

EAS 595 PROB HW2

Proof: Ehsan Esfahani

Abhishek Kumar
UB# 50910561. Chapter 2 Problems.

3. $P(\text{Fischer wins the match})$ follows a geometric distribution (sum),
hence

$$\begin{aligned}
 @ P(\text{Fischer wins the match}) &= \sum_{n=1}^{10} \underbrace{(0.4)}_{\text{wins}} \underbrace{(0.3)^{n-1}}_{\text{draws}} \\
 &= \frac{0.4(1 - 0.3^9)}{1 - 0.3} = \frac{0.4}{0.7} \times (1 - 0.3^9) \\
 &= 0.5714173
 \end{aligned}$$

(b) For $n < 10$, n matches are played if $(n-1)$ matches are drawn and 1 match is won by either Fischer or Spassky. And $n=10$ matches are played only when 9 matches are drawn and does not depend on the result of the 10th match.

$$P_N(n) = P_N(N=n) = \begin{cases} (0.7)(0.3)^{n-1} & n=1, 2, \dots, 8, 9, \\ (0.3)^9 & n=10, \\ 0, & \text{otherwise} \end{cases}$$

Q18 X will be equiprobable for $2^a, 2^{a+1}, \dots, 2^b$, so the PMF of X
will be

$$P_X(x) = \begin{cases} \frac{1}{b-a+1} & x = 2^k \text{ and } a \leq k \leq b, \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore E[X] = \sum_{k=a}^b \frac{1}{b-a+1} \cdot 2^k = \frac{1}{b-a+1} \cdot \sum_{k=a}^b 2^k$$

$$= \frac{1}{b-a+1} \left(2^a + 2^{a+1} + 2^{a+2} + \dots + 2^b \right)$$

$$= \frac{2^a}{b-a+1} \left(1 + 2 + 2^2 + \dots + 2^{b-a} \right)$$

$$= \frac{2^a}{b-a+1} \times \frac{1 \cdot (1 - 2^{b-a+1})}{(1-2)} = \frac{2^a}{b-a+1} \times (2^{b-a+1} - 1)$$

$$\therefore E[X^2] = \sum_{k=a}^b \frac{1}{b-a+1} \cdot 2^{2k} = \frac{1}{b-a+1} \sum_{k=a}^b (4^a + 4^{a+1} + \dots + 4^b)$$

$$= \frac{1}{b-a+1} \cdot 4^a (1 + 4 + 4^2 + \dots + 4^{b-a})$$

$$= \frac{4^a}{b-a+1} \times \frac{(1 - 4^{b-a+1})}{1-4} = \frac{4^a}{b-a+1} \times \frac{(4^{b-a+1} - 1)}{3}$$

(2)

$$\therefore \text{Var}(x) = E[x^2] - (E[x])^2$$

$$= \left[\frac{4^a}{b-a+1} \times \frac{(4^{b-a+1}-1)}{3} \right] - \left[\frac{2^a}{b-a+1} \times (2^{b-a+1}-1) \right]^2$$

(21) For $P(\text{tail}) = p$, the PMF of the count n , the number

of tosses till the 1st tail appears is a geometric distribution

Hence,

$$P_N(n=n) = p(1-p)^{n-1}$$

Here, for a fair coin, $p = \frac{1}{2}$

$$\therefore P_N(n=n) = \frac{1}{2} \times \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n$$

And the amount received at count n is 2^n

\therefore The expected amount to be received is $= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \cdot 2^n$

$$= \sum_{n=1}^{\infty} 1$$

Oh!! Infinite. I would pay as much as he
asked and I had at that time. :)

(36) To Show,

$$P_{x,y,z}(x,y,z) = P_x(x) P_{y|x}(y|x) P_{z|y,x}(z|y,x)$$

⇒ We can see this as a chain rule where 1st x happens with prob = $P_x(x)$, then y happens given x has happened with prob $P_{y|x}(y|x)$, and likewise further.

@ $P_{x,y,z}(x,y,z) = P(x=x) \cdot \underbrace{P(y=y, z=z | x=x)}_{y,z \text{ happens given } x \text{ has occurred}}$

$$= P(x=x) \cdot P(y=y | x=x) \cdot P(z=z | x=x, y=y)$$
$$= P_x(x) \cdot P_{y|x}(y|x) \cdot P_{z|y,x}(z|y,x)$$

(b) Multiplication rule states that:

Assuming that all of the conditioning events

have positive probability, we have

$$P(\bigcap_{i=1}^n A_i) = P(A_1) P(A_2 | A_1) P(A_3 | A_1 \cap A_2) \cdots P(A_n | \bigcap_{i=1}^{n-1} A_i)$$

which can be viewed as one event occurring after other events have occurred and its multiplication gives the final event that has occurred.

(3)

Similar to the multiplication rule, we have here:

$$P(X=x) P(Y=y | X=x) P(Z=z | X=x, Y=y) =$$

$$P_{X,Y,Z}(x,y,z)$$

and can be seen as a special case of multiplication rule.

(C) The generalization is:

$$P_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) = P_{X_1}(x_1) \cdot P_{X_2 | X_1}(x_2 | x_1) \cdots$$

$$\cdots P_{X_n | X_1, X_2, \dots, X_{n-1}}(x_n | x_1, x_2, \dots, x_{n-1})$$

2. Chapter 3.

(B) a). From total probability theorem, we have

$$\begin{aligned} F_X(x) &= P(X \leq x) = p P(Y \leq x) + (1-p) P(Z \leq x) \\ &= p F_Y(x) + (1-p) F_Z(x) \end{aligned}$$

∴ Differentiating both sides, we get

$$f_X(x) = p f_Y(x) + (1-p) f_Z(x)$$

$$b) f_x(x) = \begin{cases} pd e^{dx} & \text{if } x < 0 \\ (1-p)d e^{-dx} & \text{if } x \geq 0 \end{cases}$$

for $x < 0$, we have

$$\begin{aligned} F_x(x) &= P(X < x) \\ &= \int_{-\infty}^x pd e^{dx} = \frac{pd}{d} e^{dx} \Big|_{-\infty}^x \\ &= Pe^{dx} \end{aligned}$$

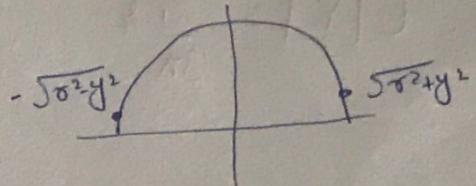
and for $x \geq 0$, we have

$$\begin{aligned} F_x(x) &= P(X < x) = \int_{-\infty}^x f_x(x) dx = \int_{-\infty}^0 f_x(x) dx + \int_0^x f_x(x) dx \\ &= \int_{-\infty}^0 pd e^{dx} + \int_0^x (1-p)d e^{-dx} \\ &= P + \frac{(1-p)d}{d} e^{-dx} \Big|_0^x \\ &= P + (P-1)(e^{-dx} - 1) = P + Pe^{-dx} - P - e^{-dx} + 1 \\ &= 1 - e^{-dx}(1-p) \end{aligned}$$

$$\therefore F_x(x) = \begin{cases} Pe^{dx} & x < 0 \\ 1 - e^{-dx}(1-p) & x \geq 0 \end{cases}$$

(15) a) Since the area under the semicircle is $\frac{\pi r^2}{2}$ and the

the point has a uniform PDF, so



$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{\pi r^2} & \text{for } x, y \text{ in the semicircle} \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{b)} f_Y(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{2}{\pi r^2} dx$$

$$= 2 \times 2 \int_0^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2} dx = \frac{4}{\pi r^2} \cdot \sqrt{r^2-y^2}$$

$$\therefore f_Y(y) = \begin{cases} \frac{4}{\pi r^2} \sqrt{r^2-y^2} & 0 \leq y \leq r, \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore E[Y] = \int_0^r y \cdot \frac{4}{\pi r^2} \sqrt{r^2-y^2} dy = \frac{4}{\pi r^2} \int_0^r \sqrt{r^2-y^2} y dy$$

$$\text{let } r^2 - y^2 = z^2 \Rightarrow -2y dy = 2z dz$$

$$\therefore E[Y] = \frac{4}{\pi r^2} \int_r^0 z (-2z) dz = \frac{4}{\pi r^2} \int_0^r z^2 dz$$

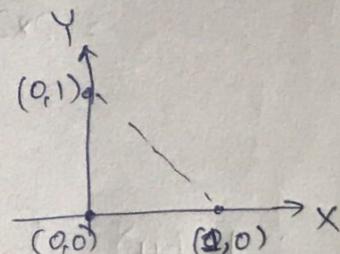
$$\therefore E[Y] = \frac{4r}{3\pi}$$

c) Without using marginal PDF of Y

$$\begin{aligned} E[Y] &= \int_0^r \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{x,y}(x,y) \cdot y \, dx \, dy \\ &= \int_0^r \frac{4}{\pi r^2} \sqrt{r^2-y^2} \cdot dy \\ &= \frac{4r}{3\pi} \end{aligned}$$

- ②3) @ Area under the triangle
is $\frac{1}{2}$ and the point has
uniform PDF over the
triangle, so

$$f_{x,y}(x,y) = \begin{cases} 2 & \text{over the } \Delta \\ 0 & \text{otherwise} \end{cases}$$



(5)

$$\textcircled{b} \quad f_Y(y) = \int_0^{1-y} 2x dy \quad \left| \begin{array}{l} y = -x + 1 \\ y \leq 1 \end{array} \right.$$

$$f_Y(y) = \begin{cases} \frac{(1-y)2}{2} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\textcircled{c} \quad f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}$$

$$= \begin{cases} \frac{1}{1-y} & 0 \leq x \leq 1-y \\ 0 & \text{otherwise.} \end{cases}$$

$$\textcircled{d} \quad E[X|Y=y] = \int_0^{1-y} x \cdot \underbrace{\frac{1}{1-y}}_{f_{X|Y}(x|y)} dx = \frac{1}{2(1-y)} x^2 \Big|_0^{1-y} \\ = \frac{(1-y)^2}{2(1-y)} = \frac{1-y}{2}$$

$$\therefore E[X|Y=y] = \begin{cases} \frac{1-y}{2} & 0 \leq y \leq 1 \quad \& y \neq 1 \end{cases}$$

Using total Expectation theorem, we have

$$E[X] = \int_0^1 E[X|Y=y] \cdot \cancel{f_Y(y)} dy \\ = \int_0^1 \frac{1-y}{2} \cdot f_Y(y) dy$$

$$\Rightarrow E[X] = \frac{1}{2} - \frac{1}{2} \int_0^1 y \cdot f_Y(y) dy$$

$$= \frac{1}{2} - \frac{1}{2} E[Y]$$

② Due to symmetry, we must have $E[X] = E[Y]$

\therefore We should have

$$E[X] = \frac{1}{2} - \frac{1}{2} E[X]$$

$$\Rightarrow \frac{3}{2} E[X] = \frac{1}{2} \Rightarrow E[X] = \frac{1}{3}$$

③ Let X be the event that a toss results in heads.

$$\begin{aligned} @ \therefore P(X) &= \int_0^1 P(X|P=p) \cdot f_P(p) dp \\ &= \int_0^1 p \cdot pe^p dp = \int_0^1 p^2 e^p dp \\ &= p^2 e^p \Big|_0^1 - \int_0^1 2p e^p dp \\ &= e - \left[2pe^p \Big|_0^1 - \int_0^1 2e^p dp \right] \\ &= e - [2e - (2e^p \Big|_0^1)] = e - [2e - (2e - 2)] \\ &= e - 2e + 2e - 2 = e - 2 \end{aligned}$$

⑤ From Bayes rule, we have

$$f_{P|X}(p) = \frac{P(p, X)}{P(X)}$$

$$f_{P|X}(p) = \frac{P(X | P=p) f_p(p)}{P(X)} \quad \text{for } 0 \leq p \leq 1$$

$$= \frac{p e^p \cdot p}{e^{-2}} = \frac{p^2 e^p}{e^{-2}}$$

$$\therefore f_{P|X}(p) = \begin{cases} \frac{p^2 e^p}{e^{-2}} & 0 \leq p \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

⑥ Let X_1 = head at first toss &
 X_2 = head at 2nd toss

$$\therefore P(X_2 | X_1) = \int_0^1 P(X_2 | P=p, X_1) f_{P|X_1}(p) dp$$

As X_1 & X_2 are independent

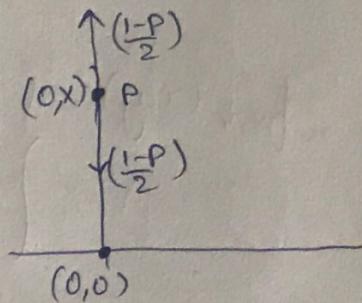
$$P(X_2 | X_1) = \int_0^1 P(X_2 | P=p) f_{P|X_1}(p) dp$$

$$= \int_0^1 p \cdot \frac{p^2 e^p}{e^{-2}} dp$$

$$\begin{aligned}
 &= \frac{1}{e-2} \int_0^1 p^3 e^p dp \\
 &= \frac{1}{e-2} \left\{ p^3 e^p \Big|_0^1 - \int_0^1 3p^2 e^p dp \right\} \\
 &= \frac{1}{e-2} \left\{ e - \left[3p^2 e^p \Big|_0^1 - \int_0^1 6p e^p dp \right] \right\} \\
 &= \frac{1}{e-2} \left\{ e - \left[3e - \left\{ 6e - 6e^p \Big|_0^1 \right\} - \int_0^1 6e^p dp \right] \right\} \\
 &= \frac{1}{e-2} \left\{ e - \left[3e - \left\{ 6e - 6e^1 + 6e^0 \right\} \right] \right\} \\
 &= \frac{1}{e-2} \left\{ e - \left[3e - \left\{ 6e - (6e - 6) \right\} \right] \right\} \\
 &= \frac{1}{e-2} \left\{ e - 3e + \left\{ 6e - 6e + 6 \right\} \right\} \\
 &= \frac{-2e + 6}{e-2} \\
 &= \frac{6-2e}{e-2} \approx 0.786
 \end{aligned}$$

OTHER QUESTIONS.

(1) Let 't' be the expected time it takes the police to capture the suspect. And let the position of police be $(0,0)$ and let the position (at $t=0$) of the suspect be $(0,x)$. Let's calculate



the expected position of the suspect after time t :

$$x + \left\{ P \cdot 0 \right\} + \left\{ \left(\frac{1-p}{2} \right) t \cdot (+1) \right\} + \left\{ \left(\frac{1-p}{2} \right) t \cdot (-1) \right\}$$

$$= x$$

∴ At time $t = t$, the expected position of the suspect is $(0,x)$.

And the police move one unit towards the suspect ~~at~~ each second

∴ The expected time by which police catches the suspect is $t = x$.

② ~~f*~~ The area of the triangle is $\frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{4}$ and
 x, y have uniform distribution over the triangle, therefore

$$f_{x,y}(x,y) = \begin{cases} 4 & \text{over the triangle,} \\ 0 & \text{otherwise,} \end{cases}$$

$$f_x(y) = \int_y^{1-y} 4 dx \quad 0 \leq y \leq \frac{1}{2}$$

$$= 4(1-y-y) = \left\{ (1) + \left(\frac{1}{2}\right) + (0 \cdot \frac{1}{2}) \right\} + y$$

$$f_x(y) = \begin{cases} 4(1-2y) & 0 \leq y \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

And $f_{x|Y} = \frac{f_{x,y}(x,y)}{f_x(y)} = \frac{4}{4(1-2y)} = \frac{1}{1-2y}$

$$f_{x|Y} = \begin{cases} \frac{1}{1-2y} & 0 \leq x \leq 1-y \\ 0 & \text{otherwise} \end{cases}$$