ECOLE INTERNATIONNALE DES SCIENCES DU TRAITEMENT DE L'INFORMATION

E.I.S.T.I. Département Mathématiques

MASTER QFRM2 2018-2019

Practical LAB 3: Numerical evaluation of European option price by the method of Monte-Carlo

The purpose of this lab is the numerical simulation of the price of Vanilla option by the method of Monte-Carlo.

Part I. Evolution of assets

The evolution of the underlying asset S is a stochastic process verified the stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S(t=0) = S_0 \tag{1}$$

- σ is la volatility of the asset
- r rate of interest
- W_t is the Brownian motion.

We use the exact solution of stochastic differential equation to evaluate the final value S_T of the underlying asset

The solution of this equation is given by the formula:

$$S(T) = S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma W(T))$$

$W(T)$ is the value of the Brownian motion at the date $t = T$. This is a random variable that follows the Normal distribution law $\mathbb{N}(0, \sqrt{T})$. **We can model the final value of the Brownian motion by the random variable:**

$$W(T) = \mathbb{N}(0,1)\sqrt{T}$$

$\mathbb{N}(0,1)$ is the random number that follows standard Normal distribution law.

$$S(T) = S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\mathbb{N}(0,1))$$

For pricing of the European option we simulate only the last value of asset $S(T)$ because the European option is path independent.

Part II. Path independent options. Price of a European option.

The price of a European option at time $t = 0$ is given by conditional expectation of the pay-off function:

$$V(S_0, 0) = e^{-rT}\mathbb{E}[max(S(T) - K, 0)/S(t = 0) = S_0] \tag{2}$$

We use the expression for the last value of asset:

$$S(T) = S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma W(T))$$

and we obtain for the price of the option:

$$V(S_0, 0) = e^{-rT}\mathbb{E}[max(S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma W(T)) - K, 0)/S(t = 0) = S_0] \tag{3}$$

For pricing of the European option we simulate only the last value of asset $S(T)$ because the European option is path independent.

We model the final value of the Brownian motion by the random variable:

$$W(T) = \mathbb{N}(0, 1)\sqrt{T}$$

The approximated value of European option at $t = 0$, for initial value of the underlying asset $S = S_0$ is given by the arithmetic mean:

$$V(S_0, 0) = e^{-rT}\sum_{n=1}^{N_{mc}} max(S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\mathbb{N}^{(n)}(0, 1)) - K, 0)/N_{mc}.$$

MatLab or Scilab programming

- **Pay-off function of European option**

$$\begin{cases} \textbf{function[f]= Payoff-Europ-Call(S)} \\ f = max(S - K, 0) \\ endfunction \end{cases}$$

- **Function which calculates the price of European option for some fixe initial value $S0$ of the underlying asset.**

$$\begin{cases} \text{function[prix]= Prix-Europ-S0-fixe}(S0) \\ sum = 0 \\ Pour \quad n = 1...N_{mc} \\ S = S0\exp((r - \frac{\sigma^2}{2})\,T + \sigma\sqrt{T}\,rand(1, 1,' n')) \\ sum = sum + \text{Payoff-Europ-Call}(S) \\ Fin \quad Pour \\ prix = e^{-rT}sum/N_{mc} \\ endfunction \end{cases}$$

- Calculate the price of Call for $S_0 = 10$.

We calculate now the price of the option for each value of $S0$. For this we discretize the interval $[0, L]$ on $M = 41$ parts and we calculate the price $V(S0, 0)$ for each value of

$$S0 = \Delta S\ (i-1), \quad i = 1 : 41 \quad \Delta S = 0.5$$

- Plot the graph

$$S_0 -> V(S_0, 0)$$

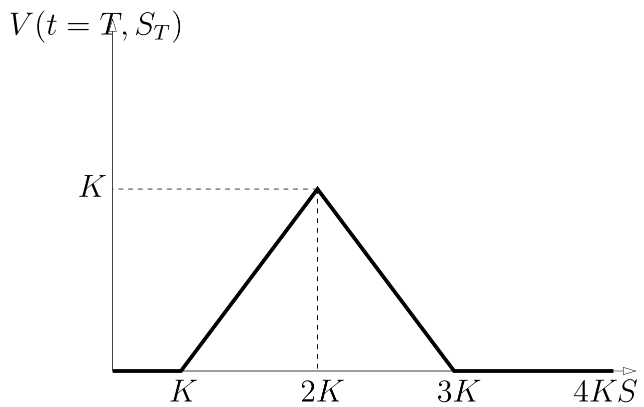
.

- **Function for the calculation of European Call for any value S_0**

$$\left\{\begin{array}{c}\text{function [prix-option]=}\textbf{Prix-Call-Europ}(\ )\\ for \quad k = 1 : 41\\ S0(k) = (k-1)\cdot 0.5\\ \text{prix-option(k)= } \textbf{Prix-Europ-S0-fixe}\text{(S0(k))}\\ end\\ plot\text{(S0,prix-option)}\\ endfunction\end{array}\right.$$

- Plot the graph for the price of Call: $S_0 \to V(t=0, S_0)$ pour $N_{mc} = 100$ et $N_{mc} = 1000$.
Use **plot (S0,prix-option)**.
- Plot in the same coordinate system the graphs of Call obtained by Monte Carlo and Binomial model. Compare.

- Calculate the price of Butterfly option by Monte-Carlo à $t = 0$ knowing the payoff function presented on FIG:



III. The price of the European option for any fixed value of asset S_t and any fixed value of time t

- We program the function that calculates the price of the option for S_t fixed and t fixed. At this moment **we know the price of the asset S_t**.

We simulate a large number N_{mc} of paths which describe of the evolution of assets over the time interval $[t, T]$, always starting from S_t. For each path we are looking for the final value S_T. The price of the European option at the time t is given by expected value of pay-off function:

$$V(t, St) = e^{-r(T-t)}\mathbb{E}[max(St\exp((r - \frac{\sigma^2}{2})(T-t) + \sigma W(T-t)) - K, 0)/S(t) = St] \qquad (4)$$

soit

$$V(t, S_t) = e^{-r(T-t)}\sum_{k=1}^{N_{mc}} max(S_T - K, 0)/N_{mc}$$

$$\left\{ \begin{array}{c} \text{function[prix]= Prix-Europ-St-fixe-t-fixe}(t, St) \\ ... \\ ... \\ endfunction \end{array} \right.$$

Plot in 2 dimensions the graph $\quad St \to V(t = T/2, St)$.

• We programm the function that calculates the option price for each S_t and each t independent.

We discretise St: $St = \Delta S \cdot (k-1)$, $\quad \Delta S = L/40$.

We discretise t: $t = \Delta t \cdot (j-1)$, $\quad \Delta t = T/10$.

• Here is the program:

$$\left\{ \begin{array}{c} \text{function [price]=}\textbf{CALL-EUROP}(\ ) \\ for \quad k = 1:41 \\ for \quad j = 1:11 \\ St(k) = (k-1)\cdot 0.5 \\ t(j) = (j-1)\cdot 0.05 \\ \text{price}(j,k) = \textbf{Prix-Europ-St-fixe-t-fixe}(t(j), St(k)) \\ end \\ end \\ \text{surf}(St, t, \text{price}) \\ endfunction \end{array} \right.$$

• Plot the surface of option price $\quad$ surf(St, t,price) and compare with that obtained by Finite Difference or Binomial methods.

Use the following values:

$$\left\{ \begin{array}{c} N_{mc} = 1000 \\ L = 20 \\ K = 10 \\ T = 0.5 \\ r = 0.1 \\ \sigma = 0.5 \\ N = 99 \end{array} \right.$$

IV. Variance. Confidence interval

Introduce a function $\Phi(W_T)$ representing a discounted price of the Call to maturity

$$\Phi(W_T) = e^{-rT} max(S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma W(T)) - K, 0).$$

Exact price of the option:

$$V(S_0, 0)_{exact} = \mathbb{E}[\Phi(W_T)/S(t=0) = S_0].$$

- **First option estimator:**

$$V(S_0, 0) = \mathbb{E}[\Phi(W_T)]$$

$$V(S_0, 0)_{estime} = \frac{1}{N_{mc}} \sum_{n=1}^{N_{mc}} (\Phi(W_T^{(n)}))$$

- We calculate the variance of Φ:

$$Var[\Phi] = \mathbb{E}[\Phi^2] - (\mathbb{E}[\Phi])^2$$

$$Var[\Phi]_{estime} = \frac{1}{N_{mc}} \sum_{n=1}^{N_{mc}} (\Phi(W_T^{(n)}))^2 - (\frac{1}{N_{mc}} \sum_{n=1}^{N_{mc}} \Phi(W_T^{(n)}))^2.$$

- Calculate the variance of the payoff function for $S_0 = 10$
- The exact price of the Call is in the confidence interval

$$[\sum_{n=1}^{N_{mc}} \Phi(W_T^{(n)})/N_{mc} - \frac{1.96 \cdot \sqrt{Var[\Phi]}}{\sqrt{N_{mc}}}, \quad \sum_{n=1}^{N_{mc}} \Phi(W_T^{(n)})/N_{mc} + \frac{1.96 \cdot \sqrt{Var[\Phi]}}{\sqrt{N_{mc}}}]$$

with the probability 0.95. Here

$$\sum_{n=1}^{N_{mc}} \Phi(W_T^{(n)})/N_{mc} = V(S_0, 0)_{estime}$$

is the price of Call that is estimated by the method of Monte Carlo. We do not know the exact value of the variance therefore it is replaced by the estimated ed variance.

- The exact price of the Call is in the confidence interval

$$[\sum_{n=1}^{N_{mc}} \Phi(W_T^{(n)})/N_{mc} - \frac{1.96 \cdot \sqrt{Var[\Phi]_{estime}}}{\sqrt{N_{mc}}}, \quad \sum_{n=1}^{N_{mc}} \Phi(W_T^{(n)})/N_{mc} + \frac{1.96 \cdot \sqrt{Var[\Phi]_{estime}}}{\sqrt{N_{mc}}}]$$

with the probability 0.95.

- Calculate the confidence interval for Call at $S_0 = 10$.

Conclusion: To reduce the confidence interval we have to increase the number N_{mc} or to decrease the variance $Var[\Phi]$.

V. Relation between the deterministic methods (Finite Difference) and stochastic (Monte-Carlo)

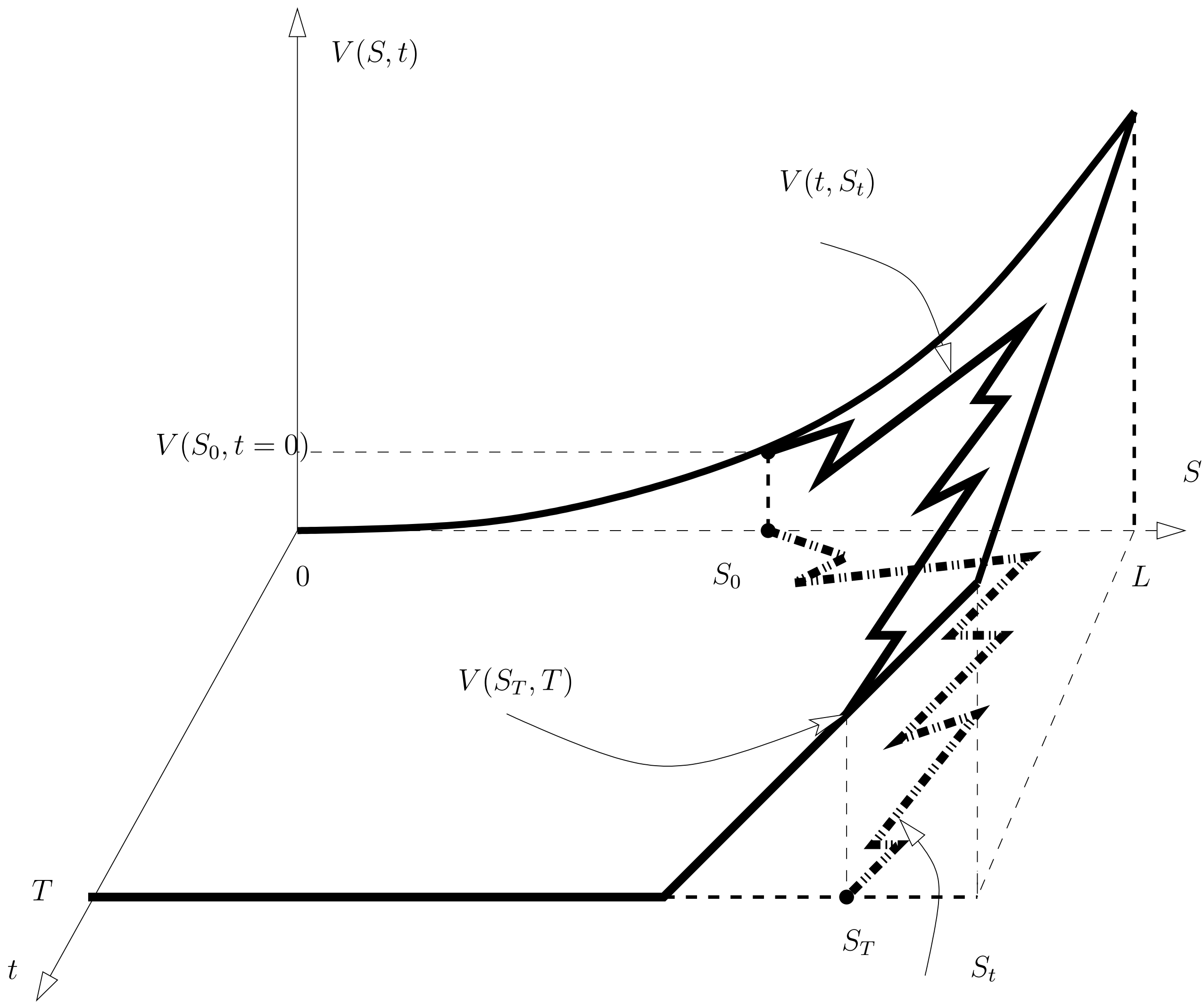


Figure 1: Evolutions of the underlying asset and the option