605-HW03-Eigenvectors

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Contents

Setup

```
##
## Attaching package: 'dplyr'

## The following objects are masked from 'package:stats':
##
##     filter, lag

## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union

##
## Attaching package: 'kableExtra'

## The following object is masked from 'package:dplyr':
##
##     group_rows
```

Part 1 - Matrix Rank

(1) What is the rank of the matrix A?

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    2    3    4
## [2,]   -1    0    1    3
## [3,]    0    1   -2    1
## [4,]    5    4   -2   -3
```

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 5 & 4 & -2 & -3 \end{bmatrix}$$

```
#B has a non-zero determinant, so it is non-singular, invertible, and full-rank:
det(A)
```

```
## [1] -9
```

```
# All 4 rows of the Row-Reduced Echelon Form are non-zero:
rref(A)
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    0
## [2,]    0    1    0    0
## [3,]    0    0    1    0
## [4,]    0    0    0    1
```

```
# The calculated matrix rank:
library(matrixcalc)
matrix.rank(A)
```

```
## [1] 4
```

Because A is a square matrix with non-zero determinant, it is full rank, i.e., $\mathbf{rank = 4}$.
This is confirmed as the Row-Reduced Echelon Form contains 4 non-zero rows.

(2) Given an mxn matrix where m > n, what can be the maximum rank?

The maximum rank is $maxrank = min(m, n)$, which in this case is the number of columns, n, as there are more rows than columns.

The minimum rank, assuming that the matrix is non-zero?

The minimum rank is 1.

(3) What is the rank of matrix B ?

```
##      [,1] [,2] [,3]
## [1,]    1    2    1
## [2,]    3    6    3
## [3,]    2    4    2
```

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 2 \end{bmatrix}$$

```r
#B has a zero determinant, so it is singular, non-invertible, and less than full-rank.
det(B)
```

```
## [1] 0
```

```r
# Row-Reduced echolon form shows a single non-zero row:
rref(B)
```

```
##      [,1] [,2] [,3]
## [1,]    1    2    1
## [2,]    0    0    0
## [3,]    0    0    0
```

```r
# calculated rank is 1:
matrix.rank(B)
```

```
## [1] 1
```

```r
Bcol1 = as.matrix(B[,1])
```

B has only a single independent column: $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

The third column is the same as the first, and the middle column is double the first.
Also, the number of non-zero rows in the row-echelon form is 1.
Therefore, the matrix B has rank 1.

Part 2 - Eigenvalues and Eigenvectors

Compute the eigenvalues and eigenvectors of the matrix A .

You'll need to show your work.
You'll need to write out the characteristic polynomial and show your solution.

```
A =
c(
1, 2, 3,
0, 4, 5,
0, 0, 6
)
A = matrix(A,nrow=3,byrow=T)
A
```

```
##      [,1] [,2] [,3]
## [1,]    1    2    3
## [2,]    0    4    5
## [3,]    0    0    6
```

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$(A - \lambda I)\, x = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} x = 0$$

$$det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = 0$$

Because the matrix is upper triangular, the determinant equals the product of the diagonal entries:

$$(1-\lambda)(4-\lambda)(6-\lambda) = 0$$

The characteristic polynomial is $24 - 34x + 11x^2 - x^3 \ = 0$, but we can easily see from the factored form above that the eigenvalues are $\lambda \in \{6, 4, 1\}$.

Eigenvector for $\lambda = 6$:

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1-6 & 2 & 3 \\ 0 & 4-6 & 5 \\ 0 & 0 & 6-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{cases} -5x_1 + 2x_2 + 3x_3 = 0 \\ 0x_1 - 2x_2 + 5x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -5x_1 + 5x_3 + 3x_3 = 0 \\ 5x_3 = 2x_2 \end{cases} \quad \Rightarrow \begin{cases} x_2 = \frac{5x_3}{2} \\ 5x_1 = 8x_3 \end{cases} \quad \Rightarrow \begin{cases} x_2 = \frac{5x_3}{2} = 2.5(x_3) \\ x_1 = \frac{8}{5}x_3 = 1.6(x_3) \end{cases} \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.6 \\ 2.5 \\ 1.0 \end{bmatrix}$$

Check:

```r
# Eigenvector for lambda = 6
x6 = c(1.6,2.5,1)
x6
```

```
## [1] 1.6 2.5 1.0
```

```r
# Ax / 6
result6 = (A %*% x6) / 6
result6
```

```
##      [,1]
## [1,]  1.6
## [2,]  2.5
## [3,]  1.0
```

```r
# Ax - 6x
result6 - x6
```

```
##                          [,1]
## [1,] -0.000000000000000222045
## [2,]  0.000000000000000000000
## [3,]  0.000000000000000000000
```

```r
# Is this result zero?
result6 - x6 < 1e-10
```

```
##      [,1]
## [1,] TRUE
## [2,] TRUE
## [3,] TRUE
```

```r
# Normalize this result
sumsq_x6 = sum(x6^2)
sumsq_x6
```

```
## [1] 9.81
```

```r
x6_norm = x6 / sqrt(sumsq_x6)
x6_norm
```

```
## [1] 0.510841 0.798189 0.319275
```

Normalized Eigenvector for lambda = 6:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.6 \\ 2.5 \\ 1.0 \end{bmatrix} \cdot \frac{1}{\sqrt{9.81}} = \begin{bmatrix} .510841 \\ .798189 \\ .319275 \end{bmatrix}$$

Eigenvector for $\lambda = 4$:

$$\begin{bmatrix} 1-4 & 2 & 3 \\ 0 & 4-4 & 5 \\ 0 & 0 & 6-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{cases} -3x_1 + 2x_2 + 3x_3 = 0 \\ 0x_1 + 0x_2 + 5x_3 = 0 \\ 0x_1 + 0x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = 0 \\ -3x_1 + 2x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = 0 \\ 3x_1 = 2x_2 \end{cases} \Rightarrow \begin{cases} x_3 = 0 \\ x_1 = \frac{2x_2}{3} \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

```r
# Eigenvector for lambda = 4
x4 = c(2,3,0)
x4
```

```
## [1] 2 3 0
```

```r
# Ax / 4
result4 = (A %*% x4) / 4
result4
```

```
##      [,1]
## [1,]    2
## [2,]    3
## [3,]    0
```

```r
# Ax - 4x
result4 - x4
```

```
##      [,1]
## [1,]    0
## [2,]    0
## [3,]    0
```

```r
# Is this result zero?
result4 - x4 < 1e-10
```

```
##      [,1]
## [1,] TRUE
## [2,] TRUE
## [3,] TRUE
```

```r
# Normalize this result
sumsq_x4 = sum(x4^2)
sumsq_x4
```

```
## [1] 13
```

```r
x4_norm = x4 / sqrt(sumsq_x4)
x4_norm
```

```
## [1] 0.55470 0.83205 0.00000
```

Normalized Eigenvector for lambda = 4:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{13}} = \begin{bmatrix} .55470 \\ .83205 \\ 0 \end{bmatrix}$$

Eigenvector for $\lambda = 1$:

$$\begin{bmatrix} 1-1 & 2 & 3 \\ 0 & 4-1 & 5 \\ 0 & 0 & 6-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{cases} 0x_1 + 2x_2 + 3x_3 = 0 \\ 0x_1 + 3x_2 + 5x_3 = 0 \\ 0x_1 + 0x_2 + 5x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = 0 \\ x_2 = 0 \\ x_1 = 1 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Check:

```
# Eigenvector for lambda = 1
x1 = c(1,0,0)
x1
```

```
## [1] 1 0 0
```

```
# Ax / 1
result1 = (A %*% x1) / 1
result1
```

```
##      [,1]
## [1,]    1
## [2,]    0
## [3,]    0
```

```
# Ax - 1x
result1 - x1
```

```
##      [,1]
## [1,]    0
## [2,]    0
## [3,]    0
```

```
# Is this result zero?
result1 - x1 < 1e-10
```

```
##      [,1]
## [1,] TRUE
## [2,] TRUE
## [3,] TRUE
```

```
# Normalize this result
sumsq_x1 = sum(x1^2)
sumsq_x1
```

```
## [1] 1
```

```r
x1_norm = x1 / sqrt(sumsq_x1)
x1_norm
```

```
## [1] 1 0 0
```

My eigenvectors

```r
myeigvecs = cbind(x6_norm, x4_norm, x1_norm)
myeigvecs
```

```
##       x6_norm x4_norm x1_norm
## [1,] 0.510841 0.55470       1
## [2,] 0.798189 0.83205       0
## [3,] 0.319275 0.00000       0
```

My normalized eigenvectors (each column): $\begin{bmatrix} 0.510840685451281 & 0.554700196225229 & 1 \\ 0.798188571017626 & 0.832050294337844 & 0 \\ 0.31927542840705 & 0 & 0 \end{bmatrix}$

Check results vs. eigen() function

```
result = eigen(A)
eigvals = result$values
eigvecs = result$vectors
```

$$Eigenvalues : \lambda = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

$$Eigenvectors : x = \begin{bmatrix} 0.510840685451281 & 0.554700196225229 & 1 \\ 0.798188571017626 & 0.832050294337844 & 0 \\ 0.31927542840705 & 0 & 0 \end{bmatrix}$$

```
# Check if Ax[i] = lambda[i] x[i]
lhs = rhs = difference = rep.int(list(NULL),3)
lhstext = rhstext = difftext = rep.int(list(NULL),3)

#whicheig = 1
for (whicheig in 1:3) {

lhstext[[whicheig]] = paste0("A x_{",whicheig,"}")
lhs[[whicheig]] = A %*% eigvecs[,whicheig]

rhstext[[whicheig]] = paste0(eigvals[whicheig]," x_{",whicheig,"}")
rhs[[whicheig]] = as.matrix(eigvals[whicheig] * eigvecs[,whicheig])

difference[[whicheig]] = lhs[[whicheig]] - rhs[[whicheig]]

}
```

$$Checking \quad Ax_1 = \lambda_1 x_1 \quad : \quad \lambda_1 = 6 \quad ; \quad x_1 = \begin{bmatrix} 0.510840685451281 \\ 0.798188571017626 \\ 0.31927542840705 \end{bmatrix}$$

$$Ax_1 = \begin{bmatrix} 3.06504411270768 \\ 4.78913142610576 \\ 1.9156525704423 \end{bmatrix} \quad ; \quad 6x_1 = \begin{bmatrix} 3.06504411270768 \\ 4.78913142610576 \\ 1.9156525704423 \end{bmatrix} \quad ; \quad diff = \begin{bmatrix} -0.000000000000000444089209850063 \\ 0 \\ 0 \end{bmatrix}$$

$$Checking \quad Ax_2 = \lambda_2 x_2 \quad : \quad \lambda_2 = 4 \quad ; \quad x_2 = \begin{bmatrix} 0.554700196225229 \\ 0.832050294337844 \\ 0 \end{bmatrix}$$

$$Ax_2 = \begin{bmatrix} 2.21880078490092 \\ 3.32820117735137 \\ 0 \end{bmatrix} \quad ; \quad 4x_2 = \begin{bmatrix} 2.21880078490092 \\ 3.32820117735137 \\ 0 \end{bmatrix} \quad ; \quad diff = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Checking \quad Ax_3 = \lambda_3 x_3 \quad : \quad \lambda_3 = 1 \quad ; \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Ax_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad 1x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad diff = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Comparing my eigenvectors with those from eigen() :

```r
### my eigenvectors
myeigvecs
```

```
##       x6_norm x4_norm x1_norm
## [1,] 0.510841 0.55470       1
## [2,] 0.798189 0.83205       0
## [3,] 0.319275 0.00000       0
```

```r
### eigvecs from eigen()
eigvecs
```

```
##          [,1]    [,2] [,3]
## [1,] 0.510841 0.55470    1
## [2,] 0.798189 0.83205    0
## [3,] 0.319275 0.00000    0
```

```r
### difference
myeigvecs - eigvecs
```

```
##                       x6_norm x4_norm x1_norm
## [1,] -0.000000000000000111022       0       0
## [2,]  0.000000000000000000000       0       0
## [3,]  0.000000000000000000000       0       0
```

```r
### Equal to zero?
myeigvecs - eigvecs < 1e-10
```

```
##      x6_norm x4_norm x1_norm
## [1,]    TRUE    TRUE    TRUE
## [2,]    TRUE    TRUE    TRUE
## [3,]    TRUE    TRUE    TRUE
```

Please show your work using an R-markdown document. Please name your assignment submission with your first initial and last name.