Validation of the convection-diffusion equation

The validation of the discretization of the convection-diffusion equation given a velocity field is done for different situation of evident or known solution.
The equation is:

$$\frac{\partial\left(\rho\phi\right)}{\partial t}+\nabla\cdot\left(\rho\vec{v}\phi\right)=\nabla\cdot\left(\Gamma\nabla\phi\right)+S \tag{1}$$

1 Unidimensional flow with a unidimensional variation of the variable solved in the same direction of the flow

A steady state analytical solution for this situation is known. The solution is an exponential function for an arbitrary value of the velocity field.

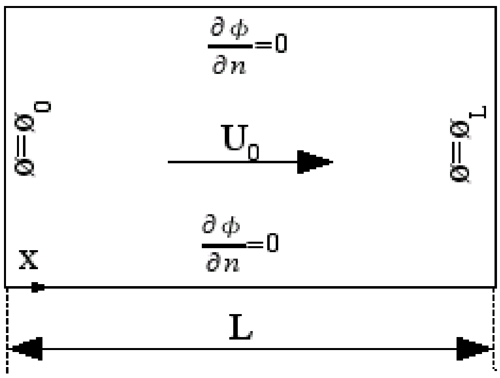

Figure 1: Unidimensional flow with a unidimensional variation of the variable solved in the same direction of the flow.

Velocity field:

$$u(x,y)=U_0$$
$$v(x,y)=0$$

Solution:

$$\frac{\phi-\phi_0}{\phi_L-\phi_0}=\frac{exp(Px/L)-1}{exp(P)-1}$$

where P is the Peclet number: $P=\rho uL/\Gamma$

Note: The numerical result has to be identical to the analytical if the numerical scheme employed for the convection terms is the exponential or the PowerLaw (the polynomial approximation of the exponential). See [1] for details about numerical schemes.

2 Unidimensional flow with a unidimensional variation of the variable solved in the perpendicular direction of the flow

This situation is the same as conduction in a moving solid, independently of the velocity field. In a rectangular geometry:

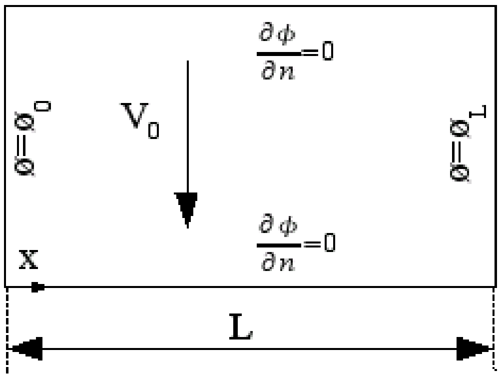

Figure 2: Unidimensional flow with a unidimensional variation of the variable solved in the perpendicular direction of the flow.

Velocity field:

$$u(x,y) = 0$$
$$v(x,y) = V_0$$

Solution:

$$\phi = \phi_0 + \frac{\phi_L - \phi_0}{L} x$$

Note: The numerical result has to be identical to the analytical independently of the numerical scheme used for the convective terms and independently of the mesh size. This situation can be generalised to a case with uniform distributed internal sources, or to a cylinder (with or without source terms) rotating at an angular velocity W.

3 Diagonal flow

If the flow is in the main diagonal and the boundary conditions of the dependent variable are the ones indicated in the figure, the solution is known for an infinite total Peclet number.

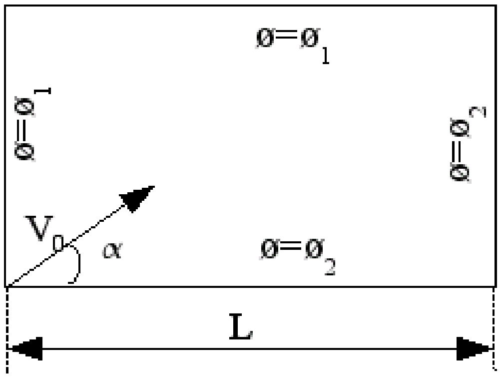

Figure 3: Diagonal flow.

Velocity field:

$$u(x,y) = V_0 \cdot cos(\alpha)$$
$$v(x,y) = V_0 \cdot sin(\alpha)$$

Solution for $P_{(\text{total})} = \infty$:

$$\phi = \phi_1 \quad \text{above the diagonal}$$
$$\phi = \phi_2 \quad \text{below the diagonal}$$

Note: The numerical solution is very sensitive to the mesh size. For coarse meshes the truncation errors are important (false diffusion). The employment of high order schemes for convective terms is specially adequate.

4 Solenoidal flow

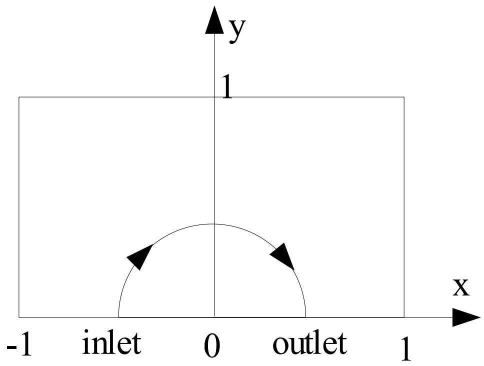

Figure 4: Solenoidal flow

In a rectangular domain with a velocity field:

$$u(x, y) = 2y(1 - x^2)$$
$$v(x, y) = -2x(1 - y^2)$$

and with the following boundary conditions of the dependent variable:

$\phi = 1 + tanh[(2x + 1)\alpha]$	at	-1<x<0	y=0	(inlet)
$\phi = 1 - tanh[\alpha]$	at	x=-1	0<y<1	
	at	-1<x<1	y=1	
	at	x=1	0<y<1	
$\frac{\partial \phi}{\partial y} = 0$	at	0<x<1	y=0	(outlet)

where $\alpha = 10$.

Note: The tanh function present in the inlet boundary condition gives a ϕ almost 0 for -1<x<-0.5 and 2 -0.5<x<0. The value of ϕ at the other boundaries 0. See [2] for more details.

Numerical results at the outlet for different values of ρ/Γ are given in the following table.

Position x	$\rho/\Gamma = 10$	$\rho/\Gamma = 1000$	$\rho/\Gamma = 1000000$
0.0	1.989	2.0000	2.000
0.1	1.402	1.9990	2.000
0.2	1.146	1.9997	2.000
0.3	0.946	1.9850	1.999
0.4	0.775	1.8410	1.964
0.5	0.621	0.9510	1.000
0.6	0.480	0 1540	0.036
0.7	0.349	0.0010	0.001
0.8	0.227	0.0000	0.000
0.9	0.111	0.0000	0.000
1.0	0.000	0.0000	0.000

Table 1: Numerical results at the outlet for different ρ/Γ numbers

References

[1] S.V. Patankar. *Numerical heat transfer and fluid flow*. Hemisphere Publishing Corporation, 1980.

[2] R.M. Smith and A.G. Hutton. The numerical treatment of advection: a performance comparison of current methods. *Numerical Heat Transfer*, 5:439–461, 1982.