PHYSICAL FORMULATION

MATHEMATICS AS THE LANGUAGE OF PHYSICS

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CHAOS, NONLINEARITY AND TURBULENCE

UNDERSTANDING ITS ORIGEN

On his death bed, Werner Heisenberg is reported to have said, «When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first.»

CHAOS AND NONLINEARITY

Chaos theory concerns about deterministic systems whose behavior can in principle be predicted, but they are highly sensitive to initial conditions (*butterfly effect*).

Thus, in practice, they appear to behave randomly.

A (finite-dimensional) chaotic system **must be nonlinear**.

$$\{\, Chaotic\ systems \,\} \subset \{\, Nonlinear\ systems \,\}$$

NONLINEAR SYSTEMS: NAVIER-STOKES

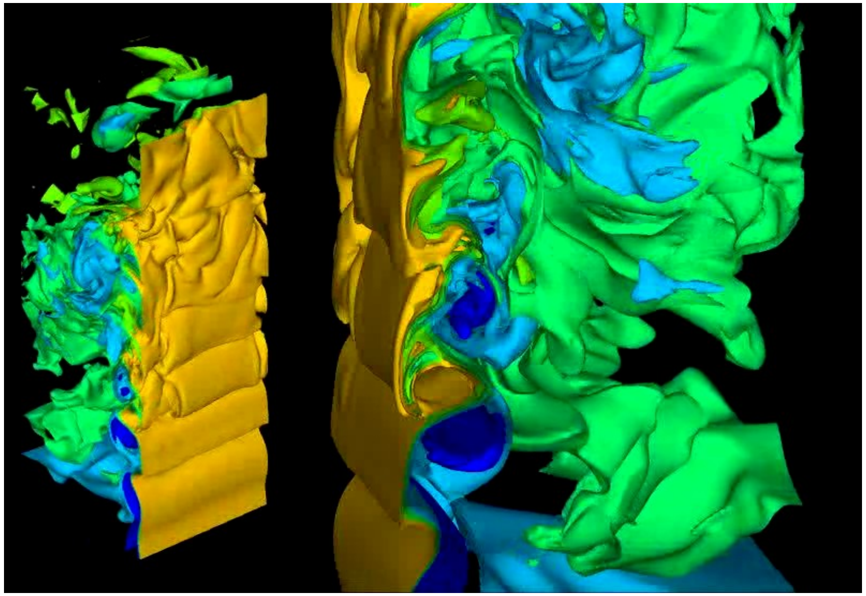
Examples of nonlinear equations [edit]

- AC power flow model
- Algebraic Riccati equation
- Ball and beam system
- Bellman equation for optimal policy
- Boltzmann transport equation
- Colebrook equation
- General relativity
- Ginzburg–Landau equation
- Navier–Stokes equations of fluid dynamics
- Korteweg–de Vries equation
- Nonlinear optics
- Nonlinear Schrödinger equation
- Richards equation for unsaturated water flow
- Robot unicycle balancing
- Sine–Gordon equation
- Landau–Lifshitz–Gilbert equation
- Ishimori equation
- Van der Pol equation
- Liénard equation
- Vlasov equation

Wikipedia contributors, "Nonlinear system," Wikipedia, The Free Encyclopedia, https://en.wikipedia.org/w/index.php?title=Nonlinear_system&oldid=757475924 (accessed December 30, 2016).

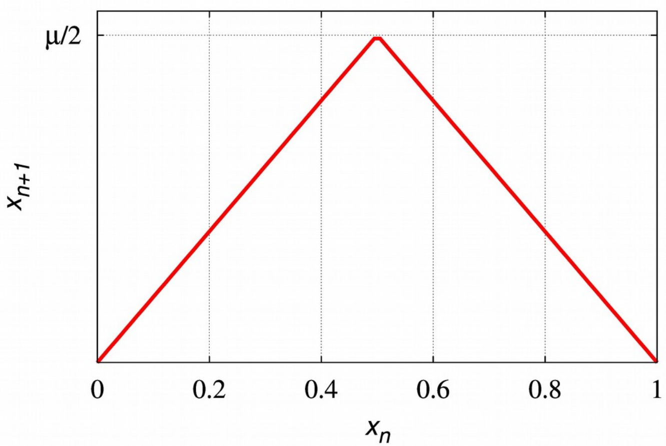
$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = \nu \nabla^2 \vec{u} - \nabla p$$

$$\nabla \cdot \vec{u} = 0$$



$$NS \in \{ Chaotic\ systems \} \subset \{ Nonlinear\ systems \}$$

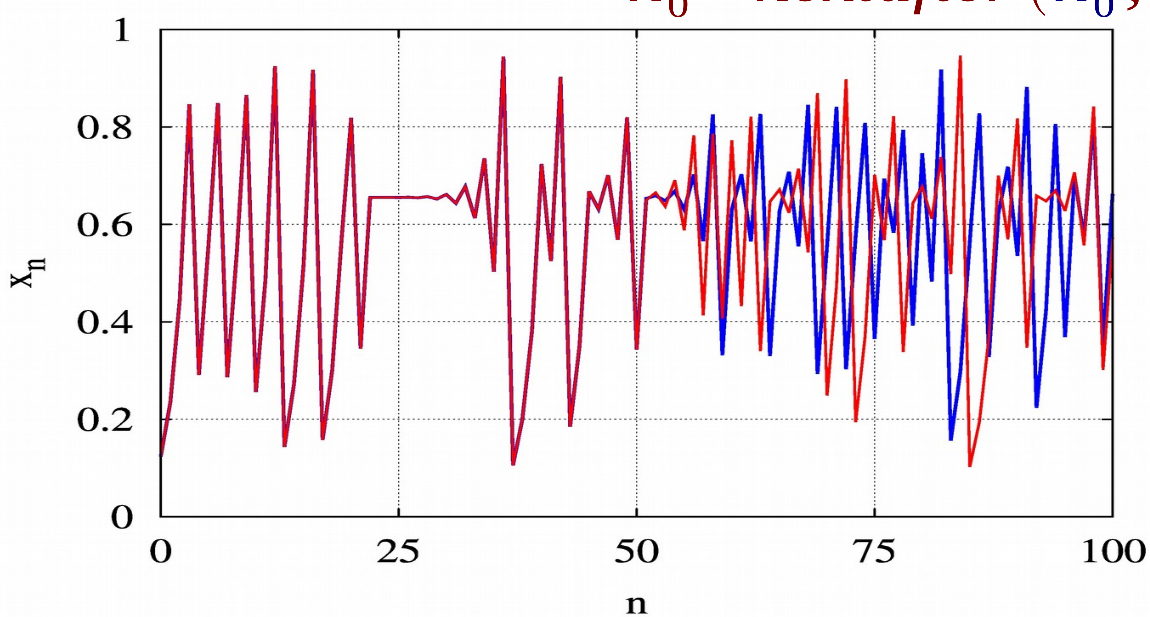
NONLINEAR SYSTEMS: TENT MAP



$$x_{n+1} = \mu\left(1/2 - |x_n - 1/2|\right)$$

$$\mu = 1.9\,;$$
$$x_0 = 0.12345678\,;$$
$$x_0 = nextafter\left(x_0, 1\right);$$

001	0.123456780000000	0.123456780000000
002	0.234567882000000	0.234567882000000
003	0.445678975800000	0.445678975800000
004	0.846790054020000	0.846790054020000
005	0.291098897362000	0.291098897362000
006	0.553087904987800	0.553087904987800
007	0.849132980523180	0.849132980523181
008	0.286647337005959	0.286647337005957
009	0.544629940311322	0.544629940311318
010	0.865203113408489	0.865203113408495
011	0.256114084523871	0.256114084523859
012	0.486616760595354	0.486616760595331
013	0.924571845131173	0.924571845131130
014	0.143313494250771	0.143313494250853
015	0.272295639076465	0.272295639076621
016	0.517361714245283	0.517361714245581
017	0.917012742933962	0.917012742933397
018	0.157675788425472	0.157675788426547
019	0.299583998008396	0.299583998010439
020	0.569209596215953	0.569209596219833
021	0.818501767189690	0.818501767182317
022	0.344846642339589	0.344846642353598
023	0.655208620445219	0.655208620471836

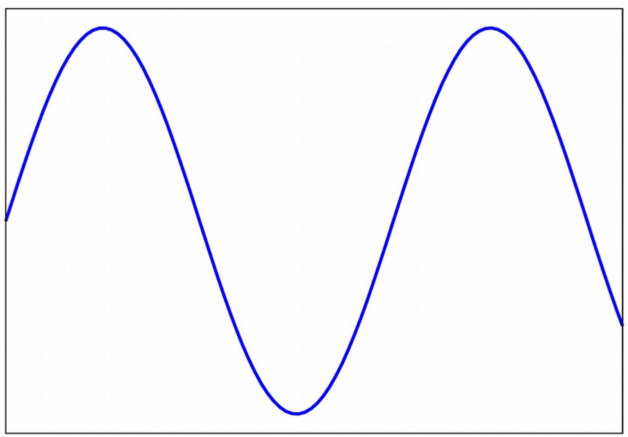


NONLINEARITY OF NS

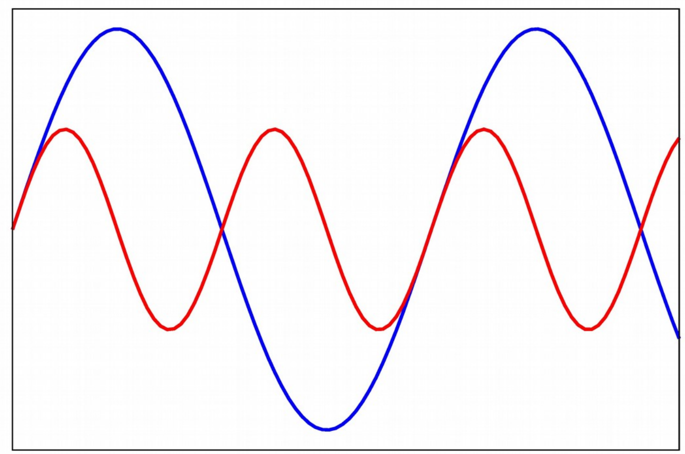
$$\frac{\partial \vec{u}}{\partial t} + \underbrace{(\vec{u}\cdot\nabla)\vec{u}}_{\text{nonlinear guy in NS}} = \nu \nabla^2 \vec{u} - \nabla p; \qquad \nabla\cdot\vec{u} = 0$$

Let us consider a one single 1D wave, $u(x) = \sin(\boldsymbol{k}\, x)$

$$u\frac{\partial u}{\partial x} = k\sin(k\,x)\cos(k\,x) = \frac{k}{2}\sin(\boldsymbol{2k}\,x)$$



$$u\frac{\partial u}{\partial x}$$

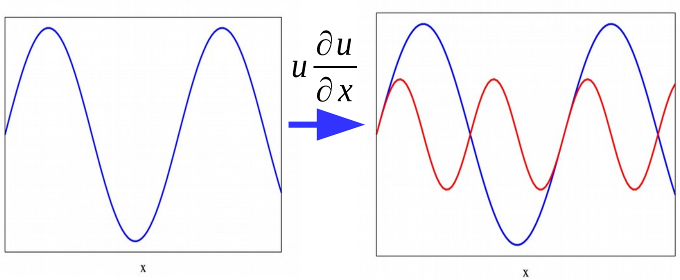


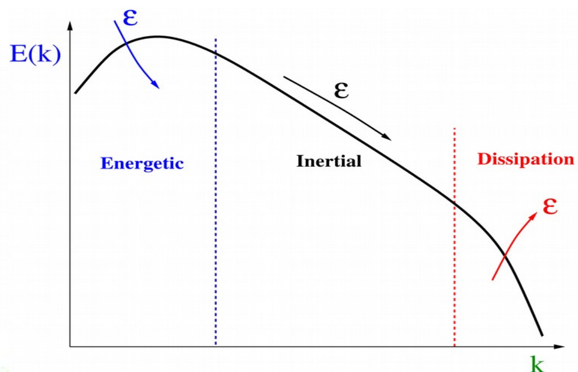
REMAINDER!!! Double-angle formula: $\sin(2x) = 2\sin(x)\cos(x)$

NONLINEARITY OF NS

$$\frac{\partial \vec{u}}{\partial t} + \underbrace{(\vec{u}\cdot\nabla)\vec{u}}_{\text{nonlinear guy in NS}} = \nu \nabla^2 \vec{u} - \nabla p; \qquad \nabla\cdot\vec{u} = 0$$

	k	2k	3k	4k	5k	6k	7k	8k	9k	10k	11k	12k	13k	14k	15k	16k	17K	18K	19K
1	x																		
2	x	x																	
3	x	x	x	x															
4	x	x	x	x	x	x	x	x											
5	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x			
...																			



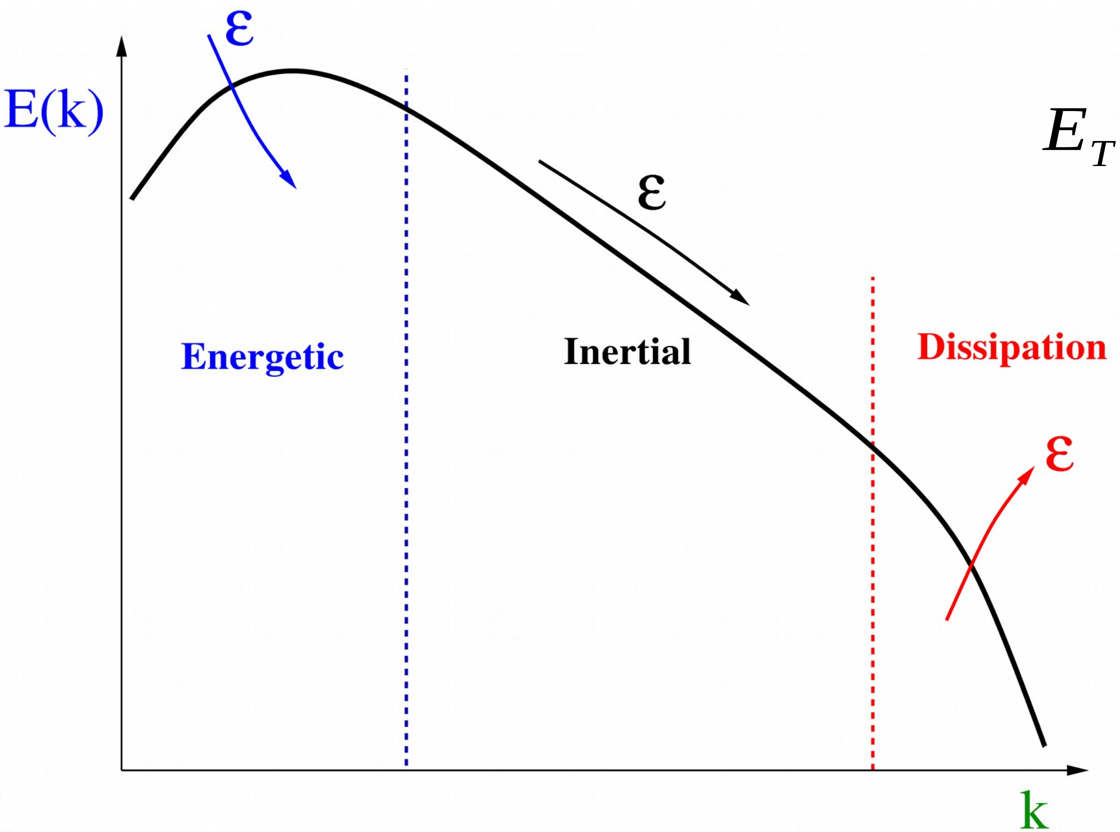


«Big whorls have little whorls,
which feed on their velocity,
And little whorls have lesser whorls,
and so on to viscosity.»

by Lewis Fry Richardson.

TURBULENCE

$$[k]=[L^{-1}] \quad [\varepsilon]=[L^2T^{-3}]$$
$$[E_T]=[L^2T^{-2}] \quad [E(k)]=[L^3T^{-2}]$$



$$E_T=\int E(k)\,dk$$

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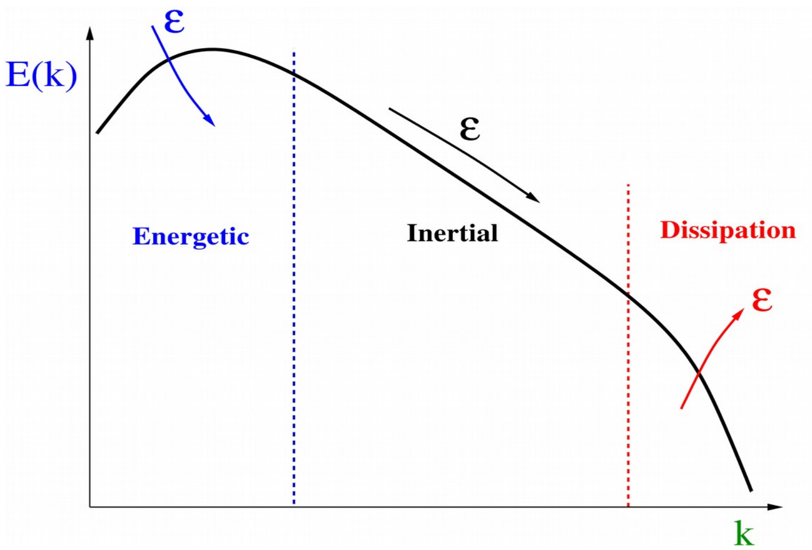


Hypothesis (K41): $E(k) \propto \varepsilon^a k^b$

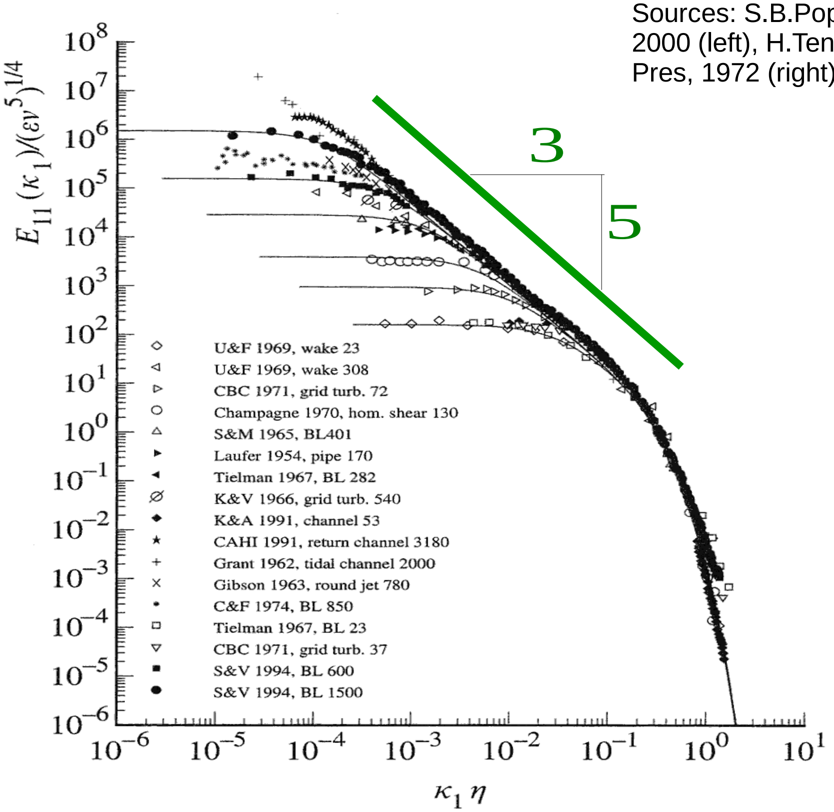
$$\begin{aligned} 2a - b &= 3 \\ -3a \quad &= -2 \end{aligned} \quad \Rightarrow \quad \begin{cases} a = 2/3 \\ b = -5/3 \end{cases}$$

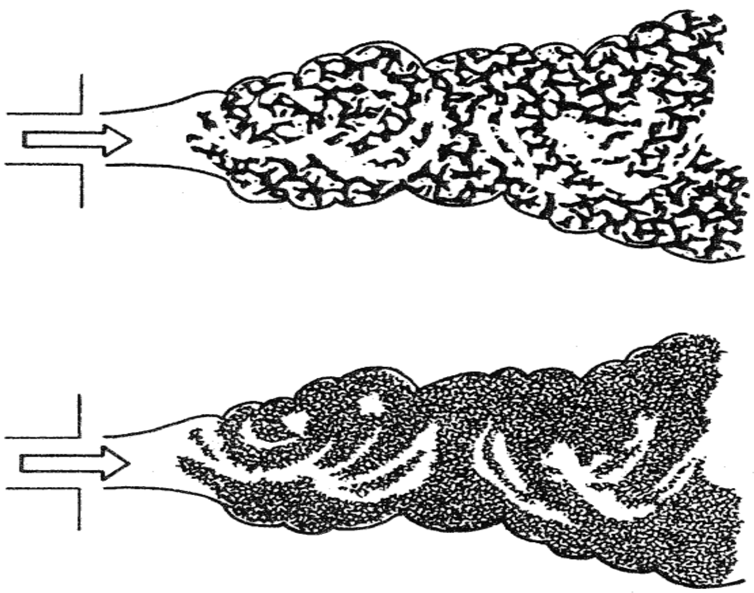
Kolmogorov energy spectrum: $\boxed{E(k)=C_K \varepsilon^{2/3} k^{-5/3}}$

Formal proof of the existence of the inertial subrange for the (incompressible) Navier-Stokes equations: C. Foias, O.P. Manley, R. Rosa and R. Temam: *Estimates for the energy cascade in three-dimensional turbulent flows*. C. R. Acad. Sci. Paris, Série I Math. 333, 499–504, (**2001**).

TURBULENCE

Sources: S.B.Pope, Turbulent Flows, Cambridge University Press, Cambridge, 2000 (left), H.Tennekes and J.L. Lumley, A First Course in Turbulence, The MIT Pres, 1972 (right)

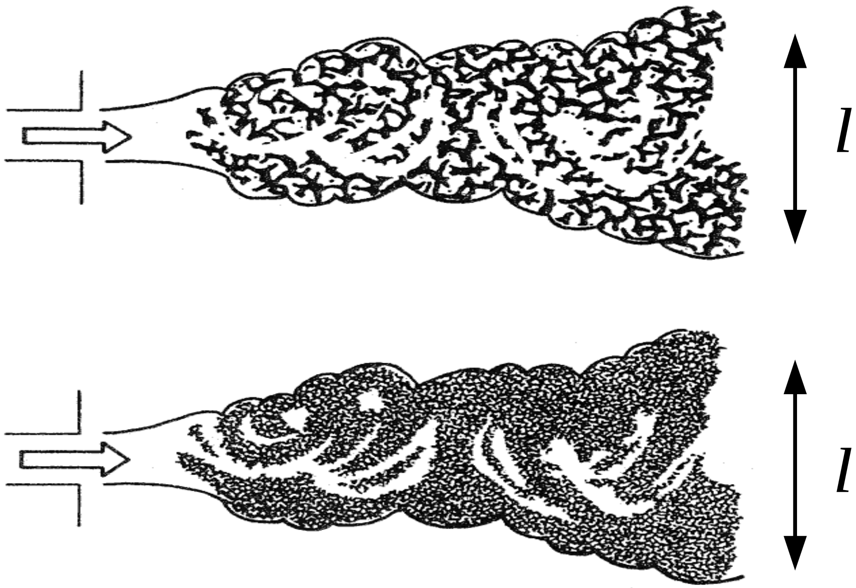




Kolmogorov energy spectrum: $\boxed{E(k) = C_K \varepsilon^{2/3} k^{-5/3}}$

l: biggest eddies (driving scale)

η: smallest eddies (Kolmogorov length scale)



Question: ?

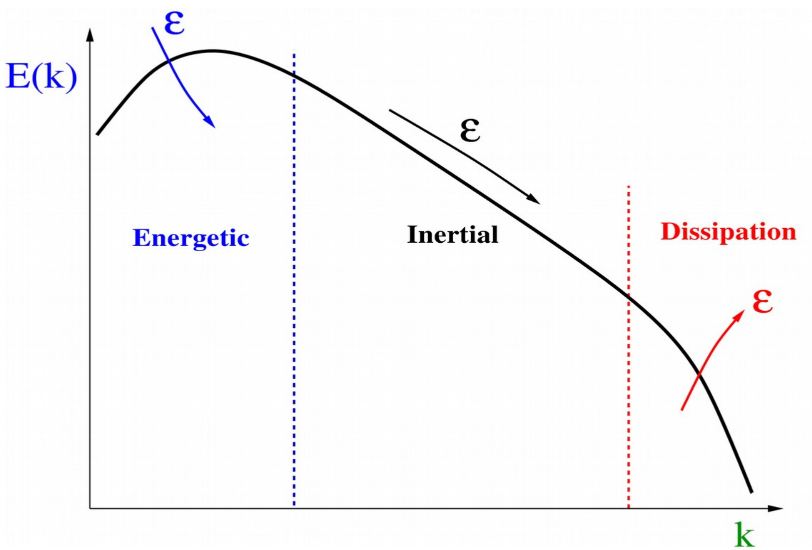
how $\frac{\eta}{l}$ decreases with Re?

Kolmogorov energy spectrum: $\boxed{E(k)=C_K\,\varepsilon^{2/3}\,k^{-5/3}}$

TURBULENCE

l: biggest eddies (driving scale) $\longrightarrow U$

η: smallest eddies (Kolmogorov length scale) $\longrightarrow u$



Associated velocities $U \gg u$

$$\varepsilon \sim \underbrace{U^2}_{energy} / (\underbrace{l/U}_{t_{eddy}}) = U^3/l$$

$$\varepsilon \sim \nu S : S \sim \nu (u^2/\eta^2)$$

$$\mathrm{Re}_l = U\,l/\nu$$

$$\mathrm{Re}_\eta = u\,\eta/\nu \sim 1$$

Kolmogorov energy spectrum: $\boxed{E(k) = C_K \varepsilon^{2/3} k^{-5/3}}$

TURBULENCE

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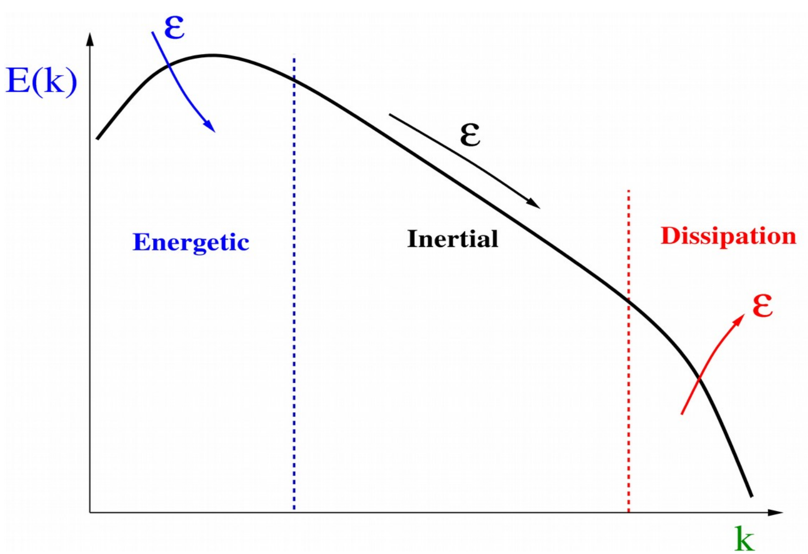

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$$Re_l = U\,l/\nu$$

$$Re_\eta = u\,\eta/\nu \sim 1$$

Hypothesis:
equilibrium $\varepsilon = \varepsilon$ $\longrightarrow$

$$\boxed{\begin{aligned} \eta/l &\sim Re_l^{-3/4} \\ u/U &\sim Re_l^{-1/4} \\ t_\eta/t_{eddy} &\sim Re_l^{-1/2} \end{aligned}}$$

$$\boxed{\begin{aligned} \eta &\sim (\nu^3/\varepsilon)^{1/4} \\ u &\sim (\nu\varepsilon)^{1/4} \\ t_\eta &\sim (\nu/\varepsilon)^{1/2} \end{aligned}}$$

l: biggest eddies (driving scale) $\longrightarrow U$

η: smallest eddies (Kolmogorov length scale) $\longrightarrow u$

(Computational) **Complexity of NS** grows very fast!!!

$$\text{In space:} \quad dim=3 \quad \Rightarrow \quad (l/\eta)^3 \sim \mathrm{Re}_l^{9/4}$$

$$\text{In time:} \quad t_{eddy}/t_\eta \sim \mathrm{Re}_l^{1/2}$$

$$\text{In space} \times \text{time:} \quad (l/\eta)^3 t_{eddy}/t_\eta \sim \mathrm{Re}_l^{11/4}$$

$$\boxed{\begin{aligned} \eta/l &\sim \mathrm{Re}_l^{-3/4} \\ u/U &\sim \mathrm{Re}_l^{-1/4} \\ t_\eta/t_{eddy} &\sim \mathrm{Re}_l^{-1/2} \end{aligned}} \qquad \boxed{\begin{aligned} \eta/l &\sim (\nu^3/\varepsilon)^{1/4} \\ u &\sim (\nu\varepsilon)^{1/4} \\ t_\eta &\sim (\nu/\varepsilon)^{1/2} \end{aligned}}$$

TAKE-AWAY MESSAGES

- **Non-linearity** is the origin of **chaos**.
- Non-linear **convective term** basically **transport** energy from large to small scales.
- **Complexity of NS** grows very fast with Reynolds.

Questions:

- How much does it **cost** to solve all relevant scales in a turbulent flow?
- Can we use a **simulation shortcut**?