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Practical LAB 4: Pricing of European option by Monte-Carlo and variance reduction.

The purpose of this Lab is to study the different methods of Variance reduction.

Part I. Antithetic Variates.

The idea of antithetic control is very simple. It is based on the symmetry property of the Brouwnian motions W_t et $-W_t$. So for a function F we have

$$E[\frac{F(W_t)+F(-W_t)}{2}] = E[F(W_t)].$$

We use the identity

$$Var[\frac{F(W_t)+F(-W_t)}{2}] = \frac{1}{4}(Var[F(W_t)] + Var[F(-W_t)] + 2Cov(F(W_t), F(-W_t))$$

and we obtain

$$Var[\frac{F(W_t)+F(-W_t)}{2}] = \frac{1}{2}(Var[F(W_t)] + Cov(F(W_t), F(-W_t)))$$

We can show that if $F: \quad x \to F(x)$ and $G: \quad x \to G(x)$ are monotonous (both increasing or both decreasing) then

$$Cov(F(W_t), G(W_t)) > 0.$$

In our case we take:

$$F(W_t) = \Phi(W_T), \quad G(W_t) = -\Phi(-W_T).$$

Then

$$Cov(\Phi(W_T), \Phi(-W_T)) < 0$$

$$Var[\frac{\Phi(W_T)+\Phi(-W_T)}{2}] <= \frac{1}{2}Var[\Phi(W_T)]$$

and we have reduced the variance of $\Phi(W_T)$.

- The price of the European option is given then by **normal estimator** of the option and we notice this price by $Call_0$:

$$Call_0(S_0, 0) = V(S_0, 0) = \mathbb{E}[\Phi(W_T)]$$

- The price of the European option with the reduced variance is given then by **new estimator** of the option and we notice this price by $Call_1$:

$$Call_1(S_0, 0) = \mathbb{E}[\frac{\Phi(W_T) + \Phi(-W_T)}{2}]$$

$$Call_1(S_0, 0)_{estim} = \frac{\sum_{n=1}^{N_{mc}} (\Phi(W_T^{(n)}) + \Phi(-W_T^{(n)}))}{2N_{mc}}$$

1) Calculate the price for $S_0 = 10$

2) Calculate the variance of

$$\frac{1}{2}(\Phi(W_T) + \Phi(-W_T))$$

What is the confidence interval for Call's price now?

3) Plot the graph

$$S_0 -> Call_0(S_0, 0)_{estim}$$

for N_{mc}=10 and 100.

4) Plot the graph

$$S_0 -> Call_1(S_0, 0)_{estim}$$

of Call with the reduced variance for N_{mc}=10 et 100. Compare these graphs, what is your observation?

Part II. Control Variates.

Instead of calculating the expectation of a random variable X we will calculate the expectation of random variable Z.

$$Z = X - b(Y - \mathbb{E}[Y])$$

Here b is a number, Y is the random variable whose expectation is known. We will show that $Var[Z] < Var[X]$.

It is obvious that

$$\mathbb{E}[X] = \mathbb{E}[Z]$$

We calculate the variance of Z.

$$Var[Z] = \mathbb{E}[(Z - \mathbb{E}[Z])^2] = \mathbb{E}[(X - b(Y - \mathbb{E}[Y]) - \mathbb{E}[X])^2] =$$

$$\mathbb{E}[(X - \mathbb{E}[X])^2] - 2b\mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] + b^2\mathbb{E}[(Y - \mathbb{E}[Y])^2] =$$

$$Var[X] - 2bCov[XY] + b^2Var[Y]$$

The variance of Z is minimal if $\frac{\partial Z}{\partial b} = 0$. Then

$$b = \frac{Cov[XY]}{Var[Y]}, \qquad Var[Z] = Var[X] - \frac{(Cov[XY])^2}{Var[Y]}$$

This idea is now applied to the calculation of the price of the European option. Let's choose for X and Y the following random variables:

$$X = e^{-rT} max(S_T - K, 0), \qquad Y = S_T, \qquad \mathbb{E}[Y] = S_0 e^{rT}$$

We notice the price of the option by $Call_2$: The price of the option is equal to

$$Call_2(S_0, 0) = \mathbb{E}[Z] = \mathbb{E}[X - b(Y - \mathbb{E}[Y])]$$

The expectation value is estimated by the arithmetic mean:

$$Call_2(S_0, 0)_{estim} = \frac{1}{N_{mc}} \sum_{n=1}^{N_{mc}} [e^{-rT} max(S_T^{(n)} - K, 0) - b \cdot (S_T^{(n)} - S_0 e^{rT})]$$

To calculate this price we will proceed in the following way:
1. We calculate b independently from $Call_2(S_0, 0)_{estim}$.
To do this:
- We simulate N_{mc} numbers of a set $\mathfrak{M}$ that follow the normal distribution.
- We estimate $\mathbb{E}[X]$ with the set $\mathfrak{M}$

$$\mathbb{E}[X]_{estime} = \frac{1}{N_{mc}} \sum_{n=1}^{N_{mc}} [e^{-rT} max(S_T^{(n)} - K, 0)$$

- Then we calculate

$$b = \frac{\sum_{n=1}^{N_{mc}} (e^{-rT} max(S_T^{(n)} - K, 0) - \mathbb{E}[X]_{estime})(S_T^{(n)} - S_0 e^{rT})}{\sum_{n=1}^{N_{mc}} (S_T^{(n)} - S_0 e^{rT})^2}$$

with the same set $\mathfrak{M}$.
2. Finally we calculate the price of Call

$$V(S_0, 0)_{estime} = \frac{1}{N_{mc}} \sum_{n=1}^{N_{mc}} [e^{-rT} max(S_T^{(n)} - K, 0) - b \cdot (S_T^{(n)} - S_0 e^{rT})]$$

with another or the same set $\mathfrak{M}$ of numbers that follow the normal distribution.

Part III. Importance Simpling technique.

Let W_t is a Brownian motion on a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, the filtration $\mathcal{F}_t$ is generated by W_t. There exists the probability measure Q defined on $(\Omega, \mathcal{F}_T)$ par

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_T} = exp(-\theta \cdot B_T - \frac{\theta^2 T}{2})$$

, where $B_T = W_T + \theta T$ and for any function Φ we have:

$$\mathbb{E}^{\mathbb{P}}[\Phi(W_T)] = \mathbb{E}^{\mathbb{Q}}[\Phi(B_T)\, exp(-\theta \cdot B_T + \frac{\theta^2 T}{2})] = \mathbb{E}^{\mathbb{P}}[\Phi(W_T + \theta T)\, exp(-\theta \cdot W_T - \frac{\theta^2 T}{2})]$$

Let $\Phi(W_T)$ pay-off function of Call actualized.

The price of the European option at time $t = 0$ is given by

$$Call_0 = \mathbb{E}[\Phi(W_T)/S(t = 0) = S_0], \qquad \Phi(W_T) = e^{-rT}\, max(S_T - K, 0)$$

Thanks to Importance Simpling technique we can introduce another estimator for the Call denoted by $Call_3$:

$$Call_3(S_0, 0) = \mathbb{E}[\Phi(W_T + \theta T)e^{-\theta \cdot W_T - \frac{\theta^2 T}{2}}]$$

Use the following values:

$$\begin{cases} T = 1 \\ r = 0.05 \\ \sigma = 0.2 \\ S_0 = 90 \\ K = 40 \end{cases}$$

Work to do

1.Calculate the price of the $Call(S_0, t = 0)$ using the formula

$$Call_0 = \mathbb{E}[\Phi(W_T)]$$

2. Calculate the price of the $Call(S_0, t = 0)$ using the formula

$$Call_3 = \mathbb{E}[\Phi(W_T + \theta T)e^{-\theta \cdot W_T - \frac{\theta^2 T}{2}}], \quad \theta = 1$$

3. Compare $Call_0$ et $Call_3$

Part IV. Variance reduction using Importance Simpling technique.

We now want to minimize the variance of the

$$Gain_3(\theta) = \Phi(W_T + \theta T)e^{-\theta \cdot W_T - \frac{\theta^2 T}{2}}$$

1. Calculate the variance of
$$Gain_0(\theta) = \Phi(W_T)$$
2. Calculate the variance of
$$Gain_3(\theta) = \Phi(W_T + \theta T)e^{-\theta \cdot W_T - \frac{\theta^2 T}{2}}, \quad \theta = 1$$
3. Calculate variance de
$$Gain_3(\theta) = \Phi(W_T + \theta T)e^{-\theta \cdot W_T - \frac{\theta^2 T}{2}}, \quad \theta = 0.34$$
4. Conclude.

Part V: Optional. Search of the parameter θ using Robbins - Monro algorithm.

Consider the minimization of the variance of

$$Gain_3(\theta) = \Phi(W_T + \theta T)e^{-\theta \cdot W_T - \frac{\theta^2 T}{2}}$$

Since the expectation of $Gain_3$ in respect of θ is a constante it suffices to minimize the following expression

$$v(\theta) = \mathbb{E}[\Phi(W_T + \theta T)^2 e^{-2\theta \cdot W_T - \theta^2 T}]$$

Using the measure Q one enable to remove the dependance on θ from the argument of Φ and to get:

$$\nabla v(\theta) = \mathbb{E}[(\theta T - W_T)\Phi^2(W)e^{-\theta W_T + \frac{\theta^2 T}{2}}]$$

We notice $\nabla v(\theta) = \mathbb{E}[U(\theta, Z)]$, where Z follows the normal distribution.

$$U(\theta, Z) = (\theta T - \sqrt{T}Z)(e^{-rT}(S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z) - K)_+)^2 e^{-\theta\sqrt{T}Z + \frac{\theta^2 T}{2}}$$

Using the Robbins - Monro algorithm we search θ^* s.a. $\mathbb{E}[U(\theta^*, Z)] = 0$

Work to do

Use the following numeric values:

$$\begin{cases} S_0 = 90 \\ r = 0.05 \\ \sigma = 0.2 \\ T = 1 \\ K = 40 \\ N_m c = 20000 \\ \theta_0 = 1 \\ s_0 = 1 \end{cases} \tag{1}$$

- Define a set $\gamma_n = \frac{\alpha}{(n+1)^\beta}$, $\quad \beta = 0.9 \quad \alpha = 0.0001$
- Simpling
-

$$U(\theta_n, Z_{n+1}) = (\theta_n T - \sqrt{T} Z_{n+1})((S_0 \exp((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T} Z_{n+1}) - K)_+)^2 e^{-\theta_n \sqrt{T} Z_{n+1} + \frac{\theta_n^2 T}{2}}$$

- $\theta_{n+1} = \theta_n - \gamma_n U(\theta_n, Z_{n+1})$
- if $\quad -ln(1+s_n) \leq Q_{n+1} \leq ln(1+s_n), \qquad \theta_{n+1} = Q_{n+1}, \quad s_{n+1} = \sigma_n$
  else $\qquad \theta_{n+1} = \theta_0, \quad s_{n+1} = s_n + 1$
- Plot the graph

$$n- > \theta_n$$

and find $\quad \theta^* = \lim_{n \mapsto \infty} \theta_n$.