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Understanding the Equations

MATHEMATICS AS THE LANGUAGE OF PHYSICS

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1. From complicated to simple equations
    1. Changing coordinates
    2. Characteristic Lines
    3. Fundamental Solutions
2. Understanding the equations and systems of equations
    1. What determines the solution properties?
    2. Heat equation, Waves equation and Poissson equation revisited
3. Towards the discretization

- I don’t know if it’s better to mix everything and explain it all together when it naturally appears.
- I mean, why not using an equation to introduce a concept and then go back to study the equation properties?
  - This would help transmit the use of each and every concept.
  - Things can become messy and inaccurate.

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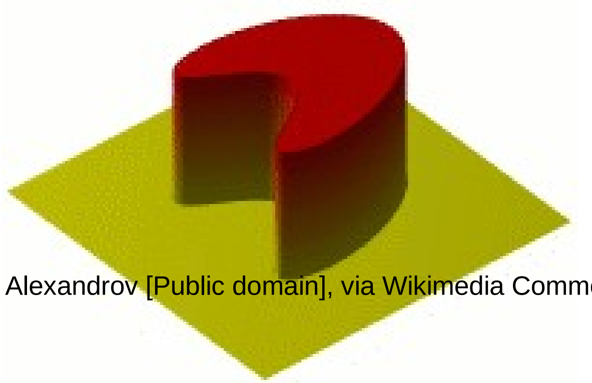
Heat transfer in an anisotropic medium

Problem statement:



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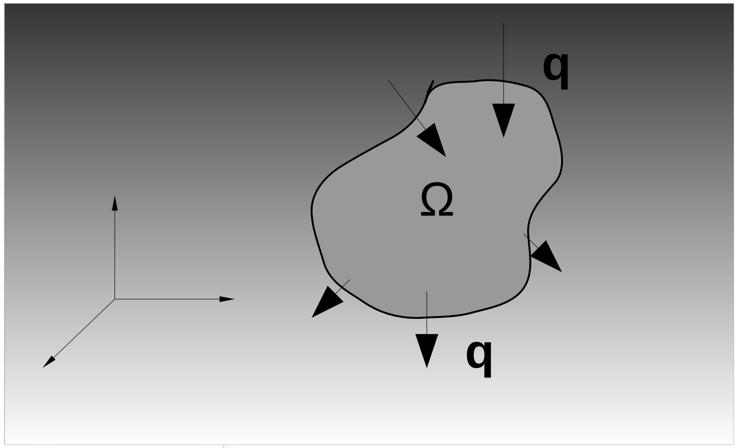
Find $\varphi(\mathbf{x}, t)$

$$(\boldsymbol{x};t)\in D\times T;\;\; D\subseteq\Re^3;\; T\subseteq[0,\infty]$$

Such that $\forall\,\Omega\subseteq D$

Positive Semidefinite ~~Matrix~~

TENSOR!



$$\frac{\partial}{\partial t}\int_{\Omega}\varphi\,\delta\Omega=\int_{\partial\Omega}\boldsymbol{q}\cdot\boldsymbol{n}\,\delta\sigma$$

$$\boldsymbol{q}=-A(\boldsymbol{x};t)\nabla\varphi;\;\; A(\boldsymbol{x};t)\succcurlyeq 0$$

$$\varphi(\boldsymbol{x},0)=\varphi_0$$

$$\Psi(\varphi,\nabla\varphi)(\boldsymbol{x_b},t)=0;\;\; \forall(\boldsymbol{x_b},t)\in\partial D\times T$$

Heat transfer in an anisotropic medium

TENSORS

Tensors are geometric objects that describe linear relations between geometric vectors, scalars, and other tensors.← LOL

When changing coordinates, TENSORS change in a very specific manner.

e.g. Viscosity tensor A(**x**, t).A is defined such that

$$\boldsymbol{q} = -A(\boldsymbol{x};t)\nabla\varphi \quad \text{or} \quad q^i = A^i_j (\nabla\varphi)^j$$

Supose we change coordinates **x**→ **ξ**

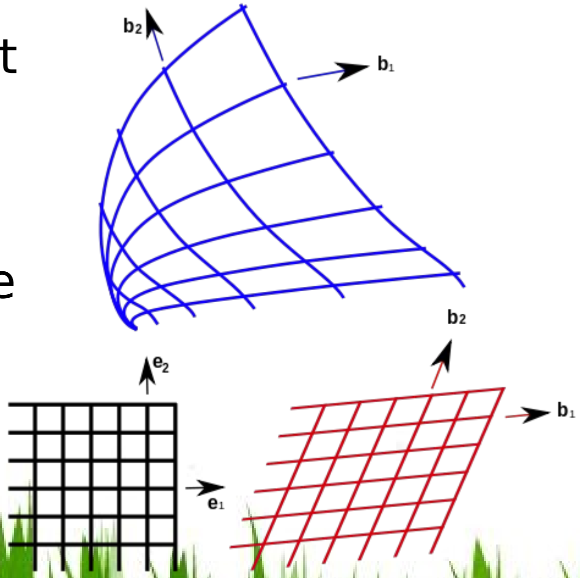
Hence, even if we changed coordinates, the heat flux **vector** should be the same at the corresponfing position.

$$q^i = -A(\boldsymbol{x};t)^i_j (\nabla\varphi_x)^j = -A(\boldsymbol{\xi}(\boldsymbol{x});t)^i_j \left(\frac{\partial \xi_k}{\partial x_j}\frac{\partial\varphi}{\partial\xi_k}\right) = \ldots$$

$$\ldots = -A(\boldsymbol{\xi}(\boldsymbol{x});t)^i_j \frac{\partial\xi_k}{\partial x_j}(\nabla_\xi\varphi)^k$$

This is the “jth” coordinate of **q** in the frame reference of **x** as a function of the derivatives of φ with respect to the **ξ** coordinates.

Now, I want to change the reference frame to be defined by **ξ**.



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Heat transfer in an anisotropic medium

$\boldsymbol{\xi}=(\xi_1, \xi_1, \xi_3)$;

The vector following the growth of ξ_i is easily calculated.

$$\boldsymbol{e}_{\xi_i}=\frac{\partial \xi_i}{\partial x_1}\boldsymbol{e}_1+\frac{\partial \xi_i}{\partial x_2}\boldsymbol{e}_2+\frac{\partial \xi_i}{\partial x_3}\boldsymbol{e}_3$$

Or, more compactly:

$$\boldsymbol{e}_{\xi_i}=\frac{\partial \xi_i}{\partial x_j}\boldsymbol{e}_j$$

We can write the same the other way around:

$$\boldsymbol{e}_i=\frac{\partial x_i}{\partial \xi_j}\boldsymbol{e}_{\xi_j}$$

And the heat flux vector does not depend on the coordinates we are using

$$\boldsymbol{q}=q^i\boldsymbol{e}_i=q^i\frac{\partial x_i}{\partial \xi_j}\boldsymbol{e}_{\xi_j}=q^{\xi_j}\boldsymbol{e}_{\xi_j}$$

Since **is a basis**, we can equalize the coorinates multiblying each of its vectors $\boldsymbol{e}_{\xi_i}$ in the former equation.

$$q^{\xi_i}=\frac{\partial x_j}{\partial \xi_i}q^j$$

We calculated q^j in the previous slide.

$$q^i=-A(\boldsymbol{x};t)^{ij}\varphi_{x_j}=-A(\boldsymbol{\xi}(\boldsymbol{x});t)^i_j\frac{\partial \xi_k}{\partial x_j}(\nabla_\xi\varphi)^k$$

So finally,

$$\boxed{q^{\xi_i}=\frac{-\partial x_p}{\partial \xi_i}A(\boldsymbol{\xi}(\boldsymbol{x});t)^p_q\left(\frac{\partial \xi_k}{\partial x_q}\varphi_{\xi_k}\right)=-A'(\boldsymbol{\xi}(\boldsymbol{x});t)^i_k(\nabla_\xi\varphi)^k}$$

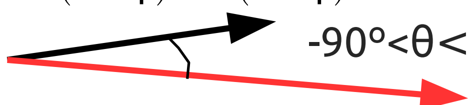
Is A' Symmetric?

Heat transfer in an anisotropic medium

What does this equation actually mean?

- Define an infinitesimal volume around each point of the domain.
- Compute $A\nabla\varphi$ at all the points in the boundary of this infinitesimal volume. REMARK:

$$A \succcurlyeq 0 \Leftrightarrow (\nabla\varphi)^T A (\nabla\varphi) \geq 0 \;\; \forall \nabla\varphi$$



-90º<θ<90º

- Integrate in the surface, divide by the volume.
- The scalar φ will evolve to reduce the average of heat fluxes in the infinitesimal volume surface. This happens in all points of the domain.

In the differential form:

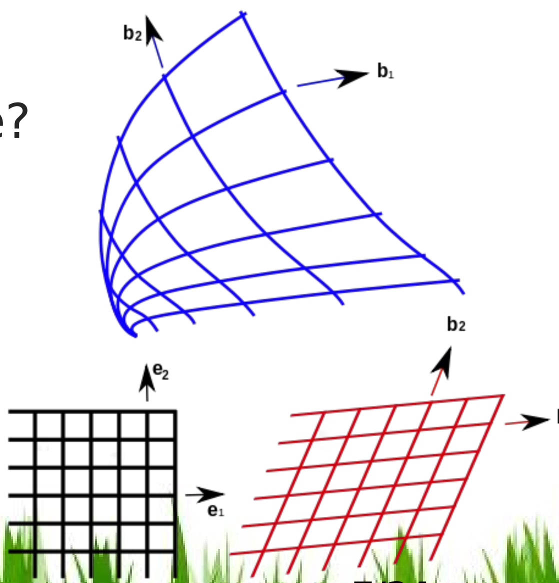
$$\frac{\partial\varphi}{\partial t} = -\nabla\cdot(A\nabla\varphi)$$

I would like to have the equation of the previous lecture to use the Separation of Variables method.

$$\frac{\partial\varphi}{\partial t} = -\alpha\nabla^2\varphi$$

Can this be done?

Maybe, changing the reference frame **X→ ξ** to a curvilinear one .



Heat transfer in an anisotropic medium

Changing coordinates and differential operators

Well constructed operators return the same value regardless of the spatial variables. e.g.

f(x,y)→ f(r, θ) with

r=√(x²+y²); θ=arctg(y/x)

Or

x=r cos(θ) ; y= r sin(θ)

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

e.g. f=sin(r²) r cos(θ)

If the operator defines an scalar, a vector or a tensor, this is true. The reason lies in the behavior of these entities when changing coordinates.

In fact, the Laplacian operator is defined over any curvilinear coordinates system as:

$$\nabla^2 f = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial u^i}\left(\sqrt{\det g}\, g^{ij}\frac{\partial f}{\partial u^j}\right)$$

Where g_{ij} is the covariant metric tensor of the coordinates system, if

$$if\;\; \boldsymbol{e_i} = \frac{\partial x_j}{\partial u_i}\boldsymbol{ec_j}$$

$$\text{then}\;\; g_{ij} = \left(\frac{\partial x_k}{\partial u_i}\boldsymbol{ec_k}\right)\cdot\left(\frac{\partial x_m}{\partial u_j}\boldsymbol{ec_m}\right) = \frac{\partial x_k}{\partial u_i}\frac{\partial x_m}{\partial u_j}\delta_{km}$$

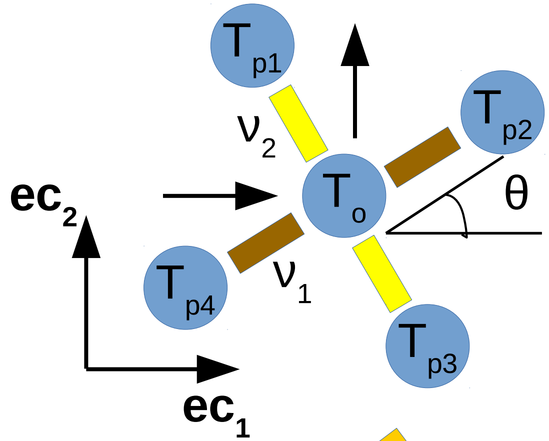
$$\text{and}\; g^{ij} = (g_{ij})^{-1}$$

Heat transfer in an anisotropic medium

Our original spatial euclidean coordinates are not the problem “natural coordinates”.

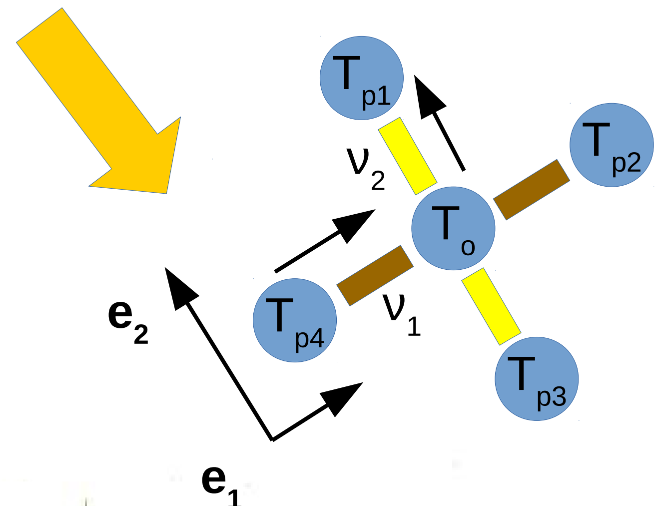
There was a coordinates system in which g for heat flux vectors was the identity and the problem was canonical, moving from it to the euclidean space, g changed, A appeared and everything got more complicated. But that ideal system exists and in it we can use Separation of Variables.

The coordinates system is a choice



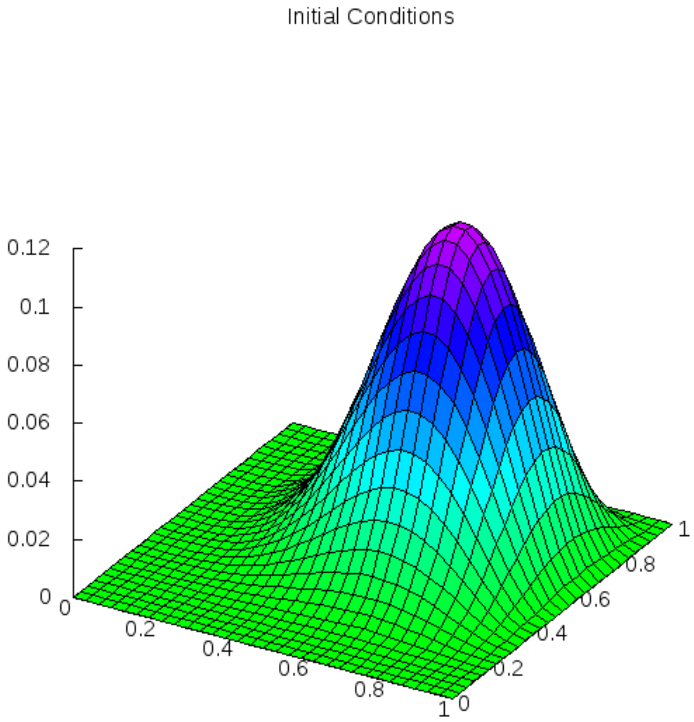
$$A = P^T \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix} P$$

$$P = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$



$$A = \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix}$$

Heat transfer in an anisotropic medium



Eigenvalue analysis

$$\left|\begin{array}{l}\dfrac{\partial\theta}{\partial t}=\alpha\nabla^2\theta \\ \theta|_{\partial\Omega}=0\end{array}\right. \rightarrow \begin{array}{c}\theta=T(t)S(\vec{s}) \\ \mu_m\in I \\ \nu_n\in I \\ \tau_{mn}\in\mathbb{R}<0\end{array}$$

$$\tau_{mn}=\alpha\left(\mu_m^2+\nu_n^2\right)$$

$$\theta=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}a_{mn}e^{(-\tau_{nm}t)}e^{(\nu_n x+\mu_m y)}$$

Heat transfer in an anisotropic medium

So equations like $\dfrac{\partial \varphi}{\partial t} = -\nabla \cdot (A \nabla \varphi)$

are equivalent to $\dfrac{\partial \varphi}{\partial t} = -\alpha \nabla^2 \varphi$

We don’t know the coordinates change that makes it, but we know that

$$\varphi = \sum_{m,n} k(m,n) \exp(-\lambda^2 t) F_{m,n}(\boldsymbol{\xi})$$

with $F_{m,n}$ orthogonal functions

Example of known $F_{m,n}$:

Bessel Functions

Can we generalize the diffusion equation result to more general equations?

e.g. $\dfrac{\partial \varphi}{\partial t} = -\nabla \cdot (B \nabla (\nabla \cdot (A \nabla \varphi)))$

$$\{A, B\} \geqslant 0$$

The real question is:

What conditions are necessary for an equation to have a solution like that of the diffusion equation?

Symmetric OPERATOR

Convection in a non-uniform velocity field

Problem statement:

Find φ(**x**, t)

$$\left(\boldsymbol{x};t\right)\in D\times T;\quad D\subseteq\Re^3;\ T\subseteq[0,\infty]$$

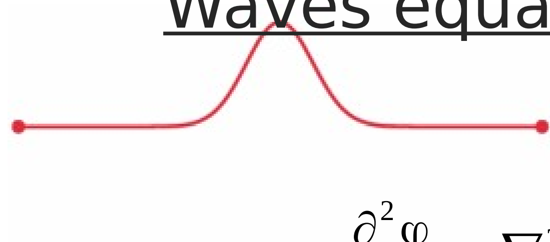
Such that

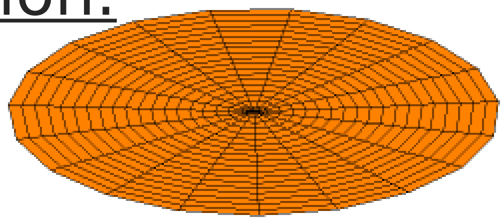
$$\frac{\partial}{\partial t}\int_{\Omega}\varphi\,\delta\Omega=-\int_{\partial\Omega}\varphi\,\boldsymbol{u}\cdot\boldsymbol{n}\,\delta\sigma$$

$$\varphi(\boldsymbol{x},0)=\varphi_0$$

$$\Psi(\varphi,\nabla\varphi)(\boldsymbol{x_b},t)=0;\quad\forall(\boldsymbol{x_b},t)\in\partial D\times T$$

Waves equation:





$$\frac{\partial^2\varphi}{\partial t^2}=c\nabla^2\varphi$$

Convection In the differential form:

$$\frac{\partial\varphi}{\partial t}=-\nabla\cdot(\varphi\boldsymbol{u})\text{ and }\nabla\cdot\boldsymbol{u}=0$$

Would I like to have the equation of the previous lecture (**u** was constant) and use the separation of variables? Not necessarily

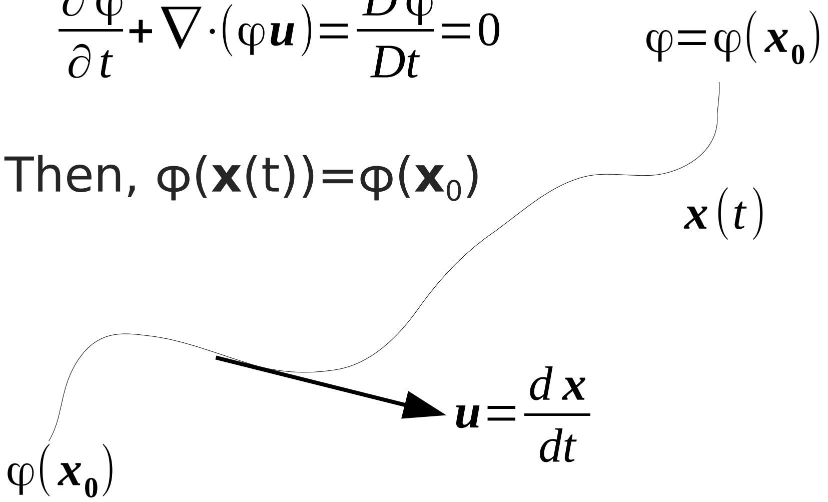
Convection in a non-uniform velocity field

The concept of characteristic lines

Suppose that exists a **x**(t) parametrization such that

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \boldsymbol{u}) = \frac{D\varphi}{Dt} = 0$$

Then, φ(**x**(t))=φ(**x**$_0$)



Do characteristic lines exist for other types of equations like Diffusion equation?

Poisson equation

Resolve φ such that:

$$\nabla^2 \varphi = f \ \forall \boldsymbol{x} \in \Omega$$
$$\Psi(\varphi, \nabla \varphi) = 0 \ \forall \boldsymbol{x} \in \partial\Omega$$

What does $\nabla^2 \varphi$ actually mean?

-Calculus of variations:

It is a comparison between the node position and the average of neighbours with changed sign.

$$\nabla^2 \varphi(\boldsymbol{x}) = \lim_{h \to 0^+} \frac{1}{h^2} \sum_{p \in N_o} \varphi_p - \varphi_o$$

$$\nabla^2 \varphi(\boldsymbol{x}) = \lim_{h \to 0^+} \frac{\varphi(\boldsymbol{x}+\boldsymbol{dx}) + \varphi(\boldsymbol{x}-\boldsymbol{dx}) - 2\varphi(\boldsymbol{x})}{h^2} + \frac{\varphi(\boldsymbol{x}+\boldsymbol{dy}) + \varphi(\boldsymbol{x}-\boldsymbol{dy}) - 2\varphi(\boldsymbol{x})}{h^2}$$

Poisson equation

Fundamental solution:

The solution of an equation on an unbounded domain when the forcing term is delta Dirac.

In Laplace's equation,

$$\nabla^2 F = \delta \ \forall \boldsymbol{x} \in \Omega = \Re^n$$
$$F \rightarrow 0 \ \forall \boldsymbol{x} \in \partial\Omega$$

$$F = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|}$$

- GENERAL Solution

PDE SYSTEMS: CLASSIFICATION

Typical transport PDE in conservation form

$$\frac{\partial \vec{U}}{\partial t} + \nabla \cdot \vec{F}(\vec{U}) = 0 \quad \rightarrow \quad \frac{\partial \vec{U}}{\partial t} + J \cdot \nabla \vec{U} = 0$$

$$M^{-1}\vec{U} = \vec{W} \qquad M^{-1} J = \Lambda$$

Chain rule!

Diagonalization of the Jacobian?

NOTE: $J = \frac{\partial \vec{F}}{\partial \vec{U}}$

PDE SYSTEMS: CLASSIFICATION

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1D, 3 variables example. Euler equations:

$$\vec{U}=\begin{pmatrix}\rho & u & e\end{pmatrix}^T \qquad \vec{F}=\begin{pmatrix}\rho u & \rho u^2+P & \rho u e\end{pmatrix}^T$$

$$\begin{pmatrix}\partial_t \rho\\ \partial_t u\\ \partial_t e\end{pmatrix}+\begin{pmatrix}\partial_x(\rho u)\\ \partial_x(\rho u^2+P)\\ \partial_x(\rho u e)\end{pmatrix}=0 \quad \begin{pmatrix}\partial_t \rho\\ \partial_t u\\ \partial_t e\end{pmatrix}+\begin{pmatrix}u & \rho & 0\\ u^2+C_s^2 & 2\rho & 0\\ u e & \rho e & \rho u\end{pmatrix}\cdot\begin{pmatrix}\partial_x \rho\\ \partial_x u\\ \partial_x e\end{pmatrix}=0$$

So far: $P=f(\rho)$ $\dfrac{\partial P}{\partial \rho}=C_s^2$ Equation of state!!

PDE SYSTEMS: CLASSIFICATION

Diagonalization:

$$\partial_t \vec{U} + J \nabla \vec{U} \quad \rightarrow \quad \partial_t \vec{W} + \Lambda \nabla \vec{W}$$

After diagonalization:

$$\Lambda = \begin{pmatrix} u - Cs & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u + Cs \end{pmatrix}$$

CONCLUSION:

The system evolves as 3 separate transport equations!

HYPERBOLIC SYSTEM OF EQUATIONS

PDE SYSTEMS: CLASSIFICATION

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Incompressible case:

$$\vec{U}=\begin{pmatrix}\rho & u & 0\end{pmatrix}^T \qquad \vec{F}=\begin{pmatrix}\rho u & \rho u^2+P & u\end{pmatrix}^T$$

$$\begin{pmatrix}\partial_t \rho\\ \partial_t u\\ 0\end{pmatrix}+\begin{pmatrix}\partial_x(\rho u)\\ \partial_x(\rho u^2+P)\\ \partial_x u\end{pmatrix}=0 \qquad \begin{pmatrix}\partial_t \rho\\ \partial_t u\\ 0\end{pmatrix}+\begin{pmatrix}u & \rho & 0\\ u^2 & 2\rho & 1\\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}\partial_x \rho\\ \partial_x u\\ \partial_x P\end{pmatrix}=0$$

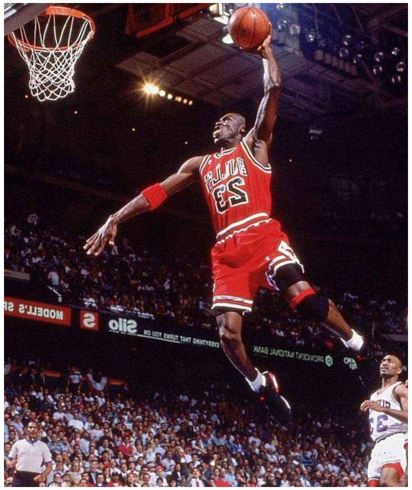
NOTE: P is not a state function anymore!

PDE SYSTEMS: CLASSIFICATION

Diagonalization:

$$\partial_t \vec{U} + J \nabla \vec{U} \quad \rightarrow \quad \partial_t \vec{W} + \Lambda \nabla \vec{W}$$

Diagonalization is not possible! Jordan Form:



$$\Lambda = \begin{pmatrix} u & 1 & 0 \\ 0 & u & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

CONCLUSION:

The system is coupled! → You will always have a coupling

ELLIPTIC SYSTEM OF EQUATIONS

NUMERICAL METHODS: TYPES

So far:

- Physical solution lies in an infinite dimensional space
- Analytical solution involves an infinite sum of functions
- Numerical solution is the truncation of this infinite summation

How are we supposed to handle an inifinite dimensional space?

NUMERICAL METHODS!

NUMERICAL METHODS: TYPES

We can project over ANY base functions we like
PROJECTION is performed thanks to the DOT PRODUCT
Instead of:

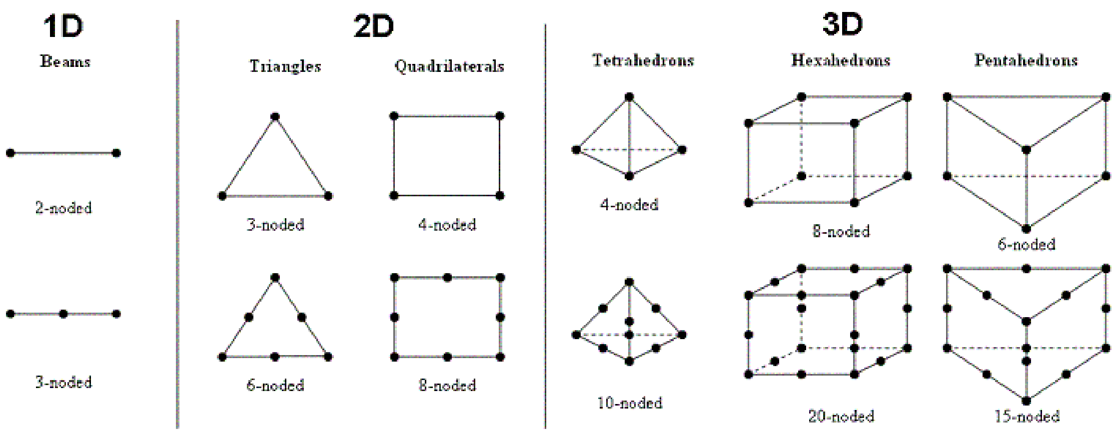
$$\theta = \sum_{n=-\infty}^{+\infty} a_n T_n(t) S_n(\vec{s})$$

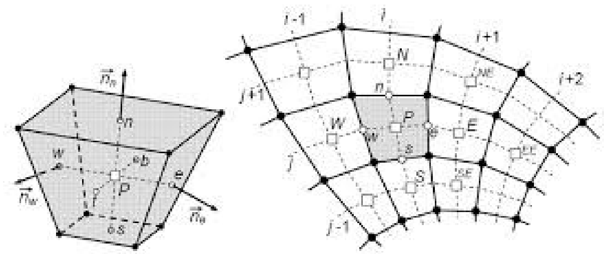
$$\theta = \sum_{n=0}^{N} T_n(t) B(\vec{s})$$

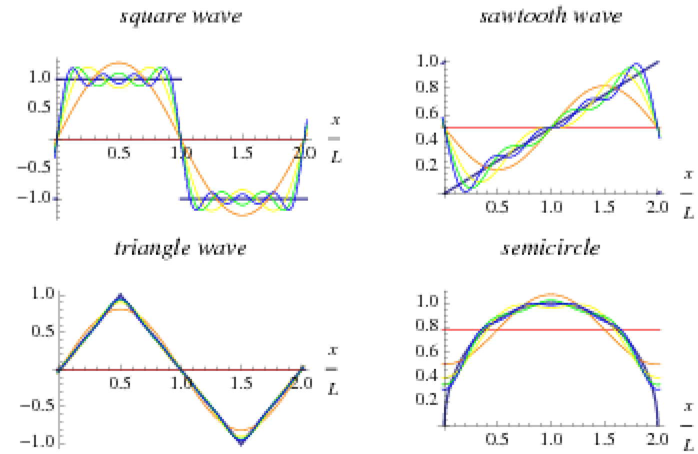
We are CHANGING the BASE of SOLUTION SPACE

What are then $B(\vec{s})$?

NUMERICAL METHODS: TYPES







$$B(\vec{s}) = \begin{cases} \delta_{xp} \rightarrow FDM \\ H_{xv} \rightarrow FVM \\ \sum_{v} w_v(x)\,\delta_{xp} \rightarrow FEM \\ \sin(k_p x) \rightarrow SPECTRAL \end{cases}$$

NOTHING else changes! (FSM, LES, Level-Set, etc)