PHYSICAL FORMULATION

MATHEMATICS AS THE LANGUAGE OF PHYSICS

A. BÁEZ, F.X. TRIAS AND N. VALLE

WHAT OPERATOR PROPERTIES CAN TELL US ABOUT NS EQUATIONS?

MORE THAN YOU PROBABLY EXPECT...

«Turbulence is the most important unsolved problem of classical Physics.» by the Nobel Laureate physicist Richard Feynman

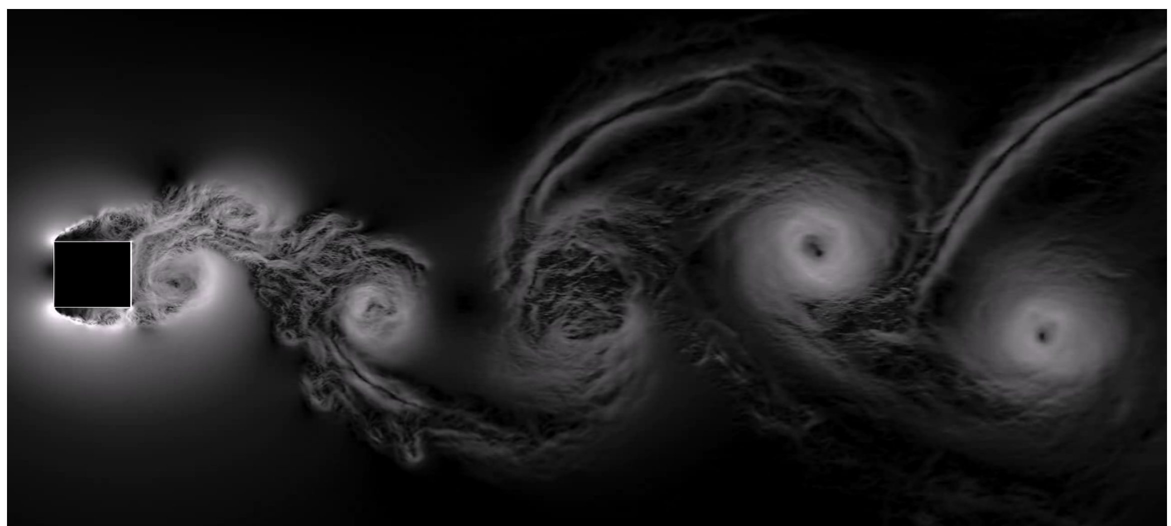
REMINDER 1: NS EQUATIONS

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = \nu \nabla^2 \vec{u} - \nabla p$$

Newton's 2nd law

$$\nabla \cdot \vec{u} = 0$$

Mass conservation



$$\langle \nabla \cdot \vec{a} | \phi \rangle = -\langle \vec{a} | \nabla \phi \rangle$$

If you are not interested in their proofs go to #Slide 8

$$\langle \nabla^2 f | g \rangle = -\langle \nabla f | \nabla g \rangle = \langle f | \nabla^2 g \rangle$$

$$\langle C(\vec{u}, \phi_1) | \phi_2 \rangle = -\langle C(\vec{u}, \phi_2) | \phi_1 \rangle \quad \text{if} \quad \nabla \cdot \vec{u} = 0$$

$$\langle \nabla \times \vec{a} | \vec{b} \rangle = \langle \vec{a} | \nabla \times \vec{b} \rangle$$

Notation:

$$\langle a | b \rangle := \int_{\Omega} ab \, d\Omega \qquad C(\vec{u}, \phi) := (\vec{u} \cdot \nabla) \phi$$

REMEMBER: we always assume **no contribution from domain boundary,** $\partial \Omega$

REMINDER 3: PROOFS

$$\langle \nabla \cdot \vec{a} | \phi \rangle = - \langle \vec{a} | \nabla \phi \rangle$$

Proof:

$$\nabla \cdot (\phi \vec{a}) = \phi \nabla \cdot \vec{a} + \vec{a} \cdot \nabla \phi$$

$$\int_{\Omega} \nabla \cdot (\phi \vec{a}) = \langle \phi | \nabla \cdot \vec{a} \rangle + \langle \vec{a} | \nabla \phi \rangle$$

$$\int_{\partial \Omega} (\phi \vec{a}) \cdot \vec{n} \, dS = \langle \phi | \nabla \cdot \vec{a} \rangle + \langle \vec{a} | \nabla \phi \rangle = 0$$

REMAINDER!!!

$$\langle a | b \rangle := \int_{\Omega} a b \, d\Omega$$

REMEMBER: we always assume **no contribution from domain boundary,** $\partial \Omega$

$$\langle \nabla \times \vec{a} | \vec{b} \rangle = \langle \vec{a} | \nabla \times \vec{b} \rangle$$

Proof:

$$\nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b})$$

$$\int_{\Omega} \nabla \cdot (\vec{a} \times \vec{b}) \, d\Omega = \langle \vec{b} | \nabla \times \vec{a} \rangle - \langle \vec{a} | \nabla \times \vec{b} \rangle$$

$$\int_{\partial\Omega} (\vec{a} \times \vec{b}) \cdot \vec{n} \, dS = \langle \vec{b} | \nabla \times \vec{a} \rangle - \langle \vec{a} | \nabla \times \vec{b} \rangle$$

REMAINDER!!!

$$\langle a | b \rangle := \int_{\Omega} a b \, d\Omega \qquad \nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b})$$

REMEMBER: we always assume **no contribution from domain boundary,** $\partial\Omega$

REMINDER 3: PROOFS

$$\langle C(\vec{u},\phi_1)|\phi_2\rangle = -\langle C(\vec{u},\phi_2)|\phi_1\rangle \quad \text{if} \quad \nabla\cdot\vec{u}=0$$

Proof:

$$\nabla\cdot(\phi_1\phi_2\vec{u}) = \underbrace{\phi_1\phi_2\nabla\cdot\vec{u}}_{=0} + (\vec{u}\cdot\nabla\phi_1)\phi_2 + (\vec{u}\cdot\nabla\phi_2)\phi_1$$

$$\int_\Omega \nabla\cdot(\phi_1\phi_2\vec{u})\,d\Omega = \langle C(\vec{u},\phi_1)|\phi_2\rangle + \langle C(\vec{u},\phi_2)|\phi_1\rangle$$

$$\int_{\partial\Omega} (\phi_1\phi_2\vec{u})\cdot\vec{n}\,dS = \langle C(\vec{u},\phi_1)|\phi_2\rangle + \langle C(\vec{u},\phi_2)|\phi_1\rangle = 0 \quad \blacksquare$$

REMAINDER!!!

$$\langle a|b\rangle := \int_\Omega ab\,d\Omega \qquad C(\vec{u},\phi) := (\vec{u}\cdot\nabla)\phi$$

REMEMBER: we always assume **no contribution from domain boundary,** $\partial\Omega$

Inviscid invariants: definitions

Definition: a quantity that does not change (invariant) in time when viscosity is set to zero (inviscid).

$\langle \vec{u} \vert \vec{u} \rangle$	Kinetic energy (in 2D/3D)
$\langle \vec{\omega} \vert \vec{\omega} \rangle$	Enstrophy (only in 2D)
$\langle \vec{u} \vert \vec{\omega} \rangle$	Helicity (in 3D)

If you are not interested in their proofs go to #Slide 14

Notation:

$$\langle a|b\rangle := \int_{\Omega} ab\, d\Omega \qquad \vec{\omega} = \nabla \times \vec{u}$$

$$\langle \vec{u} | \vec{u} \rangle \qquad \text{Kinetic energy (in 2D/3D)}$$

$$\frac{1}{2}\frac{d\langle \vec{u}|\vec{u}\rangle}{dt} = \langle \frac{\partial \vec{u}}{\partial t} | \vec{u} \rangle = -\langle C(\vec{u},\vec{u})|\vec{u}\rangle + \nu \langle \nabla^2 \vec{u} | \vec{u} \rangle - \langle \nabla p | \vec{u} \rangle$$

$$= -\nu \langle \nabla \vec{u} | \nabla \vec{u} \rangle = -\nu \|\nabla \vec{u}\|^2 \leq 0$$

$$= -\nu \langle \nabla \times \nabla \times \vec{u} | \vec{u} \rangle = -\nu \|\omega\|^2 \leq 0$$

If $\nu = 0$, then $\langle \vec{u}|\vec{u}\rangle$ remains constant!!!

Also, if the flow is irrotational, $\vec{\omega} = \vec{0}$. Remember Bernoulli!

ADDITIONAL REMAINDER!!!

$$\nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times \nabla \times \vec{u}$$

REMAINDER!!!

$$\langle \nabla \cdot \vec{a} | \phi \rangle = -\langle \vec{a} | \nabla \phi \rangle$$

$$\langle \nabla^2 f | g \rangle = -\langle \nabla f | \nabla g \rangle = \langle f | \nabla^2 g \rangle$$

$$\langle C(\vec{u},\phi_1)|\phi_2\rangle = -\langle C(\vec{u},\phi_2)|\phi_1\rangle \quad \text{if} \quad \nabla \cdot \vec{u} = 0$$

$$\langle \nabla \times \vec{a} | \vec{b} \rangle = \langle \vec{a} | \nabla \times \vec{b} \rangle$$

Inviscid invariants: enstrophy

$$\langle \vec{\omega} | \vec{\omega} \rangle \qquad \text{Enstrophy (only in 2D)}$$

$$\nabla \times NS \Rightarrow \frac{\partial \vec{\omega}}{\partial t} + C(\vec{u}, \vec{\omega}) = \nu \nabla^2 \vec{\omega} + \boldsymbol{S} \vec{\boldsymbol{\omega}} \qquad \text{vorticity eq.}$$

$$\frac{1}{2} \frac{d \langle \vec{\omega} | \vec{\omega} \rangle}{dt} = \langle \frac{\partial \vec{\omega}}{\partial t} | \vec{\omega} \rangle = -\langle C(\vec{u}, \vec{\omega}) | \vec{\omega} \rangle + \nu \langle \nabla^2 \vec{\omega} | \vec{\omega} \rangle + \langle \vec{\omega} | \boldsymbol{S} \vec{\boldsymbol{\omega}} \rangle$$

$$= -\nu \| \nabla \vec{\omega} \|^2 + \langle \vec{\omega} | \boldsymbol{S} \vec{\boldsymbol{\omega}} \rangle$$

Definition:

$$\boldsymbol{S} = \frac{1}{2} (\nabla \vec{u} + \nabla \vec{u}^T) \quad \text{rate-of-strain}$$

REMAINDER!!!

$$\langle \nabla \cdot \vec{a} | \phi \rangle = -\langle \vec{a} | \nabla \phi \rangle$$

$$\langle \nabla^2 f | g \rangle = -\langle \nabla f | \nabla g \rangle = \langle f | \nabla^2 g \rangle$$

$$\langle C(\vec{u}, \phi_1) | \phi_2 \rangle = -\langle C(\vec{u}, \phi_2) | \phi_1 \rangle \quad \text{if} \quad \nabla \cdot \vec{u} = 0$$

$$\langle \nabla \times \vec{a} | \vec{b} \rangle = \langle \vec{a} | \nabla \times \vec{b} \rangle$$

Inviscid invariants: enstrophy

In 2D, $\boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\vec{\boldsymbol{\omega}} = \begin{pmatrix} 0 \\ 0 \\ \omega_3 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{S}\vec{\boldsymbol{\omega}} = \vec{0}$

$$\frac{1}{2}\frac{d\langle\vec{\omega}|\vec{\omega}\rangle}{dt} = -\nu\|\nabla\vec{\omega}\|^2 + \underbrace{\langle\vec{\omega}|\boldsymbol{S}\vec{\boldsymbol{\omega}}\rangle}_{=0 \text{ in 2D}} \leq 0$$

If $\nu = 0$, then $\langle\vec{\omega}|\vec{\omega}\rangle$ remains constant (in 2D)!!!

$1M question: does $\langle\vec{\omega}|\vec{\omega}\rangle$ remains well-bounded in 3D?

Definition:

$\boldsymbol{S}\vec{\boldsymbol{\omega}}$ is the vortex-stretching term!!!

Inviscid invariants: helicity

$$\langle \vec{u} | \vec{\omega} \rangle$$ Helicity (in 3D) (Moffatt 1969)

$$\frac{d\langle \vec{u}|\vec{\omega}\rangle}{dt} = \langle \frac{\partial \vec{u}}{\partial t} | \vec{\omega} \rangle + \langle \frac{\partial \vec{\omega}}{\partial t} | \vec{u} \rangle = Conv + Diff + Gradp + VortStretch$$

$$Conv + VortStretch = -\langle C(\vec{u},\vec{u}) | \vec{\omega} \rangle - \langle C(\vec{u},\vec{\omega}) | \vec{u} \rangle + \langle \vec{u} | \boldsymbol{S}\vec{\omega} \rangle = 0$$

$$Gradp = -\langle \nabla p | \vec{\omega} \rangle = -\langle p | \nabla \cdot \vec{\omega} \rangle = 0$$

$$Diff = \nu \langle \nabla^2 \vec{u} | \vec{\omega} \rangle + \nu \langle \nabla^2 \vec{\omega} | \vec{u} \rangle = 2\nu \langle \nabla^2 \vec{u} | \vec{\omega} \rangle = -2\nu \langle \vec{\omega} | \nabla \times \vec{\omega} \rangle$$

ADDITIONAL REMAINDER!!!

$$C(\vec{\omega},\vec{u}) = \boldsymbol{S}\vec{\omega}$$

$$\nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times \nabla \times \vec{u}$$

$$\nabla \cdot (\nabla \times \vec{u}) = 0$$

REMAINDER!!!

$$\langle \nabla \cdot \vec{a} | \phi \rangle = -\langle \vec{a} | \nabla \phi \rangle$$

$$\langle \nabla^2 f | g \rangle = -\langle \nabla f | \nabla g \rangle = \langle f | \nabla^2 g \rangle$$

$$\langle C(\vec{u},\phi_1) | \phi_2 \rangle = -\langle C(\vec{u},\phi_2) | \phi_1 \rangle \quad \text{if} \quad \nabla \cdot \vec{u} = 0$$

$$\langle \nabla \times \vec{a} | \vec{b} \rangle = \langle \vec{a} | \nabla \times \vec{b} \rangle$$

Inviscid invariants: helicity

$\langle \vec{u} | \vec{\omega} \rangle$ Helicity (in 3D) (Moffatt 1969)

$$\frac{d\langle \vec{u} | \vec{\omega} \rangle}{dt} = -2\nu \langle \vec{\omega} | \nabla \times \vec{\omega} \rangle$$

If $\nu = 0$, then $\langle \vec{u} | \vec{\omega} \rangle$ remains constant!!!

ADDITIONAL REMAINDER!!!

$C(\vec{\omega}, \vec{u}) = \boldsymbol{S}\vec{\boldsymbol{\omega}}$

$\nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times \nabla \times \vec{u}$

$\nabla \cdot (\nabla \times \vec{u}) = 0$

Inviscid invariants: relationships

$$E := \langle \vec{u} | \vec{u} \rangle \quad \Omega := \langle \vec{\omega} | \vec{\omega} \rangle \quad H := \langle \vec{u} | \vec{\omega} \rangle$$

$$H_{\omega} := \langle \vec{\omega} | \nabla \times \vec{\omega} \rangle \quad \leftarrow \text{Vortical helicity}$$

$$P := \langle \nabla \times \vec{\omega} | \nabla \times \vec{\omega} \rangle \quad \leftarrow \text{Palinstrophy}$$

$$E_t = -2 \nu \Omega$$

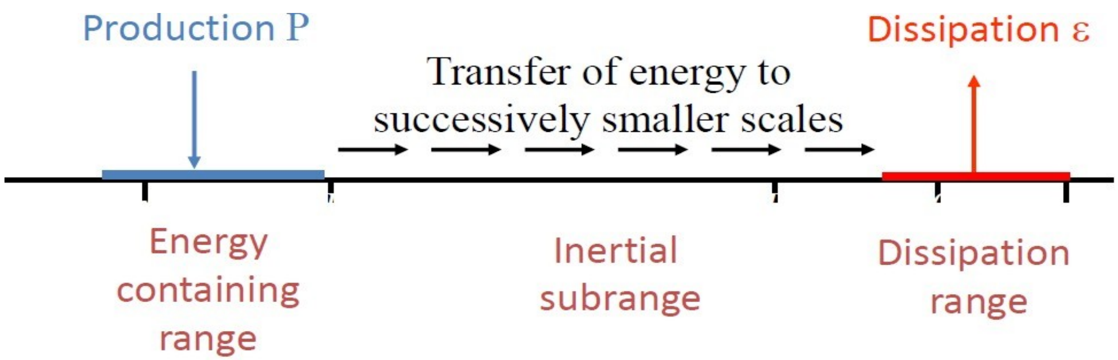
$$H_t = -2 \nu H_{\omega}$$

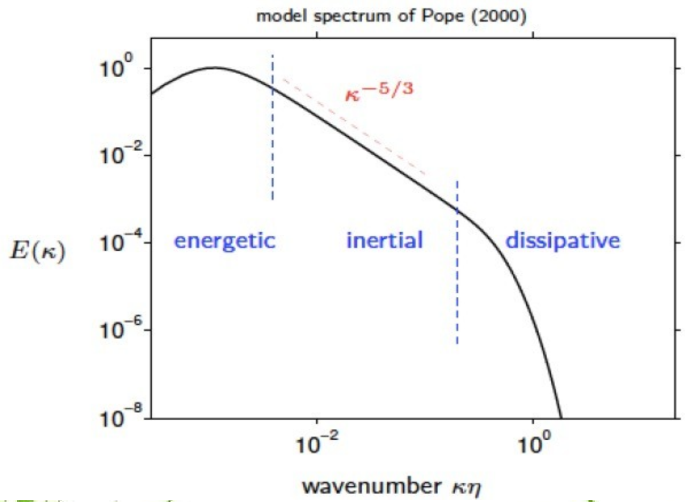
$$\Omega_t = -2 \nu P + 2 \underbrace{\langle \vec{\omega} | \boldsymbol{S} \vec{\omega} \rangle}_{=0 \text{ in 2D}}$$

What can we learn from these relationships?

Inviscid invariants: relationships

REMAINDER about the classical (K41) *energy cascade* concept in turbulence.



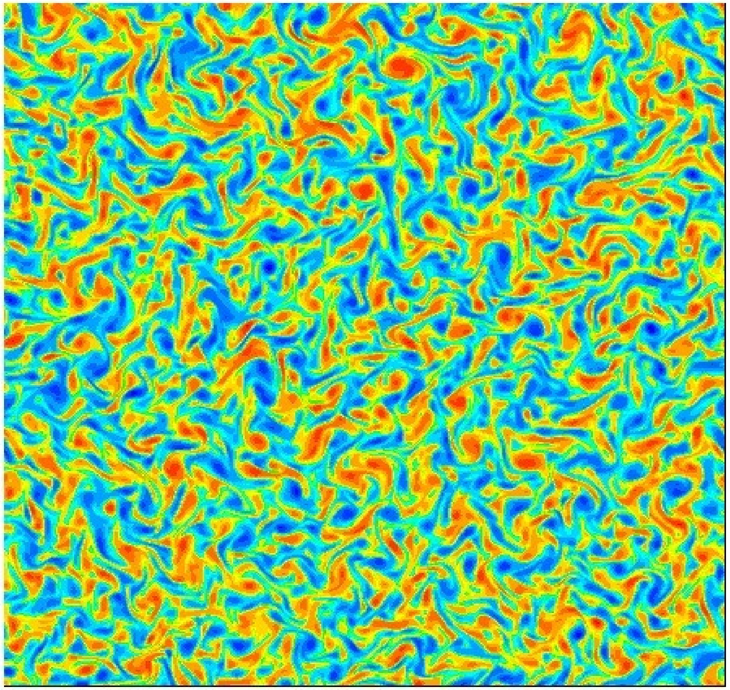


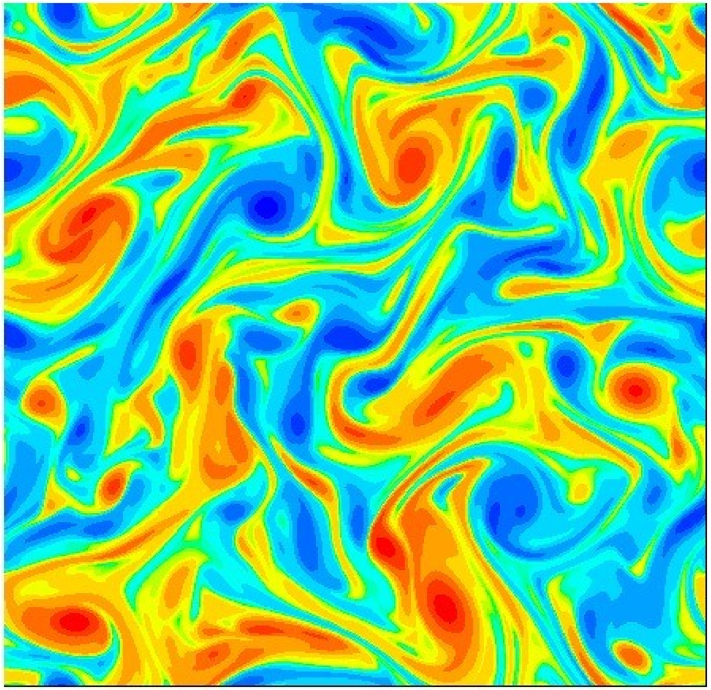
«Big whorls have little whorls,
which feed on their velocity,
And little whorls have lesser whorls,
and so on to viscosity.»

by Lewis Fry Richardson.

Inviscid invariants: relationships

A *B*





Question:

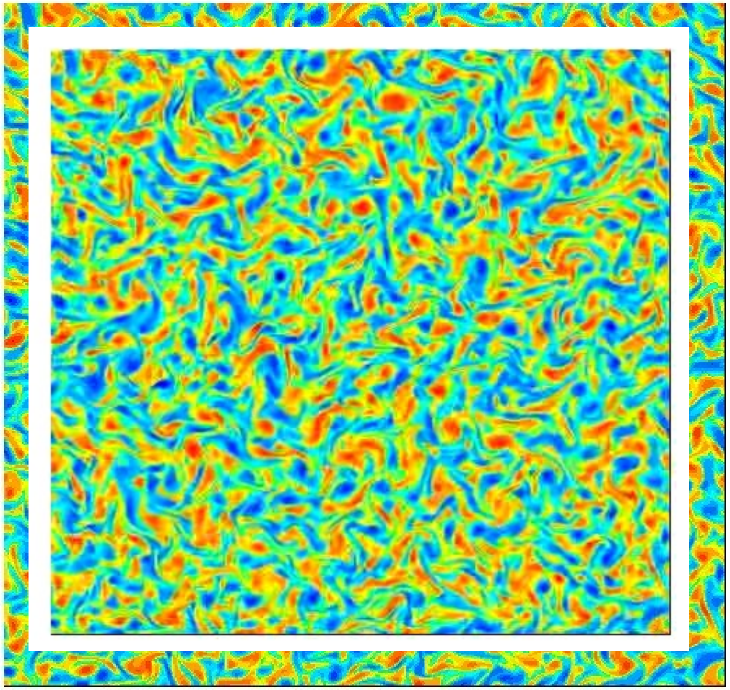
a) $t_A > t_B$

b) $t_A < t_B$

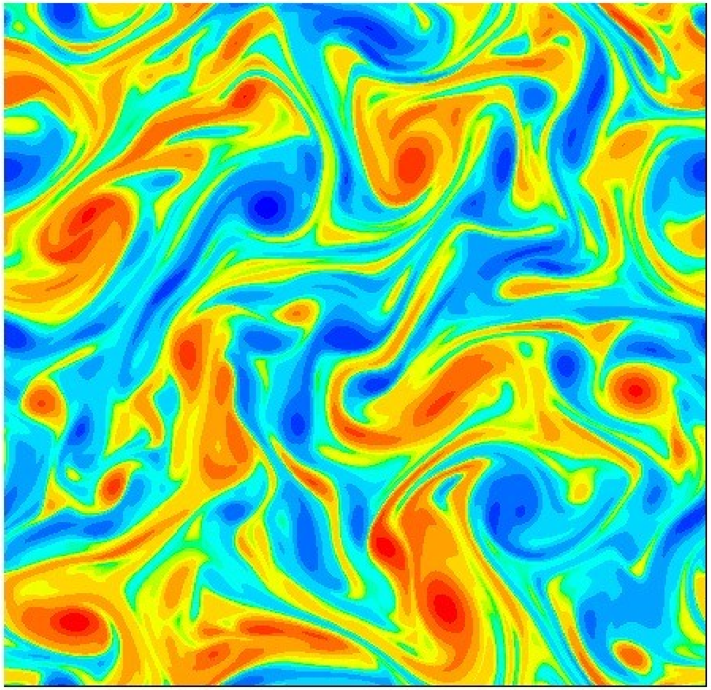
c) Both a and b may be correct

d) I don't care

A



B



Question:

a) $t_A > t_B$

b) $t_A < t_B$

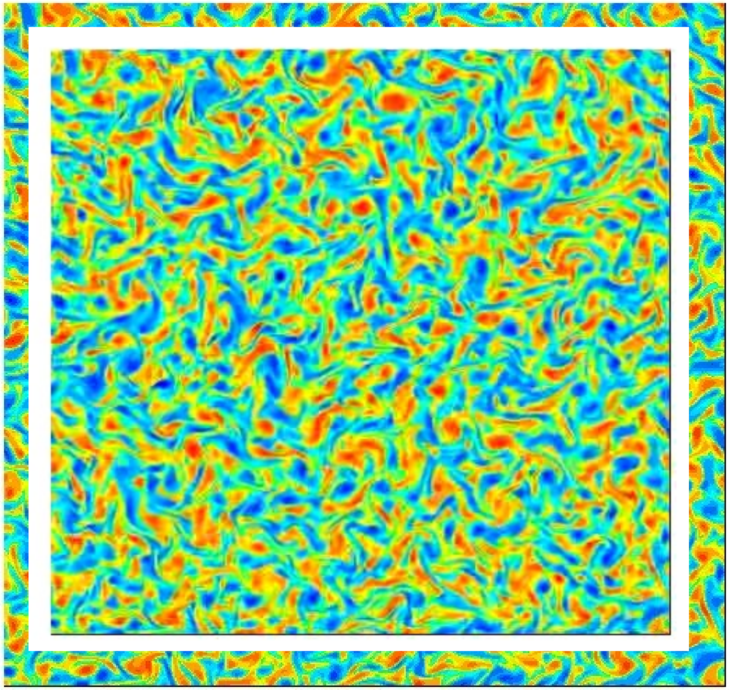
c) Both a and b may be correct

d) I don't care

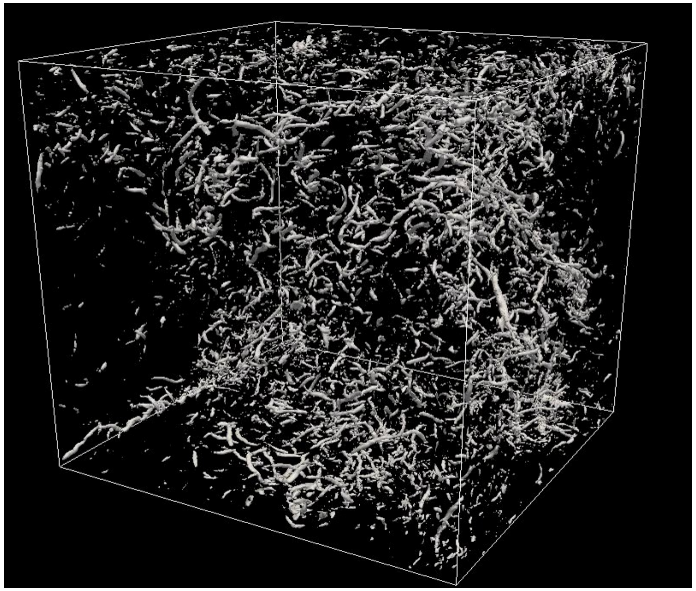
Inviscid invariants: 2D vs 3D

2D

3D



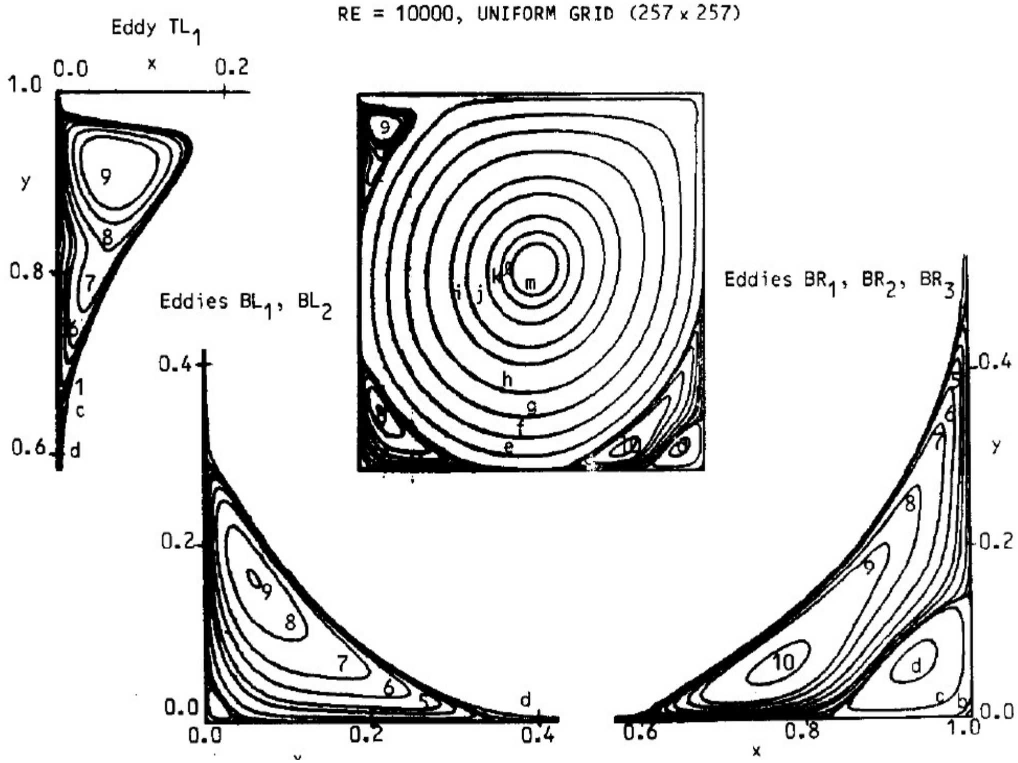
2D results by CTTC

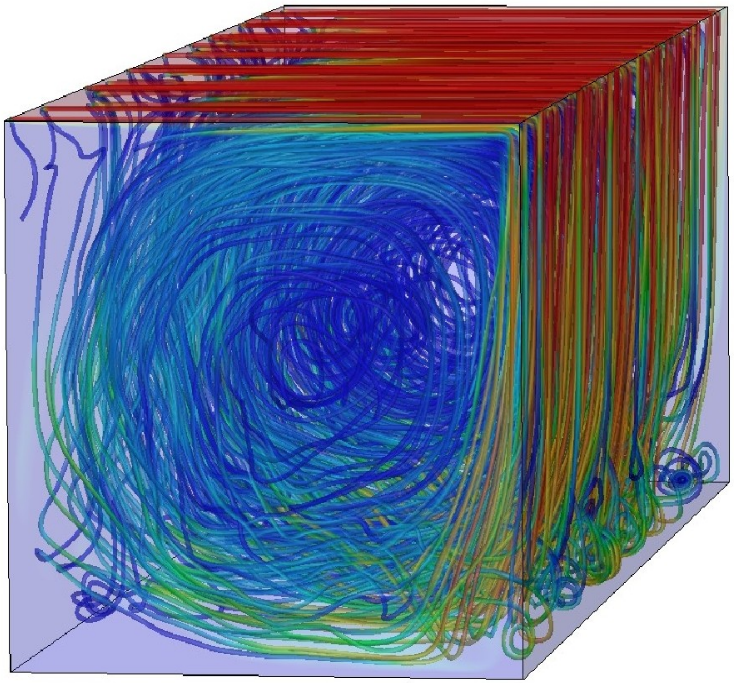


3D results by CTTC

Inviscid invariants: 2D vs 3D

Lid-Driven Cavity at Re=10000





2D results by Ghia et al. JCP, 48, 387-411 (1982)   3D results by CTTC (2011)

Air-filled differentially heated cavity

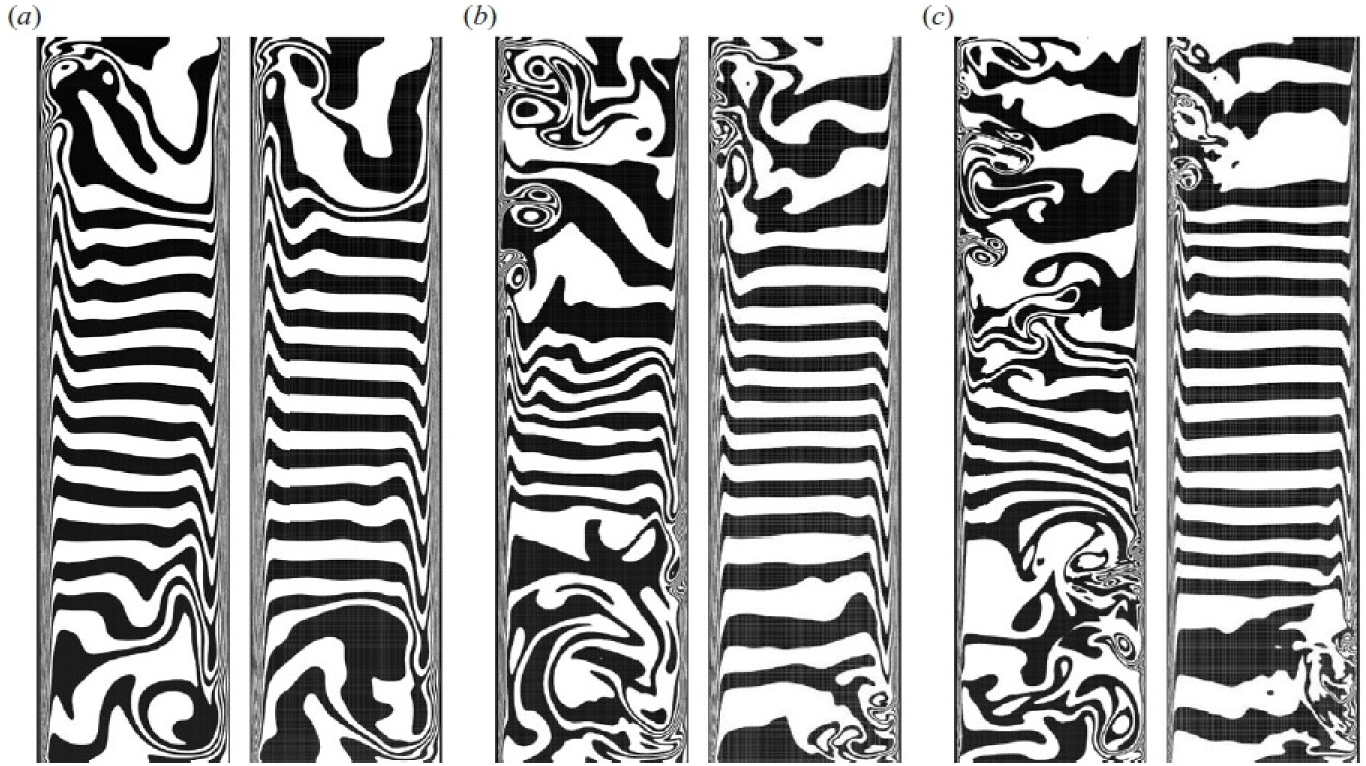


FIGURE 10. Instantaneous isotherms: (a) $Ra = 6.4 \times 10^8$, (b) 2×10^9 and (c) 10^{10}. For each solution the two-dimensional results are on the left and the three-dimensional results on the right. The isotherms are uniformly distributed from 0 to 1.

2D and 3D results by F.X.Trias *et al.* JFM, 586, 259-292 (2007)

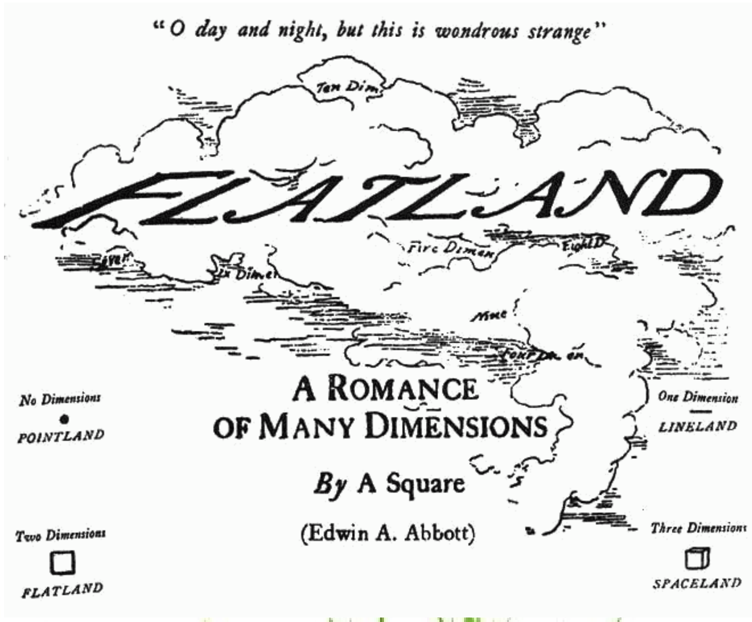
Flows in Flatland (2D)

What can we learn from these relationships?

$$E := \langle \vec{u} | \vec{u} \rangle \qquad \Omega := \langle \vec{\omega} | \vec{\omega} \rangle \qquad H := \langle \vec{u} | \vec{\omega} \rangle$$

$$E_t = -2\nu\Omega$$

$$\Omega_t = -2\nu P + 2 \underbrace{\langle \vec{\omega} | \boldsymbol{S} \vec{\boldsymbol{\omega}} \rangle}_{=0 \text{ in 2D}}$$



Flow dynamics is completely different in Flatland... but why?

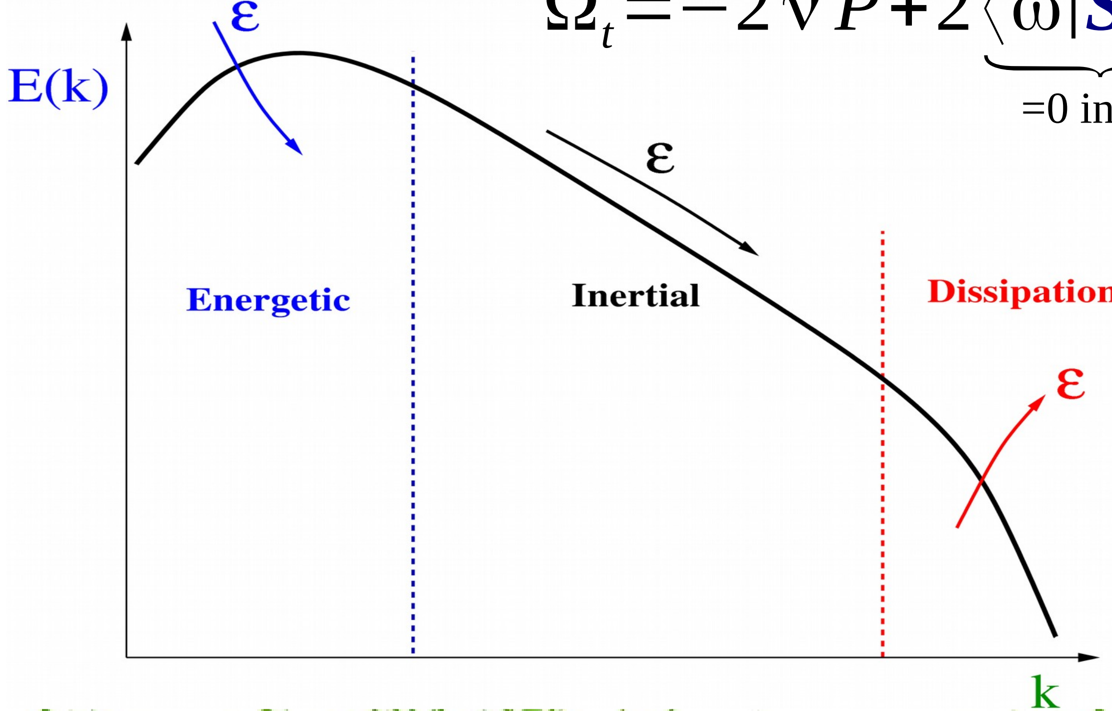
Flows in Flatland (2D)

$$E := \int E(k)\,dk \quad \Omega := \int k^2 E(k)\,dk \quad P := \int k^4 E(k)\,dk$$

$$E := \langle \vec{u} | \vec{u} \rangle \quad \Omega := \langle \vec{\omega} | \vec{\omega} \rangle \quad H := \langle \vec{u} | \vec{\omega} \rangle$$

$$E_t = -2\nu\Omega$$

$$\Omega_t = -2\nu P + 2 \underbrace{\langle \vec{\omega} | \boldsymbol{S}\vec{\boldsymbol{\omega}} \rangle}_{=0 \text{ in 2D}}$$

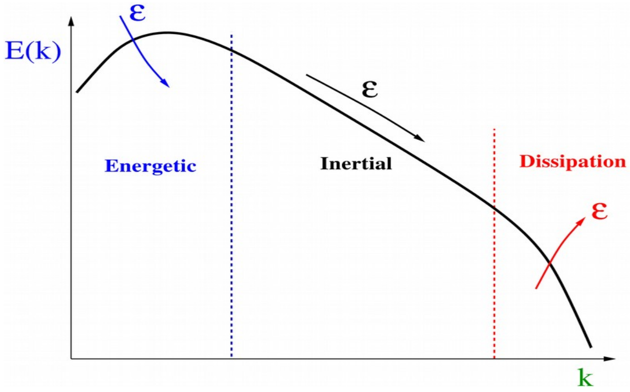


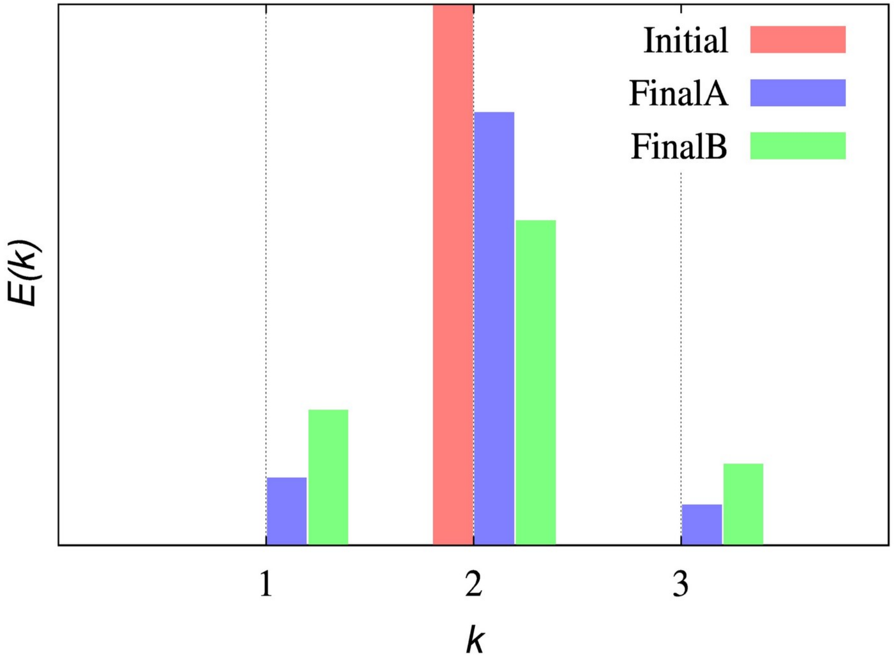
Flows in Flatland (2D)

$$E := \int E(k)\,dk \quad \Omega := \int k^2 E(k)\,dk \quad P := \int k^4 E(k)\,dk$$

$$E_t = -2\nu\Omega$$

$$\Omega_t = -2\nu P + 2\underbrace{\langle \vec{\omega} | \boldsymbol{S}\vec{\omega} \rangle}_{=0 \text{ in 2D}}$$





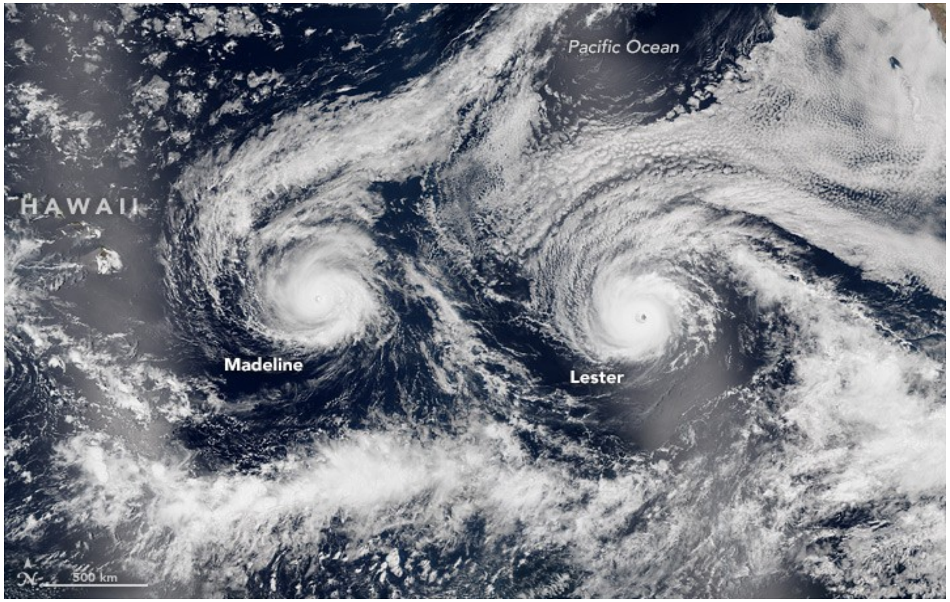
$$A\ very\ simple\ case:$$

$$E := E_1 + E_2 + E_3$$

$$\Omega := (1)^2 E_1 + (2)^2 E_2 + (3)^3 E_3$$

Inverse energy cascade!

Examples: Geophysical flows



Satellite image taken on August 29, 2016

Large scale motions and "thin" layers of fluid

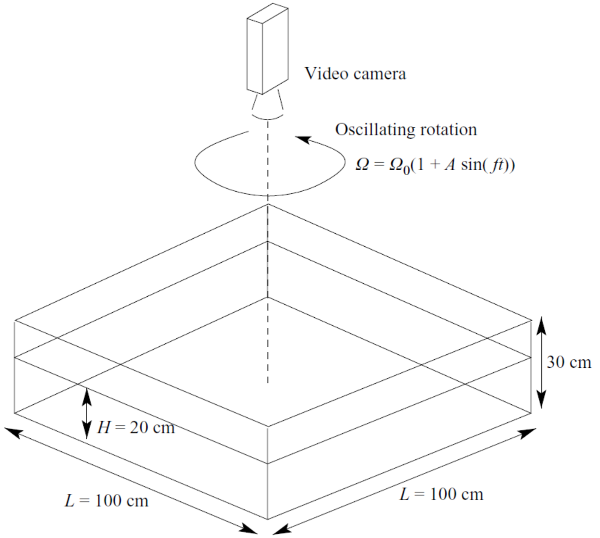
Other 2D mechanims:

- *Stratification*
- *Rotation (Coriolis)*

Taylor–Proudman theorem

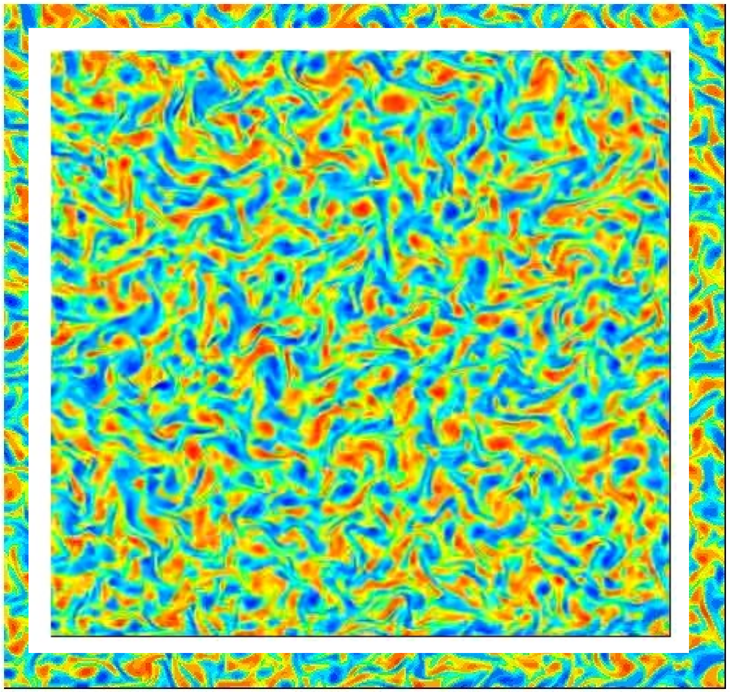
Close-up of the Jupiter's Great Red Spot (first documented observation in 1830)

Examples: in Lab



Experiment by M.G. Wells, H.J.H. Clercx, and G.J.F. van Heijst 'Vortices in oscillating spin-up'. Journal Fluid Mechanics 573, 339-369, 2007

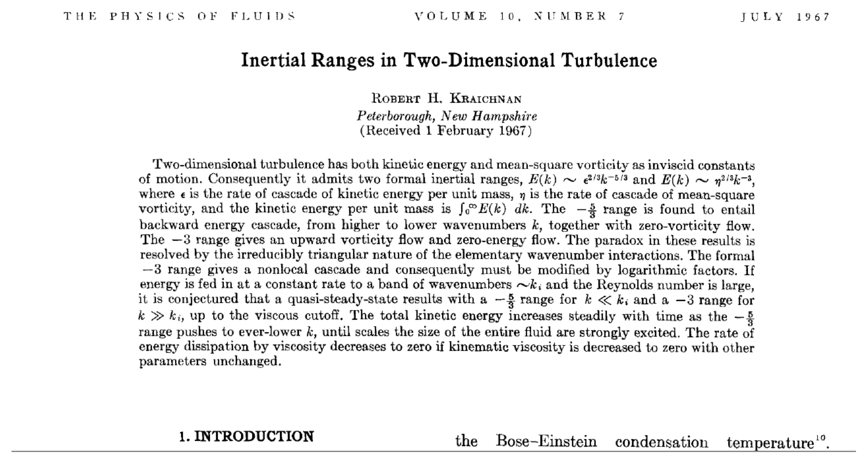
Sketch of experimental set-up. Extracted from the paper.



2D results by CTTC

Further reading:

Robert Kraichnan, 'Inertial ranges in two-dimensional turbulence', Phys. Fluids 10:1417–23, 1967



THE PHYSICS OF FLUIDS VOLUME 10, NUMBER 7 JULY 1967

Inertial Ranges in Two-Dimensional Turbulence

ROBERT H. KRAICHNAN
Peterborough, New Hampshire
(Received 1 February 1967)

Two-dimensional turbulence has both kinetic energy and mean-square vorticity as inviscid constants of motion. Consequently it admits two formal inertial ranges, $E(k) \sim \epsilon^{2/3}k^{-5/3}$ and $E(k) \sim \eta^{2/3}k^{-3}$, where ϵ is the rate of cascade of kinetic energy per unit mass, η is the rate of cascade of mean-square vorticity, and the kinetic energy per unit mass is $\int_0^\infty E(k)\, dk$. The $-\frac{5}{3}$ range is found to entail backward energy cascade, from higher to lower wavenumbers k, together with zero-vorticity flow. The -3 range gives an upward vorticity flow and zero-energy flow. The paradox in these results is resolved by the irreducibly triangular nature of the elementary wavenumber interactions. The formal -3 range gives a nonlocal cascade and consequently must be modified by logarithmic factors. If energy is fed in at a constant rate to a band of wavenumbers $\sim k_i$ and the Reynolds number is large, it is conjectured that a quasi-steady-state results with a $-\frac{5}{3}$ range for $k \ll k_i$ and a -3 range for $k \gg k_i$, up to the viscous cutoff. The total kinetic energy increases steadily with time as the $-\frac{5}{3}$ range pushes to ever-lower k, until scales the size of the entire fluid are strongly excited. The rate of energy dissipation by viscosity decreases to zero if kinematic viscosity is decreased to zero with other parameters unchanged.

1. INTRODUCTION

the Bose–Einstein condensation temperature[10].

Guido Boffetta and Robert E. Ecke, 'Two-dimensional turbulence', Annu. Rev. Fluid Mech. 44:427–51, 2012.

Take-away messages

- (Skew-)**symmetries of the operators** play a very important role in the dynamics of NS
- **2D flows** have a completely different behaviour. Not valid as a simulation shortcut!

Questions:

- How we should **discretize** the **NS** equations if we want to preserve all these features?
- What **restrictions** does it impose in our **numerical schemes**? Are they reasonable?