Chapter 3

The Numerical Flux Function

3.1 Higher-Order Extensions of the Godunov Method

They can be obtained by several techniques. One is to use a more accurate reconstruction of the function from cell averages, such as, for example, a piecewise linear function, and then solve the generalized Riemann problem (see Figure 3.1)

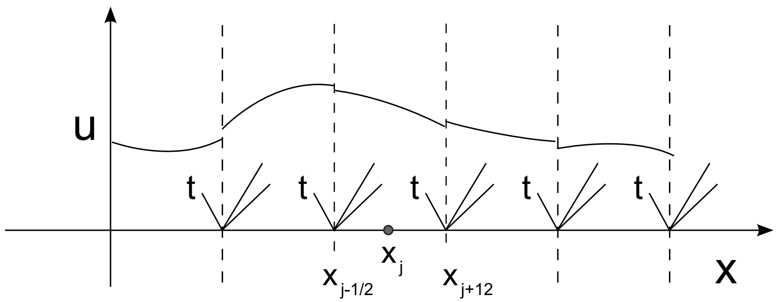

Figure 3.1: Piecewise smooth reconstruction, generalized Riemann problem and high-order extension of the Godunov method.

This approach has been used, for example, by Van Leer.

A second alternative is to use a semidiscrete scheme of the form

$$\frac{d\bar{u}_j}{dt} = -\frac{F(u^-_{j+1/2}, u^+_{j+1/2}) - F(u^-_{j-1/2}, u^+_{j-1/2})}{\Delta x}, \tag{3.1}$$

where $F(u^-, u^+)$ can be, for example, the flux function defining a Godunov scheme $F(u^-, u^+) = f(u^*(u^-, u^+))$, or some other numerical flux function, and the values $u^+_{j+1/2}$, $u^-_{j+1/2}$ are obtained by a suitable reconstruction from cell averages. Because of the relevance of this aspect, a section will be devoted to the reconstruction later. Now we shall concentrate on the properties of the numerical flux function.

3.2 The Scalar Equation and Monotone Fluxes

When dealing with nonlinear problems, linear stability is usually not enough to ensure that a numerical solution converges to a function (and therefore to a weak solution of the conservation law). The upwind scheme

$$u_j^{n+1} = u_j^n - \lambda(u_j^n - u_{j-1}^n)$$

is stable in the L^1-norm, as it is easy to check for the scalar equation, provided $0 \le \lambda \le 1$. The same can be said about the Lax-Friedrichs scheme, while the Lax-Wendroff scheme, on the contrary, is not stable in the L^1-norm, since the coefficients that appear in the three-point formula are not all positive (the L^1-stability is strictly related to the positivity of the coefficients, as we shall see).

To prove that, under the assumption that CFL condition is satisfied, i.e., that $\left|f'(u)\frac{\Delta t}{\Delta x}\right| < 1$, the Lax-Friedrichs scheme is L^1-stable for the equation

$$u_t + f(u)_x = 0\,,$$

one can act as follows: let us assume periodic boundary conditions, and let u_j^n, v_j^n be two numerical solutions.

$$\begin{aligned} u_j^{n+1} &= \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\Delta t}{2\Delta x}(f(u_{j+1}^n) - f(u_{j-1}^n))\,,\\ v_j^{n+1} &= \frac{1}{2}(v_{j+1}^n + v_{j-1}^n) - \frac{\Delta t}{2\Delta x}(f(v_{j+1}^n) - f(v_{j-1}^n))\,. \end{aligned}$$

Let us take the difference:

$$\begin{aligned} u_j^{n+1} - v_j^{n+1} = &\frac{1}{2}(u_{j+1}^n - v_{j+1}^n) - \frac{\Delta t}{2\Delta x}(f(u_{j+1}^n) - f(v_{j+1}^n))\\ &+ \frac{1}{2}(u_{j-1}^n - v_{j-1}^n) + \frac{\Delta t}{2\Delta x}(f(u_{j-1}^n) - f(v_{j-1}^n))\,. \end{aligned}$$

By Lagrange's theorem of the mean, one has

$$f(u_j^n) - f(v_j^n) = f'(\xi_j)(u_j^n - v_j^n)\,,$$

where ξ_j is between u_j^n and v_j^n. Using this relation, one has

$$u_j^{n+1} - v_j^{n+1} = \frac{1}{2}(1 - \lambda_{j+1})(u_{j+1}^n - v_{j+1}^n) + \frac{1}{2}(1 + \lambda_{j-1})(u_{j-1}^n - v_{j-1}^n)\,,$$

where

$$\lambda_j \equiv \frac{\Delta t}{\Delta x} f'(\xi_j)\,.$$