ACTSC446: Mathematics of Financial Markets

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Contents

1 Introduction to Derivatives Market 3

1.1 Financial Markets 3

1.1.1 Assets 3

1.1.2 Review: Present Value of Future Payments 3

1.2 Derivative Securities 4

1.2.1 Assumptions on a Financial Market 5

1.2.2 The Concept of (No) Arbitrage 6

1.3 Forwards and Futures 6

1.3.1 Forward Contract 7

1.3.2 Futures Contract 9

1.4 Options 9

1.4.1 Put and Call Options 10

1.4.2 Moneyness 10

1.4.3 Payoff Diagrams 11

1.4.4 Profit Diagrams 11

1.4.5 Forwards (Futures) and Options 12

1.4.6 Intrinsic and Time Value of an Option 13

1.5 Bounds on Option Prices 13

1.5.1 No-Arbitrage Bounds on Option Prices 13

1.5.2 Bounds on Non-Dividend Paying Stock 14

1.5.3 Put-Call Parity 16

1.6 Mathematical Properties of Option Prices 19

1.6.1 Review of Functions 19

1.6.2 Properties of Option Prices 19

1.7 Investment Strategies Using Options 22

1.7.1 Common Investment Strategies Using Options 22

1.8 Portfolio Insurance Strategies Using Options **23**
1.8.1 Four Insurance Strategies 23

2 Discrete Time Models **25**

2.1 One-Period Binomial Model **25**
2.1.1 Portfolio and Arbitrage 26
2.1.2 Risk-Neutral Probabilities 27
2.1.3 Risk-Neutral Valuation 28

2.2 The Multi-Period Binomial Model **30**
2.2.1 Portfolio and Arbitrage 31
2.2.2 Risk-Neutral Probabilities 32
2.2.3 Continuous Compounding 36

2.3 Option Pricing in the Binomial Model **37**

1. Introduction to Derivatives Market

1.1 Financial Markets

Basic components of a financial market:

- Money
- Assets (such as stock)
- Time
- People/organizations
- Uncertainty

Why is there a financial market?

$\Longrightarrow$ People have different financial needs

- preferences on timing
- perspectives on risks and uncertainty
- sets of information
- means of economic activities

$\Longrightarrow$ So they trade

1.1.1 Assets

In this course, an asset:

- has a current price
- its future price may be uncertain
- is tradeable

We use S_t or $S(t)$ to represent the price of an asset (stock) S at time t

- $\{S_t\}_{t\geq 0}$ (or S for short) is a **stochastic process**
- Typically S_0 represents the current ($t = 0$) price, which is **not random**
- $t \in [0, T]$ for some terminal time T

In this course, we aim to study this stochastic process S and some related quantities.

1.1.2 Review: Present Value of Future Payments

Note 1.1 — time-value of money.

A dollar today is worth more than a dollar tomorrow, because you can invest it and earn (non-negative) interest on it

The value at time $t < T$ of an amount $\$K$ in the future is $PV_t(K)$, the present value of K

Here, we assume that the (continuously compounded or annually effective) interest rate is non-negative, implying that $PV_t(K) \leq K$, for any $K \geq 0$ and $t < T$

Suppose the interest rate is r annually,

Definition 1.1 — Present value of future payment. If an asset (e.g. **zero-coupon bond**) pays K dollars in time T, then the time t $(T < t)$ value of the future payment is

$$PV_t(k) = \begin{cases} e^{-r(T-t)K} & \text{if continuously compounded} \\ \frac{K}{(1+r)^{T-t}} & \text{if annually effective} \end{cases}$$

Definition 1.2 — Risk-free asset. If the future payoff(s) of an asset is non-random, we call it **risk free**.

1.2 Derivative Securities

Definition 1.3 — Derivative. In finance, a **derivative** is a contract that derives its value from the performance of an **underlying entity**

- This underlying entity can be an asset, an index, interest rate, a basket of assets, or even another derivative
- The underlying entity is called "**the underlying asset**" or simply "the underlying"

Definition 1.4 — derivative security. A financial contract F is a **derivative security** or a **contingent claim**, whose value F_T at F_T at expiration date (maturity) T is "derived" exactly from the market price of more basic underlying primitive instruments up to and including time T.

Primitive Instruments (underlying)

- Stocks
- Currencies
- Interest rates
- Indices
- Commodities
- Bonds

Derivatives

- Futures & Forwards
- Options (Call, Puts, Caps, Floors, Bond Options, Swaptions...)
- Credit derivatives
- Swaps

Definition 1.5 — OTC and ETD.
Based on where they are traded, derivatives can be classified as **OTC (Over the Counter)** or **ETD (Exchange-traded derivatives)**
- OTC derivatives are private, tailored contracts between counterparties
- ETD's are more structured and standardized contracts where the underlying assets, the quantities and the mode of settlement are defined by an exchange house

- Being private contracts between two counterparties, OTC derivatives can be tailored and customized to suit exact risk and return needs
  - On the flip side, lack of a clearing house or exchange results in increased credit or default risk associated with each OTC contract
- Being transacted on an organized exchange, ETD transactions are governed by a set of specific terms. They are standardized and more transparent than OTC derivatives
  - Each party of an ETD contract is required to hold a margin at the clearing house to cover its unsettled positions and the clearing house will monitor this margin level to make sure that it covers outstanding trades
  - A margin is the amount of cash an investor must put up to open an account to start trading
  - Therefore, ETD's carry less credit risk than OTC derivatives in general

Some Terminology

- **Long Position**: When you buy something ...
- **Short Position**: When you sell something what you don't yet own ...
- **Model-Free**: Independent of specific assumptions (e.g. about stock price distribution, etc.) ...

Usage

Derivatives are used:
- to **manage risk** (risk-management/insurance tool)
  - e.g. a pension fund invested in a broad market index can use derivatives to obtain downside protection
  - e.g. an airline company can use derivatives to put a ceiling on the future price of jet fuel
- for **speculation**
  - e.g. for a given investment, the use of derivatives magnifies the financial consequences, i.e. we can obtain large exposures with relatively little capital
- as an important part of **compensation**
  - executive stock options

1.2.1 Assumptions on a Financial Market

(1) No transaction fee.
(2) No bid-ask spread.
(3) One can buy any amount/share of any security.
(4) One can trade at any time instantly.
(5) Buying or selling a security does not change its price.
(6) No default/credit risk.
(7) Allow naked short selling.
(8) No information difference between investors.

These assumptions are not very realistic but they help us to understand the fundamental issues of a financial market.

1.2.2 The Concept of (No) Arbitrage

Definition 1.6 — Arbitrage.
An **arbitrage opportunity** is a portfolio value process $\{V_t\}_{t\geq 0}$ such that
(1) $V_0 \leq 0$
(2) $P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$, for some time $T > 0$

In other words, an arbitrage opportunity is a portfolio that:
- costs nothing to hold, or you are paid to hold it
- generates non-negative payoff with probability 1, and positive payoffs with strictly positive probability

The Principle of No-Arbitrage
- There ain't no such thing as a free lunch
- An immediate consequence of no-arbitrage is the Law of One Price

Proposition 1.1 — Law of One Price.
In an arbitrage-free market, if two securities have exactly the same payoffs they must have the same price

Proposition 1.2
In a market, if there exists a portfolio value process $\{V_t\}_{t\geq 0}$ satisfying
(1) $V_0 \leq 0$
(2) $V_T \geq 0$ for some time $T > 0$
then there is an arbitrage opportunity in the market.

From now on, assume a market with no arbitrage in this chapter.

1.3 Forwards and Futures

Definition 1.7 — Forward.
A **forward contract** is a non-standardized agreement to buy or sell an asset at a certain future time T for a certain price K, known as the **delivery price** (or **forward price**)
- The delivery price is determined so that the value of the contract at initiation is zero

Terminology
- **Underlying asset**: The asset on which the forward contract is based
- **Expiration date**: The time at which the asset is delivered
- **Forward price**: The price the buyer will pay at the expiration date
  - This is not the price one party needs to pay the other at the initial time; there is no initial price associated with a forward contract!
- ∗ It is normally traded Over-the-Counter (OTC)
- ∗ The party that **agrees to buy the underlying** asset is said to have a **long position** in the forwards
- ∗ The party that **agrees to sell the underlying** asset is said to have a **short position** in the forwards
- ∗ At the time the contract is entered into, no exchange of money takes place
- ∗ A forward contract can be contrasted with a spot contract:
  - A *spot contract* is an agreement to buy or sell an asset today, **with immediate** cash exchange

– A *forward contract* is an agreement **with no immediate** cash exchange

1.3.1 Forward Contract

Forward Contract - Payoff

- S_t: The spot price of the underlying asset at time $t \geq 0$
- T: The expiration date
- K: The forward price (delivery price)
- Long position: the position of the buyer
- Short position: the position of the seller

Pay off to long forward $= S_T - K$
Pay off to short forward $= K - S_T$

Forward Contract - Pricing

Pricing a forward contract is model-free, using simple no-arbitrage arguments

- Suppose that a stock pays no dividend, the current stock price is S_0, and the risk-free rate is r per year continuously compounded
- Consider the following trading strategy:
  (1) Borrow S_0 at the risk-free rate for the period of T years, and buy one share
  (2) Short one forward contract on the stock with delivery price K expiring at T

The cash flows are:

	Cash flow at $t = 0$	Cash flow at $t = T$
Borrowing S_0	$+S_0$	$-S_0e^{rT}$
1 long share	$-S_0$	$+S_T$
1 short forward	0	$K - S_T$
Total	0	$K - S_0e^{rT}$

The principle of No-Arbitrage ("No free lunch") then implies that the cash flow at time T should be 0. Thus the forward price is:

$$K = S_0e^{rT}$$

Proposition 1.3
Let S denote the price process of a non-dividend-paying stock. For a forward contract F on S, issued at time t and having maturity T, the forward price K determined at t is given by

$$K = S_0e^{rT}$$

Proof. (Equivalent to the previous cash flow table)

- At time t, but F, sell S, and deposit S_t money
- At t you value is 0
- At T you have $S_T - K - S_T + S_te^{r(T-t)} = S_te^{r(T-t)} - k$
- This value is not random, and if it is not zero there is an arbitrage

■

Forward Contract - Pricing with Dividends

Dividends are the payments made by a security (e.g. stock of a corporation) to its shareholders. They can be discrete (paid at discrete time intervals) or continuous (paid continuously).

Discrete Dividends

Consider a forward on a stock St , which will pay a dividend of \$$c$ at time $t_1 \in [0, T]$, where T is the expiration date of the forward contract.
Consider the following two trading strategies:
(1) Borrow S_0 at the risk-free rate for the period of T years, and buy one share
(2) Short one forward contract on the stock with delivery price K expiring at T
The cash flows are:

	Cash flow at $t = 0$	Cash flow at $t = T$
Borrowing S_0	$+S_0$	$-S_0 e^{rT}$
1 long share	$-S_0$	$+S_T + ce^{r(T-t_1)}$
1 short forward	0	$K - S_T$
Total	0	$K - S_0 e^{rT} + ce^{r(T-t_1)}$

The principle of No-Arbitrage (“No free lunch") then implies that the cash flow at time T should be 0. Thus the forward price is:

$$K = S_0 e^{rT} - +ce^{r(T-t_1)}$$

Proposition 1.4
Let S denote the price process of a stock earning discrete dividends between time t and time T. For a forward contract F on S, issued at time t and having maturity T, the forward price K determined at t is given by

$$K = S_0 e^{rT} - \text{Accumulated value at time } T \text{ of all dividends}$$

Continuous Dividends

R When there is a continuous dividend paid by stock S in a constant rate δ, an investment of $S_t e^{-\delta(T-t)}$ in the stock at time t will yield 1 share of stock at time T (with price S_T)

Proposition 1.5
Let S denote the price of a stock earning a continuous dividend rate δ. For a forward contract F on S, issued at time t and having maturity T, the forward price K determined at t is given by

$$K = S_t e^{(r-\delta)(T-t)}$$

Proof. Consider a forward on a stock S_t, paying dividends continuously at a dividend yield of δ per annum. Consider the following two trading portfolios:

- Portfolio A:
  - At time t, enter into a forward contract to buy one share of the stock, with forward price \$$K$, maturing at time T
  - Simultaneously invest an amount \$$Ke^{-r(T-t)}$ the risk-free asset
  - At time T, the risk-free investment will accumulate to \$$K$; use this \$$K$ buy a share of stock via the forward contract.
- Portfolio B:
  - Buy $e^{-\delta(T-t)}$ shares of the stock, at the current price S_t. Reinvest dividend incomes in the stock S immediately when they are received.

The cash flows are:

Portfolio	Cash flow at t	Cash flow at T
A	$\$Ke^{-r(T-t)}$	S_T
B	$\$S_te^{-\delta(T-t)}$	S_T

Thus by the no-arbitrage principle, $\$Ke^{-r(T-t)} = \$S_te^{-\delta(T-t)}$, i.e. $\$K = S_te^{(r-\delta)(T-t)}$, when the underlying pays dividends continuously at a yield of δ per annum. ■

Continuous dividends are unusual but easy to calculate.

Prepaid Forward Contracts

Definition 1.8 — prepaid forward.
A **prepaid forward** is a forward contract which calls for payment today and delivery of the underlying asset at a future date.

In a similar fashion, an application of the no-arbitrage principle and the replication strategy yields the prepaid forward price K_0 as follows:

- No dividend: $K_0 = S_0$
- A discrete dividend of $\$c$ at time t: $K_0 = S_0 - ce^{-et}$
- Continuous dividend at a yield rate of δ: $K_0 = S_0e^{-\delta T}$

1.3.2 Futures Contract

Definition 1.9 — futures contract.
Like a forward contract, a futures contract is a costless-to-enter agreement between two parties to exchange an asset at a certain future time for a certain delivery price

However, contrary to forwards that are mainly OTC contracts, futures are ETD, hence there are various structural differences.

Futures vs. Forwards

Forward	Futures
OTC (private between 2 parties)	Exchange-traded Contract
Not Standardised	Standardised to the Exchange Rules
Settled at maturity T	Daily Settlements/Margins ("marked-to-market")
Counterparty/Credit/Default Risk	No Risk (except for the risk to meet a margin call)

Since futures contracts are marked-to-market, every day any profits or losses on the contract are calculated and traders have to cover up any losses or receive any profits in their **margin account**.
Other differences between forwards and futures:

- Futures:
  - Standard ETD
  - Ignorable default risk
  - Usually closed before maturity so delivery usually never happens
- Forwards
  - OTC derivatives
  - Substantially high default probability
  - Delivery usually happens

1.4 Options

Definition 1.10 — option.
An **option** is a contract which gives the buyer the right, but not the obligation, to buy or sell an underlying asset at a specified strike price, on or before expiration

Terminology

- **Underlying asset**: the asset on which the option is based
- **Expiration date**: the date by which the option must either be exercised or it becomes worthless
- **Exercise**: the action of carrying out the transaction specified by the option
- **Strike price**: the price for the asset at which exercise can occur

There are three common exercise styles for options:

- European-style: The option can only be exercised at maturity
- American-style: The option can be exercised at any time at or before maturity
- Bermudan-style: The option can only be exercised on a set of specified dates at or before maturity

1.4.1 Put and Call Options

Definition 1.11 — call option.
A **call option** gives its owner the right, but not the obligation, to buy the underlying asset at a specified exercise or strike price K on or before a specified exercise date T.
$\Longrightarrow$ The payoff at time T is $\max(S_T - K, 0)$

Definition 1.12 — put option.
A **put option** gives its owner the right, but not the obligation, to sell the underlying asset at a specified exercise or strike price K on or before a specified exercise date T.
$\Longrightarrow$ The payoff at time T is $\max(K - S_T, 0)$

Definition 1.13 — European & American Feature.
An option that can be exercised **only on** one particular day T is conventionally known as a European option.
If the option can be exercised **on or at any time before** day T, then it is known as an American option

Which is more expensive and why?
American options have a higher price than European options with the same characteristics (see later)

1.4.2 Moneyness

Definition 1.14 — Moneyness.

- **In-The-Money (ITM)**: an option is in the money if exercising the option immediately leads to a positive cash flow to the holder
- **At-The-Money (ATM)**: an option is at the money if exercising the option immediately leads to zero cash flow to the holder: "priced at-the-money"
- **Out-of-The-Money (OTM)**: an option is out of the money if exercising the option immediately leads to a negative cash flow to the holder

For call and put options, moneyness is related to the difference between K and S:

	$S < K$	$S = K$	$S > K$
Call	OTM	ATM	ITM
Put	ITM	ATM	OTM

1.4.3 Payoff Diagrams

Payoff Diagrams - Long Side

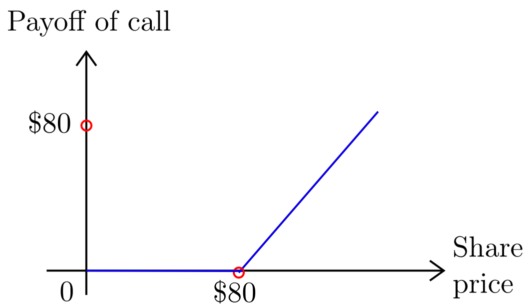
■ **Example 1.1** Consider a long European call and a long European put with:

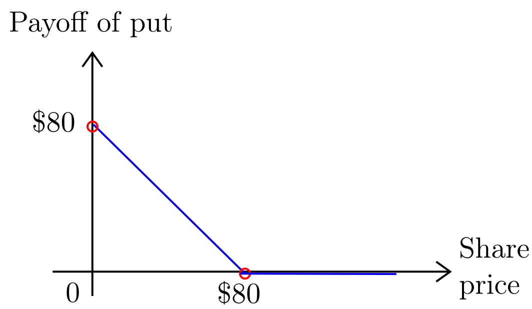
- Same underlying S
- Same strike $K = \$80$
- Same maturity T

The payoffs for the holders are:

$$\text{Call payoff} = \max(S_T - K, 0) = \max(S_T - 80, 0)$$
$$\text{Put payoff} = \max(K - S_T, 0) = \max(80 - S_T, 0)$$

Graphically, these payoffs are as follows:

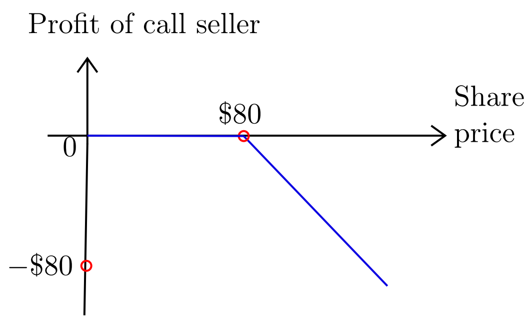


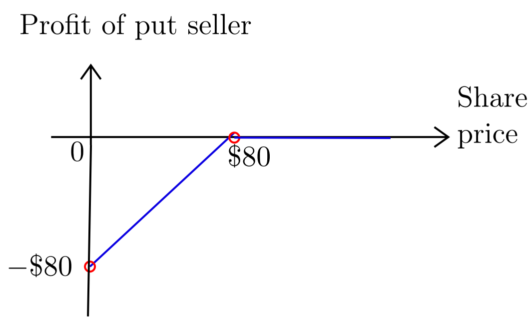


- A long call has infinite potential gain
- A long put has insurance-type features: it pays off when the firm goes bankrupt

Payoff Diagrams - Short Side

■ **Example 1.2** Consider the previous example but suppose that we are now short both options:



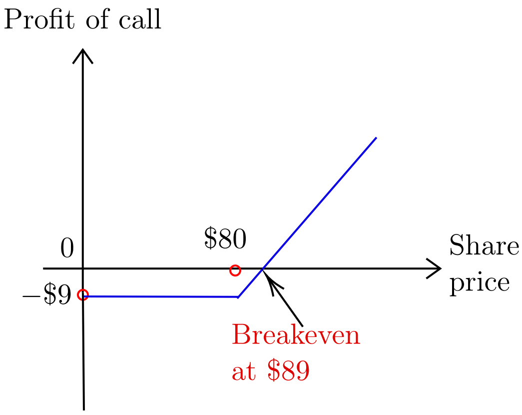


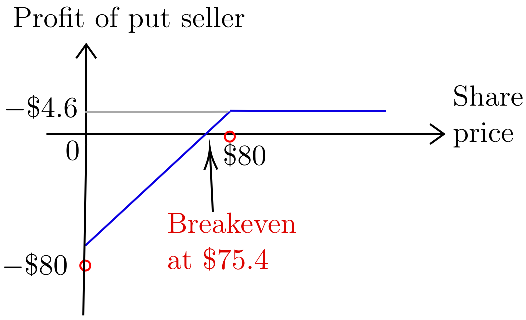
- A short call faces a potentially **infinite loss** (like a short position of a stock)

1.4.4 Profit Diagrams

Profit diagrams incorporate the costs of buying an option or the proceeds from selling one.

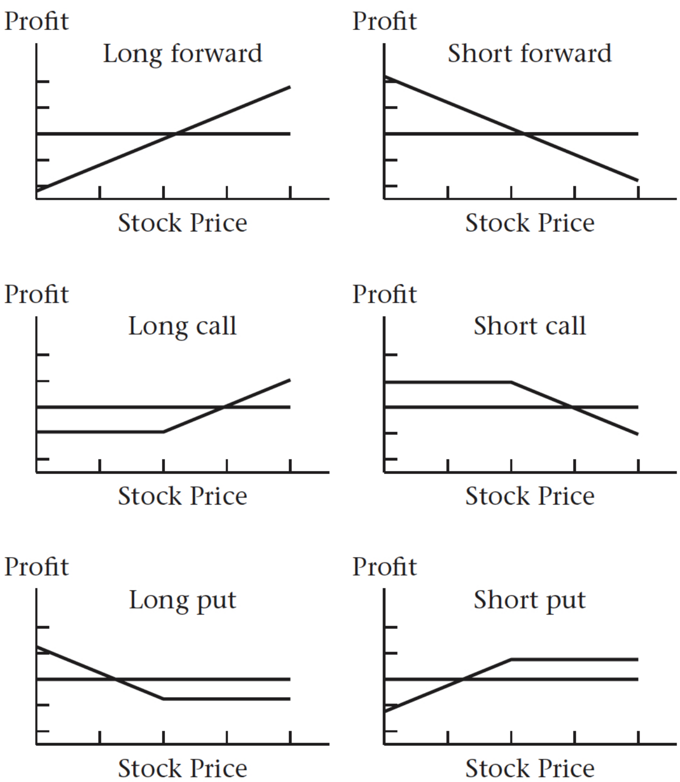
■ **Example 1.3** The investor purchased a call with strike price of \$80 at \$9 (assuming the interest rate is 0), while in the right panel, the investor sold a put option with strike of \$80 for \$4.60





The break-even price is always in the ITM region of the option

Summary



1.4.5 Forwards (Futures) and Options

Similarities

- Both are derivative
- Bothe have an expiration date and a strike price

Differences

	Forward	Option
Payoff Type	Only one	Various
Exercise	Obligation	Right but not obligation
Price	Usually zero	Positive

1.4.6 Intrinsic and Time Value of an Option

Definition 1.15 — Intrinsic Value.
The **intrinsic value** of an (American) option is defined as the payoff that could be obtained by immediate exercise of the option at time $t < T$

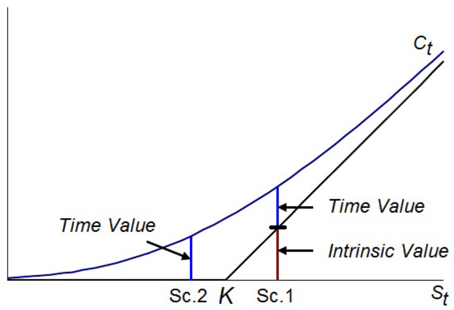
■ **Example 1.4** An American call option has intrinsic value at any time t equal to $\max(S_t - K, 0)$

Definition 1.16 — Time Value.
The **time value** of an option at any time $t < T$ is defined as the difference between the actual option price at t and its intrinsic value at t

■ **Example 1.5** An American call option C has intrinsic value at any time t equal to $C_t - \max(S_t - K, 0)$

Note 1.2
- The intrinsic value and the time value of an option are key quantities to consider when deciding on whether or not to exercise an American option early
- If/when time value is 0, one may choose to exercise immediately

■ **Example 1.6** Let's look at a long position in an American call option



We notice that
- The intrinsic value is positive when the option is ITM
- When intrinsic value = 0, the option may be selling for a positive price, because there is (almost) always positive probability that it will end up ITM at T

  ■ **Example 1.7** Suppose that a call is OTM:
  - If you exercise, you get nothing. It can't get any worse than that!
  - If the stock price rebounds, however, and exceeds the strike by expiration, we may end up with a positive payoff

1.5 Bounds on Option Prices

1.5.1 No-Arbitrage Bounds on Option Prices

- Computing option prices requires making assumptions about the evolution of the underlying asset (i.e, a model)
- However, no-arbitrage arguments can impose model-free price bounds
- Trivially, option payoffs are non-negative hence they must have **non-negative prices** as well. Can we derive sharper bounds?
- Assume for now non-dividend paying stocks as the underlying assets

  - Since most stocks pay dividends only once a year and most exchange-traded options are written with less than one-year time to expiration, the assumption of no dividends will actually be true for many real-world options

1.5.2 Bounds on Non-Dividend Paying Stock

Lower Bound on American Options

- European options:
  - Call: $c(S, K, t, T) = c_t$
  - Put: $p(S, K, t, T) = p_t$
- American options:
  - Call: $C(S, K, t, T) = C_t$
  - Put: $P(S, K, t, T) = P_t$

Trading Strategy and Portfolio

- A **portfolio** is a collection of securities
  - Under our market assumptions, you can short any portfolio
- A **trading strategy** is the dynamic organization of a portfolio, including buying, selling securities or exercising derivatives Holding a portfolio is a trading strategy.
  - You may not short a trading strategy

We use π for a trading strategy (or its corresponding portfolio) and π_t for its time t value (sometimes we use $\pi(t)$)

- As previously mentioned, since an American option can be exercised at any time, it must always be at least as valuable as an otherwise identical European option:

Proposition 1.6 — European vs. American options.
$C(S, K, t, T) \geq c(S, K, t, T)$ and $P(S, K, t, T) \geq p(S, K, t, T)$

Lower Bound on a European Call Option

Consider the following trading strategies at time $t = 0$

(1) Buy 1 European call option on a non-dividend paying stock with a strike price of K, expiring at time T

(2) Buy 1 share of the underlying stock and borrow at the risk-free rate the amount $PV_0(K) = Ke^{-rT}$ or $\frac{K}{(1+r)^T}$

The cash flows of there strategies are:

	Cash flow at $t = 0$	Cash flow at $t = T$	
		$S_T < K$	$S_T \geq K$
Strategy 1	$-c_0$	0	$S_T - K$
Strategy 2	$PV_0(K) - S_0$	$S_T - K$	$S_T - K$

Thus, no matter what happens in the future, the cash flow of Strategy 1 is always greater than or equal to the cash flow of Strategy 2. Thus:

$$c_0 \geq S_0 - PV_0(K) \geq S_0 - K$$

More generally, we have the following result (assume continuously compounded interest rate; recall that $r \geq 0$):

Proposition 1.7
At time $t \geq 0$, we have $C_t \geq c_t \geq S_t - Ke^{-r(T-t)} = S_T - PV_t(K) \geq S_t - K$

Early Exercise of an American Call

Assume no dividends; does it make sense to exercise an American call early?

- At any time $t < T$, there are two scenarios
  (1) Exercise the American call early:
  $$\text{Payoff}_1(t) = \text{Intrinsic Value}(t) = S_t - K$$
  (2) Sell the call instead of exercising it:
  $$\text{Payoff}_2(t) = C_t \geq c_t \geq c_t \geq S_t - PV_t(K) \geq S_t - K$$

$\Longrightarrow$ Clearly, we are better off selling the option since $S_t - PV_t(K) \geq S_t - K$
$\Longrightarrow$ An American call on a non-dividend paying stock should never be exercised early. Hence, with no dividends: $C_t = c_t$

Proposition 1.8 — American call vs. European call.
If S does not pay dividends in $[t, T]$, then $c(S, K, t, T) = C(S, K, t, T)$

R The above is NOT true if the underlying stock pays a dividend during the life of the option!

- If the underlying pays a dividend between t and T, we have: $C_t \geq c_t$
- Trivially, $0 \leq c_t \leq C_t \leq S_t$
  - An option to buy an asset cannot cost more than the asset itself
- Combining all the above bounds, both American and European calls on a non-dividend paying stock must satisfy the following:

Proposition 1.9
At time $t \geq 0$, $S_t \geq C_t = c_t \geq \max(S_t - PV_t(K), 0) \geq \max(S_t - K, 0) \geq 0$

Lower Bound on a European Put Option

Consider the following trading strategies at time $t = 0$:

(1) Buy 1 European put option on a non-dividend paying stock with a strike price of K, expiring at time T and 1 share of the underlying stock
(2) Deposit the amount of $PV_0(K) = Ke^{-rT}$ (or $\frac{K}{(1+r)^T}$) into your risk-free savings account

The cash flows of there strategies are:

	Cash flow at $t = 0$	Cash flow at $t = T$	
		$S_T < K$	$S_T \geq K$
Strategy 1	$-p_0 - S_0$	K	S_T
Strategy 2	$PV_0(K)$	K	K

Thus, no matter what happens in the future, the cash flow of Strategy 1 is always greater than or equal to the cash flow of Strategy 2. Thus:

$$p_0 \geq PV_0(K) - S_0$$

More generally, we have the following result:

Proposition 1.10
At time $t \geq 0$, $P_t \geq p_t \geq Ke^{-r(T-t)} - S_t = PV_t(K) - S_t$

Early Exercise of an American Put

Unlike for call options (remember the effect of dividends), the optimality of early exercise of an American put option is always a possibility. Hence, for all $t \in [0, T]$:

$$P_t \geq p_t$$

■ **Example 1.8 — An extreme scenario.**

- Suppose you hold a put option on the stock of a company that goes bankrupt before expiration
- The value of the stock is zero and there is no possibility for it to rebound! The company is dead
- An American put allows immediate exercise, hence a payoff of K
  - Putting K into a savings account for the period remaining to expiration results you will have $Ke^{r\tau}$ at maturity, where $\tau = T - t$ is the time to expiration
- A European put would only pay K at expiration (...cash delivery of course)
  - Clearly, you are better off having the American put. Therefore, $P_t > p_t$

Bounds of Put Option on Non-Dividend Paying Stock

- The American put must satisfy $0 \leq P_t \leq K$
  - An option to sell at **any time** an asset for K cannot cost more than K
- Similarly, the European put must satisfy $0 \leq p_t \leq PV_t(K)$
  - An option to sell at time T an asset for K cannot cost at time t more than $PV_t(K)$

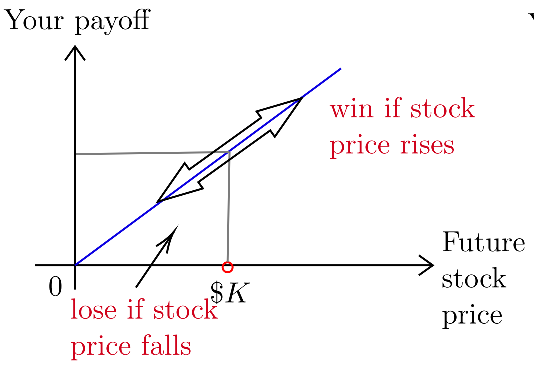
Combining the above bounds, we obtain:

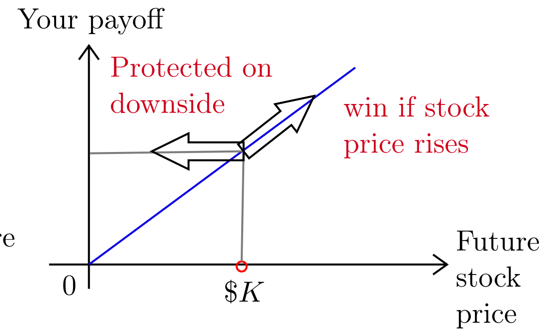
- American put: $K \geq P_t \geq \max(K - S_t, 0) \geq 0$
- European put: $PV_t(K) \geq p_t \geq \max(PV_t(K) - S_t, 0) \geq 0$

1.5.3 Put-Call Parity

Downside Protection

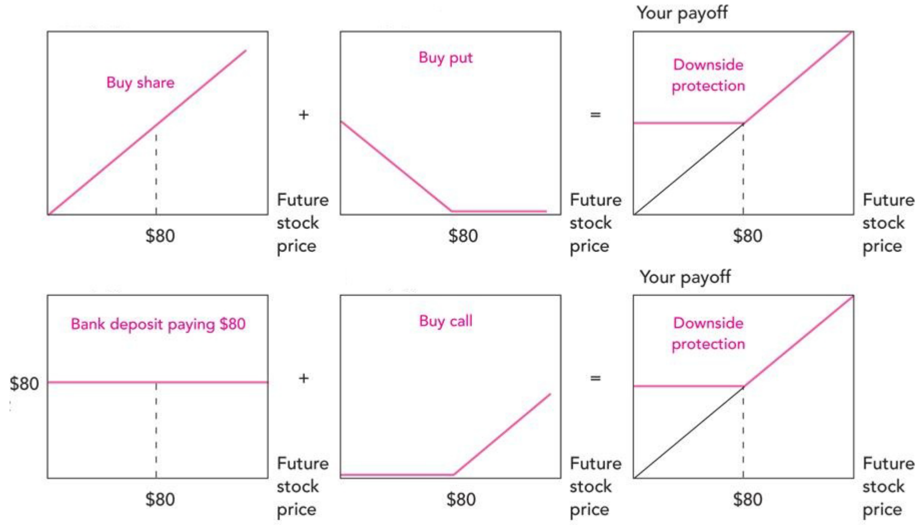
- Investing into a stock is risky because the stock price might fall
- Suppose we want to put a limit on the maximum possible loss
- Buy a **put option** on the stock, as it has insurance-type features
- No matter what happens in the future, the value of your investment cannot fall below the strike price of the put
- Such put options are called **protective puts** and are very popular risk management tools with institutional investors such as mutual and pension funds





But one could create the same payoff by lending and buying a call option

Two ways of creating Downside Protection



R The two portfolios have the same payoff! Law of One Price must apply!

Put-Call Parity for European Options

There are two ways to achieve downside protection:

(1) Buy 1 share and 1 European put on a non-dividend paying stock with strike K

(2) Deposit the present value of K in a risk-free savings account and buy 1 European call on the same stock with the same strike K

The cash flows of there strategies are:

	Cash flow at $t=0$	Cash flow at $t=T$	
		$S_T < K$	$S_T \geq K$
Strategy 1	$-p_0 - S_0$	K	S_T
Strategy 2	$PV_0(K) - c_0$	K	K

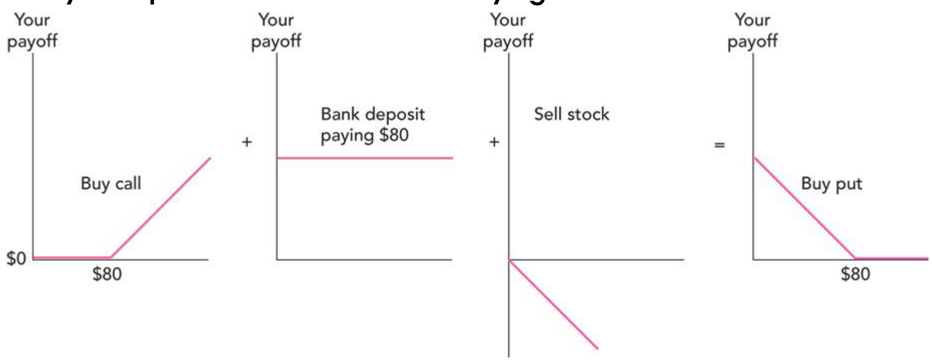
By the Law of One Price 1.1:

$$c_0 + PV_0(K) = S_0 + p_0$$

More generally, we have the following result:

Proposition 1.11 — Put-Call Parity.
At time $t \leq T$, $c_t + PV_t(K) = S_t + p_t$

Put-Call Parity for Options on Non-Dividend Paying Stock



Extending put-call parity to American options, we get:

- European: $c_t + PV_t(K) = S_t + p_t$
- American: $S_t - K \leq C_t - P_t \leq S_t - PV_t(K)$

R Put-Call Parity is a model-free result!

Violation of Put-Call Parity - Arbitrage

If the parity relation is ever violated, an arbitrage opportunity arises

■ **Example 1.9** Suppose you collect these data for a certain stock and European options written on it:

Stock Price	\$110
Call Price ($T = 1Y, K = \$105$)	\$17
Put Price ($T = 1Y, K = \$105$)	\$5
Risk-free Rate	5% per year

Clearly

$$c_0 + PV_0(K) =^? S_0 + p_0$$
$$17 + 105/1.05 =^? 110 + 5$$
$$117 > 115$$

Hence, the protective put strategy is cheaper than the call plus deposit of $PV_0(K)$. What will you do?
How to benefit from the arbitrage? - Buy the cheap, sell the expensive (buy low, sell high)

(1) Buy the cheap: buy the stock, buy the put
(2) Sell the expensive: write (sell) the call and borrow \$100 for one year

The cash flows of this strategy:

Position	Cash flow at $t = 0$	Cash flow at $t = T$	
		$S_T < K$	$S_T \geq K$
Buy Stock	-110	S_T	S_T
Borrow $\$105/1.05 = \$100$	+100	-105	-105
Sell Call	+17	0	-(S_T-105)
Buy Put	-5	105-S_T	0
Total	+2	0	0

- Immediate profit of \$2! Fully covered at time T
- In the appearance of an arbitrage opportunity, arbitrageurs will step in and the buying and selling pressure will restore the parity

Put-Call with Dividends

Proposition 1.12 — Put-Call parity with dividend.

$$c_t - p_t = S(t) - PV_t(D) - PV_t(K)$$

Proposition 1.13 — Put-Call parity with continuous dividend.

$$c_t - p_t = S(t)e^{-\delta(T-t)} - PV_t(K)$$

Factors Affecting Option Prices

option price effect from a change in one variable, while keeping the rest **fixed**:

Factor	Call		Put	
	European	American	European	American
Stock Price (S_t) $\uparrow$	$\uparrow$		$\downarrow$	
Strike Price (K) $\uparrow$	$\downarrow$		$\uparrow$	
Maturity (T) $\uparrow$	unknown ($\uparrow$ if no dividend)	$\uparrow$	unknown	$\uparrow$
Time to Maturity ($T-t$) $\downarrow$	unknown ($\downarrow$ if no dividend)	$\uparrow$	unknown	$\uparrow$
Stock Volatility (σ_t) $\uparrow$	$\uparrow$		$\uparrow$	
Risk-Free Interest Rate (r_t) $\uparrow$	$\uparrow$		$\downarrow$	

1.6 Mathematical Properties of Option Prices

1.6.1 Review of Functions

A function $f : D \to \mathbb{R}$ is

- **Continuous** if
$$\lim_{x\to a} f(x) = f(a),\ \forall a \in D$$
- **Lipschitz continuous** if
$$|f(x) - f(y)| < C|x-y|,\ \forall x, y \in D, \exists C \text{ constant}$$
- **Increasing** if
$$f(x) - f(y) \geq 0,\ \forall x \geq y$$
- **Decreasing** if
$$f(x) - f(y) \leq 0,\ \forall x \geq y$$
- **Convex** if
$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y),\ x, y \in D, \lambda \in [0,1]$$
- **Concave** if
$$\lambda f(x) + (1-\lambda) f(y) \leq f(\lambda x + (1-\lambda) y),\ x, y \in D, \lambda \in [0,1]$$

A twice-differentiable function $f : D \to \mathbb{R}$ is

- **Lipschitz continuous** if
$$|f'(x)| < C,\ x \in D, C \text{ constant}$$
- **Increasing** is
$$f'(x) \geq 0,\ x \in D$$
- **Decreasing** is
$$f'(x) \leq 0,\ x \in D$$
- **Convex** is
$$f''(x) \geq 0,\ x \in D$$
- **Concave** is
$$f''(x) \leq 0,\ x \in D$$

1.6.2 Properties of Option Prices

Suppose the stock S pays no dividend over the period $[t, T]$

Proposition 1.14 — Inequality set 1.

$$S_t \geq C(S,K,t,T) \geq c(S,K,t,T) \geq S_t - PV(K) \geq S_t - K$$
$$K \geq P(S,K,t,T) \geq p(S,K,t,T) \geq PV(K) - S_t$$

Proposition 1.15 — Inequality set 2A.
For $0 \leq K_1 \leq K_2$

$$0 \leq c(S,K_1,t,T) - c(S,K_2,t,T) \leq K_2 - K_1$$

$$0 \leq C(S,K_1,t,T) - C(S,K_2,t,T) \leq K_2 - K_1$$

That is $c(S,K,t,T)$ and $C(S,K,t,T)$ are decreasing functions of K and are (Lipschitz) continuous on $\mathbb{R}^+$

Proof. We build up trading strategies and use a non-arbitrage argument to show these inequalities. Write $c_t(K) = c(S,K,t,T)$ for short, and similarly for the other quantities.

- To show $c_t(K_1) - c_t(K_2) \geq 0$: build a portfolio π which is to long 1 unite of $c(K_1)$ and short 1 unit of $c(K_2)$ at time t. At time T, the payoff is

$$\pi_T = (S_T - K_1)_+ - (S_T - K_2)_+ = \begin{cases} K_2 - K_1 & S_T \geq K_2 \\ S_T - K_1 & K_1 \leq S_T \leq K_2 \\ 0 & S_T < K_1 \end{cases}$$

  So $\pi_T \geq 0$, we must have $\pi_t \geq 0$
- To show $C_t(K_1) - C_t(K_2) \geq 0$: build a portfolio π which is to long 1 unite of $C(K_1)$ and short 1 unit of $C(K_2)$ at time t. If $C(K_2)$ is not exercised by the counter-party, then $\pi_T = (S_T - K_1)_+ \geq 0$. If $C(K_2)$ is exercised at t_0, then we exercise $C(K_1)$ at t_0, and

$$\pi_{t_0} = (S_{t_0} - K_1)_+ - (S_{t_0} - K_2)_+ \geq 0$$

  We hold this amount of money till T, and $\pi_T \geq 0$. So no matter what, $\pi_T \geq 0$, we must have $\pi_t \geq 0$
- To show $c_t(K_1) - c_t(K_2) \leq K_2 - K_1$: build a portfolio π which is to hold the cash amount of $K_2 - K_1$, short 1 unit of $c(K_1)$, and long 1 unit of $c(K_2)$ at time t. At time T, the payoff is

$$\pi_T = e^{r(T-t)}(K_2 - K_1) - ((S_T - K_1)_+ - (S_T - K_2)_+) \geq 0$$

  by (1), SO $\pi_T \geq 0$, we must have $\pi_t \geq 0$
- To show $C_t(K_1) - C_t(K_2) \leq K_2 - K_1$: build a portfolio π which is to hold the cash amount of $K_2 - K_1$, short 1 unit of $c(K_1)$, and long 1 unit of $c(K_2)$ at time t. Suppose that at some time t_0, the counter-party (which holds $C(k_1)$) exercises her call option. We exercise $C(K_2)$ immediately. The value of the portfolio at t_0 is

$$\pi_{t_0} = e^{r(t_0-t)}(K_2 - K_1) - ((S_{t_0} - K_1)_+ - (S_{t_0} - K_2)_+) \geq 0$$

  by (1), SO $\pi(t_0) \geq 0$. We hold this amount of money till T, hence $\pi_T \geq 0$. We must have $\pi_t \geq 0$

For $K_1 \leq K_2$,

- $c(K_1) - c(K_2) \geq 0 \iff c$ decreasing in $K \iff$ a call bull spread (see later) has a positive price.
- $c(K_1) - c(K_2) \leq K_2 - K_1 \iff c$ is (Lipschitz) continuous in $K \iff$ a call bull spread has a profit less than $K_2 - K_1$

■

Proposition 1.16 — Inequality set 2B.
For $0 \leq K_1 \leq K_2$

$$0 \leq p(S, K_2, t, T) - p(S, K_1, t, T) \leq K_2 - K_1$$

$$0 \leq P(S, K_2, t, T) - P(S, K_1, t, T) \leq K_2 - K_1$$

That is $p(S, K, t, T)$ and $P(S, K, t, T)$ are increasing functions of K and are (Lipschitz) continuous on $\mathbb{R}^+$

Proposition 1.17 — Inequality set 3A (Convexity Properties).
For $K_1, K_2 \geq 0, \lambda \in [0, 1]$

$$\lambda c(S, K_1, t, T) + (1 - \lambda)c(S, K_2, t, T) \geq c(S, \lambda K_1 + (1 - \lambda)K_2, t, T)$$

$$\lambda C(S, K_1, t, T) + (1 - \lambda)C(S, K_2, t, T) \geq C(S, \lambda K_1 + (1 - \lambda)K_2, t, T)$$

That is $c(S, K, t, T)$ and $C(S, K, t, T)$ are convex functions of K on $\mathbb{R}^+$

Proof. WLOG assume $K_1 \leq K_2$, and write $K = \lambda K_1 + (1 - \lambda)K_2$

- To show $\lambda c_t(K_1) + (1 - \lambda)c_t(K_2) \geq c_t(K)$: build a portfolio $\pi = \lambda c(K_1) + (1 - \lambda)c(K_2) - c(K)$ at time t. At time T, the payoff is

$$\pi_T = \lambda(S_T - K_1)_+ + (1 - \lambda)(S_T - K_2)_+ - (S_T - K)_+$$
$$= \begin{cases} 0, & S_T \geq K_2 \\ \lambda(S_T - K_1) - (S_t - K), & K \leq S_T < K_2 \\ \lambda(S_T - K_1), & K_1 \leq S_T < K \\ 0, & S_T < K_1 \end{cases}$$

  Note that in the second case,

$$\lambda(S_T - K_1) - (S_t - K) = (1 - \lambda)(K_2 - S_T) \geq 0$$

  So $\pi_T \geq 0$ and we must have $\pi_t \geq 0$.
- To show $\lambda C_t(K_1) + (1 - \lambda)C_t(K_2) \geq C_t(K)$: similar

$$\lambda c(S, K_1, t, T) + (1 - \lambda)c(S, K_2, t, T) \geq c(S, \lambda K_1 + (1 - \lambda)K_2, t, T)$$
$$\iff c \text{ is convex}$$

This also implies (by choosing $\lambda = \frac{1}{2}$)

$$c(S, K_1, t, T) + c(S, K_2, t, T) \geq 2c(S, K, t, T)$$

■

■ **Example 1.10** Suppose we observe 3 call option prices today on the same stock and with same maturity: $c(50) = 14, c(59) = 8.9$ and $c(65) = 5$. How do we undertake arbitrage?

Solution. Observe that $K_1 = 50, K_2 = 65$. Note $0.4K_1 + 0.6K_2 = 59$. Hence the call option should satisfy $0.6c(65)+0.4c(50) \geq c(59)$. However $0.6c(65)+0.4c(50) = 0.6\times 5+0.4\times 14 = 8.6 < c(59)$. $c(59)$ is overpriced (or the other two are underpriced), this is an arbitrage.

(1) Buy 6 units $c(65)$
(2) Buy 4 units $c(50)$
(3) Sell 10 units $c(59)$

The initial value of the portfolio is \$3 and this portfolio has a terminal payoff ≥ 0 ■

Proposition 1.18 — Inequality set 3B (Convexity Properties).
For $K_1, K_2 \geq 0, \lambda \in [0,1]$

$$\lambda p(S,K_1,t,T) + (1-\lambda)p(S,K_2,t,T) \geq p(S,\lambda K_1 + (1-\lambda)K_2,t,T)$$

$$\lambda P(S,K_1,t,T) + (1-\lambda)P(S,K_2,t,T) \geq P(S,\lambda K_1 + (1-\lambda)K_2,t,T)$$

That is $p(S,K,t,T)$ and $P(S,K,t,T)$ are convex functions of K on $\mathbb{R}^+$

Proposition 1.19 — Inequality set 4.
If $T_1 \geq T_2 > t$, then

$$C(S,K,t,T_1) \geq C(S,K,t,T_2)$$
$$P(S,K,t,T_1) \geq P(S,K,t,T_2)$$

Further, if the stock S does not pay dividends, then

$$c(S,K,t,T_1) \geq c(S,K,t,T_2)$$

Proof.

- The first part is intuitive.
- The second part is due to the fact $c = C$ when no dividend is paid

■

Note 1.3 The previous inequality does not hold for European puts:

$$p(S,K,t,T_1)\ vs.\ p(S,K,t,T_2)$$

$\implies$ It is not obvious which one has a larger price

1.7 Investment Strategies Using Options

1.7.1 Common Investment Strategies Using Options

Definition 1.17 — spread.
A **spread** is a position consisting of only calls or only puts
Depending on the combination, there are various spreads:

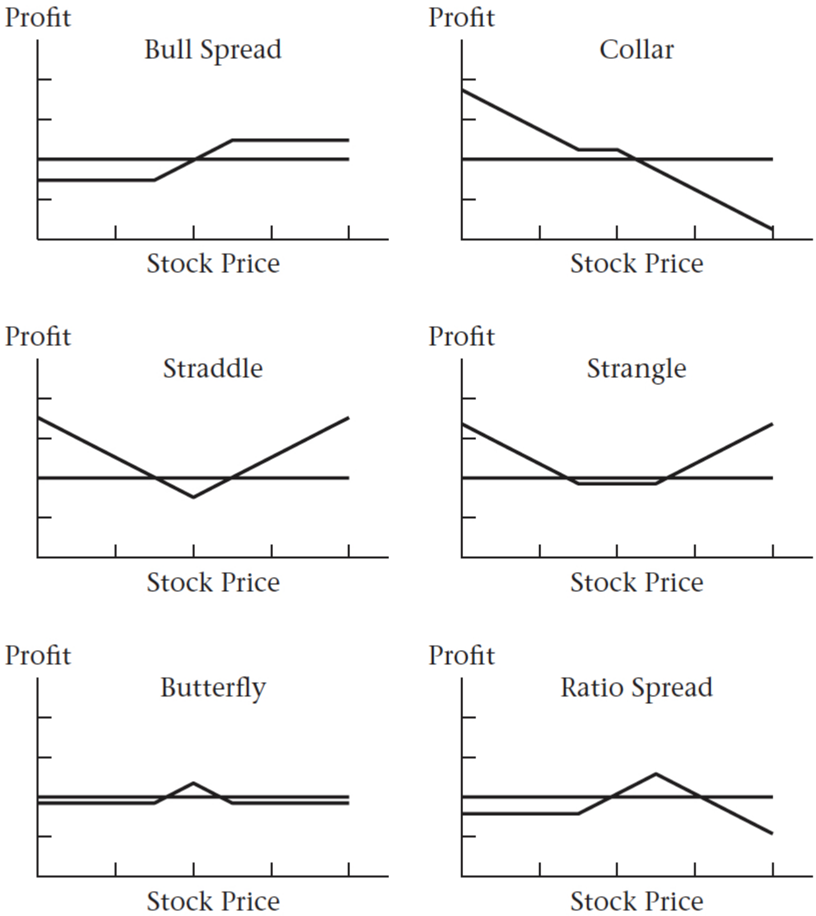
- **Bull spread**: long a call and short another call with higher strike (to bet on up movement)
- **Bear spread**: long a call and short another call with lower strike (to bet on down movement)
- **Ratio spread**: long m call options and short n call options at a different strike

Spreads can also be constructed using puts in a similar fashion!

Definition 1.18 — collar.
A **collar** is a position consisting of a long put and a short call with higher strike
Collar width: the difference between the strikes

Definition 1.19 — straddle, strangle, butterfly.

- **Straddle**: long an at-the-money call and an at-the-money put with the same strike
- **Strangle**: long an out-of-the-money call and an out-of-the-money put
- **Butterfly**: short a straddle and long a strangle



Use of straddle, strangle, and butterfly:

- Long straddle: speculation on high volatility
- Long strangle: speculation on high volatility with lower costs
- Long butterfly: speculation on low volatility

1.8 Portfolio Insurance Strategies Using Options

1.8.1 Four Insurance Strategies

- **Floor**: long a stock and long a put

$$Payoff = S_T + (K - S_T)_+$$

- **Covered call writing**: long a stock and short (write) a call

$$Payoff = S_T - (K - S_T)_+$$

- **Cap**: short a stock and long a call

$$Payoff = -S_T + (K - S_T)_+$$

- **Covered put writing**: short a stock and short (write) a put

$$Payoff = -S_T - (K - S_T)_+$$

Different positions can have the same payoff because of the identity:

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K$$

or more briefly, call - put = stock - bond

2. Discrete Time Models

Remember our aim in this course is to **price derivatives**.
To do so, we need a model of stock prices.

2.1 One-Period Binomial Model

Model Description

A stock model needs at least two basic components

(1) Something to represent randomness
(2) Something to represent the time-value of money

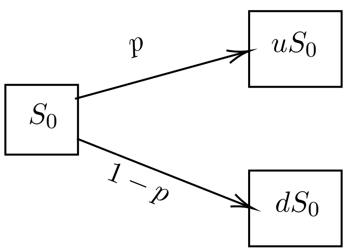
■ **Example 2.1**

- Only one period of time, therefore only two dates: $t = 0$ (now) and $t = 1$ (end of period)
- We have a stock with price S_0 today and S_1 at $t = 1$.

During the period, the stock price can either

- go up to $S_1 = uS_0$ with probability $p \in (0, 1)$
- go down to $S_1 = dS_0$ with probability $1 - p$

We assume $0 < d < u$.



We also have a risk-free (e.g. a default bond) with price B_0 today and B_1 at $t = 1$. The risk free rate (effective **per period**) is constant and equal to $r > 0$. B_0 is given (observable) and $B_1 = B_0(1 + r)$, with probability 1. We usually take $B_0 = 1$.

$$B_0 \xrightarrow{1} \boxed{B_1 = (1+r)}$$

2.1.1 Portfolio and Arbitrage

Definition 2.1 — portfolio.
In the previous model, a **portfolio** is a vector $\theta = (x, y)$, where
- $x =$ number of bonds held
- $y =$ units of stock held

Note 2.1
- x and y can be fractional
- x or y can be negative (short-selling)

Definition 2.2 — value process.
The **value process** of the portfolio $\theta = (x, y)$ is

$$V_t^\theta = xB_t + yS_t, \ t = 0, 1$$

- $V_0^\theta = x + yS_0$ is constant (S_0 is known)
- $V_0^\theta = x(1+r) + yS_1$ is a random variable

Definition 2.3 — arbitrage opportunity.
An **arbitrage opportunity** is a portfolio $\theta = (x, y)$ such that
- $V_0^\theta \leq 0$
- $V_1^\theta \geq 0$ and $\mathbb{P}(V_1^\theta > 0) > 0$

- An arbitrage opportunity is a deterministic money-making machine
- A fundamental assumption is that in a well-functioning market there are no arbitrage opportunity. (No Free Lunch!)

Proposition 2.1
The above one-period binomial model is arbitrage-free if and only if $d < 1 + r < u$

Proof. $\Rightarrow$: If the stock always outperform the risk free asset, then there is no point to buy risk-free assets, vice verse.
Suppose $1 + r \geq u$, construct $\theta = (S_0, -1)$.

$$V_0^\theta = S_0 - 1 \times S_0 = 0$$

$$V_1^\theta = S_0(1+r) - S_1 - \begin{cases} S_0(1+r-u) & \text{if } S_1 = uS_0 \\ S_0(1+r-d) & \text{if } S_1 = dS_0 \end{cases}$$

Then $\mathbb{P}(V_1^\theta > 0), V_1^\theta \geq 0$, arbitrage.

$\Leftarrow$: Suppose $d < 1 + r < u$. By contradiction, assume that there exists an arbitrage opportunity $\theta^* = (x^*, y^*) \neq (0, 0)$

$$\begin{cases} V_0^{\theta^*} = x^*B_0 + y^*S_0 = x^* + y^*S_0 \leq 0 \\ V_1^{\theta^*} = x^*B_1 + y^*S_1 = x^*(1+r) + y^*S_1 \geq 0 \end{cases}$$

Multiply $(1+r)$ to the first equation we get: $x^*(1+r) + y^*S_0(1+r) \leq 0$

$$\implies y^*S_1 \geq y^*S_0(1+r)$$

$$\implies \begin{cases} y^*uS_0 \geq y^*S_0(1+r) \\ y^*dS_0 \geq y^*S_0(1+r) \end{cases}$$

(1) If $y^* > 0$. $d \geq (1+r)$, contradiction
(2) If $y^* < 0$. $u \leq (1+r)$, contradiction
(3) If $y^* = 0$, then we have $x^* \leq 0$ and $x^*(1+r) \geq 0$, $\implies x^* = 0$, contradiction
Putting all cases together, θ^* does not exist ■

Note 2.2
- Suppose that $(1+r) \geq u$. Then it is easy to see that the portfolio $\theta_1 = (S_0, -1)$ is an arbitrage opportunity
- Suppose that $(1+r) \leq d$. Then it is easy to see that the portfolio $\theta_1 = (-_0, 1)$ is an arbitrage opportunity
- $d < 1+r < u$ means that $(1+r)$ is convex combination of d and u
- In other words, there exists some $q_u \in (0,1)$ such that

$$1 + r = q_u u + q_d d, \quad \text{where } q_d = 1 - q_u$$

Simple Example
Suppose that $p = \frac{1}{2}, r = 0, S_0 = 100, u = 1.2$ and $d = 0.9$
- A security X pays 30 at time 1 if $S_1 = 120$, and if pays 0 of $S_1 = 90$ (e.g. a call option with strike 90). What may be a fair price of X
- A security Y pays constantly 90 at time 1. What may be a fair price of Y?
- What may be a fair price of the combination $X + Y$?

Taking (discounted) simple average does not work! We need a more sophisticated method.
An intuitive reason why discounted simple average does not work:
Suppose that securities are priced by their expected values
- two securities with the same price would have the same expected return
- no risk-averse investors would be interested in any risky investments
  - It is commonly assumed that stocks should have higher return than risk-free bonds
  - Most investors are risk-averse to some extend

2.1.2 Risk-Neutral Probabilities

Recall that there exists q_u and $q_d = 1 - q_u$ such that

$$1 + r = q_u u + q_d d$$

- The numbers $q_u, q_d \in (0,1)$ can be interpreted as **probabilities**
- Let $\mathbb{P} = (p, 1-p)$ be the initial probability measure (a.k.a. **physical measure**), let $\mathbb{Q} = (q_u, q_d)$ be the new probability measure
- Then if $p \neq q_u$

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{P}}[S_1] = \frac{S_0}{1+r}(up + d(1-p)) \neq S_0$$

  and

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1] = \frac{S_0}{1+r}(uq_u + d(1-q_d)) = \frac{S_0}{1+r}(1+r) = S_0$$

Definition 2.4 — risk-neutral probability measure.
A probability measure $\mathbb{Q}$ is called a **risk-neutral probability measure** (or a martingale measure) if

$$S_0 = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1]$$

Theorem 2.2 — First Fundamental Theorem of Asset Pricing - infant version.
The above one-period binomial model is arbitrage-free if and only if there exists a risk-neutral probability measure $\mathbb{Q}$.

Proof.
$\Rightarrow$: arbitrage free $\implies$ $\exists\mathbb{Q}$ by the previous proposition:

$$\begin{aligned}
\text{arbitrage free} &\implies u > 1+r > d \\
&\implies \exists q_u, q_d \in (0,1) \text{ with } q_d = 1-q_u \text{ and } \mathbb{Q} = (q_u, q_d) \\
&\quad\ \ \text{such that } \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1] = S_0 \\
&\implies \mathbb{Q} \text{ is a RNPM}
\end{aligned}$$

$\Leftarrow$: There exists a RNPM $\mathbb{Q} = (q_u, q_d)$ $\implies$ arbitrage free.

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1] = S_0 \implies \frac{1}{1+r}(S_0 u q_u + S_0 d q_d) = S_0$$

$$\begin{aligned}
&\implies uq_u + d(1-q_u) = 1+r \\
&\quad\ \ q_u = \frac{1+r-d}{u+d} > 0 \\
&\implies 1+r > d \\
&\quad\ \ 1-q_u = \frac{u-(1+r)}{u+d} > 0 \\
&\implies u > 1+r \\
&\implies u > 1+r > d \implies \text{ arbitrage free}
\end{aligned}$$

■

We can give an explicit characterization of the **risk-neutral measure** $\mathbb{Q} = (q_u, q_d)$:

- We have one unknown: q_u (remember that $q_d = 1-q_u$).
- We have one equation: $1+r = q_u u + q_d d$
- Therefore, we can solve for q_u, and then obtain q_d:

$$q_u = \frac{(1+r)-d}{u-d} \quad \text{and} \quad q_d = \frac{u-(1+r)}{u-d}$$

Can the probability measure $\mathbb{Q}$ be used to price derivative instruments?

2.1.3 Risk-Neutral Valuation

Recall: A **contingent claim** (or derivative instrument) is a random variable $X = \Phi(S_1)$, where Φ is a deterministic function

- The value of the claim X depends only on the value of the underlying stock
- The function Φ is called the **contract function**, and it is known

- Example: for a European call option with strike K, we have $\Phi(S_1) = \max(0, S_1 - K)$

What is a **fair** price for a contingent claim?

Let $\Pi_X(t)$ = **price of the derivative X at time t, for** $t = 0, 1$. Assuming the market is arbitrage-free, the price of the derivative at $t = 1$ has to be equal to its payoff:

$$\Pi_X(1) = X = \Phi(S_1)$$

Our **goal is to determine** $\Pi_X(0)$, the price that one must pay today to purchase the derivative X, such that no arbitrage can arise.

One way to do this is using the **Law of One Price**:

- If we can find a portfolio θ that yields the same payoff as X, then the price $\Pi_X(0$ of X has to be equal to the value V_0^θ of the portfolio.
- Such a portfolio is called a **replicating portfolio**.

Definition 2.5 — attainable, replicating portfolio, complete.

- A contingent claim X is said to be attainable is there exists a portfolio θ such that $V_1^\theta = X$.
- In this case, we say that θ is a **replicating portfolio** for X
- In an arbitrage-free market, if all contingent claims are attainable, we say that the market is complete.

Note: This definition will be extended to more general market models later.

Proposition 2.3
If the binomial model is arbitrage-free then it is complete.

Proof. We have to show that any contingent claim $X = \Phi(S_1)$ is attainable by a portfolio $\theta = (x, y)$. Now, S_1 is either uS_0 or dS_0. For θ to replicate X, V_1^θ has to be equal to $\Phi(S_1)$, that is, either $\Phi(uS_0)$ (when $S_1 = uS_0$) or $\Phi(dS_0)$ (when $S_1 = dS_0$). Therefore, we simply need to solve the following 2 equations with 2 unknowns:

$$\begin{aligned} \Phi(uS_0) &= (1+r)x + yuS_0 \\ \Phi(dS_0) &= (1+r)x + ydS_0 \end{aligned}$$

Therefore, the portfolio $\theta^* = (x^*, y^*)$ replicates $\Phi(S_1)$ where

$$\begin{aligned} x^* &= \frac{1}{1+r} \frac{u\Phi(dS_0) - d\Phi(uS_0)}{u-d}, \\ y^* &= \frac{\Phi(uS_0) - \Phi(dS_0)}{(u-d)S_0} = \frac{\Delta \text{ derivative}}{\Delta \text{ underlying}} \end{aligned}$$

Therefore, the contingent claim (derivative) $X = \Phi(S_1)$ can be replicated by the portfolio $\theta^* = (x^*, y^*)$. Note also that $V_1^{\theta^*} = X$, by construction. The Law of One Price implies that the price $\Pi_X(0)$ of the derivative X is equal to the value $V_0^{\theta^*}$ of the replicating portfolio at $t = 0$. That is

$$\begin{aligned}
\Pi_X(0) = V_0^{\theta^*} &= x^* + y^* S_0 \\
&= \frac{1}{1+r}\left(\frac{u\Phi(dS_0) - d\Phi(uS_0)}{u-d}\right) + \left(\frac{\Phi(uS_0) - \Phi(dS_0)}{(u-d)S_0}\right) S_) \\
&= \frac{1}{1+r}\left(\frac{u\Phi(dS_0) - d\Phi(uS_0)}{u-d} + \frac{(1+r)(\Phi(uS_0) - \Phi(dS_0))}{u-d}\right) \\
&= \frac{1}{1+r}\left(\Phi(uS_0)\underbrace{\left(\frac{(1+r)-d}{u-d}\right)}_{q_u} + \Phi(dS_0)\underbrace{\left(\frac{u-(1+r)}{u-d}\right)}_{q_d}\right)
\end{aligned}$$

Therefore

$$\Pi_X(0) = V_0^{\theta^*} = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[\Phi(S_1)] = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[X]$$

■

Theorem 2.4 — Risk-Neutral Valuation.
If the binomial model above is arbitrage-free, then the **risk-neutral (or arbitrage-free) price of a contingent claim** $X = \Phi(S_1)$ is given by

$$\Pi_X(0) = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[X] = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[\Phi(S_1)]$$

where the probability measure $\mathbb{Q}$ is such that

$$s_0 = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1]$$

The risk-neutral price is the discounted expected payoff, where expectation is under the risk-neutral measure Q

The risk-neutral price is the **only possible price** under the assumption that the market is arbitrage-free. Not to be confused with **risk-neutral investors**, investors that are insensitive to risks and only care about expected profit.

- In the no-arbitrage pricing theory, we **do not need to assume the risk preference** of investors
- The risk-neutral pricing method only relies on the assumption of **no arbitrage** and an **idealistic market**

2.2 The Multi-Period Binomial Model

Model Description

Only T periods of time: $t = 0, 1, 2, \dots, T$, where T is fixed. We have a **stock** with price S_0 today (given and observable). For each $t = 0, 1, 2, \dots, T-1$, the stock price at time $t+1$ is

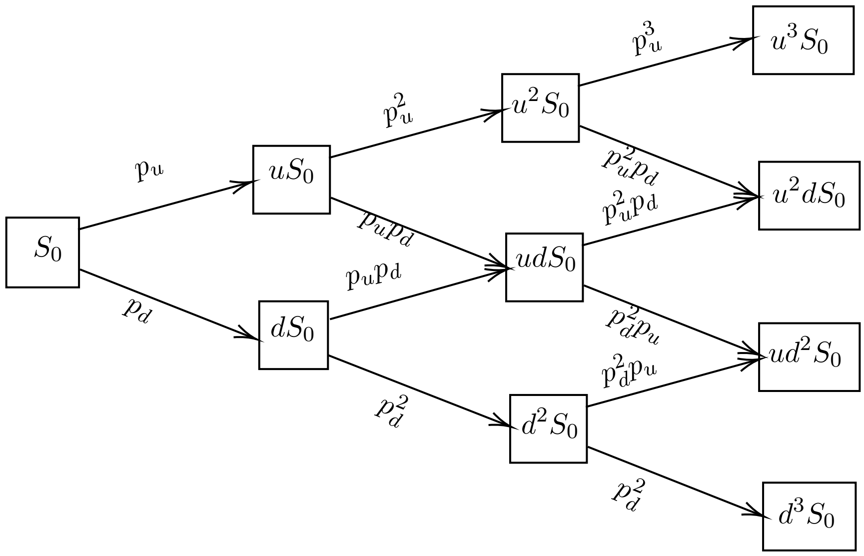
$$S_{t+1} = S_t Z_{t+1}$$

where $Z_1, Z_2, \dots, Z_T$ are i.i.d having the same distribution as

$$Z = \begin{cases} u & \text{with prob. } p_u \\ d & \text{with prob. } p_d = 1 - p_u \end{cases}$$

where $u > d$.

■ **Example 2.2** $T = 3$:



We also have a **risk-free asset** (e.g., default-free bond) with price $B_0 = 1$ today (given). For each $t = 0, 1, 2, \dots, T-1$, the price of this asset at time $t+1$ is

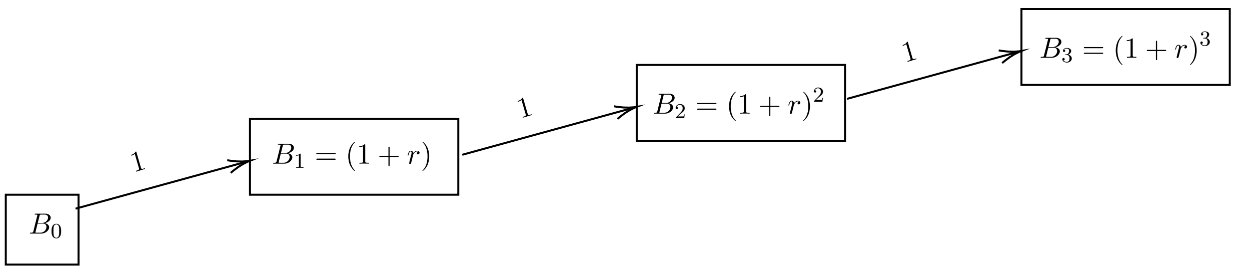
$$B_{t+1} = B_t(1+r)$$

Therefore, for each t,

$$B_t = (1+r)^t$$

where $r > 0$ is the constant risk-free rate (effective per period)

■ **Example 2.3** T=3:



2.2.1 Portfolio and Arbitrage

Definition 2.6 — portfolio strategy.
A **portfolio strategy (trading strategy)** is a stochastic process $\theta = \{\theta_t = (x_t, y_t) : t = 0, 1, 2, \dots, T-1\}$, where:

- For each t, θ_t is a function of $S_0, S_1, S_2, \dots, S_t$
- x_t = units of risk-free asset at time t and held until time $t+1$
- y_t = units of shares of the stock bought at time t and held until time $t+1$
- By convention set $\theta_T = \theta_{T-1}$

Definition 2.7 — value process.
The **value process** of the portfolio strategy $\theta = \{\theta_t = (x_t, y_t) : t = 0, 1, 2, \ldots, T-1\}$ is the stochastic process $\{V_t^\theta : t = 0, 1, \ldots, T\}$, where

$$V_t^\theta = x_{t-1}B_t + y_{t-1}S_t, \quad t = 0, 1, \ldots, T$$

- V_t^θ is the market value at time t of the portfolio position (x_{t-1}, y_{t-1}), which has been held since time $t-1$.

Definition 2.8 — self-financing.
A portfolio strategy $\theta = \{\theta_t = (x_t, y_t) : t = 0, 1, 2, \ldots, T-1\}$ is called **self-financing** if for all $t = 1, 2, \ldots, T$

$$\underbrace{x_{t-1}B_t + y_{t-1}S_t}_{\text{Money in}} = \underbrace{x_tB_t + y_tS_t}_{\text{Money out}}$$

- If at time t we sell the portfolio (held since time $t-1$), then we receive the amount of money $x_{t-1}B_t + y_{t-1}S_t$
- This money is just enough to buy the new portfolio position (x_t, y_t) at time t (without any extra cash), whose market value at time t is $x_tB_t + y_tS_t$

Definition 2.9 — arbitrage opportunity.
An arbitrage opportunity is a self-financing portfolio strategy $= \{\theta_t\}_t$ such that
- $V_0^\theta \leq 0$
- $\mathbb{P}[V_T^\theta \geq 0] = 1$
- $\mathbb{P}[V_T^\theta > 0] > 0$

We say that the multi-period binomial market is **arbitrage-free** if there are no arbitrage opportunities in this market.

2.2.2 Risk-Neutral Probabilities

Proposition 2.5
The above mutli-period binomial model is arbitrage-free if and only if $d < 1 + r < u$

Note 2.3
- $d < 1 + r < u$ means that $(1 + r)$ is a convex combination of d and u
- In other words, there exists some $q_u \in (0, 1)$ such that

$$1 + r = q_u u + qdd, \quad \text{where } q_d = 1 - q_u$$

- The numbers $q_u, q_d \in (0, 1)$ can be interpreted as **probabilities**

Let $\mathbb{Q}$ be a new probability measure, under which $Z_1, Z_2, \ldots, Z_T$ are i.i.d. having the same distribution as

$$Z \begin{cases} u & \text{with prob. } q_u \\ d & \text{with prob. } q_d \end{cases}$$

Then for each $t = 0, 1, 2, \dots, T-1$

$$\begin{aligned}
\frac{1}{r+1}\mathbb{E}^{\mathbb{Q}}[S_{t+1}|S_t = s] &= \frac{1}{r+1}\mathbb{E}^{\mathbb{Q}}[S_t Z_{t+1}|S_t = s] \\
&= s\frac{1}{r+1}\mathbb{E}^{\mathbb{Q}}[Z_{t+1}|S_t = s] \\
&= s\frac{1}{r+1}\mathbb{E}^{\mathbb{Q}}[Z_{t+1}] \\
&= \frac{s}{1+r}\underbrace{uq_u + dq_d}_{1+r} \\
&= s
\end{aligned}$$

Definition 2.10 — risk-neutral (probability) measure.
A probability measure $\mathbb{Q}$ is called a **risk-neutral (probability) measure** (or a martingale measure) if *it is positive for every event* and for each $t = 0, 1, 2, \dots, T-1$

$$S_t = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_{t+1}|S_t]$$

Theorem 2.6 — First Fundamental Theorem of Asset Pricing - toddler version.
The above mutli-period binomial model is arbitrage-free if and only if there exists a risk-neutral probability measure $\mathbb{Q}$

We can give an explicit characterization of the risk-neutral measure $\mathbb{Q}$:

- We have one unknown: q_u (remember that $q_d = 1 - q_u$)
- We have one equation $1 + r = q_u u + q_d d$
- Therefore, we can solve for q_u, and then obtain q_d

$$q_u = \frac{(1+r) - d}{u - d} \quad \text{and} \quad q_d = \frac{u - (1+r)}{u - d}$$

Definition 2.11 — contingent claim, adapted process.
A contingent claim (with maturity T) is a random variable $X = \Phi(S_0, S_1, \dots, S_T)$, where Φ is a deterministic function (the contract function).
An **adapted process** is a stochastic process $\{X_t\}_{t=0,1,\dots,T}$ where for each t, X_t is determined by $S_0, S_1, \dots, S_t$

For $t = 0, 1, \dots, T$, let $\Pi_X(t)$ be the time t price of the derivative which pays X at time T ("derivative X" for short), which is an **adapted process**. The price of derivative X at T has to be equal to its payoff $\Pi_X(T) = X$. Our goal is to determine a "reasonable" price process $\{\Pi_X(t) : t = 0, 1, \dots, T\}$ for derivative X.

Definition 2.12 — attainable, replicating portfolio, complete.
A contingent claim X is said to be **attainable** is there exists a portfolio strategy θ such that $V_T^\theta = X$.

An adapted process $\{X_t\}_t$ is said to be **attainable** is there exists a portfolio strategy θ such that $V_t^\theta = X_t$ for all $t = 0, 1, \dots, T$.

In both situations, we say that θ is a **replicating portfolio** for X or $\{X_t\}_t$.

In an arbitrage-free market, if all adapted processes are attainable, we say that the market is **complete**.

Proposition 2.7
If the binomial model is arbitrage-free then it is complete.

Proof. similar to the one-period model. ∎

Proposition 2.8
Suppose that the market is arbitrage-free. If a trading strategy θ is self-financing and $V_T^\theta = X$, then $V_t^\theta = \Pi_x(t)$ for all $t = 0, 1, \ldots, T$

Proof. a simple application of the **Law of One Price**. ∎

Proposition 2.9
If the market is complete, then there exists a self-financing trading strategy which replicates the price process of the contingent claim X.

Proof. Since the market is complete, there exists a portfolio $\theta = \{(x_t, y_t)\}_t$ such that at each time $0 \leq t < T$,

$$x_t B_{t+1} + y_t S_{t+1} = \Pi_X(t+1)$$

By no-arbitrage, we have $x_t B_t + y_t S_t = \Pi_X(t)$. Therefore,

$$x_{t-1} B_t + y_{t-1} S_t =_t B_t + y_t S_t$$

Hence by setting $\theta_T = \theta_{T-1}$, the portfolio $\theta = \{(x_t, y_t)\}_t$ is self-financing and it replicates $\{\Pi_X(t)\}_t$ ∎

Proposition 2.10 — Martingale property of a self-financing trading strategy:.
Let $\mathbb{Q}$ be the risk-neutral probability measure. For any self-financing trading strategy θ

$$(1+r)V_{t-1}^\theta = \mathbb{E}^{\mathbb{Q}}[V_t^\theta | S_0, \ldots, S_{t-1}]$$

In particular, for $t = 1, \ldots, T$

$$\begin{aligned}
V_0^\theta &= \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[V_1^\theta] \\
&= \left(\frac{1}{1+r}\right)^2 \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}[V_2^\theta | S_1]\right] \\
&= \left(\frac{1}{1+r}\right)^2 \mathbb{E}^{\mathbb{Q}}[V_2^\theta] = \cdots = \left(\frac{1}{1+r}\right)^t \mathbb{E}^{\mathbb{Q}}[V_t^\theta]
\end{aligned}$$

Proof.

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[V_t^\theta|S_0,\dots,S_{t-1}] &= \mathbb{E}^{\mathbb{Q}}[x_{t-1}B_t + y_{t-1}S_t|S_0,\dots,S_{t-1}] \\
&\text{(x_{t-1} and y_{t-1} are function of $S_1,\dots,S_{t-1}$)} \\
&= x_{t-1}B_t + y_{t-1}S_t\mathbb{E}^{\mathbb{Q}}[S_t|S_0,\dots,S_{t-1}] \\
&\text{$\mathbb{Q}$ is a risk neutral probability measure} \\
&= (1+r)x_{t-1}B_{t-1} + (1+r)y_{t-1}S_{t-1} \\
&\text{(θ is self-financing)} \\
&= (1+r)x_{t-2}B_{t-1} + (1+r)y_{t-2}S_{t-1} \\
&= (1+r)V_{t-1}^\theta
\end{aligned}
$$

■

Using the above results, at $t = 0$, the risk-neutral price is the discounted expected payoff, where expectation is under the risk-neutral measure.

Theorem 2.11 — Risk-Neutral Valuation - Part 1.
If the binomial model above is arbitrage-free, then the risk-neutral (or arbitrage-free) price of a contingent claim X at time $t = 0$ is given by:

$$\Pi_X(0) = \left(\frac{1}{1+r}\right)^T \mathbb{E}^{\mathbb{Q}}[X]$$

where $\mathbb{Q}$ is the risk-neutral measure

Proof. Let θ be a self-financing trading strategy which replicates $\{\Pi_X(t)\}_t$. Then

$$\Pi_X(0) = V_0^\theta = \left(\frac{1}{1+r}\right)^T \mathbb{E}^{\mathbb{Q}}[V_T^\theta] = \left(\frac{1}{1+r}\right)^T \mathbb{E}^{\mathbb{Q}}[X]$$

■

For $t > 0$, the risk-neutral price is also the discounted expected payoff, where expectation is under the risk-neutral measure, and conditioning on the current value of the underlying

Theorem 2.12 — Risk-Neutral Valuation - Part 2.
If the binomial model above is arbitrage-free, then the risk-neutral (or arbitrage-free) price of a contingent claim X at time $t > 0$ is given by:

$$\Pi_X(t) = \left(\frac{1}{1+r}\right)^{T-t} \mathbb{E}^{\mathbb{Q}}[X|S_0, S_1, \dots, S_t]$$

where $\mathbb{Q}$ is the risk-neutral measure

A special case: if $X = \Phi(S_T)$, then the condition $S_0, S_1, \dots, S_t$ reduces to S_t (**the Markovian property**):

$$\Pi_X(t) = \left(\frac{1}{1+r}\right)^T \mathbb{E}^{\mathbb{Q}}[\Phi(S_T)|S_t]$$

- This applies to European call and put options
- This does not apply to path-dependent options (see later: exotic options)

2.2.3 Continuous Compounding

The risk-free asset B accumulates at the constant risk-free rate r. Thus far, we have assumed that r is an effective rate compounded per period. Here, we will assume that *interest is compounded continuously at the risk-free rate* $r > 0$ **per period**. Therefore, for each $t = 0, 1, 2, \ldots, T$, we have

$$B_t = B_0 e^{rt}$$

■ **Example 2.4** T=3:

$$\boxed{B_0} \xrightarrow{1} \boxed{B_1 = e^r} \xrightarrow{1} \boxed{B_2 = e^{2r}} \xrightarrow{1} \boxed{B_3 = e^{3r}}$$

Proposition 2.13
The binomial model is arbitrage-free if and only if $d < e^r < u$

Note 2.4
- $d < e^r < u$ means that e^r is convex combination of d and u
- In other words, there exists some $q_u \in (0, 1)$ such that

$$e^r = q_u u + q_d d, \quad \text{where } q_d = 1 - q_u$$

- The numbers $q_u, q_d \in (0, 1)$ can be interpreted as **probabilities**.
- All statements in the previous setting hold true if we **replace** $1 + r$ **by** e^r

Definition 2.13 — risk-neutral probability measure.
A probability measure $\mathbb{Q}$ is called a **risk-neutral probability measure** (or a martingale measure) if *it is positive for every event* and for each $t = 0, 1, 2, \ldots, T - 1$

$$S_t = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[S_{t+1}|S_t]$$

Similar to the previous model:
- The binomial model is arbitrage-free if and only if there exists a risk-neutral probability measure $\mathbb{Q}$
- The formula of q_u is

$$q_u = \frac{e^r - d}{u - d}, \quad \text{and} \quad q_d = \frac{u - e^r}{u - d}$$

Theorem 2.14 — Risk-Neutral Valuation.
If the binomial model above is arbitrage-free, then the risk-neutral (or arbitrage-free) price of a contingent claim X at time $t > 0$ is given by:

$$\Pi_X(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[X|S_0, S_1, \ldots, S_t]$$

where $\mathbb{Q}$ is the risk-neutral measure

In particular,

- $\Pi_X(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[X]$
- $\Pi_X(T) = X$

2.3 Option Pricing in the Binomial Model

Consider a contingent claim X (derivative) on an underlying stock S, with payoff $\Phi(S_T)$ at maturity T

- T here denotes the number of periods in this model

We want to **determine the risk-neutral price of X at** $t = 0$, assuming that the binomial model is arbitrage-free.

Let $\{\Pi_X(t) : t = 0, 1, 2, \ldots, T\}$ denote the risk-neutral price process of this contingent claim.

The per-year interest rate is r, which means that the per-period **interest rate is** rh **for a period of length** h. To approximate a continuous-time model, we typically let $h \to 0$ while keeping Th fixed.

We work with continuous compounded interested rate ($B_t = e^{rht}$).

Theorem 2.15 — Recursive Valuation.
The risk-neutral price process $\{\Pi_X(t) : t = 0, 1, 2, \ldots, T\}$ of the contingent claim $X = \Phi(S_T)$ is such that

$$\pi_X(t) = e^{-rh}\mathbb{E}^{\mathbb{Q}}[\Pi_X(t+1)|S_t]$$

for $t = 0, 1, 2, \ldots, T$, where $\mathbb{Q}$ is the risk neutral measure.

Theorem 2.16
The risk-neutral price of the contingent claim $X = \Phi(S_T)$ at time t is given by:

$$\pi_X(t) = e^{-rh(T-t)}\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)|S_t]$$

where $\mathbb{Q}$ is the risk neutral measure.

We will now see how to use these two results to price an European derivative.

■ **Example 2.5 — European Put Option.** We want to price a 6-month European put option on a stock with strike $K = 160$.

The option payoff is modelled using a 2-period binomial model with one risky stock and a risk-free asset.

The stock price at $t = 0$ is $S_0 = 150$.

In each period, the stock can either go up by a factor $u = 1.3$, or go down by a factor $d = 0.7$.

The risk-free asset earns interest at a continuously compounded rate of $r = 0.06$ **per annum**.

The stock earns no dividends.

Calculate the risk-neutral price of the option at $t = 0$**.**

Solution. **Recursive Valuation**
Note that the physical probabilities are not given, but they are **not needed to price**!

We have two periods of length $h = 3/12$ of a year each.

Then the risk-neutral measure $\mathbb{Q} = (q_u, q_d)$ is given by

$$q_u = \frac{e^{rh} - d}{u - d} = \frac{e^{0.06\left(\frac{3}{12}\right)} - 0.7}{1.3 - 0.7} = 0.5252$$

and

$$q_d = \frac{u - e^{rh}}{u - d} = 1 - q_u = 0.4748$$

Let $\{\Pi_X(t) : t = 0, 1, 2\}$ denote the risk-neutral price process of this put option.

$$\begin{tikzcd}
 & & \boxed{u^2 S_0 = 253.5} \\
 & \boxed{uS_0 = 195} \arrow[ur] \arrow[dr] & \\
\boxed{S_0 = 150} \arrow[ur] \arrow[dr] & & \boxed{udS_0 = 136.5} \\
 & \boxed{dS_0 = 105} \arrow[ur] \arrow[dr] & \\
 & & \boxed{d^2 S_0 = 73.5}
\end{tikzcd}$$

The option's payoff is $\Phi(S_2) = \max(K - S_2, 0)$, and so the option's risk-neutral price at each node is given on the tree as follows:

$$\begin{tikzcd}
 & & \boxed{\Pi_{X,2}(2) = \Phi\left(u^2 S_0\right) = 0} \\
 & \boxed{\Pi_{X,1}(1)} \arrow[ur] \arrow[dr] & \\
\boxed{\Pi_X(0)} \arrow[ur] \arrow[dr] & & \boxed{\Pi_{X,1}(2) = \Phi(udS_0) = 23.5} \\
 & \boxed{\Pi_{X,0}(1)} \arrow[ur] \arrow[dr] & \\
 & & \boxed{\Pi_{X,0}(2) = \Phi\left(d^2 S_0\right) = 86.5}
\end{tikzcd}$$

$\Pi_{X,1}(1) = e^{rh}\mathbb{E}^{\mathbb{Q}}[\Pi_X(2)|S_1 = uS_0]$

$$\begin{tikzcd}
 & \boxed{\Pi_{X,2}(2) = \Phi\left(u^2 S_0\right) = 0} \\
\boxed{\Pi_{X,1}(1)} \arrow[ur, "q_u = 0.5252"] \arrow[dr, "q_d = 0.4748"'] & \\
 & \boxed{\Pi_{X,1}(2) = \Phi(udS_0) = 23.5}
\end{tikzcd}$$

Therefore

$$\Pi_{X,1}(1) = e^{-0.06\left(\frac{3}{12}\right)}[0.5252 \times 0 + 0.4748 \times 23.5] = 10.9919$$

$\Pi_{X,0}(1) = e^{rh}\mathbb{E}^{\mathbb{Q}}[\Pi_X(2)|S_1 = dS_0]$

$$\begin{tikzcd}
 & \boxed{\Pi_{X,1}(2) = \Phi(udS_0) = 23.5} \\
\boxed{\Pi_{X,0}(1)} \arrow[ur, "q_u = 0.5252"] \arrow[dr, "q_d = 0.4748"'] & \\
 & \boxed{\Pi_{X,0}(2) = \Phi\left(d^2 S_0\right) = 86.5}
\end{tikzcd}$$

Therefore

$$\Pi_{X,0}(1) = e^{-0.06\left(\frac{3}{12}\right)}[0.5252 \times 23.5 + 0.4748 \times 86.5] = 52.6179$$

$\Pi_X(0) = e^{rh}\mathbb{E}^{\mathbb{Q}}[\Pi_X(1)]$

$$\begin{tikzcd}
 & \boxed{\Pi_{X,1}(1) = 10.9919} \\
\boxed{\Pi_X(0)} \arrow[ur, "q_u = 0.5252"] \arrow[dr, "q_d = 0.4748"'] & \\
 & \boxed{\Pi_{X,0}(1) = 52.6179}
\end{tikzcd}$$

Therefore

$$\Pi_X(0) = e^{-0.06\left(\frac{3}{12}\right)}[0.5252 \times 10.9919 + 0.4748 \times 52.6179] = 30.2985$$

■

Solution. **Risk-Neutral Valuation**
The option price at time 0 is

$$\Pi_X(0) = e^{-2rh}\mathbb{E}^{\mathbb{Q}}[\max(0, K - S_2)]$$

where $h = \frac{3}{12}$.

Moreover, the random variable S_2 can take 3 values:

$$S_2 = S_0 u^k d^{2-k}, \quad \text{where } k = 0, 1, 2$$

- For $k = 0, 1, 2$, there are $\binom{2}{k}$ ways in which $S_k = S_0 u^k d^{2-k}$
- Therefore, for $k = 0, 1, 2$

$$\mathbb{Q}\left[S = S_0 u^k d^{2-k}\right] = \binom{2}{k} q_u^k q_d^{2-k}$$

Hence

$$\begin{aligned}
\Pi_X(0) &= e^{-2rh} \sum_{k=0}^{2} \binom{2}{k} q_u^k q_d^{2-k} \max\left(0, K - S_0 u^k d^{T-k}\right) \\
&= e^{-2rh}[q_u^2 \max(0, K - S_0 u^2) \\
&\qquad + q_u q_d \max(0, K - S_0 ud) + q_d^2 \max(0, K - S_0 d^2)] \\
&= e^{-0.06\left(\frac{3}{12}\right)}[(0.5252)^2 \max(0, 160 - 253.5) \\
&\qquad + 2 \times 0.5252 \times 0.4748 \times \max(0, 160 - 136.5) \\
&\qquad + (0.4748)^2 \times \max(0, 160 - 73.5)] \\
&= 30.2985
\end{aligned}$$

■

In the general T-period binomial model, the random variable S_T can take $T + 1$ values:

$$S_T = S_0 u^i d^{T-j}, \quad \text{where } j = 0, 1, 2, \ldots, T$$

For a given $j \in \{0, 1, \ldots, T\}$ in how many ways can the stock price at time T be equal to $S_0 u^i d^{T-j}$?

- For each j, there are $\binom{T}{j}$ ways in which $S_T = S_0 u^j d^{T-j}$

Therefore,

$$\mathbb{P}\left[S_T = S_0 u^j d^{T-j}\right] = \binom{T}{j} p_u^j p_d^{T-j}$$

and

$$\mathbb{Q}\left[S_T = S_0 u^j d^{T-j}\right] \binom{T}{j} q_u^j q_d^{T-j}$$