ACTSC446: Mathematics of Financial Markets

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Contents

1. Introduction to Derivatives Market

1.1 Financial Markets

Basic components of a financial market:

- Money
- Assets (such as stock)
- Time
- People/organizations
- Uncertainty

Why is there a financial market?

$\Longrightarrow$ People have different financial needs

- preferences on timing
- perspectives on risks and uncertainty
- sets of information
- means of economic activities

$\Longrightarrow$ So they trade

1.1.1 Assets

In this course, an asset:

- has a current price
- its future price may be uncertain
- is tradeable

We use S_t or $S(t)$ to represent the price of an asset (stock) S at time t

- $\{S_t\}_{t\geq 0}$ (or S for short) is a **stochastic process**
- Typically S_0 represents the current $(t = 0)$ price, which is **not random**
- $t \in [0, T]$ for some terminal time T

In this course, we aim to study this stochastic process S and some related quantities.

1.1.2 Review: Present Value of Future Payments

Note 1.1 — time-value of money.

A dollar today is worth more than a dollar tomorrow, because you can invest it and earn (non-negative) interest on it

The value at time $t < T$ of an amount $\$K$ in the future is $PV_t(K)$, the present value of K

Here, we assume that the (continuously compounded or annually effective) interest rate is non-negative, implying that $PV_t(K) \le K$, for any $K \ge 0$ and $t < T$

Suppose the interest rate is r annually,

Definition 1.1 — Present value of future payment.
If an asset (e.g. **zero-coupon bond**) pays K dollars in time T, then the time t $(T < t)$ value of the future payment is

$$PV_t(k) = \begin{cases} e^{-r(T-t)K} & \text{if continuously compounded} \\ \frac{K}{(1+r)^{T-t}} & \text{if annually effective} \end{cases}$$

Definition 1.2 — Risk-free asset.
If the future payoff(s) of an asset is non-random, we call it **risk free**.

1.2 Derivative Securities

Definition 1.3 — Derivative.
In finance, a **derivative** is a contract that derives its value from the performance of an **underlying entity**

- This underlying entity can be an asset, an index, interest rate, a basket of assets, or even another derivative
- The underlying entity is called "**the underlying asset**" or simply "the underlying"

Definition 1.4 — derivative security.
A financial contract F is a **derivative security** or a **contingent claim**, whose value F_T at F_T at expiration date (maturity) T is "derived" exactly from the market price of more basic underlying primitive instruments up to and including time T.

Primitive Instruments (underlying)

- Stocks
- Currencies
- Interest rates
- Indices
- Commodities
- Bonds

Derivatives

- Futures & Forwards
- Options (Call, Puts, Caps, Floors, Bond Options, Swaptions...)
- Credit derivatives
- Swaps

Definition 1.5 — OTC and ETD.
Based on where they are traded, derivatives can be classified as **OTC (Over the Counter)** or **ETD (Exchange-traded derivatives)**
- OTC derivatives are private, tailored contracts between counterparties
- ETD's are more structured and standardized contracts where the underlying assets, the quantities and the mode of settlement are defined by an exchange house

- Being private contracts between two counterparties, OTC derivatives can be tailored and customized to suit exact risk and return needs
  - On the flip side, lack of a clearing house or exchange results in increased credit or default risk associated with each OTC contract
- Being transacted on an organized exchange, ETD transactions are governed by a set of specific terms. They are standardized and more transparent than OTC derivatives
  - Each party of an ETD contract is required to hold a margin at the clearing house to cover its unsettled positions and the clearing house will monitor this margin level to make sure that it covers outstanding trades
  - A margin is the amount of cash an investor must put up to open an account to start trading
  - Therefore, ETD's carry less credit risk than OTC derivatives in general

Some Terminology

- **Long Position**: When you buy something ...
- **Short Position**: When you sell something what you don't yet own ...
- **Model-Free**: Independent of specific assumptions (e.g. about stock price distribution, etc.) ...

Usage

Derivatives are used:
- to **manage risk** (risk-management/insurance tool)
  - e.g. a pension fund invested in a broad market index can use derivatives to obtain downside protection
  - e.g. an airline company can use derivatives to put a ceiling on the future price of jet fuel
- for **speculation**
  - e.g. for a given investment, the use of derivatives magnifies the financial consequences, i.e. we can obtain large exposures with relatively little capital
- as an important part of **compensation**
  - executive stock options

1.2.1 Assumptions on a Financial Market

(1) No transaction fee.
(2) No bid-ask spread.
(3) One can buy any amount/share of any security.
(4) One can trade at any time instantly.
(5) Buying or selling a security does not change its price.
(6) No default/credit risk.
(7) Allow naked short selling.
(8) No information difference between investors.

These assumptions are not very realistic but they help us to understand the fundamental issues of a financial market.

1.2.2 The Concept of (No) Arbitrage

Definition 1.6 — Arbitrage.
An **arbitrage opportunity** is a portfolio value process $\{V_t\}_{t\geq 0}$ such that
(1) $V_0 \leq 0$
(2) $P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$, for some time $T > 0$

In other words, an arbitrage opportunity is a portfolio that:
- costs nothing to hold, or you are paid to hold it
- generates non-negative payoff with probability 1, and positive payoffs with strictly positive probability

The Principle of No-Arbitrage
- There ain't no such thing as a free lunch
- An immediate consequence of no-arbitrage is the Law of One Price

Proposition 1.1 — Law of One Price.
In an arbitrage-free market, if two securities have exactly the same payoffs they must have the same price

Proposition 1.2
In a market, if there exists a portfolio value process $\{V_t\}_{t\geq 0}$ satisfying
(1) $V_0 \leq 0$
(2) $V_T \geq 0$ for some time $T > 0$
then there is an arbitrage opportunity in the market.

From now on, assume a market with no arbitrage in this chapter.

1.3 Forwards and Futures

Definition 1.7 — Forward.
A **forward contract** is a non-standardized agreement to buy or sell an asset at a certain future time T for a certain price K, known as the **delivery price** (or **forward price**)
- The delivery price is determined so that the value of the contract at initiation is zero

Terminology
- **Underlying asset**: The asset on which the forward contract is based
- **Expiration date**: The time at which the asset is delivered
- **Forward price**: The price the buyer will pay at the expiration date
  - This is not the price one party needs to pay the other at the initial time; there is no initial price associated with a forward contract!

* It is normally traded Over-the-Counter (OTC)
* The party that **agrees to buy the underlying** asset is said to have a **long position** in the forwards
* The party that **agrees to sell the underlying** asset is said to have a **short position** in the forwards
* At the time the contract is entered into, no exchange of money takes place
* A forward contract can be contrasted with a spot contract:
  - A *spot contract* is an agreement to buy or sell an asset today, **with immediate** cash exchange

– A *forward contract* is an agreement **with no immediate** cash exchange

1.3.1 Forward Contract

Forward Contract - Payoff

- S_t: The spot price of the underlying asset at time $t \geq 0$
- T: The expiration date
- K: The forward price (delivery price)
- Long position: the position of the buyer
- Short position: the position of the seller

Pay off to long forward $= S_T - K$
Pay off to short forward $= K - S_T$

Forward Contract - Pricing

Pricing a forward contract is model-free, using simple no-arbitrage arguments

- Suppose that a stock pays no dividend, the current stock price is S_0, and the risk-free rate is r per year continuously compounded
- Consider the following trading strategy:
  (1) Borrow S_0 at the risk-free rate for the period of T years, and buy one share
  (2) Short one forward contract on the stock with delivery price K expiring at T

The cash flows are:

	Cash flow at $t = 0$	Cash flow at $t = T$
Borrowing S_0	$+S_0$	$-S_0e^{rT}$
1 long share	$-S_0$	$+S_T$
1 short forward	0	$K - S_T$
Total	0	$K - S_0e^{rT}$

The principle of No-Arbitrage ("No free lunch") then implies that the cash flow at time T should be 0. Thus the forward price is:

$$K = S_0e^{rT}$$

Proposition 1.3
Let S denote the price process of a non-dividend-paying stock. For a forward contract F on S, issued at time t and having maturity T, the forward price K determined at t is given by

$$K = S_0e^{rT}$$

Proof. (Equivalent to the previous cash flow table)

- At time t, but F, sell S, and deposit S_t money
- At t you value is 0
- At T you have $S_T - K - S_T + S_te^{r(T-t)} = S_te^{r(T-t)} - k$
- This value is not random, and if it is not zero there is an arbitrage

■

Forward Contract - Pricing with Dividends

Dividends are the payments made by a security (e.g. stock of a corporation) to its shareholders. They can be discrete (paid at discrete time intervals) or continuous (paid continuously).

Discrete Dividends

Consider a forward on a stock St , which will pay a dividend of $c at time $t_1 \in [0, T]$, where T is the expiration date of the forward contract.
Consider the following two trading strategies:

(1) Borrow S_0 at the risk-free rate for the period of T years, and buy one share
(2) Short one forward contract on the stock with delivery price K expiring at T

The cash flows are:

	Cash flow at $t = 0$	Cash flow at $t = T$
Borrowing S_0	$+S_0$	$-S_0e^{rT}$
1 long share	$-S_0$	$+S_T + ce^{r(T-t_1)}$
1 short forward	0	$K - S_T$
Total	0	$K - S_0e^{rT} + ce^{r(T-t_1)}$

The principle of No-Arbitrage ("No free lunch") then implies that the cash flow at time T should be 0. Thus the forward price is:

$$K = S_0e^{rT} - +ce^{r(T-t_1)}$$

Proposition 1.4
Let S denote the price process of a stock earning discrete dividends between time t and time T. For a forward contract F on S, issued at time t and having maturity T, the forward price K determined at t is given by

$$K = S_0e^{rT} - \text{Accumulated value at time } T \text{ of all dividends}$$

Continuous Dividends

R When there is a continuous dividend paid by stock S in a constant rate δ, an investment of $S_te^{-\delta(T-t)}$ in the stock at time t will yield 1 share of stock at time T (with price S_T)

Proposition 1.5
Let S denote the price of a stock earning a continuous dividend rate δ. For a forward contract F on S, issued at time t and having maturity T, the forward price K determined at t is given by

$$K = S_te^{(r-\delta)(T-t)}$$

Proof. Consider a forward on a stock S_t, paying dividends continuously at a dividend yield of δ per annum. Consider the following two trading portfolios:

- Portfolio A:
  - At time t, enter into a forward contract to buy one share of the stock, with forward price $K, maturing at time T
  - Simultaneously invest an amount $Ke^{-r(T-t)}$ the risk-free asset
  - At time T, the risk-free investment will accumulate to $K; use this $K buy a share of stock via the forward contract.
- Portfolio B:
  - Buy $e^{-\delta(T-t)}$ shares of the stock, at the current price S_t. Reinvest dividend incomes in the stock S immediately when they are received.

The cash flows are:

Portfolio	Cash flow at t	Cash flow at T
A	$\$Ke^{-r(T-t)}$	S_T
B	$\$S_t e^{-\delta(T-t)}$	S_T

Thus by the no-arbitrage principle, $\$Ke^{-r(T-t)} = \$S_t e^{-\delta(T-t)}$, i.e. $\$K = S_t e^{(r-\delta)(T-t)}$, when the underlying pays dividends continuously at a yield of δ per annum. ■

Continuous dividends are unusual but easy to calculate.

Prepaid Forward Contracts

Definition 1.8 — prepaid forward.
A **prepaid forward** is a forward contract which calls for payment today and delivery of the underlying asset at a future date.

In a similar fashion, an application of the no-arbitrage principle and the replication strategy yields the prepaid forward price K_0 as follows:

- No dividend: $K_0 = S_0$
- A discrete dividend of $\$c$ at time t: $K_0 = S_0 - ce^{-et}$
- Continuous dividend at a yield rate of δ: $K_0 = S_0 e^{-\delta T}$

1.3.2 Futures Contract

Definition 1.9 — futures contract.
Like a forward contract, a futures contract is a costless-to-enter agreement between two parties to exchange an asset at a certain future time for a certain delivery price

However, contrary to forwards that are mainly OTC contracts, futures are ETD, hence there are various structural differences.

Futures vs. Forwards

Forward	Futures
OTC (private between 2 parties)	Exchange-traded Contract
Not Standardised	Standardised to the Exchange Rules
Settled at maturity T	Daily Settlements/Margins ("marked-to-market")
Counterparty/Credit/Default Risk	No Risk (except for the risk to meet a margin call)

Since futures contracts are marked-to-market, every day any profits or losses on the contract are calculated and traders have to cover up any losses or receive any profits in their **margin account**.
Other differences between forwards and futures:

- Futures:
  - Standard ETD
  - Ignorable default risk
  - Usually closed before maturity so delivery usually never happens
- Forwards
  - OTC derivatives
  - Substantially high default probability
  - Delivery usually happens

1.4 Options

Definition 1.10 — option.
An **option** is a contract which gives the buyer the right, but not the obligation, to buy or sell an underlying asset at a specified strike price, on or before expiration

Terminology

- **Underlying asset**: the asset on which the option is based
- **Expiration date**: the date by which the option must either be exercised or it becomes worthless
- **Exercise**: the action of carrying out the transaction specified by the option
- **Strike price**: the price for the asset at which exercise can occur

There are three common exercise styles for options:

- European-style: The option can only be exercised at maturity
- American-style: The option can be exercised at any time at or before maturity
- Bermudan-style: The option can only be exercised on a set of specified dates at or before maturity

1.4.1 Put and Call Options

Definition 1.11 — call option.
A **call option** gives its owner the right, but not the obligation, to buy the underlying asset at a specified exercise or strike price K on or before a specified exercise date T.
$\implies$ The payoff at time T is $\max(S_T - K, 0)$

Definition 1.12 — put option.
A **put option** gives its owner the right, but not the obligation, to sell the underlying asset at a specified exercise or strike price K on or before a specified exercise date T.
$\implies$ The payoff at time T is $\max(K - S_T, 0)$

Definition 1.13 — European & American Feature.
An option that can be exercised **only on** one particular day T is conventionally known as a European option.

If the option can be exercised **on or at any time before** day T, then it is known as an American option

Which is more expensive and why?

American options have a higher price than European options with the same characteristics (see later)

1.4.2 Moneyness

Definition 1.14 — Moneyness.

- **In-The-Money (ITM)**: an option is in the money if exercising the option immediately leads to a positive cash flow to the holder
- **At-The-Money (ATM)**: an option is at the money if exercising the option immediately leads to zero cash flow to the holder: "priced at-the-money"
- **Out-of-The-Money (OTM)**: an option is out of the money if exercising the option immediately leads to a negative cash flow to the holder

For call and put options, moneyness is related to the difference between K and S:

	$S < K$	$S = K$	$S > K$
Call	OTM	ATM	ITM
Put	ITM	ATM	OTM

1.4.3 Payoff Diagrams

Payoff Diagrams - Long Side

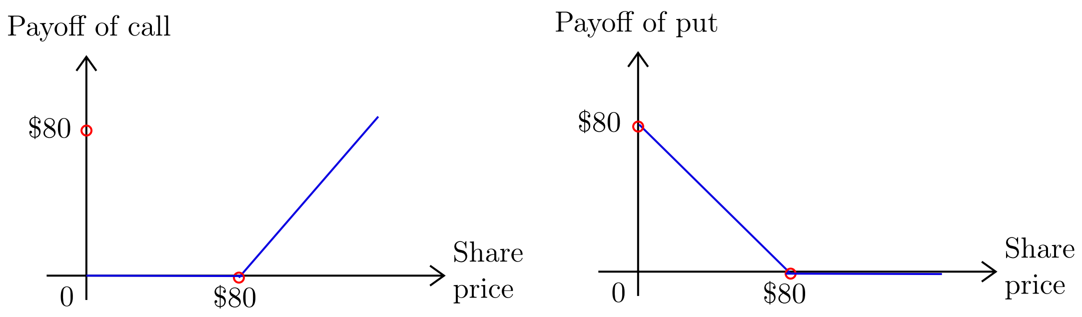
■ **Example 1.1** Consider a long European call and a long European put with:

- Same underlying S
- Same strike $K = \$80$
- Same maturity T

The payoffs for the holders are:

$$\text{Call payoff} = \max(S_T - K, 0) = \max(S_T - 80, 0)$$
$$\text{Put payoff} = \max(K - S_T, 0) = \max(80 - S_T, 0)$$

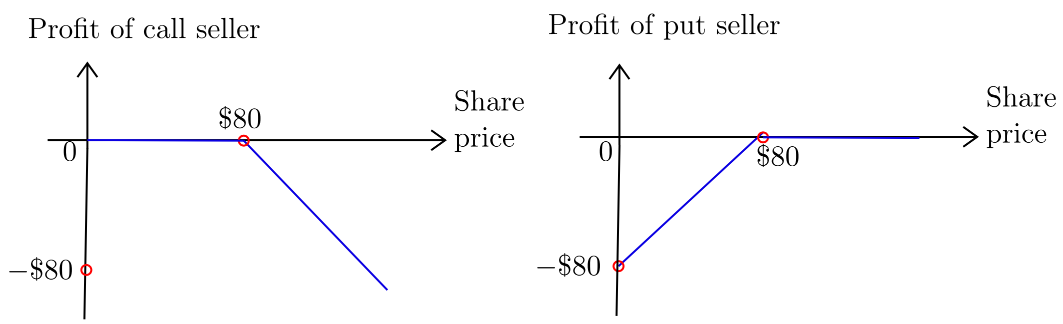
Graphically, these payoffs are as follows:



- A long call has infinite potential gain
- A long put has insurance-type features: it pays off when the firm goes bankrupt

Payoff Diagrams - Short Side

■ **Example 1.2** Consider the previous example but suppose that we are now short both options:

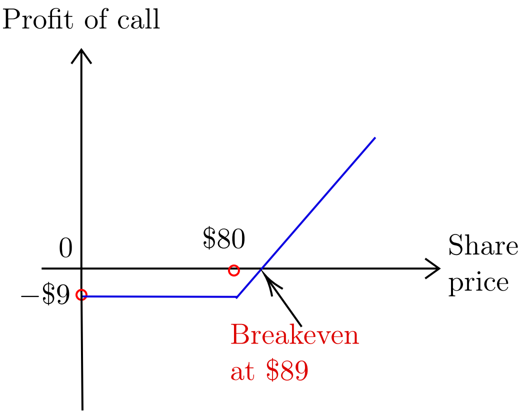


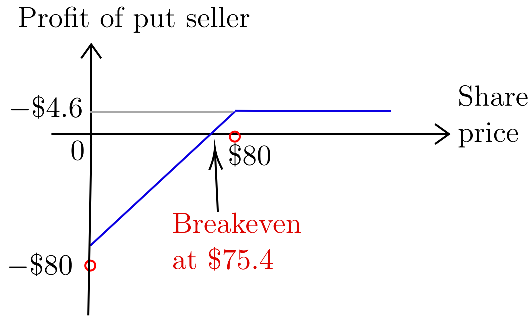
- A short call faces a potentially **infinite loss** (like a short position of a stock)

1.4.4 Profit Diagrams

Profit diagrams incorporate the costs of buying an option or the proceeds from selling one.

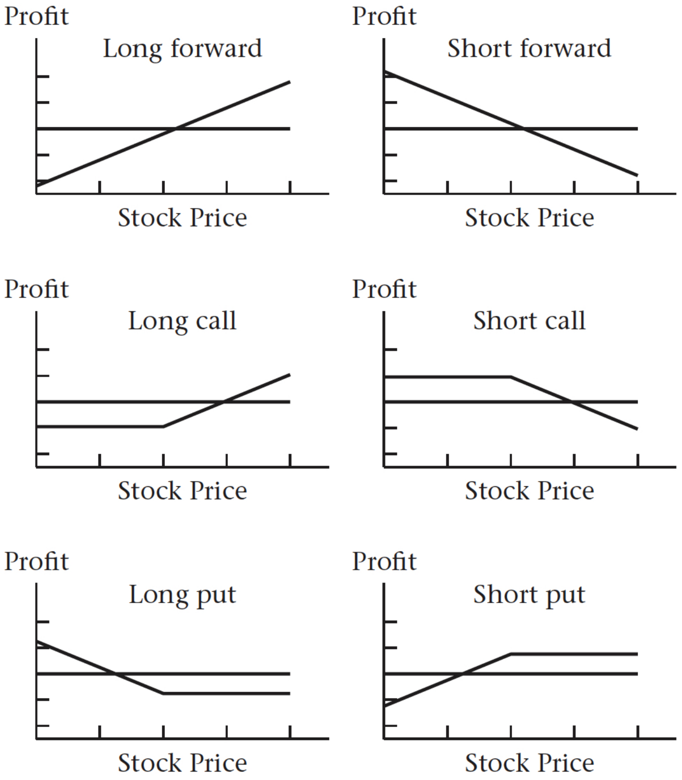
■ **Example 1.3** The investor purchased a call with strike price of \$80 at \$9 (assuming the interest rate is 0), while in the right panel, the investor sold a put option with strike of \$80 for \$4.60





The break-even price is always in the ITM region of the option

Summary



1.4.5 Forwards (Futures) and Options

Similarities

- Both are derivative
- Bothe have an expiration date and a strike price

Differences

	Forward	Option
Payoff Type	Only one	Various
Exercise	Obligation	Right but not obligation
Price	Usually zero	Positive

1.4.6 Intrinsic and Time Value of an Option

Definition 1.15 — Intrinsic Value.
The **intrinsic value** of an (American) option is defined as the payoff that could be obtained by immediate exercise of the option at time $t < T$

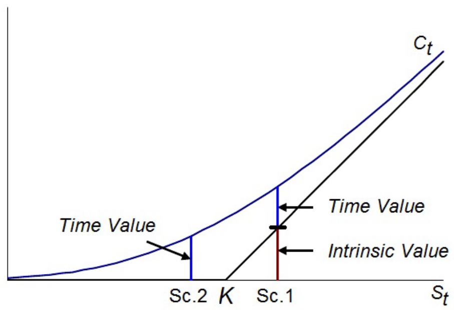
■ **Example 1.4** An American call option has intrinsic value at any time t equal to $\max(S_t - K, 0)$

Definition 1.16 — Time Value.
The **time value** of an option at any time $t < T$ is defined as the difference between the actual option price at t and its intrinsic value at t

■ **Example 1.5** An American call option C has intrinsic value at any time t equal to $C_t - \max(S_t - K, 0)$

Note 1.2
- The intrinsic value and the time value of an option are key quantities to consider when deciding on whether or not to exercise an American option early
- If/when time value is 0, one may choose to exercise immediately

■ **Example 1.6** Let's look at a long position in an American call option



We notice that
- The intrinsic value is positive when the option is ITM
- When intrinsic value = 0, the option may be selling for a positive price, because there is (almost) always positive probability that it will end up ITM at T

  ■ **Example 1.7** Suppose that a call is OTM:
  - If you exercise, you get nothing. It can't get any worse than that!
  - If the stock price rebounds, however, and exceeds the strike by expiration, we may end up with a positive payoff

1.5 Bounds on Option Prices

1.5.1 No-Arbitrage Bounds on Option Prices

- Computing option prices requires making assumptions about the evolution of the underlying asset (i.e, a model)
- However, no-arbitrage arguments can impose model-free price bounds
- Trivially, option payoffs are non-negative hence they must have **non-negative prices** as well. Can we derive sharper bounds?
- Assume for now non-dividend paying stocks as the underlying assets

  - Since most stocks pay dividends only once a year and most exchange-traded options are written with less than one-year time to expiration, the assumption of no dividends will actually be true for many real-world options

1.5.2 Bounds on Non-Dividend Paying Stock

Lower Bound on American Options

Notation 1.1

- European options:
  - Call: $c(S, K, t, T) = c_t$
  - Put: $p(S, K, t, T) = p_t$
- American options:
  - Call: $C(S, K, t, T) = C_t$
  - Put: $P(S, K, t, T) = P_t$

Trading Strategy and Portfolio

- A **portfolio** is a collection of securities
  - Under our market assumptions, you can short any portfolio
- A **trading strategy** is the dynamic organization of a portfolio, including buying, selling securities or exercising derivatives Holding a portfolio is a trading strategy.
  - You may not short a trading strategy

We use π for a trading strategy (or its corresponding portfolio) and π_t for its time t value (sometimes we use $\pi(t)$)

- As previously mentioned, since an American option can be exercised at any time, it must always be at least as valuable as an otherwise identical European option:

Proposition 1.6 — European vs. American options.
$C(S, K, t, T) \geq c(S, K, t, T)$ and $P(S, K, t, T) \geq p(S, K, t, T)$

Lower Bound on a European Call Option

Consider the following trading strategies at time $t = 0$

(1) Buy 1 European call option on a non-dividend paying stock with a strike price of K, expiring at time T

(2) Buy 1 share of the underlying stock and borrow at the risk-free rate the amount $PV_0(K) = Ke^{-rT}$ or $\frac{K}{(1+r)^T}$

The cash flows of there strategies are:

	Cash flow at $t = 0$	Cash flow at $t = T$	
		$S_T < K$	$S_T \geq K$
Strategy 1	$-c_0$	0	$S_T - K$
Strategy 2	$PV_0(K) - S_0$	$S_T - K$	$S_T - K$

Thus, no matter what happens in the future, the cash flow of Strategy 1 is always greater than or equal to the cash flow of Strategy 2. Thus:

$$c_0 \geq S_0 - PV_0(K) \geq S_0 - K$$

More generally, we have the following result (assume continuously compounded interest rate; recall that $r \geq 0$):

Proposition 1.7
At time $t \geq 0$, we have $C_t \geq c_t \geq S_t - Ke^{-r(T-t)} = S_T - PV_t(K) \geq S_t - K$

Early Exercise of an American Call

Assume no dividends; does it make sense to exercise an American call early?

- At any time $t < T$, there are two scenarios
  (1) Exercise the American call early:

$$\text{Payoff}_1(t) = \text{Intrinsic Value}(t) = S_t - K$$

  (2) Sell the call instead of exercising it:

$$\text{Payoff}_2(t) = C_t \geq c_t \geq c_t \geq S_t - PV_t(K) \geq S_t - K$$

$\Longrightarrow$ Clearly, we are better off selling the option since $S_t - PV_t(K) \geq S_t - K$

$\Longrightarrow$ An American call on a non-dividend paying stock should never be exercised early. Hence, with no dividends: $C_t = c_t$

Proposition 1.8 — American call vs. European call.
If S does not pay dividends in $[t, T]$, then $c(S, K, t, T) = C(S, K, t, T)$

R The above is NOT true if the underlying stock pays a dividend during the life of the option!

- If the underlying pays a dividend between t and T, we have: $C_t \geq c_t$
- Trivially, $0 \leq c_t \leq C_t \leq S_t$
  - An option to buy an asset cannot cost more than the asset itself
- Combining all the above bounds, both American and European calls on a non-dividend paying stock must satisfy the following:

Proposition 1.9
At time $t \geq 0$, $S_t \geq C_t = c_t \geq \max(S_t - PV_t(K), 0) \geq \max(S_t - K, 0) \geq 0$

Lower Bound on a European Put Option

Consider the following trading strategies at time $t = 0$:

(1) Buy 1 European put option on a non-dividend paying stock with a strike price of K, expiring at time T and 1 share of the underlying stock
(2) Deposit the amount of $PV_0(K) = Ke^{-rT}$ (or $\frac{K}{(1+r)^T}$) into your risk-free savings account

The cash flows of there strategies are:

	Cash flow at $t = 0$	Cash flow at $t = T$	
		$S_T < K$	$S_T \geq K$
Strategy 1	$-p_0 - S_0$	K	S_T
Strategy 2	$PV_0(K)$	K	K

Thus, no matter what happens in the future, the cash flow of Strategy 1 is always greater than or equal to the cash flow of Strategy 2. Thus:

$$p_0 \geq PV_0(K) - S_0$$

More generally, we have the following result:

Proposition 1.10
At time $t \geq 0$, $P_t \geq p_t \geq Ke^{-r(T-t)} - S_t = PV_t(K) - S_t$

Early Exercise of an American Put

Unlike for call options (remember the effect of dividends), the optimality of early exercise of an American put option is always a possibility. Hence, for all $t \in [0, T]$:

$$P_t \geq p_t$$

■ **Example 1.8 — An extreme scenario.**

- Suppose you hold a put option on the stock of a company that goes bankrupt before expiration
- The value of the stock is zero and there is no possibility for it to rebound! The company is dead
- An American put allows immediate exercise, hence a payoff of K
  - Putting K into a savings account for the period remaining to expiration results you will have Ke^{rT} at maturity, where $\tau = T - t$ is the time to expiration
- A European put would only pay K at expiration (...cash delivery of course)
  - Clearly, you are better off having the American put. Therefore, $P_t > p_t$

Bounds of Put Option on Non-Dividend Paying Stock

- The American put must satisfy $0 \leq P_t \leq K$
  - An option to sell at **any time** an asset for K cannot cost more than K
- Similarly, the European put must satisfy $0 \leq p_t \leq PV_t(K)$
  - An option to sell at time T an asset for K cannot cost at time t more than $PV_t(K)$

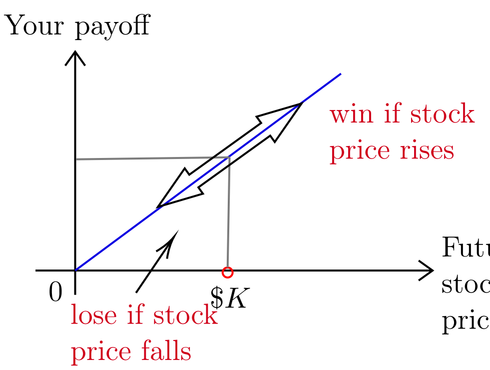
Combining the above bounds, we obtain:

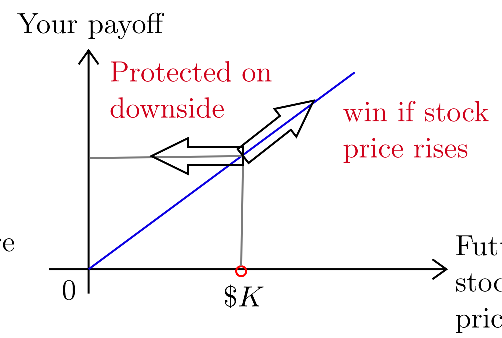
- American put: $K \geq P_t \geq \max(K - S_t, 0) \geq 0$
- European put: $PV_t(K) \geq p_t \geq \max(PV_t(K) - S_t, 0) \geq 0$

1.5.3 Put-Call Parity

Downside Protection

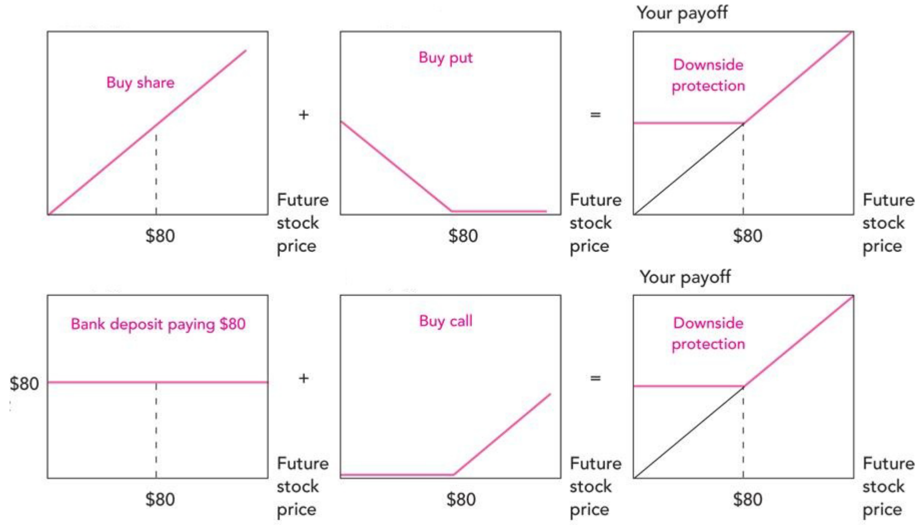
- Investing into a stock is risky because the stock price might fall
- Suppose we want to put a limit on the maximum possible loss
- Buy a **put option** on the stock, as it has insurance-type features
- No matter what happens in the future, the value of your investment cannot fall below the strike price of the put
- Such put options are called **protective puts** and are very popular risk management tools with institutional investors such as mutual and pension funds





But one could create the same payoff by lending and buying a call option

Two ways of creating Downside Protection



The two portfolios have the same payoff! Law of One Price must apply!

Put-Call Parity for European Options

There are two ways to achieve downside protection:

(1) Buy 1 share and 1 European put on a non-dividend paying stock with strike K

(2) Deposit the present value of K in a risk-free savings account and buy 1 European call on the same stock with the same strike K

The cash flows of there strategies are:

	Cash flow at $t = 0$	Cash flow at $t = T$	
		$S_T < K$	$S_T \geq K$
Strategy 1	$-p_0 - S_0$	K	S_T
Strategy 2	$PV_0(K) - c_0$	K	K

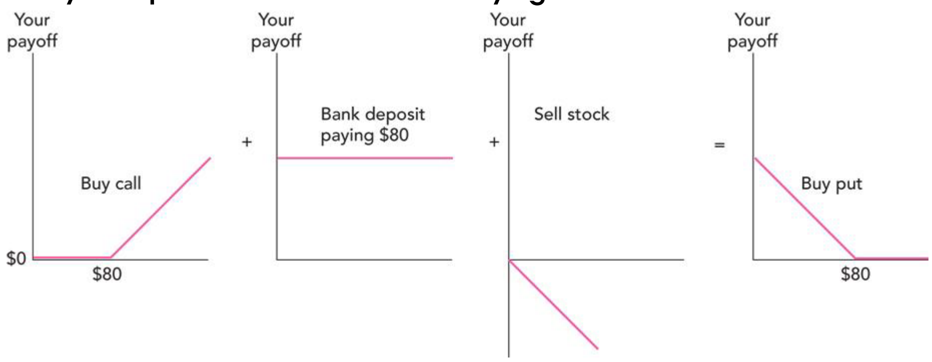
By the Law of One Price 1.1:

$$c_0 + PV_0(K) = S_0 + p_0$$

More generally, we have the following result:

Proposition 1.11 — Put-Call Parity.
At time $t \leq T$, $c_t + PV_t(K) = S_t + p_t$

Put-Call Parity for Options on Non-Dividend Paying Stock



Extending put-call parity to American options, we get:

- European: $c_t + PV_t(K) = S_t + p_t$
- American: $S_t - K \leq C_t - P_t \leq S_t - PV_t(K)$

R Put-Call Parity is a model-free result!

Violation of Put-Call Parity - Arbitrage

If the parity relation is ever violated, an arbitrage opportunity arises

Example 1.9 Suppose you collect these data for a certain stock and European options written on it:

Stock Price	\$100
Call Price ($T = 1Y, K = \$105$)	\$17
Put Price ($T = 1Y, K = \$105$)	\$5
Risk-free Rate	5% per year

Clearly

$$c_0 + PV_0(K) =^? S_0 + p_0$$
$$17 + 105/1.05 =^? 110 + 5$$
$$117 > 115$$

Hence, the protective put strategy is cheaper than the call plus deposit of $PV_0(K)$. What will you do?
How to benefit from the arbitrage? - Buy the cheap, sell the expensive (buy low, sell high)
(1) Buy the cheap: buy the stock, buy the put
(2) Sell the expensive: write (sell) the call and borrow \$100 for one year
The cash flows of this strategy:

Position	Cash flow at $t = 0$	Cash flow at $t = T$	
		$S_T < K$	$S)T \geq K$
Buy Stock	-110	S_T	S_T
Borrow $\$105/1.05 = \$100$	+100	-105	-105
Sell Call	+17	0	-(S_T-105)
Buy Put	-5	105-S_T	0
Total	+2	0	0

- Immediate profit of \$2! Fully covered at time T
- In the appearance of an arbitrage opportunity, arbitrageurs will step in and the buying and selling pressure will restore the parity

Put-Call with Dividends

Proposition 1.12 — Put-Call parity with dividend.

$$c_t - p_t = S(t) - PV_t(D) - PV_t(K)$$

Proposition 1.13 — Put-Call parity with continuous dividend.

$$c_t - p_t = S(t)e^{-\delta(T-t)} - PV_t(K)$$

Factors Affecting Option Prices

option price effect from a change in one variable, while keeping the rest **fixed**:

Factor	Call		Put	
	European	American	European	American
Stock Price (S_t) $\uparrow$	$\uparrow$		$\downarrow$	
Strike Price (K) $\uparrow$	$\downarrow$		$\uparrow$	
Maturity (T) $\uparrow$	unknown ($\uparrow$ if no dividend)	$\uparrow$	unknown	$\uparrow$
Time to Maturity ($T-t$) $\downarrow$	unknown ($\downarrow$ if no dividend)	$\uparrow$	unknown	$\uparrow$
Stock Volatility (σ_t) $\uparrow$	$\uparrow$		$\uparrow$	
Risk-Free Interest Rate (r_t) $\uparrow$	$\uparrow$		$\downarrow$	

1.6 Mathematical Properties of Option Prices

1.6.1 Review of Functions

A function $f : D \to \mathbb{R}$ is

- **Continuous** if
$$\lim_{x \to a} f(x) = f(a), \ \forall a \in D$$
- **Lipschitz continuous** if
$$|f(x) - f(y)| < C|x - y|, \ \forall x, y \in D, \exists C \text{ constant}$$
- **Increasing** if
$$f(x) - f(y) \geq 0, \ \forall x \geq y$$
- **Decreasing** if
$$f(x) - f(y) \leq 0, \ \forall x \geq y$$
- **Convex** if
$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y), \ x, y \in D, \lambda \in [0,1]$$
- **Concave** if
$$\lambda f(x) + (1-\lambda) f(y) \leq f(\lambda x + (1-\lambda) y), \ x, y \in D, \lambda \in [0,1]$$

A twice-differentiable function $f : D \to \mathbb{R}$ is

- **Lipschitz continuous** if
$$|f'(x)| < C, \ x \in D, C \text{ constant}$$
- **Increasing** is
$$f'(x) \geq 0, \ x \in D$$
- **Decreasing** is
$$f'(x) \leq 0, \ x \in D$$
- **Convex** is
$$f''(x) \geq 0, \ x \in D$$
- **Concave** is
$$f''(x) \leq 0, \ x \in D$$

1.6.2 Properties of Option Prices

Suppose the stock S pays no dividend over the period $[t, T]$

Proposition 1.14 — Inequality set 1.

$$S_t \geq C(S,K,t,T) \geq c(S,K,t,T) \geq S_t - PV(K) \geq S_t - K$$
$$K \geq P(S,K,t,T) \geq p(S,K,t,T) \geq PV(K) - S_t$$

Proposition 1.15 — Inequality set 2A.
For $0 \leq K_1 \leq K_2$

$$0 \leq c(S,K_1,t,T) - c(S,K_2,t,T) \leq K_2 - K_1$$

$$0 \leq C(S,K_1,t,T) - C(S,K_2,t,T) \leq K_2 - K_1$$

That is $c(S,K,t,T)$ and $C(S,K,t,T)$ are decreasing functions of K and are (Lipschitz) continuous on $\mathbb{R}^+$

Proof. We build up trading strategies and use a non-arbitrage argument to show these inequalities. Write $c_t(K) = c(S,K,t,T)$ for short, and similarly for the other quantities.

- To show $c_t(K_1) - c_t(K_2) \geq 0$: build a portfolio π which is to long 1 unite of $c(K_1)$ and short 1 unit of $c(K_2)$ at time t. At time T, the payoff is

$$\pi_T = (S_T - K_1)_+ - (S_T - K_2)_+ = \begin{cases} K_2 - K_1 & S_T \geq K_2 \\ S_T - K_1 & K_1 \leq S_T \leq K_2 \\ 0 & S_T < K_1 \end{cases}$$

  So $\pi_T \geq 0$, we must have $\pi_t \geq 0$
- To show $C_t(K_1) - C_t(K_2) \geq 0$: build a portfolio π which is to long 1 unite of $C(K_1)$ and short 1 unit of $C(K_2)$ at time t. If $C(K_2)$ is not exercised by the counter-party, then $\pi_T = (S_T - K_1)_+ \geq 0$. If $C(K_2)$ is exercised at t_0, then we exercise $C(K_1)$ at t_0, and

$$\pi_{t_0} = (S_{t_0} - K_1)_+ - (S_{t_0} - K_2)_+ \geq 0$$

  We hold this amount of money till T, and $\pi_T \geq 0$. So no matter what, $\pi_T \geq 0$, we must have $\pi_t \geq 0$
- To show $c_t(K_1) - c_t(K_2) \leq K_2 - K_1$: build a portfolio π which is to hold the cash amount of $K_2 - K_1$, short 1 unit of $c(K_1)$, and long 1 unit of $c(K_2)$ at time t. At time T, the payoff is

$$\pi_T = e^{r(T-t)}(K_2 - K_1) - ((S_T - K_1)_+ - (S_T - K_2)_+) \geq 0$$

  by (1), SO $\pi_T \geq 0$, we must have $\pi_t \geq 0$
- To show $C_t(K_1) - C_t(K_2) \leq K_2 - K_1$: build a portfolio π which is to hold the cash amount of $K_2 - K_1$, short 1 unit of $c(K_1)$, and long 1 unit of $c(K_2)$ at time t. Suppose that at some time t_0, the counter-party (which holds $C(k_1)$) exercises her call option. We exercise $C(K_2)$ immediately. The value of the portfolio at t_0 is

$$\pi_{t_0} = e^{r(t_0-t)}(K_2 - K_1) - ((S_{t_0} - K_1)_+ - (S_{t_0} - K_2)_+) \geq 0$$

  by (1), SO $\pi(t_0) \geq 0$. We hold this amount of money till T, hence $\pi_T \geq 0$. We must have $\pi_t \geq 0$

For $K_1 \leq K_2$,

- $c(K_1) - c(K_2) \geq 0 \iff c$ decreasing in $K \iff$ a call bull spread (see later) has a positive price.
- $c(K_1) - c(K_2) \leq K_2 - K_1 \iff c$ is (Lipschitz) continuous in $K \iff$ a call bull spread has a profit less than $K_2 - K_1$

■

Proposition 1.16 — Inequality set 2B.
For $0 \leq K_1 \leq K_2$

$$0 \leq p(S, K_2, t, T) - p(S, K_1, t, T) \leq K_2 - K_1$$

$$0 \leq P(S, K_2, t, T) - P(S, K_1, t, T) \leq K_2 - K_1$$

That is $p(S, K, t, T)$ and $P(S, K, t, T)$ are increasing functions of K and are (Lipschitz) continuous on $\mathbb{R}^+$

Proposition 1.17 — Inequality set 3A (Convexity Properties).
For $K_1, K_2 \geq 0, \lambda \in [0, 1]$

$$\lambda c(S, K_1, t, T) + (1-\lambda)c(S, K_2, t, T) \geq c(S, \lambda K_1 + (1-\lambda)K_2, t, T)$$

$$\lambda C(S, K_1, t, T) + (1-\lambda)C(S, K_2, t, T) \geq C(S, \lambda K_1 + (1-\lambda)K_2, t, T)$$

That is $c(S, K, t, T)$ and $C(S, K, t, T)$ are convex functions of K on $\mathbb{R}^+$

Proof. WLOG assume $K_1 \leq K_2$, and write $K = \lambda K_1 + (1-\lambda)K_2$

- To show $\lambda c_t(K_1) + (1-\lambda)c_t(K_2) \geq c_t(K)$: build a portfolio $\pi = \lambda c(K_1) + (1 - \lambda)c(K_2) - c(K)$ at time t. At time T, the payoff is

$$\pi_T = \lambda(S_T - K_1)_+ + (1-\lambda)(S_T - K_2)_+ - (S_T - K)_+$$
$$= \begin{cases} 0, & S_T \geq K_2 \\ \lambda(S_T - K_1) - (S_t - K), & K \leq S_T < K_2 \\ \lambda(S_T - K_1), & K_1 \leq S_T < K \\ 0, & S_T < K_1 \end{cases}$$

Note that in the second case,

$$\lambda(S_T - K_1) - (S_t - K) = (1-\lambda)(K_2 - S_T) \geq 0$$

So $\pi_T \geq 0$ and we must have $\pi_t \geq 0$.
- To show $\lambda C_t(K_1) + (1-\lambda)C_t(K_2) \geq C_t(K)$: similar

$$\lambda c(S, K_1, t, T) + (1-\lambda)c(S, K_2, t, T) \geq c(S, \lambda K_1 + (1-\lambda)K_2, t, T)$$
$$\iff c \text{ is convex}$$

This also implies (by choosing $\lambda = \frac{1}{2}$)

$$c(S, K_1, t, T) + c(S, K_2, t, T) \geq 2c(S, K, t, T)$$

■

■ **Example 1.10** Suppose we observe 3 call option prices today on the same stock and with same maturity: $c(50) = 14, c(59) = 8.9$ and $c(65) = 5$. How do we undertake arbitrage?

Solution. Observe that $K_1 = 50, K_2 = 65$. Note $0.4K_1 + 0.6K_2 = 59$. Hence the call option should satisfy $0.6c(65)+0.4c(50) \geq c(59)$. However $0.6c(65)+0.4c(50) = 0.6\times5+0.4\times14 = 8.6 < c(59)$. $c(59)$ is overpriced (or the other two are underpriced), this is an arbitrage.

(1) Buy 6 units $c(65)$
(2) Buy 4 units $c(50)$
(3) Sell 10 units $c(59)$

The initial value of the portfolio is \$3 and this portfolio has a terminal payoff ≥ 0 ■

Proposition 1.18 — Inequality set 3B (Convexity Properties).
For $K_1, K_2 \geq 0, \lambda \in [0,1]$

$$\lambda p(S,K_1,t,T) + (1-\lambda)p(S,K_2,t,T) \geq p(S,\lambda K_1 + (1-\lambda)K_2,t,T)$$

$$\lambda P(S,K_1,t,T) + (1-\lambda)P(S,K_2,t,T) \geq P(S,\lambda K_1 + (1-\lambda)K_2,t,T)$$

That is $p(S,K,t,T)$ and $P(S,K,t,T)$ are convex functions of K on $\mathbb{R}^+$

Proposition 1.19 — Inequality set 4.
If $T_1 \geq T_2 > t$, then

$$C(S,K,t,T_1) \geq C(S,K,t,T_2)$$
$$P(S,K,t,T_1) \geq P(S,K,t,T_2)$$

Further, if the stock S does not pay dividends, then

$$c(S,K,t,T_1) \geq c(S,K,t,T_2)$$

Proof.

- The first part is intuitive.
- The second part is due to the fact $c = C$ when no dividend is paid

■

Note 1.3 The previous inequality does not hold for European puts:

$$p(S,K,t,T_1) \ vs. \ p(S,K,t,T_2)$$

$\Longrightarrow$ It is not obvious which one has a larger price

1.7 Investment Strategies Using Options

1.7.1 Common Investment Strategies Using Options

Definition 1.17 — spread.
A **spread** is a position consisting of only calls or only puts
Depending on the combination, there are various spreads:

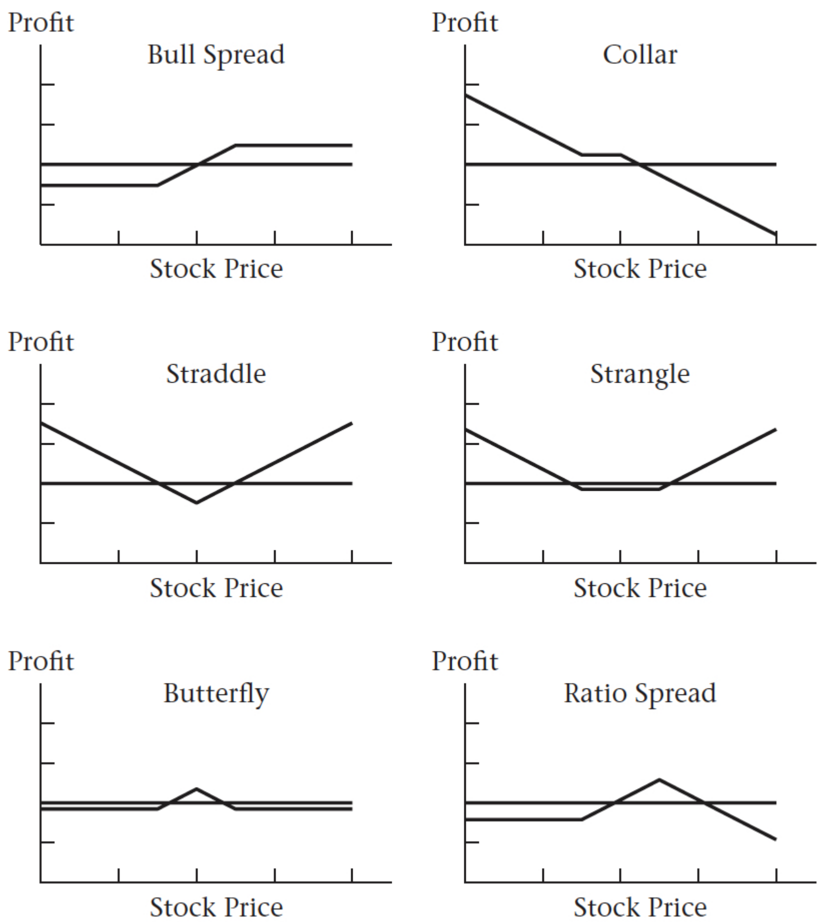
- **Bull spread**: long a call and short another call with higher strike (to bet on up movement)
- **Bear spread**: long a call and short another call with lower strike (to bet on down movement)
- **Ratio spread**: long m call options and short n call options at a different strike

Spreads can also be constructed using puts in a similar fashion!

Definition 1.18 — collar.
A **collar** is a position consisting of a long put and a short call with higher strike
Collar width: the difference between the strikes

Definition 1.19 — straddle, strangle, butterfly.

- **Straddle**: long an at-the-money call and an at-the-money put with the same strike
- **Strangle**: long an out-of-the-money call and an out-of-the-money put
- **Butterfly**: short a straddle and long a strangle



Use of straddle, strangle, and butterfly:

- Long straddle: speculation on high volatility
- Long strangle: speculation on high volatility with lower costs
- Long butterfly: speculation on low volatility

1.8 Portfolio Insurance Strategies Using Options

1.8.1 Four Insurance Strategies

- **Floor**: long a stock and long a put

$$Payoff = S_T + (K - S_T)_+$$

- **Covered call writing**: long a stock and short (write) a call

$$Payoff = S_T - (K - S_T)_+$$

- **Cap**: short a stock and long a call

$$Payoff = -S_T + (K - S_T)_+$$

- **Covered put writing**: short a stock and short (write) a put

$$Payoff = -S_T - (K - S_T)_+$$

Different positions can have the same payoff because of the identity:

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K$$

or more briefly, call - put = stock - bond