# Mathematical Problems and Solutions

Chang-ye Tu

July 8, 2016

**Problem 1** (Folklore Knowledge)**.** Derive the followings.

1. spherical coordinate, volume element, volume/surface area of $n$-ball
2. gradient/divergence/laplacian/curl in cylindrical/spherical coordinates
3. scalar/vector Green identities
4. $\int_{-\infty}^{\infty} e^{-x^2}\,dx$
5. $\int_0^{\infty} \frac{\sin x}{x}\,dx$

**Problem 2** (Theorems)**.** Describe the following theorems, their usages and proofs.

1. inverse function theorem
2. implicit function theorem
3. partition of unity
4. Fatou lemma
5. Lebesgue dominated convergence theorem
6. Radon-Nikodym theorem
7. Calderon-Zygmund decomposition
8. Rellich-Kondrachov theorem
9. Sobolev embedding theorem

**Problem 3** (Inequalities)**.** Describe the following inequalities, their usages and proofs.

1. Cauchy-Schwarz
2. Hölder
3. Minkowski
4. Hardy
5. Poincaré

**Problem 4.** Let $\mathbb{P}_l$ be the space of monomials of three variables with total degree less or equal than $l$. Then

$$\dim \mathbb{P}_l = \binom{l+3}{3}.$$

*Solution.* Use

$$\sum_{k=0}^{l} \binom{m+k}{k} = \sum_{k=0}^{l} \left\{ \binom{m+k+1}{k} - \binom{m+k}{k-1} \right\} = \binom{m+l+1}{m+1}$$

Then

$$\dim \mathbb{P}_l = \sum_{k=0}^{l} \binom{2+k}{k} = \binom{l+3}{3}$$

$\square$

**Problem 5.** Evaluate
$$\int_0^\infty \frac{1-e^{-kx}}{\sqrt{x^3}}\,\mathrm{d}x, \qquad k \geqslant 0.$$

*Solution.* Let
$$f(k) = \int_0^\infty \frac{1-e^{-kx}}{\sqrt{x^3}}\,\mathrm{d}x$$

Then
$$\begin{aligned} f'(k) &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1-e^{-kx}}{\sqrt{x^3}}\right)\mathrm{d}x \\ &= \int_0^\infty \frac{e^{-kx}}{\sqrt{x}}\,\mathrm{d}x \\ &= \frac{\sqrt{\pi}}{\sqrt{k}} \end{aligned}$$

Hence
$$f(k) = 2\sqrt{k\pi} + C$$

From $f(0) = 0$ we have $C = 0$, so $f(k) = 2\sqrt{k\pi}$. □

**Problem 6.** Let $z = \cos\theta$ and $w(z) = y(\theta)$, rewrite the equation
$$\sin\theta(\sin\theta\, y'(\theta))' - (m^2 + \lambda\sin^2\theta)y(\theta) = 0$$

into
$$(1-z^2)w''(z) - 2zw'(z) - \left(\frac{m^2}{1-z^2} + \lambda\right)w(z) = 0$$

*Solution.*
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\theta} = -y'(1-z^2) \\ \frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)\frac{\mathrm{d}z}{\mathrm{d}\theta} + \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)\frac{\mathrm{d}z}{\mathrm{d}\theta} + \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right) \\ &= y''\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 + y'\frac{\mathrm{d}^2z}{\mathrm{d}\theta^2} \\ &= (1-z^2)y'' - z\,y' \end{aligned}$$

Substitute into the equation, we have
$$-(1-z^2)z\,y' + (1-z^2)\left((1-z^2)y'' - z\,y'\right) - (m^2 + \lambda(1-z^2))y = 0$$

□

**Problem 7.** Let $u \in H^1(\Omega)$. Then $\forall\lambda > 0$, $\exists c > 0$ s.t.
$$\int_\Omega |\nabla u|^2 + \lambda\int_\Gamma u^2 \geqslant c\|u\|_{H^1(\Omega)}$$

*Solution.* `http://math.stackexchange.com/questions/604577/a-question-about-a-sobolev-spa`
`lq=1` □