PMATH 365: Differential Geometry

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Contents

1. Curves

1.1 Curves in $\mathbb{R}^n$

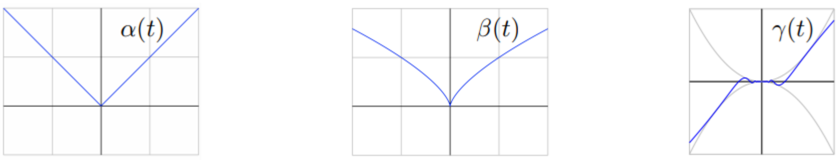
Definition 1.1 — curve, tangent vector, smooth, regular.
A (parametrized) **curve** in $\mathbb{R}^n$ is a continuous map $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ where I is a nonempty interval. We can write $\alpha(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ where each $x_k : I \subseteq \mathbb{R} \to \mathbb{R}$ is continuous.

When $a \in I$ and $\alpha'(a) = ({x_1}'(a), \cdots, {x_n}'(a))$ exists, $\alpha'(a)$ is called the **tangent vector** to α at $t = a$.

We say that α is $\mathcal{C}^k$ when the k^{th} order derivative of α exists and is continuous on I, we say that α is **smooth** or $\mathcal{C}^\infty$ when α is $\mathcal{C}^k$ for all $k \in \mathbb{Z}^+$, and we say that α is **regular** when α is $\mathcal{C}^1$ with $\alpha'(t) \neq 0$ for all $t \in I$.

Unless otherwise stated, we shall always assume curves are smooth and regular.

■ **Example 1.2** The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t, |t|)$ is not regular because $\alpha'(0)$ does not exist. The curve $\beta : \mathbb{R} \to \mathbb{R}^2$ given by $\beta(t) = (t^3, t^2)$ is not regular because $\beta'(0) = 0$. The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(0) = (0,0)$ and $\gamma(t) = \left(t, t^2 \sin \frac{1}{t}\right)$ for $t \neq 0$ is differentiable but not regular because γ' is not continuous at $t = 0$.



Theorem 1.3
Every regular curve in $\mathbb{R}^n$ is locally injective.

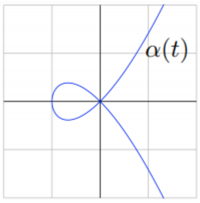
Proof. Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ be a regular curve, write $\alpha(t) = (x_1(t), \cdots, x_n(t))$, and let $a \in I$. Since $\alpha'(a) \neq 0$ we have ${x_k}'(a) \neq 0$ for some index k, say ${x_k}'(a) > 0$ (the case that ${x_k}'(a) < 0$ is similar). Since α' is continuous, ${x_k}'$ is continuous. Since ${x_k}'$ is continuous and

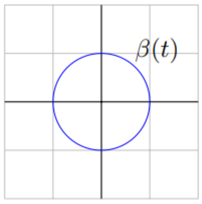
${x_k}'(a) > 0$ we can choose $\delta > 0$ so that $|t - a| < \delta \implies {x_k}'(t) > 0$. Then x_k is increasing, hence injective, in the interval $(a - \delta, a + \delta) \cap I$, and so α is injective in the same interval. ■

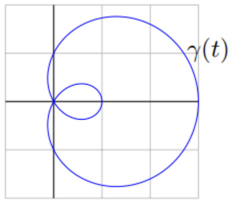
■ **Example 1.4** The curves $\alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}^2$ from Example 1.2 are not regular, but they are all injective, so a curve does not necessarily need to be regular in order to be injective.

■ **Example 1.5** The alpha curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ which is given by $\alpha(t) = \big(t^2 - 1, t(t^2 - 1)\big)$, the circle $\beta : \mathbb{R} \to \mathbb{R}^2$ which is given by $\beta(t) = (\cos t, \sin t)$, and the limcon $\gamma : \mathbb{R} \to \mathbb{R}^2$ which is given by $\gamma(t) = ((1 + 2\cos t)\cos t, (1 + 2\cos t)\sin t)$, are all regular, so they are all locally injective, but they are not (globally) injective (the alpha curve crosses itself with $\alpha(1) = \alpha(-1) = (0, 0)$, the circle is periodic with $\beta(t + 2\pi k) = \beta(t)$ for all $k \in \mathbb{Z}$, and the limcon is periodic and crosses itself).

■ **Example 1.6** The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(0) = 0$ and $\alpha(t) = \big(t^2, t^2 \sin \frac{1}{t}\big)$ for $t \neq 0$ is differentiable, but not regular since $\alpha'(0) = 0$, and (as you can verify) it is not locally injective at $t = 0$.







Definition 1.7 — length, rectifiable.
For a curve $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$, the **length** of α on $[a, b]$ is

$$L = L_\alpha\left([a, b]\right) = \sup\left\{ \sum_{j=1}^{p} \big|\alpha(t_j) - \alpha(t_{j-1})\big| \,\Big|\, a = t_0 < t_1 < t_2 < \cdots < t_p = b \right\}$$

(which can be infinite) and we say that α is **rectifiable** on $[a, b]$ when $L_\alpha\left([a, b]\right)$ is finite.

Theorem 1.8
Let $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ be a regular curve. Then α is rectifiable with length

$$L = L_\alpha\left([a, b]\right) = \int_a^b \big|\alpha'(t)\big|\, dt.$$

Proof. For a partition $P = (t_0, t_1, \cdots, t_p)$, where $a = t_0 < t_1 < \cdots < t_p = b$, let us write

$$L(\alpha, P) = \sum_{j=1}^{p} \big|\alpha(t_j) - \alpha(t_{j-1})\big| \quad \text{and} \quad S(\alpha, P) = \sum_{j=1}^{p} \big|\alpha'(t_j)\big|(t_j - t_{j-1})$$

so $L(\alpha, P)$ is the sum which approximates Length(α) and $S(\alpha, P)$ is the Riemann sum (using right endpoints) which approximates the integral $\int_a^b |\alpha'(t)|\, dt$. First note that

$$\begin{aligned} L(\alpha, P) &= \sum_{j=1}^{p} \big|\alpha(t_j) - \alpha(t_{j-1})\big| \leq \sum_{j=1}^{p}\sum_{k=1}^{n} \big|x_k(t_j) - x_k(t_{j-1})\big| = \sum_{j=1}^{n}\sum_{k=1}^{p} \big|{x_k}'(c_{j,k})\,(t_j - t_{j-1})\big| \\ &\leq \sum_{j=1}^{n}\sum_{k=1}^{p} M(t_j - t_{j-1}) = \sum_{j=1}^{n} M(b - a) = nM(b - a) \end{aligned}$$

where we used the Mean Value Theorem to choose points $c_{j,k}$ between t_{j-1} and t_j such that $(x_k(t_j) - x_k(t_{j-1})) = x_k{}'(c_{j,k})(t_j - t_{j-1})$ and we let $M = \max\left\{|x_k{}'(t)|\,\middle|\,1 \leq k \leq n, t \in [a,b]\right\}$. This shows that $L = L_\alpha([a,b])$ is finite.

Note that if $P = (t_0, t_1, \cdots, t_p)$ is a partition of $[a,b]$, and Q is a partition which is obtained by adding one more point, say $Q = (t_0, t_1, \cdots, t_{j-1}, s, t_j, \cdots, t_p)$, then we have $L(\alpha, P) \leq L(\alpha, Q)$ because $\left|\alpha(t_j) - \alpha(t_{j-1}) \leq \left|\alpha(t_j) - \alpha(s)\right| + \left|\alpha(s) - \alpha(t_{j-1})\right|\right.$. It follows (by induction) that when Q is any partition with $P \subseteq Q$ we have

$$L(\alpha, P) \leq L(\alpha, Q) \leq L.$$

Also note that for any partition P, with $c_{j,k}$ chosen as above, we have

$$\begin{aligned}
&\left|L(\alpha,P) - S(\alpha,P)\right| = \Big|\sum_{j=1}^{p}\left|\alpha(t_j) - \alpha(t_{j-1})\right| - \sum_{j=1}^{p}\left|\alpha'(t_j)\right|(t_j - t_{j-1})\Big| \\
&= \Big|\sum_{j=1}^{p}\Big|\left(x_1(t_j) - x_1(t_{j-1}), \cdots, x_n(t_j) - x_n(t_{j-1})\right)\Big| - \sum_{j=1}^{p}\left|\alpha'(t_j)\right|(x_j - t_{j-1})\Big| \\
&= \Big|\sum_{j=1}^{p}\Big|\left(x_1{}'(c_{j,1}), \cdots, x_n{}'(c_{j,n})\right)\Big|(t_j - t_{j-1}) - \sum_{j-1}^{p}\Big|\left(x_1{}'(t_j), \cdots, x_n{}'(t_j)\right)\Big|(t_j - t_{j-1})\Big| \\
&\leq \sum_{j=1}^{p}\Big|\left|\left(x_1{}'(c_{j,1}), \cdots, x_n{}'(c_{j,n})\right)\right| - \left|\left(x_1{}'(t_j), \cdots, x_n{}'(t_j)\right)\right|\Big|(t_t - t_{j-1}) \\
&\leq \sum_{j=1}^{p}\Big|\left(x_1{}'(c_{j,1}) - x_1{}'(t_j), x_n{}'(c_{j,n}) - x_n{}'(t_j)\right)\Big|(t_j - t_{j-1}) \\
&\leq \sum_{j=1}^{p}\sum_{k=1}^{n}\left|x_k{}'(c_{j,k}) - x_k{}'(t_j)\right|(t_j - t_{j-1}).
\end{aligned}$$

Let $\epsilon > 0$. Since each $x_k{}'$ is continuous (hence uniformly continuous) on $[a,b]$ and since $|\alpha'|$ is continuous (hence Riemann integrable) on $[a,b]$, we can choose $\delta > 0$ such that for all $s, t \in [a,b]$ with $|s - t| < \delta$ we have $\left|x_k{}'(s) - x_k{}'(t)\right| < \frac{\epsilon}{3n(b-a)}$ for all k , and such that for every partition $P = (t_0, t_1, \cdots, t_p)$ with $|P| < \delta$ we have $\left|\int_a^b |\alpha'(t)|\,dt - S(\alpha, P)\right| < \frac{\epsilon}{3}$ where $|P|$ is the size of the partition P, that is $|P| = \max\left\{t_j - t_{j-1}\,\middle|\,1 \leq j \leq p\right\}$. Choose a partition P_1 with $|P_1| < \delta$ and choose a partition P_2 such that $\left|L - L(\alpha, P_2)\right| < \frac{\epsilon}{3}$ then let $P = P_1 \cup P_2$. Since $P_2 \subseteq P_1$ we have $L(\alpha, P_1) \leq L(\alpha, P) \leq L$ so that $\left|L - L(\alpha, P)\right| \leq \left|L - L(\alpha, P_1)\right| < \frac{\epsilon}{3}$. Since $|P| \leq |P_1| < \delta$ we have $\left|\int_a^b |\alpha'(t)|\,dt - S(\alpha, P)\right| < \frac{\epsilon}{3}$. Also since $|P| < \delta$, for all of the points $c_{j,k}$ we have $|c_{j,k} - t_j| < \delta$ so that $\left|x_k{}'(c_{j,k}) - x_k{}'(t_j)\right| < \frac{\epsilon}{3n(b-a)}$

and hence (as shown above) $\left|L(\alpha, P) - S(\alpha, P)\right| \leq \sum\limits_{j=1}^{p}\sum\limits_{k=1}^{n}\left|x_k{}'(c_{j,k}) - x_k{}'(t_j)\right|(t_j - t_{j-1}) < \frac{\epsilon}{3}$.

Thus

$$\begin{aligned}
\left|L - \int_a^b |\alpha'(t)|\,dt\right| &\leq \left|L - L(\alpha,P)\right| + \left|L(\alpha,P) - S(\alpha,P)\right| + \left|S(\alpha,P) - \int_a^b |\alpha'(t)|\,dt\right| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $L = \int_a^b |\alpha'(t)|\,dt$, as required. ■

■ **Example 1.9** A curve which is differentiable, but not $\mathcal{C}^1$, can have infinite length. For example, consider the curve $\alpha : [0,1] \to \mathbb{R}^2$ given by $\alpha(t) = (x(t), y(t))$ where $x(t) = t$ $y(t) = t^2 \cos\frac{\pi}{t^2}$ when $t \neq 0$ with $y(0) = 0$. Note that $x(y)$ and $y(t)$ are both differentiable (with $y'(0) = 0$) but $y'(t)$ is not continuous at 0 (as you can check).

Let P be the partition $P = (t_0, t_1, \cdots t_p)$ with $t_0 = 0$ and $t_j = \frac{1}{\sqrt{p-j+1}}$, that is let $P = \left(0, \frac{1}{\sqrt{p}}, \frac{1}{\sqrt{p-1}}, \cdots, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, 1\right)$. We have $y(t_j) = \frac{1}{p-j+1}\cos(p-j+1)\pi = \frac{(-1)^{p-j+1}}{p-j+1}$ for $1 \leq j \leq p$, and hence $\left|y(t_j) - y(t_{j-1})\right| = \left|\frac{1}{p-j+1} + \frac{1}{p-j+2}\right| \geq \frac{2}{p-j+2}$ for $2 \leq j \leq p$. Letting $\ell = p - j + 2$ we have

$$\sum_{j=1}^{p} \left|\alpha'(t_j) - \alpha'(t_{j-1}\right| \geq \sum_{j=2}^{p} \left|y(t_j) - y(t_{j-1})\right| \geq \sum_{j=2}^{p} \frac{2}{p-j+2} = \sum_{\ell=2}^{p} \frac{2}{\ell}.$$

Since $\sum \frac{2}{\ell}$ diverges, it follows that $L_\alpha([a,b]) = \infty$.

Definition 1.10 — reparameterisation, change of parameter, regular, preserves direction, parameterised by arclength.
When $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ is a continuous curve and $s : I \subseteq \mathbb{R} \to J \subseteq \mathbb{R}$ is a homomorphism with inverse $t = t(s)$, the curve $\beta : J \subseteq \mathbb{R} \to \mathbb{R}^n$ defined by $\beta(s) = \alpha(t(s))$ is called a **reparameterisation** of α, and the map s is called a **change of parameter** (or a **change of coordinates**).

When s is $\mathcal{C}^1$ with $s'(t) \neq 0$ for all t, we say that s is **regular**.

By the Inverse Function Theorem, if $s = s(t)$ is smooth (or $\mathcal{C}^k$) and regular then so is its inverse $t = t(s)$.

When $s'(t) > 0$ for all t we say s **preserves direction** and when $s'(t) < 0$ for all t we say s **reverses direction**.

When α and s are both smooth (or $\mathcal{C}^k$) and regular, so is β, and for $t = t(s)$ we have $\beta'(s) = \alpha'(t(s))\, t'(s) = \frac{\alpha'(t)}{s'(t)}$.

When $|\beta'(s)| = 1$ for all $s \in J$, we say that β is **parameterised by arclength**. Unless otherwise stated, we shall assume that any change of coordinates is smooth and regular.

Theorem 1.11
Every regular curve can be reparameterised by arclength, using a regular direction-preserving change of coordinates.

Proof. Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^n$ be a regular curve. Let $a \in I$ and define $s(t) = \int_a^t \left|\alpha'(r)\right| dr$. Note that $s'(t) = \left|\alpha'(t)\right| > 0$ so $s(t)$ is regular and strictly increasing, and it maps the interval I to an interval J, and if α is $\mathcal{C}^k$ then so is $s = s(t)$. By the inverse function $t = t(s)$ satisfies $t'(s) = \frac{1}{s'(t)} = \frac{1}{|\alpha'(t)|}$. The reparameterised curve $\beta : J \to \mathbb{R}^n$ given by $\beta(s) = \alpha(t(s))$ satisfies $\beta'(s) = \alpha'(t(s))\, t'(s) = \frac{\alpha'(t(s))}{|\alpha'(t(s))|}$ so that $\left|\beta'(s)\right| = 1$ for all $s \in J$. ■

1.2 Curves in $\mathbb{R}^2$

Definition 1.12 — unit tangent vector, unit normal vector, signed curvature, scalar curvature.
Let $\beta : J \subseteq \mathbb{R} \to \mathbb{R}^2$ be a smooth regular curve parameterised by arclength. For a vector $u = (x, y) \in \mathbb{R}^2$, write $u^\times = (-y, x)$ and note that $|u^\times| = |u|$. The **unit tangent vector** and the **unit normal vector** of β at s are the vectors

$$T(s) = T_\beta(s) = \beta'(s),$$
$$N(s) = N_\beta(s) = T(s)^\times.$$

Since β is parametrized by arclength, $|T(s)| = |\beta'(s)| = 1$ and $|N(s)| = |\beta'(s)^\times| = 1$ for all s. For all s we have $\beta'(s) \times \beta'(s) = \left|\beta'(s)\right|^2 = 1$. By differentiation both sides we obtain $\frac{d}{ds}\left(\beta'(s) \times \beta'(s)\right) = 0$, that is $2\,\beta'(t) \times \beta''(t) = 0$. Thus $\beta''(s)$ is orthogonal to $\beta'(s) = T(s)$, and so $\beta''(s)$ lies in the span of $T(s)^\times = N(s)$.

We define the **signed curvature** of β at s to be the real number $k(s) = k_\beta(s)$ such that

$$\beta''(s) = k(s)\,N(s) = k_\beta(s)\,N_\beta(s).$$

Since $|N_\beta(s)| = 1$ we have $|\beta''(s)| = \left|k_\beta(s)\right|$.

The **scalar curvature** of β at s is

$$\kappa(s) = \kappa_\beta(s) = \left|k(s)\right| = \left|\beta''(s)\right|.$$

When $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ is a smooth regular curve we first reparametrize by arclength by choosing $a \in I$ and letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\,dr$, and then we define $T(t) = T_\alpha(t) = T_\beta(s(t))$, $N(t) = N_\alpha(t) = N_\beta(s(t))$, $k(t) = k_\alpha(t) = k_\beta(s(t))$ and $\kappa(t) = \kappa_\alpha(t) = \kappa_\beta(s(t))$, and we call these the unit tangent vector, the unit normal vector, the signed curvature, and the scalar curvature, of α at t. The following theorem shows that these are well-defined, that is they do not depend on the choice of $a \in I$.

Theorem 1.13
For a smooth regular curve $\alpha = \alpha(t)$ we have

$$T = \frac{\alpha'}{|\alpha'|}\ ,\ N = \left(\frac{\alpha'}{|\alpha'|}\right)^\times$$
$$k = \frac{\det_2(\alpha', \alpha'')}{|\alpha'|^3} = \frac{(\alpha' \times \alpha'') \times e_3}{|\alpha'|^3} = \frac{\det_3(\alpha', \alpha'', e_3)}{|\alpha'|^3}$$
$$\kappa = \frac{\left|\det_2(\alpha', \alpha'')\right|}{|\alpha'|^3} = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$$

where $\det_2(\alpha', \alpha'')$ is the determinant of the 2×2 matrix with columns $\alpha', \alpha'' \in \mathbb{R}^2$, and where we identify $\alpha', \alpha'' \in \mathbb{R}^2$ with $\binom{\alpha'}{0}, \binom{\alpha''}{0} \in \mathbb{R}^3$ so that $\alpha' \times \alpha''$ is the cross product of two vectors $\binom{\alpha'}{0}, \binom{\alpha''}{0} \in \mathbb{R}^3$ and $\det_3(\alpha', \alpha'', e_3)$ is the determinant of the 3×3 matrix whose first two columns are $\binom{\alpha'}{0}, \binom{\alpha''}{0} \in \mathbb{R}^3$ and whose last column is the 3^{rd} standard basis vector e_3.

Proof. First verify (easily) that when we identify $u, v \in \mathbb{R}^2$ with $\binom{u}{0}, \binom{v}{0} \in \mathbb{R}^3$ we have

$$u^\times \cdot v = \det_2(u, v) = (u \times v) \cdot e_3 = \det_3(u, v, e_3)$$

and $\left|\det_2(u, v)\right| = \left|u \times v\right|$.

Peparametrize by arclength by choosing $a \in I$ and letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\, dr$. We have $T_\beta(s) = \beta'(s)$ and $N_\beta(s) = \beta'(s)^\times$. Let us find formulas for $k_\beta(s)$ and $\kappa_\beta(s)$. By definition, $k_\beta(s)\beta'(s)^\times = k_\beta(s)N_\beta(s) = \beta''(s)$. Take the dot product of both sides with $\beta'(s)^\times$ to get

$$\begin{aligned} k_\beta(s) &= \beta'(s)^\times \times \beta''(s) \\ \kappa_\beta(s) &= \left|k_\beta(s)\right| = \left|\beta'(s) \times \beta''(s)\right|. \end{aligned}$$

Now let us find formulas for $T(t) = T_\alpha(t) = T_\beta(s(t))$, $N(t) = N_\alpha(t) = N_\beta(s(t)$, $k(t) = k_\alpha(t) = k_\beta(s(t))$ and $\kappa(t) = \kappa_\alpha(t) = \kappa_\beta(s(t))$. We have $\alpha(t) = \beta(s(t))$ so that $\alpha'(t) = \beta'(s(t))s'(t)$. Since $|\beta'(s(t))| = 1$ and $s'(t) > 0$, it follows that $\left|\alpha'(t)\right| = s'(t)$. Since $\beta''(s) = k_\beta(s)N_\beta(s)$ and $|N_\beta(s)| = |T_\beta(s)| = 1$, we have $\left|\beta''(s)\right| = \left|k_\beta(s)\right| = \kappa_\beta(s)$. Since $\beta''(s)$ is orthogonal to $\beta'(s)$ (see Definition 1.12) we have $\kappa_\beta(s) = \left|\beta''(s)\right| = \left|\beta'(s) \times \beta''(s)\right|$.

$$\begin{aligned} \alpha(t) &= \beta(s(t)) \\ \alpha'(t) &= \beta'(s(t))s'(t) \\ \alpha''(t) &= \beta''(s(t))s'(t)^2 + \beta'(s(t))s''(t) \\ \alpha'(t) \times \alpha''(t) &= \left(\beta'(s(t)) \times \beta''(s(t))\right)(s')^3. \end{aligned}$$

Since $\left|\alpha'(t)\right| = s'(t)$ this gives

$$\frac{\alpha'(t)}{|\alpha'(t)|} = \beta'(s(t)) = T_\beta(s(t)) = T(t) \quad , \quad \left(\frac{\alpha'(t)}{|\alpha'(t)|}\right)^\times = T_\beta(s(t))^\times = N_\beta(s(t)) = N(t)$$

$$\begin{aligned} \frac{\alpha'(t) \times \alpha''(t)}{|\alpha'(t)|^3} &= \beta'(s(t)) \times \beta''(s(t)) = k_\beta(s(t)) = k(t) \\ \frac{\left|\alpha'(t) \times \alpha''(t)\right|}{|\alpha'(t)|^3} &= \left|\beta'(s(t)) \times \beta''(s(t))\right| = \kappa_\beta(s(t)) = \kappa(t)\,. \end{aligned}$$

■

Theorem 1.14
For a smooth regular curve α in $\mathbb{R}^2$, the curvature of α is identically zero if and only if (the image of) α lies on a straight line.

Proof. Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a smooth regular curve. Choose $a \in I$ and reparametrize α by arclength by setting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(t)|\, dt$. Suppose that $\kappa(t) = 0$ for all t. Then we have $0 = \kappa(t(s)) = \left|\beta''(s)\right|$ for all s so that $\beta''(s) = 0$ for all s. By integrating once we obtain $\beta'(s) = u$ for some $u \in \mathbb{R}^2$ since $|\beta'(s)| = 1$, u is a unit vector) and by integrating again we obtain $\beta(s) = p + su$ for some $p \in \mathbb{R}^2$. Thus $\alpha(t) = p + s(t)u$ for all t so that α lies on the line through p in the direction of u.

Suppose, conversely, that (the image of) α lies on a straight line, say the line $p + su$ where $p, u \in \mathbb{R}^2$ and $|u| = 1$. Then for every $t \in I$ there is a (unique) $s = s(t)$ such that $\alpha(t) = p + s(t)u$. We remark that taking the dot product with u gives $s(t) = (\alpha(t) - p)$ for all t so we see that $s(t)$ is smooth. Since $\alpha(t) = p + tu$, we have $\alpha'(t) = s'(t)u$ and $\alpha''(t) = s''(t)u$ so that $\alpha(t) \times \alpha''(t) = s'(t)s''(t)\, u \times u = 0$ and kence $\kappa(t) = 0$ for all t. ■

Definition 1.15 — osculating circle.
Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a smooth regular curve, let $a \in I$, and suppose that $\kappa(a) \neq 0$. We define the **osculating circle** (or the **best-fit circle**) of α at $t = a$ as follows. Let $p = \alpha(a)$, $T = T(a)$, $N = N(a)$, $k = k(a)$ and $\kappa = \kappa(a)$. Reparametrize by arclength, letting $\beta(s) = \alpha(t(s)$ where $s(t) = \int_a^t |\alpha'(r)|\,dr$ so that we have $\beta(0) = p$. $\beta'(0) = T$ and $\beta''(0) = kN$. The osculating circle at $t = a$ is the circle given by

$$\begin{aligned}\sigma(s) &= \left(p + \frac{1}{k}N\right) - \frac{1}{k}\cos(ks)N + \frac{1}{k}\sin(ks)T\\ \sigma'(s) &= \sin(ks)N + \cos(ks)T\\ \sigma''(s) &= k\cos(ks)N - k\sin(ks)T\end{aligned}$$

which is the circle of radius $R = \frac{1}{\kappa}$ centered at $p + \frac{1}{k}N$, parametrized by arclength (since $|\sigma'(s)| = 1$ for all s), such that $\sigma(0) = p = \beta(0)$, $\sigma'(0) = T = \beta'(0)$ and $\sigma''(0) = kN = \beta''(0)$.

Note 1.16 When α is a smooth regular curve, the scalar curvature at $t = a$ is equal to the reciprocal of the radius of the best-fit circle at $t = a$.

Theorem 1.17 — Polar Coordinates.
Let $I \subseteq \mathbb{R}$ be an interval with $a \in I$, let $p \in \mathbb{R}^2$, and let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a continuous curve in $\mathbb{R}^2$ with $\alpha(t) \neq p$ for any $t \in I$. Let $r_0 = \big|\alpha(a) - p\big|$ and choose $\theta_0 \in \mathbb{R}$ such that $\alpha(a) - p = r_0(\cos\theta_0, \sin\theta_0)$ (θ_0 is unique up to an integer multiple of 2π). Then there exist unique continuous functions $r, \theta : I \to \mathbb{R}$ with $r(a) = r_0$ and $\theta(a) = \theta_0$ such that

$$\alpha(t) = p + r(t)\left(\cos\theta(t), \sin\theta(t)\right)$$

for all $t \in I$. Moreover, if α is smooth (or $\mathcal{C}^k$) then so are the functions r and θ.

Proof. We omit the proof, but we remark that it is surprisingly involved. ■

Definition 1.18 — winding number, turning number.
For a continuous curve $\alpha : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^2$ with $\alpha(t) \neq p$ for any t, we define the **winding number** $\mathrm{Wind}(\alpha, p)$ of α about p as follows. We let $r_0 = \big|\alpha(a) - p\big|$ and choose θ_0ß$[0, 2\pi)$ so that $\alpha(a) = p + r_0(\cos\theta_0, \sin\theta_0)$, then we let $r, \theta : [a, b] \subset \mathbb{R} \to \mathbb{R}^2$ be the unique continuous maps such that $\alpha(t) = p + r(t)\left(\cos\theta(t), \sin\theta(t)\right)$ for all $t \in [a, b]$, and then we define

$$\mathrm{Wind}(\alpha, p) = \frac{1}{2\pi}\left(\theta(b) - \theta(a)\right).$$

When α is regular, we define the **turning number** of α to be

$$\mathrm{Turn}(\alpha) = \mathrm{Wind}\left(\alpha', 0\right).$$

Theorem 1.19
Let $\alpha : [a,b] \subseteq \mathbb{R} \to \mathbb{R}^2$ be a curve in $\mathbb{R}^2$ and write $\alpha(t) = (x(t), y(t))$.

(1) If α is a $\mathcal{C}^1$ curve with $\alpha(t) \neq 0$ for any $t \in [a,b]$ then

$$\text{Wind}(\alpha, 0) = \frac{1}{2\pi} \int_a^b \frac{x(t)y'(t) - y(t)x'(t)}{x(t)^2 + y(t)^2}\, dt.$$

(2) If α is $\mathcal{C}^2$ regular curve then

$$\text{Turn}(\alpha) = \frac{1}{2\pi} \int_a^b k(\alpha(t))\big|\alpha'(t)\big|\, dt.$$

Proof. To prove Part 1, write α in polar coordinates as $\alpha(t) = r(t)\,(\cos\theta(t), \sin\theta(t))$, that is write $x = r\cos\theta$ and $y = r\sin\theta$ where $r = r(t)$ and $\theta = \theta(t)$ are continuous with $r(t) > 0$ for all $t \in [a,b]$ and $\theta(a) \in [0, 2\pi)$. Then

$$\begin{aligned}
\int_a^b \frac{x\,y' - y\,x'}{x^2 + y^2}\, dt &= \int_a^b \frac{(r\cos\theta)\,(r'\sin\theta + r\cos\theta\,\theta') - (r\sin\theta)\,(r'\cos\theta - r\sin\theta\,\theta')}{r^2\cos^\theta + r^2\sin^2\theta}\, dt \\
&= \int_a^b \frac{r^2\cos^2\theta\,\theta' + r^2\sin^2\theta\,\theta'}{r^2}\, dt = \int_a^b \theta'\, dt \\
&= \theta(b) - \theta(a) = 2\pi\,\text{Wind}(\alpha, 0).
\end{aligned}$$

To prove Part 2, write $\alpha'(t)$ in polar coordinates as $\alpha'(t) = |\alpha'(t)|\,(\cos\theta(t), \sin\theta(t))$ with $\theta(a) \in [0, 2\pi)$. Since α is $\mathcal{C}^2$ and regular, we note that α' is $\mathcal{C}^1$ with $\alpha'(t) \neq 0$ for all $t \in [a,b]$. Reparametrize α by arclength letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(t)|\, dt$, then write $\beta'(s)$ in polar coordinates as $\beta'(s) = \big|\beta'(s)\big|\,(\cos\phi(s), \sin\phi(s))$ with $\phi(0) \in [0, 2\pi)$. Since $|\beta'(s)| = 1$ we have $(\cos\phi(s(t)), \sin\phi(s(t))) = \beta'(s(t)) = \frac{\alpha'(t)}{|\alpha'(t)|} = (\cos\theta(t), \sin\theta(t))$ for all $t \in [a,b]$, and hence $\phi(s(t)) = \theta(t)$ for all $t \in [a,b]$ (by the uniqueness of the polar representation). Since $\beta'(s) = (\cos\phi(s), \sin\phi(s))$, we have

$$\beta''(s) = \big(-\sin\phi(s)\,\phi'(s), \cos\phi(s)\,\phi'(s)\big) = \phi'(s)\,(-\sin\phi(s), \cos\phi(s)) = \phi'(s)\,\beta'(s)^\times$$

By the definition of $k(s)$ we see that $k(s) = \phi'(s)$. Thus

$$\begin{aligned}
\int_a^b k(t)\big|\alpha'(t)\big|\, dt &= \int_a^b k(s(t))s'(t)\, dt = \int_{s(a)}^{s(b)} k(s)ds = \int_{s(a)}^{s(b)} \phi'(s)\, ds \\
&= \phi(s(b)) - \phi(s(a)) = \theta(b) - \theta(a) = 2\pi\,\text{Wind}(\alpha', 0) = 2\pi\,\text{Turn}(\alpha)
\end{aligned}$$

■

Theorem 1.20 — The Fundamental Theorem for Plane Curves.
Let $I \subseteq \mathbb{R}$ be an interval with $a \in I$, let $p, u \in \mathbb{R}^2$ with $|u| = 1$, and let $\ell : I \subseteq \mathbb{R} \to \mathbb{R}$ be a smooth function. Then there exists a unique smooth regular curve $\beta : I \subseteq \mathbb{R} \to \mathbb{R}^2$ with $|\beta'(s)| = 1$ for all $s \in I$ such that $\beta(a) = p$ and $\beta'(a) = u$ and $k(s) = \ell(s)$ for all $s \in I$.

Proof. Suppose that such a curve β exists. Since $|\beta'(s)| = 1$ for all s, we can write β' in polar coordinates as $\beta'(s) = (\cos\theta(s), \sin\theta(s))$ with $\theta(a) \in [0, 2\pi)$. Then we have $\beta''(s) = (-\sin\theta(s)\theta'(s), \cos\theta(s)\theta'(s)) = \theta'(s)\beta'(s)^\times$ so that $\theta'(s) = k(s) = \ell(s)$. We can integrate to get $\theta(s) = \theta(a) + \int_a^s \ell(t)\, dt$. Since $\beta'(s) = (\cos\theta(s), \sin\theta(s))$ we can integrate again to get

$$\beta(s) = p + \Big(\int_a^s \cos\theta(t)\, dt\,,\ \int_a^s \sin\theta(t)\, dt\Big).$$

This shows that $\beta(s)$ is uniquely determined and gives us a formula for $\beta(s)$.

Conversely, we can choose $\theta_0 \in [0, 2\pi)$ so that $(\cos\theta_0, \sin\theta_0) = u$, and then define $\theta(s) = \theta_0 + \int_a^s \ell(t)\,dt$ so that $\theta(a) = \theta_0$ and $\theta'(s) = \ell(s)$ for all $s \in I$, and then define $\beta(s) = p + \left(\int_a^s \cos\theta(t)\,dt\,,\, \int_a^s \sin\theta(t)\,dt \right)$ so that $\beta'(s) = (\cos\theta(s), \sin\theta(s))$ for all $s \in I$. Then $|\beta'(s)| = 1$ for all s and $\beta(a) = p$ and $\beta'(a) = (\cos\theta(a), \sin\theta(a)) = (\cos\theta_0, \sin\theta_0) = u$ and $\beta''(s) = \theta'(s)\beta'(s)^\times$ so that $k(s) = \theta'(s) = \ell(s)$ for all $s \in I$, as required. ∎

1.3 Curves in $\mathbb{R}^3$

Definition 1.21 — unit tangent vector, curvature vector, scalar curvature, principal normal vector, binormal vector, torsion.
Let $\beta : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$ parametrized by arclength (so $|\beta'(s)| = 1$ for all $s \in I$). The **unit tangent vector** of β at s is the unit vector $T(s) = T_\beta(s) = \beta'(s)$. The vector $\beta''(s)$ iis called the **curvature vector** of β at s. The **scalar curvature** of β at s is given by $\kappa(s) = \kappa_\beta(s) = \big|\beta''(s)\big|$.

If $\beta''(s) \neq 0$ then we define the **principal normal vector** of β at s to be the unit vector $P(s) = P_\beta(s) = \frac{\beta''(s)}{|\beta''(s)|}$, and we define the **binormal vector** of β at s to be the unit vector $B(s) = B_\beta(s) = T(s) \times P(s)$. Note that $\big\{T(s), P(s), B(s)\big\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$. Since $B = T \times P$ and $P = \frac{T'}{|T'|}$, we have

$$B' = T' \times P + T \times P' = |T'|\, P \times P + T \times P' = T \times P'.$$

Notice that B' is orthogonal to both T and B ($itisorthogonalto$T$because$B'=T×P' and it is orthogonal to B because we have $B(s)(s) = |B(s)|^2 = 1$ for all s so taking the derivative on both sides gives $2\,B' = 0. Since$ {T, P, B}$isanorthonormalbasisfor$R^3 and B' is orthogonal to both T and B, we have $B' = (B')P$. We define the **torsion** of β at s to be $\tau(s) = \tau_\beta(s) = -B'(s)(s)$ so that $B'(s) = -\tau(s)P(s)$ for all s (the negative sign is included so that the torsion of the right-handed helix is positive).

To summarize the above definitions, when $\beta : I \subseteq \mathbb{R} \to \mathbb{R}^3$ is a smooth regular curve, parametrized by arclength, with non-zero curvature vector $\beta''(s) \neq 0$, the **unit tangent vector**, the **principal normal vector,** the **binormal vector**, the **scalar curvature** and the **torsion** of β at s are given by

$$\begin{aligned}
T(s) &= T_\beta(s) = \beta'(s),\\
P(s) &= P_\beta(s) = \frac{\beta''(s)}{|\beta''(s)|},\\
B(s) &= B_\beta(s) = T(s) \times P(s),\\
\kappa(s) &= \kappa_\beta(s) = \big|\beta''(s)\big|,\\
\tau(s) &= \tau_\beta(s) = -\,B'(s)(s).
\end{aligned}$$

and $\{T,(s)P(s),B(s)\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$ for every $s \in I$. From the definition of P and κ we have $T' = \kappa P$, and as explained above, we defined $\tau = -B'$ so that $B' = -\tau P$. Since $P = B \times T$ we also have

$$P' = B' \times T + B \times T' = -\tau P \times T + \kappa B \times P = \tau B - \kappa T.$$

Thus the derivatives T', P' and B' satisfy the following matrix identity which gives a system of three equations called the **Frenet-Seret Formulas**

$$\begin{pmatrix} T' \\ P' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ P \\ B \end{pmatrix}.$$

When $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ is a smooth regular curve in $\mathbb{R}^3$, we reparametrize by arclength by choosing $a \in I$ and letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\,dr$, then we define the **unit tangent vector** of α at t to be $T(t) = T_\alpha(t) = T_\beta(s(t))$, and if $\beta''(s(t)) \neq 0$, we define the **principal normal vector**, the **binormal vector**, the **scalar curvature** and the **torsion** of α at t to be given by $P(t) = P_\alpha(t) = P_\beta(s(t))$, $B(t) = B_\alpha(t) = B_\beta(s(t))$, $\kappa(t) = \kappa_\alpha(t) = \kappa_\beta(s(t))$ and $\tau(t) = \tau_\alpha(t) = \tau_\beta(s(t))$. The following theorem shows that these are all well-defined, that is they do not depend on the choice of $a \in I$.

Theorem 1.22
Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a smooth regular curve. For all $t \in I$ for which $\alpha'(t) \times \alpha''(t) \neq 0$, we have

$$T = \frac{\alpha'}{|\alpha'|} \quad , \quad P = \frac{T'}{|T'|} \quad , \quad B = T \times P$$
$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} \quad , \quad \tau = \frac{\det_3(\alpha', \alpha'', \alpha''')}{\left|\alpha' \times \alpha''\right|^2}.$$

Proof. Choose $a \in I$ and let $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\,dr$. Then $\alpha(t) = \beta(s(t))$ and so for all $t \in I$

$$\begin{aligned}
\alpha'(t) &= \beta'(s(t))s'(t), \\
\alpha''(t) &= \beta''(s(t))s'(t)^2 + \beta'(t)s''(t), \\
\alpha'''(t) &= \beta'''(s(t))s'(t)^3 + 3\,\beta''(s(t))s'(t)s''(t) + \beta'(t)s'''(t), \\
\alpha'(t) \times \alpha''(t) &= \big(\beta'(s(t)) \times \beta''(s(t))\big)\, s'(t)^3, \\
(\alpha'(t) \times \alpha''(t)) \times \alpha'''(t) &= \big(\big(\beta'(s(t)) \times \beta''(s(t))\big)\, \beta'''(s(t))\big)\, s'(t)^6.
\end{aligned}$$

Since $\alpha' \times \alpha'' = (\beta' \times \beta'')(s')^3$, we have $\alpha' \times \alpha'' = 0 \iff \beta' \times \beta'' = 0$. Since $|\beta'(s)| = 1$ for all s, it follows $(by taking the derivative of\, 1{=}\beta'(s)\beta'(s) that \beta'$ and β'' are orthogonal, and so we have $|\beta' \times \beta''| = |\beta'|\,|\beta''| = |\beta''|$ so that $\beta' \times \beta'' = 0 \iff \beta'' = 0$. Since $T_\alpha(t) = \beta'(s(t))$ we have ${T_\alpha}'(t) = \beta''(s(t))s'(t)$ so that ${T_\alpha}'(t) = 0 \iff \beta''(s(t)) = 0$. Thus

$$\alpha'(t) \times \alpha''(t) = 0 \iff \beta'(s(t)) \times \beta''(s(t)) = 0 \iff \beta''(s(t)) = 0 \iff {T_\alpha}'(t) = 0.$$

Suppose that $\alpha'(t) \times \alpha''(t) \neq 0$. Since ${T_\alpha}'(t) = \beta''(s(t))s'(t)$ and $s'(t) = |\alpha'(t)| > 0$ we have

$$\begin{aligned}
\frac{{T_\alpha}'(t)}{\left|{T_\alpha}'(t)\right|} = \frac{\beta''(s(t))s'(t)}{\left|\beta''(s(t))s'(t)\right|} = \frac{\beta''(s(t))}{\left|\beta''(s(t))\right|} &= P_\beta(s(t) = P_\alpha(t) \text{ and} \\
B_\alpha(t) = B_\beta(s(t)) = T_\beta(s(t)) \times P_\beta(s(t)) &= T_\alpha(t) \times P_\alpha(t).
\end{aligned}$$

Since $\alpha' \times \alpha'' = (\beta' \times \beta'')(s')^3$ and $|\beta' \times \beta''| = |\beta''|$ and $s' = |\alpha'|$, we have

$$\frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} = \left|\beta''(s(t))\right| = \kappa_\beta(s(t)) = \kappa_\alpha(t).$$

To simplify notation, write β for $\beta(s(t))$ and similarly for β' and β'', and write T for $T_\beta(s(t)) = T_\alpha(t)$ and similarly for P and B, and write κ for $\kappa_\beta(s(t)) = \kappa_\alpha(t)$ and similarly for τ. Since $\beta' = T$, using the Frenet-Serre formulas and the fact that $\{T, P, B\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$, we have

$$\begin{aligned}
(\beta' \times \beta'')\beta'' &= (T \times T')'' = (T \times (\kappa P)) \times (\kappa P)' = (T \times (\kappa P)) \times (\kappa' P + \kappa P') \\
&= \kappa^2 (T \times P)' = \kappa^2 (T \times P)(-\kappa T + \tau B) = \kappa^2 \tau.
\end{aligned}$$

Since we have $\det_3(\alpha', \alpha'', \alpha''') = (\alpha' \times \alpha'') \times \alpha''' = ((\beta' \times \beta'')\beta''')\,(s')^6 = \kappa^2\tau\,|\alpha'|^6$ and we have $|\alpha' \times \alpha''| = \kappa\,|\alpha'|^3$, it follows that

$$\frac{\det_3\left(\alpha'(t), \alpha''(t), \alpha'''(t)\right)}{\left|\alpha'(t) \times \alpha''(t)\right|^2} = \frac{\kappa_\alpha(t)^2\tau_\alpha(t)\,|\alpha'(t)|^6}{\kappa_\alpha(t)^2|\alpha'(t)|^6} = \tau_\alpha(t).$$

■

■ **Example 1.23** The curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ given by $\alpha(t) = (a\cos t, a\sin t, bt)$ is called a (right-handed) **helix**. We have

$$\begin{aligned}
\alpha'(t) &= (-s\sin t, a\cos t, b)\,, \\
\alpha''(t) &= (-a\cos t, -a\sin t, 0)\,, \\
\alpha'''(t) &= (a\sin t, -a\cos t, 0) \ \text{ and} \\
\alpha'(t) \times'' (t) &= \left(ab\sin t, -ab\cos t, a^2,\right)
\end{aligned}$$

and so

$$\begin{aligned}
\left|\alpha'(t)\right| &= (a^2+b^2)^{1/2}, \\
\left|\alpha'(t) \times \alpha''(t)\right| &= a(a^2+b^2)^{1/2} \ \text{and} \\
(\alpha'(t) \times \alpha''(t))\alpha'''(t) &= a^2 b,
\end{aligned}$$

and hence

$$\begin{aligned}
\kappa(t) &= \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} = \frac{a}{a^2+b^2} \ \text{and} \\
\tau(t) &= \frac{(\alpha'(t) \times \alpha''(t))\alpha'''(t)}{|\alpha'(t)|^3} = \frac{b}{a^2+b^2}.
\end{aligned}$$

We note that the scalar curvature and the torsion of the helix are constant.

Theorem 1.24
Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$.

(1) The curvature of α is identically zero if and only if (the image of) α lies on a line.
(2) If α has non-vanishing curvature (so its torsion is defined) then the torsion of α is identically zero if and only if (the image of) α lies in a plane.

Proof. The proof of Part 1 is the same as the proof of the analogous theorem for plane curves (Theorem 1.14). To prove part 2, suppose that $\kappa_\alpha(t) \neq 0$ for all $t \in I$. Choose $a \in I$ and let $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\,dr$.

Suppose $\tau_\alpha(t) = 0$ for all t. Then $\tau_\beta(s) = \tau_\alpha(t(s)) = 0$ for all s. Write $\tau(s) = \tau_\beta(s)$. We have $B'(s) = -\tau(s)P(s) = 0$ for all s, so $B(s)$ is constant, say $B(s) = b \in \mathbb{R}^3$ for all s and note that $|b| = |B(s)| = 1$. Note that $\frac{d}{ds}(\beta(s)) = \beta'(s) = T(s)(s) = 0$ for all s, and so $\beta(s)$ is constant, say $\beta(s) = c \in \mathbb{R}$. Thus we have $\alpha(t) = \beta(s(t)) = c$ for all t and so (the image of) α lies on the plane in $\mathbb{R}^3$ given by $x = c$.

Suppose, conversely, that (the image of) α lies on a plane in $\mathbb{R}^3$, say $\alpha(t) = c$ for all $t \in I$ where $b, c \in \mathbb{R}^3$ with $|b| = 1$. Then $\beta(s) = \alpha(t(s)) = c$ for all s. Take the derivative to get $\beta'(s) = 0$ and $\beta''(s) = 0$ for all s, that is $T(s) = 0$ and $\kappa(s)P(s) = 0$ for all s. Since we are assuming that $\kappa_\alpha(t) \neq 0$ for all t, hence $\kappa(s) = \kappa_\beta(s) \neq 0$ for all s, it follows that $P(s) = 0$ for all s. Since $\{T(s), P(s), B(s)\}$ is orthonormal and $T(s) = P(s) = 0$ and $|b| = 1$, it follows that $B(s) = \pm b$ for all s. Since $B(s)$ is continuous, either we have

$B(s) = b$ for all s or we have $B(s) = -b$ for all s and, in either case, $B'(s) = 0$ for all s. Since $0 = B'(s) = -\tau(s)P(s)$ with $|P(s)| = 1$, we have $\tau(s) = 0$, that is $\tau_\beta(s) = 0$, for all s, and hence $\tau_\alpha(t) = \tau_\beta(s(t)) = 0$ for all t. ■

Definition 1.25 — osculating plane, osculating circle.
Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$, let $a \in I$, and suppose that $\kappa(a) \neq 0$ (and hence $\tau(a)$ is defined). We define the **osculating plane** of α at $t = a$ to be the plane through $\alpha(a)$ parallel to $T(a)$ and $P(a)$, that is the plane $(x - \alpha(a))(a) = 0$. We define the **osculating circle** (or the **best-fit circle**) of α at $t = a$ as we did for a planar curve (in Definition 1.15). Let $p = \alpha(a)$, $T = T(a)$, $P = P(a)$, and $\kappa = \kappa(a)$. Reparametrize by arclength, letting $\beta(s) = \alpha(t(s))$ where $s(t) = \int_a^t |\alpha'(r)|\, dr$ so that we have $\beta(0) = p$. $\beta'(0) = T$ and $\beta''(0) = \kappa P$. The osculating circle at $t = a$ is the circle given by

$$\begin{aligned}
\sigma(s) &= \left(p + \frac{1}{\kappa}P\right) - \frac{1}{\kappa}\cos(\kappa s)P + \frac{1}{\kappa}\sin(\kappa s)T \\
\sigma'(s) &= \sin(\kappa s)P + \cos(\kappa s)T \\
\sigma''(s) &= \kappa\cos(ks)P - \kappa\sin(\kappa s)T
\end{aligned}$$

which is the circle of radius $R = \frac{1}{\kappa}$ centered at $p + \frac{1}{\kappa}P$, parametrized by arclength (since $|\sigma'(s)| = 1$ for all s), such that $\sigma(0) = p = \beta(0)$, $\sigma'(0) = T = \beta'(0)$ and $\sigma''(0) = \kappa P = \beta''(0)$.

Note 1.26 When α is a smooth regular curve, the scalar curvature at $t = a$ is equal to the reciprocal of the radius of the osculating circle at $t = a$.

Theorem 1.27 — The Fundamental Theorem for Space Curves.
Given $p, u, v \in \mathbb{R}^3$ with $|u| = |v| = 1$ and given smooth functions $c, d : I \subseteq \mathbb{R} \to \mathbb{R}$ where I is an interval with $0 \in I$ and $c(s) > 0$ for all $s \in I$, there exists a unique smooth regular curve $\beta : I \to \mathbb{R}^3$ with $\beta(0) = p$, $T(0) = u$, $P(0) = v$ and $\kappa(s) = c(s)$ and $\tau(s) = d(s)$ for all $s \in I$.

Proof. We want to have $T' = \kappa P$, $P' = -\kappa T + \tau B$ and $B' = -\tau P$, so we solve the system of linear first order differential equations

$$\begin{aligned}
X' &= cY \\
Y' &= -cX + dZ \\
Z' &= -dY
\end{aligned}$$

with the initial conditions $X(0) = u$, $Y(0) = v$ and $Z(0) = u \times v$ (such a system always has a unique solution). We claim that $\{X(s), Y(s), Z(s)\}$ is a positive ordered orthonormal basis for $\mathbb{R}^3$ for all s (this is true when $s = 0$ from the initial conditions). Write $X_1 = X$, $X_2 = Y$ and $X_3 = Z$ and define $F_{k,\ell} : I \to \mathbb{R}$ by $F_{k,\ell}(s) = X_k(s)_\ell(s)$ for $1 \leq k \leq \ell \leq 3$.

Then the functions $F_{k,\ell}$ satisfy the system of differential equations

$$\begin{aligned}
\frac{d}{ds}F_{1,1} &= \frac{d}{ds}(X) = 2\,X' = 2\,(cY) = 2c\,F_{1,2}\\
\frac{d}{ds}F_{1,2} &= \frac{d}{ds}(X) = X' + X' = c\,Y + X(-cX + dZ) = -c\,F_{1,1} + d\,F_{1,3} + c\,F_{2,2}\\
\frac{d}{ds}F_{1,3} &= \frac{d}{ds}(X) = X' + X' = c\,Y + X(-dY) = -d\,F_{1,2} + c\,F_{2,3}\\
\frac{d}{ds}F_{2,2} &= \frac{d}{ds}(Y) = 2\,Y' = 2\,(-cX + dZ) = -2c\,F_{1,2} + 2d\,F_{2,3}\\
\frac{d}{ds}F_{2,3} &= \frac{d}{ds}(Y) = Y' + Y' = (-cX + dZ) + Y(-dY) = -c\,F_{1,3} - d\,F_{2,2} + d\,F_{3,3}\\
\frac{d}{ds}F_{3,3} &= \frac{d}{ds}(Z) = 2\,Z' = 2\,(-dY) = -2\,F_{2,3}
\end{aligned}$$

with the initial conditions $F_{k,k}(0) = 1$ and $F_{k,\ell}(0) = 0$ when $k \neq \ell$. Again, such a system has a unique solution, and the unique solution to this system is easily seen to be given by the constant functions $F_{k,k}(s) = 1$ and $F_{k,\ell}(s) = 0$ for all $s \in I$ and all $k \neq \ell$. Thus $\{X(s), Y(s), Z(s)\}$ is an orthonormal system for all $s \in I$, as claimed. To get $\beta'(s) = T(s) = X(s)$ with $\beta(0) = p$ we must choose $\beta(s) = p + \int_0^s X(t)\,dt$. Then we have $T = X$ and $\kappa(s) = |\beta''(s)| = |T'| = |X'| = |cY| = c$ and $P = \frac{1}{\kappa}\,T' = \frac{1}{c}X' = \frac{1}{c}(cY) = Y$ and $B = T \times P = X \times Y = Z$ and $\tau = -B' = -Z' = (dY) = d$, as required. ∎

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