Navier-Stokes equations: symmetry and conservation

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June 23^{th} 2011

Abstract —

It is usual in mechanics to discuss conservation laws together with symmetries. The first outstanding theoretical approach for this association goes back to 1918, when the mathematician Emmy Noether published her famous paper [1]. Noether's theorem is valid for conservative systems describable by a Lagrangian function and states that for each symmetry there is a corresponding conservation law. For instance, momentum conservation corresponds to the invariance of the Lagrangian under space-translations. Although Navier-Stokes equations are not conservative[1], it is useful in our context to discuss conservation laws in terms of (skew-)symmetries of the continuous differential operators.

1. Navier-Stokes equations

The Navier-Stokes equations describe the motion of a fluid in $\mathbb{R}^n$ ($n = 2$ or 3). We restrict our attention to incompressible fluids. The Navier-Stokes equations are then given by

$$\frac{\partial \boldsymbol{u}}{\partial t} = -C(\boldsymbol{u}, \boldsymbol{u}) + \frac{1}{Re}\Delta \boldsymbol{u} - \nabla p \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

where Re is the dimensionless Reynolds number and the non-linear convective term is given by

$$C(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{v} \tag{3}$$

2. Kinetic energy transport equation

The transport equation for kinetic energy, $e = 1/2\,(\boldsymbol{u} \cdot \boldsymbol{u})$, is obtained from the scalar product of the velocity vector, $\boldsymbol{u}$, and the momentum equation (1)

$$\frac{\partial e}{\partial t} = -\boldsymbol{u} \cdot C(\boldsymbol{u}, \boldsymbol{u}) + \frac{1}{Re}\boldsymbol{u} \cdot \Delta \boldsymbol{u} - \boldsymbol{u} \cdot \nabla p \tag{4}$$

This expression can be simplified with some algebra:

1. Using the identity (A.13) and taking $\boldsymbol{a} = \boldsymbol{u}$ we get

[1] The Euler equations, obtained by setting $\nu = 0$, are conservative and have various Lagrangian formulations.

$$\boldsymbol{u} \cdot \nabla \left(\frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} \right) = \boldsymbol{u} \cdot (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} + \boldsymbol{u} \cdot ((\nabla \times \boldsymbol{u}) \times \boldsymbol{u}) \tag{5}$$

and simplifying the previous expression we finally obtain

$$\boldsymbol{u} \cdot C(\boldsymbol{u}, \boldsymbol{u}) = C(\boldsymbol{u}, e) \tag{6}$$

2. Using the identity (A.6) and taking $\boldsymbol{a} = \boldsymbol{u}$ and $\mathsf{T} = \nabla \boldsymbol{u}$ we get

$$\boldsymbol{u} \cdot \Delta \boldsymbol{u} = \nabla \cdot ((\nabla \boldsymbol{u}) \cdot \boldsymbol{u}) - (\nabla \boldsymbol{u} : \nabla \boldsymbol{u}) \tag{7}$$

and using the formulas (A.14) and (A.13) we obtain

$$\boldsymbol{u} \cdot \Delta \boldsymbol{u} = \Delta e - (\nabla \boldsymbol{u} : \nabla \boldsymbol{u}) \tag{8}$$

3. Using the identity (A.5), $\nabla \cdot \boldsymbol{u} = 0$ and taking $\varphi = p$ and $\boldsymbol{a} = \boldsymbol{u}$ we get

$$\boldsymbol{u} \cdot \nabla p = \nabla \cdot (p \boldsymbol{u}) - p (\nabla \cdot \boldsymbol{u}) = \nabla \cdot (p \boldsymbol{u}) \tag{9}$$

Reducing the kinetic energy transport equation to

$$\frac{\partial e}{\partial t} = -C(\boldsymbol{u}, e) + \frac{1}{Re} \Delta e - \frac{1}{Re} (\nabla \boldsymbol{u} : \nabla \boldsymbol{u}) - \nabla \cdot (p \boldsymbol{u}) \tag{10}$$

where the pressure term represents a flux of kinetic energy along a streamline, while the rest of terms represent the convection, diffusion and destruction of kinetic energy respectively.

3. Vorticity transport equation

We can obtain the vorticity transport equation by taking the curl of the momentum equation (1)

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \times C(\boldsymbol{u}, \boldsymbol{u}) = \frac{1}{Re} \nabla \times \Delta \boldsymbol{u} - \nabla \times \nabla p \tag{11}$$

where $\mathbf{w} = \nabla \times \boldsymbol{u}$ is the vorticity. The latter expression can be simplified with some algebra:

1. Using the identity (A.9) and taking $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{u}$ we get

$$\frac{1}{2} \nabla (\boldsymbol{u} \cdot \boldsymbol{u}) = (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} + \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) \tag{12}$$

applying the curl and using the formula (A.5) we obtain

$$\nabla \times C(\boldsymbol{u}, \boldsymbol{u}) = -\nabla \times (\boldsymbol{u} \times \mathbf{w}) = \nabla \times (\mathbf{w} \times \boldsymbol{u}) \tag{13}$$

2. Using the identity (A.12), $\nabla \cdot \boldsymbol{u} = 0$ and taking $\boldsymbol{a} = \mathbf{w}$ we get

$$\nabla \times (\Delta \boldsymbol{u}) = \Delta \mathbf{w} \tag{14}$$

3. Using the identity (A.5) the pressure contribution vanishes

$$\nabla \times \nabla p = \mathbf{0} \tag{15}$$

Reducing the vorticity transport equation to

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \times (\mathbf{w} \times \boldsymbol{u}) = \frac{1}{Re} \Delta \mathbf{w} \tag{16}$$

Using the vector identities (A.2) and (A.11) with $\boldsymbol{a} = \mathbf{w}$ and $\boldsymbol{b} = \boldsymbol{u}$ and considering an incompressible fluid ($\nabla \cdot \boldsymbol{u} = 0$), we obtain the vorticity transport equation,

$$\frac{\partial \mathbf{w}}{\partial t} + C(\boldsymbol{u}, \mathbf{w}) = \frac{1}{Re} \Delta \mathbf{w} + C(\mathbf{w}, \boldsymbol{u}) \tag{17}$$

where the first term represents the temporal evolution of the vorticity, the second is the convective term, the third represents the viscous effects and the fourth is the vortex-stretching term. Note that the vortex-stretching term $C(\mathbf{w}, \boldsymbol{u})$ vanishes in 2D.

3.1. Vortex-stretching mechanism

Using the identity (A.14) we can prove that vortex-stretching vector is related to

$$C(\mathbf{w}, \boldsymbol{u}) = (\mathbf{w} \cdot \nabla) \boldsymbol{u} = \frac{1}{2} \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^* \right) \mathbf{w} \tag{18}$$

the strain tensor $\mathsf{S} = \frac{1}{2} \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^* \right)$. This tensor is symmetric and

$$trace(\mathsf{S}) = \nabla \cdot \boldsymbol{u} = 0 \tag{19}$$

Note that the trace of a symmetric matrix is the sum of its eigenvalues. Since the trace is null (19) it implies that, at least, one eigenvalue of S is positive. Hence, if vorticity, $\mathbf{w}$, aligns with an eigenvector of S corresponding to a positive eigenvalue then the term $C(\mathbf{w}, \boldsymbol{u})$ becomes positive and amplifies (by stretching) the vorticity.

Vortex-stretching is a very important mechanism in fluid dynamics: it usually corresponds to the presence of vortical structures which are much more intense that those produced by simple shear or rotating flows.

4. Global kinetic energy equation

The temporal evolution for the global kinetic energy on Ω can be obtained by differentiating $(\boldsymbol{u}, \boldsymbol{u})$ with respect to time and using Navier-Stokes equations in vector form (1)

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{u}, \boldsymbol{u}) &= \left(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{u} \right) + \left(\boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial t} \right) \\ &= -\left(C(\boldsymbol{u}, \boldsymbol{u}), \boldsymbol{u} \right) - \left(\boldsymbol{u}, C(\boldsymbol{u}, \boldsymbol{u}) \right) \\ &\quad + \frac{1}{Re} \left((\nabla \cdot \nabla \boldsymbol{u}, \boldsymbol{u}) + (\boldsymbol{u}, \nabla \cdot \nabla \boldsymbol{u}) \right) \\ &\quad - (\nabla p, \boldsymbol{u}) - (\boldsymbol{u}, \nabla p) \end{aligned} \tag{20}$$

where $(.,.)$ operator represents the usual scalar product

$$(\boldsymbol{a}, \boldsymbol{b}) = \int_{\Omega} \boldsymbol{a} \cdot \boldsymbol{b} \, dV \tag{21}$$

The skew-symmetry of the $C(.,.)$ operator (see Eq. (43)) demonstrates that the convective contribution cancels from the global kinetic energy equation. Using equality (48), we can also prove that the pressure does not contribute either (for an incompressible flow). Hence, with the help of the equality (48) the energy equation reduces to[2]

$$\frac{d}{dt}(\boldsymbol{u},\boldsymbol{u}) = -\frac{2}{Re}(\nabla\boldsymbol{u},\nabla\boldsymbol{u}) \tag{22}$$

Using the equalities (A.12) and (53) and $\nabla\cdot\boldsymbol{u}=0$, the rate of dissipation of energy can be expressed in terms of the enstrophy

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{u},\boldsymbol{u}) &= -\frac{2}{Re}(\nabla\boldsymbol{u},\nabla\boldsymbol{u})\\ &= -\frac{2}{Re}(\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{u}) = -\frac{2}{Re}(\mathbf{w},\mathbf{w})\end{aligned} \tag{23}$$

5. Enstrophy equation

Using the identities (A.12) and (53), the evolution of the enstrophy can be obtained by taking the inner product of Navier-Stokes equations with the vector field $-\Delta\boldsymbol{u}$

$$\begin{aligned}\left(\frac{\partial\boldsymbol{u}}{\partial t},-\Delta\boldsymbol{u}\right) &= -\left(C(\boldsymbol{u},\boldsymbol{u}),-\Delta\boldsymbol{u}\right)+\frac{1}{Re}(\Delta\boldsymbol{u},-\Delta\boldsymbol{u})-(\nabla p,-\Delta\boldsymbol{u})\\ \frac{d}{dt}(\mathbf{w},\mathbf{w}) &= -\frac{1}{Re}(\Delta\boldsymbol{u},\Delta\boldsymbol{u})-\left(C(\boldsymbol{u},\boldsymbol{u}),\Delta\boldsymbol{u}\right)\end{aligned} \tag{24}$$

and pressure gradient contribution is proven to be zero using identities (48), (A.12) and (A.2).

$$\begin{aligned}(\nabla p,-\Delta\boldsymbol{u}) &= -(p,\nabla\cdot(\Delta\boldsymbol{u}))\\ &= (p,\nabla\cdot(\nabla\times(\nabla\times\boldsymbol{u})))-(p,\nabla\cdot(\nabla(\nabla\cdot\boldsymbol{u})))\\ &= 0\end{aligned} \tag{25}$$

if $\nabla\cdot\boldsymbol{u}=0$.

5.1. Enstrophy equation in 2D

Let us consider the convective contribution $(C(\boldsymbol{u},\boldsymbol{u}),\Delta\boldsymbol{u})$ for 2D flows

$$\begin{aligned}(C(\boldsymbol{u},\boldsymbol{u}),\Delta\boldsymbol{u}) &= -\left(C(\boldsymbol{u},\boldsymbol{u}),\nabla\times(\nabla\times\boldsymbol{u})\right)\\ &= \left(\nabla\times(C(\boldsymbol{u},\boldsymbol{u})),\nabla\times\boldsymbol{u}\right)\end{aligned} \tag{26}$$

if $\nabla\cdot\boldsymbol{u}=0$. As we have previously seen in section 3. the curl of the convective term can be easily rewritten in two terms

$$\nabla\times(C(\boldsymbol{u},\boldsymbol{u})) = C(\boldsymbol{u},\mathbf{w})-C(\mathbf{w},\boldsymbol{u}) \tag{27}$$

where the first term at the right-hand side of (27) is the convective term of the vorticity transport equation (17). The second one is the so-called vortex-stretching term, the responsible to drive the energy to small scales of motion. In 2D, the vortex-stretching term vanishes

[2] This result could also be obtained integrating the kinetic energy transport equation (10).

$$C(\mathbf{w}, \boldsymbol{u}) = \mathbf{0} \tag{28}$$

and the convective term of the vorticity transport equation is orthogonal to the vorticity itself

$$C(\boldsymbol{u}, \mathbf{w}) \cdot \mathbf{w} = \mathbf{0} \tag{29}$$

Hence, the convective contribution vanishes for 2D flows,

$$\left(\frac{\partial \boldsymbol{u}}{\partial t}, -\Delta \boldsymbol{u}\right) = -\frac{1}{Re}(\Delta \boldsymbol{u}, \Delta \boldsymbol{u}) \ , \quad \text{(in 2D)} \tag{30}$$

and the enstrophy is conserved in absence of viscous dissipation.

6. Helicity equation

Since the vorticity has a nonzero component only in the direction orthogonal to the velocity field, the helicity $(\mathbf{w}, \boldsymbol{u})$ is identically zero in 2D. In 3D, the evolution of helicity results from the inner products of the vorticity transport equation (17) with the velocity $\boldsymbol{u}$

$$\left(\frac{\partial \mathbf{w}}{\partial t}, \boldsymbol{u}\right) = -\left(C(\boldsymbol{u}, \mathbf{w}), \boldsymbol{u}\right) + \frac{1}{Re}(\Delta \mathbf{w}, \boldsymbol{u}) + \left(C(\mathbf{w}, \boldsymbol{u}), \boldsymbol{u}\right) \tag{31}$$

and the Navier-Stokes equation (1) with the vorticity

$$\left(\frac{\partial \boldsymbol{u}}{\partial t}, \mathbf{w}\right) = -\left(C(\boldsymbol{u}, \boldsymbol{u}), \mathbf{w}\right) + \frac{1}{Re}(\Delta \boldsymbol{u}, \mathbf{w}) - (\nabla p, \mathbf{w}) \tag{32}$$

Adding these inner products (31) and (32) leads to the temporal evolution of helicity

$$\begin{aligned} \frac{d}{dt}(\mathbf{w}, \boldsymbol{u}) &= \left(C(\mathbf{w}, \boldsymbol{u}), \boldsymbol{u}\right) - \left(C(\boldsymbol{u}, \mathbf{w}), \boldsymbol{u}\right) - \left(C(\boldsymbol{u}, \boldsymbol{u}), \mathbf{w}\right) + \\ &\quad \frac{1}{Re}\left((\Delta \mathbf{w}, \boldsymbol{u}) + (\Delta \boldsymbol{u}, \mathbf{w})\right) \end{aligned} \tag{33}$$

the convective contribution vanishes as a consequence of the skew-symmetry (43) of the convective operator. Hence, using the identities (57) and (A.12) the evolution of helicity results into

$$\begin{aligned} \frac{d}{dt}(\mathbf{w}, \boldsymbol{u}) &= \frac{1}{Re}\left((\Delta \mathbf{w}, \boldsymbol{u}) + (\Delta \boldsymbol{u}, \mathbf{w})\right) \\ &= -\frac{2}{Re}(\mathbf{w}, \nabla \times \mathbf{w}) \end{aligned} \tag{34}$$

7. Summary

We introduce now some important notation

$$\mathsf{E} = (\boldsymbol{u}, \boldsymbol{u}) \tag{35}$$

$$\Omega = (\mathbf{w}, \mathbf{w}) \tag{36}$$

$$\mathsf{H} = (\boldsymbol{u}, \mathbf{w}) \tag{37}$$

$$\mathsf{H}_{\mathbf{w}} = (\mathbf{w}, \nabla \times \mathbf{w}) \tag{38}$$

$$\mathsf{P} = (\nabla \times \mathbf{w}, \nabla \times \mathbf{w}) \tag{39}$$

where E, Ω and H are the *total kinetic energy*, the *enstrophy* and the *helicity*. The quantities $\mathsf{H}_\mathbf{w}$ and P are called the *vortical helicity* and the *palinstrophy*, respectively (see [2], for instance). Using this notation, the energy (23) and helicity (34) balance equations can be written as

$$\frac{d}{dt}\mathsf{E} = -\frac{2}{Re}\Omega \tag{40}$$

$$\frac{d}{dt}\mathsf{H} = -\frac{2}{Re}\mathsf{H}_\mathbf{w} \tag{41}$$

In two dimensions, there is an additional balance equation (30) for the enstrophy given by

$$\frac{d}{dt}\Omega = -\frac{2}{Re}\mathsf{P}\ , \qquad \text{(in 2D)} \tag{42}$$

8. Theorems

Theorem 1. *The trilinear form* $(C(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w})$ *is skew-symmetric with respect the last two arguments if boundary terms are ignored and* $\nabla\cdot\boldsymbol{u}=0$.

$$(C(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}) = -(C(\boldsymbol{u},\boldsymbol{w}),\boldsymbol{v}) \tag{43}$$

Proof. Using the following identity

$$\nabla\cdot(f\boldsymbol{u}) = f\nabla\cdot\boldsymbol{u} + \nabla f\cdot\boldsymbol{u} \tag{44}$$

where f and $\boldsymbol{u}$ are any differentiable scalar and vector fields respectively. Taking $\nabla\cdot\boldsymbol{u}=0$ and $f=\boldsymbol{v}\cdot\boldsymbol{w}$ and integrating over the domain

$$\begin{aligned}\int_\Omega \nabla\cdot((\boldsymbol{v}\cdot\boldsymbol{w})\boldsymbol{u})\,d\Omega &= \int_\Omega \nabla(\boldsymbol{v}\cdot\boldsymbol{w})\cdot\boldsymbol{u}d\Omega \\ &= (C(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}) + (C(\boldsymbol{u},\boldsymbol{w}),\boldsymbol{v})\end{aligned} \tag{45}$$

Moreover, applying the Gauss Divergence Theorem,

$$\int_\Omega \nabla\cdot((\boldsymbol{v}\cdot\boldsymbol{w})\boldsymbol{u})\,d\Omega = \int_{\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{w})(\boldsymbol{u}\cdot\boldsymbol{n})\,dS \tag{46}$$

leads to the following identity

$$(C(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}) + (C(\boldsymbol{u},\boldsymbol{w}),\boldsymbol{v}) = \int_{\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{w})(\boldsymbol{u}\cdot\boldsymbol{n})\,dS \tag{47}$$

which shows that the triliniar form $(C(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w})$ is skew-symmetric with respect to $\boldsymbol{v}$ and $\boldsymbol{w}$ if the boundary terms vanishes.

□

Theorem 2. *The linear form* $(\nabla p,\boldsymbol{u})$ *satisfies the identity*

$$(\nabla p,\boldsymbol{u}) = -(p,\nabla\cdot\boldsymbol{u}) \tag{48}$$

if the boundary terms are ignored.

Proof. Using the following identity

$$\nabla \cdot (f\boldsymbol{u}) = f\nabla \cdot \boldsymbol{u} + \nabla f \cdot \boldsymbol{u} \tag{49}$$

where f and $\boldsymbol{u}$ are any differentiable scalar and vector fields respectively. Taking $f = p$ and integrating over the domain

$$\int_\Omega \nabla \cdot (p\boldsymbol{u})\, d\Omega = (p, \nabla \cdot \boldsymbol{u}) + (\nabla p, \boldsymbol{u}) \tag{50}$$

Applying the Gauss Divergence Theorem,

$$\int_\Omega \nabla \cdot (p\boldsymbol{u})\, d\Omega = \int_{\partial\Omega} p\, (\boldsymbol{u} \cdot \boldsymbol{n})\, dS \tag{51}$$

leads to the following expression

$$(p, \nabla \cdot \boldsymbol{u}) + (\nabla p, \boldsymbol{u}) = \int_{\partial\Omega} p\, (\boldsymbol{u} \cdot \boldsymbol{n})\, dS \tag{52}$$

which proves the identity (48) if the boundary terms are vanished. □

Theorem 3. *The liniar form* $(\boldsymbol{u}, \nabla \times \boldsymbol{v})$ *satisfies the identity*

$$(\boldsymbol{u}, \nabla \times \boldsymbol{v}) = (\nabla \times \boldsymbol{u}, \boldsymbol{v}) \tag{53}$$

if the boundary terms are ignored.

Proof. Using the identity (A.10) and integrating over the domain

$$\int_\Omega \nabla \cdot (\boldsymbol{u} \times \boldsymbol{v})\, d\Omega = (\boldsymbol{u}, \nabla \times \boldsymbol{v}) - (\nabla \times \boldsymbol{u}, \boldsymbol{v}) \tag{54}$$

and applying the Gauss Divergence Theorem,

$$\int_\Omega \nabla \cdot (\boldsymbol{u} \times \boldsymbol{v})\, d\Omega = \int_{\partial\Omega} (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{n} dS \tag{55}$$

leads to the following identity

$$(\boldsymbol{u}, \nabla \times \boldsymbol{v}) - (\nabla \times \boldsymbol{u}, \boldsymbol{v}) = \int_{\partial\Omega} (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{n} dS \tag{56}$$

which proves the identity (53) if the boundary terms vanished.

□

Theorem 4. *The linear form* $(\nabla \cdot \mathsf{T}, \boldsymbol{u})$ *satisfies the identity*

$$(\nabla \cdot \mathsf{T}, \boldsymbol{u}) = -(\mathsf{T}, \nabla \boldsymbol{u}) \tag{57}$$

if the boundary terms are ignored.

Proof. Using the following identity

$$\nabla \cdot (\boldsymbol{u} \cdot \mathsf{T}) = \boldsymbol{u} \cdot (\nabla \cdot \mathsf{T}) + (\nabla \boldsymbol{u}) : \mathsf{T} \tag{58}$$

where $\boldsymbol{u}$ and T are any differentiable vector and tensor fields respectively. And integrating over the domain Ω leads

$$\int_{\Omega} \nabla \cdot (\boldsymbol{u} \cdot \mathsf{T}) \, d\Omega = (\boldsymbol{u}, \nabla \cdot \mathsf{T}) + (\nabla \boldsymbol{u}, \mathsf{T}) \tag{59}$$

Applying the Gauss Divergence Theorem,

$$\int_{\Omega} \nabla \cdot (\boldsymbol{u} \cdot \mathsf{T}) \, d\Omega = \int_{\partial\Omega} (\boldsymbol{u} \cdot \mathsf{T}) \cdot \boldsymbol{n} dS \tag{60}$$

leads to the following expression

$$(\boldsymbol{u}, \nabla \cdot \mathsf{T}) + (\nabla \boldsymbol{u}, \mathsf{T}) = \int_{\partial\Omega} (\boldsymbol{u} \cdot \mathsf{T}) \cdot \boldsymbol{n} dS \tag{61}$$

which proves the identity (57) if the boundary terms are vanished.

$\square$

A Useful Formulas of Vector Analysis

$$
\begin{aligned}
\nabla \cdot \nabla \boldsymbol{a} &= \Delta \boldsymbol{a} = \nabla^2 \boldsymbol{a} && \text{(A.1)}\\
\nabla \cdot (\nabla \times \boldsymbol{a}) &= 0 && \text{(A.2)}\\
\nabla \times (\nabla \varphi) &= 0 && \text{(A.3)}\\
\nabla \cdot ((\nabla \boldsymbol{a})^*) &= \nabla (\nabla \cdot \boldsymbol{a}) && \text{(A.4)}\\
\nabla \cdot (\varphi \boldsymbol{a}) &= (\nabla \varphi) \cdot \boldsymbol{a} + \varphi (\nabla \cdot \boldsymbol{a}) && \text{(A.5)}\\
\nabla \cdot (\mathsf{T} \cdot \boldsymbol{a}) &= (\nabla \boldsymbol{a}) : \mathsf{T} + \boldsymbol{a} \cdot (\nabla \cdot \mathsf{T}) && \text{(A.6)}\\
\nabla \cdot (\boldsymbol{a} \cdot \mathsf{T}) &= (\nabla \boldsymbol{a}) : \mathsf{T} + \boldsymbol{a} \cdot (\nabla \cdot \mathsf{T}) \qquad \text{if } \mathsf{T} = \mathsf{T}^* && \text{(A.7)}\\
\nabla \times (\varphi \boldsymbol{a}) &= (\nabla \varphi) \times \boldsymbol{a} + \varphi (\nabla \times \boldsymbol{a}) && \text{(A.8)}\\
\nabla (\boldsymbol{a} \cdot \boldsymbol{b}) &= (\boldsymbol{a} \cdot \nabla) \boldsymbol{b} + (\boldsymbol{b} \cdot \nabla) \boldsymbol{a} + \boldsymbol{a} \times (\nabla \times \boldsymbol{b}) + \boldsymbol{b} \times (\nabla \times \boldsymbol{a}) && \text{(A.9)}\\
\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) &= \boldsymbol{b} \cdot (\nabla \times \boldsymbol{a}) - \boldsymbol{a} \cdot (\nabla \times \boldsymbol{b}) && \text{(A.10)}\\
\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) &= \boldsymbol{a} (\nabla \cdot \boldsymbol{b}) - \boldsymbol{b} (\nabla \cdot \boldsymbol{a}) + (\boldsymbol{b} \cdot \nabla) \boldsymbol{a} - (\boldsymbol{a} \cdot \nabla) \boldsymbol{b} && \text{(A.11)}\\
\nabla \times (\nabla \times \boldsymbol{a}) &= \nabla (\nabla \cdot \boldsymbol{a}) - \Delta \boldsymbol{a} && \text{(A.12)}\\
(\boldsymbol{a} \cdot \nabla) \boldsymbol{a} &= \frac{1}{2} \nabla |\boldsymbol{a}|^2 + (\nabla \times \boldsymbol{a}) \times \boldsymbol{a} && \text{(A.13)}\\
\boldsymbol{a} \times (\nabla \times \boldsymbol{b}) &= (\nabla \boldsymbol{b}) \cdot \boldsymbol{a} - \boldsymbol{a} \cdot (\nabla \boldsymbol{b}) && \text{(A.14)}\\
\nabla \times (\nabla \cdot (\nabla \boldsymbol{a})) &= \nabla \cdot (\nabla (\nabla \times \boldsymbol{a})) && \text{(A.15)}
\end{aligned}
$$

where φ is an scalar, $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors and T is a tensor. The following four vector equations are also often useful

$$
\begin{aligned}
\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) &= \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) && \text{(A.16)}\\
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (\boldsymbol{a} \cdot \boldsymbol{c}) \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b}) \boldsymbol{c} && \text{(A.17)}\\
(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) &= \boldsymbol{a} \cdot [\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{d})] = (\boldsymbol{a} \cdot \boldsymbol{c}) (\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d}) (\boldsymbol{b} \cdot \boldsymbol{c}) && \text{(A.18)}\\
(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{a}) &= (\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{d}) \boldsymbol{c} - (\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}) \boldsymbol{d} && \text{(A.19)}
\end{aligned}
$$

B Some famous quotes about turbulence

For centuries, the complex nature of fluid flows has captivated our attention as children, as adults and as scientists. The artist and scientist Leonardo da Vinci was the first to attempt a scientific study of turbulence (*turbolenza*) placing obstructions in water and observing the results (see Fig. 1). His famous poetic description of turbulence is dated 1513...

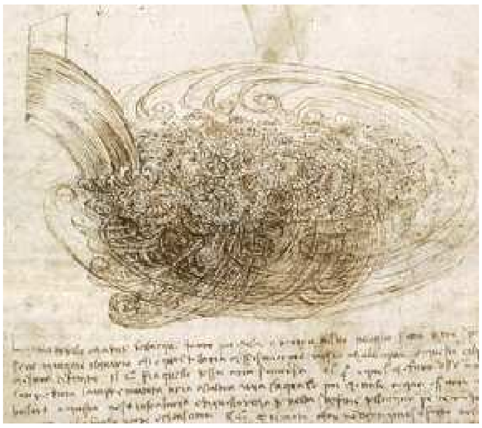

Figure 1: Leonardo da Vinci pictures of turbulent flows

"Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to random and reverse motion."

by Leonardo da Vinci.

Since then, turbulence has captured the attraction and fascination of many other notable scientifics...

"Turbulence is the most important unsolved problem of classical physics."

by Richard Feynman.

"The necessary connection between the diffusion and the supply of energy to the turbulent motion is a fundamental characteristic of turbulent flow."

by Townsend.

"Turbulence is a three dimensional time dependent motion in which vortex stretching causes velocity fluctuations to spread to all wavelengths between

a minimum determined by viscous forces and a maximum determined by the boundary conditions. It is the usual state of fluid motion except at low Reynolds numbers.”

by Bradshaw.

On his death bed, Werner Heisenberg is reported to have said, ”When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first.”

Let me finish with probably the most famous verses about turbulence pictured as a cascade of vortices

“Big whorls have little whorls,
which feed on their velocity,
And little whorls have lesser whorls,
and so on to viscosity.”

by Lewis Fry Richardson.

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