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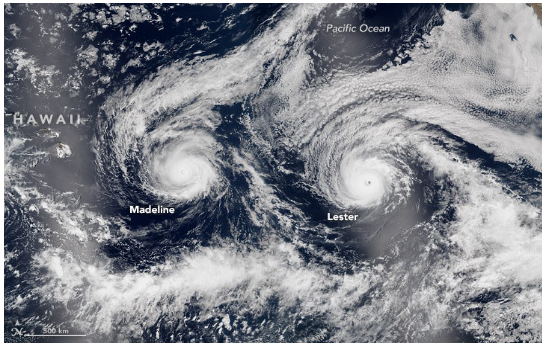
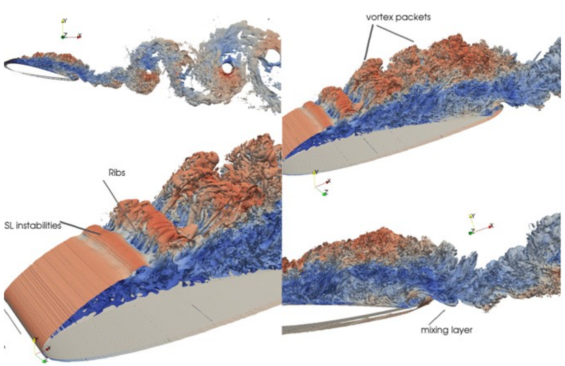
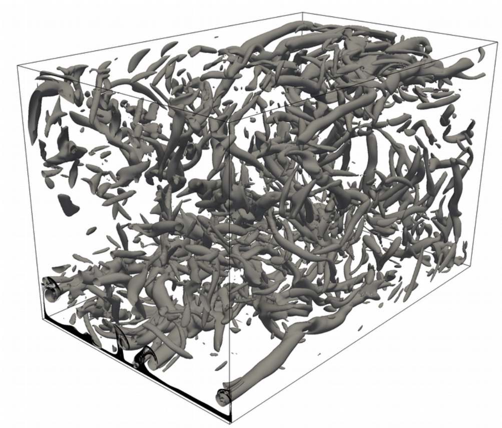


COMPUTATIONAL ENGINEERING

TACKLING TURBULENCE WITH (SUPER)COMPUTERS





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BURGULENCE

A GOOD TOY MODEL OF TURBULENCE

On his death bed, Werner Heisenberg is reported to have said, «When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first.»

REMAINDER 1: TURBULENCE IS EXPENSIVE

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l: biggest eddies (driving scale) $\longrightarrow U$

η: smallest eddies (Kolmogorov length scale) $\longrightarrow u$

(Computational) **Complexity of NS** grows very fast!!!

$$\text{In space:} \quad dim=3 \quad \Rightarrow \quad (l/\eta)^3 \sim Re_l^{9/4}$$

$$\text{In time:} \quad t_{eddy}/t_\eta \sim Re_l^{1/2}$$

$$\text{In space} \times \text{time:} \quad (l/\eta)^3 t_{eddy}/t_\eta \sim Re_l^{11/4}$$

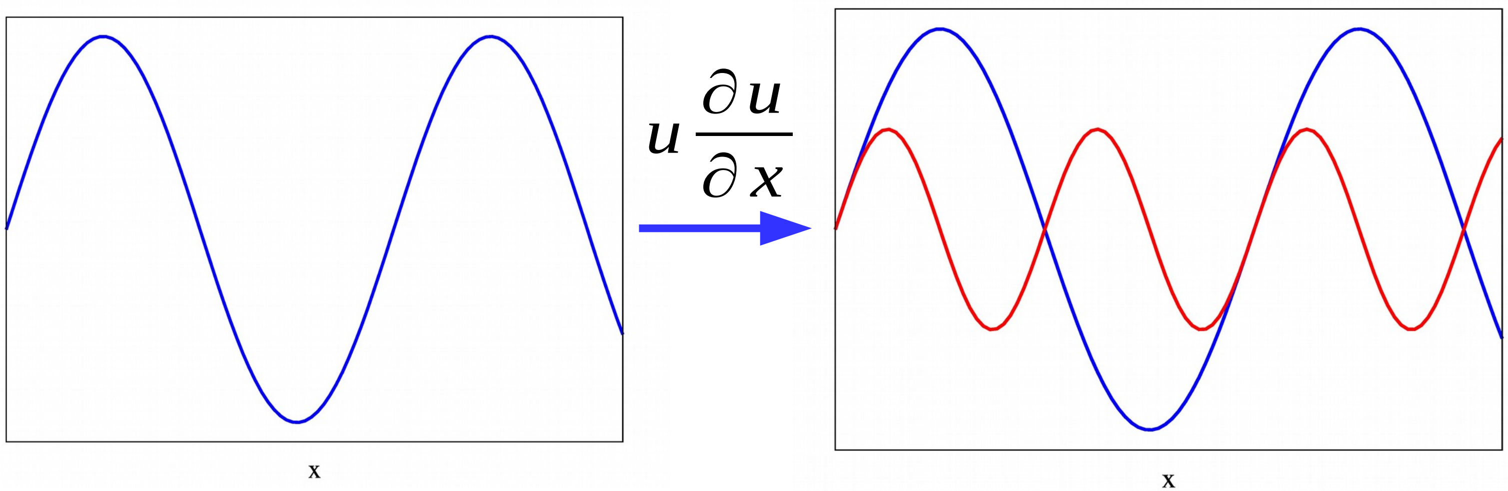
$$\boxed{\begin{aligned} \eta/l &\sim Re_l^{-3/4} \\ u/U &\sim Re_l^{-1/4} \\ t_\eta/t_{eddy} &\sim Re_l^{-1/2} \end{aligned}} \qquad \boxed{\begin{aligned} \eta/l &\sim (\nu^3/\varepsilon)^{1/4} \\ u &\sim (\nu\varepsilon)^{1/4} \\ t_\eta &\sim (\nu/\varepsilon)^{1/2} \end{aligned}}$$

REMAINDER 2: NONLINEARITY OF NS

$$\frac{\partial \vec{u}}{\partial t} + \underbrace{(\vec{u} \cdot \nabla)\vec{u}}_{\text{nonlinear guy in NS}} = \nu \nabla^2 \vec{u} - \nabla p; \qquad \nabla \cdot \vec{u} = 0$$

Let us consider a one single 1D wave, $u(x) = \sin(\boldsymbol{k}\, x)$

$$u \frac{\partial u}{\partial x} = k \sin(k\,x) \cos(k\,x) = \frac{k}{2} \sin(\boldsymbol{2k}\, x)$$



REMAINDER!!! Double-angle formula: $\sin(2x) = 2\sin(x)\cos(x)$

REMAINDER 3: ENERGY CASCADE

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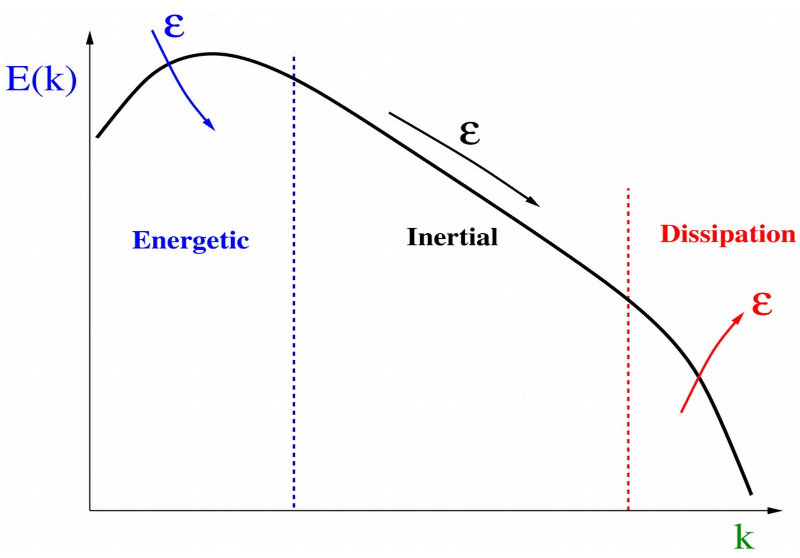
$$[k]=[L^{-1}] \quad [\varepsilon]=[L^2T^{-3}]$$
$$[E_T]=[L^2T^{-2}] \quad [E(k)]=[L^3T^{-2}]$$



Hypothesis (K41): $E(k)\propto \varepsilon^a k^b$

$$\begin{array}{rl} 2a-b= & 3 \\ -3a \quad = & -2 \end{array} \Rightarrow \begin{cases} a= & 2/3 \\ b= & -5/3 \end{cases}$$

Kolmogorov energy spectrum: $\boxed{E(k)=C_K \varepsilon^{2/3} k^{-5/3}}$

Formal proof of the existence of the inertial subrange for the (incompressible) Navier-Stokes equations: C. Foias, O.P. Manley, R. Rosa and R. Temam: *Estimates for the energy cascade in three-dimensional turbulent flows*. C. R. Acad. Sci. Paris, Série I Math. 333, 499–504, (**2001**).

BURGERS' EQUATION

$$\frac{\partial \vec{u}}{\partial t} + \underbrace{(\vec{u} \cdot \nabla)\vec{u}}_{\text{nonlinear guy in NS}} = \nu \nabla^2 \vec{u} - \nabla p; \qquad \nabla \cdot \vec{u} = 0$$

1D No continuity eq.

$$\boxed{\frac{\partial u}{\partial t} + \underbrace{u \frac{\partial u}{\partial x}}_{\text{nonlinear guy in Burgers'}} = \nu \frac{\partial^2 u}{\partial x^2} + f}$$

This is simplified model that shares many of the features of the NS equations.

BURGERS' EQUATION

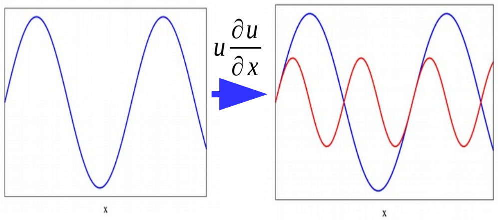
$$\frac{\partial \vec{u}}{\partial t} + \underbrace{(\vec{u}\cdot\nabla)\vec{u}}_{\text{nonlinear guy in NS}} = \nu \nabla^2 \vec{u} - \nabla p; \qquad \nabla\cdot\vec{u} = 0$$

1D $\downarrow$ No continuity eq.

$$\boxed{\frac{\partial u}{\partial t} + \underbrace{u\frac{\partial u}{\partial x}}_{\text{nonlinear guy in Burgers'}} = \nu\frac{\partial^2 u}{\partial x^2} + f}$$

$\downarrow$ Fourier Transform $\quad u(x) = \sum_{k=-N}^{k=+N} \hat{u}_k e^{ikx}$

$$\boxed{\frac{\partial \hat{u}_k}{\partial t} + \underbrace{\sum_{p+q=k} \hat{u}_p\, i q\, \hat{u}_q}_{\text{nonlinear guy in Burgers'}} = -\nu k^2 \hat{u}_k + \hat{f}_k}$$



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We consider equation (6) on an interval Ω with periodic boundary conditions. In Fourier space, the Burgers equation reads[1]

$$\partial_t \hat{u}_k + \sum_{k=p+q} \hat{u}_p i q \hat{u}_q = -\frac{k^2}{Re}\hat{u}_k + F_k \qquad k = 0, \ldots, N \tag{7}$$

where the forcing term is given by $F_k = 0$ for $k > 1$ and F_1 such that $\partial_t \hat{u}_1 = 0$ for $t > 0$. Here $\hat{u}_k(t) \in \mathbb{C}$ denotes the k-th Fourier coefficient of $u(x,t)$

$$u(x) \equiv \sum_{k=-N}^{k=+N} \hat{u}_k e^{ikx} \tag{8}$$

where N in the total number of Fourier modes. Note that since $u(x,t) \in \mathbb{R}$ the following condition must be accomplished

$$\hat{u}_k = \overline{\hat{u}_{-k}} \tag{9}$$

where $\overline{(\cdot)}$ denotes the complex conjugate. For details about the computation of the non-linear convective term the reader is referred to Appendix B.

BURGERS' EQUATION IN FOURIER SPACE

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$$\partial_t u + u\partial_x u = \frac{1}{Re}\partial_{xx} u + f \tag{A.1}$$

each of the terms can be converted into Fourier space. Transient, diffusive and forcing terms are straightforward

$$\partial_t u = \sum_{k=-N}^{k=+N} \left(\partial_t \hat{u}_k\right) e^{ikx} \tag{A.2}$$

$$\partial_{xx} u = \sum_{k=-N}^{k=+N} \hat{u}_k \partial_{xx} e^{ikx} = \sum_{k=-N}^{K=+N} \left(-k^2 \hat{u}_k\right) e^{ikx} \tag{A.3}$$

$$f = \sum_{k=-N}^{k=+N} F_k e^{ikx} \tag{A.4}$$

BURGERS' EQUATION IN FOURIER SPACE

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and non-linear convective term is given by

$$\partial_x u = \sum_{k=-N}^{k=+N} \hat{u}_k \partial_x e^{ikx} = \sum_{k=-N}^{k=+N} ik\hat{u}_k e^{ikx} \tag{A.5}$$

$$u\partial_x u = \sum_{p=-N}^{p=N} \hat{u}_p e^{ipx} \sum_{q=-N}^{q=+N} iq\hat{u}_q e^{iqx} = \sum_{p=-N,q=-N}^{p=+N,q=+N} \hat{u}_p iq\hat{u}_q e^{i(p+q)x} \tag{A.6}$$

and the condition that $u(x,t) \in \mathbb{R}$ is automatically satisfied

$$\hat{u}_{-p} i(-q)\hat{u}_{-q} = -\overline{\hat{u}_p} iq\overline{\hat{u}_p} = \overline{\hat{u}_p iq\hat{u}_q} \tag{A.7}$$

Therefore, the Burgers equation (A.1) in Fourier space results

$$\sum_{k=-N}^{k=+N} (\partial_t \hat{u}_k)\, e^{ikx} + \sum_{p=-N,q=-N}^{p=+N,q=+N} \hat{u}_p iq\hat{u}_q e^{i(p+q)x} = \frac{1}{Re} \sum_{k=-N}^{k=+N} \left(-k^2\hat{u}_k\right) e^{ikx} + \sum_{k=-N}^{k=+N} F_k e^{ikx} \tag{A.8}$$

COMPUTING TRIADIC INTERACTIONS

The contribution of the convective term to the k^{th} mode is given by

$$\sum_{k=p+q} \hat{u}_p i q \hat{u}_q \in \mathbb{C}. \tag{B.1}$$

$$\hat{u}_p = \overline{\hat{u}_{-p}}. \tag{B.2}$$

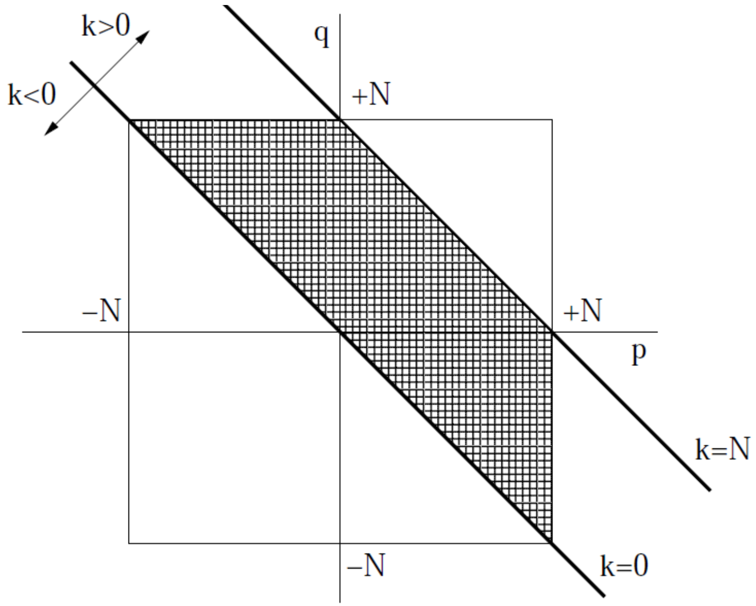


Figure 4: Representation of all possible triadic interactions between modes. Only the interactions between the straight lines $k = 0$ and $k = N$ need to be considered for the computation of the non-linear convective term.

RESULTS

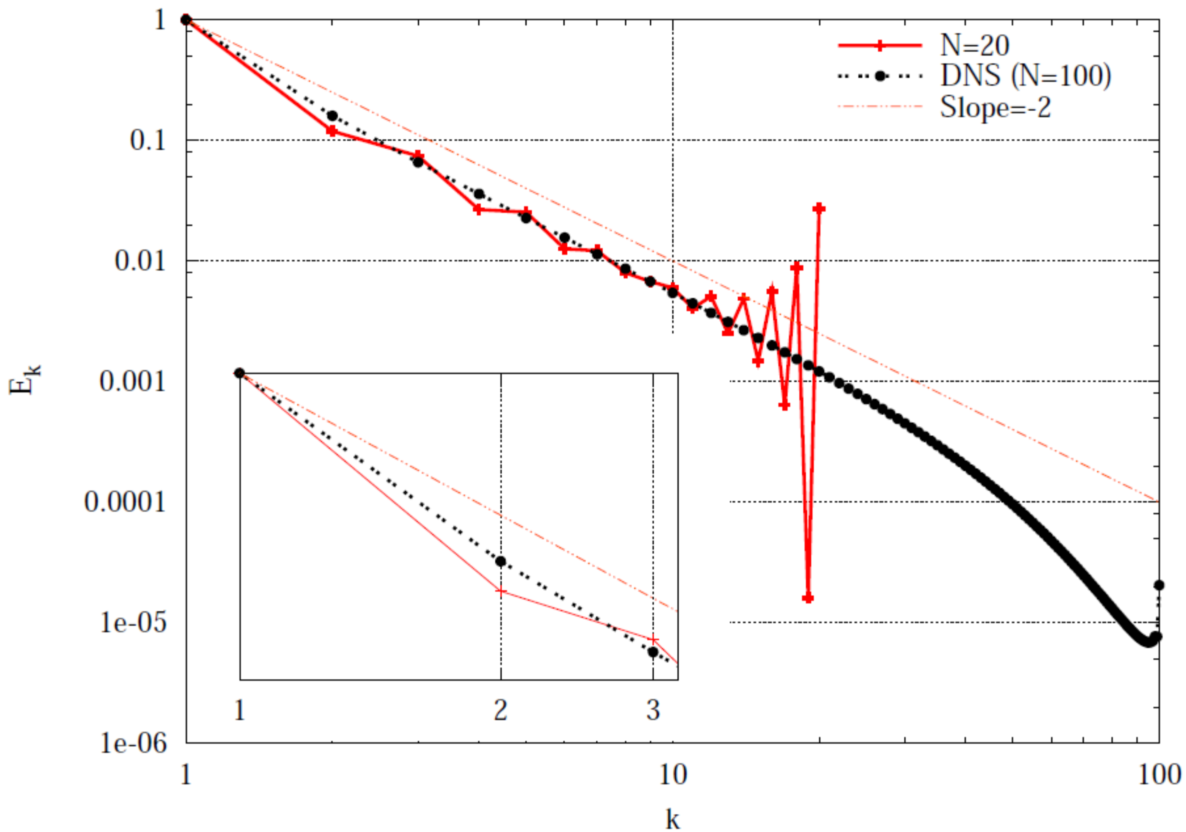
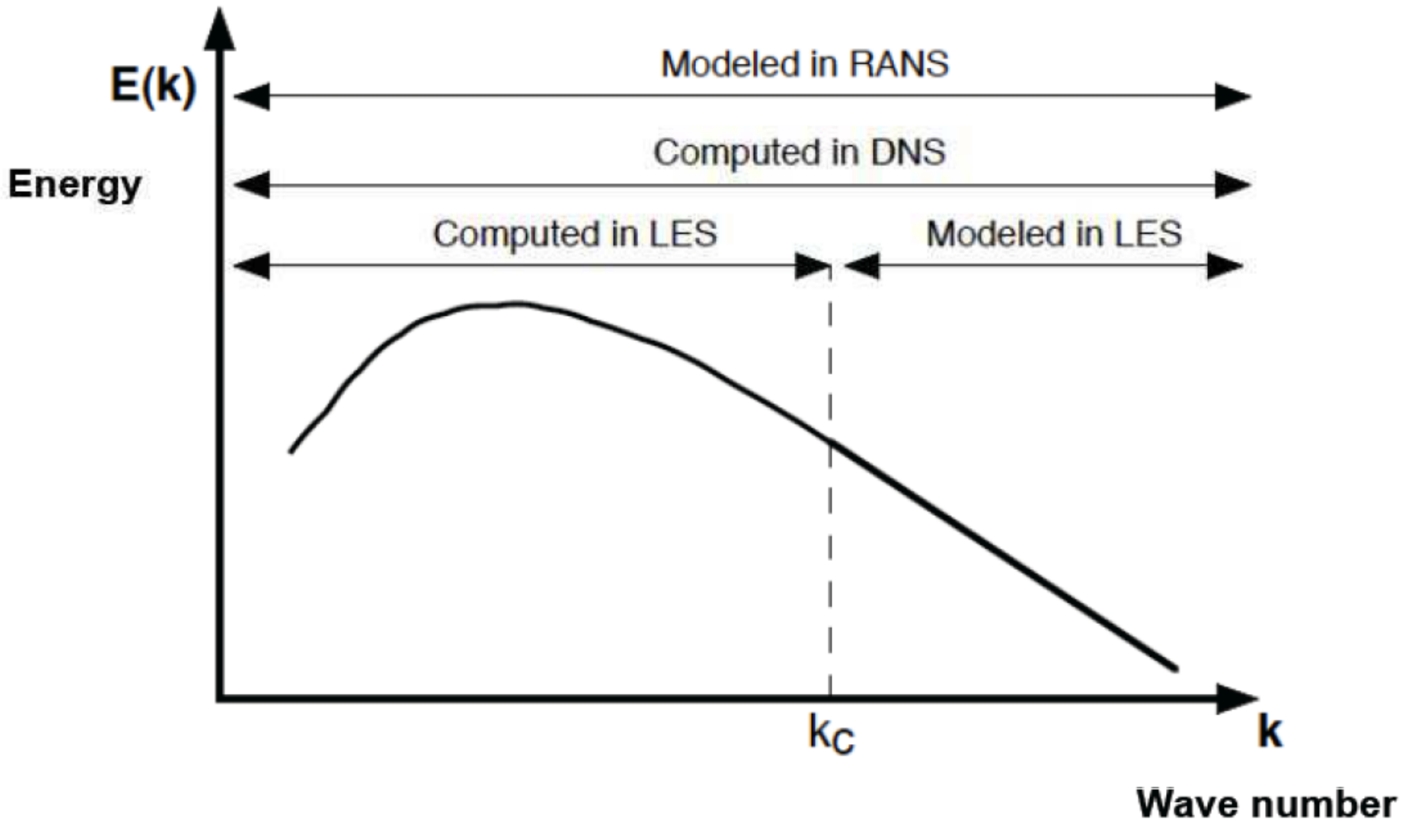


Figure 2: Energy spectrum of the steady-state solution of the Burgers equation for $N = 20$ and $N = 100$ (DNS). The steady state is reached at $t = 3$ approximately.

DNS vs LES vs RANS



TURBULENCE MODELING: A PREVIEW

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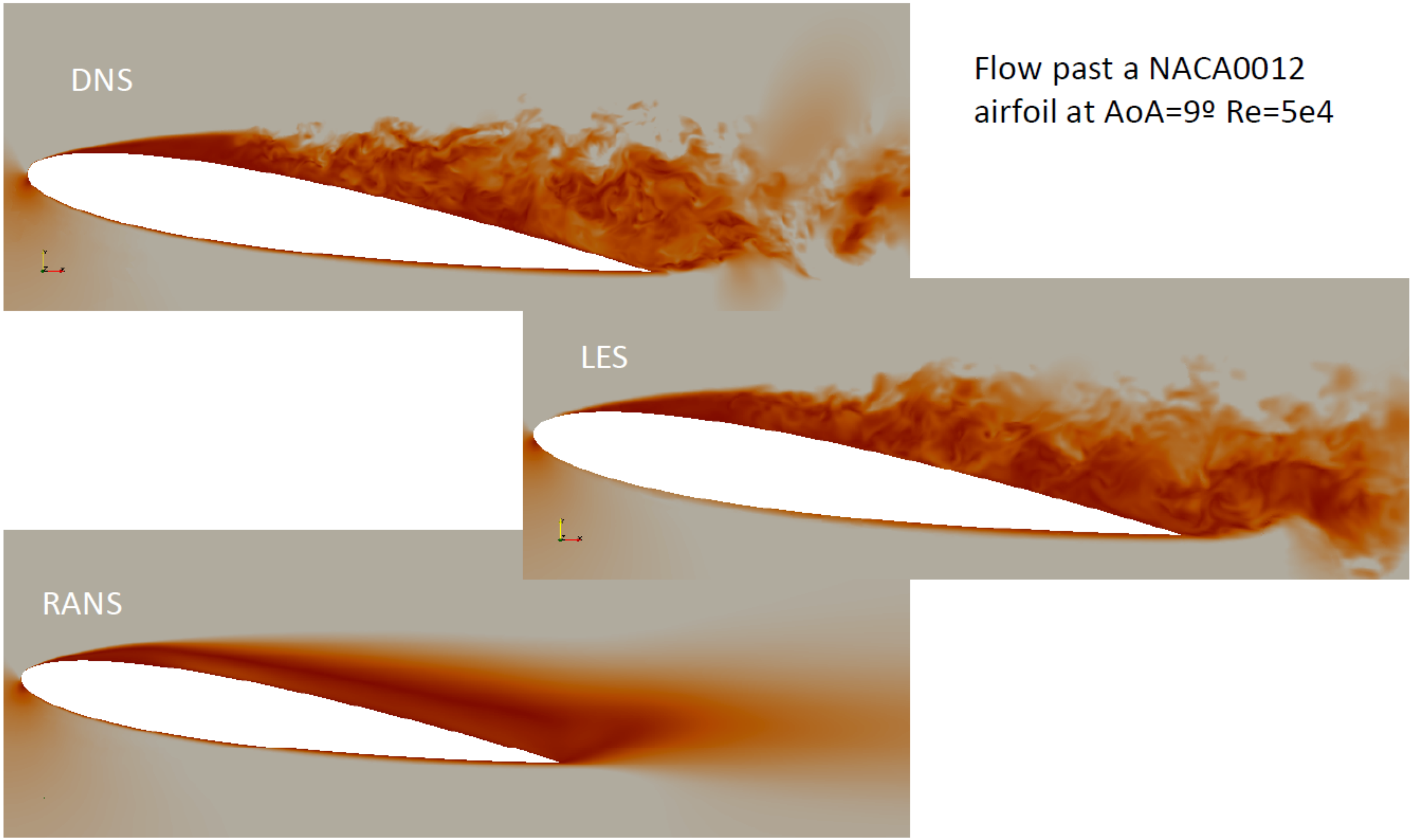
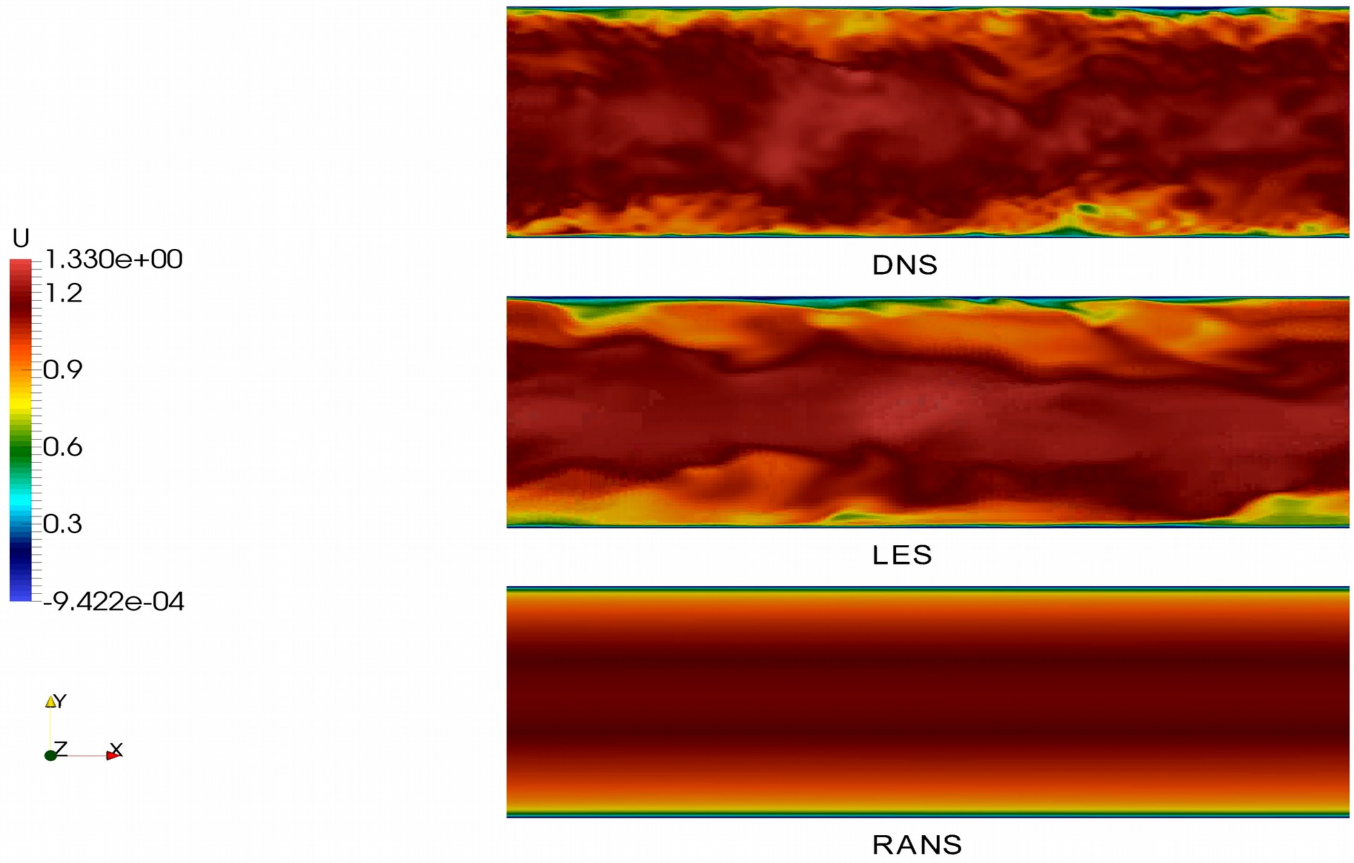


Flow past a NACA0012 airfoil at AoA=9º Re=5e4

TURBULENCE MODELING: A PREVIEW



Turbulent channel flow at $Re_{\tau}=395 \quad (Re_{bulk}=13760)$

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$$\nu_t(k/k_N) = \nu_t^{+\infty}\left(\frac{E_{k_N}}{k_N}\right)^{1/2}\nu_t^*\left(\frac{k}{k_N}\right) \tag{19}$$

with

$$\nu_t^{+\infty} = 0.31\frac{5-m}{m+1}\sqrt{3-m}C_K^{-3/2} \tag{20}$$

where m is the slope of the energy spectrum, that is k^{-m}, E_{k_N} is the energy at the cutoff frequency, k_N, and C_K is the Kolmogorov constant. ν_t^* is a non-dimensional eddy-viscosity equal to 1 for small values of k/k_N and with a strong increase for higher k up to $k/k_N = 1$; it reads

$$\nu_t^*\left(\frac{k}{k_N}\right) = 1 + 34.5e^{-3.03(k_N/k)} \tag{21}$$

Note that the classical $\nu_t(k/k_N)$ is recovered for $m = 5/3$. In our case, the energy spectrum is approximately $m = 2$ (see Figure 2) and the Kolmogorov constant (for 1D Burgers equation) is $C_K \approx 0.4523$. In our case, $\nu = Re^{-1}$, and therefore, in practice the only modification in the code implies to modify ν by $\nu_{eff}(k) = \nu + \nu_t(k)$. Try to reproduce the results reported in Figure 3.

LES RESULTS

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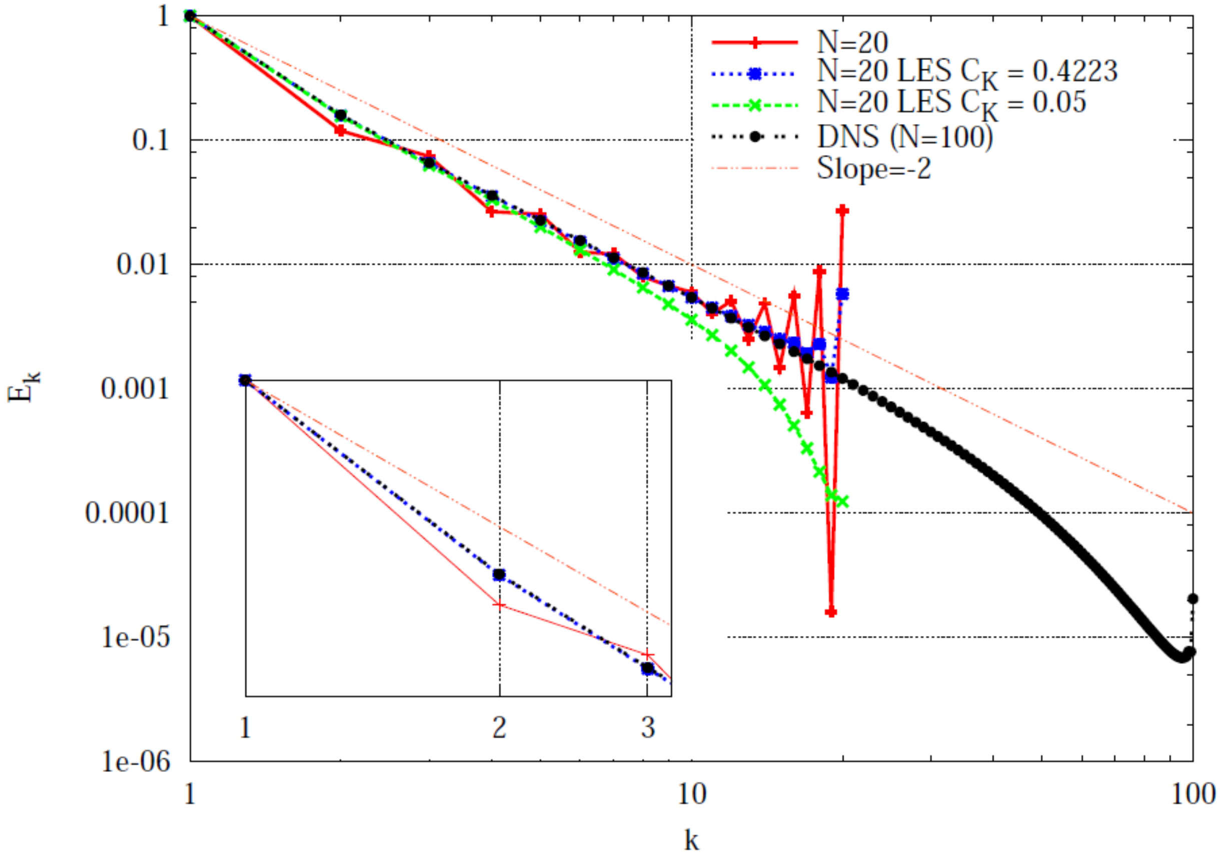

Figure 3: Energy spectrum of the steady-state solution of the Burgers equation for $N = 20$ (with spectral eddy-viscosity model taking $m = 2$ and $C_K = 0.4523$) and $N = 100$ (DNS).

SOME HINTS

- Note that although $E_k \in \mathbb{R}$, $\hat{u}_k \in \mathbb{C}$. Therefore, your code must be able to perform basic operations with complex numbers.
- You can use a fully explicit time-integration scheme. Be careful with Δt!! A CFL-like condition must be imposed[4]

$$\Delta t < C_1 \frac{Re}{N^2} \tag{22}$$

- All the results reported here correspond to the steady state solution. You can also visualize the temporal evolution, play with other Re-numbers, different initial conditions, etc... It would help you to understand better the role of each term.