PHYSICAL FORMULATION

MATHEMATICS AS THE LANGUAGE OF PHYSICS

A. BÁEZ, F.X. TRIAS AND N. VALLE

Centre Tecnològic de Transferència de Calor
UNIVERSITAT POLITÈCNICA DE CATALUNYA

EQUATIONS

CONSTRAINING SOLUTIONS

REVIEW SO FAR

- Concept of operator
- Link between algebra and calculus
- Operator properties
- Function spaces as vector spaces
- Eigenvalue analysis

AGENDA

1) Eigenvalue analysis of typical PDEs
2) Discussion: Characteristics and Green functions
3) Typical PDEs revisited
4) PDE classification

MULTIDIMENSIONAL SALE!

We can compact our multidimensional forms:

Applying SOV: $S(\vec{s}) = X(x)Y(y)\ldots = \Gamma_i(s_i)\Gamma_j(s_j)\ldots$

We end up with many: $\partial_x^2 \Gamma = \mu^2 \Gamma$

Whose solution is: $\Gamma_n = a_n e^{(\mu_n x)} + b_n e^{(-\mu_n x)} \forall n \in \mathbb{N}$

But is the same as: $\Gamma_n = c_n e^{(\mu_n x)} \forall n \in \mathbb{Z}$

Finally yelds: $S(\vec{s}) = d_n e^{(\mu_n s_1 + \nu_m s_2 + \ldots)} \forall n, m, \ldots \in \mathbb{Z}$

BONUS slide!

$$\vec{\lambda}_{mn} = (\mu_m, \nu_n)$$

$$S(\vec{s}) = d_n e^{(\vec{\lambda}_{mn} \cdot \vec{s})} \forall n, m, \ldots \in \mathbb{Z}$$

MULTIDIMENSIONAL SALE!

Multidimensional SOV:

$$Y\,\partial_x^2 X + X\,\partial_y^2 Y = \mu^2 XY \rightarrow \frac{\partial_x^2 X}{X} = \mu^2 - \frac{\partial_y^2 Y}{Y} = \psi^2$$

$$\frac{\partial_x^2 X}{X} = \psi^2 \rightarrow X = a_n e^{(\psi_n x)} + b_n e^{(-\psi_n x)} \rightarrow BC \rightarrow X_n = a_n\left(e^{(\psi_n x)} \pm e^{(-\psi_n x)}\right)$$

$$\frac{\partial_y^2 Y}{Y} = \omega^2 \rightarrow Y = c_m e^{(\omega_m y)} + d_m e^{(-\omega_m y)} \rightarrow BC \rightarrow Y_m = d_m\left(e^{(\omega_m y)} \pm e^{(-\omega_m y)}\right)$$

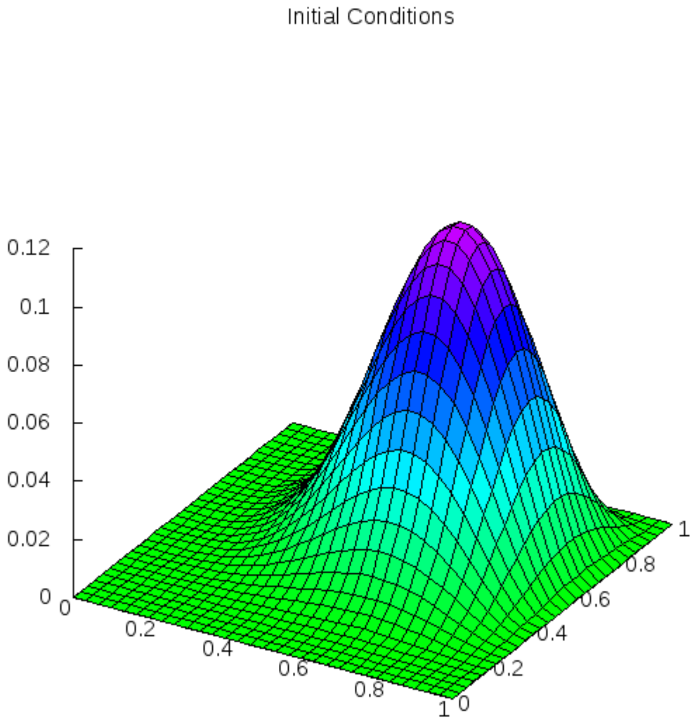
$NOTES:$

$$\omega_m^2 = \mu_{nm}^2 - \psi_n^2 \rightarrow \mu_{nm}^2 = \omega_m^2 + \psi_n^2 \quad a_n = f(n)\,\forall\, n \in \mathbb{N} \quad b_m = g(m)\,\forall\, m \in \mathbb{N}$$

$$X_n = \pm X_{-n} \rightarrow X_n = k_n e^{(\mu_n x)}\,\forall\, n \in \mathbb{Z}$$

$$Y_m = \pm Y_{-m} \rightarrow Y_m = l_m e^{(\omega_m y)}\,\forall\, m \in \mathbb{Z}$$

BASIC PDE: DIFFUSION



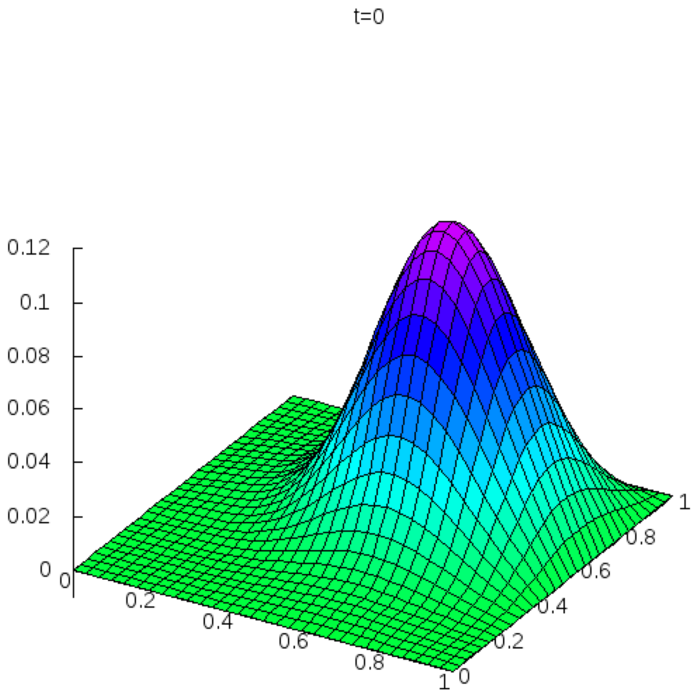
Eigenvalue analysis

$$\left| \begin{array}{l} \dfrac{\partial \theta}{\partial t} = \alpha \nabla^2 \theta \\ \theta|_{\partial\Omega} = 0 \end{array} \right. \rightarrow \begin{array}{c} \theta = T(t) S(\vec{s}) \\ \mu_m \in I \\ \nu_n \in I \\ \tau_{mn} \in \mathbb{R} < 0 \end{array}$$

$$\tau_{mn} = \alpha \left(\mu_m^2 + \nu_n^2 \right)$$

$$\theta = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{mn} e^{(-\tau_{nm} t)} e^{(\nu_n x + \mu_m y)}$$

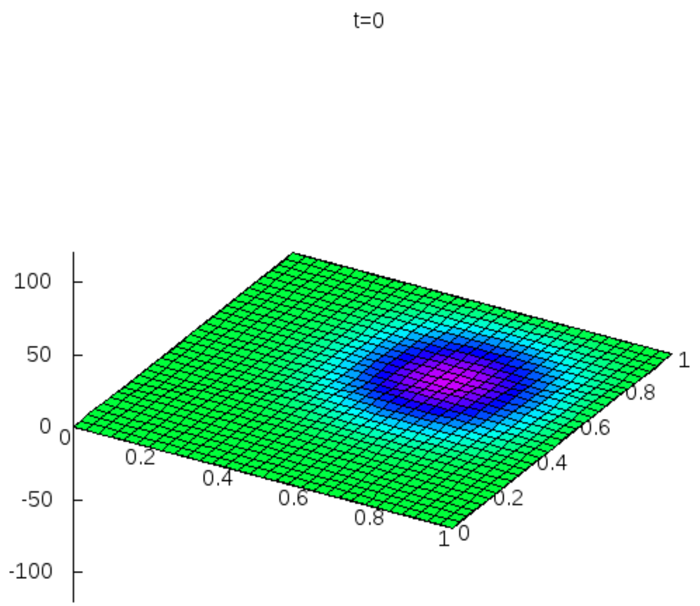
BASIC PDE: DIFFUSION



Physical conclusions

- No new maxima or minima
- Smearing of the profile
- What is the effect of negative diffusivity?

BASIC PDE: DIFFUSION

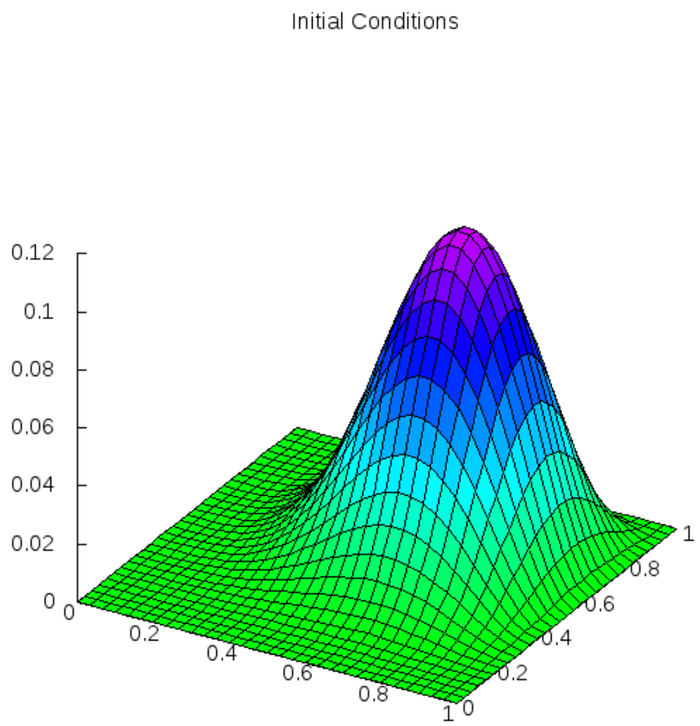


Physical conclusions

- Negative diffusivity just blows things up!

BASIC PDE: ADVECTION

Eigenvalue analysis



$$\frac{\partial \theta}{\partial t} = -\vec{u} \cdot \nabla \theta \rightarrow$$

$$\theta = T(t) S(\vec{s})$$

$$\mu_m \in I$$

$$\nu_n \in I$$

$$\tau_{mn} \in I$$

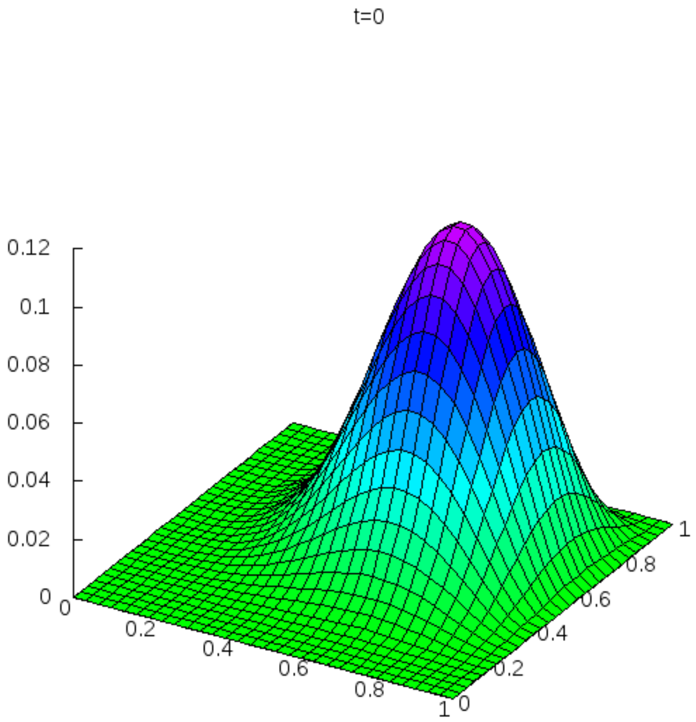
$$\theta = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{mn} e^{(-\tau_{mn} t)} e^{(\nu_m x + \mu_n y)}$$

$$\vec{\lambda}_{mn} = (\mu_m, \nu_n)$$

$$\tau_{mn} = -\mu_m u - \nu_n v = -\vec{\lambda}_{mn} \cdot \vec{u}$$

$$\theta = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{mn} e^{(\vec{\lambda}_{mn} \cdot (\vec{x} - t \vec{u}))}$$

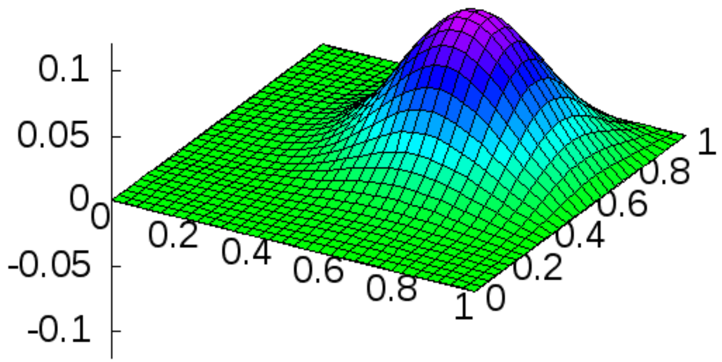
BASIC PDE: ADVECTION



Physical conclusions

- Shift translation
- Shape preserving

BASIC PDE: POISSON



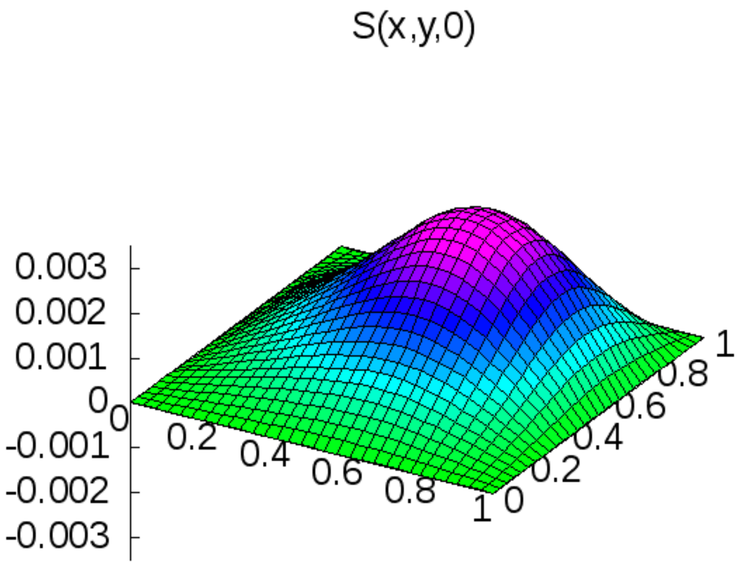
Eigenvalue analysis

$$\left| \begin{array}{l} \nabla^2 \theta = f(\vec{s}, t) \\ \theta|_{\partial\Omega} = 0 \end{array} \right. \rightarrow \begin{array}{c} \theta = T(t) S(\vec{s}) \\ \mu_m \in I \\ \nu_n \in I \end{array}$$

$$\theta = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{mn}(t) e^{(\nu_m x + \mu_n y)}$$

$$f(\vec{s}, t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_{mn}(t) e^{(\nu_m x + \mu_n y)}$$

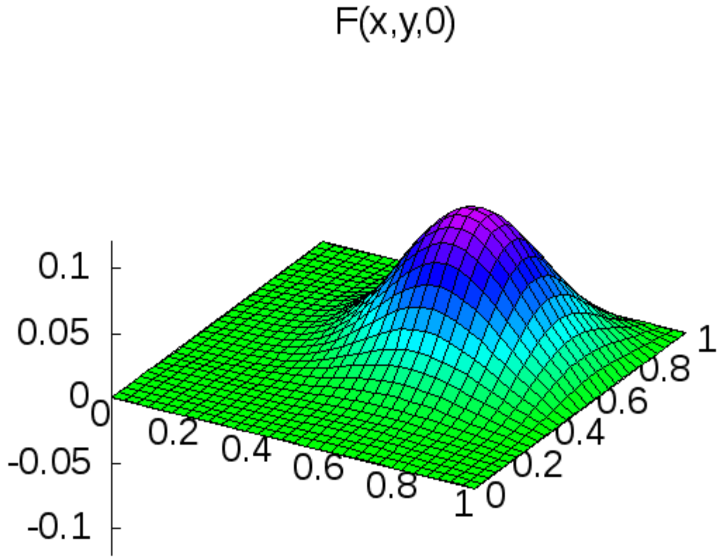
BASIC PDE: POISSON

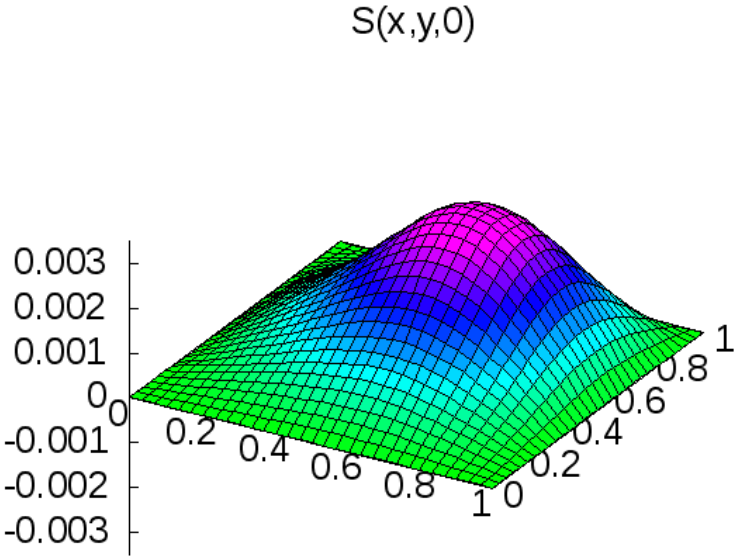


Physical conclusions

- Immediate time response
- Profile smoothing

BASIC PDE: POISSON





DISCUSSION

- Are these cases realistic?
- What can you do when dealing with variable coefficients?
- What kind of information can we obtain from the base space?