PMATH 451: Measure and Integration

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Fsll 2021

Contents

1. Chapter 1

1.1 Lecture 1 A brief review of Riemann integrals

Limitations of Riemann Integration (R-int)

(1) Heavily rely on the structure of real line $\mathbb{R}$
(2) Not many functions are R-int

> **Theorem 1.1 — Lebesgue.**
> $f[a,b] \to \mathbb{R}$ is R-int if and only if the set of discontinuity of f is Lebesgue null set (has Lebesgue measure 0). (i.e. $\exists (a_n, b_n)$ s.t. the set of discontinuities $\subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$, $\sum(b_n - a_n) < \epsilon$)

■ **Example 1.1** $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$, $x \in [0,1]$. f is nowhere continuous. f is NOT R-int

(3) NOT well behaved under limits

■ **Example 1.2** Let $\{r_k\}_{k=1}^{\infty}$ be all $\mathbb{Q}$ in $[0,1]$, $f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \ldots, r_n\} \\ 0 & \text{otherwise} \end{cases}$

- f_n is R-int
- $\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \notin \mathbb{Q} \cap [0,1] \end{cases} = f(x)$ (pointwise limit) is not R-int

Lebesgue's Idea

Ideally, define $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$

- $m([a,b]) = b - a$
- $m(A + x) = m(A)$
- $m(\bigcup_{n=1}^{\infty} A_n) = \sum m(A_n)$, A_n disjoint

Problem: m does not exists

Proof. Define $x \sim y$ if $x - y \in \mathbb{Q}$, consider $[0,1]$.

Let A = pick one x from each eq-class of $\sim$.
Let $\{r_k\}_{k=1}^{\infty}$ be all rationals in $[-1,1]$.
Let $A_k = A + r_k$

(1) A_k are disjoint. If $x \in A_k \cap A_\ell$, then

$$x = \underbrace{a}_{\in A} + r_k = \underbrace{b}_{\in A} + r_\ell \implies a - b = r_\ell - r_k \in \mathbb{Q}$$

$$\implies a \sim b, a \neq b$$

not possible

(2) $[0,1] \subseteq \bigcup_{n=1}^{\infty} A_n \subseteq [-1,2]$
  (a) $A \subseteq [0,1]$, $-1 \leq r_k \leq 1$, $-1 \leq a + r_k \leq 2, A + r_k \subseteq [-1,2]$
  (b) $\forall x \in [0,1]$, $\exists a \in A$, $a \sim x$, $x - a \in \mathbb{Q}$, $-1 \leq x - a \leq 1$, $\implies$ $x - a = r_k$ for some k, $x = a + r_k \in A + r_k \subseteq \bigcup_{n=1}^{\infty} A_n$
  (c)

$$1 = m([0,1]) \leq m(\bigcup_{n=1}^{\infty} A_n) \leq m([-1,2]) = 3$$

$$m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} m(A)$$

not possible

■

1.2 Lecture 2 Introduction to Sigma Algebra

Definition 1.1 — algebra.
Let X be a set. An **algebra** $\mathcal{A}$ is a collection of subsets of X ($\mathcal{A} \subseteq (x)$) s.t.
(1) $\varnothing \in \mathcal{A}$
(2) $X \backslash A \in \mathcal{A}$ for all $A \in \mathcal{A}$
(3) If $A_1, \ldots, A_n \in \mathcal{A}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$
We call $\mathcal{A}$ a $\sigma-$**algebra** if
(3') If $A_1, \ldots, A_n, \ldots \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

■ **Example 1.3** (1) X be any set, $\{\varnothing, X\}$ is a $\sigma-$algebra

Note 1.1 If $\mathcal{A}$ is a $\sigma-$algebra, then it's an algebra

(2) $\mathcal{P}(X)$ is a $\sigma-$algebra
(3) Let X be an uncountable set (real line, Cantor set, etc.) Let $\mathcal{A} = \{E \subseteq X :$ either E is countable, or $X \backslash E$ is countable$\}$.
Claim: $\mathcal{A}$ is a $\sigma-$algebra

Proof. (a) $\varnothing \subseteq \mathcal{A}$, $\varnothing$ is countable
(b) Let $E \in \mathcal{A}$,
- Case 1: E is countable, $X \backslash (X \backslash E) = E$ is countable, $\implies$ $X \backslash E \in \mathcal{A}$
- Case 2: $X \backslash E$ is countable, $\implies$ $X \backslash E \in \mathcal{A}$

(c) Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, consider $\bigcup_{n=1}^{\infty} E_n$
- Case 1: If all E_n are countable, $\bigcup_{n=1}^{\infty} E_n$ is countable $\implies$ $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$
- Case 2: $\exists E_N$, s.t. $X \backslash E_N$ is countable.

$$X \backslash \left(\bigcup_{n=1}^{\infty} E_n \right) = \bigcap_{n=1}^{\infty} (X \backslash E_n) \subseteq X \backslash E_N \implies \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$$

Hence $\mathcal{A}$ is a $\sigma-$algebra. ■

1.3 Lecture 3 Properties of Sigma Algebra

Basic properties of Algebra and $\sigma-$algebra

Let $\mathcal{A}$ be an algebra, $\mathcal{B}$ be a $\sigma-$algebra

- $E, F \in \mathcal{A} \implies E \cap F,\ E \backslash F \in \mathcal{A}$
- $\{E_n\} \subseteq \mathcal{B} \implies \bigcap_{n=1}^{\infty} E_n \in \mathcal{B}$

Proof.

$$X \backslash (E \cap F) = \underbrace{(X \backslash E)}_{\in \mathcal{A}} \cup \underbrace{(X \backslash F)}_{\in \mathcal{A}} \in \mathcal{A}$$

$$E \cap F = X \backslash (X \backslash (E \cup F)) \in \mathcal{A}$$

$$E \backslash F = E \cap (X \backslash F) \in \mathcal{A}$$

Exercise 1.1 $\bigcup_{n=1}^{k} E_n \in \mathcal{A}$, induction

$$X \backslash \left(\bigcap_{n=1}^{\infty} E_n \right) = \bigcup_{n=1}^{\infty} (X \backslash E_n) \in \mathcal{B}$$

■

Proposition 1.2
Let $\mathcal{A}$ be an algebra, $\forall \{A_n\}_{n=1}^{\infty}$ disjoint $\implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. Then $\mathcal{A}$ is a $\sigma-$algebra

Proof. Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, goal is to show $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$. Consider

$$E_1 : A_1 = E_1$$

$$E_2 : A_2 = E_2 \backslash E_1$$

$$E_3 : A_3 = E_3 \backslash (E_1 \cup E_2)$$

$$\vdots$$

$$E_n : A_n = E_n \backslash \left(\bigcup_{i=1}^{n-1} E_i \right)$$

$$\vdots$$

Claim:

(1) $\bigcup_{n=1}^{k} A_n = \bigcup_{n=1}^{k} E_n$ (exercise using induction)

(2) A_n are disjoint.

$A_n \cap \left(\bigcup_{i=1}^{n-1} E_i \right) = \varnothing$

$A_n \cap \left(\bigcup_{i=1}^{n-1} A_i \right) = \varnothing$

(3) $A_n \in \mathcal{A}$.

By definition, $A_n = E_n \backslash \left(\bigcup_{i=1}^{n-1} E_i \right) \in \mathcal{A}$

$\implies \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ (by assumption)

$\implies \mathcal{A}$ is $\sigma-$algebra ■

Proposition 1.3
Suppose $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ are $\sigma-$algebras (on X). Then $\bigcap_{\lambda\in\Lambda}\mathcal{B}_\lambda$ is a $\sigma-$algebra

Proof. (1) $\varnothing\in\mathcal{B}_\lambda,\ \forall\lambda \implies \varnothing\in\bigcap_{\lambda\in\Lambda}\mathcal{B}_\lambda$
(2) Take

$$\begin{aligned} A\in\bigcap_{\lambda\in\Lambda}\mathcal{B}_\lambda &\implies A\in\mathcal{B}_\lambda\ \forall\lambda \\ &\implies X\backslash A\in\mathcal{B}_\lambda\ \forall\lambda \\ &\implies X\backslash A\in\bigcap_{\lambda\in\Lambda}\mathcal{B}_\lambda\ \forall\lambda \end{aligned}$$

(3) Take

$$\{A_n\}_{n=1}^\infty\subseteq\bigcup_{\lambda\in\Lambda}\mathcal{B}_\lambda \implies \bigcup_{n=1}^\infty A_n\in\bigcap_{\lambda\in\Lambda}\mathcal{B}_\lambda \text{ (exercise)}$$

■

Definition 1.2
Let $\mathcal{F}\subseteq\mathcal{P}(X)$. Define $\sigma-$algebra generated by $\mathcal{F}$ to be $\sigma(\mathcal{F})=\bigcap_{\lambda\in\Lambda}\mathcal{B}_\lambda$, $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ are all σ-algebra containing $\mathcal{F}$

(1) $\mathcal{F}\subseteq\mathcal{P}(X)$, $\mathcal{P}(X)$ is a $\sigma-$algebra, so this intersection makes sense.
(2) Since $\mathcal{B}_\lambda$ is $\sigma-$algebra $\implies \bigcup_{\lambda\in\Lambda}\mathcal{B}_\lambda$ is a $\sigma-$algebra ($\sigma(\mathcal{F})$ is a $\sigma-$algebra)

Proposition 1.4
$\sigma(\mathcal{F})$ is the **smallest** $\sigma-$algebra containing $\mathcal{F}$.
By smallest: if $\exists\mathcal{B}$ $\sigma-$algebra, $\mathcal{B}\supseteq\mathcal{F}$, then $\sigma(\mathcal{F})\subseteq B$

Proof. By defn, $\sigma(\mathcal{F})=\bigcap_{\lambda\in\Lambda}B_\lambda$, $\sigma-$alg $\mathcal{B}\supseteq\mathcal{F}\implies\mathcal{B}=\mathcal{B}_{\lambda_0}$ for some λ_0. $\sigma(\mathcal{F})=\bigcap_{\lambda\in\Lambda}\mathcal{B}_\lambda\subseteq\mathcal{B}_{\lambda_0}=\mathcal{B}$ ■

1.3.1 Borel $\sigma-$algebra

Let X be a metric space, let $g=\{A\subseteq X : A \text{ open}\}$

Definition 1.3 — Borel $\sigma-$algebra.
the **Borel $\sigma-$alg** on X is $\sigma(\mathcal{G})$. Denote by $\mathcal{B}_X=\sigma(\mathcal{G})$

Notation 1.1.

- *$\mathcal{G}$ = set of open sets*
- *$\mathcal{F}$ = set of closed sets*
- $\mathcal{G}_\delta=\{\bigcap_{n=1}^\infty A_n :\ A_n\in\mathcal{G}\}$
- $\mathcal{F}_\sigma=\{\bigcup_{n=1}^\infty A_n :\ A_n$ *closed*$\}$
  $\mathcal{G}_\delta,\mathcal{F}_\sigma\subseteq\mathcal{B}_X$
- *a set $A\subseteq X$ is Borel if $A\in\mathcal{B}_X$*
  - *open set*
  - *closed set*
  - $\mathcal{G}_\delta,\ \mathcal{F}_\sigma$
  - $X=\mathbb{R}$, $(a,b]\in\mathcal{B}_X$