605-HW13-Calculus

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Contents

HW13 - Calculus

1. Use integration by substitution to solve the integral $\int 4e^{-7x}dx$.

F(x) $= \int 4e^{-7x}dx$.

Let $u = -7x$.
Then $du = -7dx$, so $dx = \frac{du}{-7}$.

So,

$$\begin{aligned}\int 4e^{-7x}dx &= \int \frac{-4}{7}e^{u}du \\ &= \frac{-4}{7}e^{u} + C \\ &= \frac{-4}{7}e^{-7x} + C\end{aligned}$$

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2. Biologists are treating a pond contaminated with bacteria.

The level of contamination is changing at a rate of $\frac{dN}{dt} = -\frac{3150}{t^4} - 220$ bacteria per cubic centimeter per day, where t is the number of days since treatment began.

Find a function $N(t)$ to estimate the level of contamination if the level after 1 day was 6530 bacteria per cubic centimeter.

$$\begin{aligned}
N(t) - N(0) &= \int_0^t \left(\frac{-3150}{s^4} - 220 \right) ds \\
&= \int_0^t \left(-3150 s^{-4} - 220 \right) ds \\
&= \int_0^t -3150 \cdot s^{-4} ds - \int_0^s 220 ds \\
&= -3150 \int_0^t s^{-4} ds - 220 \int_0^t ds \\
&= \left[-3150 \frac{-1}{3} s^{-3} - 220 s \right]_{s=0}^{s=t} \\
&= \frac{3150}{3} t^{-3} - 220t - 0 + 0 \\
N(t) - N(0) &= \frac{1050}{t^3} - 220t
\end{aligned}$$

We are given an initial condition: $N(t = 1) = 6530$.

So,

$$\begin{aligned}
N(1) - N(0) &= \frac{1050}{1^3} - 220 \cdot 1 = 1050 - 220 = 830 \\
6530 - N(0) &= 830 \\
N(0) &= 6530 - 830 = 5700
\end{aligned}$$

Therefore, the solution is $N(t) = \frac{1050}{t^3} - 220t + 5700$.

Note that the solution is not defined at $t = 0$.
Note also that the first term will become asymptotically zero, so the reduction in bacteria will soon become linear at -220 per cubic centimeter per day. At day 26, the amount of bacteria would become negative, which is of course not feasible . . .

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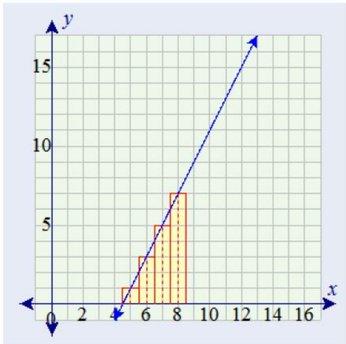
3. Find the total area of the red rectangles in the figure below, where the equation of the line is $f(x) = 2x - 9$.



By inspection, we can see that the area is 1+3+5+7 = 16 .

Using calculus, we compute as follows:

$$\int_{4.5}^{8.5} (2x - 9)dx = [x^2 - 9x]_{x=\frac{9}{2}}^{x=\frac{17}{2}}$$

$$= \left(\frac{17}{2}\right)^2 - \frac{9 \cdot 17}{2} - \left(\frac{9}{2}\right)^2 + \frac{9 \cdot 9}{2}$$

$$= \frac{289 - 81}{4} - \frac{153 - 81}{2}$$

$$= \frac{208}{4} - \frac{72}{2}$$

$$= 52 - 36$$

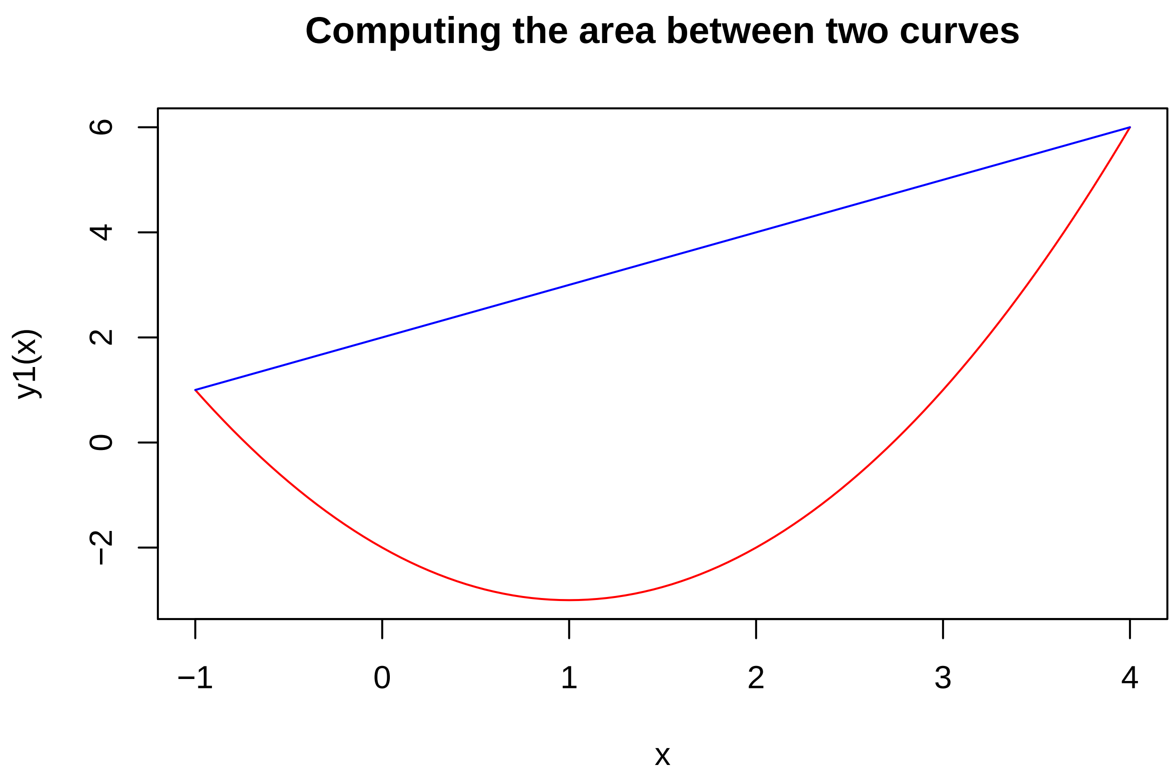
$$= 16$$

4. Find the area of the region bounded by the graphs of the given equations: $y = x^2 - 2x - 2$, $y = x + 2$

First, let's determine where these curves intersect:

$$\begin{aligned} x^2 - 2x - 2 &= x + 2 \\ x^2 - 3x - 4 &= 0 \\ (x+1)(x-4) &= 0 \\ x &\in \{-1, 4\} \end{aligned}$$

```r
y1 <- function(x) x^2 - 2*x -2
y2 <- function(x) x + 2
curve(expr = y1(x), from = -1, to = 4, col="red")
curve(expr = y2(x), from = -1, to = 4, col="blue", add=TRUE)
title(main = "Computing the area between two curves")
```



So, we clearly want to compute the following quantity:

$$
\begin{aligned}
\int_{-1}^{4} (x+2) - (x^2 - 2x - 2) &= \int_{-1}^{4} (-x^2 + 3x + 4)dx \\
&= -\frac{x^3}{3} + \frac{3x^2}{2} + 4x\Big]_{-1}^{4} \\
&= -\frac{4^3}{3} + \frac{3 \cdot 4^2}{2} + 4 \cdot 4 - \left(-\frac{(-1)^3}{3} + \frac{3 \cdot (-1)^2}{2} + 4 \cdot (-1)\right) \\
&= -\frac{64}{3} + \frac{48}{2} + 16 - \left(\frac{1}{3} + \frac{3}{2} - 4\right) \\
&= -\frac{65}{3} + \frac{45}{2} + 20 \\
&= -\frac{130}{6} + \frac{135}{6} + 20 \\
&= 20 + \frac{5}{6} \\
&= 20.8333333
\end{aligned}
$$

We can double check using the "integrate" function:

```
integrate(y2, lower = -1, upper = 4)$value - integrate(y1, lower = -1, upper = 4)$value
```

```
## [1] 20.8333333
```

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5. A beauty supply store expects to sell 110 flat irons during the next year.

It costs \$3.75 to store one flat iron for one year.
There is a fixed cost of \$8.25 for each order.
Find the lot size and the number of orders per year that will minimize inventory costs.

We will make the following assumptions:

- We will order 110 flat irons over the next year, and assume we will sell each one at a constant rate, i.e., one iron sold every $\frac{365}{110} \approx 3.32$ days.
- We are not concerned with the purchase or sales price for the irons, as no information is provided.
- During the year, we will place $numOrders$ orders, where each order will be placed for $lotSize = \frac{110}{numOrders}$ irons .
- This means that the fixed cost associated with placing the orders will be $8.25 * numOrders$ dollars .
- On average, the number of irons in storage will be $\frac{lotsize}{2}$.
- The cost associated with maintaining such storage (over the year) will be $3.75 * \frac{lotSize}{2}$ dollars.

Thus, the cost function that we seek to minimize is:

$$\begin{aligned} inventoryCost &= 8.25 * numOrders + 3.75 * \frac{lotSize}{2} \\ &= 8.25 * numOrders + 3.75 * \frac{110}{2 \cdot numOrders} \\ &= 8.25 * numOrders + \frac{3.75 * 55}{numOrders} \\ inventoryCost &= 8.25 * numOrders + \frac{206.25}{numOrders} \end{aligned}$$

To find the extremum, $\frac{d(inventoryCost)}{d(numOrders)} = 8.25 - \frac{206.25}{numOrders^2} = 0$
So, $8.25 = \frac{206.25}{numOrders^2}$, or $(numOrders)^2 = \frac{206.25}{8.25} = 25$.

Therefore, $numOrders = 5$ and $lotSize = \frac{110}{numOrders} = \frac{110}{5} = 22$.

Therefore, we should place 5 orders each year, with 22 flatirons in each order.

The cost associated with submitting the orders will be $8.25 * 5 = 41.25$ dollars,
and the cost associated with holding an average of $\frac{22}{2} = 11$ irons in inventory will also be $3.75 * 11 = 41.25$ dollars,
for a total inventory cost of \$ 82.50 .

6. Use integration by parts to solve the integral $\int ln(9x) \cdot x^6 dx$.

Let $u = ln(9x)$ and $dv = x^6 dx$.
Then $du = \frac{9dx}{9x} = \frac{dx}{x}$ and $v = \frac{x^7}{7}$.

The formula for integration by parts:

$$
\begin{aligned}
\int u \cdot dv &= u \cdot v - \int v \cdot du \\
\int ln(9x) \cdot x^6 dx &= ln(9x) \cdot \frac{x^7}{7} - \int \frac{x^7}{7} \frac{dx}{x} \\
&= \frac{x^7 \cdot ln(9x)}{7} - \int \frac{x^6 dx}{7} \\
&= \frac{x^7 \cdot ln(9x)}{7} - \frac{x^7 dx}{7 \cdot 7} + C \\
&= \frac{x^7}{7} \left[ln(9x) + \frac{1}{7} \right] + C
\end{aligned}
$$

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7. Determine whether $f(x) = \frac{1}{6x}$ is a probability density function on the interval $[1, e^6]$.

If not, determine the value of the definite integral $\int\limits_1^{e^6} \frac{1}{6x} dx$.

For the function to be a PDF on the interval,

- the value must always be non-negative, and
- the definite integral needs to equal 1.

For the first requirement:
$f(x) = \frac{1}{6x} > 0$ for $x > 0$, so this holds for $x \in [1, e^6]$.

For the second requirement:

$$\begin{aligned} \int\limits_1^{e^6} \frac{1}{6x} dx &= \frac{ln(x)}{6} \Bigg]_1^{e^6} \\ &= \frac{ln(e^6) - ln(1)}{6} \\ &= \frac{6-0}{6} \\ &= 1 \end{aligned}$$

Therefore, the function is a PDF on the interval.