Course on Numerical Methods in Heat Transfer and Fluid Dynamics

Fractional Step Method

Staggered Meshes

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Objectives

- Solve the NS equations using the FSM
- To understand the key features of the Fractional Step Method
- Study the checkerboard problem and review the different existing solutions
- Implement a CFD code for structured and staggered meshes
- Verification of the developed code using the Driven Cavity benchmark data

Introduction to Fractional Step Method

The fractional step method (**FSM**) **is a** common technique for solving the incompressible NS equations

The main reasons for this success are basically:

- Better performance than other methods such as SIMPLE-like algorithms (SIMPLE stands for Semi-Implicit Method for Pressure-Linked Equations)
- Code simplicity

Main issues to bear in mind:

- **FSM** are also referred to as **projection methods** because it can be interpreted as a projection into a divergence-free velocity space.

- The ***predictor velocity***, is an approximate solution of the momentum equations, but it cannot satisfy the incompressibility constraint at the next time level.

- The **pressure Poisson equation** determines the minimum perturbation that will make the predictor velocity incompressible.

Theoretical background: the Helmholtz-Hodge theorem

Theorem: *A given vector field* $\boldsymbol{\omega}$*, defined in a bounded domain* Ω *with smooth boundary* $\delta\Omega$*, is uniquely decomposed in a pure gradient field and a divergence-free vector parallel to* $\delta\Omega$

$$\boldsymbol{\omega} = \boldsymbol{a} + \nabla\varphi$$

where,

$$\nabla \cdot \boldsymbol{a} = \mathbf{0} \qquad \boldsymbol{a} \in \Omega$$

The theorem also applies for periodic inflow/outflow conditions.

The proof of the theorem can be found in the extra material of the course entitled: "Introduction to the Fractional Step Method".

Application of the HH theorem to NS equations (1/4)

Navier-Stokes (NS) equations for incompressible and constant viscosity flows:

$$\nabla \cdot \boldsymbol{v} = \mathbf{0}$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + (\rho \boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} p + \mu \Delta \boldsymbol{v} \quad \textit{or} \quad \rho \frac{\partial \boldsymbol{v}}{\partial t} = \mathbf{R}(\boldsymbol{v}) - \boldsymbol{\nabla} p$$

where, $\mathbf{R}(\boldsymbol{v}) = -(\rho \boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \mu \Delta \boldsymbol{v}$

Time integration of NS equations gives:

$$\nabla \cdot \boldsymbol{v}^{n+1} = \mathbf{0}$$

$$\rho \frac{\boldsymbol{v}^{n+1} - \boldsymbol{v}^{n}}{\Delta t} = \frac{3}{2}\boldsymbol{R}(\boldsymbol{v}^{n}) - \frac{1}{2}\boldsymbol{R}(\boldsymbol{v}^{n-1}) - \boldsymbol{\nabla} p^{n+1}$$

Momentm equations are integrated at time instant (n+1/2) while continuity equations is implicitly integrated.

Application of the HH theorem to NS equations (2/4)

Now, if we introduce the following unique decomposition (thanks to the HH theorem),

$$\boldsymbol{v}^p = \boldsymbol{v}^{n+1} + \frac{\Delta t}{\rho}\boldsymbol{\nabla} p^{n+1} \qquad \text{(where } \boldsymbol{\nabla}\cdot\boldsymbol{v}^{n+1} = \boldsymbol{0}\text{)}$$

we can transform the original momentum equation to the following velocity projection equation,

$$\rho\frac{\boldsymbol{v}^p - \boldsymbol{v}^n}{\Delta t} = \frac{3}{2}\boldsymbol{R}(\boldsymbol{v}^n) - \frac{1}{2}\boldsymbol{R}(\boldsymbol{v}^{n-1})$$

Application of the HH theorem to NS equations (3/4)

An equation for the pressure can be derived from the velocity decomposition equation if the divergence operator is applied,

$$\nabla \cdot \boldsymbol{v}^{n+1} = \nabla \cdot \boldsymbol{v}^{p} - \nabla \cdot \left(\frac{\Delta t}{\rho} \nabla p^{n+1}\right)$$

Since $\boldsymbol{\nabla} \cdot \boldsymbol{v}^{n+1} = \mathbf{0}$, a final Poisson equation for the pressure is found,

$$\Delta p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \boldsymbol{v}^{p}$$

Application of the HH theorem to NS equations (4/4)

Finally, $\boldsymbol{v}^{n+1}$ results from the original decomposition,

$$\boldsymbol{v}^{n+1} = \boldsymbol{v}^{p} - \frac{\Delta t}{\rho}\boldsymbol{\nabla} p^{n+1}$$

Therefore, at each time step the following equations give a unique $\boldsymbol{v}^{n+1}$ and $\boldsymbol{\nabla} p^{n+1}$. In summary:

1. $\boldsymbol{v}^{p} = \boldsymbol{v}^{n} + \frac{\Delta t}{\rho}[\frac{3}{2}\boldsymbol{R}(\boldsymbol{v}^{n}) - \frac{1}{2}\boldsymbol{R}(\boldsymbol{v}^{n-1})]$
2. $\Delta p^{n+1} = \frac{\rho}{\Delta t}\boldsymbol{\nabla}\cdot\boldsymbol{v}^{p}$
3. $\boldsymbol{v}^{n+1} = \boldsymbol{v}^{p} - \frac{\Delta t}{\rho}\boldsymbol{\nabla} p^{n+1}$

FSM

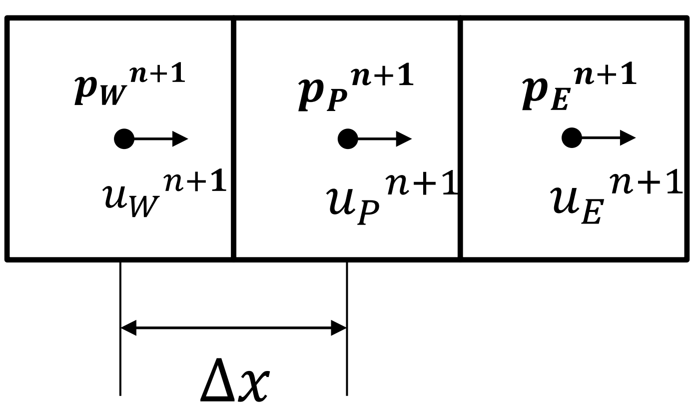
The checkerboard problem (1/3)

If we focus ourselves in the 1D spatial discretization of the step 3 of the previously described FSM, and after applying finite differences at node **P**:

$$\boldsymbol{v}^{n+1} = \boldsymbol{v}^{p} - \frac{\Delta t}{\rho}\boldsymbol{\nabla} p^{n+1}$$

For the x-component of the velocity (*$\boldsymbol{v}$=u**i**+v**j**+w**k**):*

$$u^{n+1} = u^{p} - \frac{\Delta t}{\rho}\left(\frac{p_E{}^{n+1} - p_W{}^{n+1}}{2\Delta x}\right)$$

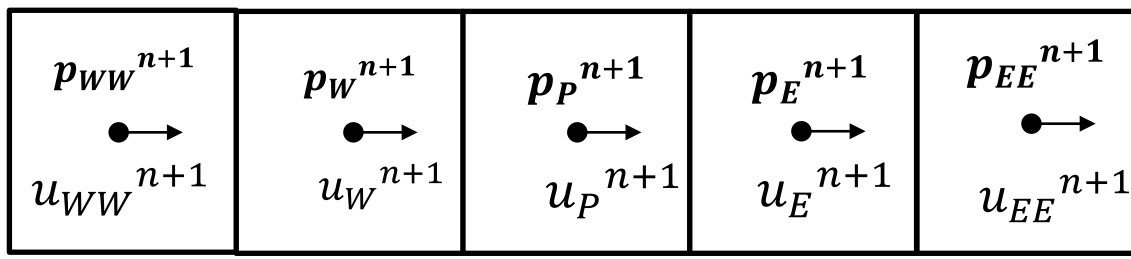


The discrete approximation of $\boldsymbol{\nabla} p^{n+1}$ at node **P** is independent of $p_P{}^{n+1}$ **!!!!**

The checkerboard problem (2/3)

We can obtain converged velocity fields for unphysical pressure distributions. For example,

$$p_{WW}{}^{n+1} = 100$$
$$p_W{}^{n+1} = 0$$
$$p_P{}^{n+1} = 100$$
$$p_E{}^{n+1} = 0$$
$$p_{EE}{}^{n+1} = 100$$



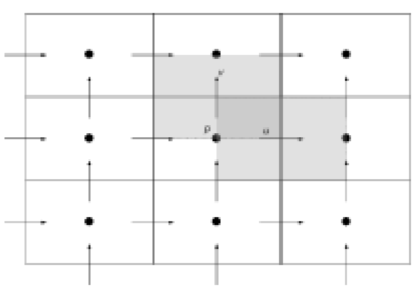
Since ∇p^{n+1} at node P is independent of $p_P{}^{n+1}$, the final velocity field will verify $\nabla p^{n+1} = \mathbf{0}$

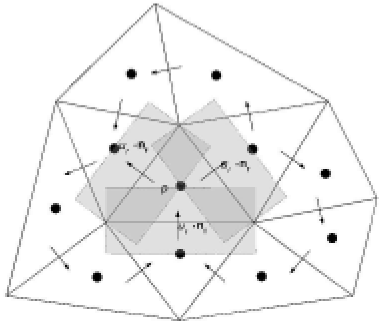
We need a smarter strategy to couple ∇p^{n+1} with the velocity field $^{n+1}$!

The checkerboard problem (3/3)

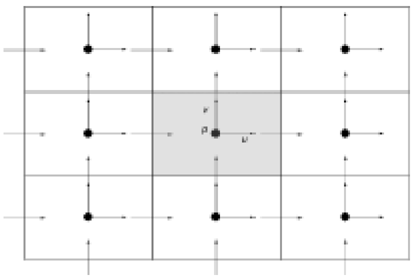
Two possible solutions have been developed to solve the checkerboard problem,

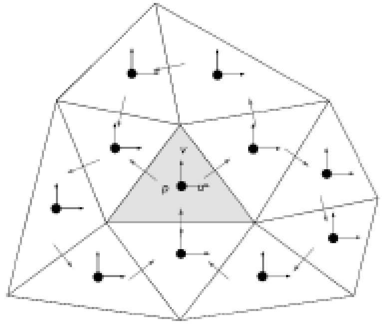
Staggered meshes



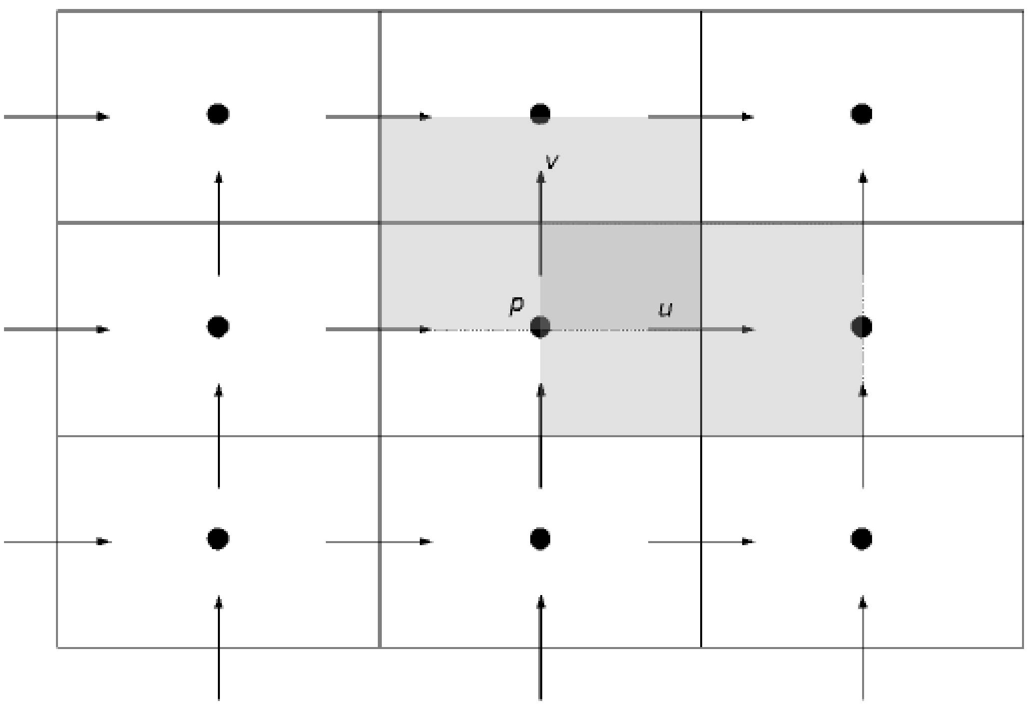


Collocated meshes





FSM for staggered meshes (1/13)



- Staggered velocity mesh solves the checkerboard problem.
- Easy to implement on structured meshes.
- But on unstructured meshes, it is difficult to implement !!
- Widely used for academic purposes.
- Next lesson will be focused on collocated arrangement (*Unit 4: FSM. Part 2: Collocated Meshes*).

FSM for staggered meshes (2/13)

$t = 0$

Summary:

1. $\boldsymbol{v}^p = \boldsymbol{v}^n + \frac{\Delta t}{\rho}[\frac{3}{2}\boldsymbol{R}(\boldsymbol{v}^n) - \frac{1}{2}R(\boldsymbol{v}^{n-1})]$
2. $\Delta p^{n+1} = \frac{\rho}{\Delta t}\boldsymbol{\nabla} \cdot \boldsymbol{v}^p$
3. $\boldsymbol{v}^{n+1} = \boldsymbol{v}^p - \frac{\Delta t}{\rho}\boldsymbol{\nabla} p^{n+1}$
4. *Choose your new* $\Delta t = min(\Delta t_c, \Delta t_d)$

At each time step …

… and finish when the steady state is reached

$t = t_{steady}$

The unsteady resolution advances with adaptive time steps until a specified condition is reached , e.g. steady state.

Next slides show the evaluation details of the different terms.

FSM Step 1: Stagg-x mesh (3/13)

Step 1 FSM (x component of $\boldsymbol{v}$): $u^P = u^n + \frac{\Delta t}{\rho}[\frac{3}{2}R(u^n) - \frac{1}{2}R(u^{n-1})]$

where :

$$R(u) = -(\rho\boldsymbol{v}\cdot\boldsymbol{\nabla})u + \mu\Delta u$$

If we integrate $\mathrm{R}(u)$ over the stagg-x control volume and then the Gauss theorem is applied:

$$\int_{\Omega_x} R(u)d\Omega_x = -\int_{\Omega_x}(\rho\boldsymbol{v}\cdot\boldsymbol{\nabla})u d\Omega_x + \int_{\Omega_x}\mu\Delta u d\Omega_x =$$

$$= -\int_{\partial\Omega_x}(\rho\boldsymbol{v})u\cdot\boldsymbol{n}dS + \int_{\partial\Omega_x}\mu\boldsymbol{\nabla}u\cdot\boldsymbol{n}dS$$

FSM Step 1: Stagg-x mesh (4/13)

$$\int_{\Omega_x} R(u)d\Omega_x = -\int_{\partial\Omega_x} (\rho\boldsymbol{v})u \cdot \boldsymbol{n}dS + \int_{\partial\Omega_x} \mu\boldsymbol{\nabla}u \cdot \boldsymbol{n}dS$$

$$R(u)\Omega_{xP} = -\,[(\rho u)_e u_e\, A_e - (\rho u)_w u_w\, A_w + (\rho v)_n u_n\, A_n - (\rho v)_s u_s\, A_s] +$$

$$[\mu_e \frac{u_E - u_P}{d_{EP}}\, A_e - \mu_w \frac{u_P - u_W}{d_{WP}}\, A_w + \mu_n \frac{u_N - u_P}{d_{NP}}\, A_n - \mu_s \frac{u_P - u_S}{d_{SP}}\, A_s]$$

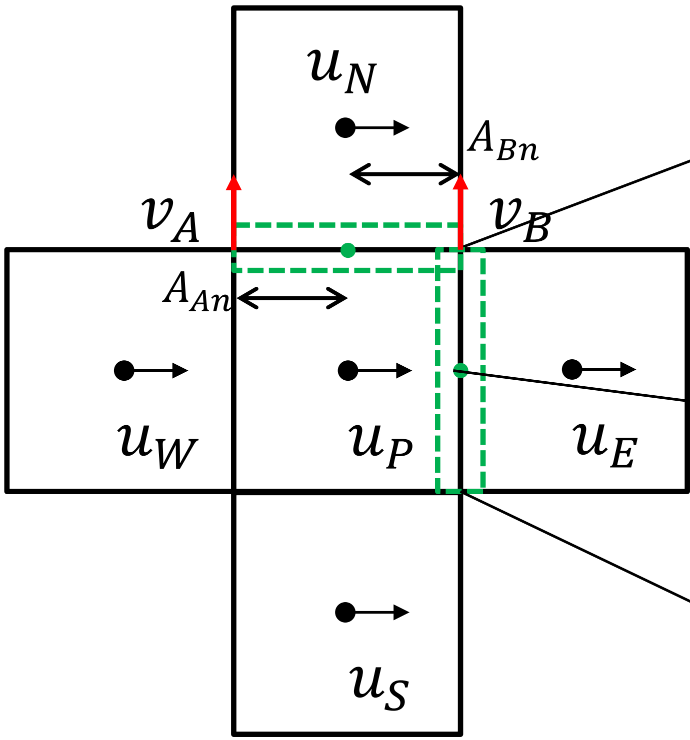
But, how can we evaluate, the volumetric flow rate and the transport property (i.e. momentum)?:

$(\rho u)_e$, $(\rho v)_n$, $(\rho u)_w$, $(\rho v)_s$???

and:

u_e, u_n, u_w, u_s ???

FSM Step 1: Stagg-x mesh (5/13)



$$(\rho v)_n = \frac{(\rho v)_A A_{An} + (\rho v)_B A_{Bn}}{A_n}$$

u_e= Upwind(u_P,u_E)

u_e= CDS(u_P,u_E)

u_e= QUICK(u_P,u_E, u_W, u_{EE}, u_{WW})

See lecture : *Numerical Solution of Convection-Diffussion equation*

$$(\rho u)_e = \frac{(\rho u)_E + (\rho u)_P}{2}$$

FSM Step 1: Stagg-y mesh (6/13)

Step 1 FSM (y component of $\boldsymbol{v}$): $v^P = v^n + \frac{\Delta t}{\rho}[\frac{3}{2}R(v^n)\ \frac{1}{2}R(v^{n-1})]$

where :

$$R(v)\Omega_{yP} \approx \text{-}\ [(\rho u)_e v_e\ A_e\ \text{-}\ (\rho u)_w v_w\ A_w + (\rho v)_n v_n\ A_n\ \text{-}\ (\rho v)_s v_s\ A_s]$$

$$[\mu_e \frac{v_E - v}{d_{EP}}\ A_e \text{-}\ \mu_w \frac{v_P - v_W}{d_{WP}}\ A_w + \mu_n \frac{v_N - v_P}{d_{NP}}\ A_n\ \text{-}\ \mu_s \frac{v_P - v_S}{d_{SP}}\ A_s]$$

and,

- $(\rho v)_e$, $(\rho v)_n$, $(\rho v)_w$, $(\rho v)_s$ are evaluated with mass conserving interpolations
- v_e, v_n, v_w, v_s are evaluated with convective numerical schemes

FSM Step 2: Main mesh (7/13)

$$\Delta p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \boldsymbol{v}^{\boldsymbol{p}}$$

$$\int_{\Omega} \Delta p^{n+1} d\Omega = \frac{\rho}{\Delta t} \int_{\Omega} \nabla \cdot \boldsymbol{v}^{\boldsymbol{p}} d\Omega$$

$$\int_{\partial\Omega} \nabla p^{n+1} \cdot \boldsymbol{n} dS = \frac{\rho}{\Delta t} \int_{\partial\Omega} \boldsymbol{v}^{\boldsymbol{p}} \cdot \boldsymbol{n} dS$$

$$\frac{p_E^{n+1} - p_P^{n+1}}{d_{EP}} A_e - \frac{p_P^{n+1} - p_W^{n+1}}{d_{WP}} A_w + \frac{p_N^{n+1} - p_P^{n+1}}{d_{NP}} A_n - \frac{p_P^{n+1} - p_S^{n+1}}{d_{SP}} A_s =$$

$$\frac{1}{\Delta t}[(\rho u^P)_e A_e - (\rho u^P)_w A_w + (\rho v^P)_n A_n - (\rho v^P)_s A_s]$$

FSM Step 2: Main mesh (8/13)

$$a_P p_P^{n+1} = a_E p_E^{n+1} + a_W p_W^{n+1} + a_N p_N^{n+1} + a_S p_S^{n+1} + b_P$$

$$a_P = a_E + a_W + a_N + a_S$$

$$a_E = \frac{A_e}{d_{EP}} \qquad a_N = \frac{A_n}{d_{NP}}$$

$$a_W = \frac{A_w}{d_{WP}} \qquad a_S = \frac{A_s}{d_{SP}}$$

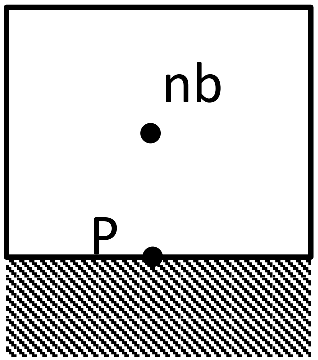
$$b_P = -\frac{1}{\Delta t}\left[\left(\rho u^{\boldsymbol{P}}\right)_{\boldsymbol{e}} A_e - \left(\rho u^{\boldsymbol{P}}\right)_{\boldsymbol{w}} A_w + \left(\rho v^{\boldsymbol{P}}\right)_{\boldsymbol{n}} A_n - \left(\rho v^{\boldsymbol{P}}\right)_{\boldsymbol{s}} A_s\right]$$

Any of the linear solvers developed for the conduction exercises can be used here (Jacobi, Gauss-Seidel or TDMA+GS)

See "Unit 2: Numerical Solution of Conduction" for more information about linear systems solvers.

FSM Step 2: Boundary conditions (9/13)

- **Wall boundary condition:**



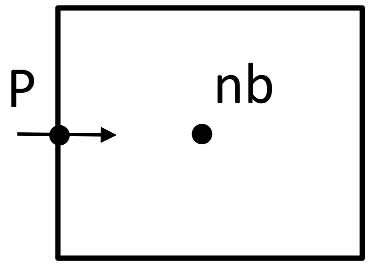
- Since a boundary layer is created at the wall $\frac{\partial p}{\partial n} = 0$

$$a_P = 1$$

$$a_{nb} = 1 \quad a_{i \neq nb} = 0$$

FSM Step 2: Boundary conditions (10/13)

- **Prescribed velocity:**



From,

$$\boldsymbol{v}^{n+1} = \boldsymbol{v}^{P} - \frac{\Delta t}{\rho}\boldsymbol{\nabla} p^{n+1}$$

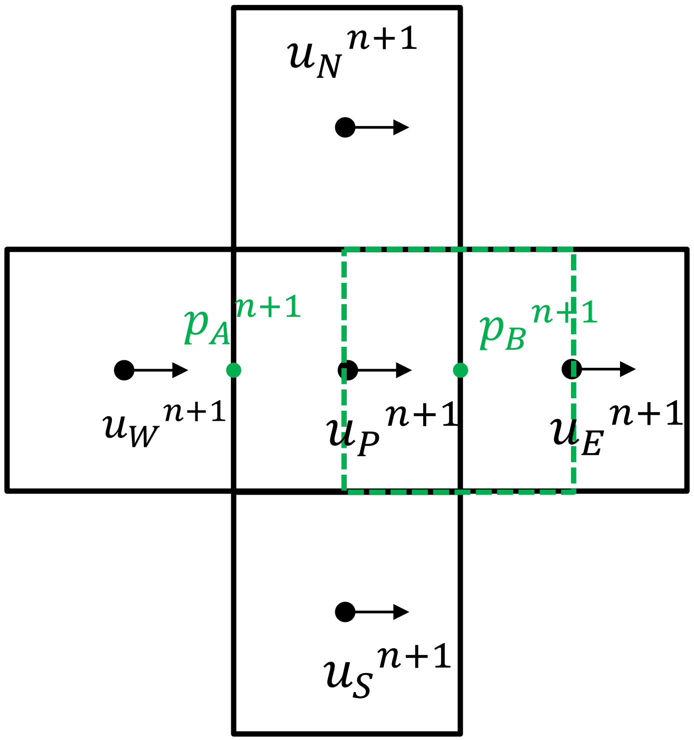
if $\boldsymbol{v}^{n+1}{}_P$ is known, we can set $\boldsymbol{v}^{P} = \boldsymbol{v}^{n+1}{}_P$, thus,

$$\frac{\partial p}{\partial n} = 0$$

$$a_P = 1$$

$$a_{nb} = 1 \quad a_{i \neq nb} = 0$$

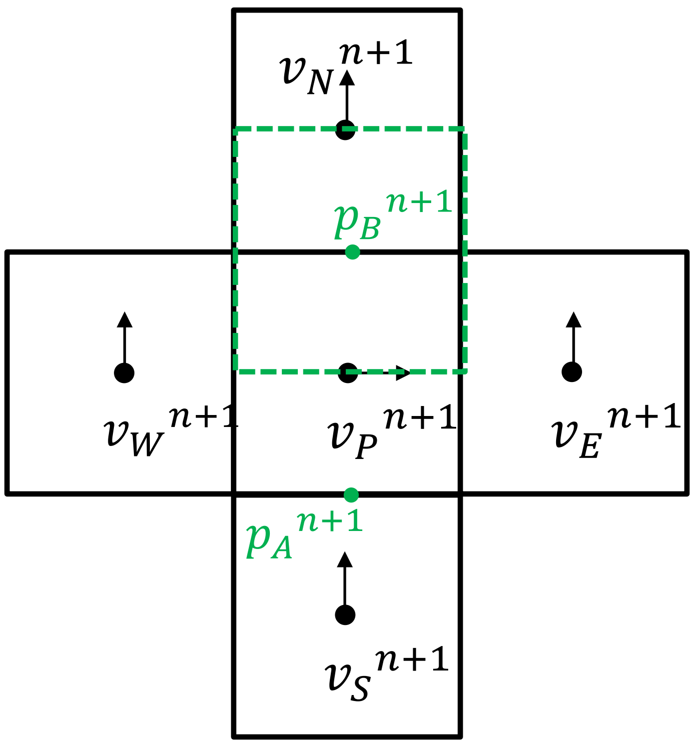
FSM Step 3: Stagg-x mesh (11/13)



$$u_P^{n+1} = u_P{}^P - \frac{\Delta t}{\rho}\left(\frac{\partial p}{\partial x}\right)_P^{n+1}$$

$$u_P^{n+1} = u_P{}^P - \frac{\Delta t}{\rho} \cdot \frac{p_B^{n+1} - p_A^{n+1}}{d_{BA}}$$

FSM Step 3: Stagg-y mesh (12/13)



$$v_P^{n+1} = v_P{}^P - \frac{\Delta t}{\rho}\left(\frac{\partial p}{\partial y}\right)_P^{n+1}$$

$$v_P^{n+1} = v_P{}^P - \frac{\Delta t}{\rho} \cdot \frac{p_B^{n+1} - p_A^{n+1}}{d_{BA}}$$

FSM Step 4: Choice of the time step (13/13)

CFL (Courant-Friedrich-Levy) condition:

$$\Delta t_c = \min\left(0.35\,\frac{\Delta x}{|\boldsymbol{v}|}\right)$$

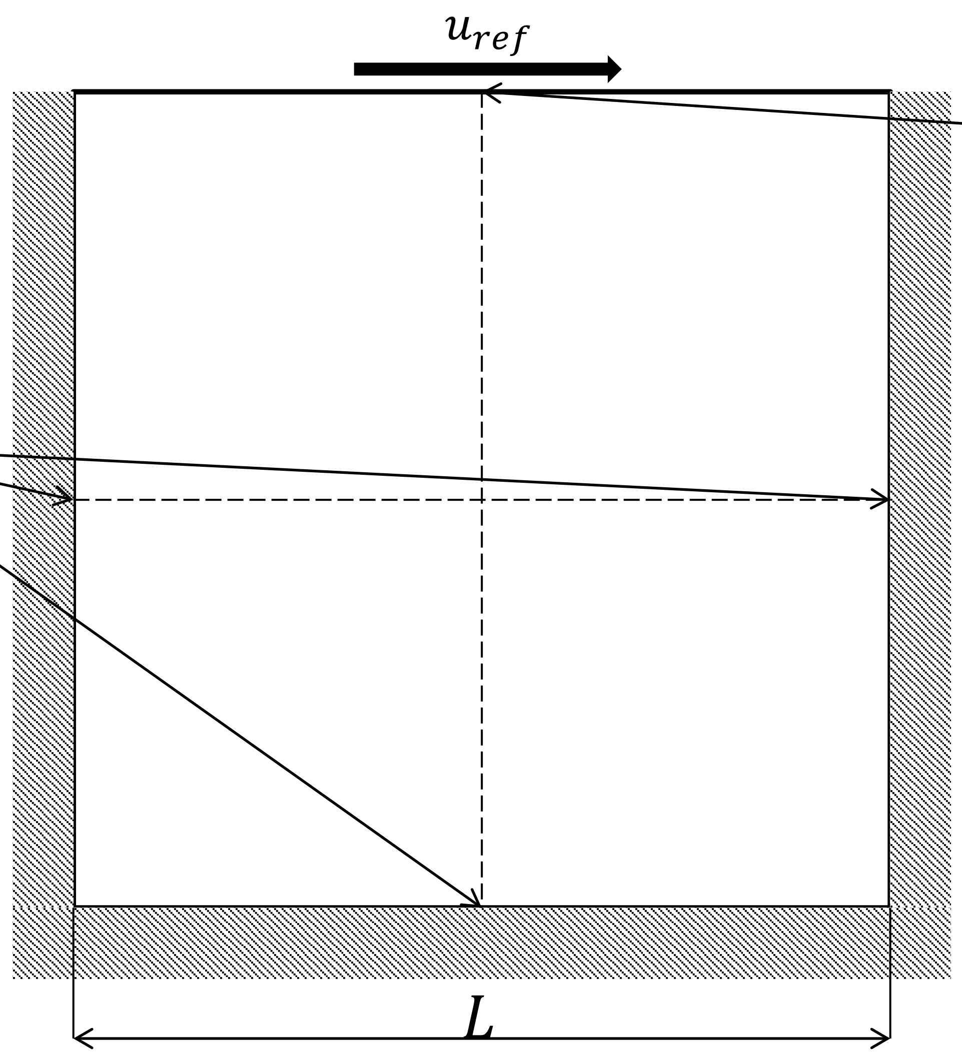
$$\Delta t_d = \min\left(0.20\,\frac{\rho \Delta x^2}{\mu}\right)$$

$$\Delta t = \min(\Delta t_c, \Delta t_d)$$

More advanced ways to find the optimal Δt can be found in: *"A self-adaptive strategy for the time integration of Navier-Stokes equations",* FX Trias, O Lehmkuhl, Numerical Heat Transfer, Part B: Fundamentals 60 (2), 116-134, 2011.

Exercise: Driven Cavity (1/3)

$$u = u_{ref}$$
$$v = 0$$
$$\frac{\partial p}{\partial x} = 0$$



$$u = 0$$
$$v = 0$$
$$\frac{\partial p}{\partial n} = 0$$

$$Re = \frac{\rho u_{ref} L}{\mu}$$

Exercise: Driven Cavity, u in the vertical center line (2/3)

129-grid pt. no.	y	Re 100	400	1000	3200	5000	7500	10,000
129	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
126	0.9766	0.84123	0.75837	0.65928	0.53236	0.48223	0.47244	0.47221
125	0.9688	0.78871	0.68439	0.57492	0.48296	0.46120	0.47048	0.47783
124	0.9609	0.73722	0.61756	0.51117	0.46547	0.45992	0.47323	0.48070
123	0.9531	0.68717	0.55892	0.46604	0.46101	0.46036	0.47167	0.47804
110	0.8516	0.23151	0.29093	0.33304	0.34682	0.33556	0.34228	0.34635
95	0.7344	0.00332	0.16256	0.18719	0.19791	0.20087	0.20591	0.20673
80	0.6172	−0.13641	0.02135	0.05702	0.07156	0.08183	0.08342	0.08344
65	0.5000	−0.20581	−0.11477	−0.06080	−0.04272	−0.03039	−0.03800	0.03111
59	0.4531	−0.21090	−0.17119	−0.10648	−0.86636	−0.07404	−0.07503	−0.07540
37	0.2813	−0.15662	−0.32726	−0.27805	−0.24427	−0.22855	−0.23176	−0.23186
23	0.1719	−0.10150	−0.24299	−0.38289	−0.34323	−0.33050	−0.32393	−0.32709
14	0.1016	−0.06434	−0.14612	−0.29730	−0.41933	−0.40435	−0.38324	−0.38000
10	0.0703	−0.04775	−0.10338	−0.22220	−0.37827	−0.43643	−0.43025	−0.41657
9	0.0625	−0.04192	−0.09266	−0.20196	−0.35344	−0.42901	−0.43590	−0.42537
8	0.0547	−0.03717	−0.08186	−0.18109	−0.32407	−0.41165	−0.43154	−0.42735
1	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Results extracted from, "High-Re Solutions for Incompressible Flow Using Navier-Stokes Equations and a Multigrid Method", Ghia et al., Journal of Computational Physics 48, 387-411 (1982).

Exercise: Driven Cavity, v in the vertical center line (3/3)

129-grid pt. no.	x	Re 100	400	1000	3200	5000	7500	10,000
129	1.0000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
125	0.9688	–0.05906	–0.12146	–0.21388	–0.39017	–0.49774	–0.53858	–0.54302
124	0.9609	–0.07391	–0.15663	–0.27669	–0.47425	–0.55069	–0.55216	–0.52987
123	0.9531	–0.08864	–0.19254	–0.33714	–0.52357	–0.55408	–0.52347	–0.49099
122	0.9453	–0.10313	–0.22847	–0.39188	–0.54053	–0.52876	–0.48590	–0.45863
117	0.9063	–0.16914	–0.23827	–0.51550	–0.44307	–0.41442	–0.41050	–0.41496
111	0.8594	–0.22445	–0.44993	–0.42665	–0.37401	–0.36214	–0.36213	–0.36737
104	0.8047	–0.24533	–0.38598	–0.31966	–0.31184	–0.30018	–0.30448	–0.30719
65	0.5000	0.05454	0.05186	0.02526	0.00999	0.00945	0.00824	0.00831
31	0.2344	0.17527	0.30174	0.32235	0.28188	0.27280	0.27348	0.27224
30	0.2266	0.17507	0.30203	0.33075	0.29030	0.28066	0.28117	0.28003
21	0.1563	0.16077	0.28124	0.37095	0.37119	0.35368	0.35060	0.35070
13	0.0938	0.12317	0.22965	0.32627	0.42768	0.42951	0.41824	0.41487
11	0.0781	0.10890	0.20920	0.30353	0.41906	0.43648	0.43564	0.43124
10	0.0703	0.10091	0.19713	0.29012	0.40917	0.43329	0.44030	0.43733
9	0.0625	0.09233	0.18360	0.27485	0.39560	0.42447	0.43979	0.43983
1	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Results extracted from, "High-Re Solutions for Incompressible Flow Using Navier-Stokes Equations and a Multigrid Method", Ghia et al., Journal of Computational Physics 48, 387-411 (1982).

Summary

- The basics concepts for solving NS equations using the FSM have been studied.
- An introduction to the checkerboard problem and its possible solutions have been presented.
- An staggered mesh code for the solution of NS equations should be developed by the student.
- The developed code must be verified through direct comparison with benchmark data of a driven cavity at different Re.

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