ACTSC446: Mathematics of Financial Markets

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1. Introduction to Derivatives Market

1.1 Financial Markets

Basic components of a financial market:
- Money
- Assets (such as stock)
- Time
- People/organizations
- Uncertainty

Why is there a financial market?
$\Longrightarrow$ People have different financial needs
- preferences on timing
- perspectives on risks and uncertainty
- sets of information
- means of economic activities

$\Longrightarrow$ So they trade

1.1.1 Assets

In this course, an asset:
- has a current price
- its future price may be uncertain
- is tradeable

We use S_t or $S(t)$ to represent the price of an asset (stock) S at time t
- $\{S_t\}_{t\geq 0}$ (or S for short) is a **stochastic process**
- Typically S_0 represents the current ($t = 0$) price, which is **not random**
- $t \in [0, T]$ for some terminal time T

In this course, we aim to study this stochastic process S and some related quantities.

1.1.2 Review: Present Value of Future Payments

Note 1.1 — time-value of money.

A dollar today is worth more than a dollar tomorrow, because you can invest it and earn (non-negative) interest on it

The value at time $t < T$ of an amount $\$K$ in the future is $PV_t(K)$, the present value of K

Here, we assume that the (continuously compounded or annually effective) interest rate is non-negative, implying that $PV_t(K) \leq K$, for any $K \geq 0$ and $t < T$

Suppose the interest rate is r annually,

Definition 1.1 — Present value of future payment.
If an asset (e.g. **zero-coupon bond**) pays K dollars in time T, then the time t $(T < t)$ value of the future payment is

$$PV_t(k) = \begin{cases} e^{-r(T-t)K} & \text{if continuously compounded} \\ \frac{K}{(1+r)^{T-t}} & \text{if annually effective} \end{cases}$$

Definition 1.2 — Risk-free asset.
If the future payoff(s) of an asset is non-random, we call it **risk free**.

1.2 Derivative Securities

Definition 1.3 — Derivative.
In finance, a **derivative** is a contract that derives its value from the performance of an **underlying entity**

- This underlying entity can be an asset, an index, interest rate, a basket of assets, or even another derivative
- The underlying entity is called "**the underlying asset**" or simply "the underlying"

Definition 1.4 — derivative security.
A financial contract F is a **derivative security** or a **contingent claim**, whose value F_T at F_T at expiration date (maturity) T is "derived" exactly from the market price of more basic underlying primitive instruments up to and including time T.

Primitive Instruments (underlying)

- Stocks
- Currencies
- Interest rates
- Indices
- Commodities
- Bonds

Derivatives

- Futures & Forwards
- Options (Call, Puts, Caps, Floors, Bond Options, Swaptions...)
- Credit derivatives
- Swaps

Definition 1.5 — OTC and ETD.
Based on where they are traded, derivatives can be classified as **OTC (Over the Counter)** or **ETD (Exchange-traded derivatives)**
- OTC derivatives are private, tailored contracts between counterparties
- ETD's are more structured and standardized contracts where the underlying assets, the quantities and the mode of settlement are defined by an exchange house

- Being private contracts between two counterparties, OTC derivatives can be tailored and customized to suit exact risk and return needs
  - On the flip side, lack of a clearing house or exchange results in increased credit or default risk associated with each OTC contract
- Being transacted on an organized exchange, ETD transactions are governed by a set of specific terms. They are standardized and more transparent than OTC derivatives
  - Each party of an ETD contract is required to hold a margin at the clearing house to cover its unsettled positions and the clearing house will monitor this margin level to make sure that it covers outstanding trades
  - A margin is the amount of cash an investor must put up to open an account to start trading
  - Therefore, ETD's carry less credit risk than OTC derivatives in general

Some Terminology

- **Long Position**: When you buy something ...
- **Short Position**: When you sell something what you don't yet own ...
- **Model-Free**: Independent of specific assumptions (e.g. about stock price distribution, etc.) ...

Usage

Derivatives are used:
- to **manage risk** (risk-management/insurance tool)
  - e.g. a pension fund invested in a broad market index can use derivatives to obtain downside protection
  - e.g. an airline company can use derivatives to put a ceiling on the future price of jet fuel
- for **speculation**
  - e.g. for a given investment, the use of derivatives magnifies the financial consequences, i.e. we can obtain large exposures with relatively little capital
- as an important part of **compensation**
  - executive stock options

1.2.1 Assumptions on a Financial Market

(1) No transaction fee.
(2) No bid-ask spread.
(3) One can buy any amount/share of any security.
(4) One can trade at any time instantly.
(5) Buying or selling a security does not change its price.
(6) No default/credit risk.
(7) Allow naked short selling.
(8) No information difference between investors.

These assumptions are not very realistic but they help us to understand the fundamental issues of a financial market.

1.2.2 The Concept of (No) Arbitrage

Definition 1.6 — Arbitrage.
An **arbitrage opportunity** is a portfolio value process $\{V_t\}_{t\geq 0}$ such that
(1) $V_0 \leq 0$
(2) $P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$, for some time $T > 0$

In other words, an arbitrage opportunity is a portfolio that:
- costs nothing to hold, or you are paid to hold it
- generates non-negative payoff with probability 1, and positive payoffs with strictly positive probability

The Principle of No-Arbitrage
- There ain't no such thing as a free lunch
- An immediate consequence of no-arbitrage is the Law of One Price

Proposition 1.1 — Law of One Price.
In an arbitrage-free market, if two securities have exactly the same payoffs they must have the same price

Proposition 1.2
In a market, if there exists a portfolio value process $\{V_t\}_{t\geq 0}$ satisfying
(1) $V_0 \leq 0$
(2) $V_T \geq 0$ for some time $T > 0$
then there is an arbitrage opportunity in the market.

From now on, assume a market with no arbitrage in this chapter.

1.3 Forwards and Futures

Definition 1.7 — Forward.
A **forward contract** is a non-standardized agreement to buy or sell an asset at a certain future time T for a certain price K, known as the **delivery price** (or **forward price**)
- The delivery price is determined so that the value of the contract at initiation is zero

Terminology
- **Underlying asset**: The asset on which the forward contract is based
- **Expiration date**: The time at which the asset is delivered
- **Forward price**: The price the buyer will pay at the expiration date
  - This is not the price one party needs to pay the other at the initial time; there is no initial price associated with a forward contract!

* It is normally traded Over-the-Counter (OTC)
* The party that **agrees to buy the underlying** asset is said to have a **long position** in the forwards
* The party that **agrees to sell the underlying** asset is said to have a **short position** in the forwards
* At the time the contract is entered into, no exchange of money takes place
* A forward contract can be contrasted with a spot contract:
  - A *spot contract* is an agreement to buy or sell an asset today, **with immediate** cash exchange

  - A *forward contract* is an agreement **with no immediate** cash exchange

1.3.1 Forward Contract - Payoff

- S_t: The spot price of the underlying asset at time $t \geq 0$
- T: The expiration date
- K: The forward price (delivery price)
- Long position: the position of the buyer
- Short position: the position of the seller

Pay off to long forward $= S_T - K$
Pay off to short forward $= K - S_T$

1.3.2 Forward Contract - Pricing

Pricing a forward contract is model-free, using simple no-arbitrage arguments

- Suppose that a stock pays no dividend, the current stock price is S_0, and the risk-free rate is r per year continuously compounded
- Consider the following trading strategy:
  (1) Borrow S_0 at the risk-free rate for the period of T years, and buy one share
  (2) Short one forward contract on the stock with delivery price K expiring at T

The cash flows are:

	Cash flow at $t = 0$	Cash flow at $t = T$
Borrowing S_0	$+S_0$	$-S_0e^{rT}$
1 long share	$-S_0$	$+S_T$
1 short forward	0	$K - S_T$
Total	0	$K - S_0e^{rT}$

The principle of No-Arbitrage (“No free lunch") then implies that the cash flow at time T should be 0. Thus the forward price is:

$$K = S_0e^{rT}$$

Proposition 1.3
Let S denote the price process of a non-dividend-paying stock. For a forward contract F on S, issued at time t and having maturity T, the forward price K determined at t is given by

$$K = S_0e^{rT}$$

Proof. (Equivalent to the previous cash flow table)

- At time t, but F, sell S, and deposit S_t money
- At t you value is 0
- At T you have $S_T - K - S_T + S_te^{r(T-t)} = S_te^{r(T-t)} - k$
- This value is not random, and if it is not zero there is an arbitrage

■

1.3.3 Forward Contract - Pricing with Dividends

Dividends are the payments made by a security (e.g. stock of a corporation) to its shareholders. They can be discrete (paid at discrete time intervals) or continuous (paid continuously).

Discrete Dividends

Consider a forward on a stock St , which will pay a dividend of $c at time $t_1 \in [0, T]$, where T is the expiration date of the forward contract.
Consider the following two trading strategies:

(1) Borrow S_0 at the risk-free rate for the period of T years, and buy one share
(2) Short one forward contract on the stock with delivery price K expiring at T

The cash flows are:

	Cash flow at $t = 0$	Cash flow at $t = T$
Borrowing S_0	$+S_0$	$-S_0e^{rT}$
1 long share	$-S_0$	$+S_T + ce^{r(T-t_1)}$
1 short forward	0	$K - S_T$
Total	0	$K - S_0e^{rT} + ce^{r(T-t_1)}$

The principle of No-Arbitrage ("No free lunch") then implies that the cash flow at time T should be 0. Thus the forward price is:

$$K = S_0e^{rT} - +ce^{r(T-t_1)}$$

Proposition 1.4
Let S denote the price process of a stock earning discrete dividends between time t and time T. For a forward contract F on S, issued at time t and having maturity T, the forward price K determined at t is given by

$$K = S_0e^{rT} - \text{Accumulated value at time } T \text{ of all dividends}$$

Continuous Dividends

R When there is a continuous dividend paid by stock S in a constant rate δ, an investment of $S_te^{-\delta(T-t)}$ in the stock at time t will yield 1 share of stock at time T (with price S_T)

Proposition 1.5
Let S denote the price of a stock earning a continuous dividend rate δ. For a forward contract F on S, issued at time t and having maturity T, the forward price K determined at t is given by

$$K = S_te^{(r-\delta)(T-t)}$$

Proof. Consider a forward on a stock S_t, paying dividends continuously at a dividend yield of δ per annum. Consider the following two trading portfolios:

- Portfolio A:
  - At time t, enter into a forward contract to buy one share of the stock, with forward price $K, maturing at time T
  - Simultaneously invest an amount $Ke^{-r(T-t)}$ the risk-free asset
  - At time T, the risk-free investment will accumulate to $K; use this $K buy a share of stock via the forward contract.
- Portfolio B:
  - Buy $e^{-\delta(T-t)}$ shares of the stock, at the current price S_t. Reinvest dividend incomes in the stock S immediately when they are received.

The cash flows are:

Portfolio	Cash flow at t	Cash flow at T
A	$\$Ke^{-r(T-t)}$	S_T
B	$\$S_te^{-\delta(T-t)}$	S_T

Thus by the no-arbitrage principle, $\$Ke^{-r(T-t)} = \$S_te^{-\delta(T-t)}$, i.e. $\$K = S_te^{(r-\delta)(T-t)}$, when the underlying pays dividends continuously at a yield of δ per annum. ■

Continuous dividends are unusual but easy to calculate.