PMATH 365: Differential Geometry

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Contents

1. Curves

1.1 Curves in $\mathbb{R}^n$

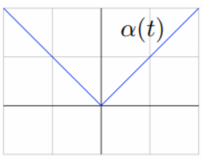
Definition 1.1 — curve, tangent vector, smooth, regular.
A (parametrized) **curve** in $\mathbb{R}^n$ is a continuous map $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ where I is a nonempty interval. We can write $\alpha(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ where each $x_k : I \subseteq \mathbb{R} \to \mathbb{R}$ is continuous.

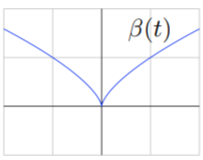
When $a \in I$ and $\alpha'(a) = ({x_1}'(a), \cdots, {x_n}'(a))$ exists, $\alpha'(a)$ is called the **tangent vector** to α at $t = a$.

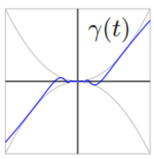
We say that α is $\mathcal{C}^k$ when the k^{th} order derivative of α exists and is continuous on I, we say that α is **smooth** or $\mathcal{C}^\infty$ when α is $\mathcal{C}^k$ for all $k \in \mathbb{Z}^+$, and we say that α is **regular** when α is $\mathcal{C}^1$ with $\alpha'(t) \neq 0$ for all $t \in I$.

Unless otherwise stated, we shall always assume curves are smooth and regular.

■ **Example 1.2** The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t, |t|)$ is not regular because $\alpha'(0)$ does not exist. The curve $\beta : \mathbb{R} \to \mathbb{R}^2$ given by $\beta(t) = (t^3, t^2)$ is not regular because $\beta'(0) = 0$. The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(0) = (0, 0)$ and $\gamma(t) = \left(t, t^2 \sin \frac{1}{t}\right)$ for $t \neq 0$ is differentiable but not regular because γ' is not continuous at $t = 0$.







Theorem 1.3
Every regular curve in $\mathbb{R}^n$ is locally injective.

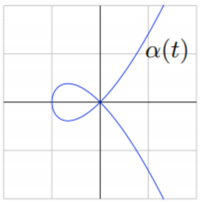
Proof. Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ be a regular curve, write $\alpha(t) = (x_1(t), \cdots, x_n(t))$, and let $a \in I$. Since $\alpha'(a) \neq 0$ we have ${x_k}'(a) \neq 0$ for some index k, say ${x_k}'(a) > 0$ (the case that ${x_k}'(a) < 0$ is similar). Since α' is continuous, ${x_k}'$ is continuous. Since ${x_k}'$ is continuous and

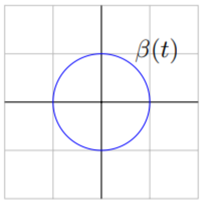
${x_k}'(a) > 0$ we can choose $\delta > 0$ so that $|t-a| < \delta \implies {x_k}'(t) > 0$. Then x_k is increasing, hence injective, in the interval $(a-\delta, a+\delta) \cap I$, and so α is injective in the same interval. ■

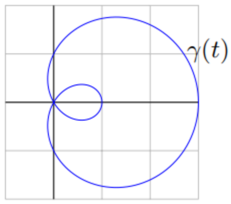
■ **Example 1.4** The curves $\alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}^2$ from Example 1.2 are not regular, but they are all injective, so a curve does not necessarily need to be regular in order to be injective.

■ **Example 1.5** The alpha curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ which is given by $\alpha(t) = \big(t^2-1, t(t^2-1)\big)$, the circle $\beta : \mathbb{R} \to \mathbb{R}^2$ which is given by $\beta(t) = (\cos t, \sin t)$, and the limcon $\gamma : \mathbb{R} \to \mathbb{R}^2$ which is given by $\gamma(t) = ((1+2\cos t)\cos t, (1+2\cos t)\sin t)$, are all regular, so they are all locally injective, but they are not (globally) injective (the alpha curve crosses itself with $\alpha(1) = \alpha(-1) = (0,0)$, the circle is periodic with $\beta(t+2\pi k) = \beta(t)$ for all $k \in \mathbb{Z}$, and the limcon is periodic and crosses itself).

■ **Example 1.6** The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(0) = 0$ and $\alpha(t) = \big(t^2, t^2 \sin \frac{1}{t}\big)$ for $t \neq 0$ is differentiable, but not regular since $\alpha'(0) = 0$, and (as you can verify) it is not locally injective at $t = 0$.







Definition 1.7 — length, rectifiable.
For a curve $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^n$, the **length** of α on $[a,b]$ is

$$L = L_\alpha([a,b]) = \sup\left\{ \sum_{j=1}^{p} \big|\alpha(t_j) - \alpha(t_{j-1})\big| \,\Big|\, a = t_0 < t_1 < t_2 < \cdots < t_p = b \right\}$$

(which can be infinite) and we say that α is **rectifiable** on $[a,b]$ when $L_\alpha([a,b])$ is finite.

Theorem 1.8
Let $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a regular curve. Then α is rectifiable with length

$$L = L_\alpha([a,b]) = \int_a^b \big|\alpha'(t)\big|\, dt.$$

Proof. For a partition $P = (t_0, t_1, \cdots, t_p)$, where $a = t_0 < t_1 < \cdots < t_p = b$, let us write

$$L(\alpha, P) = \sum_{j=1}^{p} \big|\alpha(t_j) - \alpha(t_{j-1})\big| \quad \text{and} \quad S(\alpha, P) = \sum_{j=1}^{p} \big|\alpha'(t_j)\big|(t_j - t_{j-1})$$

so $L(\alpha, P)$ is the sum which approximates Length(α) and $S(\alpha, P)$ is the Riemann sum (using right endpoints) which approximates the integral $\int_a^b |\alpha'(t)|\, dt$. First note that

$$\begin{aligned} L(\alpha, P) = \sum_{j=1}^{p} \big|\alpha(t_j) - \alpha(t_{j-1})\big| &\leq \sum_{j=1}^{p}\sum_{k=1}^{n} \big|x_k(t_j) - x_k(t_{j-1})\big| = \sum_{j=1}^{n}\sum_{k=1}^{p} \big|{x_k}'(c_{j,k})\,(t_j - t_{j-1})\big| \\ &\leq \sum_{j=1}^{n}\sum_{k=1}^{p} M(t_j - t_{j-1}) = \sum_{j=1}^{n} M(b-a) = nM(b-a) \end{aligned}$$

where we used the Mean Value Theorem to choose points $c_{j,k}$ between t_{j-1} and t_j such that $(x_k(t_j) - x_k(t_{j-1})) = x_k{}'(c_{j,k})(t_j - t_{j-1})$ and we let $M = \max\big\{|x_k{}'(t)|\,\big|1 \le k \le n, t \in [a,b]\big\}$. This shows that $L = L_\alpha([a,b])$ is finite.

Note that if $P = (t_0, t_1, \cdots, t_p)$ is a partition of $[a,b]$, and Q is a partition which is obtained by adding one more point, say $Q = (t_0, t_1, \cdots, t_{j-1}, s, t_j, \cdots, t_p)$, then we have $L(\alpha, P) \le L(\alpha, Q)$ because $\big|\alpha(t_j) - \alpha(t_{j-1}) \le \big|\alpha(t_j) - \alpha(s)\big| + \big|\alpha(s) - \alpha(t_{j-1})\big|$. It follows (by induction) that when Q is any partition with $P \subseteq Q$ we have

$$L(\alpha, P) \le L(\alpha, Q) \le L.$$

Also note that for any partition P, with $c_{j,k}$ chosen as above, we have

$$\begin{aligned}
\big|L(\alpha,P) - S(\alpha,P)\big| &= \Big|\sum_{j=1}^{p} \big|\alpha(t_j) - \alpha(t_{j-1})\big| - \sum_{j=1}^{p} \big|\alpha'(t_j)\big|(t_j - t_{j-1})\,\Big| \\
&= \Big|\sum_{j=1}^{p} \Big|\,(x_1(t_j) - x_1(t_{j-1}), \cdots, x_n(t_j) - x_n(t_{j-1})\,\Big| - \sum_{j=1}^{p} \big|\alpha'(t_j)\big|(x_j - t_{j-1})\,\Big| \\
&= \Big|\sum_{j=1}^{p} \Big|\,\big(x_1{}'(c_{j,1}), \cdots, x_n{}'(c_{j,n})\big)\,\Big|(t_j - t_{j-1}) - \sum_{j-1}^{p} \Big|\,\big(x_1{}'(t_j), \cdots, x_n{}'(t_j)\big)\,\Big|(t_j - t_{j-1})\,\Big| \\
&\le \sum_{j=1}^{p} \Big|\big|\,\big(x_1{}'(c_{j,1}), \cdots, x_n{}'(c_{j,n})\big)\,\big| - \big|\,\big(x_1{}'(t_j), \cdots, x_n{}'(t_j)\big)\,\big|\Big|(t_t - t_{j-1}) \\
&\le \sum_{j=1}^{p} \Big|\,\big(x_1{}'(c_{j,1}) - x_1{}'(t_j), x_n{}'(c_{j,n}) - x_n{}'(t_j)\big)\,\Big|(t_j - t_{j-1}) \\
&\le \sum_{j=1}^{p}\sum_{k=1}^{n} \big|x_k{}'(c_{j,k}) - x_k{}'(t_j)\big|(t_j - t_{j-1}).
\end{aligned}$$

Let $\epsilon > 0$. Since each $x_k{}'$ is continuous (hence uniformly continuous) on $[a,b]$ and since $|\alpha'|$ is continuous (hence Riemann integrable) on $[a,b]$, we can choose $\delta > 0$ such that for all $s, t \in [a,b]$ with $|s-t| < \delta$ we have $\big|x_k{}'(s) - x_k{}'(t)\big| < \frac{\epsilon}{3n(b-a)}$ for all k , and such that for every partition $P = (t_0, t_1, \cdots, t_p)$ with $|P| < \delta$ we have $\big|\int_a^b |\alpha'(t)|\,dt - S(\alpha,P)\big| < \frac{\epsilon}{3}$ where $|P|$ is the size of the partition P, that is $|P| = \max\big\{t_j - t_{j-1}\,\big|\,1 \le j \le p\big\}$. Choose a partition P_1 with $|P_1| < \delta$ and choose a partition P_2 such that $\big|L - L(\alpha, P_2)\big| < \frac{\epsilon}{3}$ then let $P = P_1 \cup P_2$. Since $P_2 \subseteq P_1$ we have $L(\alpha, P_1) \le L(\alpha, P) \le L$ so that $\big|L - L(\alpha,P)\big| \le \big|L - L(\alpha, P_1)\big| < \frac{\epsilon}{3}$. Since $|P| \le |P_1| < \delta$ we have $\big|\int_a^b |\alpha'(t)|\,dt - S(\alpha,P)\big| < \frac{\epsilon}{3}$. Also since $|P| < \delta$, for all of the points $c_{j,k}$ we have $|c_{j,k} - t_j| < \delta$ so that $\big|x_k{}'(c_{j,k}) - x_k{}'(t_j)\big| < \frac{\epsilon}{3n(b-a)}$ and hence (as shown above) $\big|L(\alpha,P) - S(\alpha,P)\big| \le \sum_{j=1}^{p}\sum_{k=1}^{n} \big|x_k{}'(c_{j,k}) - x_k{}'(t_j)\big|(t_j - t_{j-1}) < \frac{\epsilon}{3}$.

Thus

$$\begin{aligned}
\Big|L - \int_a^b |\alpha'(t)|\,dt\Big| &\le \Big|L - L(\alpha,P)\Big| + \Big|L(\alpha,P) - S(\alpha,P)\Big| + \Big|S(\alpha,P) - \int_a^b |\alpha'(t)|\,dt\Big| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $L = \int_a^b |\alpha'(t)|\,dt$, as required. ■

■ **Example 1.9** A curve which is differentiable, but not $\mathcal{C}^1$, can have infinite length. For example, consider the curve $\alpha : [0,1] \to \mathbb{R}^2$ given by $\alpha(t) = (x(t), y(t))$ where $x(t) = t$

$y(t) = t^2 \cos\frac{\pi}{t^2}$ when $t \neq 0$ with $y(0) = 0$. Note that $x(y)$ and $y(t)$ are both differentiable (with $y'(0) = 0$) but $y'(t)$ is not continuous at 0 (as you can check).

Let P be the partition $P = (t_0, t_1, \cdots t_p)$ with $t_0 = 0$ and $t_j = \frac{1}{\sqrt{p-j+1}}$, that is let $P = \left(0, \frac{1}{\sqrt{p}}, \frac{1}{\sqrt{p-1}}, \cdots, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, 1\right)$. We have $y(t_j) = \frac{1}{p-j+1}\cos(p-j+1)\pi = \frac{(-1)^{p-j+1}}{p-j+1}$ for $1 \leq j \leq p$, and hence $\left|y(t_j) - y(t_{j-1})\right| = \left|\frac{1}{p-j+1} + \frac{1}{p-j+2}\right| \geq \frac{2}{p-j+2}$ for $2 \leq j \leq p$. Letting $\ell = p - j + 2$ we have

$$\sum_{j=1}^{p} \left|\alpha'(t_j) - \alpha'(t_{j-1}\right| \geq \sum_{j=2}^{p} \left|y(t_j) - y_(t_{j-1})\right| \geq \sum_{j=2}^{p} \frac{2}{p-j+2} = \sum_{\ell=2}^{p} \frac{2}{\ell}.$$

Since $\sum \frac{2}{\ell}$ diverges, it follows that $L_\alpha\left([a,b]\right) = \infty$.

Definition 1.10 — reparameterisation, change of parameter, regular, preserves direction, parameterised by arclength.
When $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ is a continuous curve and $s : I \subseteq \mathbb{R} \to J \subseteq \mathbb{R}$ is a homomorphism with inverse $t = t(s)$, the curve $\beta : J \subseteq \mathbb{R} \to \mathbb{R}^n$ defined by $\beta(s) = \alpha\left(t(s)\right)$ is called a **reparameterisation** of α, and the map s is called a **change of parameter** (or a **change of coordinates**).

When s is $\mathcal{C}^1$ with $s'(t) \neq 0$ for all t, we say that s is **regular**.

By the Inverse Function Theorem, if $s = s(t)$ is smooth (or $\mathcal{C}^k$) and regular then so is its inverse $t = t(s)$.

When $s'(t) > 0$ for all t we say s **preserves direction** and when $s'(t) < 0$ for all t we say s **reverses direction**.

When α and s are both smooth (or $\mathcal{C}^k$) and regular, so is β, and for $t = t(s)$ we have $\beta'(s) = \alpha'\left(t(s)\right)t'(s) = \frac{\alpha'(t)}{s'(t)}$.

When $|\beta'(s)| = 1$ for all $s \in J$, we say that β is **parameterised by arclength**. Unless otherwise stated, we shall assume that any change of coordinates is smooth and regular.

Theorem 1.11
Every regular curve can be reparameterised by arclength, using a regular direction-preserving change of coordinates.

Proof. Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ be a regular curve. Let $a \in I$ and define $s(t) = \int_a^t \left|\alpha'(r)\right| dr$. Note that $s'(t) = \left|\alpha'(t)\right| > 0$ so $s(t)$ is regular and strictly increasing, and it maps the interval I to an interval J, and if α is $\mathcal{C}^k$ then so is $s = s(t)$. By the inverse function $t = t(s)$ satisfies $t'(s) = \frac{1}{s'(t)} = \frac{1}{|\alpha'(t)|}$. The reparameterised curve $\beta : J \to \mathbb{R}^n$ given by $\beta(s) = \alpha\left(t(s)\right)$ satisfies $\beta'(s) = \alpha'\left(t(s)\right)t'(s) = \frac{\alpha'(t(s))}{|\alpha'(t(s))|}$ so that $\left|\beta'(s)\right| = 1$ for all $s \in J$. ■

1.2 Curves in $\mathbb{R}^2$

Definition 1.12 — unit tangent vector, unit normal vector, signed curvature, scalar curvature.
Let $\beta : J \subseteq \mathbb{R} \to \mathbb{R}^2$ be a smooth regular curve parameterised by arclength. For a vector $u = (x, y) \in \mathbb{R}^2$, write $u^\times = (-y, x)$ and note that $|u^\times| = |u|$. The **unit tangent vector** and the **unit normal vector** of β at s are the vectors

$$\begin{aligned} T(s) &= T_\beta(s) = \beta'(s), \\ N(s) &= N_\beta(s) = T(s)^\times. \end{aligned}$$

Since β is parametrized by arclength, $|T(s)| = |\beta'(s)| = 1$ and $|N(s)| = |\beta'(s)^\times| = 1$ for all s. For all s we have $\beta'(s) \times \beta'(s) = \left|\beta'(s)\right|^2 = 1$. By differentiation both sides we obtain $\frac{d}{ds}\left(\beta'(s) \times \beta'(s)\right) = 0$, that is $2\,\beta'(t) \times \beta''(t) = 0$. Thus $\beta''(s)$ is orthogonal to $\beta'(s) = T(s)$, and so $\beta''(s)$ lies in the span of $T(s)^\times = N(s)$.

We define the **signed curvature** of β at s to be the real number $k(s) = k_\beta(s)$ such that

$$\beta''(s) = k(s)\, N(s) = k_\beta(s)\, N_\beta(s).$$

Since $|N_\beta(s)| = 1$ we have $|\beta''(s)| = \left|k_\beta(s)\right|$.

The **scalar curvature** of β at s is

$$\kappa(s) = \kappa_\beta(s) = \left|k(s)\right| = \left|\beta''(s)\right|.$$

When $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ is a smooth regular curve we first reparametrize by arclength by choosing $a \in I$ and letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\, dr$, and then we define $T(t) = T_\alpha(t) = T_\beta(s(t))$, $N(t) = N_\alpha(t) = N_\beta(s(t))$, $k(t) = k_\alpha(t) = k_\beta(s(t))$ and $\kappa(t) = \kappa_\alpha(t) = \kappa_\beta(s(t))$, and we call these the unit tangent vector, the unit normal vector, the signed curvature, and the scalar curvature, of α at t. The following theorem shows that these are well-defined, that is they do not depend on the choice of $a \in I$.

Theorem 1.13
For a smooth regular curve $\alpha = \alpha(t)$ we have

$$\begin{aligned} T &= \frac{\alpha'}{|\alpha'|}\ ,\ N = \left(\frac{\alpha'}{|\alpha'|}\right)^\times \\ k &= \frac{\det_2(\alpha', \alpha'')}{|\alpha'|^3} = \frac{(\alpha' \times \alpha'') \times e_3}{|\alpha'|^3} = \frac{\det_3(\alpha', \alpha'', e_3)}{|\alpha'|^3} \\ \kappa &= \frac{\left|\det_2(\alpha', \alpha'')\right|}{|\alpha'|^3} = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} \end{aligned}$$

where $\det_2(\alpha', \alpha'')$ is the determinant of the 2×2 matrix with columns $\alpha', \alpha'' \in \mathbb{R}^2$, and where we identify $\alpha', \alpha'' \in \mathbb{R}^2$ with $\binom{\alpha'}{0}, \binom{\alpha''}{0} \in \mathbb{R}^3$ so that $\alpha' \times \alpha''$ is the cross product of two vectors $\binom{\alpha'}{0}, \binom{\alpha''}{0} \in \mathbb{R}^3$ and $\det_3(\alpha', \alpha'', e_3)$ is the determinant of the 3×3 matrix whose first two columns are $\binom{\alpha'}{0}, \binom{\alpha''}{0} \in \mathbb{R}^3$ and whose last column is the 3^{rd} standard basis vector e_3.

Proof. First verify (easily) that when we identify $u, v \in \mathbb{R}^2$ with $\binom{u}{0}, \binom{v}{0} \in \mathbb{R}^3$ we have

$$u^\times \cdot v = \det_2(u,v) = (u \times v) \cdot e_3 = \det_3(u,v,e_3)$$

and $\big|\det_2(u,v)\big| = \big|u \times v\big|$.

Peparametrize by arclength by choosing $a \in I$ and letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\,dr$. We have $T_\beta(s) = \beta'(s)$ and $N_\beta(s) = \beta'(s)^\times$. Let us find formulas for $k_\beta(s)$ and $\kappa_\beta(s)$. By definition, $k_\beta(s)\beta'(s)^\times = k_\beta(s)N_\beta(s) = \beta''(s)$. Take the dot product of both sides with $\beta'(s)^\times$ to get

$$\begin{aligned} k_\beta(s) &= \beta'(s)^\times \times \beta''(s) \\ \kappa_\beta(s) &= \big|k_\beta(s)\big| = \big|\beta'(s) \times \beta''(s)\big|. \end{aligned}$$

Now let us find formulas for $T(t) = T_\alpha(t) = T_\beta(s(t))$, $N(t) = N_\alpha(t) = N_\beta(s(t)$, $k(t) = k_\alpha(t) = k_\beta(s(t))$ and $\kappa(t) = \kappa_\alpha(t) = \kappa_\beta(s(t))$. We have $\alpha(t) = \beta(s(t))$ so that $\alpha'(t) = \beta'(s(t))s'(t)$. Since $|\beta'(s(t))| = 1$ and $s'(t) > 0$, it follows that $\big|\alpha'(t)\big| = s'(t)$. Since $\beta''(s) = k_\beta(s)N_\beta(s)$ and $|N_\beta(s)| = |T_\beta(s)| = 1$, we have $\big|\beta''(s)\big| = \big|k_\beta(s)\big| = \kappa_\beta(s)$. Since $\beta''(s)$ is orthogonal to $\beta'(s)$ (see Definition 1.12) we have $\kappa_\beta(s) = \big|\beta''(s)\big| = \big|\beta'(s) \times \beta''(s)\big|$.

$$\begin{aligned} \alpha(t) &= \beta(s(t)) \\ \alpha'(t) &= \beta'(s(t))s'(t) \\ \alpha''(t) &= \beta''(s(t))s'(t)^2 + \beta'(s(t))s''(t) \\ \alpha'(t) \times \alpha''(t) &= \big(\beta'(s(t)) \times \beta''(s(t))\big)\,(s')^3. \end{aligned}$$

Since $\big|\alpha'(t)\big| = s'(t)$ this gives

$$\frac{\alpha'(t)}{|\alpha'(t)|} = \beta'(s(t)) = T_\beta(s(t)) = T(t) \quad , \quad \Big(\frac{\alpha'(t)}{|\alpha'(t)|}\Big)^\times = T_\beta(s(t))^\times = N_\beta(s(t)) = N(t)$$

$$\frac{\alpha'(t) \times \alpha''(t)}{|\alpha'(t)|^3} = \beta'(s(t)) \times \beta''(s(t)) = k_\beta(s(t)) = k(t)$$

$$\frac{\big|\alpha'(t) \times \alpha''(t)\big|}{|\alpha'(t)|^3} = \Big|\beta'(s(t)) \times \beta''(s(t))\Big| = \kappa_\beta(s(t)) = \kappa(t)\,.$$

■

Theorem 1.14
For a smooth regular curve α in $\mathbb{R}^2$, the curvature of α is identically zero if and only if (the image of) α lies on a straight line.

Proof. Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a smooth regular curve. Choose $a \in I$ and reparametrize α by arclength by setting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(t)|\,dt$. Suppose that $\kappa(t) = 0$ for all t. Then we have $0 = \kappa(t(s)) = \big|\beta''(s)\big|$ for all s so that $\beta''(s) = 0$ for all s. By integrating once we obtain $\beta'(s) = u$ for some $u \in \mathbb{R}^2$ since $|\beta'(s)| = 1$, u is a unit vector) and by integrating again we obtain $\beta(s) = p + su$ for some $p \in \mathbb{R}^2$. Thus $\alpha(t) = p + s(t)u$ for all t so that α lies on the line through p in the direction of u.

Suppose, conversely, that (the image of) α lies on a straight line, say the line $p + su$ where $p, u \in \mathbb{R}^2$ and $|u| = 1$. Then for every $t \in I$ there is a (unique) $s = s(t)$ such that $\alpha(t) = p + s(t)u$. We remark that taking the dot product with u gives $s(t) = (\alpha(t) - p)$ for all t so we see that $s(t)$ is smooth. Since $\alpha(t) = p + tu$, we have $\alpha'(t) = s'(t)u$ and $\alpha''(t) = s''(t)u$ so that $\alpha(t) \times \alpha''(t) = s'(t)s''(t)\,u \times u = 0$ and kence $\kappa(t) = 0$ for all t. ■

Definition 1.15 — osculating circle.
Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a smooth regular curve, let $a \in I$, and suppose that $\kappa(a) \neq 0$. We define the **osculating circle** (or the **best-fit circle**) of α at $t = a$ as follows. Let $p = \alpha(a)$, $T = T(a)$, $N = N(a)$, $k = k(a)$ and $\kappa = \kappa(a)$. Reparametrize by arclength, letting $\beta(s) = \alpha(t(s)$ where $s(t) = \int_a^t |\alpha'(r)|\, dr$ so that we have $\beta(0) = p$. $\beta'(0) = T$ and $\beta''(0) = kN$. The osculating circle at $t = a$ is the circle given by

$$\begin{aligned}
\sigma(s) &= \left(p + \frac{1}{k}N\right) - \frac{1}{k}\cos(ks)N + \frac{1}{k}\sin(ks)T \\
\sigma'(s) &= \sin(ks)N + \cos(ks)T \\
\sigma''(s) &= k\cos(ks)N - k\sin(ks)T
\end{aligned}$$

which is the circle of radius $R = \frac{1}{\kappa}$ centered at $p + \frac{1}{k}N$, parametrized by arclength (since $|\sigma'(s)| = 1$ for all s), such that $\sigma(0) = p = \beta(0)$, $\sigma'(0) = T = \beta'(0)$ and $\sigma''(0) = kN = \beta''(0)$.

Note 1.16 When α is a smooth regular curve, the scalar curvature at $t = a$ is equal to the reciprocal of the radius of the best-fit circle at $t = a$.

Theorem 1.17 — Polar Coordinates.
Let $I \subseteq \mathbb{R}$ be an interval with $a \in I$, let $p \in \mathbb{R}^2$, and let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a continuous curve in $\mathbb{R}^2$ with $\alpha(t) \neq p$ for any $t \in I$. Let $r_0 = \big|\alpha(a) - p\big|$ and choose $\theta_0 \in \mathbb{R}$ such that $\alpha(a) - p = r_0(\cos\theta_0, \sin\theta_0)$ (θ_0 is unique up to an integer multiple of 2π). Then there exist unique continuous functions $r, \theta : I \to \mathbb{R}$ with $r(a) = r_0$ and $\theta(a) = \theta_0$ such that

$$\alpha(t) = p + r(t)\,(\cos\theta(t), \sin\theta(t))$$

for all $t \in I$. Moreover, if α is smooth (or $\mathcal{C}^k$) then so are the functions r and θ.

Proof. We omit the proof, but we remark that it is surprisingly involved. ■

Definition 1.18 — winding number, turning number.
For a continuous curve $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^2$ with $\alpha(t) \neq p$ for any t, we define the **winding number** $\text{Wind}(\alpha, p)$ of α about p as follows. We let $r_0 = \big|\alpha(a) - p\big|$ and choose $\theta_0 \in [0, 2\pi)$ so that $\alpha(a) = p + r_0(\cos\theta_0, \sin\theta_0)$, then we let $r, \theta : [a,b] \subset \mathbb{R} \to \mathbb{R}^2$ be the unique continuous maps such that $\alpha(t) = p + r(t)\,(\cos\theta(t), \sin\theta(t))$ for all $t \in [a,b]$, and then we define

$$\text{Wind}(\alpha, p) = \frac{1}{2\pi}\left(\theta(b) - \theta(a)\right).$$

When α is regular, we define the **turning number** of α to be

$$\text{Turn}(\alpha) = \text{Wind}\left(\alpha', 0\right).$$

Theorem 1.19
Let $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$ be a curve in $\mathbb{R}^2$ and write $\alpha(t) = (x(t), y(t))$.
(1) If α is a $\mathcal{C}^1$ curve with $\alpha(t) \neq 0$ for any $t \in [a, b]$ then

$$\mathrm{Wind}(\alpha, 0) = \frac{1}{2\pi} \int_a^b \frac{x(t)y'(t) - y(t)x'(t)}{x(t)^2 + y(t)^2}\, dt.$$

(2) If α is $\mathcal{C}^2$ regular curve then

$$\mathrm{Turn}(\alpha) = \frac{1}{2\pi} \int_a^b k(\alpha(t))\big|\alpha'(t)\big|\, dt.$$

Proof. To prove Part 1, write α in polar coordinates as $\alpha(t) = r(t)\,(\cos\theta(t), \sin\theta(t))$, that is write $x = r\cos\theta$ and $y = r\sin\theta$ where $r = r(t)$ and $\theta = \theta(t)$ are continuous with $r(t) > 0$ for all $t \in [a, b]$ and $\theta(a) \in [0, 2\pi)$. Then

$$\begin{aligned}
\int_a^b \frac{x\,y' - y\,x'}{x^2 + y^2}\, dt &= \int_a^b \frac{(r\cos\theta)\,(r'\sin\theta + r\cos\theta\,\theta') - (r\sin\theta)\,(r'\cos\theta - r\sin\theta\,\theta')}{r^2\cos^\theta + r^2\sin^2\theta}\, dt \\
&= \int_a^b \frac{r^2\cos^2\theta\,\theta' + r^2\sin^2\theta\,\theta'}{r^2}\, dt = \int_a^b \theta'\, dt \\
&= \theta(b) - \theta(a) = 2\pi\,\mathrm{Wind}(\alpha, 0).
\end{aligned}$$

To prove Part 2, write $\alpha'(t)$ in polar coordinates as $\alpha'(t) = |\alpha'(t)|\,(\cos\theta(t), \sin\theta(t))$ with $\theta(a) \in [0, 2\pi)$. Since α is $\mathcal{C}^2$ and regular, we note that α' is $\mathcal{C}^1$ with $\alpha'(t) \neq 0$ for all $t \in [a, b]$. Reparametrize α by arclength letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(t)|\, dt$, then write $\beta'(s)$ in polar coordinates as $\beta'(s) = \big|\beta'(s)\big|\,(\cos\phi(s), \sin\phi(s))$ with $\phi(0) \in [0, 2\pi)$. Since $|\beta'(s)| = 1$ we have $(\cos\phi(s(t)), \sin\phi(s(t))) = \beta'(s(t)) = \frac{\alpha'(t)}{|\alpha'(t)|} = (\cos\theta(t), \sin\theta(t))$ for all $t \in [a, b]$, and hence $\phi(s(t)) = \theta(t)$ for all $t \in [a, b]$ (by the uniqueness of the polar representation). Since $\beta'(s) = (\cos\phi(s), \sin\phi(s))$, we have

$$\beta''(s) = \big(-\sin\phi(s)\,\phi'(s), \cos\phi(s)\,\phi'(s)\big) = \phi'(s)\,(-\sin\phi(s), \cos\phi(s)) = \phi'(s)\,\beta'(s)^\times$$

By the definition of $k(s)$ we see that $k(s) = \phi'(s)$. Thus

$$\begin{aligned}
\int_a^b k(t)\big|\alpha'(t)\big|\, dt &= \int_a^b k(s(t))s'(t)\, dt = \int_{s(a)}^{s(b)} k(s)ds = \int_{s(a)}^{s(b)} \phi'(s)\, ds \\
&= \phi(s(b)) - \phi(s(a)) = \theta(b) - \theta(a) = 2\pi\,\mathrm{Wind}(\alpha', 0) = 2\pi\,\mathrm{Turn}(\alpha)
\end{aligned}$$

■

Theorem 1.20 — The Fundamental Theorem for Plane Curves.
Let $I \subseteq \mathbb{R}$ be an interval with $a \in I$, let $p, u \in \mathbb{R}^2$ with $|u| = 1$, and let $\ell : I \subseteq \mathbb{R} \to \mathbb{R}$ be a smooth function. Then there exists a unique smooth regular curve $\beta : I \subseteq \mathbb{R} \to \mathbb{R}^2$ with $|\beta'(s)| = 1$ for all $s \in I$ such that $\beta(a) = p$ and $\beta'(a) = u$ and $k(s) = \ell(s)$ for all $s \in I$.

Proof. Suppose that such a curve β exists. Since $|\beta'(s)| = 1$ for all s, we can write β' in polar coordinates as $\beta'(s) = (\cos\theta(s), \sin\theta(s))$ with $\theta(a) \in [0, 2\pi)$. Then we have $\beta''(s) = (-\sin\theta(s)\theta'(s), \cos\theta(s)\theta'(s)) = \theta'(s)\beta'(s)^\times$ so that $\theta'(s) = k(s) = \ell(s)$. We can integrate to get $\theta(s) = \theta(a) + \int_a^s \ell(t)\, dt$. Since $\beta'(s) = (\cos\theta(s), \sin\theta(s))$ we can integrate again to get

$$\beta(s) = p + \Big(\int_a^s \cos\theta(t)\, dt\,,\ \int_a^s \sin\theta(t)\, dt\Big).$$

This shows that $\beta(s)$ is uniquely determined and gives us a formula for $\beta(s)$.

Conversely, we can choose $\theta_0 \in [0, 2\pi)$ so that $(\cos\theta_0, \sin\theta_0) = u$, and then define $\theta(s) = \theta_0 + \int_a^s \ell(t)\,dt$ so that $\theta(a) = \theta_0$ and $\theta'(s) = \ell(s)$ for all $s \in I$, and then define $\beta(s) = p + \left(\int_a^s \cos\theta(t)\,dt\,,\, \int_a^s \sin\theta(t)\,dt\right)$ so that $\beta'(s) = (\cos\theta(s), \sin\theta(s))$ for all $s \in I$. Then $|\beta'(s)| = 1$ for all s and $\beta(a) = p$ and $\beta'(a) = (\cos\theta(a), \sin\theta(a)) = (\cos\theta_0, \sin\theta_0) = u$ and $\beta''(s) = \theta'(s)\beta'(s)^\times$ so that $k(s) = \theta'(s) = \ell(s)$ for all $s \in I$, as required. ■

1.3 Curves in $\mathbb{R}^3$

Definition 1.21 — unit tangent vector, curvature vector, scalar curvature, principal normal vector, binormal vector, torsion.
Let $\beta : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$ parametrized by arclength (so $|\beta'(s)| = 1$ for all $s \in I$). The **unit tangent vector** of β at s is the unit vector $T(s) = T_\beta(s) = \beta'(s)$. The vector $\beta''(s)$ iis called the **curvature vector** of β at s. The **scalar curvature** of β at s is given by $\kappa(s) = \kappa_\beta(s) = \big|\beta''(s)\big|$.

If $\beta''(s) \neq 0$ then we define the **principal normal vector** of β at s to be the unit vector $P(s) = P_\beta(s) = \frac{\beta''(s)}{|\beta''(s)|}$, and we define the **binormal vector** of β at s to be the unit vector $B(s) = B_\beta(s) = T(s) \times P(s)$. Note that $\{T(s), P(s), B(s)\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$. Since $B = T \times P$ and $P = \frac{T'}{|T'|}$, we have

$$B' = T' \times P + T \times P' = |T'|\, P \times P + T \times P' = T \times P'.$$

Notice that B' is orthogonal to both T and B (it is orthogonal to T because $B' = T \times P'$ and it is orthogonal to B because we have $B(s)(s) = |B(s)|^2 = 1$ for all s so taking the derivative on both sides gives $2\,B' = 0$). Since $\{T, P, B\}$ is an orthonormal basis for $\mathbb{R}^3$ and B' is orthogonal to both T and B, we have $B' = (B')P$. We define the **torsion** of β at s to be $\tau(s) = \tau_\beta(s) = -B'(s)(s)$ so that $B'(s) = -\tau(s)P(s)$ for all s (the negative sign is included so that the torsion of the right-handed helix is positive).

To summarize the above definitions, when $\beta : I \subseteq \mathbb{R} \to \mathbb{R}^3$ is a smooth regular curve, parametrized by arclength, with non-zero curvature vector $\beta''(s) \neq 0$, the **unit tangent vector**, the **principal normal vector,** the **binormal vector**, the **scalar curvature** and the **torsion** of β at s are given by

$$\begin{aligned}
T(s) &= T_\beta(s) = \beta'(s),\\
P(s) &= P_\beta(s) = \frac{\beta''(s)}{|\beta''(s)|},\\
B(s) &= B_\beta(s) = T(s) \times P(s),\\
\kappa(s) &= \kappa_\beta(s) = \big|\beta''(s)\big|,\\
\tau(s) &= \tau_\beta(s) = -\,B'(s) \cdot P(s).
\end{aligned}$$

and $\{T, (s)P(s), B(s)\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$ for every $s \in I$. From the definition of P and κ we have $T' = \kappa P$, and as explained above, we defined $\tau = -B'$ so that $B' = -\tau P$. Since $P = B \times T$ we also have

$$P' = B' \times T + B \times T' = -\tau P \times T + \kappa B \times P = \tau B - \kappa T.$$

Thus the derivatives T', P' and B' satisfy the following matrix identity which gives a system of three equations called the **Frenet-Seret Formulas**

$$\begin{pmatrix} T' \\ P' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ P \\ B \end{pmatrix}.$$

When $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ is a smooth regular curve in $\mathbb{R}^3$, we reparametrize by arclength by choosing $a \in I$ and letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\, dr$, then we define the **unit tangent vector** of α at t to be $T(t) = T_\alpha(t) = T_\beta(s(t))$, and if $\beta''(s(t)) \neq 0$, we define the **principal normal vector**, the **binormal vector**, the **scalar curvature** and the **torsion** of α at t to be given by $P(t) = P_\alpha(t) = P_\beta(s(t))$, $B(t) = B_\alpha(t) = B_\beta(s(t))$, $\kappa(t) = \kappa_\alpha(t) = \kappa_\beta(s(t))$ and $\tau(t) = \tau_\alpha(t) = \tau_\beta(s(t))$. The following theorem shows that these are all well-defined, that is they do not depend on the choice of $a \in I$.

Theorem 1.22
Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a smooth regular curve. For all $t \in I$ for which $\alpha'(t) \times \alpha''(t) \neq 0$, we have

$$T = \frac{\alpha'}{|\alpha'|} \quad , \quad P = \frac{T'}{|T'|} \quad , \quad B = T \times P$$
$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} \quad , \quad \tau = \frac{\det_3(\alpha', \alpha'', \alpha''')}{\left|\alpha' \times \alpha''\right|^2}.$$

Proof. Choose $a \in I$ and let $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\,dr$. Then $\alpha(t) = \beta(s(t))$ and so for all $t \in I$

$$\begin{aligned}
\alpha'(t) &= \beta'(s(t))s'(t), \\
\alpha''(t) &= \beta''(s(t))s'(t)^2 + \beta'(t)s''(t), \\
\alpha'''(t) &= \beta'''(s(t))s'(t)^3 + 3\,\beta''(s(t))s'(t)s''(t) + \beta'(t)s'''(t), \\
\alpha'(t) \times \alpha''(t) &= \big(\beta'(s(t)) \times \beta''(s(t))\big)\, s'(t)^3, \\
(\alpha'(t) \times \alpha''(t)) \times \alpha'''(t) &= \Big(\big(\beta'(s(t)) \times \beta''(s(t))\big) \cdot \beta'''(s(t))\Big)\, s'(t)^6.
\end{aligned}$$

Since $\alpha' \times \alpha'' = (\beta' \times \beta'')(s')^3$, we have $\alpha' \times \alpha'' = 0 \iff \beta' \times \beta'' = 0$. Since $|\beta'(s)| = 1$ for all s, it follows $(bytakingthederivativeof\, 1{=}\beta'(s) \cdot \beta'(s) that \beta'$ and β'' are orthogonal, and so we have $|\beta' \times \beta''| = |\beta'|\,|\beta''| = |\beta''|$ so that $\beta' \times \beta'' = 0 \iff \beta'' = 0$. Since $T_\alpha(t) = \beta'(s(t))$ we have $T_\alpha{}'(t) = \beta''(s(t))s'(t)$ so that $T_\alpha{}'(t) = 0 \iff \beta''(s(t)) = 0$. Thus

$$\alpha'(t) \times \alpha''(t) = 0 \iff \beta'(s(t)) \times \beta''(s(t)) = 0 \iff \beta''(s(t)) = 0 \iff T_\alpha{}'(t) = 0.$$

Suppose that $\alpha'(t) \times \alpha''(t) \neq 0$. Since $T_\alpha{}'(t) = \beta''(s(t))s'(t)$ and $s'(t) = |\alpha'(t)| > 0$ we have

$$\begin{gathered}
\frac{T_\alpha{}'(t)}{\left|T_\alpha{}'(t)\right|} = \frac{\beta''(s(t))s'(t)}{\left|\beta''(s(t))s'(t)\right|} = \frac{\beta''(s(t))}{\left|\beta''(s(t))\right|} = P_\beta(s(t) = P_\alpha(t) \text{ and} \\
B_\alpha(t) = B_\beta(s(t)) = T_\beta(s(t)) \times P_\beta(s(t)) = T_\alpha(t) \times P_\alpha(t).
\end{gathered}$$

Since $\alpha' \times \alpha'' = (\beta' \times \beta'')(s')^3$ and $|\beta' \times \beta''| = |\beta''|$ and $s' = |\alpha'|$, we have

$$\frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} = \left|\beta''(s(t))\right| = \kappa_\beta(s(t)) = \kappa_\alpha(t).$$

To simplify notation, write β for $\beta(s(t))$ and similarly for β' and β'', and write T for $T_\beta(s(t)) = T_\alpha(t)$ and similarly for P and B, and write κ for $\kappa_\beta(s(t)) = \kappa_\alpha(t)$ and similarly for τ. Since $\beta' = T$, using the Frenet-Serre formulas and the fact that $\{T, P, B\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$, we have

$$\begin{aligned}
(\beta' \times \beta'') \cdot \beta'' &= (T \times T')'' = (T \times (\kappa P)) \times (\kappa P)' = (T \times (\kappa P)) \times (\kappa' P + \kappa P') \\
&= \kappa^2 (T \times P) \times P' = \kappa^2 (T \times P) \cdot (-\kappa T + \tau B) = \kappa^2 \tau.
\end{aligned}$$

Since we have $\det_3(\alpha', \alpha'', \alpha''') = (\alpha' \times \alpha'') \times \alpha''' = ((\beta' \times \beta'') \cdot \beta''') (s')^6 = \kappa^2 \tau |\alpha'|^6$ and we have $|\alpha' \times \alpha''| = \kappa |\alpha'|^3$, it follows that

$$\frac{\det_3 (\alpha'(t), \alpha''(t), \alpha'''(t))}{\left|\alpha'(t) \times \alpha''(t)\right|^2} = \frac{\kappa_\alpha(t)^2 \tau_\alpha(t) |\alpha'(t)|^6}{\kappa_\alpha(t)^2 |\alpha'(t)|^6} = \tau_\alpha(t).$$

■

■ **Example 1.23** The curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ given by $\alpha(t) = (a \cos t, a \sin t, bt)$ is called a (right-handed) **helix**. We have

$$\begin{aligned} \alpha'(t) &= (-s \sin t, a \cos t, b)\,, \\ \alpha''(t) &= (-a \cos t, -a \sin t, 0)\,, \\ \alpha'''(t) &= (a \sin t, -a \cos t, 0) \text{ and} \\ \alpha'(t) \times \alpha''(t) &= \left(ab \sin t, -ab \cos t, a^2, \right) \end{aligned}$$

and so

$$\begin{aligned} \left|\alpha'(t)\right| &= (a^2 + b^2)^{1/2}, \\ \left|\alpha'(t) \times \alpha''(t)\right| &= a(a^2 + b^2)^{1/2} \text{ and} \\ (\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t) &= a^2 b, \end{aligned}$$

and hence

$$\begin{aligned} \kappa(t) &= \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} = \frac{a}{a^2 + b^2} \text{ and} \\ \tau(t) &= \frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{|\alpha'(t)|^3} = \frac{b}{a^2 + b^2}. \end{aligned}$$

We note that the scalar curvature and the torsion of the helix are constant.

Theorem 1.24
Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$.

(1) The curvature of α is identically zero if and only if (the image of) α lies on a line.
(2) If α has non-vanishing curvature (so its torsion is defined) then the torsion of α is identically zero if and only if (the image of) α lies in a plane.

Proof. The proof of Part 1 is the same as the proof of the analogous theorem for plane curves (Theorem 1.14). To prove part 2, suppose that $\kappa_\alpha(t) \neq 0$ for all $t \in I$. Choose $a \in I$ and let $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\, dr$.

Suppose $\tau_\alpha(t) = 0$ for all t. Then $\tau_\beta(s) = \tau_\alpha(t(s)) = 0$ for all s. Write $\tau(s) = \tau_\beta(s)$. We have $B'(s) = -\tau(s)P(s) = 0$ for all s, so $B(s)$ is constant, say $B(s) = b \in \mathbb{R}^3$ for all s and note that $|b| = |B(s)| = 1$. Note that $\frac{d}{ds}(\beta(s)) = \beta'(s) = T(s)(s) = 0$ for all s, and so $\beta(s)$ is constant, say $\beta(s) = c \in \mathbb{R}$. Thus we have $\alpha(t) = \beta(s(t)) = c$ for all t and so (the image of) α lies on the plane in $\mathbb{R}^3$ given by $x = c$.

Suppose, conversely, that (the image of) α lies on a plane in $\mathbb{R}^3$, say $\alpha(t) = c$ for all $t \in I$ where $b, c \in \mathbb{R}^3$ with $|b| = 1$. Then $\beta(s) = \alpha(t(s)) = c$ for all s. Take the derivative to get $\beta'(s) = 0$ and $\beta''(s) = 0$ for all s, that is $T(s) = 0$ and $\kappa(s)P(s) = 0$ for all s. Since we are assuming that $\kappa_\alpha(t) \neq 0$ for all t, hence $\kappa(s) = \kappa_\beta(s) \neq 0$ for all s, it follows that $P(s) = 0$ for all s. Since $\{T(s), P(s), B(s)\}$ is orthonormal and $T(s) = P(s) = 0$ and $|b| = 1$, it follows that $B(s) = \pm b$ for all s. Since $B(s)$ is continuous, either we have

$B(s) = b$ for all s or we have $B(s) = -b$ for all s and, in either case, $B'(s) = 0$ for all s. Since $0 = B'(s) = -\tau(s)P(s)$ with $|P(s)| = 1$, we have $\tau(s) = 0$, that is $\tau_\beta(s) = 0$, for all s, and hence $\tau_\alpha(t) = \tau_\beta(s(t)) = 0$ for all t. ■

Definition 1.25 — osculating plane, osculating circle.
Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$, let $a \in I$, and suppose that $\kappa(a) \neq 0$ (and hence $\tau(a)$ is defined). We define the **osculating plane** of α at $t = a$ to be the plane through $\alpha(a)$ parallel to $T(a)$ and $P(a)$, that is the plane $(x - \alpha(a))(a) = 0$. We define the **osculating circle** (or the **best-fit circle**) of α at $t = a$ as we did for a planar curve (in Definition 1.15). Let $p = \alpha(a)$, $T = T(a)$, $P = P(a)$, and $\kappa = \kappa(a)$. Reparametrize by arclength, letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\, dr$ so that we have $\beta(0) = p$. $\beta'(0) = T$ and $\beta''(0) = \kappa P$. The osculating circle at $t = a$ is the circle given by

$$\begin{aligned}
\sigma(s) &= \left(p + \frac{1}{\kappa}P\right) - \frac{1}{\kappa}\cos(\kappa s)P + \frac{1}{\kappa}\sin(\kappa s)T \\
\sigma'(s) &= \sin(\kappa s)P + \cos(\kappa s)T \\
\sigma''(s) &= \kappa\cos(ks)P - \kappa\sin(\kappa s)T
\end{aligned}$$

which is the circle of radius $R = \frac{1}{\kappa}$ centered at $p + \frac{1}{\kappa}P$, parametrized by arclength (since $|\sigma'(s)| = 1$ for all s), such that $\sigma(0) = p = \beta(0)$, $\sigma'(0) = T = \beta'(0)$ and $\sigma''(0) = \kappa P = \beta''(0)$.

Note 1.26 When α is a smooth regular curve, the scalar curvature at $t = a$ is equal to the reciprocal of the radius of the osculating circle at $t = a$.

Theorem 1.27 — The Fundamental Theorem for Space Curves.
Given $p, u, v \in \mathbb{R}^3$ with $|u| = |v| = 1$ and given smooth functions $c, d : I \subseteq \mathbb{R} \to \mathbb{R}$ where I is an interval with $0 \in I$ and $c(s) > 0$ for all $s \in I$, there exists a unique smooth regular curve $\beta : I \to \mathbb{R}^3$ with $\beta(0) = p$, $T(0) = u$, $P(0) = v$ and $\kappa(s) = c(s)$ and $\tau(s) = d(s)$ for all $s \in I$.

Proof. We want to have $T' = \kappa P$, $P' = -\kappa T + \tau B$ and $B' = -\tau P$, so we solve the system of linear first order differential equations

$$\begin{aligned}
X' &= cY \\
Y' &= -cX + dZ \\
Z' &= -dY
\end{aligned}$$

with the initial conditions $X(0) = u$, $Y(0) = v$ and $Z(0) = u \times v$ (such a system always has a unique solution). We claim that $\{X(s), Y(s), Z(s)\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$ for all s (this is true when $s = 0$ from the initial conditions). Write $X_1 = X$, $X_2 = Y$ and $X_3 = Z$ and define $F_{k,\ell} : I \to \mathbb{R}$ by $F_{k,\ell}(s) = X_k(s)_\ell(s)$ for $1 \le k \le \ell \le 3$.

Then the functions $F_{k,\ell}$ satisfy the system of differential equations

$$\begin{aligned}
\frac{d}{ds}F_{1,1} &= \frac{d}{ds}(X) = 2\,X' = 2\,(cY) = 2c\,F_{1,2}\\
\frac{d}{ds}F_{1,2} &= \frac{d}{ds}(X) = X' + X' = c\,Y + X\cdot(-cX + dZ) = -c\,F_{1,1} + d\,F_{1,3} + c\,F_{2,2}\\
\frac{d}{ds}F_{1,3} &= \frac{d}{ds}(X) = X' + X' = c\,Y + X\cdot(-dY) = -d\,F_{1,2} + c\,F_{2,3}\\
\frac{d}{ds}F_{2,2} &= \frac{d}{ds}(Y) = 2\,Y' = 2\,(-cX + dZ) = -2c\,F_{1,2} + 2d\,F_{2,3}\\
\frac{d}{ds}F_{2,3} &= \frac{d}{ds}(Y) = Y' + Y' = (-cX + dZ) + Y\cdot(-dY) = -c\,F_{1,3} - d\,F_{2,2} + d\,F_{3,3}\\
\frac{d}{ds}F_{3,3} &= \frac{d}{ds}(Z) = 2\,Z' = 2\,(-dY) = -2\,F_{2,3}
\end{aligned}$$

with the initial conditions $F_{k,k}(0) = 1$ and $F_{k,\ell}(0) = 0$ when $k \neq \ell$. Again, such a system has a unique solution, and the unique solution to this system is easily seen to be given by the constant functions $F_{k,k}(s) = 1$ and $F_{k,\ell}(s) = 0$ for all $s \in I$ and all $k \neq \ell$. Thus $\big\{X(s), Y(s), Z(s)\big\}$ is an orthonormal system for all $s \in I$, as claimed. To get $\beta'(s) = T(s) = X(s)$ with $\beta(0) = p$ we must choose $\beta(s) = p + \int_0^s X(t)\,dt$. Then we have $T = X$ and $\kappa(s) = |\beta''(s)| = |T'| = |X'| = |cY| = c$ and $P = \frac{1}{\kappa}\,T' = \frac{1}{c}X' = \frac{1}{c}(cY) = Y$ and $B = T \times P = X \times Y = Z$ and $\tau = -B' = -Z' = (dY) = d$, as required. ∎

2. Surfaces

2.1 Surfaces in $\mathbb{R}^n$

Definition 2.1 — surface, smooth, regular.
A (local parametrized) **surface** in $\mathbb{R}^n$ is a continuous map $\sigma : U \subseteq \mathbb{R} \to \mathbb{R}^n$ where U is an open set. We can write $\sigma(u,v) = (x_1(u,v), x_2(u,v), \cdots, x_n(u,v))$ where each $x_k : U \subseteq \mathbb{R} \to \mathbb{R}$ is continuous. We say that σ is $\mathcal{C}^k$ when all of the k^{th} order partial derivatives exist and are continuous in U, and we say σ is **smooth**, or $\mathcal{C}^\infty$, when σ is $\mathcal{C}^k$ for every $k \in \mathbb{Z}^+$. Recall that when σ is $\mathcal{C}^1$, it is also differentiable and its derivative (or Jacobian) matrix is

$$D\sigma = \begin{pmatrix}\sigma_u, \sigma_v\end{pmatrix} = \begin{pmatrix}\frac{\partial \sigma}{\partial u} & \frac{\partial \sigma}{\partial v}\end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \vdots & \vdots \\ \frac{\partial x_n}{\partial u} & \frac{\partial x_n}{\partial v} \end{pmatrix}.$$

We say that σ is **regular** when σ is $\mathcal{C}^1$ and its derivative matrix is of rank 2, that is when the two columns σ_u and σ_v are linearly independent. In this case, the **tangent plane** to the surface at $(u,v) = (a,b)$ is the plane through $\sigma(a,b)$ parallel to $\sigma_u(a,b)$ and $\sigma_v(a,b)$. Unless otherwise stated, we shall assume all surfaces are smooth and regular.

■ **Example 2.2** When $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is a $\mathcal{C}^2$ function, its **graph** $z = f(x,y)$ is the image of the local parametrized surface $\sigma : U \to \mathbb{R}^3$ given by $(x,y,z) = \sigma(u,v) = (u,v,f(u,v))$. The derivative matrix is

$$D\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}$$

which has rank 2, and the tangent plane at (a,b) is the plane through $(a,b,f(a,b))$ parallel to $\left(1, 0, \frac{\partial f}{\partial x}(a,b)\right)$ and $\left(0, 0, \frac{\partial f}{\partial y}(a,b)\right)$.

■ **Example 2.3** The **paraboloid** $z = x^2 + y^2$ in $\mathbb{R}^3$ can be given parametrically in Cartesian coordinates by $\sigma(u,v) = \big(u, v, u^2+v^2\big)$ or it can be given parametrically in polar coordinates by $\rho(r,\theta) = \big(r\cos\theta, r\sin\theta, r^2\big)$. Using polar coordinates, the derivative matrix is

$$D\rho = \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \\ z_r & z_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 2r & 0 \end{pmatrix}$$

which has rank 2 except when $r=0$, so this parametrization is regular when $r \neq 0$.

■ **Example 2.4** The top half of the **sphere** $x^2+y^2+z^2=r^2$ of radius $r>0$ is the graph of the function $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(x,x) = \sqrt{r^2-x^2-y^2}$ where U is the open disc $x^2+y^2<r^2$, so it can be given parametrically in Cartesian coordinates by $\sigma(u,v) = \Big(u, v, \sqrt{r^2-u^2-v^2}\Big)$. The entire sphere can be given parametrically using spherical coordinates by

$$\rho(\phi,\theta) = (r\sin\phi\cos\theta\,,\ r\sin\phi\sin\theta\,,\ r\cos\phi)\,.$$

The derivative matrix is

$$D\rho = \begin{pmatrix} r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ -r\sin\phi & 0 \end{pmatrix}$$

which has rank 2 except when $\sin\phi = 0$, that is when $\phi = k\pi$ for some $k \in \mathbb{Z}$.

■ **Example 2.5** The **torus** obtained by revolving the circle $(x-R)^2+z^2=r^2$ (in the xz-plane) about the z-axis, where $0<r<R$, can be given parametrically by

$$\begin{aligned}\sigma(\theta,\phi) &= R\,(\cos\theta,\sin\theta,0) + r\cos\phi\,(\cos\theta,\sin\theta,0) + r\sin\phi\,(0,0,1) \\ &= ((R+r\cos\phi)\cos\theta\,,\ (R+r\cos\phi)\sin\theta\,,\ r\sin\phi)\end{aligned}$$

Verify, as an exercise, that the derivative matrix has rank 2 everywhere, so the surface is regular everywhere.

Definition 2.6 — Riemannian metric, first fundamental form.
A **Riemannian metric** on an open set $U \subseteq \mathbb{R}^n$ is a smooth map $g : U \to M_{n\times n}(\mathbb{R})$ such that $g(p)$ is a positive-definite symmetric matrix for every $p \in U$, so that for each $p \in U$, $g(p)$ defines an inner-product on $\mathbb{R}^n$ given by

$$I(A,B) = \langle A,B\rangle = B^T g(p)\,A\,.$$

Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^n$ be a smooth regular surface in $\mathbb{R}^n$. The **first fundamental form** of σ is the smooth map $g = g_\sigma : U \subseteq \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$ given by

$$g = g_\sigma = D\sigma^T D\sigma = \begin{pmatrix} \sigma_u\cdot\sigma_u & \sigma_u\cdot\sigma_v \\ \sigma_u\cdot\sigma_v & \sigma_v\cdot\sigma_v \end{pmatrix}.$$

For each $p \in U$, $g = g(p) = D\sigma(p)^T D\sigma(p)$ is a positive-definite symmetric matrix, so g is a Riemannian metric on U. It is traditional to write

$$E = g_{1,1} = \sigma_u\cdot\sigma_u \quad , \quad F = g_{1,2} = g_{2,1} = \sigma_u\cdot\sigma_v \quad , \quad G = \sigma_v\cdot\sigma_v\,.$$

■ **Example 2.7 — The length of a curve on a surface.** Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^n$ be a regular surface in $\mathbb{R}^n$ and let $\alpha : [a,b] \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^2$ be a regular curve which takes values in U.

Then the composite $\gamma(t) = \sigma(\alpha(t))$ is a regular curve on the surface in $\mathbb{R}^n$. Let us find a formula for the length of γ on $[a,b]$. We have $\gamma'(t) = D\sigma(\alpha(t))\,\alpha'(t)$ and so

$$|\gamma'|^2 = \big(D\sigma\,\alpha'\big)\cdot\big(D\sigma\,\alpha'\big) = \big(D\sigma\cdot\alpha'\big)^T\big(D\sigma\,\alpha'\big) = (\alpha')^T g\,\alpha'$$

hence

$$L_\gamma([a,b]) = \int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{\alpha'(t)^T g(\alpha(t))\ \alpha'(t)}\ dt = \int_a^b \|\alpha'(t)\|\,dt\,,$$

where we are using the Riemannian inner product $\langle X,Y\rangle = Y^T gX$ and its associated norm $\|X\| = \sqrt{\langle X,X\rangle}$. We also consider this to be the length of the curve α on $[a,b]$ in U with respect to the Riemann metric g.

■ **Example 2.8 — The angle between curves on a surface.** Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^n$ be a regular surface in $\mathbb{R}^n$ and let $p \in U$. Let $\alpha,\beta : I \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^2$ be two regular curves with $0 \in U$ and $\alpha(0) = \beta(0) = p$. Then $\gamma(t) = \sigma(\alpha(t))$ and $\delta(t) = \sigma(\beta(t))$ are two regular curves on the surface in $\mathbb{R}^n$ which intersect at $t = 0$. Let us calculate the angle between the two curves γ at δ at $t = 0$ (that is the angle between $\gamma'(0)$ and $\delta'(0)$). We have $\gamma(t) = D\sigma(\gamma(t))$ so $\gamma'(0) = D\sigma(p)\alpha'(0)$ and similarly $\delta'(0) = D\sigma(p)\beta'(0)$, and so

$$\gamma'(0)\cdot\delta'(0) = \big(D\sigma(p)\alpha'(0)\big)\cdot\big(D\sigma(p)\beta'(0)\big) = \beta'(0)^T g(p)\,\alpha'(0).$$

Similarly, we have $|\gamma'(0)|^2 = \alpha'(0)^T g(p)\,\alpha'(0)$ and $|\delta'(0)|^2 = \beta'(0)^T g(p)\,\beta'(0)$, and so the angle $\theta \in [0\pi]$ between $\gamma'(0)$ and $\delta'(0)$ is given by

$$\cos\theta = \frac{\gamma'(0)\cdot\delta'(0)}{|\gamma'(0)|\,|\delta'(0)|} = \frac{\beta'(0)^T g(p)\,\alpha'(0)}{\sqrt{\alpha'(0)^T g(p)\,\alpha'(0)}\,\sqrt{\beta'(0)^T g(p)\,\beta'(0)}} = \frac{\langle\alpha'(0),\beta'(0)\rangle}{\|\alpha'(0)\|\,\|\beta'(0)\|}.$$

This is also the angle between $\alpha'(0)$ and $\beta'(0)$ in $\mathbb{R}^2$, with respect to the Riemanninan metric g.

Remark 2.9 We shall define the area of a portion of a regular surface, but before stating the definition let us provide some informal motivation for the definition. Recall that when $p,u,v \in \mathbb{R}^n$, the area of the parallelogram with vertices at p, $p+u$, $p+v$ and $p+u+v$ is equal to $A = \sqrt{\det(P^TP)}$ where $P = (u,v) \in M_{n\times 2}(\mathbb{R})$. Indeed the angle θ between u and v is given by $\theta = \cos^{-1}\frac{u\cdot v}{|u|\,|v|}$ and the parallelogram has base $|u|$ and height $|v|\sin\theta$ so its area is $A = |u|\,|v|\sin\theta$ and hence

$$\begin{aligned} A^2 &= |u|^2|v|^2\left(1-\cos^2\theta\right) = |u|^2|v|^2\left(1-\left(\frac{u\cdot v}{|u|\,|v|}\right)^2\right) = |u|^2|v|^2 - (u\cdot v)^2 \\ &= \det\begin{pmatrix} u\cdot u & u\cdot v \\ u\cdot v & v\cdot v\end{pmatrix} = \det(P^TP). \end{aligned}$$

Given a closed Jordan region $R \subseteq U$ (that is a region with a well defined area), we can approximate its area (arbitrarily closely) by covering it with finitely many closed rectangles which are each contained in U. If the k^{th} rectangle has vertices at $p = (u,v)$, $(u+du,v)$, $(u,v+dv)$ and $(u+du,v+dv)$ (where du and dv are small positive real numbers) then the image $\sigma(R_k)$ can be approximated by the parallelogram in $\mathbb{R}^n$ with vertices at $\sigma(p)$, $\sigma(p)+\sigma_u(p)du$, $\sigma(p)+\sigma_v(p)dv$ and $\sigma(p)+\sigma_u(p)du+\sigma_v(p)dv$ whose area is $dA_k = \sqrt{\det(P^TP)}$ where $P = (\sigma_u(p)du, \sigma_v(p)dv)$, that is

$$dA_k = \sqrt{\det\begin{pmatrix}\sigma_u\cdot\sigma_u\,du\,du & \sigma_u\cdot\sigma_v\,du\,dv \\ \sigma_u\cdot\sigma_v\,du\,dv & \sigma_v\cdot\sigma_v\,dv\,dv\end{pmatrix}} = \sqrt{\det g(p)}\,dudv\,.$$

The total area of $\sigma(R)$ is approximated by the sum of the areas of these parallelograms, which is a Riemann sum for the function $\sqrt{\det g}$, and the limit of these Riemann sums is

$$\int\int_R \sqrt{\det g(u,v)}\, du\, dv\,.$$

Definition 2.10

Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^n$ be a regular surface in $\mathbb{R}^n$ and let $R \subseteq U$ be a closed Jordan region (for example, a region of the form $a \le u \le b$, $f(u) \le v \le g(u)$ where f and g are continuous). We write $dA = \sqrt{\det g(u,v)}\, du\, dv$ and we define the area of σ on R to be

$$A_\sigma(R) = \int\int_R dA = \int\int_R \sqrt{\det g(u,v)}\, du\, dv.$$

More generally, for a continuous function $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ we define

$$\int\int_R f\, dA = \int\int_R f(u,v)\, \sqrt{\det g(u,v)}\, du\, dv.$$

Definition 2.11 — regular change of coordinates, positive, preserves orientation.

Let $U, V \subseteq \mathbb{R}^n$ be open sets. A **regular change of coordinates** from U to V is a bijective map $\phi : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^n$ such that ϕ and its inverse $\psi = \phi^{-1} : V \to U$ are both $\mathcal{C}^1$. Note that by the Chain Rule, for all $p \in U$ we have $D\psi(\phi(p))D\phi(p) = I$ (so the derivative matrices of ϕ and ψ are invertible at all points). We say ϕ is **positive**, or ϕ **preserves orientation**, when $\det D\phi(p) > 0$ for all $p \in U$. Unless otherwise stated, we assume that any change of coordinates is smooth and regular.

Theorem 2.12

(Change of Coordinates) Let $U, V \subseteq \mathbb{R}^2$ be open sets and let $\phi : U \to V$ be a smooth regular change of coordinates from U to V with inverse $\psi = \phi^{-1} : V \to U$. Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^n$ be a smooth regular surface in $\mathbb{R}^n$ and let $\rho : V \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be the smooth regular surface given by $\rho(q) = \sigma(\psi(q))$. For points $p \in U$ and $q = \phi(p) \in V$ and for Jordan regions $R \subseteq U$ and $Q = \phi(R) \subseteq V$ we have 1 $g_\rho(q) = D\psi(q)^T g_\sigma(p)\, D\psi(q)$ and

2 $A_\rho(Q) = A_\sigma(R)$.

Proof. To prove Part 1, note that since $\rho(q) = \sigma(\psi(q))$ we have $D\rho = D\sigma\, D\psi$ and so

$$g_\rho = D\rho^T D\rho = (D\sigma\, D\psi)^T\, (D\sigma\, D\psi) = D\psi^T D\sigma^T D\sigma\, D\psi = D\psi^T g_\sigma\, D\psi.$$

Let us prove Part 2. Since $g_\rho = D\psi^T g_\sigma\, D\psi$, we have $\det g_\rho = (\det D\psi)^2 \det g_\sigma$ and so, since $\det D\psi > 0$, we have

$$\sqrt{\det g_\rho} = \det D\psi \sqrt{\det g_\sigma}.$$

If $R \subseteq U$ and $Q = \phi(R) \subseteq V$ are Jordan regions then, writing $(u,v) = \psi(s,t)$, the change of variables formula for integration gives

$$\begin{aligned} A_\sigma(R) &= \int\int_R \sqrt{\det g_\sigma(u,v)}\, du\, dv = \int\int_Q \sqrt{\det g_\sigma(\psi(s,t))}\, \det D\psi(s,t)\, ds\, dt \\ &= \int\int_Q \sqrt{g_\rho(s,t)}\, ds\, dt = A_\rho(Q). \end{aligned}$$

This proves Part 2 and confirms our intuitive expectation that the area of a surface does not change when we use a change of coordinates to obtain an alternate parametrization. ■

2.2 Surfaces in $\mathbb{R}^3$

Definition 2.13 — unit normal vector, Gauss map, curvature.
Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$. Since $D\sigma = (\sigma_u, \sigma_v)$ has rank 2, it follows that $\sigma_u \times \sigma_v \neq 0$. For $p \in U$, since the tangent plane to σ at p is parallel to $\sigma_u(p)$ and $\sigma_v(p)$, it has normal vector $\sigma_u \times \sigma_v$. We define the **unit normal vector** to σ at p to be $n = n(p)$ where

$$n = n_\sigma = \frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|} .$$

Notice that $n : U \subseteq \mathbb{R}^2 \to \S^2 \subseteq \mathbb{R}^3$ where $\S^2$ is the unit sphere $\S^2 = \left\{x \in \mathbb{R}^3 \, |x| = 1\right\}$, and this map is called the **Gauss map** of σ on U.

Given a point $p \in U$ and a nonzero vector $0 \neq A \in \mathbb{R}^2$, we define the (directional) **curvature** $k(p)(A) = k_\sigma(p)(A)$ of σ in the direction of A at p as follows: choose any smooth regular curve $\alpha : I \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^2$ with $0 \in I$ and $\alpha(0) = p$ and $\alpha'(0) = A$, let $\gamma(t) = \sigma(\alpha(t))$ (so γ is a curve on the surface in $\mathbb{R}^3$), reparametrize by arcclength by letting $\delta(s) = \gamma(t(s))$ where $s(t) = \int_0^t |\gamma'(r)| \, dr$, let $N(s) = n\left(\alpha(t(s))\right)$, and then define

$$k_\sigma(p)(A) = \delta''(0) \cdot N(0).$$

The following theorem shows that this definition does not depend on the choice of α. It only depends on σ and p and the direction of the vector A.

Theorem 2.14 — Directional Curvature.
Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$. For $p \in U$ and $0 \neq A \in \mathbb{R}^2$, the curvature of σ in the direction of A at p is

$$k_\sigma(p)(A) = \frac{A^T h(p) \, A}{A^T g(p) \, A}$$

where $g = D\sigma^T D\sigma$ and

$$h = -Dn^T D\sigma = -\begin{pmatrix} \sigma_u \cdot n_u & \sigma_v \cdot n_u \\ \sigma_u \cdot n_v & \sigma_v \cdot n_v \end{pmatrix} = \begin{pmatrix} \sigma_{uu} \cdot n & \sigma_{uv} \cdot n \\ \sigma_{uv} \cdot n & \sigma_{vv} \cdot n \end{pmatrix} .$$

Proof. Let α, γ, δ, β and N be as in the definition. Since δ is a curve on the surface so δ' is tangent to the surface, we expect, intuitively, that $\delta' \cdot N = 0$. Let us verify this algebraically. Since

$$N(s) = n(\beta(s)) = \frac{\sigma_u(\beta(s)) \times \sigma_v(\beta(s))}{|\sigma_u(\beta(s)) \times \sigma_v(\beta(s))|},$$

it follows that $N(s) \cdot \sigma_u(\beta(s)) = N(s) \cdot \sigma_v(\beta(s)) = 0$ and so $N(s)^T D\sigma(\beta(s)) = 0$. Since $\delta(s) = \gamma(t(s)) = \sigma\left(\alpha(t(s))\right)$, we have $\delta'(s) = D\sigma\left(\alpha(t(s))\right) \alpha'(t(s)) \, t'(s)$ and so

$$\delta'(s) \cdot N(s) = N(s)^T \delta'(s) = N(s)^T D\sigma\left(\alpha(t(s))\right) \alpha'(t(s)) \, t'(s) = 0,$$

as we expected. Differentiating gives $0 = \frac{d}{ds}\left(\delta'(s) \cdot N(s)\right) = \delta''(s) \cdot N(s) + \delta'(s) \cdot N'(s)$, so

$$k(p, A) = \delta''(0) \cdot n(0) = -\delta'(0) \cdot n'(0).$$

Since $\gamma(t) = \sigma(\alpha(t))$ we have $\gamma'(t) = D\sigma(\alpha(t))\,\alpha'(t)$. Since $s(t) = \int_0^t |\gamma'(r)|\,dr$ we have $s'(t) = |\gamma'(t)|$ hence $|t'(s)| = \frac{1}{|\gamma'(t(s))|}$. Since $\delta(s) = \gamma(t(s))$ we have

$$\delta'(s) = \gamma'(t(s))\,t'(s) = \frac{\gamma'(t(s))}{|\gamma'(t(s))|} = \frac{D\sigma\left(\alpha(t(s))\right)\,\alpha'(t(s))}{|\sigma\left(\alpha(t(s))\right)\,\alpha'(t(s))|}.$$

Since $N(s) = n\left(\alpha(t(s))\right)$ we have

$$N'(s) = Dn\left(\alpha(t(s))\right)\,\alpha'(t(s))\,t'(s) = \frac{Dn\left(\alpha(t(s))\right)\alpha'(t(s))}{|\sigma\left(\alpha(t(s))\right)\,\alpha'(t(s))|}.$$

Thus we have

$$\delta' \cdot N' = \frac{D\sigma\,\alpha'}{|D\sigma\,\alpha'|} \cdot \frac{Dn\,\alpha'}{|D\sigma\,\alpha'|} = \frac{(\alpha')^T Dn^T D\sigma\,\alpha'}{(\alpha')^T D\sigma^T D\sigma\,\alpha'}$$

so, in particular,

$$k_\sigma(p)(A) = -\delta'(0) \cdot N(0) = -\frac{A^T Dn^T(p) D\sigma(p)\,A}{A^T D\sigma(p)^T D\sigma(p)\,A} = \frac{A^T h(p)\,A}{A^T g(p)\,A}$$

where $g = D\sigma^T D\sigma$ and $h = -Dn^T D\sigma$.

Finally, note that since $\sigma_u \cdot n = 0$ and $\sigma_v \cdot n = 0$, we can differentiate with respect to u and v to get $\sigma_{uu} \cdot n + \sigma_u \cdot n_u = 0$, $\sigma_{uv} \cdot n + \sigma_u \cdot n_v = 0$, $\sigma_{vu} \cdot n + \sigma_v \cdot n_u = 0$ and $\sigma_{vv} \cdot n + \sigma_v \cdot n_v = 0$ and so

$$h = -Dn^T D\sigma = -\begin{pmatrix} \sigma_u \cdot n_u & \sigma_v \cdot n_u \\ \sigma_u \cdot n_v & \sigma_v \cdot n_v \end{pmatrix} = \begin{pmatrix} \sigma_{uu} \cdot n & \sigma_{uv} \cdot n \\ \sigma_{uv} \cdot n & \sigma_{vv} \cdot n \end{pmatrix}.$$

■

Definition 2.15 — second fundamental form.
For a smooth regular surface $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ in $\mathbb{R}^3$, the **second fundamental form** of σ is the smooth map $h = h_\sigma : U \subseteq \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$ given by

$$h = -Dn^T D\sigma = -\begin{pmatrix} \sigma_u \cdot n_u & \sigma_v \cdot n_u \\ \sigma_u \cdot n_v & \sigma_v \cdot n_v \end{pmatrix} = \begin{pmatrix} \sigma_{uu} \cdot n & \sigma_{uv} \cdot n \\ \sigma_{uv} \cdot n & \sigma_{vv} \cdot n \end{pmatrix}.$$

For each $p \in U$, $h(p) = Dn(p)^T D\sigma(p)$ is a symmetric matrix so it defines a symmetric bilinear form on $\mathbb{R}^2$ given by $II(A, B) = B^T h(p)\,A$. It is traditional to write

$$L = h_{1,1}\ ,\ \ M = h_{1,2} = h_{2,1}\ ,\ \ N = h_{2,2}.$$

Theorem 2.16 — Change of Coordinates.
Let $U, V \subseteq \mathbb{R}^2$ be open, and let $\phi : U \to V$ be a smooth regular change of coordinates. Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface, and let $\rho : V \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be the corresponding surface given by $\rho(q) = \sigma(\psi(q))$. Then for points $p \in U$ and $q = \phi(p) \in V$ and nonzero vectors $0 \neq A \in \mathbb{R}^2$ and $B = D\phi(p)A$, we have
(1) $n_\rho(q) = n_\sigma(p)$,
(2) $h_\rho(q) = D\psi(q)^T h_\sigma(p)\,D\psi(q)$, and
(3) $k_\rho(q)(B) = k_\sigma(p)(A)$.

Proof. Write $p = (u, v)$ and $q = (s, t) = \phi(u, v)$. Then $\rho(s, t) = \sigma\left(\psi(s, t)\right)$ so we have $\rho_s = \sigma_u\,u_s + \sigma_v\,v_s$ and $\rho_t = \sigma_u\,u_t + \sigma_v\,v_t$, hence

$$\rho_s \times \rho_t = (\sigma_u\,u_s + \sigma_v\,v_s) \times (\sigma_u\,u_t + \sigma_v\,v_t) = (\sigma_u \times \sigma_v)(u_s v_t - u_t v_s) = \det D\psi\,(\sigma_u \times \sigma_v),$$

that is $\rho_s(q) \times \rho_t(q) = \det \psi(q)(\sigma_u(p) \times \sigma_v(p))$. Since $D\psi(p) > 0$ for all p, we have

$$n_\rho(q) = \frac{\rho(p) \times \rho(p)}{|\rho(p) \times \rho(p)|} = \frac{\det D\psi(q)(\sigma_u(p) \times \sigma_v(p))}{|\det D\psi(q)(\sigma_u(p) \times \sigma_v(p))|} = \frac{\sigma_u(p) \times \sigma_v(p)}{|\sigma_u(p) \times \sigma_v(p)|} = n_\sigma(p).$$

This proves Part 1.

Since $\rho(q) = \sigma(\psi(q))$ so that $D\rho(q) = D\sigma(p)D\psi(q)$, and $n_\rho(q) = n_\sigma(\psi(q))$ so that $Dn_\rho(q) = Dn_\sigma(p)D\psi(q)$, we have

$$h_\rho(q) = Dn_\rho(q)^T D\rho(q) = D\psi(q)^T Dn_\sigma(p)^T D\sigma(p)D\psi(q) = D\psi(q)^T h_\sigma(p)D\psi(q).$$

This proves Part 2.

To prove Part 3, let $\alpha : I \in \mathbb{R} \to U \subseteq \mathbb{R}^2$ be a regular $\mathcal{C}^2$ curve with $\alpha(0) = p$ and $\alpha'(0) = A$, and let $\beta : I \subseteq \mathbb{R} \to V \subseteq \mathbb{R}^2$ be the corresponding curve $\beta(t) = \phi(\alpha(t))$, and note that $\beta(0) = \phi(\alpha(0)) = \phi(p) = q$. Since $\beta(t) = \phi(\alpha(t)$ we have $\beta'(t) = D\phi(\alpha(t))\,\alpha'(t)$ so that $\beta'(0) = D\phi(p)\,A = B$. Let $\gamma_\sigma(t) = \sigma(\alpha(t))$, reparametrize by arclength to get $\delta_\sigma(s) = \gamma_\sigma(t(s))$, and let $N_\sigma(s) = n_\sigma(\alpha(t(s))$ so that $k_\sigma(p)(A) = \delta_\sigma{}''(0) \cdot N_\sigma(0)$. Similarly, let $\gamma_\rho(t) = \rho(\beta(t))$, reparametrize by arclength to get $\delta_\rho(s) = \gamma_\rho(t(s))$, and let $N_\rho(s) = n_\rho\left(\beta(t(s))\right)$ so that $k_\rho(q)(B) = \delta_\rho{}''(0) \cdot N_\rho(0)$. Since ϕ and ψ are inverses we have

$$\gamma_\rho(t) = \rho(\beta(t) = \rho\left(\phi(\alpha(t))\right) = \sigma\left(\psi\left(\phi(\alpha(t))\right)\right) = \sigma(\alpha(t)) = \gamma_\alpha(t)$$

for all t, hence $\delta_\rho(s) = \delta_\rho(t(s)) = \gamma_\sigma(t(s)) = \delta_\sigma(s)$ for all s, so $\delta_\rho{}''(0) = \delta_\sigma{}''(0)$. Also, we have $N_\rho(0) = n_\rho(\beta(0)) = n_\rho(q) = n_\sigma(p) = n_\sigma(\alpha(0)) = N_\sigma(0)$, and so

$$k_\rho(q)(B) = \delta_\rho{}''(0) \cdot N_\rho(0) = \delta_\sigma{}''(0) \cdot N_\sigma(0) = k_\sigma(p)(A).$$

■

Remark 2.17 When $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is a smooth regular surface, for each $p \in U$ the directional curvature $k_\sigma(p)$ is a smooth function $k_\sigma(p) : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ with the property that $k_\sigma(p)(tA) = k_\sigma(p)(A)$ for all $0 \neq t \in \mathbb{R}$ and all $0 \neq A \in \mathbb{R}^2$. Thus we can consider $k_\sigma(p)$ to be a function $k_\sigma(p) : \mathbb{P}^1(\mathbb{R}) \to \mathbb{R}$ where $\mathbb{P}^1(\mathbb{R})$ is the real **projective space**, which is the set of 1-dimensional subspaces of $\mathbb{R}^2$.

Theorem 2.18 — Principal Curvature Directions.
Let $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$, and let $p \in U$.

(1) The directional curvature $k_\sigma(p)(A)$ attains its maximum and minimum values in two directions which are orthogonal with respect to the inner product $\langle X, Y\rangle = Y^T g_\sigma(p)\,X$.

(2) The maximum and minimum values k_1 and k_2 of $k_\sigma(p)(A)$ for $0 \neq A \in \mathbb{R}^2$ are the eigenvalues of $g_\sigma(p)^{-1}h_\sigma(p)$, and the directions $A_1, A_2 \in \mathbb{R}^2$ in which they occur are the corresponding eigenvalues.

(3) The maximum and minimum values k_1 and k_2 are roots of the quadratic polynomial

$$0 = f(k) = \det\left(h_\sigma(p) - k\, g_\sigma(p)\right)$$

and the directions $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ in which they occur are roots of the quadratic form

$$0 = f(x, y) = \det\left(h_\sigma(p)\begin{pmatrix} x \\ y \end{pmatrix}, g_\sigma(p)\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Proof. Let $p=(a,b)$. We begin by changing coordinates so that, in the new coordinates, at the point q corresponding to p, the inner product becomes the standard dot product. We do this as follows: Apply the Gram-Schmidt procedure to the standard basis for $\mathbb{R}^2$ to obtain a positive ordered basis $\{A,B\}$ for $\mathbb{R}^2$ which is orthonormal with respect to the inner product $\langle X,Y\rangle = Y^T g(p)\,X$. Let $P=(A,B)\in M_{2\times 2}(\mathbb{R})$ and define $\psi:\mathbb{R}^2\to\mathbb{R}^2$ by

$$\psi(s,t)=\begin{pmatrix}a\\b\end{pmatrix}+P\begin{pmatrix}s\\t\end{pmatrix}.$$

Note that $\psi(0)=p$ and $D\psi(s,t)=P$ for all s,t. Let $V=\phi(U)$ where $\phi=\psi^{-1}$, and let $\rho:V\subseteq\mathbb{R}^2\to\mathbb{R}^3$ be the corresponding surface given by $\rho(s,t)=\sigma\left(\psi(s,t)\right)$. Then at $q=0$

$$g_\rho = D\psi^T g_\sigma\, D\psi = P^T g_\sigma P = \begin{pmatrix}A^T g_\sigma A & A^T g_\sigma B\\ B^T g_\sigma A & B^T g_\sigma B\end{pmatrix} = I,$$

so in the new coordinates, at the point $q=\phi(p)=0$, the inner product is the standard inner product. We have

$$k_\sigma(p)(A)=k_\rho(q)(B)=\frac{B^T h_\rho\, B}{B^T g_\rho\, B}=\hat{B}^T h_\rho\,\hat{B}$$

where $B=D\phi(p)A=P^{-1}A$ and $\hat{B}$ is the unit vector $\frac{B}{|B|}$. Recall from linear algebra (or verify using the fact that symmetric matrices are orthogonally diagonalizable) that h_ρ has real eigenvalues $k_1,k_2\in\mathbb{R}$ with $k_1\geq k_2$, with orthogonal unit eigenvectors $B_1,B_2\in\mathbb{R}^2$, and the maximum and minimum values of the quadratic form given by $Q(X)=X^T h_\rho X$, over all unit vectors $X\in\mathbb{R}^2$, are $Q(B_1)=k_1$ and $Q(B_2)=k_2$. Note that the unit vectors B_1 and B_2 are orthogonal with respect to the standard inner product, and the corresponding vectors $A_1=D\psi(0)B_1=PB_1$ and $A_2=D\psi(0)B_2=PB_2$ are orthogonal unit vectors with respect to the inner product given by $g_\sigma(p)$. This proves Part 1.

For $k\in\mathbb{R}$ and $0\neq B\in\mathbb{R}^2$, recall that k is an eigenvalue of h_ρ with eigenvector B if and only if $(h_\rho-kI)B=0$. For $A=PB$, we have

$$\begin{aligned}(h_\rho-kI)B=0 &\iff (P^T h_\sigma P - k\,P^T g_\sigma P)B=0 \iff P^T(h_\sigma-kg_\sigma)PB=0\\ &\iff P^T(h_\sigma-kg_\sigma)A=0 \iff (h_\sigma-kg_\sigma)A=0\\ &\iff {g_\sigma}^{-1}(h_\sigma-kg_\sigma)A=0 \iff ({g_\sigma}^{-1}h_\sigma-kI)A=0.\end{aligned}$$

Thus k_1 and k_2 are the eigenvalues of h_ρ with eigenvectors B_1 and B_2 if and only if k_1 and k_2 are eigenvalues of ${g_\sigma}^{-1}h_\sigma$ with eigenvectors A_1 and A_2. This proves Part 2.

For $k\in\mathbb{R}$, k is an eigenvalue of h_ρ if and only if there exists $0\neq A\in\mathbb{R}^2$ such that $(h_\sigma-kg_\sigma)A=0$, if and only if $\det(h_\sigma-kg_\sigma)=0$. This gives the first formula in Part 3. For $0\neq A\in\mathbb{R}^2$ we have $A=PB$ where B is an eigenvector of h_ρ if and only if there exists $k\in\mathbb{R}$ such that $0=(h_\sigma-kg_\sigma)A=(h_\sigma A,g_\sigma A)\begin{pmatrix}1\\-k\end{pmatrix}$ if and only if $\det\left(h_\sigma A,g_\sigma A\right)=0$. This gives the second formula in Part 3. ■

Definition 2.19 — principal curvatures, principal directions, mean curvature, Gaussian curvature.
For a smooth regular surface $\sigma:U\subseteq\mathbb{R}^2\to\mathbb{R}^3$ in $\mathbb{R}^3$, the maximum and minimum values k_1 and k_2 of the directional curvature $k_\sigma(p)(A)$ where $0\neq A\in\mathbb{R}^2$, are called the **principal curvatures** of σ at p, and the directions $0\neq A_1,A_2\in\mathbb{R}^2$ in which the maximum and minimum values occur are called the **principal directions** for σ at p. The **mean curvature** $H(p)=H_\sigma(p)$ of σ at p and the **Gaussian curvature**

$K(p) = K_\sigma(p)$ of σ at p are define by

$$H(p) = H_\sigma(p) = \frac{1}{2}(k_1 + k_2)\,,$$
$$K(p) = K_\sigma(p) = k_1 k_2\,.$$

By Part 2 of the above theorem, k_1 and k_2 are the roots of the characteristic polynomial of $g^{-1}h$, so we have

$$(x - k_1)(x - k_2) = \det(g^{-1}h - xI) = x^2 - (g^{-1}h)\,x + \det(g^{-1}h).$$

Comparing coefficients gives

$$H(p) = \frac{1}{2}(k_1 + k_2) = \ \mathrm{trace}\left(g(p)^{-1}h(p)\right),$$
$$K(p) = k_1 k_2 = \det\left(g(p)^{-1}h(p)\right) = \frac{\det h(p)}{\det g(p)}$$

Theorem 2.20 — The Gauss-Weingarten Equations.
For a smooth regular surface σ in $\mathbb{R}^3$, we have

$$\begin{pmatrix} \sigma_{uu} \\ \sigma_{uv} \\ \sigma_{vv} \\ n_u \\ n_v \end{pmatrix} = \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} & h_{11} \\ \Gamma^1_{12} & \Gamma^2_{12} & h_{12} \\ \Gamma^1_{22} & \Gamma^2_{22} & h_{22} \\ b^1_1 & b^2_1 & 0 \\ b^1_2 & b^2_2 & 0 \end{pmatrix} \begin{pmatrix} \sigma_u \\ \sigma_v \\ n \end{pmatrix}$$

where

$$b = \begin{pmatrix} b^1_1 & b^1_2 \\ b^2_1 & b^2_2 \end{pmatrix} = -g^{-1}h \text{ , and}$$
$$\Gamma = \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} = \frac{1}{2}\,g^{-1} \begin{pmatrix} (g_{11})_u & (g_{11})_v & 2(g_{12})_v - (g_{22})_u \\ 2(g_{12})_u - (g_{11})_v & (g_{22})_u & (g_{22})_v \end{pmatrix}$$

Proof. First we note that since $\{\sigma_u, \sigma_v, n\}$ is a basis for $\mathbb{R}^3$, such a 5×3 matrix exists, and we just need to determine the entries. We do not yet know the entries in the final column, so let us let us say the final column is $(a_1, a_2, \cdots, a_5)^T$. We shall calculate the entries on the first and last rows (the other calculations are similar). To find the entries on the first row, we shall need a formula for $\sigma_{uu} \cdot \sigma_u$ and $\sigma_{uu} \cdot \sigma_v$ in terms of the entries of g. Since $g_{11} = \sigma_u \cdot \sigma_u$, we can differentiate with respect to u and v to get $(g_{11})_u = 2\sigma_{uu} \cdot \sigma_u$ and $(g_{11})_v = 2\sigma_{uv} \cdot \sigma_u$. Since $g_{12} = \sigma_u \cdot \sigma_v$, we have $(g_{12})_u = \sigma_{uu} \cdot \sigma_v + \sigma_u \cdot \sigma_{vu}$. Thus we obtain the formulas

$$\sigma_{uu} \cdot \sigma_u = \frac{1}{2}(g_{11})_u \quad , \quad \sigma_{uu} \cdot \sigma_v = (g_{12})_u - \frac{1}{2}(g_{11})_u\,.$$

The first row gives the equation $\Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + a_1 n = \sigma_{uu}$. Take the dot product with n on both sides to get $a_1 = \sigma_{uu} \cdot n = h_{11}$. Take the dot product with σ_u and with σ_v to get

$$\Gamma^1_{11} g_{11} + \Gamma^2_{11} g_{12} = \sigma_{uu} \cdot \sigma_u = \frac{1}{2}(g_{11})_u$$
$$\Gamma^1_{11} g_{12} + \Gamma^2_{11} g_{22} = \sigma_{uu} \cdot \sigma_v = (g_{12})_u - \frac{1}{2}(g_{22})_u$$

These two equations can be written as $g\begin{pmatrix}\Gamma^1_{22}\\ \Gamma^2_{11}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}(g_{11})_u\\ (2(g_{12})_2-(g_{11})_v\end{pmatrix}$, and so we obtain

$$\begin{pmatrix}\Gamma^1_{11}\\ \Gamma^2_{11}\end{pmatrix} = \frac{1}{2}g^{-1}\begin{pmatrix}(g_{11})_u\\ 2(g_{12})_2-(g_{11})_v\end{pmatrix}.$$

Now consider the final row. By differentiating $1 = n\cdot n$ with respect to v we see that n_v is orthogonal to n, and hence n_v is in the span of σ_u and σ_v, so $a_5 = 0$ and the final row gives the equation $b^1_2\sigma_u + b^2_2\sigma_v = n_v$. Taking the dot product on both sides with σ_u and with σ_v gives the equations $b^1_2 g_{11} + b^2_2 g_{12} = -h_{12}$, and σ_v gives $b^1_2 g_{12} + b^2_2 g_{22} = -h_{22}$. These can be written as $g\begin{pmatrix}b^2_1\\ b^2_2\end{pmatrix} = -\begin{pmatrix}h_{12}\\ h_{22}\end{pmatrix}$, and so we obtain

$$\begin{pmatrix}b^2_1\\ b^2_2\end{pmatrix} = -g^{-1}\begin{pmatrix}h_{12}\\ h_{22}\end{pmatrix}.$$

■

Definition 2.21 — Christoffel symbols.
For a smooth regular surface $\sigma : U\subseteq\mathbb{R}^2\to\mathbb{R}^3$, the functions Γ^j_{kl} which appear in the above theorem are called the **Christoffel symbols** of σ on U.

Theorem 2.22 — The Gauss-Codazzi Equations.
Let $\sigma : U\subseteq\mathbb{R}^2\to\mathbb{R}^3$ be a smooth regular surface in $\mathbb{R}^3$. Then the entries of g and h satisfy the Codazzi equations 1 $(h_{11})_v - (h_{12})_u = h_{11}\Gamma^1_{12} + h_{12}(\Gamma^2_{12}-\Gamma^1_{11}) - h_{22}\Gamma^2_{11}$

2 $(h_{12})_v - (h_{22})_u = h_{11}\Gamma^1_{22} + h_{12}(\Gamma^2_{22}-\Gamma^1_{12}) - h_{22}\Gamma^2_{12}$ and the Gauss equations

(1) $g_{11}K = (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^2_{11}\Gamma^2_{22} + \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12}$

(2) $g_{12}K = (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11}$
$\quad = (\Gamma^2_{12})_v - (\Gamma^2_{22})_u + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11}$

(3) $g_{22}K = (\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{11}\Gamma^1_{22} + \Gamma^1_{12}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^1_{12} - \Gamma^1_{22}\Gamma^2_{12}$.

Proof. Let us use the fact that $\sigma_{uuv} = \sigma_{uvu}$. Using the Gauss-Weingarten equations, we have

$$\begin{aligned}
\sigma_{uuv} &= (\Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + h_{11}n)_v\\
&= (\Gamma^1_{11})_v\sigma_u + \Gamma^1_{11}\sigma_{uv} + (\Gamma^2_{11})_v\sigma_v + \Gamma^2_{11}\sigma_{vv} + (h_{11})_v n + h_{11}n_v,\\
\sigma_{uvu} &= (\Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_v + h_{12}n)_u\\
&= (\Gamma^1_{12})_u\sigma_u + \Gamma^1_{12}\sigma_{uu} + (\Gamma^2_{12})_u\sigma_v + \Gamma^2_{12}\sigma_{vu} + (h_{12})_n n + h_{12}n_u
\end{aligned}$$

In the above expressions for σ_{uuv} and σ_{uvu}, use the Gauss-Weingarten equations again to write σ_{uu}, σ_{uv}, σ_{vv}, n_u and n_v as linear combinations of σ_u, σ_v and n, then expand and equate coefficients. Equating the coefficient of n gives

$$\Gamma^1_{11}h_{12} + \Gamma^2_{11}h_{22} + (h_{11})_v = \Gamma^1_{12}h_{11} + \Gamma^2_{12}h_{12} + (h_{12})_u$$

hence

$$(h_{11})_v - (h_{12})_u = \Gamma^1_{12}h_{11} + (\Gamma^2_{12}-\Gamma^1_{11})h_{12} - \Gamma^2_{11}h_{22},$$

which is the first Codazzi equation. Equating the coefficient of σ_u gives

$$(\Gamma^1_{11})_v + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} + h_{11}b^1_2 = (\Gamma^1_{12})_u + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} + h_{12}b^1_1$$

so we have

$$h_{11}b^1_2 - h_{12}b^1_1 = (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22}.$$

Notice that $h_{11}b_2^1 - h_{12}b_1^1$ is equal to the $(1,2)$-entry of the matrix

$$\begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix}\begin{pmatrix} h_{22} & -h_{12} \\ h_{21} & h_{11} \end{pmatrix} = \det h\, bh^{-1} = \det h(-g^{-1}h)h^{-1} = -\det h\, g^{-1},$$

that is $h_{11}b_2^1 - h_{12}b_1^1 = -\det h\left(g^{-1}\right)_{1,2} = g_{12}\frac{\det h}{\det g}$, and so we have

$$g_{12}\frac{\det h}{\det g} = (\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22},$$

which is one of the second Gauss equations. Equating the coefficient of σ_v and performing a similar calculation gives

$$g_{11}\frac{\det h}{\det g} = (\Gamma^2_{11})_v - (\Gamma^2_{12})_u + \Gamma^2_{11}\Gamma^2_{22} + \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12},$$

which is the first Gauss equation.

Similar calculations, using the fact that $\sigma_{vvu} = \sigma_{vuv}$, produce similar formulas, but with u and v interchanged, and with the indices 1 and 2 interchanged. In this way we obtain the second Codazzi equation, and the other Gauss equations. ∎

Theorem 2.23 — Gauss' Theorema Egregium.
For a smooth regular surface in $\mathbb{R}^3$, the Gausian curvature $K = \frac{\det h}{\det g}$ can be expressed only in terms of g (without using h).

Proof. We can express K in terms of g using either the first and second, or the second and third, Gauss equations. For example, using the first, and one of the second Gauss equations, we have

$$\begin{aligned}(\det g)K &= g_{11}(g_{22}K) - g_{12}(g_{12}K) \\ &= g_{11}\left((\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{11}\Gamma^1_{22} + \Gamma^1_{12}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^1_{12} - \Gamma^1_{22}\Gamma^2_{12}\right) \\ &- g_{12}\left((\Gamma^1_{12})_u - (\Gamma^1_{11})_v + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11}\right).\end{aligned}$$

Note that the Christoffel symbols are all expressed in terms of g. ∎

Theorem 2.24 — The Fundamental Theorem for Surfaces in $\mathbb{R}^3$, or the Bonnet Theorem.
Given a connected open set $U \subseteq \mathbb{R}^2$ with $0 \in U$, given a point $p \in \mathbb{R}^3$ and orthogonal unit vectors $A, B \in \mathbb{R}^3$, and given smooth functions $g_{11}, g_{12}, g_{22}, h_{11}, h_{12}, h_{22} : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ with $g_{11} > 0$ and $g_{11}g_{22} - g_{12}^2 > 0$ such that all of the Gauss-Codazi equations hold for the given functions, there exists a unique smooth surface $\sigma : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ which has the given functions as the entries of its first and second fundamental forms such that $\sigma(0) = p$, $\sigma_u(0) \in \{A\}$ and $\sigma_v(0) \in \mathrm{Span}\{A, B\}$.

Proof. We shall not supply the proof, but we make some remarks. The idea of the proof is similar to the proof of the fundamental theorem for curves in $\mathbb{R}^3$. In the proof of the fundamental theorem for curves, we used the fact that there exists a solution to the system of ordinary differential equations which is obtained by requiring that the Frenet-Seret formulas hold. To prove Bennet's theorem, we obtain a system a partial differential equations by requiring that the Gauss-Weingarten equations hold. But such a system of partial differential equations does not always admit a solution. In order for a solution to exist, the coefficients of the partial differential equations must satisfy certain compatibility requirements. In the case of the system which comes from the Gauss-Weingarten equations, it so happens that the compatibility requirements are satisfied when the Gauss-Codazzi equations hold. ∎

Remark 2.25 The fundamental theorem for surfaces tells us that, up to isometry, a surface is determined from its first and second fundamental forms g and h. So all geometric properties of a surface should be expressible in terms of g and h. We say that a geometric property is **intrinsic** when it can be expressed only in term of g which, we recall is a Riemannian metric (that is an inner product at each point), otherwise we say the property is **extrinsic**. Properties such as the length of a curve on a surface, or the angle between two curves on a surface, or the area of a portion of the surface are intrinsic. Gauss' Theorema Egregium (which is Latin for Gauss' Remarkable Theorem) states that the Gaussian curvature K is an intrinsic property. By contrast, the mean curvature H is extrinsic.

■ **Example 2.26** When a flat rectangle is bent to form a cylinder of radius r, you can verify that the Riemannian metric at each point does not change so the intrinsic geometry does not change. For the rectangle, the maximum and minimum directional curvatures at each point are $k_1 = k_2 = 0$ so that $K = k_1 k_2 = 0$ and $H = \frac{1}{2}(k_1 + k_2) = 0$, and for the cylinder the maximum and minimum curvatures at each point are $k_1 = \frac{1}{r}$ and $k_2 = 0$ so that $K = 0$ and $H = \frac{1}{2r}$.

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