

605-HW08-Sums-of-RVs-Law-of-Large-Numbers

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Contents

| | |
|--|----------|
| HW8 - Sums of RVs; Law of Large Numbers | 2 |
| Problem 11 on page 303: | 2 |
| Exponential lifetime of lightbulbs | 2 |
| Problem 14 on page 303: | 3 |
| Distribution of difference of two exponentials | 3 |
| Problem 1 on page 320-321: Chebychev's inequality. | 5 |
| (a) $Pr(X - 10 \geq 2)$ | 5 |
| (b) $Pr(X - 10 \geq 5)$ | 5 |
| (c) $Pr(X - 10 \geq 9)$ | 5 |
| (d) $Pr(X - 10 \geq 20)$ | 5 |

HW8 - Sums of RVs; Law of Large Numbers

Problem 11 on page 303:

Exponential lifetime of lightbulbs

A company buys 100 lightbulbs, each of which has an exponential lifetime of 1000 hours.

What is the expected time for the first of these bulbs to burn out? (See Exercise 10.)

According to exercise 10:

Let X_1, X_2, \dots, X_n be n independent random variables each of which has an exponential density with mean μ .

Let M be the minimum value of the X_j .

Show that the density for M is exponential with mean $\frac{\mu}{n}$.

Hint: Use cumulative distribution functions.

So, we have $n = 100$ exponential variables, each representing the lifetime of a lightbulb, where the average lightbulb has a lifetime of 1000 hours, which is distributed exponentially across the set of lightbulbs (independent and identically distributed.)

The time for the first lightbulb to burn out is thus the minimum of this set.

Because the lifetime of each lightbulb is exponentially distributed, we know that the probability that any individual lightbulb has **not** burned out by time $M = m$ is $Pr(X_i > m) = e^{-\lambda m}$, where $\lambda = \frac{1}{\mu} = \frac{1}{1000}$.

We know that the probability that **no** lightbulb has burned out by time $M = m$ is

$$\begin{aligned} Pr(X_1 > m) \cdot Pr(X_2 > m) \cdot \dots \cdot Pr(X_n > m) &= e^{-\lambda m} \cdot e^{-\lambda m} \cdot \dots \cdot e^{-\lambda m} \\ &= (e^{-\lambda m})^n \\ &= e^{-\lambda mn} \\ &= e^{-\frac{m \cdot n}{\mu}} \\ &= e^{-\frac{m \cdot 100}{1000}} \\ &= e^{-\frac{m}{10}} \end{aligned}$$

The above is the probability that **no** bulb has burned out by time $M = m$.

Therefore the probability that the **first** bulb has burned out by time $M = m$ is $Pr(M \leq m) = 1 - e^{-\frac{m}{10}}$.

This is an exponential distribution with $\lambda = \frac{1}{\mu} = \frac{1}{10}$.

Therefore, the expected time for the first of these bulbs to burn out is 10 hours.

Problem 14 on page 303:

Distribution of difference of two exponentials

Assume that X_1 and X_2 are independent random variables, each having an exponential density with parameter λ .

Show that $Z = X_1 - X_2$ has density $f_Z(z) = \frac{1}{2}\lambda e^{-\lambda|z|}$.

$$\text{We have } f(x_1; \lambda) = \begin{cases} \lambda e^{-\lambda x_1} & x_1 \geq 0, \\ 0 & x_1 < 0. \end{cases} \text{ and } f(x_2; \lambda) = \begin{cases} \lambda e^{-\lambda x_2} & x_2 \geq 0, \\ 0 & x_2 < 0. \end{cases}.$$

$$\text{By independence, the joint PDF is } f(x_1, x_2; \lambda) = \begin{cases} \lambda^2 e^{-\lambda x_1} e^{-\lambda x_2} & x_1, x_2 \geq 0, \\ 0 & (x_1 < 0) \vee (x_2 < 0). \end{cases}.$$

Let $Z = X_1 - X_2$. Because this is a difference, Z can be positive or negative.

We seek $F_Z(z) = Pr(Z \leq z)$ where $-\infty < z < \infty$.

In the case where $z \leq 0$, the CDF is:

$$\begin{aligned} F_Z(z) &= Pr(Z \leq z) \\ &= Pr(X_1 - X_2 \leq z) \\ &= Pr(X_2 \geq X_1 - z) \\ &= \int_0^\infty \int_{x_1-z}^\infty f_{X_1, X_2}(x_1, x_2) dx_2 dx_1 \\ &= \int_0^\infty \int_{x_1-z}^\infty \lambda^2 e^{-\lambda x_1} e^{-\lambda x_2} dx_2 dx_1 \\ &= \int_0^\infty \lambda e^{-\lambda x_1} \left[\int_{x_1-z}^\infty \lambda e^{-\lambda x_2} dx_2 \right] dx_1 \\ &= \int_0^\infty \lambda e^{-\lambda x_1} \left[-e^{-\lambda x_2} \right]_{x_1-z}^\infty dx_1 \\ &= \int_0^\infty \lambda e^{-\lambda x_1} \left[-0 + e^{-\lambda(x_1-z)} \right] dx_1 \\ &= \int_0^\infty \lambda e^{-\lambda x_1} e^{-\lambda(x_1-z)} dx_1 \\ &= \int_0^\infty \lambda e^{-\lambda(2x_1-z)} dx_1 \\ &= e^{\lambda z} \int_0^\infty \lambda e^{-\lambda(2x_1)} dx_1 \\ &= e^{\lambda z} \cdot \frac{1}{2} \end{aligned}$$

Differentiating, $f_Z(z) = \frac{\lambda e^{\lambda z}}{2}$ for $z < 0$.

In the case where $z > 0$, the CDF is:

$$\begin{aligned}
F_Z(z) &= Pr(Z \leq z) \\
&= 1 - Pr(Z > z) \\
&= 1 - Pr(X_1 - X_2 > z) \\
&= Pr(X_2 \geq X_1 + z) \\
&= \int_0^\infty \int_0^{x_1+z} f_{X_1, X_2}(x_1, x_2) dx_2 dx_1 \\
&= \int_0^\infty \int_0^{x_1+z} \lambda^2 e^{-\lambda x_1} e^{-\lambda x_2} dx_2 dx_1 \\
&= \int_0^\infty \lambda e^{-\lambda x_1} \left[\int_0^{x_1+z} \lambda e^{-\lambda x_2} dx_2 \right] dx_1 \\
&= \int_0^\infty \lambda e^{-\lambda x_1} [1 - e^{-\lambda x_2}]_0^{x_1+z} dx_1 \\
&= \int_0^\infty \lambda e^{-\lambda x_1} [1 - e^{-\lambda(x_1+z)} - (1 - 1)] dx_1 \\
&= \int_0^\infty \lambda e^{-\lambda x_1} [1 - e^{-\lambda(x_1+z)}] dx_1 \\
&= \int_0^\infty \lambda e^{-\lambda x_1} - \lambda e^{-\lambda x_1} e^{-\lambda(x_1+z)} dx_1 \\
&= \int_0^\infty \lambda e^{-\lambda x_1} - \lambda e^{-2\lambda x_1} e^{-\lambda z} dx_1 \\
&= \int_0^\infty \lambda e^{-\lambda x_1} - \int_0^\infty \lambda e^{-2\lambda x_1} e^{-\lambda z} dx_1 \\
&= 1 - e^{-\lambda z} \int_0^\infty \lambda e^{-2\lambda x_1} dx_1 \\
&= 1 - e^{-\lambda z} \int_0^\infty \lambda e^{-2\lambda x_1} dx_1 \\
&= 1 - e^{-\lambda z} \frac{1}{2} \\
&= 1 - \int_0^\infty \lambda e^{-\lambda x_1} e^{-\lambda(x_1+z)} dx_1 \\
&= 1 - \int_0^\infty \lambda e^{-\lambda(2x_1+z)} dx_1 \\
&= 1 - e^{\lambda z} \int_0^\infty \lambda e^{-\lambda(2x_1)} dx_1 \\
&= 1 - e^{-\lambda z} \cdot \frac{1}{2}
\end{aligned}$$

Differentiating, $f_Z(z) = \frac{\lambda e^{-\lambda z}}{2}$ for $z \geq 0$.

Thus, the density for all z is $f_Z(z) = \frac{\lambda e^{-\lambda|z|}}{2}$

This is the Laplace distribution, also known as the “Double Exponential” distribution.

Problem 1 on page 320-321: Chebychev's inequality.

Let X be a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = \frac{100}{3}$.

Chebychev's Inequality states that under the above conditions, $Pr(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}, \forall \epsilon > 0$.

If $\epsilon = k \cdot \sigma$, this can be rewritten as

$$Pr(|X - \mu| \geq k \cdot \sigma) \leq \frac{\sigma^2}{k^2 \cdot \sigma^2} = \frac{1}{k^2}, \forall \epsilon > 0$$

Using Chebyshev's Inequality, find an upper bound for the following probabilities:

(a) $Pr(|X - 10| \geq 2)$

$$\epsilon = 2 = k \cdot \sigma = k \cdot \sqrt{\frac{100}{3}} \quad \epsilon^2 = 2^2 = 4 = k^2 \cdot \frac{100}{3} \quad \frac{1}{k^2} = \frac{100}{3 \cdot 4} = \frac{100}{12} = 8.33333$$

Because probabilities cannot be greater than 1, the upper bound is 1.

(b) $Pr(|X - 10| \geq 5)$

$$\epsilon = 5 = k \cdot \sigma = k \cdot \sqrt{\frac{100}{3}} \quad \epsilon^2 = 5^2 = 25 = k^2 \cdot \frac{100}{3} \quad \frac{1}{k^2} = \frac{100}{3 \cdot 25} = \frac{100}{75} = 1.33333$$

Because probabilities cannot be greater than 1, the upper bound is 1.

(c) $Pr(|X - 10| \geq 9)$

$$\epsilon = 9 = k \cdot \sigma = k \cdot \sqrt{\frac{100}{3}} \quad \epsilon^2 = 9^2 = 81 = k^2 \cdot \frac{100}{3} \quad \frac{1}{k^2} = \frac{100}{3 \cdot 81} = \frac{100}{243} \approx 0.4115$$

Thus, the upper bound is $\frac{100}{243} \approx 0.4115$.

(d) $Pr(|X - 10| \geq 20)$

$$\epsilon = 20 = k \cdot \sigma = k \cdot \sqrt{\frac{100}{3}} \quad \epsilon^2 = 20^2 = 400 = k^2 \cdot \frac{100}{3} \quad \frac{1}{k^2} = \frac{100}{3 \cdot 400} = \frac{100}{1200} \approx 0.083333$$

Thus, the upper bound is $\frac{1}{12} \approx 0.083333$.