PMATH 450: Lesbegue Integration and Fourier Analysis

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Contents

3. Splitting Fields

Definition 3.1 — splits over.
Let E/F be a field extension. We say $f(x) \in F[x]$ **split over** E if E contains all roots of $f(x)$, i.e., $f(x)$ is a product of linear factors in $E[x]$

Definition 3.2 — splitting field.
Let $\tilde{E}/F$ be a field extension, $f(x) \in F[x]$, and $FE \subseteq \tilde{E}$. If

(1) $f(x)$ splits over E

(2) there is no proper subfield of E such that $f(x)$ splits over

Then we say E a **splitting field of** $f(x) \in F[x]$ **in** $\tilde{E}$

3.1 Existence of Splitting Field

Theorem 3.1.1
Let $p(x) \in F[x]$ be irreducible. The quotient ring $F[x]/\langle p(x)\rangle$ is a field containing F and a root of $p(x)$

Theorem 3.1.2 — Kronecker.
Let $f(x) \in F[x]$. There exists a field E containing F such that $f(x)$ splits over E

Theorem 3.1.3
Every $f(x) \in F[x]$ has a splitting field, which is a finite extension of F

3.2 Uniqueness of Splitting Fields

Definition 3.3 — extend.
Let $\phi : R \to R_1$ be a ring homomorphism, and $\Phi : R[x] \to R_1[x]$ be the unique ring

homomorphism satisfying $\Phi|_R = \phi$ and $\Phi(x) = x$. In this case, we say Φ **extends** ϕ.

More generally, if $R \subseteq S$ and $R_1 \subseteq S_1$ and $\Phi : S \to S_1$ is a ring homomorphism with $\Phi|_R = \phi$, we say Φ **extends** ϕ.

Theorem 3.2.1
Let $\phi : F \to F_1$ be an isomorphism of fields and $f(x) \in F[x]$. Let $F[x] \to F_1[x]$ be the unique ring isomorphism which extends ϕ. Let $f_1(x) = \Phi(f(x))$ and E/F and E_1/F_1 be splitting fields of $f(x)$ and $f_1(x)$ respectively. Then there exists an isomorphism $\psi : E \to E_1$ which extends ϕ.

Corollary 3.1
Any two splitting fields of $f(x) \in F[x]$ over F are F-isomorphic. Thus we can now say 'the' splitting field of $f(x)$ over F

3.3 Degrees of Splitting Fields

Theorem 3.3.1
Let F be a field and $f(x) \in F[x]$ with $\deg(f) = n \geq 1$. If E/F is the splitting field of $f(x)$, then $[E : F] | n!$