PMATH 450: Lesbegue Integration and Fourier Analysis

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Contents

1. Week 1

Goals of PMATH 450:

(1) Develop a theory of integration for functions $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ which is
  (a) more flexible (than Riemann) (applicable to more functions)
  (b) more rich (nicer results)
  (c) still extends Riemann integration
(2) Introduce Harmonic Analysis

General outline (first half):

(1) Which sets should we integrate over?
  - Measurable sets
(2) Which functions should we try to integrate?
  - Measurable functions

1.1 Borel Sets

Definition 1.1 — σ-algebra.
Consider a set X, we call $\mathcal{A} \subseteq \mathcal{P}(X)$ (which is the power set of X) a **σ-algebra** of subsets of X if
(1) $\varnothing \in \mathcal{A}$
(2) $A \in \mathcal{A} \implies X \backslash A \in \mathcal{A}$
(3) $A_1, A_2, A_3, \ldots \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

So a σ-algebra is a collection of subsets of X which contains the empty set, is closed under set difference and is closed under countable unions.

R Consider $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-algebra
(1) $X \in \mathcal{A}$
$X \backslash \varnothing = X \in \mathcal{A}$
(2) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$
$A \cup B = A \cup B \cup \varnothing \cup \varnothing \cup \cdots \in \mathcal{A}$

(3) $A_1, A_2, \ldots \in \mathcal{A} \implies \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$

$\bigcap_{i=1}^{\infty} A_i = X \setminus \left(\bigcup_{i=1}^{\infty} (X \setminus A_i) \right)$

(4) $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$

■ **Example 1.1**

- $\{\varnothing, X\}$ is the smallest σ-algebra you could have given X
- $\mathcal{A} = \mathcal{P}(X)$ is a σ-algebra
- $\mathcal{A} = \{A \subseteq \mathbb{R} : A \text{ is open}\}$ is **NOT** a σ-algebra.
  it is not closed under set difference, consider $A = (0,1) \in \mathcal{A}$, $\mathbb{R} \setminus A = (-\infty, 0] \cup [1, \infty) \notin \mathcal{A}$ because it is not open.
- $\mathcal{A} = \{A \subseteq \mathbb{R} : A \text{ open or closed}\}$ is **NOT** a σ-algebra.
  Consider $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \notin \mathcal{A}$ since $\mathbb{Q}$ is neither open nor closed

Proposition 1.1
Given a set X and $\mathcal{C} \subseteq \mathcal{P}(X)$, then $\mathcal{A} = \cap\{\mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{-algebra }, \mathcal{C} \subseteq \mathcal{B}\}$ is a σ-algebra. It is the smallest σ-algebra containing $\mathcal{C}$.

Definition 1.2 — Borel Sets.
Consider $\mathcal{C} = \{A \subseteq \mathbb{R} : A \text{ is open}\}$ (this is s subset of power set of $\mathbb{R}$), then $\mathcal{A} = \bigcap\{\mathcal{B} : \mathcal{C} \subseteq \mathcal{B}, \mathcal{B} \text{ is } \sigma\text{-algebra}\}$ is called **Borel σ-algebra**. The elements of $\mathcal{A}$ is are called the **Borel sets**.

R

(1) All the open sets are Borel. i.e. open $\implies$ Borel.
(2) All the closed sets are Borel. i.e. closed $\implies$ Borel.
since σ-algebra are closed under set difference, and $\mathbb{R}$ take away open is closed
(3) $\{X_1, X_2, \ldots\} = \bigcup_{i=1}^{\infty} \{X_i\}$ is Borel. i.e. countable $\implies$ Borel.
In particular, $\mathbb{Q}$ is a Borel set which is neither open nor closed.
(4) $[a, b) = [a, b] \setminus \{b\} = [a, b] \cap (\mathbb{R} \setminus \{b\})$ is Borel.

It is actually very hard to construct a set that is not Borel. The Borel sets are the propriate sets to integrate over.

1.2 Outer Measure

Idea

(1) Given $A \subseteq \mathbb{R}$, how should we "measure" the "size" of A
(2) Some sets have "sizes" which "measure" more nicely then others. Which ones? Borel sets?

Goal

Define a function $m : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{\infty\}$ (called a measure) such that

(1) $m((a,b)) = m([a,b]) = m((a,b]) = b - a$ (the measure of an interval I equals the length of I)
(2) $m(A \cup B) \leq m(A) + m(B)$
(3) If $A \cap B = \varnothing$, then $m(A \cup B) = m(A) + m(B)$

It will be shown later in the course that we may not use $\mathcal{P}(\mathbb{R})$

Idea

Given $A \subseteq \mathbb{R}$, there exists bounded, open intervals $I_i = (a_i, b_i)$ s.t. $A \subseteq \bigcup_{i=1}^{\infty} I_i$. We want:

$$m(A) \leq \sum_{i=1}^{\infty} m(I_i) = \sum_{i=1}^{\infty} \underbrace{\mathscr{L}(I_i)}_{\text{the length of } I_i} = \sum_{i=1}^{\infty}(b_i - a_i)$$

Cover A by bounded, open, intervals as finely as possible.

Definition 1.3 — Outer Measure.
We define (Lebesgue) **outer measure** by $m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{\infty\}$ with

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \mathscr{L}(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i, I_i \text{ is a bounded, open interval} \right\}$$

■ **Example 1.2** Consider the $\varnothing$ (we would like to the size of it been zero). For any $\epsilon > 0$, $\varnothing \subseteq (0, \epsilon)$, by definition $m^*(\varnothing) \leq \mathscr{L}((0, \epsilon)) = \epsilon$. Since $m^*(\varnothing) \geq 0$, $m^*(\varnothing) = 0$.

■ **Example 1.3** Consider a countable set $A = \{x_1, x_2, x_3, \ldots\}$, given any $\epsilon > 0$, then $A \subseteq \bigcup_{i=1}^{\infty}(x_i - \frac{\epsilon}{2^{i+1}}, x_i + \frac{\epsilon}{2^{i+1}})$. $m^*(A) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \frac{\epsilon}{2} \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = \frac{\epsilon}{2}(\frac{1}{1-\frac{1}{2}}) = \epsilon$. Since $\epsilon > 0$ was arbitrary, $m^*(A) = 0$

Follow a similar proof, we can show that the finite sets also have outer measure zero

Goal

Prove that if I is an interval, then $m^*(I) = \mathscr{L}(I)$

Proposition 1.2
If $A \subseteq B$, then $m^*(A) \leq m^*(B)$

Proof. Sketch:
$X = \left\{ \sum_{i=1}^{\infty} \mathscr{L}(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i \right\}$
$Y = \left\{ \sum_{i=1}^{\infty} \mathscr{L}(I_i) : B \subseteq \bigcup_{i=1}^{\infty} I_i \right\}$
Clearly if $A \subseteq B$ then $Y \subseteq X$ (if you have intervals cover B then they must cover A), hence $\inf X \leq \inf Y$ using the ordering of the extended real numbers. i.e. $m^*(A) \leq m^*(B)$ ■

Lemma 1.3
If $a, b \in \mathbb{R}$ with $a \leq b$, then $m^*([a, b]) = b - a$

We start with the closed bounded intervals because they are compact, so as soon as you cover this closed interval with countable union of open intervals, then you only need to take finitely many of them because you are guaranteed to have a finite subcover to cover the interval.

Proof. Let $\epsilon > 0$ be given. Since $[a, b] \subseteq (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$, we see that $m^*([a, b]) \leq b - a + \epsilon$. Since $\epsilon > 0$ was arbitrary, $m^*([a, b]) \leq b - a$.

Let I_i $(i \in \mathbb{N})$ be bounded open intervals s.t. $[a, b] \subseteq \bigcup_{i=1}^{\infty} I_i$. Since $[a, b]$ is compact, there exists $n \in \mathbb{N}$ s.t. $[a, b] \subseteq \bigcup_{i=1}^{n} I_i$, hence $b - a \leq \sum_{i=1}^{n} \mathscr{L}(I_i) \leq \sum_{i=1}^{\infty} \mathscr{L}(I_i)$ (the first inequality can be proved by induction), and so $m^*([a,b]) \geq b-a$ (since $m^*([a,b])$ is the greatest lower bound) ■

Proposition 1.4
If I is an interval then $m^*(I) = \mathscr{L}(I)$

Proof.
(1) Suppose I is bounded with endpoints $a \leq b$
Given $\epsilon > 0$, note $I \subseteq [a,b] \implies m^*(I) \leq m^*([a,b]) = b-a$. Also $[a+\frac{\epsilon}{2}, b-\frac{\epsilon}{2}] \subseteq I \implies b-a-\epsilon \leq m^*(I) \implies b-a \leq m^*(I)$.
(2) Suppose I is unbounded.
For all $n \in \mathbb{N}$, there exists I_n s.t. $I_n \subseteq I$, $\mathscr{L}(I_n) = n$. Then $m^*(I) \geq m^*(I_n) = n$, hence $m^*(I) = \infty = \mathscr{L}(I)$

■

1.3 Basic Properties of Outer Measure

Outer measure is
(1) Translation Invariant
(2) Countably Subadditivity

Notation

Given $x \in \mathbb{R}, A \subseteq \mathbb{R}$, then $x + A = \{x + a : a \in A\}$

Proposition 1.5 — Translation Invariant.
$m^*(x+A) = m^*(A)$

Proof. Sketch:

$$\begin{aligned} m^*(x+A) &= \inf\left\{\sum \mathscr{L}(I_i) : x+A \subseteq \bigcup I_i\right\} \\ &= \inf\left\{\sum \mathscr{L}(I_i) : A \subseteq \bigcup (-x+I_i)\right\} \\ &= \inf\left\{\sum \mathscr{L}(-x+I_i) : A \subseteq \bigcup (-x+I_i)\right\} \\ &= \inf\left\{\sum \mathscr{L}(J_i) : A \subseteq \bigcup (J_i)\right\} \\ &= m^*(A) \end{aligned}$$

■

Proposition 1.6 — Countable Subadditivity.
If we take countably many subset $A_i \subseteq \mathbb{R} (i \in \mathbb{N})$ then $m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$

Proof. We may assume each $m^*(A_i) < \infty$ (otherwise the result will be trivial). Let $\epsilon > 0$ be given and fix $i \in \mathbb{N}$. There exists open, bounded intervals $I_{i,j}$ s.t. $A_i \subseteq \bigcup_{i=1}^{\infty} I_{i,j}$ and $\sum_{j=1}^{\infty} \mathscr{L}(I_{i,j}) \leq m^*(A_i) + \frac{\epsilon}{2^i}$ (Note that we add a little bit on the out measure which makes it no longer a lower bound hence we can find the $I_{i,j}$, this is a common technique when working with outer measure). We see that $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i,j} I_{i,j}$ and so $m^*(A_i) \leq \sum_{i,j} \mathscr{L}(I_{i,j}) =$

$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathscr{L}(I_{i,j}) \leq \sum_{i=1}^{\infty} m^*(A_i) + \frac{\epsilon}{2^i} = \sum_{i=1}^{\infty} m^*(A_i) + \epsilon$. Since ϵ is arbitrary, the proposition follows. ∎

Corollary 1.7 — Finite Subadditivity.
If $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{R})$, then $m^*\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} m^*(A_i)$

Proof. Sketch:
$A_1 \cup \cdots \cup A_n = A_1 \cup \cdots \cup A_n \cup \varnothing \cup \varnothing \cup \cdots$ ∎

Problem

There exists $A, B \subseteq \mathbb{R}$ s.t. $A \cap B = \varnothing$ and $m^*(A \cup B) < m^*(A) + m^*(B)$. i.e. outer measure is not finitely additive. (We would like $m^*(A \cup B) = m^*(A) + m^*(B)$ for disjoint sets A, B)

Solution

Restrict the domain of m^* to only include sets which measure "nicely" (which are called measurable).

2. Week 2

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2.1 Measurable Sets

Goal

Restrict the domain of m^* to only include sets s.t. whenever $A \cap B = \varnothing$, $m^*(A \cup B) = m^*(A) + m^*(B)$

Definition 2.1 — Measurable.
We say $A \subseteq \mathbb{R}$ is **measurable** if $\forall X \subseteq \mathbb{R}$, $m^*(X) = m^*(X \cap A) + m^*(X \backslash A)$

R By finite subadditivity, we always have $m^*(X) \leq m^*(X \cap A) + m^*(X \backslash A)$ because $X = (X \cap A) \cup (X \backslash A)$

R If $\subseteq \mathbb{R}$ is measurable and $B \subseteq \mathbb{R}$ with $A \cap B = \varnothing$, then consider $X = A \cup B$, $m^*(A \cup B) = m^*(X \cap A) + m^*(X \backslash A) = m^*(A) + m^*(B)$.

Goal

Show a lot of sets are measurable.

Proposition 2.1
If $m^*(A) = 0$, then A is measurable.

Proof. Let $X \subseteq \mathbb{R}$. Since $X \cap A \subseteq A$, we have $0 \geq m^*(X \cap A) \leq m^*(A) = 0$, and so $m^*(X \cap A) = 0$. Then $m^*(X \cap A) + m^*(X \backslash A) = m^*(X \backslash A) \leq m^*(X)$. (The other inequality is trivial). ■

Proposition 2.2

If $A_1, A_2, \ldots, A_n$ are measurable, then $\bigcup_{i=1}^{n} A_i$ is measurable.

Proof. It suffices to prove the result when $n = 2$ (then the rest is a trivial induction). Let $A, B \subseteq \mathbb{R}$ be measurable. Let $X \subseteq \mathbb{R}$. Then

$$\begin{aligned} m^*(X) &= m^*(X \cap A) + m^*(X \backslash A) \\ &= m^*(X \cap A) + m^*((X \backslash A) \cap B) + m^*((X \backslash A) \backslash B) \\ &= m^*(X \cap A) + m^*((X \backslash A) \cap B) + m^*(X \backslash (A \cup B)) \\ &\geq m^*((X \cap A) \cup ((X \backslash A) \cap B)) + m^*(X \backslash (A \cup B)) \\ &= m^*(X \cap (A \cup B)) + +m^*(X \backslash (A \cup B)) \end{aligned}$$

■

Proposition 2.3

Let $A_1, A_2, \ldots, A_n$ are measurable, $A_i \cap A_j = \varnothing$ for $i \neq j$. Let $A = A_1 \cup \cdots \cup A_n$. If $X \subseteq \mathbb{R}$, then $m^*(X \cap A) = \sum_{i=1}^{b} m^*(X \cap A_i)$

Proof. It suffices to prove the result when $n = 2$ (then the rest is a trivial induction). Let $A, B \subseteq \mathbb{R}$ be measurable with $A \cap B = \varnothing$. Let $X \subseteq \mathbb{R}$. Then

$$\begin{aligned} m^*(X \cap (A \cup B)) &= m^*((X \cap (A \cup B)) \cap A) + m^*((X \cap (A \cup B)) \backslash A) \\ &= m^*(X \cap A) + m^*(X \cap B) \end{aligned}$$

We only used the measurability of A, so in some sense our assumption is stronger than it can be. When actually writing out the induction, you will need $A_1, \ldots, A_{n-1}$ to be measurable. It doesn't matter the last one A_n is measurable or not. ■

Corollary 2.4 — Finite additivity.

If $A_1, A_2, \ldots, A_n$ are measurable, $A_i \cap A_j = \varnothing$ for $i \neq j$. Then $m^*(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} m^*(A_i)$

Proof. Take $X = \mathbb{R}$ in the previous prove. ■

2.2 Countable Additivity

Lemma 2.5

Consider $A_i \subseteq \mathbb{R}$ are measurable for $i \in \mathbb{N}$. If $A_i \cap A_j = \varnothing$ for $i \neq j$. Then $A := \bigcup_{i=1}^{\infty} A_i$ is measurable.

Proof. Sketch: Consider $B_n = A_1 \cup A_2 \cup \cdots \cup A_n$ and $X \subseteq \mathbb{R}$. Then

$$\begin{aligned} m^*(X) &= m^*(X \cap B_n) + m^*(X \backslash B_n) \\ &\geq m^*(X \cap B_n) + m^*(X \backslash A) \\ &= \sum_{i=1}^{n} m^*(X \cap A_i) + m^*(X \backslash A) \text{ (Using the proposition 2.3)} \end{aligned}$$

Taking $n \to \infty$,

$$\begin{aligned} m^*(X) &\geq \sum_{i=1}^{\infty} m^*(X \cap A_i) + m^*(X \backslash A) \\ &\geq m^*\left(\bigcup_{i=1}^{\infty}(X \cap A_i)\right) + m^*(X \backslash A) \\ &= m^*(X \cap A) + m^*(X \backslash A) \end{aligned}$$

■

Proposition 2.6
If $A \subseteq \mathbb{R}$ is measurable, then $\mathbb{R} \backslash A$ is measurable.

Proof. Sketch: Take $X \subseteq \mathbb{R}$

$$\begin{aligned} m^*(X \cap (\mathbb{R} \backslash A)) + m^*(X \backslash (\mathbb{R} \backslash A)) &= m^*(X \backslash A) + m^*(X \cap A) \\ &= m^*(X) \text{ (Since } A \text{ is measurable)} \end{aligned}$$

■

Proposition 2.7
If $A_i \subseteq \mathbb{R}$ are measurable with $i \in \mathbb{N}$, then $A = \bigcup\limits_{i=1}^{\infty} A_i$ is measurable.

Proof. Sketch: Define $B_1 = A_1$ and $B_n = A_n \backslash (A_1 \cup A_2 \cup \cdots \cup A_{n-1})$ for $n \geq 2$. Then $B_n = A_n \cap (\mathbb{R} \backslash (A_1 \cup A_2 \cup \cdots \cup A_{n-1}))$ (these are two separate measurable sets). Therefore B_n is measurable and by construction, for $i \neq j$, $B_i \cap B_j = \varnothing$. Also $\bigcup\limits_{i=1}^{\infty} B_i = \bigcup\limits_{i=1}^{\infty} A_i$, which are measurable. ■

Corollary 2.8
The collection $\mathscr{L}$ of (Lebesgue) measurable sets is a σ-algebra of sets in $\mathbb{R}$

Corollary 2.9 — Countable additivity.
If $A_i \in \mathbb{R}$ are measurable with $i \in \mathbb{N}$, $A_i \cap A_j = \varnothing$ for $i \neq j$. Then $m^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \sum\limits_{i=1}^{\infty} m^*(A_i)$

Proof. Sketch: The countable subadditivity 1.6 tells us $m^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) \leq \sum\limits_{i=1}^{\infty} m^*(A_i)$.
Note that for any $n \in \mathbb{N}$

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geq m^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m^*(A_i)$$

Take $n \to \infty$, $m^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) \geq \sum\limits_{i=1}^{n} m^*(A_i)$ ■

2.3 Borel Implies Measurable

Goal

Show Borel sets are measurable.

Proposition 2.10
If $a \in \mathbb{R}$ then (a, ∞) is measurable.

Proof. Let $X \subseteq \mathbb{R}$. We want to show that $m^*(X \cap (a, \infty)) + m^*(X \backslash (a, \infty)) \leq m^*(X)$.

(1) Assume $a \notin X$. We want to show $m^*(X \cap (a, \infty)) + m^*(X \cap (-\infty, a)) \leq m^*(X)$. Let $X_1 = (X \cap (a, \infty))$ and $X_2 = (X \cap (-\infty, a))$. Let (I_i) be a sequence of bounded open intervals such that $X \subseteq \cup I_i$. Define for each i, $I_i' = I_i = \cap(a, \infty)$ and $I_i'' = I_i \cap (-\infty, a)$. Note that $X_1 \subseteq \cup I_i'$ and $X_2 \subseteq \cup I_i''$, ans so $m^*(X_1) \leq \sum \mathscr{L}(I_i')$ and $m^*(X_2) \leq \sum \mathscr{L}(I_i'')$. We then see that

$$
\begin{aligned}
m^*(X_1) + m^*(X_2) &\leq \sum \mathscr{L}(I_i') + \sum \mathscr{L}(I_i'') \\
&= \sum [\mathscr{L}(I_i') + \mathscr{L}(I_i'')] \\
&= \sum \mathscr{L}(I_i)
\end{aligned}
$$

Note that the $m^*(X)$ is the infimum of $\sum \mathscr{L}(I_i)$. By the definition of infimum, $m^*(X_1) + m^*(X_2) \leq m^*(X)$

(2) Assume $a \in X$. Let $X' = X \backslash \{a\}$. Then by the previous case, $m^*(X' \cap (a, \infty)) + m^*(X' \cap (-\infty, a)) \leq m^*(X')$. Also note $m^*(\{a\}) = 0$. Then

$$
\begin{aligned}
m^*(X \cap (a, \infty)) + m^*(X \backslash (a, \infty)) &= m^*(X' \cap (a, \infty)) + m^*((X' \backslash (a, \infty)) \cup \{a\}) \\
&= m^*(X' \cap (a, \infty)) + m^*((X' \cap (-\infty, a)) \cup \{a\}) \\
&\leq m^*(X' \cap (a, \infty)) + m^*(X' \cap (-\infty, a)) + m^*(\{a\}) \\
&= m^*(X' \cap (a, \infty)) + m^*(X' \cap (-\infty, a)) \\
&\leq m^*(X') \\
&\leq m^*(X)
\end{aligned}
$$

■

Theorem 2.11
Every Borel set is measurable.

Proof. Sketch: We will show that every open set is measurable, then the σ-algebra of measurable sets contains all the open sets, and so by smallness of Borel set it would be contained in the σ-algebra of Lebesgue measurable sets.

We have (a, ∞) is measurable, then $\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty)$ is measurable. Note $\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty) = [a, \infty$. Then $\mathbb{R} \backslash [a, \infty) = (-\infty, a]$ is also measurable. Then any open interval $(a, b) = (a, \infty) \cap (-\infty, b)$ is measurable. Hence, every open set in $\mathbb{R}$ is measurable. Recall the Borel σ-algebra is the σ-algebra generated by open sets, meaning it is the smallest σ-algebra of sets in $\mathbb{R}$ which contain open sets. Since the collection of measurable sets is a σ-algebra, now containing all open sets, we have the Borel σ-algebra has to be subset of the σ-algebra of the measurable sets. ■

Definition 2.2 — Lebesgue measurable.
We call $m : \mathscr{L} \to [0, \infty) \cup \{\infty\}$ where $\mathscr{L}$ is the σ-algebra of measurable sets given by $m(A) = m^*(A)$ Lebesgue measurable

Exercise 2.1 Prove that if $A \subseteq \mathbb{R}$ is measurable, then $x + A$ is measurable for any $x \in \mathbb{R}$

Solution. Let $X \subseteq \mathbb{R}$, we want to show $m^*(X) = m^*(X \setminus (x + A)) + m^*(X \cap (x + A)) = m^*(X \cap (x + A)^c)) + m^*(x \cap (x + A))$.

$$\begin{aligned} y \in X \cap (x + A) &\iff y \in X \text{ and } y \in x + A \\ &\iff y - x \in X - x \text{ and } y - x \in A \\ &\iff y - x \in (X - x) \cap A \\ &\iff y \in (X - x) \cap A + x \end{aligned}$$

Hence $X \cap (x+A) = (X-x) \cap A + x$. Similarly, one also has that $X \cap (x+A)^c = (X-x) \cap A^c + x$. Now, from the RHS, by the translation invariant property for outer measure, we have that

$$\begin{aligned} m^*(X \cap (x + A)^c)) + m^*(x \cap (x + A)) &= m^*((X - x) \cap A^c + x) + m^*((X - x) \cap A + x) \\ &= m^*((X - x) \cap A^c) + m^*((X - x) \cap A) \end{aligned}$$

Since A is measurable, $m^*((X - x) \cap A^c) + m^*((X - x) \cap A) = m^*(X - x) = m^*(X)$. It follows that $x + A$ is measurable. ■

2.4 Basic Properties of Lebesgue Measure

Proposition 2.12 — Excision Property.
If $A \subseteq B$ with A being measurable and $m(A) < \infty$, then $m^*(B \backslash A) = m^*(B) - m(A)$

Proof. Sketch: By the definition of measurable

$$\begin{aligned} m^*(B) &= m^*(B \cap A) + m^*(B \backslash A) \\ &= m(A) + m^*(B \backslash A) \end{aligned}$$

We want $m(A) > \infty$ to avoid the case $\infty - \infty$. ■

Theorem 2.13 — Continuity of Measure.

(1) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ are measurable, then $m\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} m(A_n)$

(2) If $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ are measurable, with $B_1 < \infty$, then $m\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} m(B_n)$

Proof. (1) Since $m(A_k) \leq m(\bigcup A_i)$ for all $k \in \mathbb{N}$, we have $\lim_{n\to\infty} m(A_n) \leq m(\bigcup A_i)$. If $\exists k \in \mathbb{N}$ s.t. $m(A_k) = \infty$, then $\lim_{n\to\infty} m(A_n) = \infty$ and we are done. Hence we may assume each $m(A_k) < \infty$ (so to use Excision Property 2.12).
(We would like to use Countable additivity 2.9, so we will replace A_i with a sequence of disjoint sets. For each $k \in \mathbb{N}$, let $D_k = A_k \backslash A_{k-1}$, $A_0 = \varnothing$.

Note 2.1

- The $D_k's$ are measurable
- The $D_k's$ are pariwise disjoint
- $\bigcup D_i = \bigcup A_i$

$$
\begin{aligned}
m\left(\bigcup A_i\right) = m\left(\bigcup D_i\right) &= \sum_{i=1}^{\infty} m(D_i) && \text{(by countable additivity)} \\
&= \sum_{i=1}^{\infty} (m(A_i) - m(A_{i-1})) && \text{(by excision property)} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (m(A_i) - m(A_{i-1})) \\
&= \lim_{n\to\infty} m(A_n) - m(A_0) \\
&= \lim_{n\to\infty} m(A_n)
\end{aligned}
$$

(2) For $k \in \mathbb{N}$, define $D_k = B_1 \backslash B_k$

Note 2.2

- $D_k's$ are measurable
- $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$

By (1), $m(\bigcup D_i) = \lim_{n\to\infty} m(D_n)$. We see that

$$\bigcup D_i = \bigcup_{i=1}^{\infty} (B_1 \backslash B_i) = B_1 \backslash \left(\bigcap_{i=1}^{\infty} B_i\right)$$

and so

$$\lim_{n\to\infty} m(D_n) = m\left(\bigcup D_i\right) = m\left(B_1 \backslash \left(\bigcap B_i\right)\right) = m(B_1) - m\left(\bigcap B_i\right)$$

Since $m(B_1) < \infty$, $B_i \subseteq B_1$ for all $i > 1$, hence $m(\bigcap B_i) < \infty$. However,

$$\lim_{n\to\infty} m(D_n) = \lim_{n\to\infty} m(B_1) - m(B_n) = m(B_1) - \lim_{n\to\infty} m(B_n)$$

■

■ **Example 2.1** $B_i = (i, \infty)$, $m\left(\bigcap B_i\right) = m(\varnothing) = 0$, however $\lim_{n\to\infty} m(B_n) = \infty$.

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