DXE_EMTR Assignment 1

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## Loading required package: ggplot2
## -- Attaching packages --------------------------------------- tidyverse 1.3.2 --
## v tibble  3.1.8      v dplyr   1.0.10
## v tidyr   1.2.1      v stringr 1.4.1
## v readr   2.1.3      v forcats 0.5.2
## v purrr   0.3.5
## -- Conflicts ------------------------------------------ tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
```

1 Regression basics

2 Maximum likelihood

We have $y_i \sim Bin(n_i, p_i)$ and $\ln(\frac{p_i}{1-p_i}) = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip}$. The likelihood function can be written as:

$$\mathbf{y} = \ln(\frac{\mathbf{p}}{1-\mathbf{p}}) \sim N(0, \sigma^2)$$

$$p(y|\mathbf{X}, \theta) = (y|\mathbf{X}, \beta, \sigma^2)$$

$$\mathbf{L}(\beta, \sigma^2|\mathbf{X}, \mathbf{y}) = f(\mathbf{y}|\mathbf{X}, \beta, \sigma^2) = \prod_{i=1}^{n} f(y_i|\mathbf{x_i}, \beta, \sigma^2)$$

and the maximum likelihood estimator is:

$$\hat{\beta} = \arg\max_{\beta} \mathbf{L}(\beta, \sigma^2|\mathbf{X}, \mathbf{y})$$

$$l(\beta, \sigma^2) = \ln(\mathbf{L}(\beta, \sigma^2|\mathbf{X}, \mathbf{y})) = \sum_{i=1}^{n} \ln(f(y_i|\mathbf{x}_i, \beta, \sigma^2))$$

Log-likelihood:

$$\begin{aligned}
l(\beta, \sigma^2) &= \sum_{i=1}^{n} \ln\left[\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(y_i - \beta^T \mathbf{x}_i)^2\right)\right] \\
&= \sum_{i=1}^{n} \frac{1}{2} \ln\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}(y_i - \beta^T \mathbf{x}_i)^2 \\
&= \frac{n}{2} \ln\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta^T \mathbf{x}_i)^2 \\
&= \frac{n}{2} \ln\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2} (\mathbf{y} - \mathbf{X}\beta)^T (\mathbf{y} - \mathbf{X}\beta)
\end{aligned}$$

Score function:

$$s(\beta) = \frac{\partial}{\partial \beta} l(\beta, \sigma^2) = -\frac{1}{2\sigma^2}(-2\mathbf{X}^T(\mathbf{y} - \mathbf{X}\beta))$$

Maximization of the log-likelihood:

$$-2\mathbf{X}^T(\mathbf{y} - \mathbf{X}\beta) = 0$$

$$-2\mathbf{X}^T\mathbf{y} + 2\mathbf{X}^T\mathbf{X}\beta = 0$$

$$\hat{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

Given that $\mathbf{y} = \ln(\frac{\mathbf{p}}{1-\mathbf{p}})$,

$$\hat{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\ln(\frac{\mathbf{p}}{1-\mathbf{p}})$$

```
#animate(p, fps = 25, duration = 20, end_pause = 100, renderer = gifski_renderer())
```

3 Bootstrap