when $A \subseteq B(a,r)$ for some $a \in X$ and some $0 < r \in \mathbb{R}$.

Exercise 2.1 Draw a picture of the open balls $B_1(0,1)$, $B_2(0,1)$ and $B_\infty(0,1)$ in $\mathbb{R}^2$ (using the metrics d_1, d_2 and d_∞).

Definition 2.7 — open, closed.
Let X be a metric space. For $A \subseteq X$, we say that A is **open** (in X) when for every $a \in A$ there exists $r > 0$ such that $B(a,r) \subseteq A$, and we say that A is closed (in X) when its complement $A^c = \mathbb{R}^n \backslash A$ is open in $\mathbb{R}^n$.

■ **Example 2.6** Let X be a metric space. Show that for $a \in X$ and $0 < r \in \mathbb{R}$, the set $B(a,r)$ is open and the set $\overline{B}(a,r)$ is closed.

Proof. Let $a \in X$ and Let $r > 0$. We claim that $B(a,r)$ is open. We need to show that for all $b \in B(a,r)$ there exists $s > 0$ such that $B(b,s) \subseteq B(a,r)$. Let $b \in B(a,r)$ and note that $d(a,b) < r$. Let $s = r - d(a,b)$ and note that $s > 0$. Let $x \in B(b,s)$, so we have $d(x,b) < s$. Then, by the Triangle Inequality, we have

$$d(x,a) \le d(x,b) + d(b,a) < s + d(a,b) = r$$

and so $x \in B(a,r)$. This shows that $B(b,s) \subseteq B(a,r)$ and hence $B(a,r)$ is open.
Next we claim that $\overline{B}(a,r)$ is closed, that is $\overline{B}(a,r)^c$ is open. Let $b \in \overline{B}(a,r)^c$, that is let $b \in X$ with $b \notin \overline{B}(a,r)$. Since $b \notin \overline{B}(a,r)$ we have $d(a,b) > r$. Let $s = d(a,b) - r > 0$. Let $x \in B(b,s)$ and not that $d(x,b) < s$. Then, by the Triangle Inequality, we have

$$d(a,b) \le d(a,x) + d(x,b) < d(x,a) + s$$

and so $d(x,a) > d(a,b) - s = r$. Since $d(x,a) > r$ we have $b \notin \overline{B}(a,r)$ and so $x \in \overline{B}(a,r)^c$. This shows that $B(b,s) \subseteq \overline{B}(a,r)^c$ and it follows that $\overline{B}(a,r)^c$ is open and hence that $\overline{B}(a,r)$ is closed. ■

Theorem 2.3 — Basic Properties of Open Sets.
Let X be a metric space
(1) The sets $\varnothing$ and X are open in X
(2) If S is a set of open sets in X then the union $\bigcup S = \bigcup_{U \in S} U$ is open in X
(3) If S is a finite set of open sets in X then the intersection $\bigcap S = \bigcap_{U \in S} U$ is open in X

Proof. (1) The empty set is open because any statement of the form "for all $x \in \varnothing, F$" (where F is any statement) is considered to be true (by convention). The set X is open because given $a \in X$ we can choose any value of $r > 0$ and then $B(a,r) \subseteq X$ by the definition of $B(a,r)$.

(2) Let S be any set of open sets in X. Let $a \in \bigcup S = \bigcup_{U \in S} U$. Choose an open set $U \in S$ such that $a \in U$. Since U is open we can choose $r > 0$ such that $B(a,r) \subseteq U$. Since $U \in S$ we have $U \subseteq \bigcup S$. Since $B(a,r) \subseteq U$ and $U \subseteq \bigcup S$ we have $B(a,r) \subseteq \bigcup S$. Thus $\bigcup S$ is open.

(3) Let S be a finite set of open sets in X. If $S \neq \varnothing$ then we use the convention that $\bigcap S = X$, which is open. Suppose that $S \neq \varnothing$. say $S = \{U_1, U_2, \cdots, U_m\}$ where

each U_k is an open set. Let $a \in \bigcap S = \bigcap_{k=1}^m U_k$. For each index k, since $a \in U_k$ we can choose $r_k > 0$ so that $B(a, r_k) \subseteq U_k$. Let $r = \min\{r_1, r_2, \cdots, r_m\}$. Then for each index k we have $B(a,r) \subseteq B(a, r_k) \subseteq U_k$. Since $B(a,r)U_k$ for every index k, it follows that $B(a,r) \subseteq \bigcap_{k=1}^m U_k = \bigcap S$

■

Theorem 2.4 — Basic Properties of Closed Sets.
Let X be a metric space
(1) The sets $\varnothing$ and X are closed in X
(2) If S is a set of closed sets in X then the union $\bigcap S = \bigcap\limits_{U \in S} U$ is open in X
(3) If S is a finite set of closed sets in X then the intersection $\bigcup S = \bigcup\limits_{U \in S} U$ is open in X

Proof. The proof is left as exercise ■

Definition 2.8 — topology, topology space, metric topology, finer, coarser.
A **topology** on a set X is a set T of subsets of X such that
(1) $\varnothing \in T$ and $X \in T$
(2) For every set $S \subseteq T$ we have $\bigcup S \in T$
(3) For every finite subset $S \subseteq T$ we have $\bigcap S \in T$
A **topology space** is a set X with a topology T. When X is a metric space, the set of all open sets in X is a topology on X. which we call the **metric topology** (or the topology **induced** by the metric). When X is any topological space, the sets in the topology T are called the **open sets** in X and their complements are called the **closed sets** in X. When S and T are both topologies on a set X with $S \subseteq T$, we say that the topology T is **finer** than the topology S, and the topology S is **coarser** than the topology T.

■ **Example 2.7** Show that in $\mathbb{R}^n$, the metrics d_1, d_2 and d_∞ all induce the same topology

Proof. For $a, x \in \mathbb{R}^n$ we have

$$\max_{a \le i \le n} |x_i - a_i| \le (\sum_{i=1}^n |x_i - a_i|^2)^{\frac{1}{2}} \le \sum_{i=1}^n |x_i - a_i| \le n \max_{a \le i \le n} |x_i - a_i|$$

and so

$$d_\infty(a,x) \le d_2(a,x) \le d_1(a,x) \le n d_\infty(a,x).$$

It follows that for all $a \in \mathbb{R}^n$ and $r > 0$ we have

$$B_\infty(a,x) \supseteq B_2(a,r) \supseteq B_1(a,r) \supseteq B_\infty(a, \frac{r}{n}).$$

Thus for $U \subseteq \mathbb{R}^n$, if U is open in $\mathbb{R}^n$ using d_∞ then it is open using d_2. and if U is open using d_2 then it is open using d_1, and if U is open using d_1 then it is open using d_∞. ■

■ **Example 2.8** Show that on the space $\mathcal{C}[a,b]$, the topology induced by the metric d_∞ is strictly finer than the topology induced by the metric d_1

Proof. For $f, g \in \mathcal{C}[a,b]$ we have

$$d_1(f,g) = \int_a^b |f - g| \le \int_a^b \max_{a \le x \le b} |f(x) - g(x)| = (b-a) d_\infty(f,g)$$

It follows that for $f \in \mathcal{C}[a,b]$ and $r > 0$ we have

$$B_\infty(f,r) \subseteq B_1(f,(b-a)r)$$

Thus for $U \subseteq \mathcal{C}[a,b]$, if U is open using d_1 then U is also open using d_∞, and so the topology induced by the metric d_∞ is finer (or equal to) the topology induced by d_1.
On the other hand, we claim that for $f \in \mathcal{C}[a,b]$ and $r > 0$, the set $B_\infty(f,r)$ is not open in the topology induced by d_1. Fix $g \in B_\infty(f,r)$ and let $s > 0$. Choose a bump function $h \in \mathcal{C}[a,b]$ with $h \geq 0$, $\int_a^b h < h$ and $\max\limits_{a \leq x \leq b} h(x) > 2r$. Then we have $g + h \in B_1(g,s)$ but $g + h \notin B_\infty(f,r)$. It follows that $B_\infty(f,r)$ is not open in the topology induced by d_1, as claimed. ■

■ **Example 2.9** For any set X, the **trivial topology** on X is the topology in which the only open sets in X are the sets $\varnothing$ and X, and the **discrete topology** on X is the topology in which every subset of X is open. Note that the discrete metric on a nonempty set X induces the discrete topology on x.

Definition 2.9 — interior, closure, dense.
Let X be a metric space (or a topological space) and let $A \subseteq X$. The **interior** and the **closure** of A (in X) are the sets

$$A^\circ = \bigcup \{U \subseteq X \mid U \text{ is open, and } U \subseteq A\}$$

$$\overline{A} = \bigcap \{K \subseteq X \mid K \text{ is closed, and } A \subseteq K\}$$

We say that A is **dense** in X when $\overline{A} = X$.

Theorem 2.5 Let X be a metric space (or a topological space) and let $A \subseteq X$
(1) The interior of A is the largest open set which is contained in A. In other words, $A^\circ \subseteq A$ and A° is open, and for every open set U with $U \subseteq A$ we have $U \subseteq A^\circ$
(2) The closure of A is the smallest closed set which contains A. In other words, $A \subseteq \overline{A}$ and $\overline{A}$ is closed, and for every closed set K with $A \subseteq K$ we have $\overline{A} \subseteq K$

Proof.
(1) Let $L = \{U \subseteq X \mid U \text{ is open, and } U \subseteq A\}$. Note that A° is open (by Part 2 of Theorem 2.3 or by Part 2 of Definition 2.8) because A° is equal to the union if S, which is a set of open sets. Also note that $A^\circ \subseteq A$ because A° is equal to the union of S, which is a set of subsets of A. Finally note that for any open set U with $U \subseteq A$ we have $U \in S$ so that $U \subseteq \bigcup S = A^\circ$.
(2) The proof is similar to (1)

■

Corollary 2.6
Let X be a metric space (or a topological space) and let $A \subseteq X$
(1) $(A^\circ)^\circ = A^\circ$ and $\overline{\overline{A}} = A$
(2) A is open $\iff$ $A = A^\circ$
(3) A is closed $\iff$ $A = \overline{A}$

Proof. The proof is left as exercise ■

Definition 2.10 — interior point, limit point, isolated point, boundary point, boundary. isolated point Let X be a metric space and let $A \subseteq X$.
An **interior point** of A is a point $a \in A$ such that for some $r > 0$ we have $B(a, r) \subseteq A$.
A **limit point** of A is a point $a \in X$ such that for every $r > 0$ we have $B^*(a, r) \cap A \neq \varnothing$.
An **isolated point** of A is a point $a \in A$ which is not a limit point of A.
A **boundary point** of A is a point of A is a point $a \in X$ such that for every $r > 0$ we have $B(a, r) \cap A \neq \varnothing$ and $B(a, r) \cap A^c \neq \varnothing$.
The set of all limit points of A is denoted by A'. The **boundary** of A, is the set of all boundary points of A.

Theorem 2.7 — Properties of Interior, Limit and Boundary Points. Let X be a metric space and let $A \subseteq X$
(1) A° is equal to the set of all interior points of A
(2) A is closed $\iff A' \subseteq A$
(3) $\overline{A} = A \cup A'$
(4) $\partial A = \overline{A} \setminus A^\circ$

Proof. We leave the proofs of (1) and (4) as exercise.
(2) Note that when $a \notin A$ we have $B(a, r) \cap A = B^*(a, r) \cap A$ and so

$$\begin{aligned}
A \text{ is closed} &\iff A^c \text{ is open} \\
&\iff \forall a \in A^c,\ \exists r > 0,\ B(a, r) \subseteq A^c \\
&\iff \forall a \in \mathbb{R}^n,\ (a \notin A \implies \exists r > 0,\ B(a, r) \subseteq A^c) \\
&\iff \forall a \in \mathbb{R}^n,\ (a \notin A \implies \exists r > 0,\ B(a, r) \cap A = \varnothing) \\
&\iff \forall a \in \mathbb{R}^n,\ (a \notin A \implies \exists r > 0,\ B^*(a, r) \cap A = \varnothing) \\
&\iff \forall a \in \mathbb{R}^n,\ (\forall r > 0, B^*(a, r) \cap A \neq \varnothing \implies a \in A) \\
&\iff \forall a \in \mathbb{R}^n,\ (a \in A' \implies a \in A) \\
&\iff A' \subseteq A.
\end{aligned}$$

(3) We shall prove that $A \cup A'$ is the smallest closed set which contains A. It is clear that $A \cup A'$ is closed, that is $(A \cup A')^c$ is open. Let $a \in (A \cup A')^c$ with, that is let $a \in X$ with $a \notin A$ and $a \notin A'$. Since $a \notin A'$ we can choose $r > 0$ so that $B(a, r) \cap A = \varnothing$. We claim that because $B(a, r) \cap A = \varnothing$ it follows that $B(a, r) \cap A'$. Since $b \in B(a, r)$ and $B(a, r)$ is open, we can choose $s > 0$ so that $B(b, s) \subseteq B(a, r)$. Since $b \in A'$ it follows that $B(b, s) \cap A \neq \varnothing$. Choose $x \in B(b, s) \cap A$. Then we have $x \in B(b, s) \subseteq B(a, r)$ and $x \in A$ and so $x \in B(a, r) \cap A$, which contradicts the fact that $B(a, r) \cap A = \varnothing$. Thus $B(a, r) \cap A' \neq \varnothing$ as claimed. Since $B(a, r) \cap A = \varnothing$ and $B(a, r) \cap A' = \varnothing$, it follows that $B(a, r) \cap (A \cup A') = \varnothing$, hence $B(a, r) \subseteq (A \cup A')^c$. Thus proves that $(A \cup A')^c$ is open, and hence $A \cup A'$ is closed.
It remains to show that for every closed set K in X with $A \subseteq K$ we have $A \cup A' \subseteq K$. Let K be a closed set in X with $A \subseteq K$. Note that since $A \subseteq K$ it follows that $A' \subseteq K'$ because if $a \in A'$ then for all $r > 0$ we have $B(a, r) \cap A \neq \varnothing$ hence $B(a, r) \cap K \neq \varnothing$ and so $a \in K'$. Since K is closed we have $K' \subseteq K$ by (2). Since $A' \subseteq K'$ and $K' \subseteq K$, we have $A' \subseteq K$. Since AK and $A' \subseteq K$ we have $A \cup A' \subseteq K$ as required. ∎

R Let X be a topological space and let $A \subseteq X$, An **interior point** of A is a point $a \in A^\circ$. A **limit point** of A is a point $a \in X$ such that for every open set U in X with $a \in U$ there exists a point $b \in U \cap A$ with $b \neq a$. The **boundary** of A in X is the set $\partial A = \overline{A} \setminus A^\circ$, and a **boundary point** of A is a point $a \in \partial A$.

Note 2.2 Let X be a metric space and let $P \subseteq X$. Note that P is also a metric space using (the restriction of) the metric used in X. For $a \in P$ and $0 < r \in \mathbb{R}$, note that the open and closed balls in P, centered at a and of radius r, are related to the open and closed balls in X by

$$B_P(a,r) = \{x \in P \mid d(x,a) < r\} = B_X(a,r) \cap P$$

$$\overline{B}_P(a,r) = \{x \in P \mid d(x,a) < r\} = \overline{B}_X(a,r) \cap P$$

Theorem 2.8 Let X be a metric space and let $A \subseteq P \subseteq X$
(1) A is open in $P \iff$ there exists an open set U in X such that $A = U \cap P$
(2) A is closed in $P \iff$ there exists a closed set K in X such that $A = K \cap P$

Proof.
(1) Suppose that A is open in P. For each $a \in A$, choose $r_a > 0$ so that $B_P(a, r_a) \subseteq A$, that is $B_X(a, r_a) \cap P \subseteq A$, and let $U = \bigcup_{a \in A} B_X(a, r_a)$. Since U is equal to the union of a set of open sets in X, it follows that U is open in X. Note that $A \subseteq U \cap P$, and, since $B_X(a, r_a) \cap P \subseteq A$ for every $a \in A$, we also have $U \cap P = (\bigcup_{a \in U} B_X(a, r_a)) \cap P = \bigcup_{a \in A}(B_X(a, r_a) \cap P) \subseteq A$. Thus $A = U \cap P$ as required.
Conversely, suppose that $A = U \cap P$ with U open in X. Let $a \in A$. Since we have $a \in A = U \cap P$, we also have $a \in U$. Since $a \in U$ and U is open in X we can choose $r > 0$ so that $B_X(a, r) \subseteq U$. Since $B_X(a, r) \subseteq U$ and $U \cap P = A$ we have $B_P(a, r) = B_X(a, r) \cap P \subseteq U \cap P = A$. Thus A is open as required.
(2) Suppose that A is closed in P. Let B be the complement of A in P, that is $B = P \setminus A$. Then B is open in P. Choose an open set U in X such that $B = U \cap P$. Let K be the complement of U in X, that is $K = X \setminus U$. Then $A = K \cap P$ since for $x \in X$ we have

$$x \in A \iff (x \in P \text{ and } x \notin U \cap P) \iff (x \in P \text{ and } x \notin U)$$

$$\iff (x \in P \text{ and } x \in K) \iff (x \in K \cap P)$$

Conversely, suppose that K is a closed set in P with $A = K \cap P$. Let B be the complement of A in P, that is $B = P \setminus A$, and let U be the complement of K in P, that is $U = P \setminus K$, and note that U is open in P. Then we have $B = U \cap P$ since for $x \in P$ we have

$$x \in B \iff (x \in P \text{ and } x \notin A) \iff (x \in P \text{ and } x \notin K \cap P)$$

$$\iff (x \in P \text{ and } x \notin K) \iff (x \in P \text{ and } x \in U) \iff (x \in U \cap P)$$

Since U us open in P and $B = U \cap P$ we know that B is open in P. Since B is open in P, its complement $A = P \setminus B$ is closed in P.

■

R Let X be a topological space and let $P \subseteq X$. Verify, as an exercise, that we can use the topology on X to define a topology on P as follows. Given a set $A \subseteq P$, we define A to be **open** in P when $A = U \cap P$ for some open set U in X. The resulting topology on P is called the **subspace topology**.

3. Limits and Continuity

Definition 3.1 — bounded, converge, limit, diverge, Cauchy.
Let $(x_n)_{n\geq p}$ be a sequence in a metric space X. We say that the sequence $(x_n)_{n\geq p}$ is **bounded** when the set $\{x_n\}_{n\geq p}$ is bounded, that is when there exists $a \in X$ and $r > 0$ such that $x_n \in B(a,r)$ for all indices $n \geq p$

For $a \in X$, we say that the sequence $(x_n)_{n\geq p}$ **converges** to a (or that the **limit** of x_n is equal to a) and we write $\lim_{n\to\infty} x_n = a$ (or write $x_n \to a$) when for every $\epsilon > 0$ there exists an index $m \geq p$ such that $d(x_n, a) < \epsilon$ for all indices $n \geq m$. We say that the sequence $(x_n)_{n\geq p}$ **converges** (in X) when it converges to some point $a \in X$, and otherwise we say that $(x_n)_{n\geq p}$ **diverges** (in X).

We say that the $(x_n)_{n\geq p}$ is **Cauchy** when for every $\epsilon > 0$ there exists an index $m \geq p$ such that $d(x_k, x_l) < \epsilon$ for all indices $k, l \geq m$.

R When $(x_n)_{n\geq p}$ is a sequence in a topological space X and $a \in X$, we say that $(x_n)_{n\geq p}$ **converges** to a (or we say the **limit** of $(x_n)_{n\geq p}$ is equal to a) and we write $\lim_{n\to\infty} x_n = a$ (or we write $x_n \to a$) when for every open set U in X with $a \in U$ there exists an index $m \geq p$ such that $x_n \in U$ for every index $n \geq m$.

Theorem 3.1 — Basic Properties of Limits of Sequences. Let $(x_n)_{n\geq p}$ be a sequence in a metric space X, and let $a \in X$

(1) If $(x_n)_{n\geq p}$ converges then its limit is unique
(2) If $q \geq p$ and $y_n = x_n$ for all $n \geq q$, then $(x_n)_{n\geq p}$ converges if and only if $(y_n)_{n\geq p}$ converges and, in this case, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$
(3) If $(x_n)_{n\geq p}$ converges then it is bounded
(4) If $(x_n)_{n\geq p}$ converges then it is Cauchy
(5) We have $\lim_{n\to\infty} = a$ if and only if for every open set U in X with $a \in U$ there exists an index $m \geq p$ such that $x_n \in U$ for every index $n \geq m$

Proof. The proof is left as exercise. ■

Note 3.1 Because of Part 2 of the above theorem, the initial index p of a sequence $(x_n)_{n\geq p}$ does not affect whether or not the sequence converges and it does not affect the limit. For this reason, we often omit the initial index p from our notation and write (x_n) for the sequence $(x_n)_{n\geq p}$. Also, we often choose a specific value of p, usually $p=1$, in the statements or the proofs of various theorems with the understanding that any other initial value would work just as well.

Exercise 3.1 For each $n \in \mathbb{Z}^+$, let $x_n \in \mathbb{R}^\infty$ be the sequence given by $x_n = \frac{1}{n}\sum_{k=1}^{n} e_k$, that is by $(x_{n,k})_{k\geq 1} = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}, 0, 0, 0, \cdots)$ with n non-zero terms. Show that (x_n) converges in $(\mathbb{R}^\infty, d_1)$.

Exercise 3.2 For each $n \in \mathbb{Z}^+$, let $f_n \in \mathcal{C}[0,1]$ be given by $f_n(x) = \sqrt{n}x^n$. Show that $(f_n)_{n\geq 1}$ converges in $(\mathcal{C}[0,1], d_1)$ but diverges in $(\mathcal{C}[0,1], d_2)$.

Note 3.2 Recall that $\mathcal{B}[a,b]$ denotes the space of bounded functions $f:[a,b]\to\mathbb{R}$. Let (f_n) be a sequence of bounded functions in $\mathcal{B}[a,b]$ and let $g\in\mathcal{B}[a,b]$. Note that (f_n) converges in the metric space $(\mathcal{B}[a,b], d_\infty)$, if and only if (f_n) converges uniformly on $[a,b]$. Indeed for $\epsilon>0$ we have $d_\infty(f_n,g)<\epsilon$ if and only if $\sup_{a\leq x\leq b}\big|f_n(x)-g(x)\big|<\epsilon$ if and only if $\big|f_n(x)-g(x)\big|<\epsilon$ for all $x\in[a,b]$. The same is true for a sequence (f_n) in $\mathcal{C}[a,b]$: (f_n) converges in the metric space $(\mathcal{C}[a,b], d_\infty)$ if and only if (f_n) converges uniformly on $[a,b]$.

Theorem 3.2 — The Sequential Characterization of Limit Points and Closed Sets.
Let X be a metric space, let $a\in X$, and let $A\subseteq X$.

(1) $a\in A'$ if and only if there exists a sequence (x_n) in $A\setminus\{a\}$ with $\lim_{n\to\infty} x_n = a$.

(2) $a\in\overline{A}$ if and only if there exists a sequence (x_n) in A with $\lim_{n\to\infty} x_n = a$.

(3) A is closed in X if and only if for every sequence (x_n) in A which converges in X, we have $\lim_{n\to\infty} x_n\in A$.

Proof.

(1) Suppose that $a\in A'$ (which means that for every $r>0$ we have $B^*(a,r)\cap A\neq\varnothing$). For each $n\in\mathbb{Z}^+$, choose $x_n\in B^*\left(a,\frac{1}{n}\right)\cap A$, that is choose $x_n\in A\setminus\{a\}$ with $d(x_n,a)<\frac{1}{n}$. Then $(x_n)_{n\geq 1}$ is a sequence in $A\setminus\{a\}$ with $\lim_{n\to\infty}x_n=a$.
Suppose, conversely, that $(x_n)_{n\geq 1}$ is a sequence in $A\setminus\{a\}$ with $\lim_{n\to\infty} x_n = a$. Let $r>0$. Choose $m\in\mathbb{Z}^+$ such that $d(x_n,a)<r$ for all $n\geq m$. Since $x_m\in A\setminus\{a\}$ with $d(x_m,a)<r$, we have $x_m\in B^*(a,r)\cap A$ and so $B^*(a,r)\cap A\neq\varnothing$.

(2) Left as exercise

(3) To prove Part 3, suppose that A is closed in X. Let $(x_n)_{n\geq 1}$ be a sequence in A which converges in X, and let $a=\lim_{n\to\infty} x_n\in X$. Suppose, for a contradiction, that $a\notin A$. Since $a\notin A$ we have $A=A\setminus\{a\}$ so in fact (x_n) is a sequence in $A\setminus\{a\}$. Since (x_n) is a sequence in $A\setminus\{a\}$ with $\lim_{n\to\infty} x_n = a$, it follows from Part 1 that $a\in A'$. Since A is closed we have $A'\subseteq A$ and so $a\in A$ giving the desired contradiction.
Suppose, conversely, that for every sequence in A which converges in X, the limit

of the sequence lies in A. Let $a \in A'$. By Part 1, we can choose a sequence (x_n) in $A \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$. Then (x_n) is a sequence in A which converges in X, so its limit lies in A, that is $a \in A$. Since $a \in A'$ was arbitrary, this shows that $A' \subseteq A$, and so A is closed.

■

■ **Example 3.1** Note that $\mathcal{C}[a,b]$ is closed in the metric space $(\mathcal{B}[a,b], d_\infty)$. We can see this using Note 3.7 together with the above theorem. Indeed, given a sequence (f_n) with each $f_n \in \mathcal{C}[a,b]$, if the sequence (f_n) converges in $(\mathcal{B}[a,b], d_\infty)$ to the function $g \in \mathcal{B}[a,b]$, then (f_n) converges uniformly to g on $[a,b]$, and so (from MATH 148) we know that g must be continuous, hence $g \in \mathcal{C}[a,b]$.

Exercise 3.3 Let

$$\begin{aligned}
\mathcal{R}[a,b] &= \big\{ f \in \mathcal{B}[a,b] \,\big|\, f \text{ is Riemann integrable} \big\}, \\
\mathcal{P}[a,b] &= \big\{ f \in \mathcal{B}[a,b] \,\big|\, f \text{ is a polynomial} \big\}, \\
\mathcal{C}^1[a,b] &= \big\{ f \in \mathcal{B}[a,b] \,\big|\, f \text{ is continuously differentiable} \big\}.
\end{aligned}$$

Determine which of the above spaces are closed in the metric space $\mathcal{B}[a,b]$, using the supremum metric d_∞.

■ **Example 3.2** Recall that $\mathbb{R}^\infty$ denotes the set of sequences with only finitely many non-zero terms. Show that $\mathbb{R}^\infty$ is dense in the metric space (ℓ_1, d_1).

Proof. Since the closure of $\mathbb{R}^\infty$ in ℓ_1 is contained in ℓ_1 (by the definition of closure), it suffices to show that $\ell_1 \subseteq \overline{\mathbb{R}^\infty}$. Let $a = (a_n)_{n\geq 1} \in \ell_1$, so we have $\sum_{n=1}^{\infty} |a_n| < \infty$. For each $n \in \mathbb{Z}^+$ let $x_n = (x_{n,k})_{k\geq 1}$ be the sequence given by $x_{n,k} = a_k$ for $1 \leq k \leq n$ and $x_{n,k} = 0$ for $k > n$, that is

$$(x_{n,k})_{k\geq 1} = (x_{n,1}, x_{n,2}, \cdots, x_{n,n}, x_{n,n+1}, \cdots) = (a_1, a_2, \cdots, a_n, 0, 0, 0, \cdots).$$

Then each $x_n \in \mathbb{R}^\infty$ and, in the metric space ℓ_1, we have $x_n \to a$ because given $\epsilon > 0$ we can choose an index m so that $\sum_{k>m} |a_k| < \epsilon$ and then for all $n \geq m$ we have

$$\|x_n - a\|_1 = \sum_{k=1}^{\infty} |x_{n,k} - a_k| = \sum_{k>n} |a_k| \leq \sum_{k>m} |a_k| < \epsilon.$$

It follows, from Part 2 of Theorem 3.8, that $a \in \overline{\mathbb{R}^\infty}$ and so we have $\ell_1 \subseteq \overline{\mathbb{R}^\infty}$, as claimed. ■

Exercise 3.4 Find the closure of $\mathbb{R}^\infty$ in the metric space ℓ_2 using the metric d_2, and find the closure of $\mathbb{R}^\infty$ in the metric space ℓ_∞ using the metric d_∞.

Definition 3.2 — limit.
Let (X, d_X) and (Y, d_Y) be metric spaces. Let $A \subseteq X$, let $f : A \to Y$, let $a \in A'$, and let $b \in Y$. We say that the **limit** of $f(x)$ as x tends to a is equal to b, when for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in A$, if $0 < d_X(x,a) < \delta$ then $d_Y(f(x), b) < \epsilon$.

Theorem 3.3 — The Sequential Characterization of Limits.
Let X and Y be metric spaces, let $A \subseteq X$, let $f : A \to Y$, let $a \in A' \subseteq X$, and let $b \in Y$. Then $\lim\limits_{x \to a} f(x) = b$ if and only if for every sequence (x_n) in $A \setminus \{a\}$ with $x_n \to a$ we have $\lim\limits_{n\to\infty} f(x_n) = b$.

Proof. Suppose that $\lim\limits_{x\to A} f(x) = b$. Let (x_n) be a sequence in $A \setminus \{a\}$ with $x_n \to a$. Let $\epsilon > 0$. Since $\lim\limits_{x\to a} f(x) = b$ we can choose $\delta > 0$ such that $0 < d(x, a) < \delta \implies d(f(x), b) < \epsilon$. Since $x_n \to a$ we can choose $m \in \mathbb{Z}^+$ such that $n \ge m \implies d(x_n, a) < \delta$. For $n \ge m$ we have $d(x_n, a) < \delta$ and we have $x_n \neq a$ $(since (\mathrm{x}_n)$ is a sequence in $A \setminus \{a\}, so that 0 <$ d$(\mathrm{x}_n, a) < \delta$, and hence $d(f(x_n), b) < \epsilon$. Thus $\lim\limits_{n\to\infty} f(x_n) = b$, as required.

Suppose, conversely, that $\lim\limits_{x\to a} f(x) \neq b$. Choose $\epsilon > 0$ such that for every $\delta > 0$ there exists $x \in A$ such that $0 < d(x, a) < \delta$ and $d(f(x), b) \ge \epsilon$. For each $n \in \mathbb{Z}^+$, choose $x_n \in A$ such that $0 < d(x_n, a) < \frac{1}{n}$ and $d(f(x_n), b) \ge \epsilon$. For each n, since $0 < d(x_n, a)$ we have $x_n \neq a$ so the sequence (x_n) lies in $A \setminus \{a\}$. Since $d(x_n, a) < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$, it follows that $x_n \to a$. Since $d(f(x_n), b) \ge \epsilon$ for all $n \in \mathbb{Z}^+$, it follows that $\lim\limits_{x\to a} f(x) \neq b$. Thus we have found a sequence (x_n) in $A \setminus \{a\}$ with $x_n \to a$ such that $\lim\limits_{x\to a} f(x_n) \neq b$. ■

Definition 3.3 — continuous, uniformly continuous, Lipschitz continuous, Lipschitz constant.
Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$. For $a \in X$, we say that f is **continuous** at a when for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$, if $d_X(x, a) < \delta$ then $d_Y(f(x), f(a)) < \epsilon$. We say that f is **continuous** (on X) when f is continuous at every point $a \in X$. We say that f is **uniformly continuous** (on X) when for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$, if $d_X(x, y) < \epsilon$ then $d_Y(f(x), f(y)) < \epsilon$. We say that f is **Lipschitz continuous** (on X) when there is a constant $\ell \ge 0$, called a **Lipschitz constant** for f, such that for all $x, y \in X$ we. have $d(f(x), f(y)) \le \ell \cdot d(x, y)$. Note that if f is Lipschitz continuous then f is also uniformly continuous (indeed we can take $\delta = \frac{\epsilon}{\ell}$ in the definition of uniform continuity).

Note 3.3 Let X and Y be metric spaces and let $a \in X$. If a is a limit point of X then f is continuous at a if and only if $\lim\limits_{x\to a} f(x) = f(a)$. If a is an isolated point of X then f is necessarily continuous at a, vacuously.

Theorem 3.4 — The Sequential Characterization of Continuity.
Let X and Y be metric spaces using metrics d_X and d_Y, let $f : X \to Y$, and let $a \in X$. Then f is continuous at a if and only if for every sequence (x_n) in X with $x_n \to a$ we have $\lim\limits_{n\to\infty} f(x_n) = f(a)$.

Proof. The proof is left as an exercise. ■

Theorem 3.5 — Composition of Continuous Functions.
Let X, Y and Z be metric spaces, let $f : X \to Y$, let $g : Y \to Z$. If f is continuous at the point $a \in X$ and g is continuous at the point $f(a) \in Y$ then the composite function $g \circ f$ is continuous at a.

Proof. The proof is left as an exercise. ■

Theorem 3.6 — The Topological Characterization of Continuity.
Let X and Y be metric spaces and let $f : X \to Y$. Then f is continuous (on X) if an only if $f^{-1}(V)$ is open in X for every open set V in Y.

Proof. Suppose that f is continuous in X. Let V be open in Y. Let $a \in f^{-1}(V)$ and let $f(a) \in V$. Since V is open, we can choose $\epsilon > 0$ such that $B(f(a), \epsilon) \subseteq V$. Since f is continuous at a we can choose $\delta > 0$ such that for all $x \in X$ with $d(x,a) < \delta$ we have $d(f(x), f(a)) < \epsilon$. Then we have $f(B(a,\delta)) \subseteq B(f(a),\epsilon) \subseteq V$ and so $B(a,\delta) \subseteq f^{-1}(V)$. Thus $f^{-1}(V)$ is open in X, as required.

Suppose, conversely, that $f^{-1}(V)$ is open in X for every open set V in Y. Let $a \in X$ and let $\epsilon > 0$. Taking $V = B(f(a), \epsilon)$, which is open in Y, we see that $f^{-1}(B(f(a),\epsilon))$ is open in X. Since $a \in f^{-1}(B(f(a),\epsilon)$ and $f^{-1}(B(f(a),\epsilon))$ is open in X, we can choose $\delta > 0$ such that $B(a,\delta) \subseteq f^{-1}(B(f(a),\epsilon))$. Then we have $f(B(a,\delta)) \subseteq B(f(a),\epsilon)$ or, in other words, for all $x \in X$, if $d(x,a) < \delta$ then $d(f(x), f(a)) < \epsilon$. Thus f is continuous at a hence, since a was arbitrary, f is continuous on X. ∎

Definition 3.4 — continuous.
Let X and Y be topological spaces and let $f : X \to Y$. We say that f is **continuous** (on X) when $f^{-1}(V)$ is open in X for every open set V in Y. A bijective map $f : X \to Y$ such that both f and f^{-1} are continuous is called a **homomorphism**.

Note 3.4 If U and V are inner product spaces and $L : U \to V$ is an inner product space isomorphism, then L and its inverse preserve distance so they are both continuous (we can take $\delta = \epsilon$ in the definition of continuity), hence L is a homomorphism.

If U and V are finite-dimensional inner product spaces with say $\dim U = n$ and $\dim V = m$, and if $\phi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^m$ are inner product space isomorphisms (obtained by choosing orthonormal bases for U and V) then a map $F : U \to V$ is continuous if and only if the composite map $\psi F \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. In particular, if F is linear then F is continuous (since $\psi F \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is linear, hence continuous).

We shall see below that the same is true for finite dimensional normed linear spaces: every linear map between finite dimensional normed linear spaces is continuous. But this is not always true for infinite dimensional spaces.

■ **Example 3.3** Recall from Example 2.24 that every set $U \subseteq \mathcal{C}[a,b]$ which is open using the metric d_1 is also open using the metric d_∞, but not vice versa. It follows that the identity map $I : \mathcal{C} \to \mathcal{C}[a,b]$ given by $I(f) = f$ is continuous as a map from the metric space $(\mathcal{C}[a,b], d_\infty)$ to the metric space $(\mathcal{C}[a,b], d_1)$, but not vice versa.

Theorem 3.7 Let U and V be normed linear spaces and let $F : U \to V$ be a linear map. Then the following are equivalent:

(1) F is Lipschitz continuous on U,
(2) F is continuous at some point $a \in U$,
(3) F is continuous at 0, and
(4) $F\left(\overline{B}(0,1)\right)$ is bounded.

In this case, if $m \geq 0$ with $F\left(\overline{B}(0,1)\right) \subseteq B(0,m)$ then m is a Lipschitz constant for F.

Proof. It is clear that if F is Lipschitz continuous on U then F is continuous at some point $a \in U$ (indeed F is continuous at every point $a \in U$). Let us show that if F is continuous at some point $a \in U$ then F is continuous at 0. Suppose that F is continuous at $a \in U$.

Let $\epsilon > 0$. Since F is continuous at $a \in U$, we can choose $\delta_1 > 0$ such that for all $u \in U$ we have $\|u - a\| \leq \delta_1 \implies \|F(u) - F(a)\| \leq 1$. Choose $\delta = \delta_1 \epsilon$. Let $x \in U$ with $\|x - 0\| < \delta$. If $x = 0$ then $\|F(x) - F(0)\| = \|0\| = 0$. Suppose that $x \neq 0$. Then for $u = a + \frac{\delta_1 x}{\|x\|}$ we have $\|u - a\| = \left\|\frac{\delta_1 x}{\|x\|}\right\| = \delta_1$ and so $\|F(u-a)\| = \|F(u) - F(a)\| \leq 1$, that is $\left\|F\left(\frac{\delta_1 x}{\|x\|}\right)\right\| \leq 1$ hence, by the linearity of F and the scaling property of the norm, we have

$$\|F(x) - F(0)\| = \|F(x)\| = \frac{\|x\|}{\delta_1}\left\|F\left(\frac{\delta_1 x}{\|x\|}\right)\right\| \leq \frac{\|x\|}{\delta_1} < \frac{\delta_1 \epsilon}{\delta_1} = \epsilon.$$

Thus F is continuous at 0, as required

Next we show that if F is continuous at 0 then $F\left(\overline{B}(0,1)\right)$ is bounded. Suppose that F is continuous at 0. Choose $\delta > 0$ so that for all $u \in U$ we have $\|u\| \leq \delta \implies \|F(u)\| \leq 1$. Let $m = \frac{1}{\delta}$. For $x \in U$, when $x = 0$ we have $\|F(x)\| = 0 \leq m$ and when $0 < \|x\| \leq 1$ we have

$$\|F(x)\| = \left\|\frac{\|x\|}{\delta} F\left(\frac{\delta x}{\|x\|}\right)\right\| = \frac{\|x\|}{\delta}\left\|F\left(\frac{\delta x}{\|x\|}\right)\right\| \leq \frac{\|x\|}{\delta} = m\|x\| \leq m.$$

Thus $F\left(\overline{B}(0,1)\right)$ is bounded, as required.

Finally we show that if $F\left(\overline{B}(0,1)\right)$ is bounded then F is Lipschitz continuous. Suppose that $F\left(\overline{B}(0,1)\right)$ is bounded. Choose $m > 0$ so that $\|F(u)\| \leq m$ for all $u \in U$ with $\|u\| \leq 1$. Let $x, y \in U$. If $x = y$ then $\|F(x) - F(y)\| = 0$. Suppose that $x \neq y$. Then we have $\left\|\frac{x-y}{\|x-y\|}\right\| = 1$ so that $\left\|F\left(\frac{x-y}{\|x-y\|}\right)\right\| \leq m$ and so

$$\|F(x) - F(y)\| = \|F(x-y)\| = \|x-y\|\left\|F\left(\frac{x-y}{\|x-y\|}\right)\right\| \leq m\|x-y\|.$$

Thus F is Lipschitz continuous with Lipschitz constant m, as required. ■

■ **Example 3.4** Define $L : (\mathcal{C}[a,b], d_\infty) \to (\mathcal{C}[a,b], d_\infty)$ by $L(f) = \int_a^x f(t)dt$. Show that L is Lipschitz continuous.

Proof. Let $f \in \mathcal{C}[a,b]$ with $\|f\|_\infty \leq 1$, that is with $\max\limits_{a\leq x\leq b} |f(x)| \leq 1$. Then

$$\|F(f)\|_\infty = \max_{a\leq x\leq b}\left|\int_a^x f(t)\,dt\right| \leq \max_{a\leq x\leq b}\int_a^x 1\,dt = \max_{a\leq x\leq b}|x-a| = |b-a|.$$

Thus $F\left(\overline{B}(0,1)\right)$ is bounded and so F is uniformly continuous. ■

■ **Example 3.5** Define $D : \left(\mathcal{C}^1[0,1], d_\infty\right) \to (\mathcal{C}[0,1], d_\infty)$ by $D(f) = f'$. Show that D is not continuous.

Proof. For $n \in \mathbb{Z}^+$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$. Then $f_n \in \mathcal{C}^1[a,b]$, and $\|f_n\|_\infty = \max\limits_{0\leq x\leq 1}|x^n| = 1$ so that $f_n \in B(0,1)$, and $\|D(f_n)\|_\infty = \max\limits_{0\leq x\leq 1}\left|n\,x^{n-1}\right| = n$. Thus $D\left(\overline{B}(0,1)\right)$ is not bounded, so D is not continuous (at any point $g \in \mathcal{C}[0,1]$). ■

■ **Example 3.6** Let X be a metric space and let $\varnothing \neq A \subseteq X$. Define $F : X \to \mathbb{R}$ by

$$F(x) = \mathrm{dist}(x,A) = \inf\left\{d(x,a)\middle| a \in A\right\}.$$

Show that F is uniformly continuous.

Proof. Given $\epsilon > 0$, chose $\delta = \frac{\epsilon}{2}$. Let $x, y \in X$ with $d(x,y) < \delta = \frac{\epsilon}{2}$. Since $\mathrm{dist}(y,A) = \inf\left\{d(y,a)\middle| a \in A\right\}$ we can choose $a \in A$ such that $d(y,a) < \mathrm{dist}(y,A) + \frac{\epsilon}{2}$. Then we have

$$\mathrm{dist}(x,A) \leq d(x,a) \leq d(x,y) + d(y,a) < \frac{\epsilon}{2} + \mathrm{dist}(y,A) + \frac{\epsilon}{2}$$

so that $\operatorname{dist}(x,A)-\operatorname{dist}(y,A)<\epsilon$. Similarly, we have $\operatorname{dist}(y,A)-\operatorname{dist}(x,A)<\epsilon$ and so

$$\left|F(y)-F(x)\right|=\left|\operatorname{dist}(x,A)-\operatorname{dist}(y,A)\right|<\epsilon.$$

■

Theorem 3.8 Let U be an n-dimensional normed linear space over $\mathbb{R}$. Let $\{u_1,\cdots,u_n\}$ be any basis for U and let $F:\mathbb{R}\to U$ be the associated vector space isomorphism given by $F(t)=\sum\limits_{k=1}^{n}t_ku_k$. Then both F and F^{-1} are Lipschitz continuous.

Proof. Let $M=\left(\sum\limits_{k=1}^{n}\|u_k\|^2\right)^{1/2}$. For $t\in\mathbb{R}^n$ we have

$$\begin{aligned}\|F(t)\|&=\left\|\sum_{k=1}^{n}t_ku_k\right\|\le\sum_{k=1}^{n}|t_k|\,\|u_k\|\text{ , by the Triangle Inequality,}\\&\le\left(\sum_{k=1}^{n}{t_k}^2\right)^{1/2}\left(\sum_{k=1}^{n}\|u_k\|^2\right)^{1/2}\text{ , by the Cauchy-Schwarz Inequality,}\\&=M\|t\|.\end{aligned}$$

For all $s,t\in\mathbb{R}^n$, $\|F(s)-F(t)\|=\|F(s-t)\|\le M\,\|s-t\|$, so F is Lipschitz continuous.

Note that the map $N:U\to\mathbb{R}$ given by $N(x)=\|x\|$ is (uniformly) continuous, indeed we can take $\delta=\epsilon$ in the definition of continuity. Since F and N are both continuous, so is the composite $G=N\circ F:\mathbb{R}^n\to\mathbb{R}$, which given by $G(t)=\|F(t)\|$. By the Extreme Value Theorem, the map G attains its minimum value on the unit sphere $\left\{t\in\mathbb{R}^n\,\middle|\,\|t\|=1\right\}$, which is compact. Let $m=\min\limits_{\|t\|=1}G(t)=\min\limits_{\|t\|=1}\|F(t)\|$. Note that $m>0$ because when $t\neq0$ we have $F(t)\neq0$ (since F is a bijective linear map) and hence $\|F(t)\|\neq0$. For $t\in\mathbb{R}^n$, if $\|t\|>1$ then we have $\left\|\frac{t}{\|t\|}\right\|=1$ so, by the choice of m,

$$\|F(t)\|=\|t\|\left\|F\left(\frac{t}{\|t\|}\right)\right\|\ge\|t\|\cdot m>m.$$

It follows that for all $t\in\mathbb{R}^n$, if $\|F(t)\|\le m$ then $\|t\|\le1$. Since F is bijective, it follows that for $x\in U$, if $\|x\|\le m$ then $\left\|F^{-1}(x)\right\|\le1$. Thus for all $x\in U$, if $x=0$ then $\|F^{-1}(x)\|=0=\frac{\|x\|}{m}$ and if $x\neq0$ then since $\left\|\frac{mx}{\|x\|}\right\|=m$ we have

$$\left\|F^{-1}(x)\right\|=\frac{\|x\|}{m}\left\|F^{-1}\left(\frac{mx}{\|x\|}\right)\right\|\le\frac{\|x\|}{m}.$$

For all $x,y\in U$, we have $\left\|F^{-1}(x)-F^{-1}(y)\right\|=\left\|F^{-1}(x-y)\right\|\le\frac{1}{m}\|x-y\|$, so F^{-1} is Lipschitz continuous. ■

Corollary 3.9
When U and V are finite-dimensional normed linear spaces, every linear map $F:U\to V$ is Lipschitz continuous.

Corollary 3.10
Any two norms on a finite-dimensional vector space U induce the same topology on U.

4. Separability and Completeness

Definition 4.1 — dense, separable.
Let X be a topological space. Recall that for $A \subseteq X$ we say that A is **dense** in X when $\overline{A} = X$. We say that X is **separable** when it has a finite or countable dense subset.

Definition 4.2 — basis, base.
Let X be a topological space. A **basis** (or a **base**) for the topology on X is a set $\mathcal{B}$ of open sets in X with the property that for every subset $A \subseteq X$, A is open if and only if for every point $a \in A$ there exists a basic set $U \in \mathcal{B}$ with $a \in U \subseteq A$.

■ **Example 4.1** In a metric space X, the set of open balls $\mathcal{B} = \left\{ B(a,r) \middle| a \in X, 0 < r \in \mathbb{R} \right\}$ is a basis for the metric topology on X.

Theorem 4.1 Let X be a metric space. 1 If X is separable then there is a finite or countable basis for the metric topology on X.

2 If every infinite subset of X has a limit point then X is separable.

3 If X is separable then every subspace of X is separable.

Proof. The proof is left as an exercise. ■

■ **Example 4.2** Euclidean space $\left(\mathbb{R}^n, d_2\right)$ is separable with $\mathbb{Q}^n$ as a countable dense subset. Every subspace of Eucliden space is also separable.

■ **Example 4.3** As an exercise, show that $\left(\ell_\infty, d_\infty\right)$ is not separable (consider characteristic functions χ_A for subsets $A \subseteq \mathbb{N}$.

■ **Example 4.4** As an exercise, show that the set $\left(c, d_\infty\right)$ of convergent sequences of real (or complex) numbers is separable. Every subspace of c is also separable, for example the space c_0 of sequences which converge to 0.

■ **Example 4.5** As an exercise, show that the space $\left(\mathcal{B}[a,b], d_\infty\right)$ of bounded functions on the interval $[a,b]$ is not separable (consider characteristic functions χ_A for appropriate sets

$A \subseteq [a,b]\big)$.

■ **Example 4.6** Later (using the Weierstrass Approximation Theorem) we will show that the space $\big(\mathcal{C}[a,b], d_\infty\big)$ of continuous real (or complex) valued functions on the interval $[a,b]$ is separable. Once we have proven this, it will follow that every subspace of $\mathcal{C}[a,b]$ is separable.

Definition 4.3 — Cauchy sequence.
Recall that a sequence $(x_n)_{n\geq 1}$ in a metric space X is called a **Cauchy sequence** when it has the property that for all $\epsilon > 0$ there exists an index $m \in \mathbb{Z}^+$ such that for all indices $k, \ell \geq m$ we have $d(x_k, x_\ell) < \epsilon$.

Theorem 4.2 Let X be a metric space. 1 Every Cauchy sequence in X is bounded.

2 Every convergent sequence in X is Cauchy.

3 If some subsequence of a Cauchy sequence (x_n) converges, then (x_n) converges.

Proof. To prove Part 1, let $(x_n)_{n\geq 1}$ be a Cauchy sequence in X. Choose $m \in \mathbb{Z}^+$ such that $k, \ell \geq m d(x_k, x_\ell) \leq 1$ and note that, in particular, we have $d(x_k, x_m) \leq 1$ for all $k \geq m$. Let $a = x_m$ and choose $r > \max\big\{d(x_1,a), d(x_2,a), \cdots, d(x_{m-1},a), 1\big\}$. Then for all $n \in \mathbb{Z}^+$ we have $d(x_n, a) < r$ so the sequence (x_n) is bounded, as required.

We remark that Part 2 of this theorem was stated earlier, without proof, as Part 5 of Theorem 3.2. We give the proof here. Let $(x_n)_{n\geq 1}$ be a convergent sequence in X and let $a = \lim\limits_{n\to\infty} x_n$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ such that $n \geq m d(x_n, a) < \frac{\epsilon}{2}$. Then for all $k, \ell \geq m$ we have

$$d(x_k, x_\ell) \leq d(x_k, a) + d(a, x_\ell) <$$

Definition 4.4 — complete, Hilbert space, Banach space.
A metric space X is called **complete** when every Cauchy sequence in X converges in X. A complete inner product space is called a **Hilbert space**, and a complete normed linear space is called a **Banach space**.

Theorem 4.3 Let X be a complete metric space and let $A \subseteq X$. Then A is complete if and only if A is closed in X

Proof. Suppose that A is closed in X. Let (x_n) be a Cauchy sequence in A. Since X is complete, (x_n) converges in X. Since A is closed in X and (x_n) is a sequence in A which converges in X, we have $\lim\limits_{n\to\infty} x_n \in A$ by Theorem 3.5 (The Sequential Characterization of Closed Sets). Thus every Cauchy sequence in A converges in A, so A is complete.

Suppose, conversely, that A is complete. Let $a \in A'$, that is let $a \in X$ be a limit point of A. Since $a \in A'$, by Theorem 3.5 (The Sequential Characterization of Limit Points) we can choose a sequence (x_n) in A (indeed in $A \setminus \{a\}$) with $\lim n \to \infty x_n = a$. Since (x_n) converges in X, it is Cauchy. Since (x_n) is Cauchy and A is complete, (x_n) converges in A, that is $a = \lim_{n\to\infty} x_n \in A$. ■

■ **Example 4.7** Recall, from MATH 247 or PMATH 333, that $\big(\mathbb{R}^n, d_2\big)$ is complete. It follows that every closed subset $A \subseteq \mathbb{R}^n$ is complete (using the standard metric d_2).

■ **Example 4.8** Note that completeness is not invariant under homeomorphism. For example, $\mathbb{R}$ is homeomorphic to $(0,1) \subseteq \mathbb{R}$, but $\mathbb{R}$ is complete while $(0,1)$ is not.

Theorem 4.4 Every finite-dimensional normed linear space is complete.

Proof. Let U be an n-dimensional normed linear space. Let $\{u_1, \cdots, u_n\}$ be a basis for the vector space U and let $F : \mathbb{R}^n \to U$ be the associated vector space isomorphism given by $F(t) = \sum_{k=1}^{n} t_k u_k$. Recall, from Theorem 3.25, that both F and F^{-1} are Lipschitz continuous. Let L be a Lipshitz constant for F and let M be a Lipschitz constant for F^{-1}. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in U. For each $n \in \mathbb{Z}^+$, let $t_n = F^{-1}(x_n) \in \mathbb{R}^n$. Note that (t_n) is a Cauchy sequence in $\mathbb{R}^n$ because

Corollary 4.5
The metric spaces $(\mathbb{R}^n, d_1)$, $(\mathbb{R}^n, d_2)$ and $(\mathbb{R}^n, d_\infty)$ are all complete.

Theorem 4.6 The metric spaces (ℓ_1, d_1), (ℓ_2, d_2) and (ℓ_∞, d_∞) are all complete.

Proof. We prove that (ℓ_1, d_1) is complete and we leave the proof that (ℓ_2, d_2) and (ℓ_∞, d_∞) are complete as an exercise. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence in ℓ_1. For each $n \in \mathbb{Z}^+$, write $a_n = (a_{n,k})_{k\geq 1} = (a_{n,1}, a_{n,2}, a_{n,3}, \cdots)$. Since $a_n \in \ell_1$ we have $\sum_{k=1}^{\infty} |a_{n,k}| < \infty$. Since $(a_n)_{n\geq 1}$ is Cauchy, for every $\epsilon > 0$ we can choose $N \in \mathbb{Z}^+$ such that for all $n, m \geq N$ we have $\|a_n - a_m\|_1 < \epsilon$, that is $\sum_{k=1}^{\infty} |a_{n,k} - a_{m,k}| < \epsilon$. For each fixed $k \in \mathbb{Z}^+$, note that for $n, m \geq N$ we have $|a_{n,k}, -a_{m,k}| \leq \sum_{j=1}^{\infty} |a_{n,j} - a_{m,j}| < \epsilon$, and so the sequence $(a_{n,k})_{n\geq 1}$ is Cauchy in $\mathbb{R}$, so it converges. For each $k \in \mathbb{Z}^+$, let $b_k = \lim_{n\to\infty} a_{n,k} \in \mathbb{R}$ and let $b = (b_k)_{k\geq 1}$.

We claim that $b \in \ell_1$. Since $(a_n)_{n\geq 1}$ is Cauchy, for every $\epsilon > 0$ we can choose $N \in \mathbb{Z}^+$ such that for all $n, m \geq N$ we have $\|a_n - a_m\|_1 < \epsilon$, that is $\sum_{k=1}^{\infty} |a_{n,k} - a_{m,k}| < \epsilon$. By the Triangle Inequality, for $n, m \geq N$ we have $\big|\|a_n\|_1 - \|a_m\|_1\big| \leq \|a_n - a_m\|_1 < \epsilon$ It follows that the sequence $\big(\|a_n\|\big)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, so it converges. Let $M = \lim_{n\to\infty} \|a_n\|_1 \in \mathbb{R}$. For each fixed $K \in \mathbb{Z}^+$ we have

$$\sum_{k=1}^{K} |b_k| = \sum_{k=1}^{K} \big|\lim_{n\to\infty} a_{n,k}\big| = \lim_{n\to\infty} \sum_{k=1}^{K} |a_{n,k}| \leq \lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k}| = \lim_{n\to\infty} \|a_n\|_1 = M.$$

Since $\sum_{k=1}^{K} |b_k| \leq M$ for all $K \in \mathbb{Z}^+$ it follows that $\sum_{k=1}^{\infty} |b_k| \leq M$, so $b \in \ell_1$, as claimed.

Finally, we claim that $\lim_{n\to\infty} a_n = b$ in ℓ_1. Let $\epsilon > 0$. Choose $N \in \mathbb{Z}^+$ such that for all $n, m \geq N$ we have $\|a_n - a_m\|_1 < \epsilon$. Then for each $K \in \mathbb{Z}^+$ we have

$$\begin{aligned}\sum_{k=1}^{K} |a_{n,k} - b_k| &= \sum_{k=1}^{K} \big|a_{n,k} - \lim_{m\to\infty} a_{m,k}\big| = \lim_{m\to\infty} \sum_{k=1}^{K} |a_{n,k} - a_{m,k}| \\ &\leq \lim_{m\to\infty} \sum_{k=1}^{\infty} |a_{n,k} - a_{m,k}| = \lim_{m\to\infty} \|a_n - a_m\|_1 \leq \epsilon\end{aligned}$$

Since $\sum_{k=1}^{K} |a_{n,k} - b_k| \leq \epsilon$ for all $K \in \mathbb{Z}^+$ it follows that $\|a_n - b\|_1 = \sum_{k=1}^{\infty} |a_{n,k} - b_k| \leq \epsilon$. ∎

Exercise 4.1 Show that (ℓ_1, d_∞) and (ℓ_2, d_∞) are not closed in (ℓ_∞, d_∞) and so they are not complete.

Exercise 4.2 Show that the metric spaces $\big(\mathcal{C}[a,b], d_1\big)$ and $\big(\mathcal{C}[a,b], d_2\big)$ are not complete. Hint: in the case $[a,b] = [-1,1]$, consider $f_n : [-1,1] \to \mathbb{R}$ given by $f_n(x) = x^{1/2n-1}$ for $n \in \mathbb{Z}^+$. Show that if (f_n) did converge, either in $\big(\mathcal{C}[-1,1], d_1\big)$ or in $\big(\mathcal{C}[-1,1], d_2\big)$, then it would necessarily converge to a function g with $g(x) = 1$ when $x > 0$ and $g(x) = -1$ when $x < 0$, but such a function g cannot be continuous.

Definition 4.5 — supremum norm, supremum metric.
Let $\mathbf{F} = \mathbb{R}$ or $\mathbb{C}$. For a metric space X, we define

$$\begin{aligned}
\mathcal{F}(X,\mathbf{F}) &= \mathbf{F}^X = \big\{f : X \to \mathbf{F}\big\} \\
\mathcal{B}(X,\mathbf{F}) &= \big\{f : X \to \mathbf{F} \big| \ f \text{ is bounded}\big\} \\
\mathcal{C}(X,\mathbf{F}) &= \big\{f : X \to \mathbf{F} \big| \ f \text{ is continuous}\big\}, \\
\mathcal{C}_b(X,\mathbf{F}) &= \big\{f : X \to \mathbf{F} \big| \ f \text{ is bounded and continuous}\big\}.
\end{aligned}$$

Since we usually take $\mathbf{F} = \mathbb{R}$ we write

$$\mathcal{F}(X) = \mathcal{F}(X,\mathbb{R}) \ , \ \mathcal{B}(X) = \mathcal{B}(X,\mathbb{R}) \ , \ \mathcal{C}(X) = \mathcal{C}(X,\mathbb{R}) \ \text{ and } \ \mathcal{C}_b(X) = \mathcal{C}_b(X,\mathbb{R}).$$

Note that $\mathcal{B}(X,\mathbf{F})$ is a normed linear space using the **supremum norm** given by

$$\|f\|_\infty = \sup_{x\in X} \big|f(x)\big|$$

and a metric space using the **supremum metric** given by $d_\infty(f,g) = \sup_{x\in X} \big|f(x) - g(x)\big|$. These do not determine a well-defined norm and metric on $\mathcal{C}(X,\mathbf{F})$ since $\|f\|_\infty = \sup_{x\in X} |f(x)|$ might not be finite, but they do determine a well-defined norm and metric on $\mathcal{C}_b(X,\mathbf{F})$.

Definition 4.6 — converges uniformly.
For a sequence (f_n) in $\mathcal{F}(X)$ and for $g \in \mathcal{F}(X)$, we say that (f_n) **converges uniformly** to g on X, and write $f_n \to g$ uniformly on X, when for every $\epsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that $|f_n(x) - g(x)| < \epsilon$ for every $n \geq m$ and every $x \in X$.

Note 4.1 For a sequence $(f_n) \in \mathcal{B}(X)$ and for $g \in \mathcal{B}(X)$, note that $|f_n(x) - g| < \epsilon$ for every $x \in X$ if and only if $\|f_n - g\|_\infty < \epsilon$. It follows that $f_n \to g$ uniformly on X if and only if $f_n \to g$ in the metric space $\big(\mathcal{B}(X), d_\infty\big)$.

Theorem 4.7 Let X be a metric space. Then the metric spaces $\big(\mathcal{B}(X), d_\infty\big)$ and $\big(\mathcal{C}_b(X), d_\infty\big)$ are complete.

Proof. Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $\big(\mathcal{B}(X), d_\infty\big)$. Note that for each $x \in X$, we have $\big|f_n(x) - f_m(x)\big| \leq \sup_{y\in X} \big|f_n(y) - f_m(y)\big| = \|f_n - f_m\|_\infty$, and so the sequence $\big(f_n(x)\big)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, so it converges. Thus we can define a function $g : X \to \mathbb{R}$ by $g(x) = \lim_{n\to\infty} f_n(x)$ and then we have $f_n \to g$ pointwise in X.

We claim that $g \in \mathcal{B}(X)$, that is we claim that g is bounded. Since (f_n) is a Cauchy sequence in $\mathcal{B}(X)$, it is bounded (by Part 1 of Theorem 4.11) so we can choose $M \geq 0$ such that $\|f_n\|_\infty \leq M$ for all indices n. Then for all $x \in X$ we have $|f_n(x)| \leq \|f_n\|_\infty \leq M$ and hence $|g(x)| = \lim\limits_{n\to\infty} |f_n(x)| \leq M$. Thus g is a bounded function, that is $g \in \mathcal{B}(X)$.

We know that $f_n \to g$ pointwise on X. We must show that $f_n \to g$ uniformly on X. Let $\epsilon > 0$. Since (f_n) is Cauchy we can choose $m \in \mathbb{Z}^+$ such that $\|f_k - f_\ell\|_\infty < \epsilon$ for all $k, \ell \geq m$. Then for all $k \geq m$ and for all $x \in X$ we have

$$\big|f_k(x) - g(x)\big| = \lim_{\ell\to\infty} \big|f_k(x) - f_\ell(x)\big| \leq \epsilon.$$

It follows that $f_n \to g$ uniformly on X, that is $f_n \to g$ in the metric space $\big(\mathcal{B}(X), d_\infty\big)$. Thus $\big(\mathcal{B}(X), d_\infty\big)$ is complete.

To show that $\big(\mathcal{C}_b(X), d_\infty\big)$ is complete, it suffices (by Theorem 4.13) to show that $\mathcal{C}_b(X)$ is closed in $\mathcal{B}(X)$. Let (f_n) be a sequence in $\mathcal{C}_b(X)$ which converges in $\big(\mathcal{B}(X), d_\infty\big)$. Let $g = \lim\limits_{n\to\infty} f_n \in \mathcal{B}(X)$. We need to show that g is continuous. Let $\epsilon > 0$ and let $a \in X$. Since $f_n \to g$ in $\big(\mathcal{B}(X), d_\infty\big)$ we know that $f_n \to g$ uniformly on X, so we can choose $m \in \mathbb{Z}^+$ such that $\big|f_m(x) - g(x)\big| < \frac{\epsilon}{3}$ for all $n \geq m$ and all $x \in X$. Since f_m is continuous at a we can choose $\delta > 0$ such that for all $x \in X$ with $d(x,a) < \delta$ we have $\big|f_m(x) - f_m(a)\big| < \frac{\epsilon}{3}$. Then for all $x \in X$ with $d(x,a) < \delta$ we have

$$\big|g(x) - g(a)\big| \leq \big|g(x) - f_m(x)\big| + \big|f_m(x) - f_m(a)\big| + \big|f_m(a) - g(a)\big| <$$

Corollary 4.8
The metric space $\big(\mathcal{C}[a,b], d_\infty\big)$ is complete.

Proof. Since every continuous function $f : [a,b] \to \mathbb{R}$ is bounded, we have $\mathcal{C}[a,b] = \mathcal{C}_b[a,b]$. ∎

■ **Example 4.9** In the metric space $\big(\mathcal{C}[a,b], d_\infty\big)$, the space $\mathcal{R}[a,b]$ of Riemann integrable functions is closed, hence complete, and the spaces $\mathcal{P}[a,b]$ of polynomial functions, and $\mathcal{C}^1[a,b]$ of continuously differentiable functions, are not closed, and hence not complete.

Theorem 4.9 (Metric Completion) Every metric space X is isometric to a dense subspace of a complete metric space.

Proof. Let X be a metric space. Fix $a \in X$. For each $x \in X$, define $f_x : X \to \mathbb{R}$ by $f_x(t) = d(t,x) - d(t,a)$. Note that f_x is bounded since, by the Triangle Inequality, $\big|f_x(t)\big| = \big|d(x,t) - d(a,t)\big| \leq d(a,x)$. Note that f_x is continuous (indeed f_x Lipschitz continuous) because for $s,t \in X$ we have

$$\begin{aligned}\big|f_x(s) - f_x(t)\big| &= \big|d(s,x) - d(s,a) - d(t,x) + d(t,a)\big| \\ &\leq \big|d(s,x) - d(t,x)\big| + \big|d(s,a) - d(t,a)\big| \\ &\leq d(s,t) + d(s,t) = 2\,d(s,t)\end{aligned}$$

Define $F : X \to \mathcal{C}_b(X)$ by $F(x) = f_x$. We claim that F preserves distance, using the d_∞ metric on $\mathcal{C}_b(X)$. For all $x,y,t \in X$ we have

$$\big|f_x(t) - f_y(t)\big| = \big|d(x,t) - d(a,t) - d(y,t) + d(a,t)\big| = \big|d(x,t) - d(y,t)\big| \leq d(x,y)$$

hence for all $x, y \in X$ we have

$$\|f_x - f_y\|_\infty = \sup_{t\in X}\big|f_x(t) - f_y(y)\big| \leq d(x,y).$$

On the other hand, for all $x, y \in X$ we also have

$$\|f_x - f_y\|_\infty = \sup_{t\in X}\big|f_x(t) - f_y(t)\big| \geq \big|f_x(y) - f_y(y)\big| = \big|d(x,y) - d(y,y)\big| = d(x,y),$$

and so F preserves distance, as claimed. Thus X is isometric to the image $F(X) \subseteq \mathcal{C}_b(X)$, which is dense in its closure $\overline{F(X)}$, which is complete because it is a closed subspace of the complete metric space $\mathcal{C}_b(X)$. ■

R When X is a metric space and $F : X \to \mathcal{C}_b(X)$ is the distance preserving map in the proof of the above theorem, we often identify X with its isometric image $F(X)$ and think of X as a dense subspace of the complete metric space $Y = \overline{F(X)}$. Alternatively we can do some cutting and pasting operations on sets to obtain a complete metric space Y which actually contains X as a dense subspace. Here is an outline of one possible way of constructing such a set Y. Choose a set Z which is disjoint from X and has the same cardinality as $\mathcal{C}_b(X)$ (a bit of set theory is required to prove that such a set Z exists). Choose a bijection $G : \mathcal{C}_b(X) \to Z$ and give Z the metric which makes G an isometry. Then Z is complete and the composite $H = G \circ F : X \to Z$ is distance preserving so that X is isometric to the image $H(X)$, and $H(X)$ is dense in the complete space $\overline{H(X)}$, and $\overline{H(X)}$ is disjoint from X. Then let $Y = \big(\overline{H(X)} \setminus H(X)\big) \cup X$ so that we have $X \subseteq Y$. Let $K : Y \to \overline{H(X)}$ be the bijection given by $K(x) = h(x)$ if $x \in X$ and $K(y) = y$ if $h \notin X$, and give Y the metric for which K is an isometry. Then Y is complete and X is dense in Y.

Definition 4.7 — metric spaces.
When X and Y are metric spaces with $X \subseteq Y$ such that X is dense in Y and Y is complete, we say that Y is the **metric completion** of X. The metric completion of X is unique in the sense of the following theorem.

Theorem 4.10 (Uniqueness of the Metric Completion) Let X, Y and Z be metric spaces with Y and Z complete such that $X \subseteq Y$ with $\overline{X} = Y$ and $X \subseteq Z$ with $\overline{X} = Z$. Then there is a (unique) isometry $F : Y \to Z$ with $F(x) = x$ for all $x \in X$.

Proof. Let $a \in Y$. Since $\overline{X} = Y$ we can choose a sequence (x_n) in X with $x_n \to a$ in Y. Then (x_n) is Cauchy in Y, hence also in X, hence also in Z. Since (x_n) is Cauchy in Z, it converges in Z, say $x_n \to b$ in Z. In order for a map $F : Y \to Z$ to be continuous with $F(x) = x$ for every $x \in X$, we must have

$$F(a) = F\big(\lim_{n\to\infty} x_n\big) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_n = b.$$

This shows that if such a map F exists, it is unique, and it must be given by the following procedure: given $a \in Y$ we choose a sequence (x_n) in X with $x_n \to a$ and then we define $F(a) = \lim_{n\to\infty} x_n \in Z$.

We claim that the above procedure does determine a well-defined map whose value $F(a)$ does not depend on the choice of the sequence (x_n). Let $a \in Y$ and let (x_n) and (y_n) be two sequences in X with $x_n \to a$ and $y_n \to a$ in Y. Let $b = \lim_{n\to\infty} x_n$ in Z and let $c = \lim_{n\to\infty}$ in Z. We need to show that $b = c$. Let $\epsilon > 0$. Choose $m \in \mathbb{Z}^+$ such that for all indices $n \geq m$ we have $d_Y(x_n, a) < \frac{\epsilon}{4}$, $d_Y(y_n, a) < \frac{\epsilon}{4}$, $d_Z(x_n, b) < \frac{\epsilon}{4}$. and $d_Z(y_n, c) < \frac{\epsilon}{4}$. Then since $d_Z(x_n, y_n) = d_X(x_n, y_n) = d_Y(x_n, y_n)$ we have

$$\begin{aligned} d_Z(b,c) &\leq d_Z(b,x_n) + d_Z(x_n,y_n) + d_Z(y_n,c) \\ &= d_Z(b,x_n) + d_Y(x_n,y_n) + d_Z(y_n,c) \\ &\leq d_Z(b,x_n) + d_Y(x_n,a) + d_Y(a,y_n) + d_Z(y_n,c) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon \end{aligned}$$

Since $d_Z(b,c) < \epsilon$ for every $\epsilon > 0$ we must have $d_Z(b, c_=0$ hence $b = c$, as required.

Note that F is bijective with its inverse G given by the same construction: given $c \in Z$ we choose a sequence (x_n) in X with $x_n \to b$ in Z and define $G(c) = b = \lim_{n\to\infty} x_n$ in Y.

It remains to prove that F preserves distance. Let $a, b \in Y$. Chooose sequences (x_n) and (y_n) in Y with $x_n \to a$ and $y_n \to b$ in Y. Let $c, d \in Z$ with $x_n \to c$ and $y_n \to d$ in Z. We need to show that $d_Y(a,b) = d_Z(c,d)$. Since

$$\begin{aligned} d_Y(a,b) &\leq d_Y(a,x_n) + d_Y(x_n,y_n) + d_Y(y_n,b) \\ d_Y(x_n,y_n) &\leq d_Y(x_n,a) + d_Y(a,b) + d_Y(b,y_n) \end{aligned}$$

it follows that

$$\left| d_Y(a,b) - d_Y(x_n,y_n) \right| \leq d_Y(a,x_n) + d_Y(y_n,b).$$

Taking the limit as $n \to \infty$ gives $\left| d_Y(a,b) - \lim_{n\to\infty} d_Y(x_n,y_n) \right| = 0$ so that

$$d_Y(a,b) = \lim_{n\to\infty} d_Y(x_n,y_n) = \lim_{n\to\infty} d_X(x_n,y_n).$$

Similarly, we have $d_Z(c,d) = \lim_{n\to\infty} d_X(x_n,y_n)$ and hence $d_Y(a,b) = d_Z(c,d)$, as required. ∎

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