PHYSICAL FORMULATION

MATHEMATICS AS THE LANGUAGE OF PHYSICS

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EQUATIONS

CONSTRAINING SOLUTIONS

REVIEW SO FAR

- Concept of operator
- Link between algebra and calculus
- Operator properties
- Function spaces as vector spaces

AGENDA

1) Definition
2) Physical space
3) Analytical solutions

DEFINITION

Equations impose an equality condition to the solution.

They can be expressed with one or more operators.

$$Ax = b$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (u \rho) = 0$$

$$\ldots$$

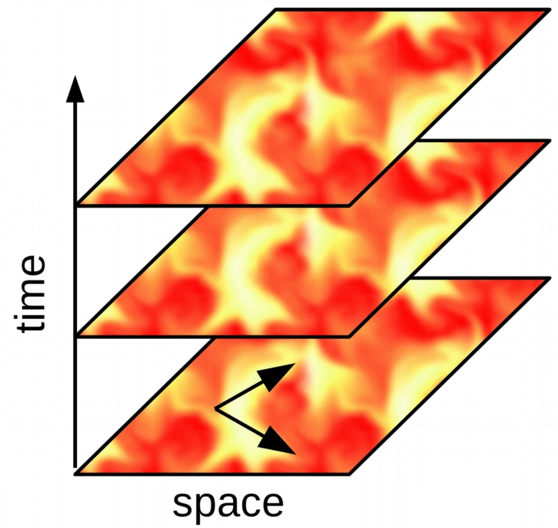
PHYSICAL SPACE

When approaching physical problems, our solution will depend, at least, on **space** and **time**.

$$\frac{\partial \theta}{\partial t} - \nabla^2 \theta = 0$$

Solution is the combination of **spatial and temporal evolution**.

$$S = T \otimes X$$



1) Relativistic effects are neglected. Measures are independent!
2) Operators are naturally split. Temporal and spatial operators.

ANALYTICAL SOLUTION

Let's take a diffusion equation as an example:

$$\frac{\partial \theta}{\partial t} - \nabla^2 \theta = 0$$

$$\theta\big|_{x=\partial\Omega} = 0 \quad \forall\, t \geqslant 0$$

$$\theta\big|_{t=0} = u_0(x) \quad \forall\, x \in \Omega$$

BC: Boundary Conditions

IC: Initial conditions

Important:

- BC: Defined for all time at space boundaries only.
- IC: Defined for all space at time boundaries (i.e., initial time) only.
- More details in

Introduction to PDE | MIT

ANALYTICAL SOLUTION

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Separation of variables:

$$\theta = T(t)\,X(x)$$

$$X\frac{\partial T}{\partial t} - T\nabla^2 X = 0$$

$$X\partial_t T = T\nabla^2 X$$

$$\frac{\partial_t T}{T} = \frac{\nabla^2 X}{X} = \lambda$$

Important:

- λ must be a scalar.
- We end up with 2 separate eigenvalue problems:

$$\partial_t T = \lambda T$$
$$\nabla^2 X = \lambda X$$

- So we are **diagonalizing the operators**!

PHYSICAL SPACE

Eigenvalue problem:

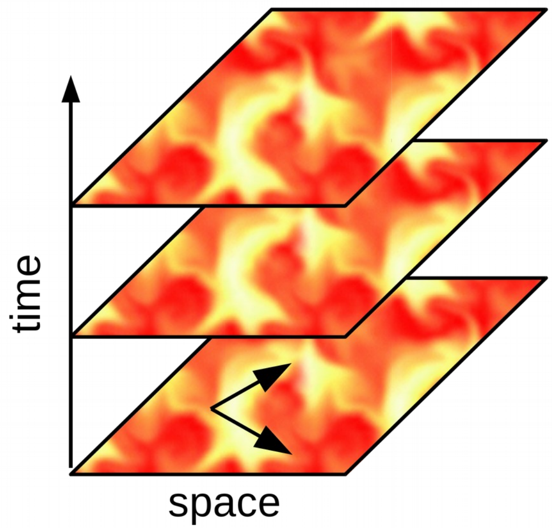
$$\partial_t T = \lambda T$$

$$\nabla^2 X = \lambda X$$

As with algebraic operators, we try to express it in a diagonal form.

$$Ax = P \Lambda P^{-1} x$$

Subject to a base P.



Every operator has an optimal way to look at it. It is the diagonal form.

ANALYTICAL SOLUTION

Eigenvalue problem:

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$$\nabla^2 X = \lambda X$$

As with algebraic operators, we try to express it in a diagonal form.

$$Ax = P \Lambda P^{-1} x$$

Subject to a base P.

Important:

- P is the base spanned by **eigenvectors.**
- Λ components are the **eigenvalues.**
- Differential operators always diagonalize with base ***e***.

$$\frac{d^n}{dt^n} \exp(\nu t) = \nu^n \exp(\nu t)$$

ANALYTICAL SOLUTION

X eigenvalue problem:

$$\nabla^2 X = \lambda X$$

Eigenfunctions look like:

$$X(x) = a_n e^{\left(\sqrt{\lambda_n} x\right)} + b_n e^{\left(-\sqrt{\lambda_n} x\right)}$$

T eigenvalue problem:

$$\partial_t T = \lambda T$$

Eigenfunctions looks like:

$$T(t) = d_n e^{\left(\lambda_n t\right)}$$

Finally, the **solution space** will look like:

$$\theta(x,t) = T(t) X(x) = \sum_{n=-\infty}^{+\infty} d_n e^{\left(\lambda_n t\right)} \left(a_n e^{\left(\sqrt{\lambda_n} x\right)} + b_n e^{\left(-\sqrt{\lambda_n} x\right)}\right)$$

ANALYTICAL SOLUTION

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Important:

- We have composed time with space to obtain our solution.
- The solution is an ***infinite-dimensional vector space!***
- Values of λ are, potentially, the full set of reals.
- We still have to determine the **coefficients** and the **eigenvalues**.

ANALYTICAL SOLUTION

Eigenvalues

Boundary Conditions: $\theta\big|_{x=\partial\Omega}=0\ \forall t$

In 1D: $\theta\big|_{x=0}=\theta\big|_{x=L}=0\ \forall t$

By expanding solution: $X(0)=X(L)=0$

For convenience: $\pm\sqrt{\lambda_n}=\pm\mu_n$

We end up with:
$$\begin{pmatrix} X(0) \\ X(L) \end{pmatrix}=\underbrace{\begin{bmatrix} e^{(\mu_n 0)} & e^{(-\mu_n 0)} \\ e^{(\mu_n L)} & e^{(-\mu_n L)} \end{bmatrix}}_{[B]}\underbrace{\begin{bmatrix} a_n \\ b_n \end{bmatrix}}_{[c_n]}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[B] \qquad [c_n]=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ANALYTICAL SOLUTION

Eigenvalues

To find a compatible solution:

$$0=\exp(-\mu_n L)-\exp(\mu_n L)\rightarrow \qquad \boxed{\mu_n=n\frac{\pi}{L}i}$$

Which, substituting back in *B* gives:

$$\begin{bmatrix} 1 & 1 \\ e^{(n\pi i)} & e^{(-n\pi i)} \end{bmatrix}\begin{bmatrix} a_n \\ b_n \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}\rightarrow a_n=-b_n$$

And so:

$$X(x)=a_n\left(e^{(\mu_n x)}-e^{(-\mu_n x)}\right)=2a_n i\sin(\mu_n x)$$

ANALYTICAL SOLUTION

BONUS

Real case: $\mu_n \in \Re$

$$\begin{pmatrix} 1 & 1 \\ e^{\mu_n L} & e^{-\mu_n L} \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$a_n = b_n = 0$$

Trivial solution!

We end up with:

$$X(x) = 0$$

Imaginary case: $i\mu_n \in \Im$

$$\begin{pmatrix} \cos(0) & \sin(0) \\ \cos(\mu_n L) & \sin(\mu_n L) \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$a_n = 0 \quad b_n \neq 0$$

$$\mu_n = n\frac{\pi}{L} \; \forall \, n \in \mathbb{N} > 0$$

We end up with:

$$X(x) = \sum_{n=1}^{+\infty} b_n \sin\left(n\pi\frac{x}{L}\right)$$

ANALYTICAL SOLUTION

At this point the eigenvectors look like:

$$\boxed{\theta(x,t)=\sum_{n=1}^{+\infty} d_n e^{-\left(\frac{n\pi}{L}\right)^2 t} \sin\left(n\pi\frac{x}{L}\right) \quad \forall n \in \mathbb{N}>0}$$

We can obtain **qualitative information** at this point:

- Solution is going to smooth down with time.
- Higher spatial frequencies will dissipate faster.
- Characteristic time is $\tau=\left(\frac{L}{n\pi}\right)^2$
- There is no interaction between frequencies

ANALYTICAL SOLUTION

Coefficients

Initial conditions: $$\theta_0(x)=\theta(x,0)=\sum_{n=1}^{+\infty} d_n \sin\left(n\pi\frac{x}{L}\right)$$

Idea: $$\langle\theta_0(x)|\theta_n(x,0)\rangle=\langle\theta(x,0)|\theta_n(x,0)\rangle$$

Project initial conditions over every base vector of the solution space

Because the solution space is orthogonal:

$$\boxed{d_n=\frac{2}{L}\int_0^L \theta_0(x)\sin\left(n\pi\frac{x}{L}\right)dx}$$

ANALYTICAL SOLUTION

Coefficients: Development details.

$$\langle \theta(x,0) | \theta_n(x,0) \rangle = \int_0^L \sum_{m=1}^{+\infty} d_m \sin\left(m\pi \frac{x}{L}\right) \sin\left(n\pi \frac{x}{L}\right) dx$$

$$= \sum_{m=1}^{+\infty} d_m \int_0^L \sin\left(m\pi \frac{x}{L}\right) \sin\left(n\pi \frac{x}{L}\right) dx = d_n \int_0^L \sin^2\left(n\pi \frac{x}{L}\right) dx$$

$$\int_0^L \sin\left(m\pi \frac{x}{L}\right) \sin\left(n\pi \frac{x}{L}\right) dx = \begin{cases} 0 \Leftrightarrow m \neq n \\ \sin^2\left(n\pi \frac{x}{L}\right) \Leftrightarrow m = n \end{cases}$$

BONUS

$$\langle \theta(x,0) | \theta_n(x,0) \rangle = d_n \frac{L}{2} = \langle \theta_0(x) | \theta_n(x,0) \rangle$$

$$d_n = \frac{2}{L} \int_0^L \theta_0(x) \sin\left(n\pi \frac{x}{L}\right) dx$$

ANALYTICAL SOLUTION

Finally, the solution will look like:

$$\boxed{\begin{gathered}\theta(x,t)=\sum_{n=1}^{+\infty} d_n e^{-\left(\frac{n\pi}{L}\right)^2 t}\sin\left(n\pi\frac{x}{L}\right)\quad \forall\, n\in\mathbb{N}>0\\ d_n=\frac{2}{L}\int_0^L \theta_0(x)\sin\left(n\pi\frac{x}{L}\right)dx\end{gathered}}$$

ANALYTICAL SOLUTION

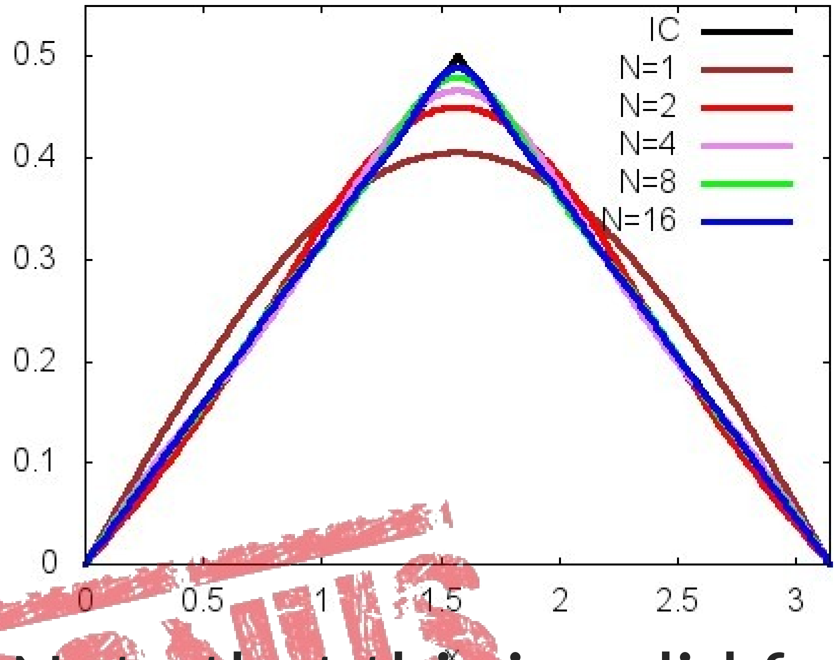
Important facts:

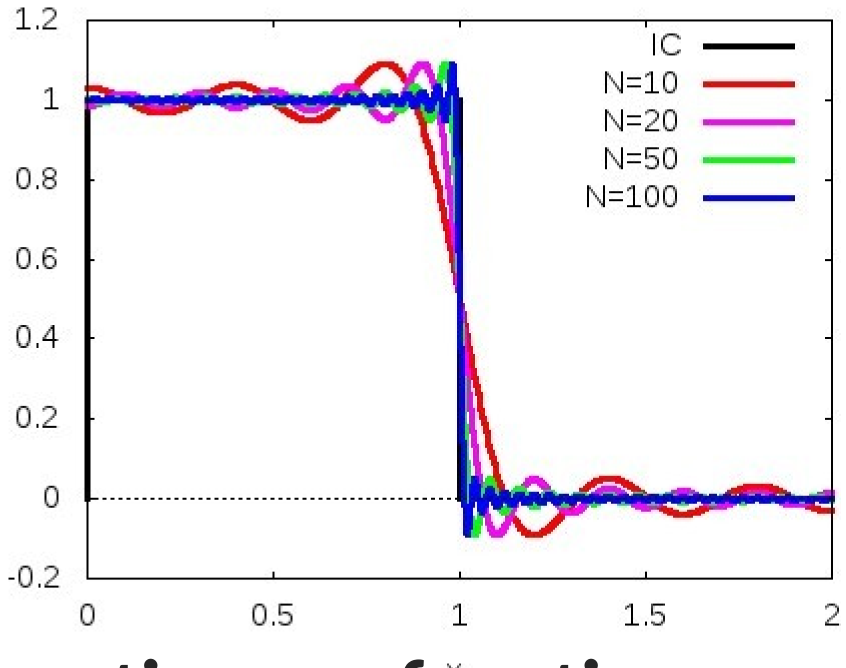
- Eigenvalues determine the **shape of the solution.**
- Eigenvalues are still infinite but **discrete.**
- Eigenvalues are determined by the **equation**

  (i.e., equality, operators and boundary conditions)
- The solution base we are working with is **just a convenient base** of the solution space. We could use others!

ANALYTICAL SOLUTION

Coefficients: Graphical example.





Note that this is valid for **continuous functions** only! Otherwise it appears **Gibbs phenomena**.

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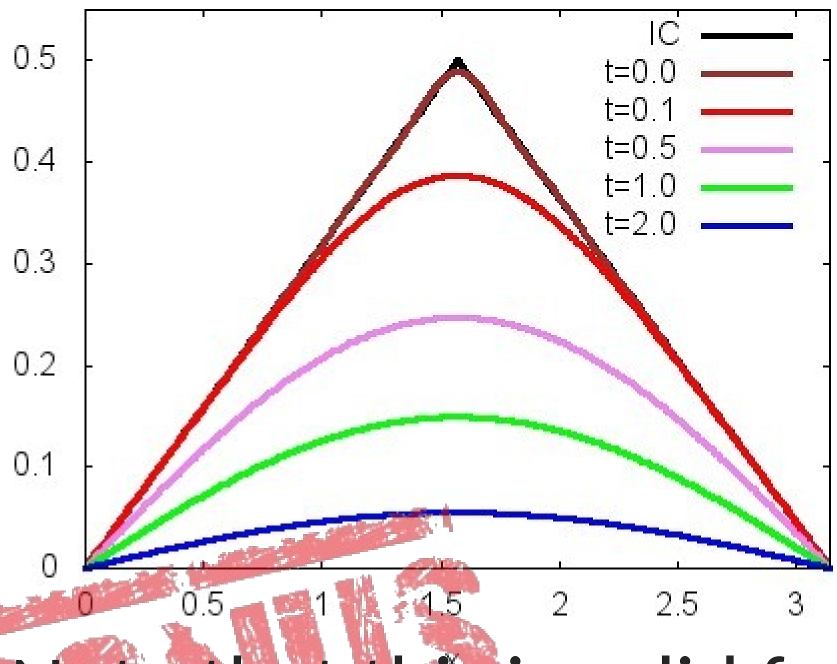
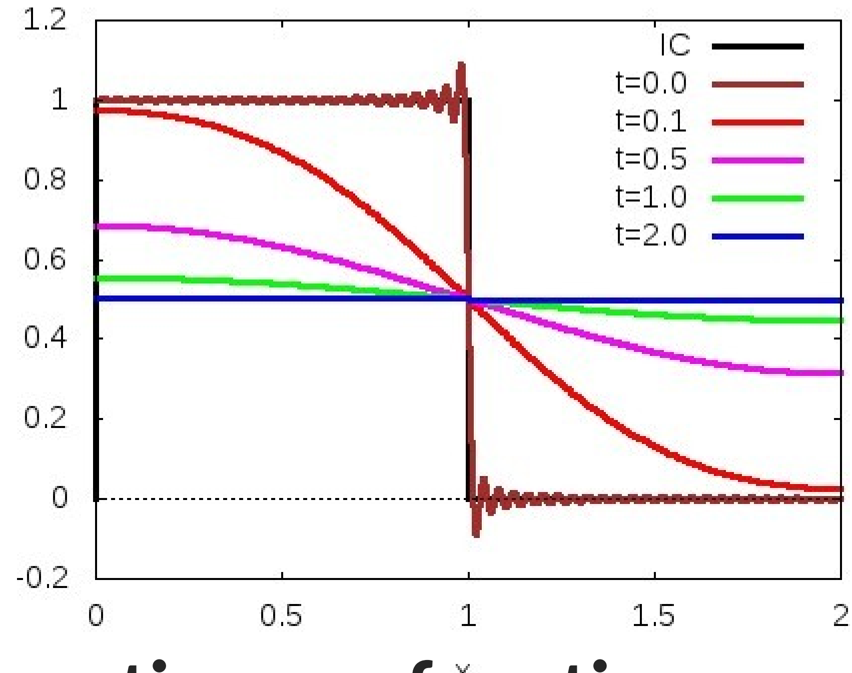
Final solution evolution: Graphical example.





Note that this is valid for **continuous functions** only! Otherwise it appears **Gibbs phenomena**.

ANALYTICAL SOLUTION

Summary:

- Operators can be diagonalized in a **convenient base**.
- Solution space is going to be **infinite dimensional**.
- The basis is **guaranteed to be orthogonal** by the Sturm-Liouville theorem
- B.C. provide information about the solution structure.
- I.C. provide information about the scaling of such structure.

ANALYTICAL SOLUTION

Coming soon…

What kind of information can we obtain from the following equations?

$$\frac{\partial \theta}{\partial t} - \nabla^2 \theta = 0$$

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = 0$$

$$\nabla^2 \theta = 0$$

What can eigenvalues and eigenfunctions tell us?

See you in next session!