Burgers equation in Fourier space

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Abstract —
This document is concerned about the Burgers equation. Despite its simplicity important aspects of the 3D Navier-Stokes equations remain. Its spectral representation in Fourier space is presented and commented. Finally, several exercises are proposed

1. Introduction

The Navier-Stokes (NS) equations provide an appropriate model for the nonlinear dynamics of turbulence. For an incompressible flow, the dimensionless NS equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = \frac{1}{Re} \Delta \boldsymbol{u} - \nabla p \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2}$$

where Re is the dimensionless Reynolds number defined as

$$Re = \frac{\rho V_0 L}{\mu} \tag{3}$$

where ρ and μ are the density and the dynamic viscosity of the working fluid. L and V_0 are the characteristic length and velocity, respectively.

However, their direct numerical simulation (DNS) is difficult and very expensive in terms of computational cost because the convective term produces far too many dynamically relevant scales of motion. According to Kolmogorov K41 (see [1], for instance), the smallest time/space scales to be solved scale with Re-number

$$\delta t \sim Re^{-1/2} \tag{4}$$

$$\delta x \sim Re^{-3/4} \tag{5}$$

Hence, assuming a perfect algorithm scaling, the DNS memory requirements grow $\sim Re^{9/4}$ and the computational cost $\sim Re^{11/4}$!!!

Figure 1: Triadic interactions. Left: from large scales to small scales. Right: from small scales to large scales.

Alternatively, here we propose to study the Burgers equation

$$\partial_t u + u\partial_x u = \frac{1}{Re}\partial_{xx} u + f \tag{6}$$

a simplified model that shares many of the aspects of the NS equations.

2. Burgers equation in Fourier space

We consider equation (6) on an interval Ω with periodic boundary conditions. In Fourier space, the Burgers equation reads[1]

$$\partial_t \hat{u}_k + \sum_{k=p+q} \hat{u}_p iq \hat{u}_q = -\frac{k^2}{Re}\hat{u}_k + F_k \qquad k = 0, \ldots, N \tag{7}$$

where the forcing term is given by $F_k = 0$ for $k > 1$ and F_1 such that $\partial_t \hat{u}_1 = 0$ for $t > 0$. Here $\hat{u}_k(t) \in \mathbb{C}$ denotes the k-th Fourier coefficient of $u(x,t)$

$$u(x) \equiv \sum_{k=-N}^{k=+N} \hat{u}_k e^{ikx} \tag{8}$$

where N in the total number of Fourier modes. Note that since $u(x,t) \in \mathbb{R}$ the following condition must be accomplished

$$\hat{u}_k = \overline{\hat{u}_{-k}} \tag{9}$$

where $\overline{(\cdot)}$ denotes the complex conjugate. For details about the computation of the non-linear convective term the reader is referred to Appendix B.

2.1. Kinetic energy transport equation

The energy E_k of the k-th mode is obtained by taking the product of $\hat{u}_k$ with its complex conjugate $\overline{\hat{u}_k}$. Then the kinetic energy transport equation is given by[2]

$$\partial_t E_k = -\frac{2k^2}{Re}E_k - \left(\overline{\hat{u}_k}\mathcal{C}_k(\hat{u}_p, \hat{u}_q) + \hat{u}_k\overline{\mathcal{C}_k(\hat{u}_p, \hat{u}_q)}\right) + \overline{\hat{u}_k}F_k + \overline{F_k}\hat{u}_k \tag{10}$$

where $\mathcal{C}_k(\hat{u}_p, \hat{u}_q) \in \mathbb{C}$ is the convective contribution

$$\mathcal{C}_k(\hat{u}_p, \hat{u}_q) \equiv \sum_{k=p+q} \hat{u}_p iq \hat{u}_q \tag{11}$$

[1] See Appendix A, for details.
[2] $\partial_t E_k = \partial_t(\hat{u}_k\overline{\hat{u}_k}) = \hat{u}_k\partial_t\overline{\hat{u}_k} + \overline{\hat{u}_k}\partial_t\hat{u}_k$.

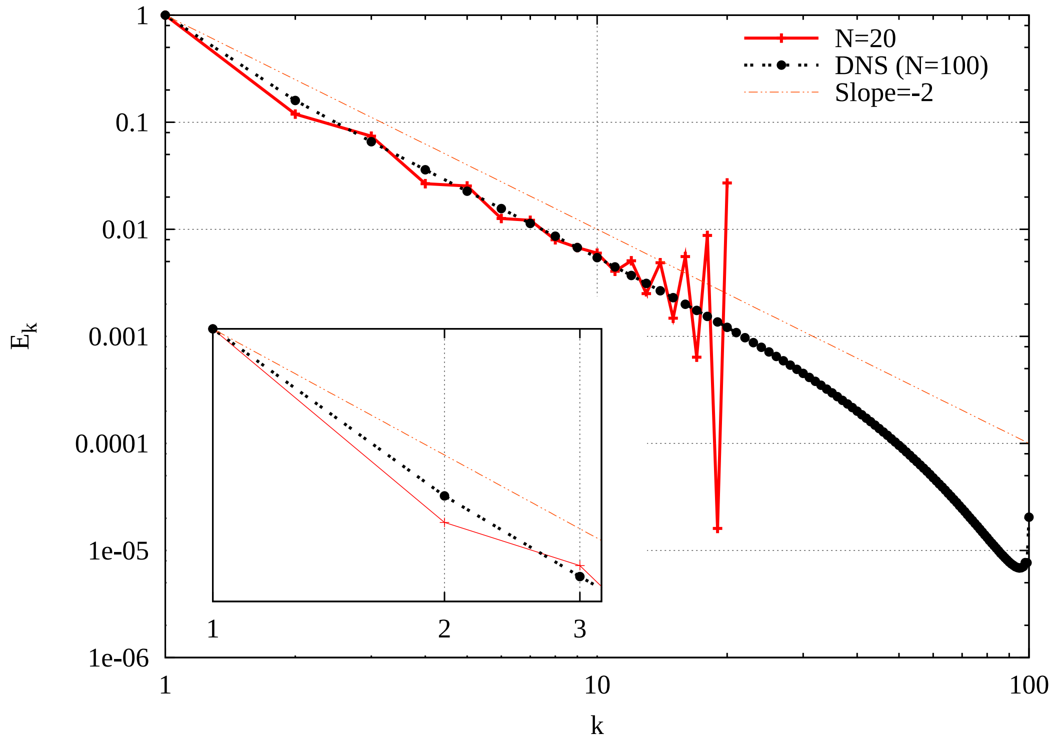

Figure 2: Energy spectrum of the steady-state solution of the Burgers equation for $N = 20$ and $N = 100$ (DNS). The steady state is reached at $t = 3$ approximately.

2.2. Role of diffusive term

Diffusive term in the kinetic energy transport equation (10) is given by

$$-\frac{2k^2}{Re} E_k \in \mathbb{R}^- \tag{12}$$

Therefore the diffusive term is damping energy (and not transporting at all!). This damping effect becomes more effective for the high-frequency modes (small scales).

2.3. Role of convection: energy cascade and triadic interactions

The (non-linear) convective term given by

$$-\left(\overline{\hat{u}_k} \mathcal{C}_k(\hat{u}_p, \hat{u}_q) + \hat{u}_k \overline{\mathcal{C}_k(\hat{u}_p, \hat{u}_q)}\right) \in \mathbb{R} \tag{13}$$

is transporting the energy mainly from large scales (low-frequency modes) to small scales (high-frequency modes) (see Fig. 1 , left); but also from small scales to large scales (energy backscattering) (see Fig. 1 , right).

3. Proposed problem

Solve the Burgers equation with $Re = 40$. As initial condition take $\hat{u}_k = k^{-1}$. Since mode $k = 0$ has no interactions with other modes, we assume $\hat{u}_0 = 0$ (no mean flow). Figure 2 shows the energy spectrum of the steady state for $N = 20$ (clearly underresolved) and a DNS with $N = 100$. Try to reproduce these results.

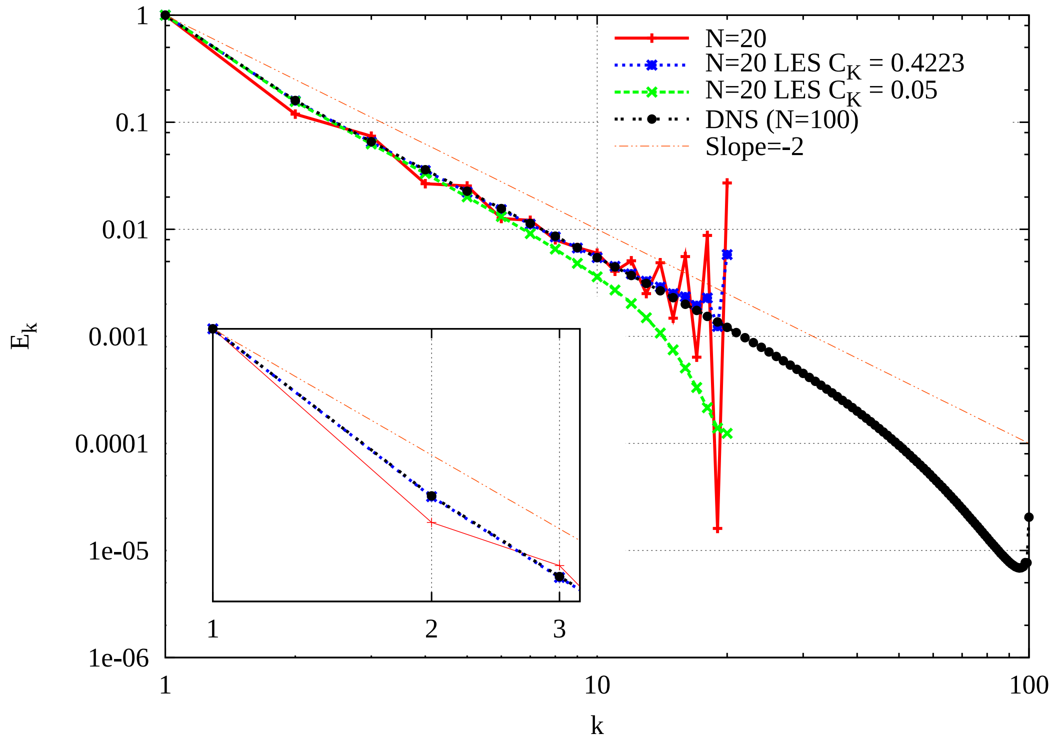

Figure 3: Energy spectrum of the steady-state solution of the Burgers equation for $N = 20$ (with spectral eddy-viscosity model taking $m = 2$ and $C_K = 0.4523$) and $N = 100$ (DNS).

3.1. And one step further..... Large-Eddy Simulation (LES)!

At this point it is easier to implement a LES model that should improve our results for the coarse mesh. The simplest LES model goes back to the mid-60s: the Smagorinsky model [2]. For a 1D problem it results[3]

$$\partial_t \overline{u} + \overline{u} \partial_x \overline{u} = \frac{1}{Re} \partial_{xx} \overline{u} + f - \partial_x \tau(u) \tag{14}$$

where

$$\tau(u) \equiv \overline{u^2} - \overline{u}^2 \tag{15}$$

To close the filtered equations, the subfilter tensor $\tau(u)$ has to be modeled in terms of the filtered velocity $\overline{u}$. This closure for the Smagorinsky model is given by

$$\tau(u) \approx \nu_t \partial_x \overline{u} \tag{16}$$

and the eddy-viscosity ν_t is modeled as

$$\nu_t = l_s^2 \sqrt{2 (\partial_x \overline{u})^2} \tag{17}$$

where l_s is the Smagorinsky lengthscale, which is proportional to the equivalent filter width Δ

$$l_s = C_s \Delta \tag{18}$$

[3]Here, the overline $\overline{(\cdot)}$ represents a linear filter.

where the Smagorinsky constant usually take values $C_s = 0.1 - 0.2$.

However, the Smagorinsky model cannot be applied in Fourier space. Instead, we propose to use an spectral eddy-viscosity model: basically the function $\nu_t(k)$ is determined assuming some *a priori* properties of the energy spectrum. The concept of k-dependent eddy viscosity was first introduced by Kraichnan [3] in the mid-70s for three-dimensional isotropic turbulence with a classical $-5/3$ slope in the energy spectrum. To take into account energy spectra with different slopes, Métais and Lesieur [4] proposed the following improvement

$$\nu_t(k/k_N) = \nu_t^{+\infty} \left(\frac{E_{k_N}}{k_N}\right)^{1/2} \nu_t^* \left(\frac{k}{k_N}\right) \tag{19}$$

with

$$\nu_t^{+\infty} = 0.31 \frac{5-m}{m+1} \sqrt{3-m} C_K^{-3/2} \tag{20}$$

where m is the slope of the energy spectrum, that is k^{-m}, E_{k_N} is the energy at the cutoff frequency, k_N, and C_K is the Kolmogorov constant. ν_t^* is a non-dimensional eddy-viscosity equal to 1 for small values of k/k_N and with a strong increase for higher k up to $k/k_N = 1$; it reads

$$\nu_t^* \left(\frac{k}{k_N}\right) = 1 + 34.5 e^{-3.03(k_N/k)} \tag{21}$$

Note that the classical $\nu_t(k/k_N)$ is recovered for $m = 5/3$. In our case, the energy spectrum is approximately $m = 2$ (see Figure 2) and the Kolmogorov constant (for 1D Burgers equation) is $C_K \approx 0.4523$. In our case, $\nu = Re^{-1}$, and therefore, in practice the only modification in the code implies to modify ν by $\nu_{eff}(k) = \nu + \nu_t(k)$. Try to reproduce the results reported in Figure 3.

4. Hints

- Note that although $E_k \in \mathbb{R}$, $\hat{u}_k \in \mathbb{C}$. Therefore, your code must be able to perform basic operations with complex numbers.

- You can use a fully explicit time-integration scheme. Be careful with Δt!! A CFL-like condition must be imposed[4]

$$\Delta t < C_1 \frac{Re}{N^2} \tag{22}$$

- All the results reported here correspond to the steady state solution. You can also visualize the temporal evolution, play with other Re-numbers, different initial conditions, etc... It would help you to understand better the role of each term.

[4] Value of $0 < C_1 < 1$ would depend on your time integration scheme. Play with it! ;-)

A From Physical space to Fourier space

Recalling again the Burgers equation in physical space

$$\partial_t u + u\partial_x u = \frac{1}{Re}\partial_{xx} u + f \tag{A.1}$$

each of the terms can be converted into Fourier space. Transient, diffusive and forcing terms are straightforward

$$\partial_t u = \sum_{k=-N}^{k=+N} \left(\partial_t \hat{u}_k\right) e^{ikx} \tag{A.2}$$

$$\partial_{xx} u = \sum_{k=-N}^{k=+N} \hat{u}_k \partial_{xx} e^{ikx} = \sum_{k=-N}^{K=+N} \left(-k^2 \hat{u}_k\right) e^{ikx} \tag{A.3}$$

$$f = \sum_{k=-N}^{k=+N} F_k e^{ikx} \tag{A.4}$$

and non-linear convective term is given by

$$\partial_x u = \sum_{k=-N}^{k=+N} \hat{u}_k \partial_x e^{ikx} = \sum_{k=-N}^{k=+N} ik\hat{u}_k e^{ikx} \tag{A.5}$$

$$u\partial_x u = \sum_{p=-N}^{p=N} \hat{u}_p e^{ipx} \sum_{q=-N}^{q=+N} iq\hat{u}_q e^{iqx} = \sum_{p=-N,q=-N}^{p=+N,q=+N} \hat{u}_p iq\hat{u}_q e^{i(p+q)x} \tag{A.6}$$

and the condition that $u(x,t) \in \mathbb{R}$ is automatically satisfied

$$\hat{u}_{-p} i(-q)\hat{u}_{-q} = -\overline{\hat{u}_p} iq\overline{\hat{u}_p} = \overline{\hat{u}_p iq\hat{u}_q} \tag{A.7}$$

Therefore, the Burgers equation (A.1) in Fourier space results

$$\sum_{k=-N}^{k=+N} \left(\partial_t \hat{u}_k\right) e^{ikx} + \sum_{p=-N,q=-N}^{p=+N,q=+N} \hat{u}_p iq\hat{u}_q e^{i(p+q)x} = \frac{1}{Re} \sum_{k=-N}^{k=+N} \left(-k^2 \hat{u}_k\right) e^{ikx} + \sum_{k=-N}^{k=+N} F_k e^{ikx} \tag{A.8}$$

Then, we can solve each Fourier mode separately[5]

$$\partial_t \hat{u}_k + \sum_{k=p+q} \hat{u}_p iq\hat{u}_q = -\frac{k^2}{Re}\hat{u}_k + F_k \qquad k = 0, \ldots, N \tag{A.9}$$

For further information about Spectral Methods the reader is referred to [5].

[5] This is the underlying idea of Spectral Methods.

B Computing all the triadic interactions

The contribution of the convective term to the k^{th} mode is given by

$$\sum_{k=p+q} \hat{u}_p iq \hat{u}_q \in \mathbb{C}. \tag{B.1}$$

As mentioned before, only the positive modes, $k > 0$, need to be solved. Negative modes follow straightforwardly from the condition (9). Of course, interactions where $p+q > N$ should also be discarded because our Fourier series is truncated for $|k| \leq N$. In this regard, Figure 4 represents the $(2N)^2$ possible interactions between modes. Among them, only the interactions between the straight lines $k = 0$ and $k = N$ need to be considered. Notice that some of these interactions involve modes with $p < 0$ (or $q < 0$). In this case, negatives modes are related with positive one via Eq.(9). Namely,

$$\hat{u}_p = \overline{\hat{u}_{-p}}. \tag{B.2}$$

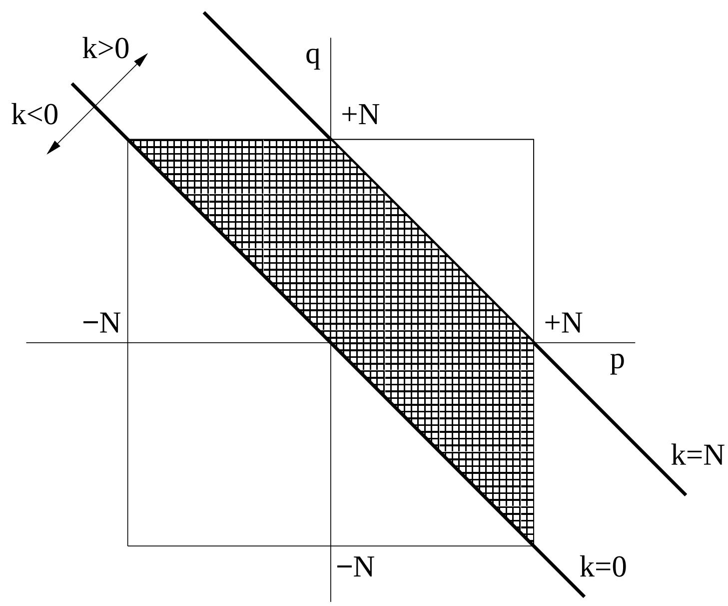

Figure 4: Representation of all possible triadic interactions between modes. Only the interactions between the straight lines $k = 0$ and $k = N$ need to be considered for the computation of the non-linear convective term.

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