HWO Answers

A. $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

1. $2A - B = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$

2. $||A|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

3. $\hat{A} = \dfrac{A}{||A||} = \dfrac{(1, 2, 3)}{\sqrt{14}} = \left(\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\right)$

4. Let α, β, γ be the angles of $\vec{A}$ with positive X, Y & Z axes respectively

$\cos\alpha = \dfrac{x}{|\vec{A}|} = \dfrac{1}{\sqrt{14}} \Rightarrow \cos\alpha = 0.267 \Rightarrow \alpha = \cos^{-1}(0.267) = 74.5^\circ$

$\cos\beta = \dfrac{y}{|\vec{A}|} = \dfrac{2}{\sqrt{14}} \Rightarrow \cos\beta = 0.5345 \Rightarrow \beta = \cos^{-1}(0.5345) = 57.7^\circ$

$\cos\gamma = \dfrac{z}{|\vec{A}|} = \dfrac{3}{\sqrt{14}} = 0.8017 \Rightarrow \gamma = \cos^{-1}(0.8017) = 36.7^\circ$

5. $A \cdot B = (1, 2, 3) \cdot (4, 5, 6) = 1 \times 4 + 2 \times 5 + 3 \times 6 = 4 + 10 + 18 = 32$

$B \cdot A = (4, 5, 6) \cdot (1, 2, 3) = 4 \times 1 + 5 \times 2 + 6 \times 3 = 4 + 10 + 18 = 32$

6. Angle between A & B: $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{||\vec{A}|| \, ||\vec{B}||}$

$= \dfrac{32}{\left(\sqrt{1^2 + 2^2 + 3^2}\right)\left(\sqrt{4^2 + 5^2 + 6^2}\right)} =$

$= \dfrac{32}{\sqrt{14} \times \sqrt{77}} = \dfrac{32}{\sqrt{1078}} = \dfrac{32}{32.833}$

$\Rightarrow \cos\theta = 0.9746 \Rightarrow \theta = 12.9^\circ$

7. Vector perpendicular to $\vec{A}$

Let the perpendicular vector be (x, y, z)

$(x, y, z) \cdot (1, 2, 3) = 0.$

$x + 2y + 3z = 0.$

If $x = 1$, $y = 1$

$1 + 2 \cdot 1 + 3 \cdot z = 0 \Rightarrow 3 + 3z = 0 \Rightarrow z = -1$

so, $(1, 1, -1)$ is a perpendicular vector to $(1, 2, 3)$

8. $A \times B$

$\cancel{(1,2,3) \times (4,5,6)}$

$A = 1i + 2j + 3k$ & $B = 4i + 5j + 6k$.

$i \times i = j \times j = k \times k = 0$

$i \times j = k$
$j \times k = i$
$k \times i = j$

$j \times i = -k$
$k \times j = -i$
$i \times k = -j$

$(i + 2j + 3k) \times (4i + 5j + 6k)$

$= (1\times4)(i \times i) + (1\times5)(i\times j) + (1\times6)(i\times k)$
$+ (2\times4)(j\times i) + (2\times5)(j\times j) + (2\times6)(j\times k)$
$+ (3\times4)(k\times i) + (3\times5)(k\times j) + (3\times6)(k\times k)$

$= 0 + 5k + 6(-j)$
$+ 8(-k) + 0 + 12(i)$
$+ 12(j) + 15(-i) + 0$

$= 5k - 6j - 8k + 12i + 12j - 15i = -3k + 6j - 3i$

$B \times A$

$= (4i + 5j + 6k) \times (i + 2j + 3k)$
$= (4\times1)(i\times i) + (4\times2)(i\times j) + (4\times3)(i\times k)$
$+ (5\times1)(j\times i) + (5\times2)(j\times j) + (5\times3)(j\times k)$
$+ (6\times1)(k\times i) + (6\times2)(k\times j) + (6\times3)(k\times k)$
$= 0 + 8k + 12(-j) + 5(-k) + 0 + 15i + 6j + 12(-i) + 0$

$= 3k + 3i - 6j$

9. Vector perpendicular to both A and B

$A \times B$ is perpendicular to both A & B

10. Linear dependency between A, B & C is $3A - B = C$

11. $A^T B = [1, 2\ 3]\begin{bmatrix}4\\5\\6\end{bmatrix} = 1\times4 + 2\times5 + 3\times6 = 32$

$A B^T = \underset{3\times1}{\begin{bmatrix}1\\2\\3\end{bmatrix}} \underset{1\times3}{[4\ 5\ 6]} = \begin{bmatrix}1\times4 & 1\times5 & 1\times6\\ 2\times4 & 2\times5 & 2\times6 \\ 3\times4 & 3\times5 & 3\times6\end{bmatrix} = \begin{bmatrix}4 & 5 & 6\\8 & 10 & 12\\12 & 15 & 18\end{bmatrix}$

B. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$

1. $$2A - B = 2\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & -4 & 6 \\ 0 & 10 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 4-2 & 6-1 \\ 8-2 & -4-1 & 6-(-4) \\ 0-3 & 10-(-2) & -2-1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}$$

2. $$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times2+3\times3 & 1\times2+2\times1+3\times-2 & 1\times1+2\times-4+3\times1 \\ 4\times1+(-2\times2)+3\times3) & 4\times2+(-2\times1)+3\times-2 & 4\times1+(-2\times-4)+3\times1 \\ 0\times1+5\times2+(-1\times3) & 0\times2+5\times1+(-1\times-2) & 0\times1+5\times-4+(-1\times1) \end{bmatrix}$$

$$AB = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times4+1\times0 & 1\times2+2\times-2+1\times5 & 1\times3+2\times3+1\times-1 \\ 2\times1+1\times4+(-4\times0) & 2\times2+1\times-2+(-4\times5) & (2\times3)+1\times3+(-4\times-1) \\ 3\times1+(-2\times4)+1\times0 & 3\times2+(-2\times-2)+1\times5 & 3\times3+(-2\times3)+1\times-1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}$$

3. $$(AB)^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$$ and $(AB)^T = B^T A^T$

4. $$|A| = 1(-2\times-1 - 5\times3) - 2(4\times-1 - 3\times0) + 3(4\times5 - (-2)\times0))$$
$$= 1(2-15) - 2(-4-0) + 3(20-0) = -13 + 8 + 60 = 55$$

$$|C| = 1(5\times3 - (6\times1)) - 2(4\times3 - (6\times-1)) + 3(4\times1 - (5\times-1))$$
$$= 1(15-6) - 2(12-(-6)) + 3(4-(-5)) = 9 - 36 + 27 = 0$$

5. Orthogonal set means that all vectors are mutually perpendicular to each other in the set.
so, Only in B are all row vectors mutually perpendicular to each other i.e, there inner (dot) product are 0.

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6) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$

$\det(A) = 1(2-15) - (2)(-4-0) + 3(20-0)$

$= -13 + 8 + 60$

$= 55$

$A^T = \begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{bmatrix}$

$\begin{vmatrix} 1 & 4 \\ 2 & -2 \end{vmatrix} = -10$ $\quad \begin{vmatrix} 4 & 0 \\ -2 & 5 \end{vmatrix} = 20$ $\quad \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} = 5$

$\begin{vmatrix} 2 & -2 \\ 3 & 3 \end{vmatrix} = 12$ $\quad \begin{vmatrix} -2 & 5 \\ 3 & -1 \end{vmatrix} = -13$ $\quad \begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} = -17$

$\begin{vmatrix} 1 & 4 \\ 3 & 3 \end{vmatrix} = -9$ $\quad \begin{vmatrix} 4 & 0 \\ 3 & -1 \end{vmatrix} = -4$ $\quad \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} = -1$

$Adj(A) = \begin{bmatrix} -13 & -17 & 12 \\ -4 & -1 & -9 \\ 20 & 5 & -10 \end{bmatrix} \times \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$A^{-1} = \frac{1}{55} \begin{bmatrix} -13 & 17 & 12 \\ 4 & -1 & 9 \\ 20 & -5 & -10 \end{bmatrix}$

$= \begin{bmatrix} \cancel{-1/3} & \cancel{-5/13} & \cancel{4/13} \\ \cancel{-4/39} & \cancel{-1/39} & \cancel{-3/13} \\ \cancel{20/39} & 5/39 & -10/39 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -13/55 & 17/55 & 12/55 \\ 4/55 & -1/55 & 9/55 \\ 4/11 & 1/11 & -2/11 \end{bmatrix}$

Since B is a matrix with all its row vectors orthogonal to each other, inverse of B is its transpose itself.

i.e. $B^{-1} = B^T$

C. $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. $B = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$

Since the matrices are symmetric $AX = \lambda X$, where λ : matrix of eigen value
X : matrix of corresponding eigen vectors.

1. $AX - \lambda X = 0$

If X is an Identity matrix.

$A - \lambda I = 0$

$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$

$\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} = 0$

$|A - \lambda I| = (1-\lambda)(2-\lambda) - 6 = 0$

$= 2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$

$= 2 - 3\lambda + \lambda^2 - 6 = 0$

$= \lambda^2 - 3\lambda - 4 = 0$

$= \lambda^2 - 4\lambda + \lambda - 4 = 0$

$= \lambda(\lambda - 4) + 1(\lambda - 4) = 0$

$= (\lambda - 4)(\lambda + 1) = 0$

$\Rightarrow \lambda = 4$ & $\underline{\lambda = -1}$ are the eigen values.

when $\lambda = 4$, $A - \lambda I = \begin{bmatrix} 1-4 & 2 \\ 3 & 2-4 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}$

$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right]$

$R_1 + R_2 \rightarrow R_2$

$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ -3+3 & 2-2 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$-3x_1 + 2x_2 = 0$

$2x_2 = 3x_1$

When $x_1 = 1$

$x_2 = 3/2$

so $\begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$ is a corresponding eigen vector for eigenvalue 4

Similarly when $\lambda = -1$

$A - \lambda I = \begin{bmatrix} 1+1 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right]$

$R_2 \leftarrow R_1 - R_2 + 1$

$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$2x_1 + 2x_2 = 0$

$2x_1 = -2x_2$

when $x_2 = -1$

$x_1 = 1$

so $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a corresponding eigen vector for eigen value -1

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2. $V = \begin{bmatrix} 1 & 1 \\ 3/2 & -1 \end{bmatrix}$

$$V^{-1} = \frac{1}{1 \times -1 - 1 \times \frac{3}{2}} \begin{bmatrix} -1 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$$

$$= \frac{1}{-1 - \frac{3}{2}} \begin{bmatrix} -1 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix}$$

$$V^{-1} A V = \begin{bmatrix} 2/5 & 2/5 \\ 3/5 & -2/5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3/2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{5} \times 1 + \frac{2}{5} \times 3 & \frac{2}{5} \times 2 + \frac{2}{5} \times 2 \\ 3/5 \times 1 + \frac{3}{5} \times 3 & \frac{3}{5} \times 2 + \frac{3}{5} \times 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3/2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{5} + \frac{6}{5} & \frac{4}{5} + \frac{4}{5} \\ \frac{3}{5} + \frac{9}{5} & \frac{6}{5} + \frac{6}{5} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{3}{2} & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 8/5 & 8/5 \\ 12/5 & 12/5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3/2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \times 1 + \frac{\cancel{8}^{4}}{5} \times \frac{3}{\cancel{2}} & \frac{8}{5} \times 1 + \frac{8}{5} \times -1 \\ \frac{12}{5} \times 1 + \frac{\cancel{12}^{6}}{5} \times \frac{3}{\cancel{2}} & \frac{12}{5} \times 1 + \frac{12}{5} \times -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ \frac{24}{5} & 0 \end{bmatrix}$$

3. eigen vectors of A : $\begin{bmatrix} 1 \\ 3/2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$(1, 3/2) \cdot (1, -1) = 1 \times 1 + \frac{3}{2} \times -1$$

$$= 1 - \frac{3}{2} = -1/2$$

4. $B = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$

$B - \lambda I$

$$\begin{bmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix}$$

$|B - \lambda I| = 0$

$(2-\lambda)(5-\lambda) - (-2) \cdot (-2) = 0$

$10 - 2\lambda - 5\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 7\lambda + 6 = 0$

$\lambda^2 - 6\lambda - \lambda + 6 = 0$

$\lambda(\lambda - 6) - 1(\lambda - 6) = 0$

$(\lambda - 6)(\lambda - 1) \Rightarrow \lambda = 1, \lambda = 6$ are the eigen values of B

Substituting $\lambda = 1$ in $B - \lambda I$

$$\begin{bmatrix} 2-1 & -2 \\ -2 & 5-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right]$$

$R_2 \leftarrow 2R_1 + R_2$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - 2x_2 = 0$

$x_1 = 2x_2$

When $x_2 = 1$,

$x_1 = 2$

Eigenvector corresponding to eigen value 1 is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Substituting $\lambda = 6$ in $B - \lambda I$

$$\begin{bmatrix} 2-6 & -2 \\ -2 & 5-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -4 & -2 & 0 \\ -2 & -1 & 0 \end{array}\right]$$

$R_2 \leftarrow -2R_2 + R_1$

$$\left[\begin{array}{cc|c} -4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-4x_1 - 2x_2 = 0$

$-4x_1 = 2x_2$

When $x_2 = 1$,

$x_1 = -1/2$

Eigen vector corresponding to eigen value 6 is $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

dot product of eigen vector of B:

$$(1,2) \cdot (-1/2, 1) = -\frac{1}{2} + 2 = \frac{3}{2}$$

5. Eigen vector of B are orthogonal to each other as the matrix itself contains orthogonal vectors to each other.

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D. $f(x) = x^2 + 3$ $g(x,y) = x^2 + y^2$

1. $f'(x) = 2x$

$f''(x) = 2$

2. $\frac{dg}{dx} = 2x$ $\frac{dg}{dy} = 2y$

3. gradient vector $\nabla g(x,y) = \begin{bmatrix} dg/dx \\ dg/dy \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

4. $f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$