PMATH 450: Lesbegue Integration and Fourier Analysis

Professor Blake Madill
LaTeXer Iris Jiang

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Contents

1. Week 1

Goals of PMATH 450:

(1) Develop a theory of integration for functions $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ which is
   (a) more flexible (than Riemann) (applicable to more functions)
   (b) more rich (nicer results)
   (c) still extends Riemann integration

(2) Introduce Harmonic Analysis

General outline (first half):

(1) Which sets should we integrate over?
   - Measurable sets

(2) Which functions should we try to integrate?
   - Measurable functions

1.1 Borel Sets

Definition 1.1 — σ-algebra.
Consider a set X, we call $\mathcal{A} \subseteq \mathcal{P}(X)$ (which is the power set of X) a **σ-algebra** of subsets of X if

(1) $\varnothing \in \mathcal{A}$

(2) $A \in \mathcal{A} \implies X \backslash A \in \mathcal{A}$

(3) $A_1, A_2, A_3, \ldots \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

So a σ-algebra is a collection of subsets of X which contains the empty set, is closed under set difference and is closed under countable unions.

R Consider $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-algebra

(1) $X \in \mathcal{A}$
   $X \backslash \varnothing = X \in \mathcal{A}$

(2) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$
   $A \cup B = A \cup B \cup \varnothing \cup \varnothing \cup \cdots \in \mathcal{A}$

(3) $A_1, A_2, \ldots \in \mathcal{A} \implies \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$

$\bigcap_{i=1}^{\infty} A_i = X \backslash \left(\bigcup_{i=1}^{\infty} (X \backslash A_i) \right)$

(4) $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$

■ **Example 1.1**

- $\{\varnothing, X\}$ is the smallest σ-algebra you could have given X
- $\mathcal{A} = \mathcal{P}(X)$ is a σ-algebra
- $\mathcal{A} = \{A \subseteq \mathbb{R} : A \text{ is open}\}$ is **NOT** a σ-algebra.
  it is not closed under set difference, consider $A = (0,1) \in \mathcal{A}$, $\mathbb{R} \backslash A = (-\infty, 0] \cup [1, \infty) \notin \mathcal{A}$ because it is not open.
- $\mathcal{A} = \{A \subseteq \mathbb{R} : A \text{ open or closed}\}$ is **NOT** a σ-algebra.
  Consider $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \notin \mathcal{A}$ since $\mathbb{Q}$ is neither open nor closed

Proposition 1.1
Given a set X and $\mathcal{C} \subseteq \mathcal{P}(X)$, then $\mathcal{A} = \cap\{\mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{-algebra }, \mathcal{C} \subseteq \mathcal{B}\}$ is a σ-algebra. It is the smallest σ-algebra containing $\mathcal{C}$.

Definition 1.2 — Borel Sets.
Consider $\mathcal{C} = \{A \subseteq \mathbb{R} : A \text{ is open}\}$ (this is s subset of power set of $\mathbb{R}$), then $\mathcal{A} = \bigcap\{\mathcal{B} : \mathcal{C} \subseteq \mathcal{B}, \mathcal{B} \text{ is } \sigma\text{-algebra}\}$ is called **Borel σ-algebra**. The elements of $\mathcal{A}$ is are called the **Borel sets**.

R

(1) All the open sets are Borel. i.e. open $\implies$ Borel.
(2) All the closed sets are Borel. i.e. closed $\implies$ Borel.
since σ-algebra are closed under set difference, and $\mathbb{R}$ take away open is closed
(3) $\{X_1, X_2, \ldots\} = \bigcup_{i=1}^{\infty} \{X_i\}$ is Borel. i.e. countable $\implies$ Borel.
In particular, $\mathbb{Q}$ is a Borel set which is neither open nor closed.
(4) $[a,b) = [a,b] \backslash \{b\} = [a,b] \cap (\mathbb{R} \backslash \{b\})$ is Borel.

It is actually very hard to construct a set that is not Borel. The Borel sets are the propriate sets to integrate over.

1.2 Outer Measure

Idea

(1) Given $A \subseteq \mathbb{R}$, how should we "measure" the "size" of A
(2) Some sets have "sizes" which "measure" more nicely then others. Which ones? Borel sets?

Goal

Define a function $m : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{\infty\}$ (called a measure) such that

(1) $m((a,b)) = m([a,b]) = m((a,b]) = b - a$ (the measure of an interval I equals the length of I)
(2) $m(A \cup B) \leq m(A) + m(B)$
(3) If $A \cap B = \varnothing$, then $m(A \cup B) = m(A) + m(B)$

It will be shown later in the course that we may not use $\mathcal{P}(\mathbb{R})$

Idea

Given $A \subseteq \mathbb{R}$, there exists bounded, open intervals $I_i = (a_i, b_i)$ s.t. $A \subseteq \bigcup_{i=1}^{\infty} I_i$. We want:

$$m(A) \leq \sum_{i=1}^{\infty} m(I_i) = \sum_{i=1}^{\infty} \underbrace{\mathscr{L}(I_i)}_{\text{the length of } I_i} = \sum_{i=1}^{\infty} (b_i - a_i)$$

Cover A by bounded, open, intervals as finely as possible.

Definition 1.3 — Outer Measure.
We define (Lebesgue) **outer measure** by $m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{\infty\}$ with

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \mathscr{L}(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i, I_i \text{ is a bounded, open interval} \right\}$$

■ **Example 1.2** Consider the $\varnothing$ (we would like to the size of it been zero). For any $\epsilon > 0$, $\varnothing \subseteq (0, \epsilon)$, by definition $m^*(\varnothing) \leq \mathscr{L}((0, \epsilon)) = \epsilon$. Since $m^*(\varnothing) \geq 0$, $m^*(\varnothing) = 0$.

■ **Example 1.3** Consider a countable set $A = \{x_1, x_2, x_3, \ldots\}$, given any $\epsilon > 0$, then $A \subseteq \bigcup_{i=1}^{\infty} (x_i - \frac{\epsilon}{2^{i+1}}, x_i + \frac{\epsilon}{2^{i+1}})$. $m^*(A) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \frac{\epsilon}{2} \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = \frac{\epsilon}{2}(\frac{1}{1-\frac{1}{2}}) = \epsilon$. Since $\epsilon > 0$ was arbitrary, $m^*(A) = 0$
Follow a similar proof, we can show that the finite sets also have outer measure zero

Goal

Prove that if I is an interval, then $m^*(I) = \mathscr{L}(I)$

Proposition 1.2
If $A \subseteq B$, then $m^*(A) \leq m^*(B)$

Proof. Sketch:
$X = \left\{ \sum_{i=1}^{\infty} \mathscr{L}(I_i) : A \subseteq \bigcup_{i=1}^{\infty} I_i \right\}$
$Y = \left\{ \sum_{i=1}^{\infty} \mathscr{L}(I_i) : B \subseteq \bigcup_{i=1}^{\infty} I_i \right\}$
Clearly if $A \subseteq B$ then $Y \subseteq X$ (if you have intervals cover B then they must cover A), hence $\inf X \leq \inf Y$ using the ordering of the extended real numbers. i.e. $m^*(A) \leq m^*(B)$ ■

Lemma 1.3
If $a, b \in \mathbb{R}$ with $a \leq b$, then $m^*([a, b]) = b - a$

We start with the closed bounded intervals because they are compact, so as soon as you cover this closed interval with countable union of open intervals, then you only need to take finitely many of them because you are guaranteed to have a finite subcover to cover the interval.

Proof. Let $\epsilon > 0$ be given. Since $[a, b] \subseteq (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$, we see that $m^*([a, b]) \leq b - a + \epsilon$. Since $\epsilon > 0$ was arbitrary, $m^*([a, b]) \leq b - a$.

Let I_i ($i \in \mathbb{N}$) be bounded open intervals s.t. $[a, b] \subseteq \bigcup_{i=1}^{\infty} I_i$. Since $[a, b]$ is compact, there exists $n \in \mathbb{N}$ s.t. $[a, b] \subseteq \bigcup_{i=1}^{n} I_i$, hence $b - a \leq \sum_{i=1}^{n} \mathscr{L}(I_i) \leq \sum_{i=1}^{\infty} \mathscr{L}(I_i)$ (the first inequality can be proved by induction), and so $m^*([a,b]) \geq b-a$ (since $m^*([a,b])$ is the greatest lower bound) ∎

Proposition 1.4
If I is an interval then $m^*(I) = \mathscr{L}(I)$

Proof.
(1) Suppose I is bounded with endpoints $a \leq b$
Given $\epsilon > 0$, note $I \subseteq [a,b] \implies m^*(I) \leq m^*([a,b]) = b-a$. Also $[a+\frac{\epsilon}{2}, b-\frac{\epsilon}{2}] \subseteq I \implies b-a-\epsilon \leq m^*(I) \implies b-a \leq m^*(I)$.
(2) Suppose I is unbounded.
For all $n \in \mathbb{N}$, there exists I_n s.t. $I_n \subseteq I$, $\mathscr{L}(I_n) = n$. Then $m^*(I) \geq m^*(I_n) = n$, hence $m^*(I) = \infty = \mathscr{L}(I)$

∎

1.3 Basic Properties of Outer Measure

Outer measure is
(1) Translation Invariant
(2) Countably Subadditivity

Notation
Given $x \in \mathbb{R}, A \subseteq \mathbb{R}$, then $x + A = \{x + a : a \in A\}$

Proposition 1.5 — Translation Invariant.
$m^*(x+A) = m^*(A)$

Proof. Sketch:

$$\begin{aligned}
m^*(x+A) &= \inf\left\{\sum \mathscr{L}(I_i) : x+A \subseteq \bigcup I_i\right\} \\
&= \inf\left\{\sum \mathscr{L}(I_i) : A \subseteq \bigcup (-x+I_i)\right\} \\
&= \inf\left\{\sum \mathscr{L}(-x+I_i) : A \subseteq \bigcup (-x+I_i)\right\} \\
&= \inf\left\{\sum \mathscr{L}(J_i) : A \subseteq \bigcup (J_i)\right\} \\
&= m^*(A)
\end{aligned}$$

∎

Proposition 1.6 — Countable Subadditivity.
If we take countably many subset $A_i \subseteq \mathbb{R} (i \in \mathbb{N})$ then $m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$

Proof. We may assume each $m^*(A_i) < \infty$ (otherwise the result will be trivial). Let $\epsilon > 0$ be given and fix $i \in \mathbb{N}$. There exists open, bounded intervals $I_{i,j}$ s.t. $A_i \subseteq \bigcup_{i=1}^{\infty} I_{i,j}$ and $\sum_{j=1}^{\infty} \mathscr{L}(I_{i,j}) \leq m^*(A_i) + \frac{\epsilon}{2^i}$ (Note that we add a little bit on the out measure which makes it no longer a lower bound hence we can find the $I_{i,j}$, this is a common technique when working with outer measure). We see that $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i,j} I_{i,j}$ and so $m^*(A_i) \leq \sum_{i,j} \mathscr{L}(I_{i,j}) =$

$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathscr{L}(I_{i,j}) \leq \sum_{i=1}^{\infty} m^*(A_i) + \frac{\epsilon}{2^i} = \sum_{i=1}^{\infty} m^*(A_i) + \epsilon$. Since ϵ is arbitrary, the proposition follows. ■

Corollary 1.7 — Finite Subadditivity.
If $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{R})$, then $m^*\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} m^*(A_i)$

Proof. Sketch:
$A_1 \cup \cdots \cup A_n = A_1 \cup \cdots \cup A_n \cup \varnothing \cup \varnothing \cup \cdots$ ■

Problem

There exists $A, B \subseteq \mathbb{R}$ s.t. $A \cap B = \varnothing$ and $m^*(A \cup B) < m^*(A) + m^*(B)$. i.e. outer measure is not finitely additive. (We would like $m^*(A \cup B) = m^*(A) + m^*(B)$ for disjoint sets A, B)

Solution

Restrict the domain of m^* to only include sets which measure "nicely" (which are called measurable).

2. Week 2

2.1 Measurable Sets

Goal

Restrict the domain of m^* to only include sets s.t. whenever $A \cap B = \varnothing$, $m^*(A \cup B) = m^*(A) + m^*(B)$

Definition 2.1 — Measurable.
We say $A \subseteq \mathbb{R}$ is **measurable** if $\forall X \subseteq \mathbb{R}$, $m^*(X) = m^*(X \cap A) + m^*(X \backslash A)$

R By finite subadditivity, we always have $m^*(X) \leq m^*(X \cap A) + m^*(X \backslash A)$ because $X = (X \cap A) \cup (X \backslash A)$

R If $\subseteq \mathbb{R}$ is measurable and $B \subseteq \mathbb{R}$ with $A \cap B = \varnothing$, then consider $X = A \cup B$, $m^*(A \cup B) = m^*(X \cap A) + m^*(X \backslash A) = m^*(A) + m^*(B)$.

Goal

Show a lot of sets are measurable.

Proposition 2.1
If $m^*(A) = 0$, then A is measurable.

Proof. Let $X \subseteq \mathbb{R}$. Since $X \cap A \subseteq A$, we have $0 \geq m^*(X \cap A) \leq m^*(A) = 0$, and so $m^*(X \cap A) = 0$. Then $m^*(X \cap A) + m^*(X \backslash A) = m^*(X \backslash A) \leq m^*(X)$. (The other inequality is trivial). ■

Proposition 2.2
If $A_1, A_2, \ldots, A_n$ are measurable, then $\bigcup_{i=1}^{n} A_i$ is measurable.

Proof. It suffices to prove the result when $n = 2$ (then the rest is a trivial induction). Let $A, B \subseteq \mathbb{R}$ be measurable. Let $X \subseteq \mathbb{R}$. Then

$$\begin{aligned} m^*(X) &= m^*(X \cap A) + m^*(X \backslash A) \\ &= m^*(X \cap A) + m^*((X \backslash A) \cap B) + m^*((X \backslash A) \backslash B) \\ &= m^*(X \cap A) + m^*((X \backslash A) \cap B) + m^*(X \backslash (A \cup B)) \\ &\geq m^*((X \cap A) \cup ((X \backslash A) \cap B)) + m^*(X \backslash (A \cup B)) \\ &= m^*(X \cap (A \cup B)) + +m^*(X \backslash (A \cup B)) \end{aligned}$$

∎

Proposition 2.3
Let $A_1, A_2, \ldots, A_n$ are measurable, $A_i \cap A_j = \varnothing$ for $i \neq j$. Let $A = A_1 \cup \cdots \cup A_n$. If $X \subseteq \mathbb{R}$, then $m^*(X \cap A) = \sum\limits_{i=1}^{n} m^*(X \cap A_i)$

Proof. It suffices to prove the result when $n = 2$ (then the rest is a trivial induction). Let $A, B \subseteq \mathbb{R}$ be measurable with $A \cap B = \varnothing$. Let $X \subseteq \mathbb{R}$. Then

$$\begin{aligned} m^*(X \cap (A \cup B)) &= m^*((X \cap (A \cup B)) \cap A) + m^*((X \cap (A \cup B)) \backslash A) \\ &= m^*(X \cap A) + m^*(X \cap B) \end{aligned}$$

We only used the measurability of A, so in some sense our assumption is stronger than it can be. When actually writing out the induction, you will need $A_1, \ldots, A_{n-1}$ to be measurable. It doesn't matter the last one A_n is measurable or not. ∎

Corollary 2.4 — Finite additivity.
If $A_1, A_2, \ldots, A_n$ are measurable, $A_i \cap A_j = \varnothing$ for $i \neq j$. Then $m^*(A_1 \cup \cdots \cup A_n) = \sum\limits_{i=1}^{n} m^*(A_i)$

Proof. Take $X = \mathbb{R}$ in the previous prove. ∎

2.2 Countable Additivity

Lemma 2.5
Consider $A_i \subseteq \mathbb{R}$ are measurable for $i \in \mathbb{N}$. If $A_i \cap A_j = \varnothing$ for $i \neq j$. Then $A := \bigcup\limits_{i=1}^{\infty} A_i$ is measurable.

Proof. Sketch: Consider $B_n = A_1 \cup A_2 \cup \cdots \cup A_n$ and $X \subseteq \mathbb{R}$. Then

$$\begin{aligned} m^*(X) &= m^*(X \cap B_n) + m^*(X \backslash B_n) \\ &\geq m^*(X \cap B_n) + m^*(X \backslash A) \\ &= \sum_{i=1}^{n} m^*(X \cap A_i) + m^*(X \backslash A) \text{ (Using the proposition 2.3)} \end{aligned}$$

Taking $n \to \infty$,

$$\begin{aligned} m^*(X) &\geq \sum_{i=1}^{\infty} m^*(X \cap A_i) + m^*(X \backslash A) \\ &\geq m^*\left(\bigcup_{i=1}^{\infty}(X \cap A_i)\right) + m^*(X \backslash A) \\ &= m^*(X \cap A) + m^*(X \backslash A) \end{aligned}$$

■

Proposition 2.6
If $A \subseteq \mathbb{R}$ is measurable, then $\mathbb{R} \backslash A$ is measurable.

Proof. Sketch: Take $X \subseteq \mathbb{R}$

$$\begin{aligned} m^*(X \cap (\mathbb{R} \backslash A)) + m^*(X \backslash (\mathbb{R} \backslash A)) &= m^*(X \backslash A) + m^*(X \cap A) \\ &= m^*(X) \text{ (Since } A \text{ is measurable)} \end{aligned}$$

■

Proposition 2.7
If $A_i \subseteq \mathbb{R}$ are measurable with $i \in \mathbb{N}$, then $A = \bigcup\limits_{i=1}^{\infty} A_i$ is measurable.

Proof. Sketch: Define $B_1 = A_1$ and $B_n = A_n \backslash (A_1 \cup A_2 \cup \cdots \cup A_{n-1})$ for $n \geq 2$. Then $B_n = A_n \cap (\mathbb{R} \backslash (A_1 \cup A_2 \cup \cdots \cup A_{n-1}))$ (these are two separate measurable sets). Therefore B_n is measurable and by construction, for $i \neq j$, $B_i \cap B_j = \varnothing$. Also $\bigcup\limits_{i=1}^{\infty} B_i = \bigcup\limits_{i=1}^{\infty} A_i$, which are measurable. ■

Corollary 2.8
The collection $\mathscr{L}$ of (Lebesgue) measurable sets is a σ-algebra of sets in $\mathbb{R}$

Corollary 2.9 — Countable additivity.
If $A_i \in \mathbb{R}$ are measurable with $i \in \mathbb{N}$, $A_i \cap A_j = \varnothing$ for $i \neq j$. Then $m^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) = \sum\limits_{i=1}^{\infty} m^*(A_i)$

Proof. Sketch: The countable subadditivity 1.6 tells us $m^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) \leq \sum\limits_{i=1}^{\infty} m^*(A_i)$.
Note that for any $n \in \mathbb{N}$

$$m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geq m^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m^*(A_i)$$

Take $n \to \infty$, $m^*\left(\bigcup\limits_{i=1}^{\infty} A_i\right) \geq \sum\limits_{i=1}^{n} m^*(A_i)$ ■

2.3 Borel Implies Measurable

Goal

Show Borel sets are measurable.

Proposition 2.10
If $a \in \mathbb{R}$ then (a, ∞) is measurable.

Proof. Let $X \subseteq \mathbb{R}$. We want to show that $m^*(X \cap (a, \infty)) + m^*(X \backslash (a, \infty)) \leq m^*(X)$.

(1) Assume $a \notin X$. We want to show $m^*(X \cap (a, \infty)) + m^*(X \cap (-\infty, a)) \leq m^*(X)$. Let $X_1 = (X \cap (a, \infty))$ and $X_2 = (X \cap (-\infty, a))$. Let (I_i) be a sequence of bounded open intervals such that $X \subseteq \cup I_i$. Define for each i, $I_i' = I_i = \cap(a, \infty)$ and $I_i'' = I_i \cap (-\infty, a)$. Note that $X_1 \subseteq \cup I_i'$ and $X_2 \subseteq \cup I_i''$, ans so $m^*(X_1) \leq \sum \mathscr{L}(I_i')$ and $m^*(X_2) \leq \sum \mathscr{L}(I_i'')$. We then see that

$$
\begin{aligned}
m^*(X_1) + m^*(X_2) &\leq \sum \mathscr{L}(I_i') + \sum \mathscr{L}(I_i'') \\
&= \sum [\mathscr{L}(I_i') + \mathscr{L}(I_i'')] \\
&= \sum \mathscr{L}(I_i)
\end{aligned}
$$

Note that the $m^*(X)$ is the infimum of $\sum \mathscr{L}(I_i)$. By the definition of infimum, $m^*(X_1) + m^*(X_2) \leq m^*(X)$

(2) Assume $a \in X$. Let $X' = X \backslash \{a\}$. Then by the previous case, $m^*(X' \cap (a, \infty)) + m^*(X' \cap (-\infty, a)) \leq m^*(X')$. Also note $m^*(\{a\}) = 0$. Then

$$
\begin{aligned}
m^*(X \cap (a, \infty)) + m^*(X \backslash (a, \infty)) &= m^*(X' \cap (a, \infty)) + m^*((X' \backslash (a, \infty)) \cup \{a\}) \\
&= m^*(X' \cap (a, \infty)) + m^*((X' \cap (-\infty, a)) \cup \{a\}) \\
&\leq m^*(X' \cap (a, \infty)) + m^*(X' \cap (-\infty, a)) + m^*(\{a\}) \\
&= m^*(X' \cap (a, \infty)) + m^*(X' \cap (-\infty, a)) \\
&\leq m^*(X') \\
&\leq m^*(X)
\end{aligned}
$$

■

Theorem 2.11
Every Borel set is measurable.

Proof. Sketch: We will show that every open set is measurable, then the σ-algebra of measurable sets contains all the open sets, and so by smallness of Borel set it would be contained in the σ-algebra of Lebesgue measurable sets.

We have (a, ∞) is measurable, then $\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty)$ is measurable. Note $\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty) = [a, \infty$. Then $\mathbb{R} \backslash [a, \infty) = (-\infty, a]$ is also measurable. Then any open interval $(a, b) = (a, \infty) \cap (-\infty, b)$ is measurable. Hence, every open set in $\mathbb{R}$ is measurable. Recall the Borel σ-algebra is the σ-algebra generated by open sets, meaning it is the smallest σ-algebra of sets in $\mathbb{R}$ which contain open sets. Since the collection of measurable sets is a σ-algebra, now containing all open sets, we have the Borel σ-algebra has to be subset of the σ-algebra of the measurable sets. ■

Definition 2.2 — Lebesgue measurable.
We call $m : \mathscr{L} \to [0, \infty) \cup \{\infty\}$ where $\mathscr{L}$ is the σ-algebra of measurable sets given by $m(A) = m^*(A)$ Lebesgue measurable

Exercise 2.1 Prove that if $A \subseteq \mathbb{R}$ is measurable, then $x + A$ is measurable for any $x \in \mathbb{R}$

Solution. Let $X \subseteq \mathbb{R}$, we want to show $m^*(X) = m^*(X \setminus (x + A)) + m^*(X \cap (x + A)) = m^*(X \cap (x + A)^c)) + m^*(x \cap (x + A))$.

$$\begin{aligned} y \in X \cap (x + A) &\iff y \in X \text{ and } y \in x + A \\ &\iff y - x \in X - x \text{ and } y - x \in A \\ &\iff y - x \in (X - x) \cap A \\ &\iff y \in (X - x) \cap A + x \end{aligned}$$

Hence $X \cap (x+A) = (X-x) \cap A+x$. Similarly, one also has that $X \cap (x+A)^c = (X-x) \cap A^c+x$. Now, from the RHS, by the translation invariant property for outer measure, we have that

$$\begin{aligned} m^*(X \cap (x + A)^c)) + m^*(x \cap (x + A)) &= m^*((X - x) \cap A^c + x) + m^*((X - x) \cap A + x) \\ &= m^*((X - x) \cap A^c) + m^*((X - x) \cap A) \end{aligned}$$

Since A is measurable, $m^*((X - x) \cap A^c) + m^*((X - x) \cap A) = m^*(X - x) = m^*(X)$. It follows that $x + A$ is measurable. ■

2.4 Basic Properties of Lebesgue Measure

Proposition 2.12 — Excision Property.
If $A \subseteq B$ with A being measurable and $m(A) < \infty$, then $m^*(B \backslash A) = m^*(B) - m(A)$

Proof. Sketch: By the definition of measurable

$$\begin{aligned} m^*(B) &= m^*(B \cap A) + m^*(B \backslash A) \\ &= m(A) + m^*(B \backslash A) \end{aligned}$$

We want $m(A) > \infty$ to avoid the case $\infty - \infty$. ■

Theorem 2.13 — Continuity of Measure.

(1) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ are measurable, then $m\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} m(A_n)$

(2) If $B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$ are measurable, with $B_1 < \infty$, then $m\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} m(B_n)$

Proof. (1) Since $m(A_k) \leq m(\bigcup A_i)$ for all $k \in \mathbb{N}$, we have $\lim_{n\to\infty} m(A_n) \leq m(\bigcup A_i)$. If $\exists k \in \mathbb{N}$ s.t. $m(A_k) = \infty$, then $\lim_{n\to\infty} m(A_n) = \infty$ and we are done. Hence we may assume each $m(A_k) < \infty$ (so to use Excision Property 2.12).
(We would like to use Countable additivity 2.9, so we will replace A_i with a sequence of disjoint sets. For each $k \in \mathbb{N}$, let $D_k = A_k \backslash A_{k-1}$, $A_0 = \varnothing$.

Note 2.1
- The $D_k's$ are measurable
- The $D_k's$ are pariwise disjoint
- $\bigcup D_i = \bigcup A_i$

$$\begin{aligned} m\left(\bigcup A_i\right) = m\left(\bigcup D_i\right) &= \sum_{i=1}^{\infty} m(D_i) && \text{(by countable additivity)} \\ &= \sum_{i=1}^{\infty}(m(A_i) - m(A_{i-1})) && \text{(by excision property)} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n}(m(A_i) - m(A_{i-1})) \\ &= \lim_{n\to\infty} m(A_n) - m(A_0) \\ &= \lim_{n\to\infty} m(A_n) \end{aligned}$$

(2) For $k \in \mathbb{N}$, define $D_k = B_1 \backslash B_k$

Note 2.2
- $D_k's$ are measurable
- $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$

By (1), $m(\bigcup D_i) = \lim_{n\to\infty} m(D_n)$. We see that

$$\bigcup D_i = \bigcup_{i=1}^{\infty}(B_1 \backslash B_i) = B_1 \backslash \left(\bigcap_{i=1}^{\infty} B_i\right)$$

and so

$$\lim_{n\to\infty} m(D_n) = m\left(\bigcup D_i\right) = m\left(B_1 \backslash \left(\bigcap B_i\right)\right) = m(B_1) - m\left(\bigcap B_i\right)$$

Since $m(B_1) < \infty$, $B_i \subseteq B_1$ for all $i > 1$, hence $m(\bigcap B_i) < \infty$. However,

$$\lim_{n\to\infty} m(D_n) = \lim_{n\to\infty} m(B_1) - m(B_n) = m(B_1) - \lim_{n\to\infty} m(B_n)$$

■

■ **Example 2.1** $B_i = (i, \infty)$, $m\left(\bigcap B_i\right) = m(\varnothing) = 0$, however $\lim_{n\to\infty} m(B_n) = \infty$.

3. Week 3

Goals of the week:

(1) Construct an example of a non measurable set
(2) Construct an element in $\mathscr{L} \setminus \mathcal{B}$

3.1 A Non-measurable Set

Lemma 3.1
Assume we have a bounded, measurable set $A \subseteq \mathbb{R}$, and a bounded, countably infinite set $\Lambda \subseteq \mathbb{R}$. If $\lambda + A$, $\lambda \in \Lambda$, are pairwise disjoint, then $m(A) = 0$.

Proof. Sketch: Consider $\bigcup\limits_{\lambda}(\lambda + A)$ which is bounded, measurable. $m\left(\bigcup_{\lambda}(\lambda + A)\right) < \infty$ (since it is bounded).

$$m\left(\bigcup_{\lambda}(\lambda + A)\right) = \sum_{\lambda} m(\lambda + A) = \sum_{\lambda}(A) < \infty \implies m(A) = 0$$

■

Construction

Start with $\varnothing \neq A \subseteq \mathbb{R}$. Consider $a \sim b \iff a - b \in \mathbb{Q}$. Then N is an equivalence relation. Let C_A denote a single choice of equivalence class representatives for A relative to $\sim$.

R The sets $\lambda + C_A$, $\lambda \in \mathbb{Q}$, are pariwise disjoint.

Proof.

$$\begin{aligned} x \in (\lambda_1 + C_A) \cap (\lambda_2 + C_A) &\implies x = \lambda_1 + a = \lambda_2 + b,\ a, b \in C_A \\ &\implies a - b = \lambda_2 - \lambda_1 \in \mathbb{Q} \\ &\implies a \sim b \\ &\implies a = b \\ &\implies \lambda_1 = \lambda_2 \end{aligned}$$

■

Theorem 3.2 — Vitali Theorem.
Every set $A \subseteq \mathbb{R}$ with $m^*(A) > 0$ contains a non-measurable subset.

Proof. By Quiz 1, we may assume A is bounded, say $A \subseteq [-N, N]$, for some $N \in \mathbb{N}$. Claim: C_A is non-measurable. For contradiction, assume C_A is measurable. Let $\Lambda \subseteq \mathbb{Q}$ be bounded and infinite. By the lemma and the remark, $m(C_A) = 0$. Let $a \in A$. Then $a \sim b$ for some $b \in C_A$. In particular, $a - b = \lambda \in \mathbb{Q}$. Moreover, $\lambda \in [-2N, 2N]$. Taking $\Lambda_0 = \mathbb{Q} \cap [-2N, 2N]$, we have that $A \subseteq \bigcup\limits_{\lambda \in \Lambda_0} (\lambda + C_A)$. Contradiction to $m^*(A) > 0$. ■

Corollary 3.3
There exists $A, B \subseteq \mathbb{R}$ s.t.
(1) $A \cap B = \varnothing$
(2) $m^*(A \cup B) < m^*(A) + m^*(B)$

Proof. Sketch: Let C be a non-measurable set. By the definition, $\exists X \subseteq \mathbb{R}$, $m^*(X) < m^*(X \cap C) + m^*(X \backslash C)$ ■

3.2 Cantor-Lebesgue Function

Recall: Cantor Set

$$\begin{aligned} &I = [0, 1] \\ &C_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1] \\ &C_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{3}{9}] \cup [\tfrac{6}{9}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1] \\ &\vdots \\ &C = \bigcap_{k=1}^{\infty} C_k \end{aligned}$$

The Cantor set is:
- uncountable
- closed

Proposition 3.4
The Cantor set if Borel and has measure zero.

Proof. Sketch: The Cantor set is closed so Borel. $C = \bigcap\limits_{k=1}^{\infty} C_k$, C_k's are measruable and

$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$, $m(C_1) < \infty$. by the continuity of measure,

$$m(C) = \lim_{k\to\infty} m(C_k) = \lim_{k\to\infty} \frac{2^k}{3^k} = 0$$

■

Construction

Uncountable set which has measure zero: Cantor-Lebesgue function

(1) For $k \in \mathbb{N}$, U_k = union of open intervals deleted in the process of constructing $C_1, C_2, \ldots, C_k$, i.e. $U_k = [0,1] \backslash C_k$

(2) $U = \bigcup_{k=1}^{\infty} U_k$, i.e. $U = [0,1] \backslash C$

(3) Say $U_k = I_{k,1} \cup I_{k,2} \cup \cdots_{k,2^k-1}$ (in order) it can be checked that each U_k can be written as the union of $2^k - 1$ disjoint open intervals.

Define $\varphi : U_k \to [0,1]$ by $\varphi|_{I_{k,i}} = \frac{i}{2^k}$

■ Example 3.1

$U_1 = (\frac{1}{3}, \frac{2}{3}) \mapsto \frac{1}{2}$

$U_2 = \underbrace{(\frac{1}{9}, \frac{2}{9})}_{\mapsto \frac{1}{4}} \cup \underbrace{(\frac{1}{3}, \frac{2}{3})}_{\mapsto \frac{2}{4} = \frac{1}{2}} \cup \underbrace{(\frac{7}{9}, \frac{8}{9})}_{\mapsto \frac{3}{4}}$

etc.

(4) Define $\varphi : [0,1] \to [0,1]$ by: for $0 \neq x \in C$, $\varphi(x) = \sup\{\varphi(t) : t \in U \cap [0,x)\}$ and $\varphi(0) = 0$. This is the **Cantor-Lebesgue Function**

Things to know about φ:

(1) φ is increasing

(2) φ is continuous

(3) $\varphi : U \to [0,1]$ is differentiable and $\varphi' = 0$

(4) φ is onto

Proof. (2):

- φ is continuous on U. (φ is constant on each little intervals.)
- $x \in C$, $x \neq 0, 1$. For large k, $\exists a_k \in I_{k,i}$, $\exists b_k \in I_{k,i+1}$ s.t. $a_k < x < b_k$. But, $\varphi(b_k) - \varphi(a_k) = \frac{i+1}{2^k} - \frac{i}{2^k} = \frac{1}{2^k} \to 0$. No jump between , $\varphi(a_k), \varphi(b_k)$
- $x \in \{0, 1\}$ can be proved similarly.

(4): $\varphi(0) = 0$, $\varphi(1) = 1$, by the IVT, φ hits all values in between ■

3.3 A non Borel Set

Let φ be the C-L function. Consider $\psi : [0,1] \to [0,2]$ defined by

$$\psi(x) = x + \varphi(x)$$

(1) ψ is **strictly** increasing

(the sum of a strictly increasing function and an increasing function)

(2) ψ is continuous

(3) ψ is onto (IVT)

$\Longrightarrow$ ψ is invertible.

Proposition 3.5 (1) $\psi(C)$ is measurable and has **positive** measure.
It is turning a measure zero set to positive measure
(2) ψ maps a particular (measurable) subset of C to a non-measurable set

Proof.
(1) By Assignment 1, ψ^{-1} is continuous, then we have $\psi(C) = (\psi^{-1})^{-1}(C)$ is closed. $\implies$ $\psi(C)$ is measurable. Note that $[0,1] = C\dot\cup U$ for disjoint C and U, $\implies$ $[0,2] = \psi(C)\dot\cup\psi(U) \implies 2 = m(\psi(C)) + m(\psi(U))$. It suffices to show that $m(\psi(U)) = 1$.
(U is nicer to work with because by construction we can write it as a union of disjoint open intervals, as we know Lebesgue measure works quite well with disjoint union of measurable sets.)
Say $U = \dot\bigcup_{i=1}^{\infty} I_i$, where I_i's are disjoint open intervals. (remember that U is defined to be the countable union of U_k's, for each U_k is a finite union of disjoint open intervals).
Then, $\psi(U) = \dot\bigcup_{i=1}^{\infty}\psi(I_i) \implies m(\psi(U)) = \sum m(\psi(I_i))$. Note that for all $i \in \mathbb{N}$, $\exists r \in \mathbb{R}$, s.t. $\psi(x) = r$ for all $x \in I_i$. In particular, $\psi(x) = x + r$ for all $x \in I_i$, and so $\psi(I_i) = r + I_i$. Hence, $m(\psi(U)) = \sum m(I_i) = m(\dot\bigcup I_i) = m(U)$. Since $[0,1] = U\dot\cup C$, we have that $1 = m(U) + m(C) = m(U)$ (the measure of Cantor set is 0). Hence, $m(\psi(U)) = m(U) = 1 > 0$.
(2) By Vitali theorem 3.2, $\psi(C)$ contains a subset $A \subset \psi(C)$ which is non measurable. Let $B = \psi^{-1}(A) \subseteq C$. Then, $\psi(B) = A$ is non measurable as required.

■

Theorem 3.6
The Cantor set contains an element of $\mathscr{L}\backslash\mathcal{B}$

Proof. Sketch: $B \subseteq C \implies B$ measurable. But we know $\psi(B)$ is non measurable. By Assignment 1, if B is Borel, then $\psi(B)$ is Borel. Therefore, B is NOT Borel.

■

4. Week 4

4.1 Measurable Functions

Question

Which functions are suitable for integration?

Definition 4.1 — measurable function.
Let $A \subseteq \mathbb{R}$ be measurable, we say $f : A \to \mathbb{R}$ is **measurable** if and only if for all open $\mathcal{U} \subseteq \mathbb{R}$, $f^{-1}(\mathcal{U})$ is **measurable**.

Proposition 4.1
If $A \subseteq \mathbb{R}$ is measurable and $f : A \to \mathbb{R}$ is continuous, then f is measurable.

Proof. Sketch: If $\mathcal{U} \subseteq \mathbb{R}$ is open, then $f^{-1}(\mathcal{U})$ is open, which is Borel, which is measurable ■

Proposition 4.2
If $A \subseteq \mathbb{R}$ is measurable and $\mathcal{X}_A : \mathbb{R} \to \mathbb{R}$, $\mathcal{X}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$, then $\mathcal{X}_A$ is measurable.

(This is called the characteristic function of A. In particular, if $A \neq \mathbb{R}$, then $\mathcal{X}_A$ is an example of measurable function which is not continuous)

Proof. Sketch: Take open set $\mathcal{U} \subseteq \mathbb{R}$, for $\mathcal{X}_A^{-1}(\mathcal{U}) = \mathbb{R}, A, \mathbb{R}\backslash A, \varnothing$ which are all measurable. ■

Proposition 4.3
Let $A \subseteq \mathbb{R}$ be measurable, $f : A \to \mathbb{R}$, the following are equivalent
(1) f is measurable
(2) $\forall a \in \mathbb{R}$, $f^{-1}((a, \infty))$ is measurable (This is always the given definition for measur-

able function)

(3) $\forall a < b$, $f^{-1}((a,b))$ is measurable

Proof.

(1) $\implies$ (2) Trivial.

(2) $\implies$ (3) Let $b \in \mathbb{R}$ so that $f^{-1}(b,\infty)$ is measurable. Then

$$\mathbb{R}\backslash f^{-1}((b,\infty)) = f^{-1}(\mathbb{R}\backslash(b,\infty)) = f^{-1}((-\infty,b])$$

is measurable as well. We see that $(-\infty,b) = \bigcup_{n=1}^{\infty}(-\infty, b-\frac{1}{n}]$ and so $f^{-1}(-\infty,b) = \bigcup_{n=1}^{\infty} f^{-1}((-\infty, b-\frac{1}{n}])$ is measurable. Finally, for $a < b$, $(a,b) = (a,\infty) \cap (-\infty,b) \implies f^{-1}((a,b)) = f^{-1}((a,\infty)) + f^{-1}((-\infty,b))$ is measurable

(3) $\implies$ (1) Trivial.

∎

4.2 Properties of Measurable Functions

Proposition 4.4

If $A \subseteq \mathbb{R}$ is measurable, $f,g : A \to \mathbb{R}$ are measurable

(1) For all $a,b \in \mathbb{R}$, $af + bg$ is measurable

(2) The function fg is measurable (fg denotes $f(x)g(x)$, not the composite function)

Proof.

(1) Let $a \in \mathbb{R}$.For $\alpha \in \mathbb{R}$, $(af)^{-1}((\alpha,\infty)) = \{x \in A : af(x) > \alpha\}$

(a) $a > 0$,

$$(af)^{-1}((a,\infty)) = \{x \in A : f(x) > \frac{\alpha}{a}\} = f^{-1}((\frac{\alpha}{a},\infty))$$

which is measurable

(b) $a < 0$,

$$(af)^{-1}((a,\infty))f^{-1}((-\infty,\frac{\alpha}{a}))$$

(c) $a = 0$, af is continuous $\implies$ measurable

We now show that $f+g$ is measurable. For $\alpha \in \mathbb{R}$,

$$\begin{aligned}(f+g)^{-1}((a,\infty)) &= \{x \in A : f(x) + g(x) > \alpha\} \\ &= \{x \in A : f(x) > \alpha - g(x)\} \\ &= \{x \in A : \exists 1 \in \mathbb{Q}, f(x) > q > \alpha - g(x)\} \\ &= \bigcup_{q\in\mathbb{Q}} (\{x \in A : f(x) > q\} \cap \{x \in A : g(x) > \alpha - q\}) \\ &= \bigcup_{q\in\mathbb{Q}} (f^{-1}((q,\infty)) \cap g^{-1}((\alpha-1,\infty)))\end{aligned}$$

which is measurable. Hence $f+g$ is measurable

(2) By the quiz, $|f|$ is measurable. For $\alpha \in \mathbb{R}$,

$$(f^2)^{-1}((\alpha,\infty)) = \{x \in A : f^2(x) > \alpha\}$$
$$= \begin{cases} A, & \alpha < 0 \\ \{x : A : |f|(x) > \sqrt{\alpha}\}, & \alpha \geq 0 \end{cases}$$
$$= \begin{cases} A, & \alpha < 0 \\ |f|^{-1}((\sqrt{\alpha},\infty)), & \alpha \geq 0 \end{cases}$$

which is measurable. Therefor f^2 is measurable.
Since $(f+g)^2 = f^2 + 2fg + g^2$ is measurable, we have that $2fg$ is measurable. By (1), fg is measurable as well.

■

■ **Example 4.1 — Not all compositions of measurable functions are measurable.** $\psi : [0,1] \to \mathbb{R}$, $\psi(x) = x + \varphi(x)$ (where φ is the Cantor-Lebesgue function) is not measurable. $\exists A \subseteq [0,1]$ s.t. A is measurable but $\psi(A)$ is not measurable. Extend $\psi : \mathbb{R} \to \mathbb{R}$ continuously to a strictly increasing surjective function s.t. ψ^{-1} is continuous.

Consider $\mathcal{X} \circ \psi^{-1}$ (we know $\mathcal{X}$ is measurable, be the continuity, ψ^{-1} is also measurable). Then

$$(\mathcal{X} \circ \psi^{-1})^{-1}(\frac{1}{2},\frac{2}{3}) = \psi(\mathcal{X}_A^{-1}(\frac{1}{2},\frac{2}{3})) = (A)$$

which is not measurable. Hence $\mathcal{X} \circ \psi^{-1}$ is not measurable.

Proposition 4.5
Given $A \subseteq \mathbb{R}$ is measurable. If $g : A \to \mathbb{R}$ is measurable and $f : \mathbb{R} \to \mathbb{R}$ is **continuous** then $f \circ g$ is measurable.

Proof. Sketch: Take $\mathcal{U} \subseteq \mathbb{R}$ be open. $(f \circ g)^{-1}(\mathcal{U}) = g^{-1}(f^{-1}(\mathcal{U}))$ is measurable since continuity gives $f^{-1}(\mathcal{U})$ is open. ■

Definition 4.2 — almost everywhere.
We say a property $P(x)$ $(x \in A)$ is true **almost everywhere** (ae) if $m(\{x \in A : P(x) \text{ is false}\}) = 0$

Proposition 4.6
$f : A \to \mathbb{R}$ is a measurable function. If $g : A \to \mathbb{R}$ is a function and $f = g$ almost everywhere, then g is measurable.
when we say a function is measurable, we implicitly say the domain is measurable

Proof. Sketch: Let $B = \{x \in A : f(x) \neq g(x)\}$, then $m(B) = 0$. Let $\alpha \in \mathbb{R}$,

$$\begin{aligned} g^{-1}((\alpha,\infty)) &= \{x : A : g(x) > \alpha\} \\ &= \{x \in A \backslash B : g(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ &= \{x \in A \backslash B : f(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ &= (f^{-1}(\alpha,\infty) \cap (A \backslash B)) \cup \{x \in B : g(x) > \alpha\} \end{aligned}$$

$(f^{-1}(\alpha,\infty) \cap (A\backslash B))$ is clearly measurable, $\{x \in B : g(x) > \alpha\}$ is a subset of B which also has measure zero.
Hence the $g^{-1}((\alpha,\infty))$ is measurable ■

Proposition 4.7
Given A is measurable, $B \subseteq A$ is measurable. A function $f : A \to B$ is measurable if and only if $f|_B$ and $f|_{A\backslash B}$ are measurable.

Proof.
$\implies$ Suppose $f : A \to \mathbb{R}$ is measurable. Let $\alpha \in \mathbb{R}$, then

$$\begin{aligned}(f|_B)^{-1}(\alpha, \infty) &= \{x \in B : f(X) > \alpha\} \\ &= f^{-1}(\alpha, \infty) \cap B\end{aligned}$$

which is measurable. Therefore $f|_B$ is measurable.
The proof for $f|_{A\to B}$ is identical.
$\Leftarrow$ Suppose $f|_B$ and $f|_{A\backslash B}$ are measurable. For $\alpha \in \mathbb{R}$,

$$\begin{aligned}f(\alpha, \infty)^{-1} &= \{x \in A : f(x) > \alpha\} \\ &= \{x \in B : f(x) > \alpha\} \cup \{x \in A\backslash B : f(x) > \alpha\} \\ &= (f|_B)^{-1}(\alpha, \infty) \cup (f|_{A\backslash B})^{-1}(\alpha, \infty)\end{aligned}$$

is measurable, and so f is a measurable function.

■

Proposition 4.8
A sequence of measurable functions (f_n), $f_n : A \to \mathbb{R}$. If $f_n \to f$ pointwise almost everywhere, then f is measurable.

Proof. Let $B = \{x \in A : f_n(x) f(x)\}$, so that $m(B) = 0$. For $\alpha \in \mathbb{R}$,

$$(f|_B)^{-1}(\alpha, \infty) = f^{-1}(\alpha, \infty) \cap B$$

which is measurable.
A function whose domain has measure zero is always measurable.
If suffices to show that $f|_{A\backslash B}$ is measurable. By replacing f be $f|_{A\backslash B}$, we may assume $f_n \to f$ pointwise. Let $\alpha \in \mathbb{R}$. Since $f_n \to f$ pointwise, we see that for $x \in A$

$$f(x) > \alpha \iff \exists n, N \in \mathbb{N}, \forall i \geq N, f_i(x) > \alpha + \frac{1}{n}$$

We then see that

$$f^{-1}(\alpha, \infty) = \bigcup_{n\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{i=N}^{\infty} f_i^{-1}(\alpha + \frac{1}{n}, \infty)$$

is measurable. Hence f is measurable.

■

4.3 Simple Approximation

Definition 4.3 — simple.
A function $\varphi : A \to \mathbb{R}$ is called **simple** if
(1) φ is measurable (the domain is measurable)
(2) $\varphi(A)$ is finite

Canonical Representation
$\varphi : A \to \mathbb{R}$ is a simple function. $\varphi(A)$ is finite, say $\varphi(A) = \{c_1, c_3, \dots, c_k\}$ for distinct c_i's. For every i, let $A_i = \varphi^{-1}(\{c_i\})$ which is measurable (by Quiz 4, the preimage of a Borel set under a measurable function is measurable).

Then $A = \dot{\bigcup}_{i=1}^{K} A_i$ and $\varphi = \sum\limits_{i=1}^{K} c_i \mathcal{X}_{A_i}$

(we can write any simple function as a finite linear combination of characteristic functions, note $\mathcal{X}_{A_i}$'s are restricted to A)

Goal

Show measurable functions can be approximated by simple functions

Lemma 4.9
Let $f : A \to \mathbb{R}$ be measurable and **bounded**. For all $\epsilon > 0$, there exists simple $\varphi_\epsilon, \psi_\epsilon : A \to \mathbb{R}$ such that
(1) $\varphi_\epsilon \le f \le \psi_\epsilon$
(2) $0 \le \psi_\epsilon - \varphi_\epsilon < \epsilon$

Proof. Sketch: $f(A) \subseteq [a, b]$ since f is bounded. Given $\epsilon > 0$, $a = y_0 < y_1 < y_2 < \cdots < y_n = b$, where every $y_{i+1} - y_i < \epsilon$. Let $I_k = [y_{k-1}, y_k)$ for $0 \le k \le n$. Let $A_k = f^{-1}(I_k)$ which is measurable since I_k is Borel. Define $\varphi_\epsilon, \psi_\epsilon : A \to \mathbb{R}$ as

$$\varphi_\epsilon = \sum_{k=1}^{n} y_{k-1} \mathcal{X}_{A_k}$$

$$\psi_\epsilon = \sum_{k=1}^{n} y_k \mathcal{X}_{A_k}$$

Each of these are measurable since each A_k is measurable and they are finite linear combination of measurable functions. And they are simple by design.
Let $x \in A$. Since $f(x) \in [a, b]$, there exists $k \in \{0, \dots, n\}$ such that $f(x) \in I_k$. i.e. $y_{k-1} \le f(x) < y_k$, $x \in A_k$. Moreover,

$$\varphi_\epsilon(x) = y_{k-1} \le f(x) < y_k = \psi_\epsilon(x)$$

and so $\varphi_\epsilon \le f < \psi_\epsilon$.
For the same x,

$$0 \le \psi_\epsilon(x) - \varphi_\epsilon(x) = y_k - y_{k-1} < \epsilon$$

■

Theorem 4.10 — Simple Approximation Theorem.
Let $A \subseteq \mathbb{R}$ be measurable. A function $f : A \to \mathbb{R}$ is measurable if and only if there is a sequence (φ_n) of simple functions on A such that
(1) $\varphi_n \to f$ pointwise
(2) $\forall n$, $|\varphi_n| \le |f|$

Proof.
$\Leftarrow$ Simple functions are measurable, pointwise limit of measurable functions is measurable
$\Longrightarrow$ Suppose $f : A \to \mathbb{R}$ is measurable.
Case 1. $f \ge 0$. For each $n \in \mathbb{N}$, define

$$A_n = \{x \in A : f(x) \le n\}$$

so that A_n is measurable and $f|_{A_n}$ is measurable and bounded. By the lemma, there exist simple functions (φ_n), (ψ_n) such that $\varphi_n \le f \le \psi_n$ on A_n and $0 \le \psi_n - \varphi_n < \frac{1}{n}$ for every n.
Fix $n \in \mathbb{N}$. Extend $\varphi_n : A \to \mathbb{R}$ be setting $\varphi_n(x) = n$ if $x \notin A_n$. Hence $0 \le \varphi_n \le f$. For each $n \in \mathbb{N}$, $\varphi_n : A \to \mathbb{R}$ is simple.
Claim: $\varphi_n \to f$ pointwise. Let $x \in A$ and let $N \in \mathbb{N}$ such that $f(x) \le N$ (i.e. $x \in A_n$). For $n \ge N$, $x \in A_n$ and so

$$0 \le f(x) - \varphi_n(x) \le \psi_n(x) - \varphi_n(x) < \frac{1}{n}$$

Case 2. $f : A \to \mathbb{R}$ is measurable. Let $B = \{x \in A : f(x) \ge 0\}$ and $C = \{x \in A : f(x) \le 0\}$ wich are both measurable. Define $g, h : A \to \mathbb{R}$

$$g = \mathcal{X}_B f,\ h = -\mathcal{X}_C f$$

both are products of measurable functions hence are measurable and nonnegative. By Case 1, there exist sequences (φ_n), (ψ_n) of simple functions such that $\varphi_n \to f$ pointwise, $\psi_n \to h$ pointwise, $\le \varphi_n \le g$ and $0 \le \psi_n \le h$. Then

$$_{\varphi_n - \psi_n} \text{simple} \to g - h = f \quad \text{pointwise}$$

and

$$|\varphi_n - \psi_n| \le |\varphi_n| + |\psi_n| = \varphi_n + \psi_n \le g + h = |f|$$

■

5. Week 5: Littlewood's Principles

Up to certain finiteness conditions:

(1) Measurable sets are "almost" finite, disjoint, unions of bounded intervals
(2) Measurable functions are "almost" continuous
(3) Pointwise limits of measurable functions are "almost" uniform limits.

5.1 Littlewood 1

Theorem 5.1 — Littlewood 1.
Let A be a measurable set with $m(A) < \infty$. For all $\epsilon > 0$, there exists finitely many open, bounded, disjoint intervals $I_1, I_2, \ldots, I_n$ such that

$$m(A\Delta\mathcal{U}) < \epsilon$$

where $\mathcal{U} = I_1 \cup I_2 \cup \cdots \cup I_n$.
Note: $m(A\Delta\mathcal{U}) = m(A\backslash\mathcal{U}) + m(\mathcal{U}\backslash A)$

Proof. Let $\epsilon > 0$ be given. We may find an open set $\mathcal{U}$ such that $A \subseteq \mathcal{U}$ and

$$m(\mathcal{U}\backslash A) < \frac{\epsilon}{2}$$

(by Assignment 1). By PMATH351, there exist bounded, open, disjoint intervals I_i ($i \in \mathbb{N}$), such that

$$\mathcal{U} = \dot{\bigcup}_{i=1}^{\infty} I_i$$

Note that

$$\sum_{i=1}^{\infty} \ell(I_i) = m(\mathcal{U}) < \infty$$

In particular, there exists $N \in \mathbb{N}$ such that

$$\sum_{N+1}^{\infty} \ell(I_i) < \frac{\epsilon}{2}$$

Take $V = I_1 \cup \cdots \cup I_N$. We see that

$$m(A \backslash V) \leq m(\mathcal{U} \backslash V) = m(\bigcup_{i=N+1}^{\infty} I_i) = \sum_{i=N+1}^{\infty} \ell(I_i) < \frac{\epsilon}{2}$$

and

$$m(V \backslash A) \leq m(\mathcal{U} \backslash A) < \frac{\epsilon}{2}$$

Hence $m(A \Delta V) < \epsilon$ ■

5.2 Egoroff's Theorem (Littlewood 3)

Lemma 5.2
Let A be measurable, $m(A) < \infty$, (f_n) are measurable functions, $f_n : A \to \mathbb{R}$. Assume $f : A \to \mathbb{R}$ such that $f_n \to f$ pointwise. For all $\alpha, \beta > 0$, there exists a measurable subset $B \subseteq A$ and $N \in \mathbb{N}$ such that

(1) $|f_n(x) - f(x)| < \alpha$ for all $x \in B$, $n \geq N$
(2) $m(A \backslash B) < \beta$

Proof. Let $\alpha, \beta > 0$ be given. For $n \in \mathbb{N}$, define

$$\begin{aligned} A_n &= \{x \in A : |f_k(x) - f(x)| < \alpha\} \text{ for all } k \geq n \\ &= \bigcap_{k=n}^{\infty} |f_k - f|^{-1}(-\infty, \alpha) \end{aligned}$$

Therefore, every A_n is measurable. Since $f_n \to f$ pointwise,

$$A = \bigcup_{n=1}^{\infty} A_n$$

Since (A_n) is ascending $(A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots)$, by the continuity of measure,

$$m(A) = \lim_{n \to \infty} m(A_n) < \infty$$

We may find $N \in \mathbb{N}$ such that for all $n \geq N$,

$$m(A) - m(A_n) < \beta$$

Pick $B = A_N$. ■

Theorem 5.3 — Egoroff's Theorem (Littlewood 3).
Let A be measurable, $m(A) < \infty$, (f_n) are measurable functions, $f_n : A \to \mathbb{R}$. Assume $f : A \to \mathbb{R}$ such that $f_n \to f$ pointwise. For all $\epsilon > 0$, there exists a closed set $C \subseteq A$ such that

(1) $f_n \to f$ uniformly on C
(2) $m(A \backslash C) < \epsilon$

Proof. Let $\epsilon > 0$ be given. By the lemma, for every $n \in \mathbb{N}$, there exists measurable $A_n \subseteq A$ and $N(n) \in \mathbb{N}$ such that

(1) For all $x \in A_n$ and $k \geq N(n)$, $|f_k(x) - f|$
(2) $m(A \backslash A_n) < \frac{\epsilon}{2^{n+1}}$

Take $B = \bigcap_{n=1}^{\infty} A_n$ which is measurable. For $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$, $k \geq N(n)$ and $x \in B$, we have

$$|f_k(x) - f(x)| < \frac{1}{n} < \epsilon$$

Therefore $f_n \to f$ uniformly on B. Moreover,

$$m(A \backslash B) = m(A \backslash \cap A_n) = m(\cup (A \backslash A_n)) \leq \sum m(A \backslash A_n) < \sum \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2}$$

By A1, there exists a closed set C such that $C \subseteq B$ and $m(B \backslash C) < \frac{\epsilon}{2}$. Therefore

(1) Since $C \subseteq B$, $f_k \to f$ uniformly on C

(2) $m(A \backslash C) = m(A \backslash B) + m(B \backslash C) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

■

Exercise 5.1 — Warning. Consider $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{x}{n}$. We see $f_n = 0$ pointwise, show $f_n \not\to 0$ uniformly on any measurable $B \subseteq \mathbb{R}$ such that $m(\mathbb{R} \backslash B) < 1$.

R To use Egoroff's Theorem, we need $m(A) < \infty$

5.3 Lusin's Theorem (Littlewood 2)

Lemma 5.4

Let $f : A \to \mathbb{R}$ be a simple function. For all $\epsilon > 0$ there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed $C \subseteq A$ such that

(1) $f = g$ on C

(2) $m(A \backslash C) < \epsilon$

Proof. Sketch: We can always write a simple function $f = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$ which is the canonical representation where $A_i = \{x \in A : f(x) = a_i\}$ is measurable. By A1, since A_i is measurable, we can find closed $C_i \subseteq A_i$ such that

$$m(A_i \backslash C_i) < \frac{\epsilon}{n}$$

Note $A = \dot{\bigcup}_{i=1}^{n} A_i$, we define $C := \dot{\bigcup}_{i=1}^{n} C_i$ is closed.

(1) For all $x \in C_i$, $f(x) = a_i$. By A_1, f is continuous on C, we then extend $f|_C$ to a continuous function $g : \mathbb{R} \to \mathbb{R}$

(2)

$$m(A \backslash C) = m\left(\dot{\bigcup}_{i=1}^{n} (A_i \backslash C_i)\right) = \sum_{i=1}^{n} m(A_i \backslash C_i) < \epsilon$$

■

Theorem 5.5 — Lusin's Theorem (Littlewood 2).

Let $f : A \to \mathbb{R}$ be a measurable function. For all $\epsilon > 0$ there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed $C \subseteq A$ such that

(1) $f = g$ on C

(2) $m(A \backslash C) < \epsilon$

Proof. Let $\epsilon > 0$ be given.

Case 1. $m(A) < \infty$. Let $f : A \to \mathbb{R}$ be measurable. By the Simple Approximation Theorem (4.10), there exists a sequence of simple functions (f_n) such that $f_n \to f$ pointwise. By the lemmar, there exists continuous $g_n : \mathbb{R} \to \mathbb{R}$ and claose $C_n \subseteq A$ such that

(1) $f_n = g_n$ on C_n and

(2) $m(A \backslash C_n) < \frac{\epsilon}{2^{n+1}}$

By Egoroff's Theorem (5.3), there exists closed set $C_0 \subseteq A$ such that $f_n \to f$ uniformly on C_0 and

$$m(A \backslash C_0) < \frac{\epsilon}{2}$$

Let $C = \bigcap_{i=0}^{\infty} C_i$. Note that

(1) $g_n = f_n \to f$ uniformly on $C \subseteq C_0$. Each g_n is continuous, therefore f is continuous on C. By A1, we may extend $f|_C$ to a continuous $g : \mathbb{R} \to \mathbb{R}$.

(2)

$$\begin{aligned} m(A \backslash C) &= m(A \backslash \bigcap_{i=0}^{\infty} C_i) = m(\bigcup_{i=1}^{\infty}(A \backslash C_i)) \\ &\leq \sum_{i=0}^{\infty} m(A \backslash C_i) = m(A \backslash C_0) + \sum_{i=1}^{\infty} m(A \backslash C_i) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Case 2. $m(A) = \infty$. For $n \in \mathbb{N}$, define $A_n := \{a \in A : |a| \in [n-1, n)\}$ so that $A = \dot{\bigcup}_{n=1}^{\infty} A_n$. By Case 1, there exists continuous $g_n : \mathbb{R} \to \mathbb{R}$ and closed $C_n \subseteq A_n$ such that

(1) $f = g_n$ on C_n

(2) $m(A_n \backslash C_n) < \frac{\epsilon}{2^n}$

Consider $C = \dot{\bigcup}_{n=1}^{\infty} C_n$. (exercise: show C is closed.)

(1)

$$m(A \backslash C) = m\left(\dot{\bigcup}(A_n \backslash C_n)\right) = \sum m(A_n \backslash C_n) < \epsilon$$

(2) $g : C \to \mathbb{R}$. Let $x \in C$ so that $c \in C_n$ for exactly one $n \in \mathbb{N}$. Define $g(x) = g_n(x) = f(x)$, remember each g_n is continuous. (exercise) Then g is continuous. By A1, extend g continuously to all $\mathbb{R}$.

■