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Symmetry-preserving discretization of Navier-Stokes equations

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Abstract —

In this document, a general operator-based symmetry-preserving discretization method is presented. Here, a collocated-mesh scheme is preferred over a staggered one due to its simpler form for non-Cartesian meshes. The basic idea behind remains the same: mimicking the crucial symmetry properties of the underlying differential operators, *i.e.*, the convective operator is approximated by a skew-symmetric matrix and the diffusive operator by a symmetric, positive-definite matrix.

1. Introduction

We consider the simulation of turbulent, incompressible flows of Newtonian fluids. Under these assumptions, the dimensionless governing equations in primitive variables are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = \frac{1}{Re}\Delta \boldsymbol{u} - \nabla p \qquad (1a)$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad (1b)$$

where Re is the dimensionless Reynolds number.

The basic physical properties of the Navier-Stokes equations can be deduced from the symmetries of the differential operators (see [1], for instance). In the following sections we will see that it is strictly necessary to preserve such operator symmetries, at a discrete level, to preserve the analogous (invariant) properties of the continuous equations. It may be argued, specially if the method is to be used on unstructured meshes, that accuracy may need to take precedence over the properties of the operator. However, in this work, we have adopted the same philosophy followed by Verstappen and Veldman [2, 3]: symmetries of the convective and diffusive operators are critical to the dynamics of turbulence and must be preserved.

2. Discrete Navier-Stokes equations

The finite volume discretization of the Navier-Stokes and continuity equations on an arbitrary mesh can be written by

$$
\begin{aligned}
\Omega_h \frac{d\boldsymbol{u}_h}{dt} + \mathsf{C}\left(\boldsymbol{u}_h\right)\boldsymbol{u}_h + \mathsf{D}\boldsymbol{u}_h + \Omega_h \mathsf{G}_h \boldsymbol{p}_h &= \mathbf{0}_h &\text{(2a)}\\
\mathsf{M}_h \boldsymbol{u}_h &= \mathbf{0}_h &\text{(2b)}
\end{aligned}
$$

where the matrix Ω_h is a (positive-definite) diagonal matrix representing the sizes of control volumes associated with the discrete velocity field, $\boldsymbol{u}_h$. The matrices $\mathsf{C}\left(\boldsymbol{u}_h\right)$ and D are the convective and diffusive operators respectively. Note the $\boldsymbol{u}_h$-dependence of the convective operator (non-linear operator). Finally, G_h represents the discrete gradient operator and the matrix M_h is the divergence operator.

Since now, no assumption about the grid (structured/unstructured) or the location (centered/staggered) of the discrete variables on such mesh has been considered yet. Hence, this discrete operator representation is generic and fits for almost all the existing spatial discretization based on a finite volume (finite differences and finite elements also) formulations.

The conservative nature of the Navier-Stokes equations is intimately tied up with the symmetries of the differential operators (see [1, 3], for instance). In the following subsections, we will see that retaining the symmetry properties of the continuous operators when discretizing equations is necessary in order to exactly conserve the inviscid invariants in a discrete sense.

2.1. Kinetic energy conservation

The global discrete energy is “naturally” defined as

$$
||\boldsymbol{u}_h||^2 \equiv \boldsymbol{u}_h^* \Omega_h \boldsymbol{u}_h \tag{3}
$$

The evolution equation of $||\boldsymbol{u}_h||^2$ can be obtained by left-multiplying Eq.(2a) by $\boldsymbol{u}_h^*$ and summing the resulting expression with its conjugate transpose

$$
\begin{aligned}
\frac{d}{dt}||\boldsymbol{u}_h||^2 &= -\boldsymbol{u}_h^* \left(\mathsf{C}\left(\boldsymbol{u}_h\right) + \mathsf{C}^*\left(\boldsymbol{u}_h\right)\right)\boldsymbol{u}_h - \boldsymbol{u}_h^* \left(\mathsf{D} + \mathsf{D}^*\right)\boldsymbol{u}_h\\
&\quad -\boldsymbol{u}_h^* \Omega_h \mathsf{G}_h \boldsymbol{p}_h - \boldsymbol{p}_h^* \mathsf{G}_h^* \Omega_h^* \boldsymbol{u}_h &\text{(4)}
\end{aligned}
$$

In absence of diffusion, that is $\mathsf{D} = 0$, the global kinetic energy $||\boldsymbol{u}_h||^2$ is conserved if both, the convective and pressure terms, vanish (for any $\boldsymbol{u}_h$, $\mathsf{M}_h \boldsymbol{u}_h = \mathbf{0}_h$) in the discrete energy equation,

$$
\begin{aligned}
\boldsymbol{u}_h^* \left(\mathsf{C}\left(\boldsymbol{u}_h\right) + \mathsf{C}^*\left(\boldsymbol{u}_h\right)\right)\boldsymbol{u}_h &= 0 &\text{(5a)}\\
\boldsymbol{u}_h^* \Omega_h \mathsf{G}_h \boldsymbol{p}_h + \boldsymbol{p}_h^* \mathsf{G}_h^* \Omega_h^* \boldsymbol{u}_h &= 0 &\text{(5b)}
\end{aligned}
$$

These conservation properties are held, if and only if, the discrete convective operator is skew-symmetric and if the negative conjugate transpose of the discrete gradient operator is exactly equal to the divergence operator.

$$
\begin{aligned}
\mathsf{C}\left(\boldsymbol{u}_h\right) &= -\mathsf{C}^*\left(\boldsymbol{u}_h\right) &\text{(6a)}\\
-\left(\Omega_h \mathsf{G}_h\right)^* &= \mathsf{M}_h &\text{(6b)}
\end{aligned}
$$

Therefore, if the convective and gradient operators are properly chosen, the global kinetic energy equation (4) reduces to

$$\frac{d}{dt}||\boldsymbol{u}_h||^2 = -\boldsymbol{u}_h^* \left(\mathsf{D} + \mathsf{D}^*\right) \boldsymbol{u}_h \leq 0 \tag{7}$$

where the inequality follows from the condition that diffusive terms must be strictly dissipative. Thus, the matrix $(\mathsf{D} + \mathsf{D}^*)$ must be positive-definite. Finally, we would also like that D be a purely diffusive operator: that is, it must be represented by a symmetric matrix. The latter two conditions are satisfied if, and only if, the diffusive operator,D, is symmetric and positive-definite.

2.2. Global momentum conservation equation

Defining the total amount of momentum by taking the scalar product of the velocity vector $\boldsymbol{u}_h$ with the vector $\Omega_h \mathbf{1}_h$, the equation for the temporal evolution of global momentum is obtained by left-multiplying Eq.(2a) by vector $\mathbf{1}_h^*$

$$\frac{d}{dt}\left(\mathbf{1}_h^* \Omega_h \boldsymbol{u}_h\right) = -\mathbf{1}_h^* \left(\mathsf{C}\left(\boldsymbol{u}_h\right) + \mathsf{D}\right) \boldsymbol{u}_h - \mathbf{1}_h^* \Omega_h \mathsf{G}_h \boldsymbol{p}_h \tag{8}$$

Therefore, momentum is conserved if

$$\left(\mathsf{C}\left(\boldsymbol{u}_h\right) + \mathsf{D}\right)^* \mathbf{1}_h = \mathbf{0}_h \tag{9}$$

Since the convective operator is skew-symmetric and the diffusive operator is symmetric we can leave the *'s away. Thus, it suffices to check that the constant vector $\mathbf{1}_h$ lies in the null space of both the convective and the diffusive operators

3. Symmetry-preserving discretization on a collocated formulation

No assumption about how the discrete variables are arranged has been made so far. In this section, we will analyse how the symmetry (6b), that relates the discrete gradient and the divergence operator, may constrain the variable arrangement. As it is well-known, the discrete variables can be defined and arranged in different ways. The two most common are the staggered and the collocated arrangements (see Fig. 1). The staggered arrangement, proposed in the pioneering work by Harlow and Welch [4], stores the pressure center of the control volume and face-normal velocities at the faces. In the collocated mesh scheme, both the pressure and the velocities are stored at the center of the control volume. In the collocated arrangement a secondary velocity field at the faces is necessary to enforce mass conservation.

For simplicity, in this section, the spatially discretized Navier-Stokes equations (2) are particularised for a general collocated formulation while keeping the formulation fully-conservative.

3.1. Definition of basic collocated operators

The finite volume discretization of the Navier-Stokes and continuity equations on an arbitrary collocated mesh can be written by

$$\Omega \frac{d\boldsymbol{u}_c}{dt} + \mathsf{C}\left(\boldsymbol{u}_s\right) \boldsymbol{u}_c + \mathsf{D}\boldsymbol{u}_c + \Omega \mathsf{G}_c \boldsymbol{p}_c = \mathbf{0}_c \tag{10a}$$

$$\mathsf{M}\boldsymbol{u}_s = \mathbf{0}_c \tag{10b}$$

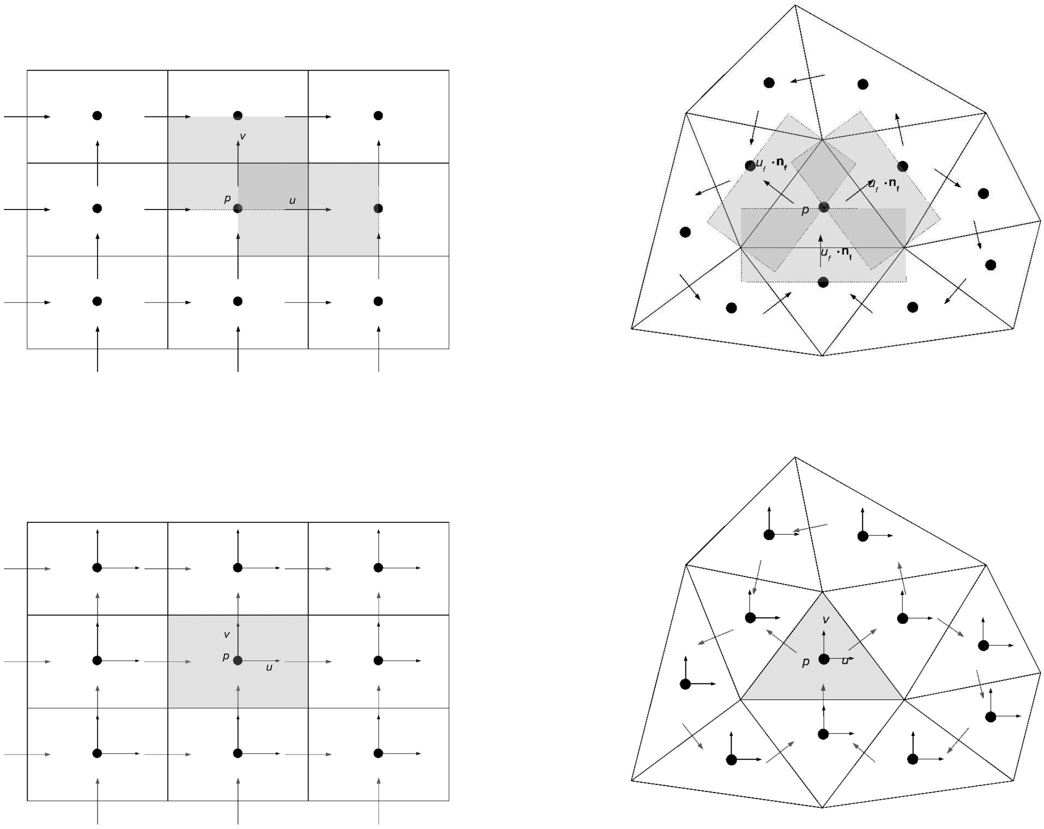

Figure 1: Variable arrangements for the staggered (top) and the collocated (bottom) schemes.

where $\boldsymbol{p}_c$ is the cell-centered pressure scalar field

$$\boldsymbol{p}_c = \left(p_1, p_2, \ldots, p_n\right)^* \in \mathbb{R}^n \tag{11}$$

where n is the number of control volumes. Here, the subindices c and s refer whether the variables are cell-centered or staggered at the faces. Then, centered velocity field is defined as column vector and arranged as follows

$$\boldsymbol{u}_c = \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \end{pmatrix} \in \mathbb{R}^{3n} \tag{12}$$

where $\boldsymbol{u}_i = \left((u_i)_1, (u_i)_2, \ldots, (u_i)_n\right)^*$ are the vectors with the velocity components corresponding to the x_i-spatial direction. The auxiliary discrete staggered velocity field

$$\boldsymbol{u}_s = \left((u_s)_1, (u_s)_2, (u_s)_3, \ldots, (u_s)_m\right)^* \in \mathbb{R}^m \tag{13}$$

where m is the number of faces on the computational domain, is related with the centered velocity field via a linear shift transformation $\Gamma_{c\rightarrow s} \in \mathbb{R}^{m\times 3n}$

$$\boldsymbol{u}_s \equiv \Gamma_{c\rightarrow s}\boldsymbol{u}_c \tag{14}$$

The matrix Ω is a block diagonal matrix given by

$$\Omega = \begin{pmatrix} \Omega_c & 0 & 0 \\ 0 & \Omega_c & 0 \\ 0 & 0 & \Omega_c \end{pmatrix} \in \mathbb{R}^{3n \times 3n} \tag{15}$$

where $\Omega_c \in \mathbb{R}^{n \times n}$ is a diagonal matrix with the cell centered control volumes.

The convective and diffusive operators $\mathsf{C}(\boldsymbol{u}_s) \in \mathbb{R}^{3n \times 3n}$ and $\mathsf{D} \in \mathbb{R}^{3n \times 3n}$ are block diagonal matrices given by

$$\mathsf{C}(\boldsymbol{u}_s) = \begin{pmatrix} \mathsf{C}_c(\boldsymbol{u}_s) & 0 & 0 \\ 0 & \mathsf{C}_c(\boldsymbol{u}_s) & 0 \\ 0 & 0 & \mathsf{C}_c(\boldsymbol{u}_s) \end{pmatrix}; \quad \mathsf{D} = \begin{pmatrix} \mathsf{D}_c & 0 & 0 \\ 0 & \mathsf{D}_c & 0 \\ 0 & 0 & \mathsf{D}_c \end{pmatrix} \tag{16}$$

where $\mathsf{C}_c(\boldsymbol{u}_s) \in \mathbb{R}^{n \times n}$ and $\mathsf{D}_c \in \mathbb{R}^{n \times n}$ are the cell-centered convective and diffusive operators for a discrete scalar field. Note the $\boldsymbol{u}_s$-dependence of the convective operator (non-linear operator). Finally, $\mathsf{G}_c \in \mathbb{R}^{3n \times n}$ represents the discrete gradient operator and the matrix $\mathsf{M} \in \mathbb{R}^{n \times m}$ is the face-to-center discrete divergence operator.

3.2. (Skew-)symmetries of the collocated operators

In section 2 global operator properties were examined. With the new collocated formulation the basic operator properties (6) become

$$\begin{aligned} \mathsf{C}(\boldsymbol{u}_s) &= -\mathsf{C}^*(\boldsymbol{u}_s) && \text{(17a)} \\ -(\Omega \mathsf{G}_c)^* &= \mathsf{M}\Gamma_{c \to s} && \text{(17b)} \end{aligned}$$

Therefore, if the convective and gradient operators are properly chosen, the global kinetic energy equation reduces to

$$\frac{d}{dt}||\boldsymbol{u}_c||^2 = -\boldsymbol{u}_c^* (\mathsf{D} + \mathsf{D}^*) \boldsymbol{u}_c \tag{18}$$

where the global discrete energy is now defined as $||\boldsymbol{u}_c||^2 \equiv \boldsymbol{u}_c^* \Omega \boldsymbol{u}_c$. Moreover, conservation of momentum demands that constant vector $\mathbf{1}_c$ lies on the null space of convective and diffusive operators

$$\begin{aligned} \mathsf{C}(\boldsymbol{u}_s)\mathbf{1}_c &= \mathbf{0}_c && \text{(19a)} \\ \mathsf{D}\mathbf{1}_c &= \mathbf{0}_c && \text{(19b)} \end{aligned}$$

3.3. Solving the pressure-velocity coupling

In order to simplify the notation, spatially discrete momentum equation (10a) can be rewritten as

$$\frac{d\boldsymbol{u}_c}{dt} = \mathsf{R}(\boldsymbol{u}_c) - \mathsf{G}_c \boldsymbol{p}_c \tag{20}$$

where $\mathsf{R}(\boldsymbol{u}_c) \equiv -\Omega^{-1}(\mathsf{C}(\boldsymbol{u}_s)\boldsymbol{u}_c + \mathsf{D}\boldsymbol{u}_c)$ represents the convective and diffusive terms. For the temporal discretization, a one-parameter fully explicit second-order one-leg scheme for $\mathsf{R}(\boldsymbol{u}_c)$

and a first-order backward Euler scheme for the pressure-gradient term. Incompressibility constraint is treated implicit. Thus, we obtain the fully-discretized Navier-Stokes equations

$$\mathsf{M}\boldsymbol{u}_s^{n+1} = \mathbf{0}_c \tag{21a}$$

$$\frac{(\kappa+1/2)\,\boldsymbol{u}_c^{n+1} - 2\kappa\boldsymbol{u}_c^n + (\kappa-1/2)\,\boldsymbol{u}_c^{n-1}}{\Delta t} = \mathsf{R}\left((1+\kappa)\,\boldsymbol{u}_c^n - \kappa\boldsymbol{u}_c^{n-1}\right) - \mathsf{G}_c\boldsymbol{p}_c^{n+1} \tag{21b}$$

where the parameter κ is computed each time-step to adapt the linear stability domain of the time-integration scheme to the instantaneous flow conditions in order to use the maximum Δt possible. For further details about the time-integration method the reader is referred to [5, 6].

To solve the velocity-pressure coupling a classical fractional step projection method [7, 8] is used. In the projection methods, solutions of the unsteady Navier-Stokes equations are obtained by first time-advancing the velocity field $\boldsymbol{u}_c$ without regard for its solenoidality constraint (21a), then recovering the proper solenoidal velocity field, $\boldsymbol{u}_c^{n+1}$ ($\mathsf{M}\boldsymbol{u}_s^{n+1} = \mathbf{0}_c$). This projection is derived from the Helmholtz-Hodge vector decomposition theorem [9], whereby the velocity $\boldsymbol{u}_s^{n+1}$ can be uniquely decomposed into a solenoidal vector, $\boldsymbol{u}_s^p$, and a curl-free vector, expressed as the gradient of a scalar field, $\mathsf{G}\boldsymbol{p}_c$. This decomposition is written as

$$\boldsymbol{u}_s^p = \boldsymbol{u}_s^{n+1} + \mathsf{G}\tilde{\boldsymbol{p}}_c \tag{22}$$

where $\mathsf{G} \in \mathbb{R}^{m\times n}$ is the staggered gradient operator which is related with the divergence operator

$$\mathsf{G} \equiv -\Omega_s^{-1}\mathsf{M}^* \tag{23}$$

and $\boldsymbol{u}_s^p$ is related with the cell-centered predictor velocity $\boldsymbol{u}_c^p$ via

$$\boldsymbol{u}_s^p = \Gamma_{c\rightarrow s}\boldsymbol{u}_c^p \tag{24}$$

which is subsequently given by

$$\boldsymbol{u}_c^p = \frac{2\kappa\boldsymbol{u}_c^n - (\kappa-1/2)\,\boldsymbol{u}_c^{n-1}}{\kappa+1/2} + \frac{\Delta t}{\kappa+1/2}\mathsf{R}\left((1+\kappa)\,\boldsymbol{u}_c^n - \kappa\boldsymbol{u}_c^{n-1}\right) \tag{25}$$

and the pseudo-pressure is $\tilde{\boldsymbol{p}}_c = \Delta t/\,(\kappa+1/2)\,\boldsymbol{p}_c^{n+1}$. Thus, taking the divergence of (22) yields a discrete Poisson equation for $\tilde{\boldsymbol{p}}_c$

$$\mathsf{M}\boldsymbol{u}_s^p = \mathsf{M}\boldsymbol{u}_s^{n+1} + \mathsf{M}\mathsf{G}\tilde{\boldsymbol{p}}_c \longrightarrow \mathsf{M}\mathsf{G}\tilde{\boldsymbol{p}}_c = \mathsf{M}\boldsymbol{u}_s^p \tag{26}$$

Finally, using the definition of G given in (23) previous equation becomes

$$\mathsf{L}\tilde{\boldsymbol{p}}_c = \mathsf{M}\boldsymbol{u}_s^p \tag{27}$$

where the discrete Laplacian operator $\mathsf{L} \in \mathbb{R}^{n\times n}$ is, by construction, a symmetric negative-definite matrix

$$\mathsf{L} \equiv -\mathsf{M}\Omega_s^{-1}\mathsf{M}^* \tag{28}$$

Note that since the Laplacian operator inherits the boundary conditions from the divergence operator, no additional boundary conditions for pressure need to be prescribed. Once the solution is obtained, $\boldsymbol{u}_s^{n+1}$ results from the correction

$$\boldsymbol{u}_s^{n+1} = \boldsymbol{u}_s^p - \mathsf{G}\tilde{\boldsymbol{p}}_c \tag{29}$$

Finally, cell-centered velocity at the next time step, $\boldsymbol{u}_c^{n+1}$ is related to $\boldsymbol{u}_s^{n+1}$ via a linear shift transformation $\Gamma_{s\to c} \in \mathbb{R}^{3n\times m}$.

$$\boldsymbol{u}_c^{n+1} \equiv \Gamma_{s\to c}\boldsymbol{u}_s^{n+1} \tag{30}$$

3.3.1. Constraints on the shift operator $\Gamma_{s\to c}$

According to previous expressions (29)-(30) and the staggered gradient definition (23), the cell-centered discrete gradient operator results

$$\mathsf{G}_c = -\Gamma_{s\to c}\Omega_s^{-1}\mathsf{M}^* \tag{31}$$

therefore, face-to-cell shift operator $\Gamma_{s\to c}$ is restricted by (17b)

$$\left(\Omega\Gamma_{s\to c}\Omega_s^{-1}\mathsf{M}^*\right)^* = \mathsf{M}\Gamma_{c\to s} \longrightarrow \Gamma_{s\to c} = \Omega^{-1}\Gamma_{c\to s}^*\Omega_s \tag{32}$$

to force pressure gradient contribution to global kinetic energy vanishes exactly.

4. Constructing discrete operators

Discretization of operators preserving the global properties defined in the previous section is a difficult task. In general, the constraints imposed by operator (skew-)symmetries strongly restrict the form of the local approximations limiting, in some cases, the local truncation error.

4.1. Reynolds transport theorem

To prepare for the finite volume symmetry-preserving discretization, we recall the Reynolds transport theorem for a function ϕ

$$\frac{d}{dt}\int_{(\Omega_c)_{k,k}} \phi dV = \int_{(\Omega_c)_{k,k}} \frac{\partial \phi}{\partial t} dV + \int_{\partial(\Omega_c)_{k,k}} \phi \boldsymbol{u}\cdot\boldsymbol{n} dS \tag{33}$$

on an arbitrary centered cell k of volume $(\Omega_c)_{k,k}$. Note that, the function ϕ can have several meanings depending on what is transported.

4.2. Collocated convective operator

Second (convective) term of the right-hand-side of (33) can be expressed exactly as

$$\int_{\partial(\Omega_c)_{k,k}} \phi \boldsymbol{u}\cdot\boldsymbol{n} dS = \sum_{f\in F_f(k)} \int_{S_f} \phi \boldsymbol{u}\cdot\boldsymbol{n} dS \tag{34}$$

where $F_f(k)$ is the set of faces bordering the cell k. Assuming that the discrete normal velocities, $[\boldsymbol{u}_s]_f \approx \boldsymbol{u}_f\cdot\boldsymbol{n}_f$, are located at the centroid of the face, then a second-order discretization of the integral (34) is given by

$$\int_{\partial(\Omega_c)_{k,k}} \phi \boldsymbol{u}\cdot\boldsymbol{n} dS \approx \sum_{f\in F_f(k)} \phi_f[\boldsymbol{u}_s]_f A_f \tag{35}$$

where A_f is the area of the face f.

Hence, the collocated convective operator is defined by its action on an arbitrary centered scalar field $\boldsymbol{\phi}_c \in \mathbb{R}^n$ at some cell k as

$$[\mathsf{C}_c(\boldsymbol{u}_s)\,\boldsymbol{\phi}_c]_k = \sum_{f \in F_f(k)} \phi_f [\boldsymbol{u}_s]_f A_f \tag{36}$$

4.2.1. Skew-symmetry of the convective operator

In section 2.1, we have seen that the convective contribution to the global kinetic energy vanishes if and only if the convective operator is skew-symmetric (Eq. 17a). Such condition is verified in two steps. First, we consider the off-diagonal elements. The matrix $\mathsf{C}_c(\boldsymbol{u}_s) - diag(\mathsf{C}_c(\boldsymbol{u}_s))$ is skew-symmetric if the interpolation weights of the adjacent discrete variables are taken equal to $1/2$, hence independent of the grid location,

$$\phi_f \approx [\Pi_{c\to s}\boldsymbol{\phi}_c]_f = \frac{\phi_{c1} + \phi_{c2}}{2} \tag{37}$$

where $c1$ and $c2$ are the cells adjacent to the face f (see Fig. 2, left) and $\Pi_{c\to s} \in \mathbb{R}^{m\times n}$ is the shift operator that interpolates a cell-centered scalar field to the faces. To illustrate this, we consider the general interpolation rule

$$\phi_f = (1-\alpha_f)\,\phi_{c1} + \alpha_f \phi_{c2} \tag{38}$$

where the coefficient α_f may depend on the local mesh sizes. Substituting this interpolation into Eq. (36), we see that corresponding off-diagonal coefficients of matrix $\mathsf{C}_c(\boldsymbol{u}_s)$ become

$$[\mathsf{C}_c(\boldsymbol{u}_s)]_{c1,c2} = (1-\alpha_f)[\boldsymbol{u}_s]_f A_f \tag{39a}$$

$$[\mathsf{C}_c(\boldsymbol{u}_s)]_{c2,c1} = -\alpha_f\,[\boldsymbol{u}_s]_f A_f \tag{39b}$$

Thus, for skew-symmetry, these two coefficients should be of opposite sign,

$$(1-\alpha_f)[\boldsymbol{u}_s]_f A_f = \alpha_f [\boldsymbol{u}_s]_f A_f \tag{40}$$

As we stated, this property can only be achieved if and only if weights α_f are taken equal to $1/2$.

Next, we consider the diagonal elements of $\mathsf{C}_c(\boldsymbol{u}_s)$. For skew-symmetry of the collocated convective operator $\mathsf{C}_c(\boldsymbol{u}_s)$ they have to be zero. The interpolation rule defined in (37) determines that

$$[\mathsf{C}_c(\boldsymbol{u}_s)]_{k,k} = \frac{1}{2}\sum_{f \in F_f(k)} [\boldsymbol{u}_s]_f A_f = 0 \tag{41}$$

In conclusion, the collocated convective operator $\mathsf{C}_c(\boldsymbol{u}_s)$ is skew-symmetric if the discrete centered variable ϕ is interpolated to the faces of the control volumes using the rule defined in (37), and the Eq.(41) is accomplished. In the following section, we will see that the latter condition would hold if the divergence operator is consistently defined.

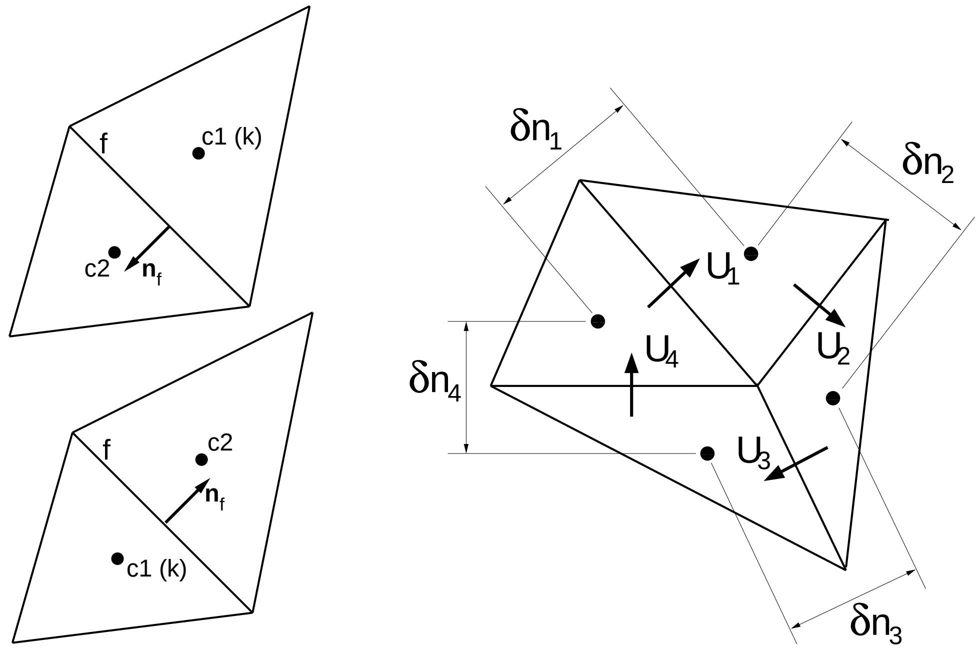

Figure 2: Left: face normal and neighbour labelling criteria. Right: definition of volumes of the face-normal velocity cell.

4.3. Gradient, divergence and Laplacian operators

Integrating the continuity equation (1b) over an arbitrary centered cell k of volume $(\Omega_c)_{k,k}$ yields

$$\int_{(\Omega_c)_{k,k}} \nabla \cdot \boldsymbol{u} dV = \int_{\partial(\Omega_c)_{k,k}} \boldsymbol{u} \cdot \boldsymbol{n} dS = \sum_{f \in F_f(k)} \int_{S_f} \boldsymbol{u} \cdot \boldsymbol{n} dS \tag{42}$$

Note that taking ϕ equal to the unity, the Reynolds transport theorem (33) also gives the continuity equation in integral form. Therefore, Eq.(35) is particularised to define the proper integrated divergence operator,

$$[\mathsf{M}\boldsymbol{u}_s]_k = \sum_{f \in F_f(f)} [\boldsymbol{u}_s]_f A_f = 0 \tag{43}$$

Doing so, we are forcing that diagonal elements of collocated convective operator be equal to zero (41). At this stage, it must be noted that the collocated convective operator defined in Eq.(36) can be rewritten by using more basic operators as

$$\mathsf{C}_c(\boldsymbol{u}_s)\boldsymbol{\phi}_c = \mathsf{M}\left(\text{diag}\left(\Pi_{c\to s}\boldsymbol{\phi}_c\right)\boldsymbol{u}_s\right) \tag{44}$$

In section 2.1, it has been shown (Eq. 23) that the integrated pressure gradient operator, $\Omega_s \mathsf{G}$, must be equal to the negative conjugate transpose of the divergence operator, $-\mathsf{M}$. Hence, the discretization of the pressure gradient at the face f follows from (43)

$$[\Omega_s \mathsf{G}\boldsymbol{p}_c]_f = (p_{c1} - p_{c2}) A_f \tag{45}$$

where $c1$ and $c2$ are the cells adjacent to the face f. The order of accuracy of the discretization of the gradient operator defined in Eq.(45) is, in general, $\mathcal{O}(1)$. Since for incompressible flows, the role of the pressure gradient is to project the velocity field into a divergence-free space, this lack of accuracy becomes irrelevant in our context.

Note that since the discrete gradient inherits the boundary conditions from the discrete divergence operator, we need not specify boundary conditions for the pressure. Finally, we compute the pressure from a Poisson equation, which arises from the incompressibility constraint. The Laplacian operator is approximated by the matrix,

$$\mathsf{L} = -\mathsf{M}\Omega_s^{-1}\mathsf{M}^* \tag{46}$$

which is symmetric and negative-definite, like the continuous Laplacian operator.

4.4. Diffusive operator

Again, a diffusive operator is easily constructed on a collocated mesh. The same method for discretization the Laplacian operator is also applied: the diffusive operator is viewed as the product of two first-order differential operators, the divergence, M, of a gradient, G,

$$\mathsf{D}_c = -\frac{1}{Re}\mathsf{MG} = \frac{1}{Re}\mathsf{M}\Omega_s^{-1}\mathsf{M}^* \tag{47}$$

The Reynolds numbers have been introduced in order to simplify the notation. Note that the collocated diffusive operator is, by definition (47), symmetric and positive-definite and its action on a cell-centered variable is given by,

$$[\mathsf{D}_c\phi_c]_k = \frac{1}{Re}\sum_{f\in F_f(k)} \frac{(\phi_{c2}-\phi_{c1})\,A_f}{\delta n_f} \tag{48}$$

where the length δn_f is an approximation of the distance between the centroids of cells $c1$ and $c2$ given by

$$\delta n_f = \frac{(\Omega_s)_f}{A_f} \tag{49}$$

where $(\Omega_s)_f$ is the volume of the face-normal velocity cell at the face f (see Fig. 2, right). However, on general unstructured meshes, the error term of the diffusive discretization is indeed $\mathcal{O}(1)$. Therefore, at first sight this operator definition might seem inappropriate. However, supraconvergence for the global truncation error has been observed in numerical experiments.

4.5. Shift operators

The linear shift transformation, $\Gamma_{c\to s} \in \mathbb{R}^{m\times 3n}$, is given by

$$(\Gamma_{c\to s}\boldsymbol{u}_c)_f = \mathsf{N}_s\,(\Pi\boldsymbol{u}_c) = \frac{1}{2}\left([\boldsymbol{u}_c]_{c1} + [\boldsymbol{u}_c]_{c2}\right)\cdot\boldsymbol{n}_f \tag{50}$$

where matrices $\mathsf{N}_s \in \mathbb{R}^{m\times 3m}$ and $\Pi \in \mathbb{R}^{3m\times 3n}$ are respectively given by

$$\mathsf{N}_s = \begin{pmatrix} \mathsf{N}_{s,1} & \mathsf{N}_{s,2} & \mathsf{N}_{s,3} \end{pmatrix}; \quad \Pi = \begin{pmatrix} \Pi_{c\to s} & 0 & 0 \\ 0 & \Pi_{c\to s} & 0 \\ 0 & 0 & \Pi_{c\to s} \end{pmatrix} \tag{51}$$

where $\mathsf{N}_{s,i} \in \mathbb{R}^{m\times m}$ are diagonal matrices containing the x_i-spatial components of the face normal vectors, and $\Pi_{c\to s} \in \mathbb{R}^{m\times n}$ is the operator that interpolates a cell-centered scalar field to the faces defined in (37). Finally, face-to-cell shift transformation, $\Gamma_{s\to c} \in \mathbb{R}^{3n\times m}$, follows straightforwardly from (32).

4.6. Boundary conditions

Since now we have not considered the numerical treatment of the boundary conditions. Obviously, symmetry properties have to be preserved. In the case of periodic boundary conditions, their discretization can be easily extended without breaking the symmetries of the matrices C and D.

For non-periodic boundary conditions it is not so clear. The operators have to be discretized is such a way that their symmetry properties are preserved.

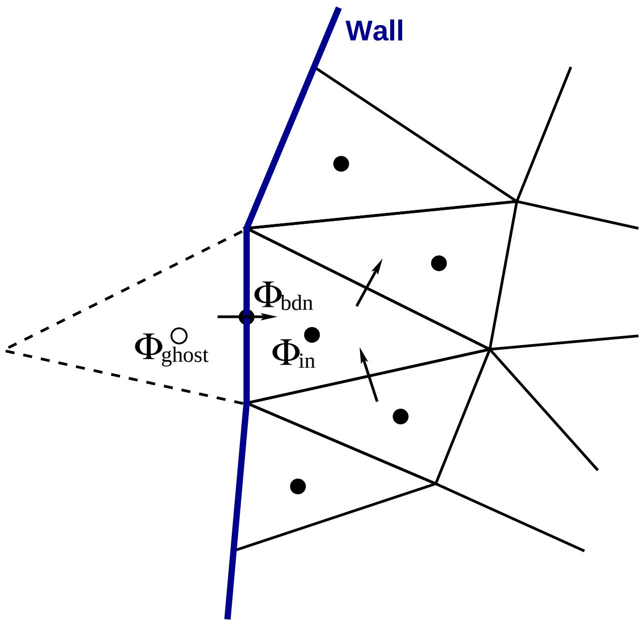

Figure 3: Boundary conditions treatment

4.6.1. Boundary conditions for convective operator C

The discretization near the boundary is not constructed to minimise the local truncation error. Instead, we have to take care to build it without violating the skew-symmetry of matrix C.

The collocated convective operator at cell ϕ_{in} (see figure 3) is defined according to expressions (36) and (37). To do so, we need to evaluate the out-of-domain variable ϕ_{ghost}. First, we consider that such *ghost* variable only depends of ϕ_{in} and the boundary value ϕ_{bdn},

$$\phi_{ghost} = f^{\mathsf{C}}\left(\phi_{bdn}, \phi_{in}\right) \tag{52}$$

Therefore, the off-diagonal coefficients remain unchanged. Then, we have to construct a linear extrapolation function f^{C} in such a way that skew-symmetry of C be exactly preserved,

$$\phi_{ghost} = a_{bdn}\phi_{bdn} + a_{in}\phi_{in} \tag{53}$$

then, the diagonal element becomes

$$[\mathsf{C}_c(\boldsymbol{u}_s)]_{k,k} = \frac{1}{2}\sum_{f\in F_f(k)}[\boldsymbol{u}_s]_f A_f - \frac{1}{2}\left([\boldsymbol{u}_s]_f A_f\right)_{bdn}(1+a_{in}) \tag{54}$$

Thus, it would remain null if and only if the coefficient a_{in} is taken equal to -1. Finally, imposing that interpolated face variable at the boundary be equal to ϕ_{bdn} yields

$$\phi_{ghost} = 2\phi_{bdn} - \phi_{in} \tag{55}$$

4.6.2. Diffusive operator

The diffusive fluxes through the wall faces must be discretized such that the resulting matrix D_c is symmetric and positive-definite. However, the coefficient matrix D_c is defined as $\mathsf{M}\Omega_s^{-1}\mathsf{M}^*$ (see Eq. 47) where the matrix Ω_s^{-1} is a positive diagonal matrix. Hence, symmetry and positive-definiteness occurs naturally. Moreover, condition (55) is implicitly imposed when constructing the divergence operator M.

5. Concluding remarks

The essence of turbulence are the smallest scales of motion. They result from a subtle balance between convective transport and diffusive dissipation. Mathematically, these terms are governed by two differential operators differing in symmetry: the convective operator is skew-symmetric, whereas the diffusive is symmetric and positive-definite. On the other hand, accuracy and stability need to be reconciled for numerical simulations of turbulent flows around complex configurations.

With this in mind, a fully-conservative discretization method for general grids (structured/unstructured) has been presented here. It exactly preserves the symmetries of the underlying differential operators on a collocated mesh. Therefore, unlike other formulations, the proposed method does not introduce any artificial dissipation which would affect the balance interplayed at the smallest scales.

In summary, the method is based on only five basic operators: the cell-centered and staggered control volumes, Ω_c and Ω_s, the matrix containing the face normal vectors, N_s, the cell-to-face scalar field interpolation, $\Pi_{c\to s}$ and the divergence operator, M. Once these operators are constructed, the rest follows straightforwardly from them. Therefore, the proposed method constitutes a robust and easy-to-implement approach to solve incompressible turbulent flow in complex configurations.

Moreover, we also consider that the symmetry-preserving discretization method presented here forms an excellent starting point for large-eddy simulation (LES). For coarse grids, the energy of the resolved scales of motion is convected in a stable manner: that is, the discrete convective operator transports energy from a resolved scale of motion to other resolved scales without dissipating any energy, as it should do from a physical point of view. Therefore, we think that it forms a solid basic for testing sub-grid scale models.

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