DISCRETE PHYSICAL REALM

ALGEBRAIC FORMULATION OF PHYSICS

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PRESERVING OPERATOR SYMMETRIES?

THAT IS THE QUESTION

«Computer science is not about computers, in the same way that astronomy is not about telescopes. There is an essential unity of mathematics and computer science»
— Edsger Dijkstra

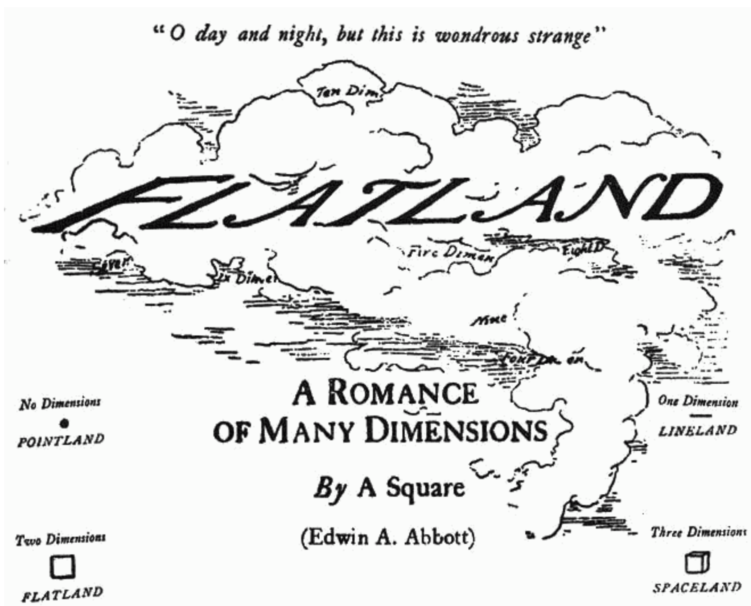
REMINDER 1: Flows in Flatland (2D)

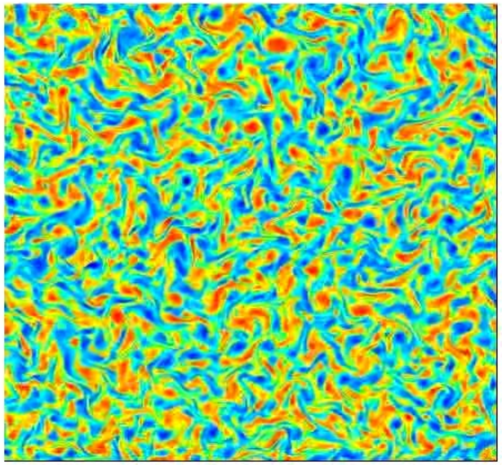
What can we learn from these relationships?

$$E := \langle \vec{u} | \vec{u} \rangle \qquad \Omega := \langle \vec{\omega} | \vec{\omega} \rangle \qquad H := \langle \vec{u} | \vec{\omega} \rangle$$

$$E_t = -2\nu\Omega$$

$$\Omega_t = -2\nu P + 2 \underbrace{\langle \vec{\omega} | \boldsymbol{S} \vec{\boldsymbol{\omega}} \rangle}_{=0 \text{ in 2D}}$$





REMINDER 2: OPERATOR SYMMETRIES

$$\langle \nabla \cdot \vec{a} | \phi \rangle = -\langle \vec{a} | \nabla \phi \rangle$$

$$\langle \nabla^2 f | g \rangle = -\langle \nabla f | \nabla g \rangle = \langle f | \nabla^2 g \rangle$$

$$\langle C(\vec{u}, \phi_1) | \phi_2 \rangle = -\langle C(\vec{u}, \phi_2) | \phi_1 \rangle \quad \text{if} \quad \nabla \cdot \vec{u} = 0$$

$$\langle \nabla \times \vec{a} | \vec{b} \rangle = \langle \vec{a} | \nabla \times \vec{b} \rangle$$

Notation:

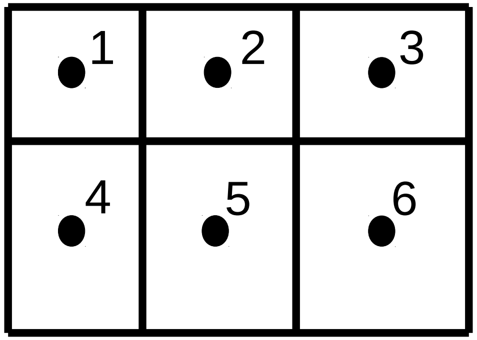
$$\langle a | b \rangle := \int_{\Omega} a b \, d\Omega \qquad C(\vec{u}, \phi) := (\vec{u} \cdot \nabla) \phi$$

REMEMBER: we always assume **no contribution from domain boundary,** $\partial \Omega$

From Calculus to Algebra (C2A)

$$\langle a|b\rangle := \int_{\Omega} ab\, d\Omega \ \in \mathbb{R}$$

$$\langle a_h|b_h\rangle := a_h^T \mathbf{\Omega}\, b_h \ \in \mathbb{R}$$



$$a_h = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} \qquad \mathbf{\Omega} = \begin{pmatrix} \Omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_6 \end{pmatrix} \qquad b_h = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix}$$

From Calculus to Algebra (C2A)

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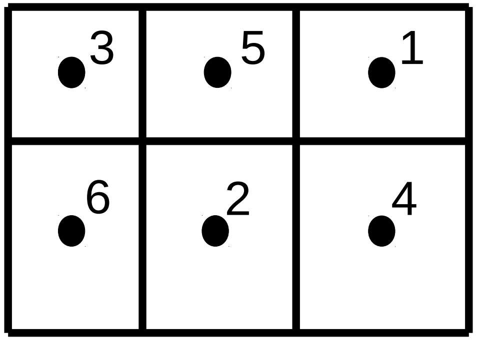
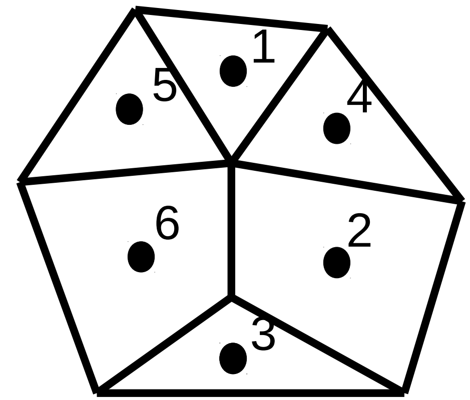


$$a_h = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} \qquad \boldsymbol{\Omega} = \begin{pmatrix} \Omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_6 \end{pmatrix} \qquad b_h = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix}$$

From Calculus to Algebra (C2A)

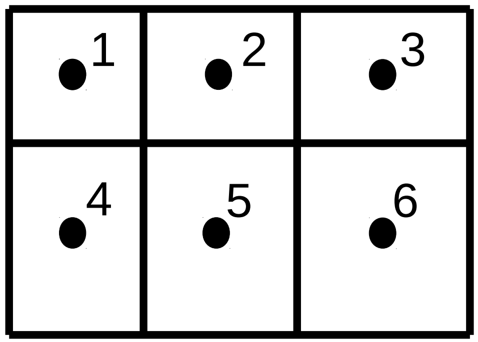
$$\langle a | b \rangle := \int_{\Omega} a b \, d\Omega \quad \in \mathbb{R}$$

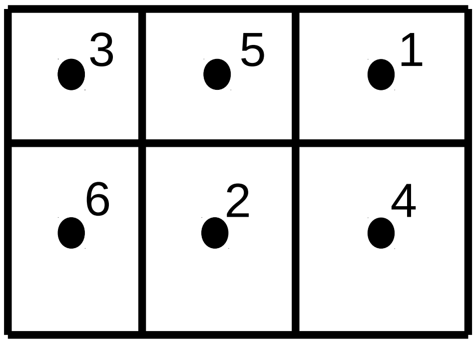


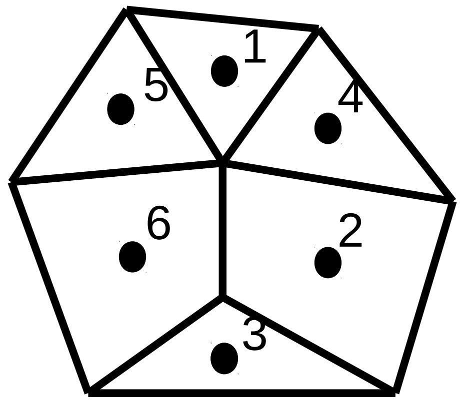
$$\langle a_h | b_h \rangle := a_h^T \mathbf{\Omega} \, b_h \quad \in \mathbb{R}$$

$$a_h = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} \qquad \mathbf{\Omega} = \begin{pmatrix} \Omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_6 \end{pmatrix} \qquad b_h = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix}$$

From Calculus to Algebra (C2A)







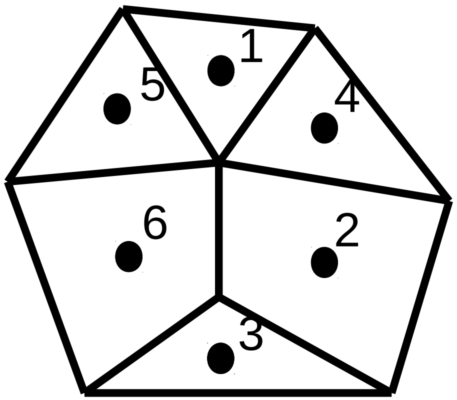
$$\boldsymbol{T}=\begin{pmatrix} \times & \times & 0 & \times & 0 & 0 \\ \times & \times & \times & 0 & \times & 0 \\ 0 & \times & \times & 0 & 0 & \times \\ \times & 0 & 0 & \times & \times & 0 \\ 0 & \times & 0 & \times & \times & \times \\ 0 & 0 & \times & 0 & \times & \times \end{pmatrix} \boldsymbol{T}=\begin{pmatrix} \times & 0 & 0 & \times & \times & 0 \\ 0 & \times & 0 & \times & \times & \times \\ 0 & 0 & \times & 0 & \times & \times \\ \times & \times & 0 & \times & 0 & 0 \\ \times & \times & \times & 0 & \times & 0 \\ 0 & \times & \times & 0 & 0 & \times \end{pmatrix} \boldsymbol{T}=\begin{pmatrix} \times & 0 & 0 & \times & \times & 0 \\ 0 & \times & \times & \times & 0 & \times \\ 0 & \times & \times & 0 & 0 & \times \\ \times & \times & 0 & \times & 0 & 0 \\ \times & 0 & 0 & 0 & \times & \times \\ 0 & \times & \times & 0 & \times & \times \end{pmatrix}$$

From Calculus to Algebra (C2A)

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = \nu \nabla^2 \vec{u} - \nabla p \qquad \nabla \cdot \vec{u} = 0$$

$$\boldsymbol{\Omega} \frac{d u_h}{d t} + \boldsymbol{C}(u_h) u_h = \boldsymbol{D} u_h - \boldsymbol{\Omega} \boldsymbol{G} p_h \qquad \boldsymbol{M} u_h = 0_h$$

$$p_h(t) = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix} \qquad u_h(t) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}$$



$$\boxed{\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Omega}_u & \\ & \boldsymbol{\Omega}_v \end{pmatrix}}$$

$$\boldsymbol{T} = \begin{pmatrix} \times & 0 & 0 & \times & \times & 0 \\ 0 & \times & \times & \times & 0 & \times \\ 0 & \times & \times & 0 & 0 & \times \\ \times & \times & 0 & \times & 0 & 0 \\ \times & 0 & 0 & 0 & \times & \times \\ 0 & \times & \times & 0 & \times & \times \end{pmatrix}$$

$$\boldsymbol{\Omega}_u = \boldsymbol{\Omega}_v = \begin{pmatrix} \Omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_6 \end{pmatrix}$$

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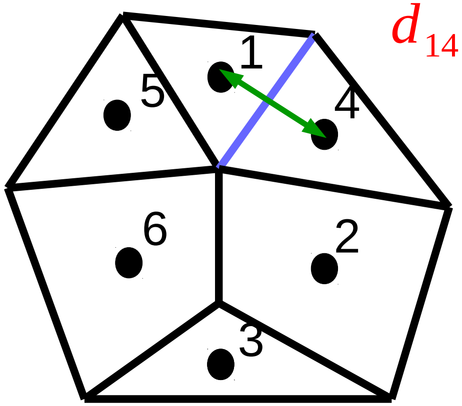


$$d_{14} = \nu A_{14} / \delta_{14}$$

$$\boxed{\boldsymbol{D} = \begin{pmatrix} \boldsymbol{D}_u & \\ & \boldsymbol{D}_v \end{pmatrix}}$$

$$\boldsymbol{T} = \begin{pmatrix} \times & 0 & 0 & \times & \times & 0 \\ 0 & \times & \times & \times & 0 & \times \\ 0 & \times & \times & 0 & 0 & \times \\ \times & \times & 0 & \times & 0 & 0 \\ \times & 0 & 0 & 0 & \times & \times \\ 0 & \times & \times & 0 & \times & \times \end{pmatrix}$$

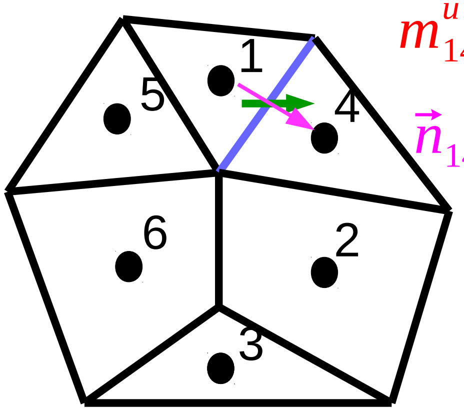
$$\boldsymbol{D}_u = \boldsymbol{D}_v = \begin{pmatrix} d_{11} & 0 & 0 & d_{14} & d_{15} & 0 \\ 0 & d_{22} & d_{23} & d_{24} & 0 & d_{26} \\ 0 & d_{23} & d_{33} & 0 & 0 & d_{36} \\ d_{14} & d_{24} & 0 & d_{44} & 0 & 0 \\ d_{15} & 0 & 0 & 0 & d_{55} & d_{56} \\ 0 & d_{26} & d_{36} & 0 & d_{56} & d_{66} \end{pmatrix}$$

From Calculus to Algebra (C2A)

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = \nu\nabla^2\vec{u} - \nabla p \qquad \nabla\cdot\vec{u} = 0$$

$$\boldsymbol{\Omega}\frac{d u_h}{d t} + \boldsymbol{C}(u_h)u_h = \boldsymbol{D}u_h - \boldsymbol{\Omega}\boldsymbol{G}p_h \qquad \boldsymbol{M}u_h = 0_h$$

$$p_h(t) = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix} \qquad u_h(t) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}$$



$$m_{14}^u = A_{14} n_{14}^x / 2$$

$$\vec{n}_{14} = \begin{pmatrix} n_{14}^x & n_{14}^y \end{pmatrix}$$

$$\boxed{\boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}^u & \boldsymbol{M}^v \end{pmatrix}}$$

$$\boldsymbol{T} = \begin{pmatrix} \times & 0 & 0 & \times & \times & 0 \\ 0 & \times & \times & \times & 0 & \times \\ 0 & \times & \times & 0 & 0 & \times \\ \times & \times & 0 & \times & 0 & 0 \\ \times & 0 & 0 & 0 & \times & \times \\ 0 & \times & \times & 0 & \times & \times \end{pmatrix}$$

$$\boldsymbol{M}^u = \begin{pmatrix} 0 & 0 & 0 & m_{14}^u & m_{15}^u & 0 \\ 0 & 0 & m_{23}^u & m_{24}^u & 0 & m_{26}^u \\ 0 & -m_{23}^u & 0 & 0 & 0 & m_{36}^u \\ -m_{14}^u & -m_{24}^u & 0 & 0 & 0 & 0 \\ -m_{15}^u & 0 & 0 & 0 & 0 & m_{56}^u \\ 0 & -m_{26}^u & -m_{36}^u & 0 & -m_{56}^u & 0 \end{pmatrix}$$

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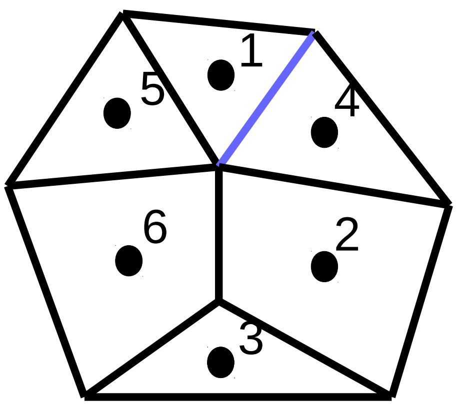

$$\boxed{\boldsymbol{G}=\begin{pmatrix}\boldsymbol{G}^x\\ \boldsymbol{G}^y\end{pmatrix}}$$

$$\boldsymbol{T}=\begin{pmatrix} \times & 0 & 0 & \times & \times & 0\\ 0 & \times & \times & \times & 0 & \times\\ 0 & \times & \times & 0 & 0 & \times\\ \times & \times & 0 & \times & 0 & 0\\ \times & 0 & 0 & 0 & \times & \times\\ 0 & \times & \times & 0 & \times & \times \end{pmatrix}$$

$$\boldsymbol{G}^x=\begin{pmatrix} 0 & 0 & 0 & g_{14}^x & g_{15}^x & 0\\ 0 & 0 & g_{23}^x & g_{24}^x & 0 & g_{26}^x\\ 0 & -g_{23}^x & 0 & 0 & 0 & g_{36}^x\\ -g_{14}^x & -g_{24}^x & 0 & 0 & 0 & 0\\ -g_{15}^x & 0 & 0 & 0 & 0 & g_{56}^x\\ 0 & -g_{26}^x & -g_{36}^x & 0 & -g_{56}^x & 0 \end{pmatrix}$$

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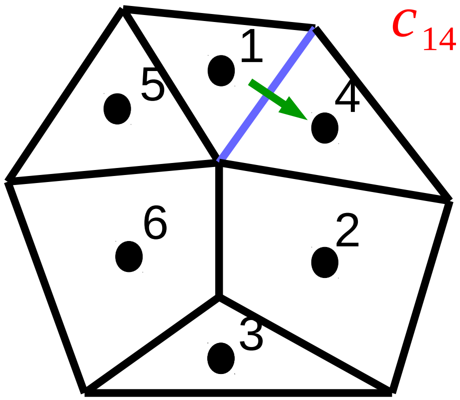

$$c_{14} = A_{14} U_{14}$$

$$\boxed{\mathbf{C} = \begin{pmatrix} \mathbf{C}_u & \\ & \mathbf{C}_v \end{pmatrix}}$$

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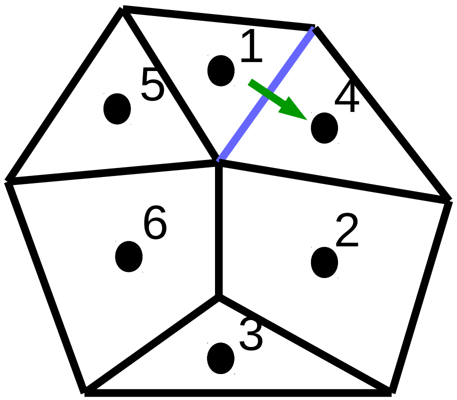
Algebraic operators

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = \nu \nabla^2 \vec{u} - \nabla p \qquad \nabla \cdot \vec{u} = 0$$

$$\mathbf{\Omega} \frac{d u_h}{d t} + \mathbf{C}(u_h) u_h = \mathbf{D} u_h - \mathbf{\Omega} \mathbf{G} p_h \qquad \mathbf{M} u_h = 0_h$$



$$\mathbf{\Omega} = \begin{pmatrix} \mathbf{\Omega}_u & \\ & \mathbf{\Omega}_v \end{pmatrix}$$

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}^x \\ \mathbf{G}^y \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}^u & \mathbf{M}^v \end{pmatrix}$$

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$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_u & \\ & \mathbf{C}_v \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} \mathbf{D}_u & \\ & \mathbf{D}_v \end{pmatrix}$$

Let us consider square matrices, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$:

- Eigenvalues&eigenvectors: $\boldsymbol{A}\vec{v}_i = \lambda_i \vec{v}_i, \quad i = 1, \ldots, n$

  ...or equivalently $(\boldsymbol{A} - \lambda \boldsymbol{I})\vec{v} = \vec{0}$

  $|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$ characteristic equation of $\boldsymbol{A}$

- $\boldsymbol{A} = \underbrace{\frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)}_{symmetric} + \underbrace{\frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T)}_{skew-symmetric}$

Algebraic operators: basic properties

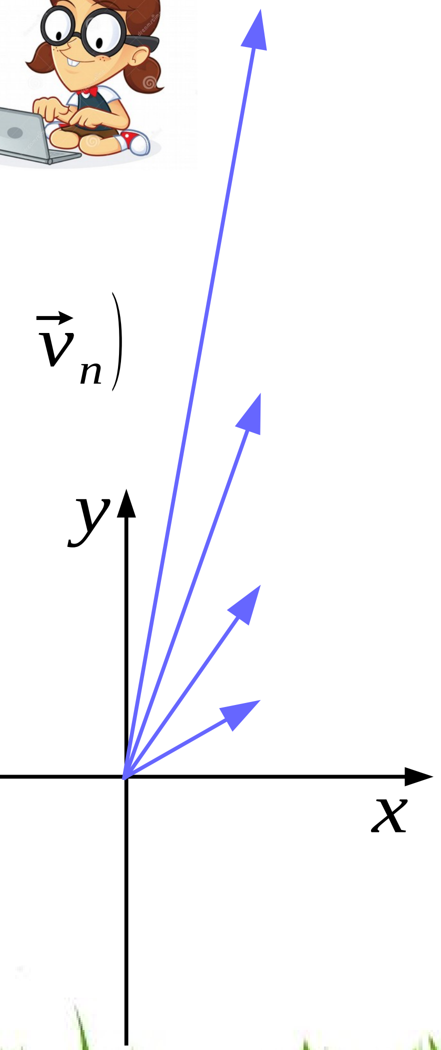
Symmetric matrices, $\boldsymbol{A}=\boldsymbol{A}^T$:

$$\boldsymbol{A}\vec{v}_i=\lambda_i\vec{v}_i, \qquad \lambda_i\in\mathbb{R} \quad \vec{v}_i\in\mathbb{R}^n$$

$$\boldsymbol{\Lambda}=\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} \quad \text{where} \quad \boldsymbol{P}=\begin{pmatrix}\vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n\end{pmatrix}$$

Example: $\vec{x},\boldsymbol{\Lambda}\vec{x},\boldsymbol{\Lambda}^2\vec{x},\boldsymbol{\Lambda}^3\vec{x},\dots$

$$\boldsymbol{\Lambda}=\begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix} \quad \vec{x}=\begin{pmatrix}3\\ 16\end{pmatrix}$$



It resembles a diffusive process!

Algebraic operators: basic properties

Skew-symmetric matrices, $\boldsymbol{A}=-\boldsymbol{A}^{T}$:

$$\boldsymbol{A}\vec{v}_i=\lambda_i\vec{v}_i, \quad \lambda_i\in I \quad \vec{v}_i\in I^n$$

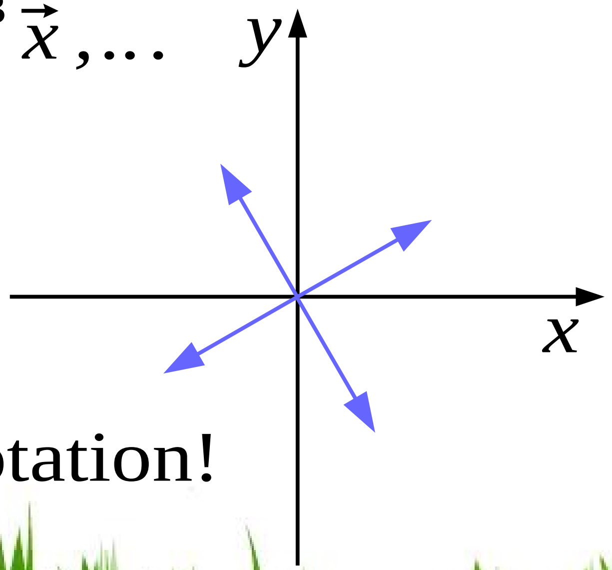
Example:

$$\vec{x}, \boldsymbol{A}\vec{x}, \boldsymbol{A}^2\vec{x}, \boldsymbol{A}^3\vec{x}, \ldots$$

$$\boldsymbol{A}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \vec{x}=\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

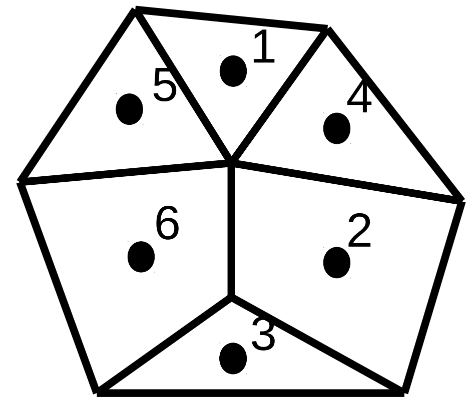
$$\boldsymbol{R}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

It is a 90º rotation!



Algebraic operators: basic properties

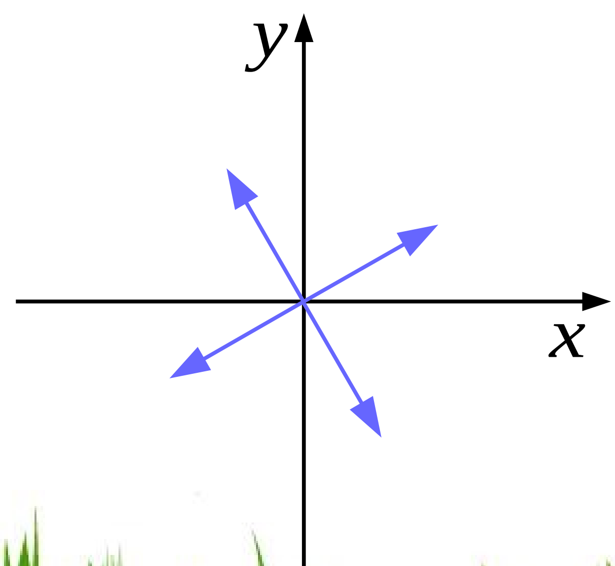
Skew-symmetric matrices, $\boldsymbol{A}=-\boldsymbol{A}^T$:



$$\boldsymbol{A}\vec{v}_i=\lambda_i\vec{v}_i\,,\qquad \lambda_i\in I\quad \vec{v}_i\in I^n$$

And it is always a 90º rotation!!!

$$\vec{x}^T\boldsymbol{A}\vec{x}=0\,,\qquad \forall\,\vec{x}\in\mathbb{R}^n$$

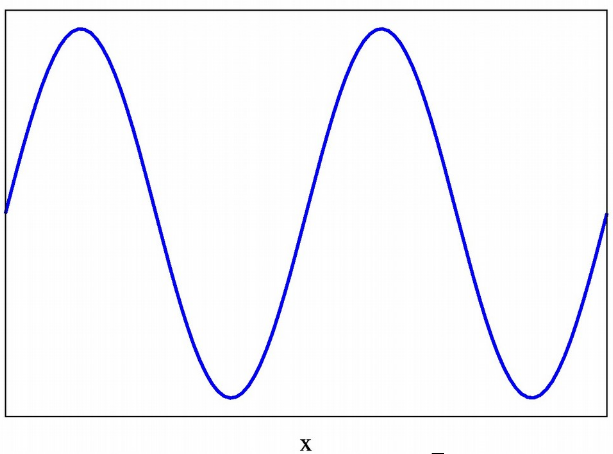


Algebraic operators: basic properties

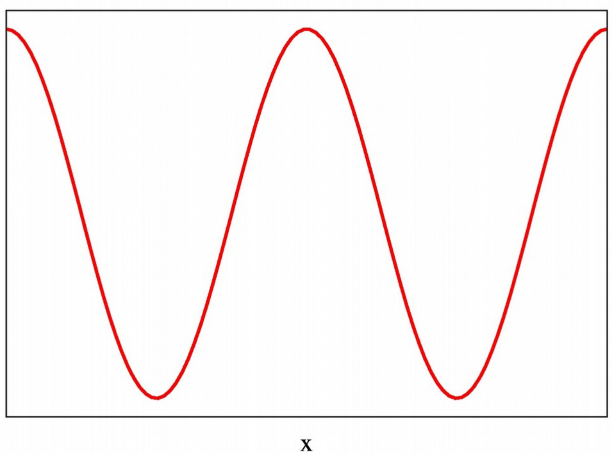
This works for all operators with $\lambda \in I$

Example: $\dfrac{\partial}{\partial x}$, $\lambda = i\boldsymbol{k}$, $e^{i\boldsymbol{k}}$

... and a 1D wave, $u(x) = \sin(\boldsymbol{k}x) \longrightarrow \dfrac{\partial u}{\partial x} = \boldsymbol{k}\cos(\boldsymbol{k}x)$



$$\xrightarrow{\dfrac{\partial u}{\partial x}}$$



They are orthogonal!

REMAINDER!!! Double-angle formula: $\sin(2x) = 2\sin(x)\cos(x)$

REMINDER 3: ENERGY PRESERVING

$$\langle \vec{u} | \vec{u} \rangle \qquad \text{Kinetic energy (in 2D/3D)}$$

$$\frac{1}{2}\frac{d\langle \vec{u}|\vec{u}\rangle}{dt} = \langle \frac{\partial \vec{u}}{\partial t} | \vec{u} \rangle = -\langle C(\vec{u},\vec{u}) | \vec{u} \rangle + \nu \langle \nabla^2 \vec{u} | \vec{u} \rangle - \langle \nabla p | \vec{u} \rangle$$

$$= -\nu \langle \nabla \vec{u} | \nabla \vec{u} \rangle = -\nu \| \nabla \vec{u} \|^2 \leq 0$$

$$= -\nu \langle \nabla \times \nabla \times \vec{u} | \vec{u} \rangle = -\nu \| \omega \|^2 \leq 0$$

If $\nu = 0$, then $\langle \vec{u} | \vec{u} \rangle$ remains constant!!!

Also, if the flow is irrotational, $\vec{\omega} = \vec{0}$. Remember Bernoulli!

ADDITIONAL REMAINDER!!!

$$\nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times \nabla \times \vec{u}$$

REMAINDER!!!

$$\langle \nabla \cdot \vec{a} | \phi \rangle = -\langle \vec{a} | \nabla \phi \rangle$$

$$\langle \nabla^2 f | g \rangle = -\langle \nabla f | \nabla g \rangle = \langle f | \nabla^2 g \rangle$$

$$\langle C(\vec{u}, \phi_1) | \phi_2 \rangle = -\langle C(\vec{u}, \phi_2) | \phi_1 \rangle \quad \text{if} \quad \nabla \cdot \vec{u} = 0$$

$$\langle \nabla \times \vec{a} | \vec{b} \rangle = \langle \vec{a} | \nabla \times \vec{b} \rangle$$

Algebraic operators

$$\frac{\partial \vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}=\nu\nabla^2\vec{u}-\nabla p \qquad \nabla\cdot\vec{u}=0 \qquad \langle a|b\rangle:=\int_\Omega ab\,d\Omega$$

$$\boldsymbol{\Omega}\frac{d u_h}{d t}+\boldsymbol{C}(u_h)u_h=\boldsymbol{D}u_h-\boldsymbol{\Omega}\boldsymbol{G}p_h \qquad \boldsymbol{M}u_h=0_h \qquad \langle a_h|b_h\rangle:=a_h^T\boldsymbol{\Omega}b_h$$

Let's consider the time evolution of $1/2\langle u_h|u_h\rangle\ldots$

$$\frac{1}{2}\frac{d\langle u_h|u_h\rangle}{dt}=u_h^T\boldsymbol{\Omega}\frac{d u_h}{dt}=-u_h^T\boldsymbol{C}(u_h)u_h+u_h^T\boldsymbol{D}u_h-u_h^T\boldsymbol{\Omega}\boldsymbol{G}p_h$$

$$=u_h^T\boldsymbol{D}u_h\leq 0$$

...mimicking the properties of continuous NS eqs leads to

REMAINDER!!!

$$\frac{1}{2}\frac{d\langle\vec{u}|\vec{u}\rangle}{dt}=\langle\frac{\partial\vec{u}}{\partial t}|\vec{u}\rangle=-\langle C(\vec{u},\vec{u})|\vec{u}\rangle+\nu\langle\nabla^2\vec{u}|\vec{u}\rangle-\langle\nabla p|\vec{u}\rangle$$

$$=-\nu\langle\nabla\vec{u}|\nabla\vec{u}\rangle=-\nu\|\nabla\vec{u}\|^2\leq 0$$

$$=-\nu\langle\nabla\times\nabla\times\vec{u}|\vec{u}\rangle=-\nu\|\omega\|^2\leq 0$$

Numerical stability!!!

Algebraic operators

$$\frac{1}{2}\frac{d\langle u_h|u_h\rangle}{dt}=u_h^T\boldsymbol{\Omega}\frac{d\,u_h}{dt}=-u_h^T\boldsymbol{C}(u_h)u_h+u_h^T\boldsymbol{D}u_h-u_h^T\boldsymbol{\Omega}\boldsymbol{G}p_h$$

$$=u_h^T\boldsymbol{D}u_h\leq 0\ ,\qquad \text{if}\ \ \boldsymbol{M}u_h=0_h,\ \forall u_h,p_h$$

$$u_h^T\boldsymbol{C}(u_h)u_h=0 \longrightarrow \boldsymbol{C}(u_h)=-\boldsymbol{C}^T(u_h)$$

$$u_h^T\boldsymbol{\Omega}\boldsymbol{G}p_h=0 \longrightarrow \boldsymbol{\Omega}\boldsymbol{G}=-\boldsymbol{M}^T$$

$$u_h^T\boldsymbol{D}u_h\leq 0 \longrightarrow \boldsymbol{D}=\boldsymbol{D}^T\ \ \text{def-}$$

REMAINDER!!!

$$\frac{1}{2}\frac{d\langle\vec{u}|\vec{u}\rangle}{dt}=\langle\frac{\partial\vec{u}}{\partial t}|\vec{u}\rangle=-\langle C(\vec{u},\vec{u})|\vec{u}\rangle+\nu\langle\nabla^2\vec{u}|\vec{u}\rangle-\langle\nabla p|\vec{u}\rangle$$

$$=-\nu\langle\nabla\vec{u}|\nabla\vec{u}\rangle=-\nu\|\nabla\vec{u}\|^2\leq 0$$

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REMAINDER!!!

$$\langle\nabla\cdot\vec{a}|\phi\rangle=-\langle\vec{a}|\nabla\phi\rangle$$

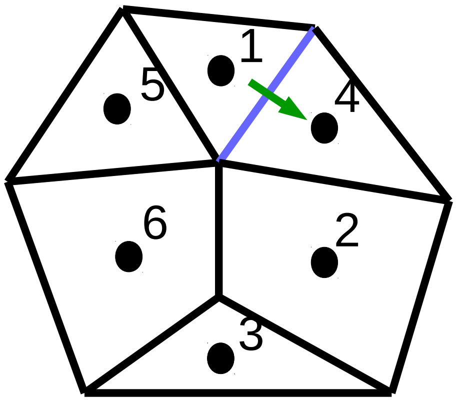
$$\langle\nabla^2 f|g\rangle=-\langle\nabla f|\nabla g\rangle=\langle f|\nabla^2 g\rangle$$

$$\langle C(\vec{u},\phi_1)|\phi_2\rangle=-\langle C(\vec{u},\phi_2)|\phi_1\rangle\quad \text{if}\ \ \nabla\cdot\vec{u}=0$$

$$\langle\nabla\times\vec{a}|\vec{b}\rangle=\langle\vec{a}|\nabla\times\vec{b}\rangle$$

Algebraic operators

$$\mathbf{\Omega}\frac{d u_h}{d t}+\mathbf{C}(u_h)u_h=\mathbf{D}u_h-\mathbf{\Omega}\mathbf{G}p_h \qquad \mathbf{M}u_h=0_h$$



$$\mathbf{T}=\begin{pmatrix} \times & 0 & 0 & \times & \times & 0 \\ 0 & \times & \times & \times & 0 & \times \\ 0 & \times & \times & 0 & 0 & \times \\ \times & \times & 0 & \times & 0 & 0 \\ \times & 0 & 0 & 0 & \times & \times \\ 0 & \times & \times & 0 & \times & \times \end{pmatrix}$$

$$\boxed{\mathbf{\Omega}=\begin{pmatrix} \mathbf{\Omega}_u & \\ & \mathbf{\Omega}_v \end{pmatrix}}$$

$$\boxed{\mathbf{M}=\begin{pmatrix} \mathbf{M}^u & \mathbf{M}^v \end{pmatrix}}$$

$$\mathbf{\Omega}\mathbf{G}=-\mathbf{M}^T$$

$$\boxed{\mathbf{G}=\begin{pmatrix} \mathbf{G}^x \\ \mathbf{G}^y \end{pmatrix}}$$

$$\mathbf{D}_u=\mathbf{D}_v=\mathbf{M}\mathbf{G}=-\mathbf{M}\mathbf{\Omega}^{-1}\mathbf{M}^T$$

$$\boxed{\mathbf{C}=\begin{pmatrix} \mathbf{C}_u & \\ & \mathbf{C}_v \end{pmatrix}} \qquad \boxed{\mathbf{D}=\begin{pmatrix} \mathbf{D}_u & \\ & \mathbf{D}_v \end{pmatrix}}$$

From Calculus to Algebra (C2A)

C2A

$$\frac{\partial \vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}=\nu\nabla^2\vec{u}-\nabla p; \quad \nabla\cdot\vec{u}=0 \longrightarrow \boldsymbol{\Omega}\frac{d u_h}{d t}+\boldsymbol{C}(u_h)u_h=\boldsymbol{D}u_h-\boldsymbol{\Omega}\boldsymbol{G}p_h; \quad \boldsymbol{M}u_h=0_h$$

$$\langle a|b\rangle:=\int_\Omega ab\,d\Omega \longrightarrow \langle a_h|b_h\rangle:=a_h^T\boldsymbol{\Omega}b_h$$

$$\langle C(\vec{u},\phi_1)|\phi_2\rangle=-\langle C(\vec{u},\phi_2)|\phi_1\rangle \longrightarrow \boldsymbol{C}(u_h)=-\boldsymbol{C}^T(u_h)$$

$$\langle\nabla\cdot\vec{a}|\phi\rangle=-\langle\vec{a}|\nabla\phi\rangle \longrightarrow \boldsymbol{\Omega}\boldsymbol{G}=-\boldsymbol{M}^T$$

$$\langle\nabla^2 f|g\rangle=\langle f|\nabla^2 g\rangle \longrightarrow \boldsymbol{D}=\boldsymbol{D}^T \ \text{def-}$$

REMAINDER!!!

$$\langle\nabla\cdot\vec{a}|\phi\rangle=-\langle\vec{a}|\nabla\phi\rangle$$

$$\langle\nabla^2 f|g\rangle=-\langle\nabla f|\nabla g\rangle=\langle f|\nabla^2 g\rangle$$

$$\langle C(\vec{u},\phi_1)|\phi_2\rangle=-\langle C(\vec{u},\phi_2)|\phi_1\rangle \quad \text{if} \quad \nabla\cdot\vec{u}=0$$

$$\langle\nabla\times\vec{a}|\vec{b}\rangle=\langle\vec{a}|\nabla\times\vec{b}\rangle$$

$$u_h^T\boldsymbol{C}(u_h)u_h=0 \longrightarrow \boldsymbol{C}(u_h)=-\boldsymbol{C}^T(u_h)$$

$$u_h^T\boldsymbol{\Omega}\boldsymbol{G}p_h=0 \longrightarrow \boldsymbol{\Omega}\boldsymbol{G}=-\boldsymbol{M}^T$$

$$u_h^T\boldsymbol{D}u_h\leq 0 \longrightarrow \boldsymbol{D}=\boldsymbol{D}^T \ \text{def-}$$

Take-away messages

- Differential **calculus** and **linear algebra** are intimately connected. All CFD-ers MUST be aware of this.
- Preserving operators symmetries leads to **numerical stability** (in the L2-norm sense).

Questions:

- What **restrictions** does it impose in our **numerical schemes**? Are they reasonable?
- What type of **conflicts** appear then? *E.g*. high-order schemes, dispersion errors, solution boundness, ...