Taylor Polynomials

Worcester Polytechnic Institute
Department of Mathematical Sciences

Calculus Labs

Taylor Series

Definition

Let f be a function with derivatives of all orders throughout some interval containing a.
The **Taylor series** generated by f at $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots.$$

Taylor Polynomial

Definition

Let f be a function with derivatives of order n for $n = 1, 2, \ldots, N$ throughout some interval containing a. Then for any integer $K \in \{0, 1, 2, \ldots, N\}$ the **Taylor polynomial of order** K generated by f at $x = a$ is the polynomial

$$P_K(x) = \sum_{n=0}^{K} \frac{f^{(n)}(a)}{n!}(x-a)^n, \tag{1}$$

or

$$P_K(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(K)}(a)}{K!}(x-a)^K. \tag{2}$$

Example: Maclaurin Series for $\sin(bx)$

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Example

$f(x) = \sin(bx)$ has Maclaurin series:

$$\sin(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}$$

$$= u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \frac{u^9}{9!} - \dots \text{ for all } u \text{ in } (-\infty, \infty)$$

Substitute $u = bx$

$$\sin(bx) = \sum_{n=0}^{\infty} (-1)^n \frac{(bx)^{2n+1}}{(2n+1)!}$$

$$= bx - \frac{b^3x^3}{3!} + \frac{b^5x^5}{5!} - \frac{b^7x^7}{7!} + \frac{b^9x^9}{9!} - \dots$$

$$= bx - \frac{b^3}{3!}x^3 + \frac{b^5}{5!}x^5 - \frac{b^7}{7!}x^7 + \frac{b^9}{9!}x^9 - \dots$$

Maclaurin Polynomials for $\sin(bx)$

$a = 0$

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Example

$$\sin(bx) = \sum_{n=0}^{\infty} (-1)^n \frac{b^{2n+1}}{(2n+1)!} \cdot x^{2n+1}$$

$$= bx - \frac{b^3}{3!}x^3 + \frac{b^5}{5!}x^5 - \frac{b^7}{7!}x^7 + \frac{b^9}{9!}x^9 - \dots$$

$$T_1(x) = bx$$

$$T_3(x) = bx - \frac{b^3}{3!}x^3$$

$$T_5(x) = bx - \frac{b^3}{3!}x^3 + \frac{b^5}{5!}x^5$$

$$T_7(x) = bx - \frac{b^3}{3!}x^3 + \frac{b^5}{5!}x^5 - \frac{b^7}{7!}x^7$$

Maclaurin Polynomials for $\sin(bx)$

$$T_1(x) = bx$$

$$T_3(x) = bx - \frac{b^3}{3!}x^3$$

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Graph in Desmos: https://www.desmos.coms

Approximations to $\sin(2x)$ at $x = 1.5$

$$T_1(1.5) = 3$$

$$T_3(1.5) = -1.5$$

$$T_5(x) = 0.525$$

$$T_7(x) = 0.0911$$

$$\sin(2 \cdot 1.5) \approx 0.14112001$$

Errors of Approximations to $\sin(2x)$ at $x = 1.5$

$$|\text{Error}| = |\text{True Value} - \text{Approximation}| \tag{3}$$

Polynomial Approximation	Error
$T_1(1.5)$	$\lvert \sin(2 \cdot 1.5) - T_1(1.5) \rvert = 2.8589$
$T_3(1.5)$	$\lvert \sin(2 \cdot 1.5) - T_3(1.5) \rvert = 1.6411$
$T_5(1.5)$	$\lvert \sin(2 \cdot 1.5) - T_5(1.5) \rvert = 0.3839$
$T_7(1.5)$	$\lvert \sin(2 \cdot 1.5) - T_7(1.5) \rvert = 0.0500$