PMATH 351: Real Analysis

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1. Cardinality

Definition 1.1 — domain, range, image, inverse image.
Let X and Y be sets and let $f : X \to Y$. Recall the **domain** of f and the **range** of f are the sets

$$\text{Domain}(f) = X, \ \text{Range}(f) = f(X) = \{f(x)|x \in X\}$$

for $A \subseteq X$, the **image** of A under f is the set

$$f(A) = \{f(x)|x \in A\}$$

For $B \subseteq Y$, the **inverse image** of B under f is the set

$$f^{-1}(B) = \{x \in X|f(x) \in B\}$$

Definition 1.2 — Composite.
Let X, Y and Z be sets, let $f : X \to Y$ and let $g : Y \to Z$. We define the **composite** function $(g \circ f)(x) = g(f(x))$ for all $x \in X$

Definition 1.3 — injective, surjective, bijective.
We say that f is **injective** (or **one-to-one**) when for every $y \in Y$ there exists at most one $x \in X$ such that $f(x) = y$. Equivalently, f is injective when for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
We say that f is **surjective** (or **onto**) when for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$. Equivalently, f is surjective when $\text{Range}(f) = Y$
We say that f is **bijective** (or **invertible**) when f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f(x) = y$. When f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f^{-1} : Y \to X$ such that for all $y \in Y$, $f^{-1}(y)$ is equal to the unique element $x \in X$ such that $f(x) = y$. Note that when f is bijective so is f^{-1}, and in this case we have $(f^{-1})^{-1} = f$

Theorem 1.1 Let $f : X \to Y$ and let $g : Y \to Z$. Then
(1) If f and g are both injective then so is $g \circ f$
(2) If f and g are both surjective then so is $g \circ f$
(3) If f and g are both invertible then so is $g \circ f$, and in this case $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof.
(1) Suppose that f and g are both injective. Let $x_1, x_2 \in X$. If $g(f(x_1)) = g(f(x_2))$ then since g is injective we have $f(x_1) = f(x_2)$, and then since f is injective we have $x_1 = x_2$. Thus $g \circ f$ is injective.
(2) Suppose that f and g are both injective. Given $z \in Z$, since g is surjective we can choose $y \in Y$ so that $g(y) = z$, then since f is surjective we can choose $x \in X$ so that $f(x) = y$, and then we have $g(f(x)) = g(y) = z$. Thus $g \circ f$ is surjective.
(3) Follows (1) and (2).

■

Definition 1.4 — identity function.
For a set X, we define the **identity function** on X to be the function $I_X : X \to X$ given by $I_X(x) = x$ for all $x \in X$. Note that for $f : X \to Y$ we have $f \circ I_X = f$ and $I_Y \circ f = f$.

Definition 1.5 — inverse.
Let X and Y be sets and let $f : X \to Y$. A **left inverse** of f is a function $g : Y \to X$ given by $g \circ f = I_X$. Equivalently, a function $g : Y \to X$ is a left inverse of f when $g(f(x)) = x$ for all $x \in X$.
A **right inverse** of f is a function $h : Y \to X$ such that $f \circ h = I_Y$. Equivalently, a function $h : Y \to X$ is a right inverse of f when $f(h(y)) = y$ for all $y \in Y$.

Theorem 1.2 Let X and Y be nonempty sets and let $f : X \to Y$. Then
(1) f is injective $\iff$ f has a left inverse.
(2) f is surjective $\iff$ f has a right inverse.
(3) f is bijective $\iff$ f has a left inverse g and a right inverse h, and in this case we have $g = h = f^{-1}$.

Proof.
(1) Suppose first that f is injective. Since $X \neq \varnothing$ we can choose $a \in X$ and then define $g : Y \to X$ as follows: if $y \in \text{Range}(f)$ then (using the fact the f is injective) we define $g(y)$ to be the unique element $x_y \in X$ with $f(x_y) = y$, and if $y \notin \text{Range}(f)$, then we define $g(y) = a$. Then for every $x \in X$ we have $y = f(x) \in \text{Range}(f)$, so $g(y) = x_y = x$, that is $g(f(x)) = x$. Conversely, if f has a left inverse, say g, then f is injective since for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x = g(f(x_1)) = g(f(x_2)) = x_2$.
(2) Suppose first that f is onto. For each $y \in Y$, choose $x_y \in X$ with $f(x_y) = y$, then define $g : X \to Y$ by $g(y) = x_y$ (We need the Axiom of Choice for this). Then g is a right inverse of f since for every $y \in Y$ we have $f(g(y)) = f(x_y) = y$. Conversely, if f has a right inverse, say g, then f is onto since given any $y \in Y$ we can choose $x = g(y)$ and then we have $f(x) = f(g(y)) = y$.
(3) Suppose first that f is bijective. The inverse function $f^{-1} : Y \to X$ is a left inverse for f because given $x \in X$ we can let $y = f(x)$ and then $f^{-1}(y) = x$ so that $f^{-1}(f(x)) = f^{-1}(y) = x$. Similarly, f^{-1} is a right inverse for f because given $y \in Y$ we can let x be the unique element in X with $y = f(x)$ and then we have $x = f^{-1}(y)$ so that $f(f^{-1}(y)) = f(x) = y$. Conversely, suppose that g is a left inverse for f and h

is a right inverse for f. Since f has a left inverse, it is injective by (1). Since f has a right inverse, it is surjective by (2). Since f is injective and surjective, it is bijective. As shown above, the inverse function f^{-1} is both a left inverse and a right inverse. Finally, note that $g = f^{-1} = h$ because for all $y \in Y$ we have

$$g(y{=}g(f(f^{-1}(y))) = f^{-1}(y) = f^{-1}(f(h(y))) = h(y)$$

■

Corollary 1.3
Let X and Y be sets. Then there exists an injective map $f : X \to Y$ if and only if there exists a surjective map $g : Y \to X$.

Proof. Suppose $f : X \to Y$ is an injective map. Then f has a left inverse. Let g be a left inverse of f. Since $g \circ f = I_X$, we see that f is a right inverse of g. Since g has a right inverse, g is surjective. Thus, there is a surjective map $g : Y \to X$. Similarly, if $g : Y \to X$ is surjective, then it has a right inverse $f : X \to Y$ which is injective. ■

Definition 1.6 — same cardinality, less than or equal to, less than.
Let A and B be sets. We say that A and B have the **same cardinality**, and write $|A| = |B|$, when there exists a bijective map: $f : A \to B$ (or equivalently when there exists a bijective map $g : B \to A$).
We say that the cardinality of A is **less than or equal to** the cardinality of B, and write $|A| \le |B|$, when there exists an injective map $f : A \to B$ (or equivalently a surjective map $g : B \to A$).
We say that the cardinality of A is **less than** the cardinality of B, and write $|A| < |B|$, when $|A| \le |B|$ and $|A| \ne |B|$, (that is when there exists an injective map $f : A \to B$ but there does not exist a bijective map $g : A \to B$).
We also write $|A| \ge |B|$ when $|B| \le |A|$; and $|A| > |B|$ when $|B| < |A|$.

■ **Example 1.1** Let $\mathbb{N} = \{n \in \mathbb{Z} | n \ge 0\} = \{0, 1, 2, \cdots\}$.
(1) The map $f : \mathbb{N} \to 2\mathbb{N}$ given by $f(k) = 2k$ is bijective, so $|2\mathbb{N}| = |\mathbb{N}|$.
(2) The map $g : \mathbb{N} \to \mathbb{Z}$ given by $g(2k) = k$ and $g(2k + 1) = -k - 1$ for $k \in \mathbb{N}$ is bijective, so we have $|\mathbb{Z}| = |\mathbb{N}|$.
(3) The map $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $h(k, l) = 2^k(2l + 1) - 1$ is bijective, so we have $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Theorem 1.4 For all sets A, B and C
(1) $|A| = |A|$
(2) If $|A| = |B|$ then $|B| = |A|$
(3) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$
(4) $|A| \le |B| \iff (|A| = |B| \text{ or } |A| < |B|)$
(5) If $|A| \le |B|$ and $|B| \le |C|$, then $|A| \le |C|$

Proof.
(1) holds because the identity function $I_A : A \to A$ is bijective.
(2) holds because if $f : A \to B$ is bijective then so is $f^{-1} : B \to A$.
(3) holds because if $f : A \to B$ and $g : B \to C$ are bijective then so is the composite $g \circ f : A \to C$

■

Definition 1.7 — finite, infinite, countable.
Let A be a set. For each $n \in \mathbb{N}$, let $S_n = \{0, 1, 2, \cdots, n-1\}$. For $n \in \mathbb{N}$, we say that the cardinality of A is equal to n, or that A has n **elements**, and we write $|A| = n$, when $|A| = |S_n|$.
We say that A is **finite** when $|A| = n$ for some $n \in \mathbb{N}$. We say A is **infinite** when A is not finite. We say that A is countable when $|A| = |\mathbb{N}|$

Note 1.1 When a set A is finite with $|A| = n$, and when $f : A \to S_n$ is a bijection, if we let $a_k = f^{-1}(k)$ for each $k \in S_n$ then we have $A = \{a_0, a_1, \cdots, a_{k-1}\}$ with the elements a_k distinct. Conversely, if $A = \{a_0, a_1, \cdots, a_{k-1}\}$ with the elements a_k all distinct, then we define a bijection $f : A \to S_n$ by $f(a_k) = k$. Thus we see that A is finite with $|A| = n$ if and only if A is of the form $A = \{a_0, a_1, \cdots, a_{n-1}\}$ with the elements a_k all distinct. Similarly, a set A is countable if and only if A is of the form $A = \{a_0, a_1, a_2, \cdots\}$ with the elements a_k all distinct.

Note 1.2 For $n \in \mathbb{N}$, if A is a finite set with $|A| = n+1$ and $a \in |A \backslash \{a\}| = n$. Indeed, if $A = \{a_0, a_1, \cdots, a_n\}$ with the elements a_i distinct, and if $a = a_k$ so that we have $A \backslash \{a\} = \{a_0, a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n\}$, then we can define a bijection $f : S_n \to A \backslash \{a\}$ by $f(i) = a_i$ for $0 \le i < k$ and $f(i) = a_{i+1}$ for $k \le i < n$.

Theorem 1.5 Let A be a set. Then the following are equivalent:
(1) A is infinite
(2) A contains a countable subset
(3) $|\mathbb{N}| \le |A|$
(4) There exists a map $f : A \to A$ which is injective but not surjective

Proof.

$(1) \implies (2)$ Suppose A is infinite. Since $A \neq \varnothing$ we can choose an element $a_0 \in A$. Since $A \neq \{a_0\}$ we can choose an element $a_1 \in A \backslash \{a_0\}$. Since $A \neq \{a_0, a_1\}$ we can choose $a_3 \in A \backslash \{a_0, a_1\}$. Continue this procedure: having chosen distinct elements $a_0, a_1, \cdots, a_{n-1} \in A$, since $A \neq \{a_0, a_1, \cdots, a_{n-1}\}$ we can choose $a_n \in A \backslash \{a_0, a_1, \cdots, a_{n-1}\}$. In this way we obtain $\{a_0, a_1, a_2, \cdots\} \subseteq A$.

$(2) \iff (3)$ Suppose that A contains a countable subset, say $\{a_0, a_1, a_2, \cdots\} \subseteq A$ with the element a_i distinct. Since a_i are distinct, the map $f : \mathbb{N} \to A$ given by $f(k) = a_k$ is injective, and so we have $|\mathbb{N}| \le |A|$. Conversely as a map from $\mathbb{N} \to f(\mathbb{N})$ where f is bijective, so we have $|\mathbb{N}| = |f(\mathbb{N})|$ hence $f(\mathbb{N})$ is a countable subset of A.

$(2) \implies (4)$ Suppose that A has a countable subset, say $\{a_0, a_1, a_2, \cdots\} \subseteq A$ with the element a_i distinct. Define $f : A \to A$ by $f(a_k) = a_{k+1}$ for all $k \in \mathbb{N}$ and by $f(b) = b$ for all $b \in A \backslash \{a_0, a_1, a_2, \cdots\}$. Then f is injective but not surjective (the element a_0 is not in the range of f).

$(4) \implies (1)$ To prove this we shall prove that if A is finite then every injective map $f : A \to A$ is surjective. We prove this by induction on the cardinality of A.
The only set A with $|A| = 0$ is the set $A \neq \varnothing$, and then the only function $f : A \to A$ is the empty function, which is surjective.
Since that base case may appear too trivial, let us consider the next case. Let $n = 1$ and let A be a set with $|A| = 1$, say $A = \{a\}$. The only function $f : A \to A$ is the function given by $f(a) = a$, which is surjective.
Let $n \ge 1$ and suppose, inductively, that for every set A with $|A| = n$, every injective

map $f : A \to A$ is surjective. Let B be a set with $|B| = n + 1$ and let $g : B \to B$ be injective.

Suppose, for a contradiction, that g is not surjective. Choose an element $b \in B$ which is not in the range of g so that we have $g : B \to B \backslash \{b\}$. Let $A = B \backslash \{b\}$ and let $f : A \to A$ be given by $f(x) = g(x)$ for all $x \in A$. Since $g : B \to A$ is injective and $f(x) = g(x)$ for all $x \in A$, f is also injective. Again since g is injective, there is no element $x \in B \backslash \{b\}$ with $g(x) = g(b)$, so there is no element $x \in A$ with $f(x) = g(b)$, and so f is not surjective. Since $|A| = n$, this contradicts the induction hypothesis. Thus g must be surjective.

By the Principle of Induction, for every $n \in \mathbb{N}$ and for every set A with $|A| = n$, every injective function $f : A \to A$ is surjective.

■

Corollary 1.6

Let A and B be sets.

(1) If A is countable then A is infinite

(2) When $|A| \le |B|$, if B is finite so is A (equivalently if A is infinite then so is B)

(3) If $|A| = n$ and $|B| = m$ then $|A| = |B|$ if and only if $n = m$

(4) If $|A| = n$ and $|B| = m$ then $|A| \le |B|$ if and only if $n \le m$

(5) When one of the two sets A and B is finite, if $|A| \le |B|$ and $|B| \le |A|$ then $|A| = |B|$

Proof.

(1) If A is countable then A contains a countable subset (itself), so A is infinite by Theorem 1.5.

(2) Suppose that $|A| \le |B|$ and that $|A|$ is infinite. Since A is infinite, we have $|\mathbb{N}| \le |A|$ (by Theorem 1.5). Since $|\mathbb{N}| \le |A|$ and $|A| \le |B|$ we have $|\mathbb{N}| \le |B|$ (by Theorem 1.4). Since $|\mathbb{N}| \le |B|$, B is infinite (by Theorem 1.5).

(3) Suppose that $|A| = n$ and $|B| = m$. If $n = m$ then we have $S_n = S_m$ and so $|A| = |S_n| = |S_m| = |B|$. Conversely, suppose that $|A| = |B|$. Suppose, for a contradiction, that $n \ne m$, say $n > m$, and note that $S_m \subsetneq S_n$. Since $|A| = |B|$ we have $|S_n| = |A| = |B| = |S_m|$ so we must have $n = m$.

(4) Suppose $|A| = n$ and $|B| = m$. If $n \le m$ then $S_n \subseteq S_m$ so the inclusion map $I : S_n \to S_m$ is injective and we have $|A| = |S_n| \le |S_m| = |B|$. Conversely, suppose that $|A| \le |B|$ and suppose, for a contradiction, that $n > m$. Since $|A| \le |B|$ we have $|S_n| = |A| \le |B| = |S_m|$ so we can choose an injective map $f : S_n \to S_m$. Since $n > m$ we have $S_m \subsetneq S_n$ so we can consider f as a map $f : S_n \to S_m$, and this map is injective but not surjective. This contradicts Theorem 1.5, and so $n \le m$.

(5) Suppose that one of the two sets A and B is finite, and that $|A| \le |B|$ and $|B| \le |A|$. If $A$is finite then, since $|B| \le |A|$, (2) implies that B is finite. If B is finite then, since $|A| \le |B|$, (2) implies that A is finite. Thus, in either case, we see that A and B are both finite. Since A and B are both finite with $|A| \le |B|$ and $|B| \le |A|$, we must have $|A| = |B|$ by (3) and (4).

■

Theorem 1.7 Let A be a set. Then $|A| \le |\mathbb{N}| \iff A$ is finite or countable.

Proof. First we claim that every subset of $\mathbb{N}$ is either finite or countable. Let $A \subseteq \mathbb{N}$ and suppose that A is not finite.

Since $A \ne \varnothing$, we can set $a_0 = \min\{A\}$ (using the Well-Ordering Property of $\mathbb{N}$). Note that

$\{0,10,\cdots,a_0\} \cap A = \{a_0\}$.
Since $A \neq \{a_0\}$ (so the set $A \backslash \{a_0\}$ is nonempty), we can set $a_1 = \min\{A \backslash \{a_0\}\}$. Then we have $a_0 < a_1$ and $\{0,1,\cdots,a_1\} \cap A = \{a_0,a_1\}$.
Since $A \neq \{a_0,a_1\}$ we can set $a_2 = \min\{A \backslash \{a_0,a_1\}\}$. Then we have $a_0 < a_1 < a_2$ and $\{0,1,2,\cdots,a_3\} \cap A = \{a_0,a_1,a_2\}$
We continue the procedure: having chosen $a_0,a_1,\cdots,a_{n-1} \in A$ with $a_0 < a_1 < \cdots < a_{n-1}$ such that $\{0,1,\cdots,a_{n-1}\} \cap A = \{a_0,a_1,\cdots,a_{n-1}\}$. Since $A \neq \{a_0,a_1,\cdots,a_{n-1}\}$, we can set $a_n = \min\{A \backslash \{a_0,a_1,\cdots,a_{n-1}\}\}$ and then we have $a_0 < a_1 < \cdots < a_{n-1} < a_n$ and $\{0,1,\cdots,a_n\} \cap A = \{a_0,a_1,\cdots,a_n\}$.
In this way, we obtain a countable set $\{a_0,a_1,a_2,\cdots\} \subseteq A$ with $a_0 < a_1 < a_2 < \cdots$ with the property that for all $m \in \mathbb{N}$, $\{0,1,2,\cdots,a_m\} \cap A = \{a_0,a_1,\cdots,a_m\}$.
Since $0 \leq a_0 < a_1 < a_2 < \cdots$, it follows (by induction) that $a_k \geq k$ for all $k \in \mathbb{N}$. It follows in turn that $A \subseteq \{a_0,a_1,a_2,\cdots\}$ because given $m \in A$, since $m \leq a_m$ we have

$$m \in \{0,1,2,\cdots,m\} \cap A \subseteq \{0,1,2,\cdots,a_m\} \cap A = \{a_0,a_1,\cdots,a_m\}.$$

Thus $A = \{a_0,a_1,a-2,\cdots\}$ and the elements a_i are distinct, so A is countable. This proves our claim that every subset of $\mathbb{N}$ is either finite or countable.
Suppose that $|A| \leq |\mathbb{N}|$ and choose an injective map $f : A \to \mathbb{N}$. Since f is injective, when we consider it as a map $f : A \to f(A)$, it is bijective, and so $|A| = |f(A)\}$. Since $f(A) \subseteq \mathbb{N}$, the previous paragraph shows that $f(A)$ is either finite or countable. If $f(A)$ is finite with $|f(A)| = n$ then $|A| = |f(A)| = |S_n|$, and if $f(A)$ is countable then we have $|A| = |f(A)| = |\mathbb{N}|$. Thus A is finite or countable.

■

Theorem 1.8 Let A be a set. Then
(1) $|A| < |\mathbb{N}| \iff A$ is finite
(2) $|\mathbb{N}| < |A| \iff A$ is neither finite nor countable
(3) if $|A| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |A|$ then $|A| = |\mathbb{N}|$

Proof.
(1) By Theorem 1.5

$$\begin{aligned} |A| < |\mathbb{N}| &\iff (|A| \leq |\mathbb{N}| \text{ and } |A| \neq |\mathbb{N}|) \\ &\iff (A \text{ if finite or countable and } A \text{ is not countable}) \\ &\iff A \text{ is finite} \end{aligned}$$

(2) By Theorem 1.7

$$\begin{aligned} |\mathbb{N}| < |A| &\iff (|\mathbb{N}| \leq |A| \text{ and } |\mathbb{N}| \neq |A|) \\ &\iff (A \text{ is not finite and } A \text{ is not countable}) \end{aligned}$$

(3) Suppose that $|A| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |A|$. Since $|A| \leq |\mathbb{N}|$, we know that A is finite or countable by Theorem 1.7. Since $|N| \leq |A|$, we know that A is infinite by Theorem 1.5. Since A is finite or countable and A is not finite, it follows that A is countable. Thus $|A| = |\mathbb{N}|$

■

Definition 1.8 — at most countable, uncountable.
Let A be a set. When A is countable we write $|A| = \aleph_0$. When A is finite we write $|A| < \aleph_0$. When A is infinite we write $|A| \geq \aleph_0$. When A is either finite or countable we write $|A| \leq \aleph_0$ and we say that A is **at most countable**. When A is neither finite nor

countable we write $|A| > \aleph_0$ and we say that A is **uncountable**.

Theorem 1.9
(1) If A and B are countable sets, then so is $A \times B$
(2) If A and B are countable sets, then so is $A \cup B$
(3) If $A_0, A_1, A_2, \cdots$ are countable sets, then so is $\bigcap_{k=0}^{\infty} A_k$
(4) $\mathbb{Q}$ is countable

Proof.
(1) Let $A = \{a_0, a_1, a_2, \cdots\}$ with the a_i distinct and let $B = \{b_0, b_1, b_2, \cdots\}$ with b_i distinct. Since every positive integer can be written uniquely in the form $2^k(2l+1)$ with $k, l \in \mathbb{N}$, the map $f : A \times B \to \mathbb{N}$ given by $f(a_k, b_l) = 2^k(2l+1) - 1$ is bijective, and so $|A \times B| = |\mathbb{N}|$
(2) Similar to (1), since the map $g : \mathbb{N} \to A \cup B$ given by $g(k) = a_k$ is injective, we have $|\mathbb{N}| \leq |A \cup B|$. Since the map $h : \mathbb{N} \to A \cup B$ given by $h(2k) = a_k$ and $h(2k+1) = b_k$ is surjective, we have $|A \cup B| \leq |\mathbb{N}|$. Since $|\mathbb{N}| \leq |A \cup B|$ and $|A \cup B| \leq |\mathbb{N}|$, we have $|A \cup B| = |\mathbb{N}|$ by Theorem 1.8
(3) For each $k \in \mathbb{N}$, let $A_k = \{a_{k0}, a_{k1}, a_{k2}, \cdots\}$ with the a_{ki} distinct. Since the map $f : \mathbb{N} \to \bigcap_{k=0}^{\infty} A_k$ given by $f(k) = a_{0,k}$ is injective, $|\mathbb{N}| \leq \left|\bigcap_{k=0}^{\infty} A_k\right|$. Since $\mathbb{N} \times \mathbb{N}$ is countable by (1), and since the map $g : \mathbb{N} \times \mathbb{N} \to \bigcap_{k=0}^{\infty} A_k$ given by $g(k,l) = a_{k,l}$ is surjective, we have $\left|\bigcap_{k=0}^{\infty} A_k\right| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. By Theorem 1.8, we have $\left|\bigcap_{k=0}^{\infty} A_k\right| = |\mathbb{N}|$.
(4) Since the map $f : \mathbb{N} \to \mathbb{Q}$ given by $f(k) = k$ is injective, we have $|\mathbb{N}| \leq |\mathbb{Q}|$. Since the map $g : \mathbb{Q} \to \mathbb{Z} \times \mathbb{Z}$ given by $g(\frac{a}{b}) = (a, b)$ for all $a, b \in \mathbb{Z}$ with $b > 0$ and $\gcd(a, b) = 1$, is injective, and since $\mathbb{Z} \times \mathbb{Z}$ is countable, we have $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$. Since $|\mathbb{N}| \leq |\mathbb{Q}|$ and $|\mathbb{Q}| \leq |\mathbb{N}|$, we have $|\mathbb{Q}| = |\mathbb{N}|$

■

Exercise 1.1 Let A be a countable set. Show that the set of finite sequences with terms in A is countable. Show that the set of all finite subsets of A is countable.

Definition 1.9 — power set.
For a set A, let $\mathcal{P}(A)$ denote the **power set** of A, that is the set of all subsets of A, and let 2^A denote the set of all functions from A to $S_2 = \{0, 1\}$

Theorem 1.10
(1) For every set A, $\mathcal{P}(A) = |2^A|$
(2) For every set A, $|A| < \mathcal{P}(A)$
(3) $\mathbb{R}$ is uncountable

Proof.
(1) Let A be any set. Define a map $g : \mathcal{P}(A) \to 2^A$ as follows: given $S \in \mathcal{P}(A)$, that is given $S \subseteq A$, we define $g(S) \in 2^A$ to be the map $g(S) : A \to \{0, 1\}$ given by

$$g(S)(a) = \begin{cases} 1 & \text{if } a \in S, \\ 0 & \text{if } a \notin S. \end{cases}$$

Define map $h : 2^A \to \mathcal{P}(A)$ as follows: given $f \in 2^A$, that is given a map: $f : A \to \{0, 1\}$, we define $h(f) \in (A)$ to be the subset

$$h(f) = \{a \in A | f(a) = 1\} \subseteq A$$

This maps g and h are the inverses of each other because for every $S \subseteq A$ and every $f : A \to \{0, 1\}$ we have

$$\begin{aligned} f = g(S) \quad &\iff \quad \forall a \in A, f(a) = g(S)(a) \iff \forall a \in A, f(a) = \begin{cases} 1 & \text{if } a \in S, \\ 0 & \text{if } a \notin S. \end{cases} \\ &\iff \quad \forall a \in A, (f(a) = 1 \iff a \in S) \iff \{a \in A | f(a) = 1\} = S \\ &\iff \quad h(f) = S \end{aligned}$$

(2) Let A be any set. Since the map $f : A \to \mathcal{P}(A)$ given by $f(a) = \{a\}$ is injective, we have $|A| \le |\mathcal{P}(A)|$. We need to show that $|A| \ne |\mathcal{P}(A)|$. Let $g : A \to \mathcal{P}(A)$ be any map. Let $S = \{a \in A | a \notin g(a)\}$. Note that S cannot be in the range of g because we could choose $a \in A$ so that $g(a) = S$ then, by the definition of S, we would have

$$a \in S \iff a \notin g(a) \iff a \notin S$$

which is impossible. Since S is not in the range of g, the map g is not surjective. Since g was an arbitrary map from A to $\mathcal{P}(A)$, it follows that there is no surjective map from A to $\mathcal{P}(A)$. Thus there is no bijective map from A to $\mathcal{P}(A)$ and so we have $|A| \ne |\mathcal{P}(A)|$.

(3) We prove $\mathbb{R}$ is uncountable using the fact that every real number has a unique decimal expansion which does not end with an infinite string of 9's. Define a map $g : 2^{\mathbb{N}} \to \mathbb{R}$ as follows: given $f \in 2^{\mathbb{N}}$, that is given a map $f : \mathbb{N} \to \{0, 1\}$, we define $g(f)$ to be the real number of $g(f) \in [0, 1)$ with the decimal expansion $g(f) = 0.f(1)f(2)f(3)\cdots$, that is $g(f) = \sum_{k=0}^{\infty} f(k)10^{-k-1}$. By the uniqueness of decimal expansions, the map g is injective, so we have $|2^{\mathbb{N}}| \le |\mathbb{R}|$. Thus $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}| \le |\mathbb{R}|$, and so $\mathbb{R}$ is uncountable by Theorem 1.8.

■

Theorem 1.11 — Cantor-Schroeder-Bernstein.
Let A and B be sets. Suppose that $|A| \le |B|$ and $|B| \le |A|$. Then $|A| = |B|$

Proof. We sketch a proof. Choose injective functions $f : A \to B$ and $g : B \to A$. Since the functions $f : A \to f(A)$, $g : B \to g(B)$ and $f : g(B) \to f(g(B))$ are bijective, we have $|A| = |f(A)|$ and $|B| = |g(B)| = |f(g(B))|$. Also note that $f(g(B)) \subseteq f(A) \subseteq B$. Let $X = f(g(B))$, $Y = f(A)$ and $Z = B$. Then we have $X \subseteq Y \subseteq Z$ and we have $|x| = |z|$ and we need to show that $|Y| = |Z|$. The composite $h = f \circ g : Z \to X$ is a bijective.
Define sets Z_n and Y_n for $n \in \mathbb{N}$ recursively by

$$Z_0 = Z, Z_n = h(Z_{n-1}) \text{ and } Y_0 = Y, Y_n = h(Y_{n-1})$$

Since $Y_0 = Y$, $Z_0 = Z$, $Z_1 = h(Z_0) = h(Z) = X$ and $X \subseteq Y \subseteq Z$, we have

$$Z_1 \subseteq Y \subseteq Z_0$$

Also note that for $1 \le n \in \mathbb{N}$,

$$Z_n \subseteq Y_{n-1} \subseteq Z_{n-1} \implies h(Z_n) \subseteq h(Y_{n-1}) \subseteq h(Z_{n-1}) \implies Z_{n+1} \subseteq Y_n \subseteq Z_n$$

By the Induction Principle, it follows that $Z_n \subseteq Y_{n-1} \subseteq Z_{n-1}$ for all $n \geq 1$, so we have

$$Z_0 \supseteq Y_0 \supseteq Z_1 \supseteq Y_1 \supseteq Z_2 \supseteq Y_2 \supseteq \cdots$$

Let $U_n = \frac{Z_n}{Y_n}$, $U = \bigcup_{n=1}^{\infty} U_n$ and $V = \frac{Z}{U}$. Define $H : Z \to Y$ by

$$H(x) = \begin{cases} h(x) & \text{if } x \in U \\ x & \text{if } x \in V \end{cases}$$

Verify that H is bijective. ■

Exercise 1.2 Show that $|\mathbb{R}| = |2^{\mathbb{N}}|$

Solution. $g : 2^{\mathbb{N}} \to \mathbb{R}$ as follows: for $f \in 2^{\mathbb{N}}$ we let $g(f)$ be the real number $g(f) \in [0, 1)$ with decimal expansion $g(f) = 0.f(1)f(2)\cdots$. Then g is injective so $|2^{\mathbb{N}}| \leq \mathbb{R}$. Define $h : 2^{\mathbb{N}} \to [0, 1)$ as follows: for $f \in 2^{\mathbb{N}}$ let $h(f)$ be the real number $h(f) \in [0, 1]$ with binary expansion $h(f) = 0.f(0)f(1)f(2)\cdots$. Then h is surjective so we have $|[0, 1]| \leq |2^{\mathbb{N}}|$. The map $k : \mathbb{R} \to [0, 1]$ given by $k(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$ is injective, so we have $|\mathbb{R}| \leq |[0, 1]|$. Since $|\mathbb{R}| \leq |[0, 1]| \leq |2^{\mathbb{N}}|$ and $|2^{\mathbb{N}}| \leq \mathbb{R}$, we have $|\mathbb{R}| = |2^{\mathbb{N}}|$ by the Cantor-Schroeder-Bernstein Theorem (1.11) ■

Notation 1.1 For sets A and B, we write A^B to denote the set of functions $f : B \to A$

Theorem 1.12 Let A and B be finite sets and let $\mathcal{P}(A)$ is the power set of A (that is the set of all subsets of A). Then

(1) if A and N are disjoint then $|A \cup B| = |A| + |B|$
(2) $|A \times B| = |A| \cdot |B|$
(3) $|A^B| = |A|^{|B|}$
(4) $|\mathcal{P}| = 2^{|A}$

Proof. The proof is left as an exercise ■

Theorem 1.13 Let A, B, C and D be sets with $|A| = |C|$ and $|B| = |D|$. Then

(1) if $A \cap B = \varnothing$ and $C \cap D = \varnothing$ then $|A \cup B| = |C \cup D|$
(2) $|A \times B| = |c \times D|$
(3) $|A^B| = |C^D|$

Proof. The proof is left as an exercise ■

R It is possible to define certain specific sets called **cardinals** such that for every set A there exists a unique cardinal κ with $|A| = |\kappa|$. We can then define the **cardinality** of a set A to be equal to the unique cardinal κ such that $|A| = ||$ and, in this case, we define the **cardinality** of the set A to be $|A| = \kappa$. In foundational set theory, the natural numbers are defined, formally, to be equal to the sets $0 = \varnothing$, $1 = \{0\} = \{\varnothing\}$, $2 = \{0, 1\} = \{\varnothing, \{\varnothing\}\}$ and, in general,$n+1 = n \cup \{n\}$ so that the natural number n is equal to the set that we previously denoted by S_n, that is $n = S_n = \{0, 1, \cdots, n-1\}$. The finite cardinals are equal to the natural numbers and the countable cardinal $\aleph_0$ is equal to the set of natural numbers. The previous theorem allows us to define **arithmetic operations** on cardinals which extend the usual arithmetic operations on the natural numbers. Given cardinals κ and λ we define $\kappa + \lambda$, $\kappa \cdot \lambda$ and κ^λ to be the cardinals such that

- $\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$
- $\kappa \cdot \lambda = |\kappa \times \lambda|$
- $\kappa^\lambda = |\kappa^\lambda|$

Theorem 1.14 Let κ, λ and μ be cardinals. Then

(1) $\kappa + \lambda = \lambda + \kappa$
(2) $(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu)$
(3) $\kappa + 0 = \kappa$
(4) $\lambda \leq \mu \implies \kappa + \lambda \leq \kappa + \mu$
(5) $\kappa \cdot \lambda = \lambda \cdot \kappa$
(6) $(\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu)$
(7) $\kappa \cdot 1 = \kappa$
(8) $\kappa \cdot (\lambda + \mu) = (\kappa \cdot \lambda) + (\kappa \cdot \lambda)$
(9) $\lambda \leq \mu \implies \kappa \cdot \lambda \leq \kappa \cdot \mu$
(10) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$
(11) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$
(12) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$
(13) $\lambda \leq \mu \implies \kappa^\lambda \leq \kappa^\mu$
(14) $\kappa \leq \lambda \implies \kappa^\mu \leq \lambda^\mu$

Proof. We sketch a proof for (9) and (11) and leave the rest as an exercise.

(9) Let A, B and C be sets with $|A| = \kappa$, $|B| = \lambda$ and $|C| = \mu$ and suppose that $|B| \leq |C|$. We need to show that $|A \times B| \leq |A \times C|$. Let $f : B \to C$ be an injective map. Define $F : A \times B \to A \times C$ by $F(a, b) = (a, f(b))$ then verify that F is injective.

(11) Let A, B and C be sets with $|A| = \kappa$, $|B| = \lambda$ and $|C| = \mu$. We need to show $|(A^B)^C| = |A^{B \times C}|$. Define $F : (A^B)^C \to A^{B \times C}$ by $F(f)(b, c) = f(c)(b)$. Verify that F is bijective with inverse $G : A^{B \times C} \to (A^B)^C$ given by $G(g)(c)(b) = g(b, c)$

■

Exercise 1.3 Show that $\left| \bigcup_{n=0}^{\infty} \mathbb{R}^n \right| = 2^{\aleph_0}$

Exercise 1.4 Find $\left|\mathbb{R}^{[0,1]}\right|$