

PMATH 351: Real Analysis

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Chapter 1

Cardinality

Definition 1.1 (domain, range, image, inverse image).

Let X and Y be sets and let $f : X \rightarrow Y$. Recall the **domain** of f and the **range** of f are the sets

$$\text{Domain}(f) = X, \text{Range}(f) = f(X) = \{f(x) | x \in X\}$$

for $A \subseteq X$, the **image** of A under f is the set

$$f(A) = \{f(x) | x \in A\}$$

For $B \subseteq Y$, the **inverse image** of B under f is the set

$$f^{-1}(B) = \{x \in X | f(x) \in B\}$$

Definition 1.2 (Composite).

Let X, Y and Z be sets, let $f : X \rightarrow Y$ and let $g : Y \rightarrow Z$. We define the **composite** function $(g \circ f)(x) = g(f(x))$ for all $x \in X$

Definition 1.3 (injective, surjective, bijective).

We say that f is **injective** (or **one-to-one**) when for every $y \in Y$ there exists **at most** one $x \in X$ such that $f(x) = y$. Equivalently, f is injective when for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

We say that f is **surjective** (or **onto**) when for every $y \in Y$ there exists **at least** one $x \in X$ such that $f(x) = y$. Equivalently, f is surjective when $\text{Range}(f) = Y$

We say that f is **bijective** (or **invertible**) when f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f(x) = y$. When f is both injective and surjective, that is when for every $y \in Y$ there exists exactly one $x \in X$ such that $f^{-1} : Y \rightarrow X$ such that for all $y \in Y$, $f^{-1}(y)$ is equal to the unique element $x \in X$ such that $f(x) = y$. Note that when f is bijective so is f^{-1} , and in this case we have $(f^{-1})^{-1} = f$

Theorem 1.1. Let $f : X \rightarrow Y$ and let $g : Y \rightarrow Z$. Then

- (1) If f and g are both injective then so is $g \circ f$
- (2) If f and g are both surjective then so is $g \circ f$
- (3) If f and g are both invertible then so is $g \circ f$, and in this case $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof.

- (1) Suppose that f and g are both injective. Let $x_1, x_2 \in X$. If $g(f(x_1)) = g(f(x_2))$ then since g is injective we have $f(x_1) = f(x_2)$, and then since f is injective we have $x_1 = x_2$. Thus $g \circ f$ is injective.
- (2) Suppose that f and g are both surjective. Given $z \in Z$, since g is surjective we can choose $y \in Y$ so that $g(y) = z$, then since f is surjective we can choose $x \in X$ so that $f(x) = y$, and then we have $g(f(x)) = g(y) = z$. Thus $g \circ f$ is surjective.
- (3) Follows (1) and (2).

■

Definition 1.4 (identity function).

For a set X , we define the **identity function** on X to be the function $I_X : X \rightarrow X$ given by $I_X(x) = x$ for all $x \in X$. Note that for $f : X \rightarrow Y$ we have $f \circ I_X = f$ and $I_Y \circ f = f$.

Definition 1.5 (inverse).

Let X and Y be sets and let $f : X \rightarrow Y$. A **left inverse** of f is a function $g : Y \rightarrow X$ given by $g \circ f = I_X$. Equivalently, a function $g : Y \rightarrow X$ is a left inverse of f when $g(f(x)) = x$ for all $x \in X$.

A **right inverse** of f is a function $h : Y \rightarrow X$ such that $f \circ h = I_Y$. Equivalently, a function $h : Y \rightarrow X$ is a right inverse of f when $f(h(y)) = y$ for all $y \in Y$.

Theorem 1.2. Let X and Y be nonempty sets and let $f : X \rightarrow Y$. Then

- (1) f is injective $\iff f$ has a left inverse.
- (2) f is surjective $\iff f$ has a right inverse.
- (3) f is bijective $\iff f$ has a left inverse g and a right inverse h , and in this case we have $g = h = f^{-1}$.

Proof.

- (1) Suppose first that f is injective. Since $X \neq \emptyset$ we can choose $a \in X$ and then define $g : Y \rightarrow X$ as follows: if $y \in \text{Range}(f)$ then (using the fact the f is injective) we define $g(y)$ to be the unique element $x_y \in X$ with $f(x_y) = y$, and if $y \notin \text{Range}(f)$, then we define $g(y) = a$. Then for every $x \in X$ we have $y = f(x) \in \text{Range}(f)$, so $g(y) = x_y = x$, that is $g(f(x)) = x$. Conversely, if f has a left inverse, say g , then f is injective since for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x = g(f(x_1)) = g(f(x_2)) = x_2$.
- (2) Suppose first that f is onto. For each $y \in Y$, choose $x_y \in X$ with $f(x_y) = y$, then define $g : X \rightarrow Y$ by $g(y) = x_y$ (We need the Axiom of Choice for this). Then g is a right inverse of f since for every $y \in Y$ we have $f(g(y)) = f(x_y) = y$. Conversely, if f has a right inverse, say g , then f is onto since given any $y \in Y$ we can choose $x = g(y)$ and then we have $f(x) = f(g(y)) = y$.
- (3) Suppose first that f is bijective. The inverse function $f^{-1} : Y \rightarrow X$ is a left inverse for f because given $x \in X$ we can let $y = f(x)$ and then $f^{-1}(y) = x$ so that $f^{-1}(f(x)) = f^{-1}(y) = x$. Similarly, f^{-1} is a right inverse for f because given $y \in Y$ we can let x be the unique element in X with $y = f(x)$ and then we have $x = f^{-1}(y)$ so that $f(f^{-1}(y)) = f(x) = y$.

Conversely, suppose that g is a left inverse for f and h is a right inverse for f . Since f has a left inverse, it is injective by (1). Since f has a right inverse, it is surjective by (2). Since f is injective and surjective, it is bijective. As shown above, the inverse function f^{-1} is both a left inverse and a right inverse. Finally, note that $g = f^{-1} = h$ because for all $y \in Y$ we have

$$g(y) = g(f(f^{-1}(y))) = f^{-1}(y) = f^{-1}(f(h(y))) = h(y)$$

■

Corollary. *Let X and Y be sets. Then there exists an injective map $f : X \rightarrow Y$ if and only if there exists a surjective map $g : Y \rightarrow X$.*

Proof. Suppose $f : X \rightarrow Y$ is an injective map. Then f has a left inverse. Let g be a left inverse of f . Since $g \circ f = I_X$, we see that f is a right inverse of g . Since g has a right inverse, g is surjective. Thus, there is a surjective map $g : Y \rightarrow X$. Similarly, if $g : Y \rightarrow X$ is surjective, then it has a right inverse $f : X \rightarrow Y$ which is injective. ■

Definition 1.6 (same cardinality, less than or equal to, less than).

Let A and B be sets. We say that A and B have the **same cardinality**, and write $|A| = |B|$, when there exists a bijective map: $f : A \rightarrow B$ (or equivalently when there exists a bijective map $g : B \rightarrow A$).

We say that the cardinality of A is **less than or equal to** the cardinality of B , and write $|A| \leq |B|$, when there exists an injective map $f : A \rightarrow B$ (or equivalently a surjective map $g : B \rightarrow A$).

We say that the cardinality of A is **less than** the cardinality of B , and write $|A| < |B|$, when $|A| \leq |B|$ and $|A| \neq |B|$, (that is when there exists an injective map $f : A \rightarrow B$ but there does not exist a bijective map $g : A \rightarrow B$).

We also write $|A| \geq |B|$ when $|B| \leq |A|$; and $|A| > |B|$ when $|B| < |A|$.

Example 1.1. Let $\mathbb{N} = \{n \in \mathbb{Z} | n \geq 0\} = \{0, 1, 2, \dots\}$.

- (1) The map $f : \mathbb{N} \rightarrow 2\mathbb{N}$ given by $f(k) = 2k$ is bijective, so $|2\mathbb{N}| = |\mathbb{N}|$.
- (2) The map $g : \mathbb{N} \rightarrow \mathbb{Z}$ given by $g(2k) = k$ and $g(2k+1) = -k-1$ for $k \in \mathbb{N}$ is bijective, so we have $|\mathbb{Z}| = |\mathbb{N}|$.
- (3) The map $h : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ given by $h(k, l) = 2^k(2l+1) - 1$ is bijective, so we have $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Theorem 1.3. For all sets A , B and C

- (1) $|A| = |A|$
- (2) If $|A| = |B|$ then $|B| = |A|$
- (3) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$
- (4) $|A| \leq |B| \iff (|A| = |B| \text{ or } |A| < |B|)$
- (5) If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$

Proof.

- (1) holds because the identity function $I_A : A \rightarrow A$ is bijective.
- (2) holds because if $f : A \rightarrow B$ is bijective then so is $f^{-1} : B \rightarrow A$.

(3) holds because if $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijective then so is the composite $g \circ f : A \rightarrow C$ ■

Definition 1.7 (finite, infinite, countable).

Let A be a set. For each $n \in \mathbb{N}$, let $S_n = \{0, 1, 2, \dots, n-1\}$. For $n \in \mathbb{N}$, we say that the cardinality of A is equal to n , or that A has n **elements**, and we write $|A| = n$, when $|A| = |S_n|$.

We say that A is **finite** when $|A| = n$ for some $n \in \mathbb{N}$. We say A is **infinite** when A is not finite. We say that A is countable when $|A| = |\mathbb{N}|$

Note. When a set A is finite with $|A| = n$, and when $f : A \rightarrow S_n$ is a bijection, if we let $a_k = f^{-1}(k)$ for each $k \in S_n$ then we have $A = \{a_0, a_1, \dots, a_{n-1}\}$ with the elements a_k distinct. Conversely, if $A = \{a_0, a_1, \dots, a_{n-1}\}$ with the elements a_k all distinct, then we define a bijection $f : A \rightarrow S_n$ by $f(a_k) = k$. Thus we see that A is finite with $|A| = n$ if and only if A is of the form $A = \{a_0, a_1, \dots, a_{n-1}\}$ with the elements a_k all distinct. Similarly, a set A is countable if and only if A is of the form $A = \{a_0, a_1, a_2, \dots\}$ with the elements a_k all distinct.

Note. For $n \in \mathbb{N}$, if A is a finite set with $|A| = n + 1$ and $a \in A \setminus \{a\} = n$. Indeed, if $A = \{a_0, a_1, \dots, a_n\}$ with the elements a_i distinct, and if $a = a_k$ so that we have $A \setminus \{a\} = \{a_0, a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n\}$, then we can define a bijection $f : S_n \rightarrow A \setminus \{a\}$ by $f(i) = a_i$ for $0 \leq i < k$ and $f(i) = a_{i+1}$ for $k \leq i < n$.

Theorem 1.4. Let A be a set. Then the following are equivalent:

- (1) A is infinite
- (2) A contains a countable subset
- (3) $|\mathbb{N}| \leq |A|$
- (4) There exists a map $f : A \rightarrow A$ which is injective but not surjective

Proof.

- (1) \implies (2) Suppose A is infinite. Since $A \neq \emptyset$ we can choose an element $a_0 \in A$. Since $A \neq \{a_0\}$ we can choose an element $a_1 \in A \setminus \{a_0\}$. Since $A \neq \{a_0, a_1\}$ we can choose $a_2 \in A \setminus \{a_0, a_1\}$. Continue this procedure: having chosen distinct elements $a_0, a_1, \dots, a_{n-1} \in A$, since $A \neq \{a_0, a_1, \dots, a_{n-1}\}$ we can choose $a_n \in A \setminus \{a_0, a_1, \dots, a_{n-1}\}$. In this way we obtain $\{a_0, a_1, a_2, \dots\} \subseteq A$.
- (2) \iff (3) Suppose that A contains a countable subset, say $\{a_0, a_1, a_2, \dots\} \subseteq A$ with the element a_i distinct. Since a_i are distinct, the map $f : \mathbb{N} \rightarrow A$ given by $f(k) = a_k$ is injective, and so we have $|\mathbb{N}| \leq |A|$. Conversely as a map from $\mathbb{N} \rightarrow f(\mathbb{N})$ where f is bijective, so we have $|\mathbb{N}| = |f(\mathbb{N})|$ hence $f(\mathbb{N})$ is a countable subset of A .
- (2) \implies (4) Suppose that A has a countable subset, say $\{a_0, a_1, a_2, \dots\} \subseteq A$ with the element a_i distinct. Define $f : A \rightarrow A$ by $f(a_k) = a_{k+1}$ for all $k \in \mathbb{N}$ and by $f(b) = b$ for all $b \in A \setminus \{a_0, a_1, a_2, \dots\}$. Then f is injective but not surjective (the element a_0 is not in the range of f).
- (4) \implies (1) To prove this we shall prove that if A is finite then every injective map $f : A \rightarrow A$ is surjective. We prove this by induction on the cardinality of A . The only set A with $|A| = 0$ is the set $A \neq \emptyset$, and then the only function $f : A \rightarrow A$ is the empty function, which is surjective.

Since that base case may appear too trivial, let us consider the next case. Let $n = 1$ and let A be a set with $|A| = 1$, say $A = \{a\}$. The only function $f : A \rightarrow A$ is the function given by $f(a) = a$, which is surjective.

Let $n \geq 1$ and suppose, inductively, that for every set A with $|A| = n$, every injective map $f : A \rightarrow A$ is surjective. Let B be a set with $|B| = n + 1$ and let $g : B \rightarrow B$ be injective.

Suppose, for a contradiction, that g is not surjective. Choose an element $b \in B$ which is not in the range of g so that we have $g : B \rightarrow B \setminus \{b\}$. Let $A = B \setminus \{b\}$ and let $f : A \rightarrow A$ be given by $f(x) = g(x)$ for all $x \in A$. Since $g : B \rightarrow A$ is injective and $f(x) = g(x)$ for all $x \in A$, f is also injective. Again since g is injective, there is no element $x \in B \setminus \{b\}$ with $g(x) = g(b)$, so there is no element $x \in A$ with $f(x) = g(b)$, and so f is not surjective. Since $|A| = n$, this contradicts the induction hypothesis. Thus g must be surjective.

By the Principle of Induction, for every $n \in \mathbb{N}$ and for every set A with $|A| = n$, every injective function $f : A \rightarrow A$ is surjective. ■

Corollary. *Let A and B be sets.*

- (1) *If A is countable then A is infinite*
- (2) *When $|A| \leq |B|$, if B is finite so is A (equivalently if A is infinite then so is B)*
- (3) *If $|A| = n$ and $|B| = m$ then $|A| = |B|$ if and only if $n = m$*
- (4) *If $|A| = n$ and $|B| = m$ then $|A| \leq |B|$ if and only if $n \leq m$*
- (5) *When one of the two sets A and B is finite, if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$*

Proof.

- (1) If A is countable then A contains a countable subset (itself), so A is infinite by Theorem 1.4.
- (2) Suppose that $|A| \leq |B|$ and that $|A|$ is infinite. Since A is infinite, we have $|\mathbb{N}| \leq |A|$ (by Theorem 1.4). Since $|\mathbb{N}| \leq |A|$ and $|A| \leq |B|$ we have $|\mathbb{N}| \leq |B|$ (by Theorem 1.3). Since $|\mathbb{N}| \leq |B|$, B is infinite (by Theorem 1.4).
- (3) Suppose that $|A| = n$ and $|B| = m$. If $n = m$ then we have $S_n = S_m$ and so $|A| = |S_n| = |S_m| = |B|$. Conversely, suppose that $|A| = |B|$. Suppose, for a contradiction, that $n \neq m$, say $n > m$, and note that $S_m \subsetneq S_n$. Since $|A| = |B|$ we have $|S_n| = |A| = |B| = |S_m|$ so we must have $n = m$.
- (4) Suppose $|A| = n$ and $|B| = m$. If $n \leq m$ then $S_n \subseteq S_m$ so the inclusion map $I : S_n \rightarrow S_m$ is injective and we have $|A| = |S_n| \leq |S_m| = |B|$. Conversely, suppose that $|A| \leq |B|$ and suppose, for a contradiction, that $n > m$. Since $|A| \leq |B|$ we have $|S_n| = |A| \leq |B| = |S_m|$ so we can choose an injective map $f : S_n \rightarrow S_m$. Since $n > m$ we have $S_m \subsetneq S_n$ so we can consider f as a map $f : S_n \rightarrow S_m$, and this map is injective but not surjective. This contradicts Theorem 1.4, and so $n \leq m$.
- (5) Suppose that one of the two sets A and B is finite, and that $|A| \leq |B|$ and $|B| \leq |A|$. If A is finite then, since $|B| \leq |A|$, (2) implies that B is finite. If B is finite then, since $|A| \leq |B|$, (2) implies that A is finite. Thus, in either case, we see that A and B are both finite. Since A and B are both finite with $|A| \leq |B|$ and $|B| \leq |A|$, we must have $|A| = |B|$ by (3) and (4). ■