



An Explicit Formula for the k th Prime Number

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Source: *Mathematics Magazine*, Vol. 48, No. 4 (Sep., 1975), pp. 230-232

Published by: Mathematical Association of America

Stable URL: <http://www.jstor.org/stable/2690354>

Accessed: 03/08/2009 07:43

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Let the join of C to the center of circle \mathcal{C} intersect this circle in P and Q . Then C is a fixed point on the fixed segment PQ , or PQ produced, where P moves on OP and Q moves on the perpendicular line OQ . Hence C describes an ellipse with principal axes OP and OQ . If triangle ABC is such that C lies either on OP or OQ , then C moves on that line.

Reference

1. Dan Pedoe, *Geometry and the Liberal Arts*, Penguin Books, London, in the press.

AN EXPLICIT FORMULA FOR THE k th PRIME NUMBER

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This paper will develop an explicit formula for the k th prime, where k is any positive integer. The formula will be elementary, finite, and dependent only upon the choice of k .

First consider the function

$$g(n) = \sum_{i=1}^{n-1} \left[\frac{\left[\frac{n}{i} \right]}{\frac{n}{i}} \right]$$

where $[x]$ in all cases denotes the greatest integer $\leq x$, and $n \geq 2$. Now the terms in this finite sum are equal either to 0 or to 1 since for any $x > 0$,

$$0 < \frac{[x]}{x} \leq 1 \Rightarrow \left[\frac{[x]}{x} \right] = 0 \quad \text{or} \quad 1.$$

Therefore, $g(n) = 1$ if and only if there exists exactly one $i \in \{1, 2, 3, \dots, n-1\}$ such that

$$\left[\frac{\left[\frac{n}{i} \right]}{\frac{n}{i}} \right] = 1 \Leftrightarrow \left[\frac{n}{i} \right] = \frac{n}{i} \Leftrightarrow i \mid n;$$

but since $i = 1$ divides n for all positive integers n , then $g(n) = 1$ if and only if $i = 1$ is the only value of $i \in \{1, 2, 3, \dots, n-1\}$ such that i divides n . But this means that n is prime. Thus, $g(n) = 1$ if and only if n is prime. Now if n is composite, then there exists some $i \in \{2, 3, 4, \dots, n-1\}$ such that i divides n , and this implies that $g(n) > 1$. Therefore,

$$g(n) \begin{cases} = 1, & n \text{ prime} \\ > 1, & n, \text{ composite.} \end{cases}$$

Now let $f(n) = [1/(g(n))]$, then it is clear that for $n \geq 2$,

$$f(n) = \begin{cases} 1, & n \text{ prime} \\ 0, & n \text{ composite.} \end{cases}$$

Note that $f(n)$ is the characteristic function of the set of primes.

Consider for $m \geq 2$,

$$\pi(m) = \sum_{n=2}^m f(n);$$

since $f(n)$ is the characteristic function of the set of primes, then $\pi(m)$ is the number of primes less than or equal to m , and thus for $m \geq 2$,

$$f(m)\pi(m) = \begin{cases} \text{number of primes } \leq m, & m \text{ prime,} \\ 0, & m \text{ composite.} \end{cases}$$

Let k be any positive integer, and the following equality holds:

$$\left[\frac{1}{1 + |k - f(m)\pi(m)|} \right] m = \begin{cases} m, & m \text{ the } k \text{th prime,} \\ 0, & \text{otherwise.} \end{cases}$$

This is true since the above function is equal to m if and only if $k = f(m)\pi(m)$, which implies that m is prime and there are k primes less than or equal to m ; but this means that m is the k th prime. If m is not the k th prime or if m is composite, then $|k - f(m)\pi(m)| > 0$, and the above function equals 0.

Now by Bertrand's postulate as proved by Hardy and Wright [1], there exists at least one prime p such that for all positive integers n , $n < p \leq 2n$. Therefore, it follows that if p_k denotes the k th prime, then for all positive integers k , $p_k < p_{k+1} \leq 2p_k$ and this supplies the induction step to prove that for all positive integers k , $2^k \geq p_k$. Now with this result it follows that for all $k \geq 1$, the k th prime is given by the formula

$$p_k = \sum_{m=2}^{2^k} \left[\frac{1}{1 + |k - f(m)\pi(m)|} \right] m$$

since summing to 2^k guarantees that we have summed past the k th prime. Now rewriting in terms of the original expressions, we have:

$$p_k = \sum_{m=2}^{2^k} \left[\frac{1}{1 + \left| k - \sum_{i=1}^{m-1} \left[\frac{\left[\frac{m}{i} \right]}{\frac{m}{i}} \right] \sum_{n=2}^m \sum_{i=1}^{n-1} \left[\frac{\left[\frac{n}{i} \right]}{\frac{n}{i}} \right] \right|} \right] m.$$

Note that this method can be generalized to yield an expression for the k th smallest element of an arbitrary infinite subset A of the positive integers. The

finiteness of this extended formula will depend not only upon the possibility of writing a finite expression for the characteristic function of the set A , but also upon the existence of a finitely expressible function $h(k)$ such that $h(k) \geq$ the k th smallest element of A for all $k \geq 1$.

Reference

1. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, London, 1960, pp. 343-4.

THE GENERAL CAYLEY-HAMILTON THEOREM VIA THE EASIEST REAL CASE

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Let A be a square matrix with elements in a field, and let $G(\lambda)$ denote the characteristic polynomial $\det(\lambda I - A)$. When A can be reduced to diagonal form by a similarity transformation, the proof that A satisfies its characteristic equation is particularly transparent, since each eigenspace is annihilated by a factor $A - \lambda_0$ in $G(A)$, where λ_0 is the characteristic value of the eigenspace. As it stands, this method is capable of giving the result only when a complete set of eigenspaces for A can be found, and general proofs usually depend either on canonical forms or on the use of adjoint matrices.

The proof of the Cayley-Hamilton theorem that we give here demonstrates how the simplest and most transparent method of diagonalization can be used to provide a proof in the general case of a square matrix A with elements that belong to an arbitrary commutative ring R . Apart from some elementary algebra, the only result that is required is the easy proof of the theorem for real matrices which have real and distinct characteristic roots.

Suppose that A is the $n \times n$ matrix $[a_{ij}]$, where $a_{ij} \in R$ for $1 \leq i, j \leq n$. Now let $X = [x_{ij}]$, $1 \leq i, j \leq n$, where each x_{ij} is an *indeterminate*, and let P denote the domain of polynomials which are generated over the integers by $\{x_{ij}\}$ and contain no 'constant' terms. A ring homomorphism $\phi: P \rightarrow R$ may be defined which 'evaluates' each polynomial at the point $x_{11} = a_{11}$, $x_{12} = a_{12}$, \dots , $x_{nn} = a_{nn}$. Thus if $f(x_{11}, x_{12}, \dots, x_{nn}) \in P$, this is mapped to $f(a_{11}, a_{12}, \dots, a_{nn}) \in R$. The characteristic polynomials of X and A are defined over P and R as follows:

$$F(\lambda) = \det(\lambda I - X) = \lambda^n - p_1 \lambda^{n-1} + \dots + (-1)^n p_n \quad \text{and}$$

$$G(\lambda) = \det(\lambda I - A) = \lambda^n - r_1 \lambda^{n-1} + \dots + (-1)^n r_n,$$

where each $p_j \in P$ and $r_j \in R$. If $f_{kl} \in P$ and $g_{kl} \in R$ denote the entries in the kl -position of the matrices $F(X)$ and $G(A)$, it follows from the relations $r_j = \phi(p_j)$ that $g_{kl} = \phi(f_{kl})$, and hence that $G(A)$ is zero if $F(X)$ is zero. Thus, to