Electromagnetic Scattering Problems in Chiral Media

Chang-ye Tu

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Contents

Chapter 1

Introduction

1.1 Preamble

In this work, we study the scattering problem of time-harmonic electromagnetic waves by a bounded obstacle embedded in another medium.

1.2 Electromagnetic Field Equations

In the absence of electrical charges and currents, the macroscopic time-dependent Maxwell equations of electromagnetism are

$$\begin{aligned} &\operatorname{div} \mathcal{D} = 0 \\ &\operatorname{div} \mathcal{B} = 0 \\ &\operatorname{curl} \mathcal{E} + \frac{\partial \mathcal{B}}{\partial t} = 0 \\ &\operatorname{curl} \mathcal{H} - \frac{\partial \mathcal{D}}{\partial t} = 0 \end{aligned} \tag{1.1}$$

where $\mathcal{D}$ denotes the electric displacement, $\mathcal{B}$ the magnetic induction, $\mathcal{E}$ the electric field, and $\mathcal{H}$ the magnetic field. By "time-harmonic" the fields are of the form

$$\begin{aligned} \mathcal{D}(x,t) &= \Re\{D(x)e^{-i\omega t}\} \\ \mathcal{B}(x,t) &= \Re\{B(x)e^{-i\omega t}\} \\ \mathcal{E}(x,t) &= \Re\{E(x)e^{-i\omega t}\} \\ \mathcal{H}(x,t) &= \Re\{H(x)e^{-i\omega t}\} \end{aligned}$$

where $x \in \mathbb{R}^3$ and ω denotes the frequency. Under the time-harmonic assumption, the Maxwell equations (1.1) become

$$\begin{aligned} &\operatorname{div} D = 0 \\ &\operatorname{div} B = 0 \\ &\operatorname{curl} E - i\omega B = 0 \\ &\operatorname{curl} H + i\omega D = 0 \end{aligned} \tag{1.2}$$

In order to proceed, the constitutive relations between D, E and B, H, which specify the response of bound charge and current to the applied fields, should be introduced.

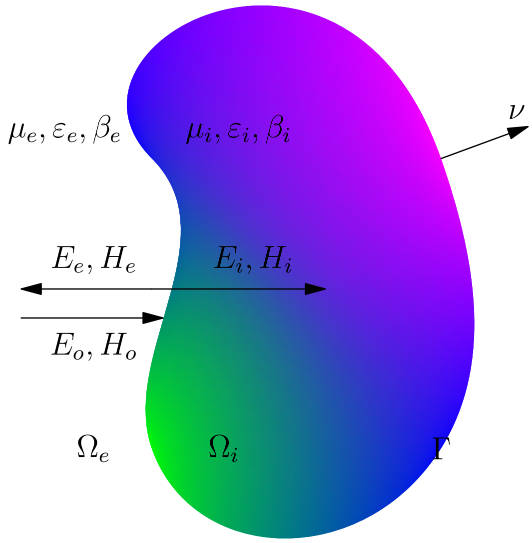

Figure 1.1: Problem Settings

For chiral media which obey the Drude-Born-Fedorov constitutive relations

$$\begin{aligned} D &= \varepsilon(E + \beta \operatorname{curl} E), \\ B &= \mu(H + \beta \operatorname{curl} H), \end{aligned} \tag{1.3}$$

where ε denotes the electric permittivity, μ the magnetic permittivity and β the chirality measure, the Maxwell equations (1.2) become

$$\begin{aligned} &\operatorname{div} E = 0 \\ &\operatorname{div} H = 0 \\ &\operatorname{curl} E - i\omega\mu(H + \beta \operatorname{curl} H) = 0 \\ &\operatorname{curl} H + i\omega\varepsilon(E + \beta \operatorname{curl} E) = 0 \end{aligned} \tag{1.4}$$

In this work

$$\begin{aligned} \nu \times (E_{\mathrm{e}} + E_{\mathrm{o}}) &= \nu \times E_{\mathrm{i}} \\ \nu \times (H_{\mathrm{e}} + H_{\mathrm{o}}) &= \nu \times H_{\mathrm{i}} \end{aligned} \tag{1.5}$$

1.2.1 Homogeneous Media

For homogeneous media with constant μ, ε independent of location, we substitute

$$\begin{aligned} E &:= \sqrt{\mu} E \\ H &:= \sqrt{\varepsilon} H \end{aligned} \tag{1.6}$$

into (1.4) to get

$$\begin{aligned} &\operatorname{div} E = 0 \\ &\operatorname{div} H = 0 \\ &\operatorname{curl} E - ik(H + \beta \operatorname{curl} H) = 0 \\ &\operatorname{curl} H + ik(E + \beta \operatorname{curl} E) = 0 \end{aligned} \tag{1.7}$$

where

$$k = \omega\sqrt{\mu\varepsilon}.$$

It is convenient to introduce the auxilliary notation U, U' as follows:

$$\begin{aligned} &\text{if } U = E, \text{ then } U' = iH \\ &\text{if } U = H, \text{ then } U' = -iE \end{aligned} \tag{1.8}$$

Here E, H are the solutions of (1.7). Note that

$$(U')' = U.$$

With the U notation we can summarize the last two equations of (1.7) as

$$\operatorname{curl} U = kU' + k\beta \operatorname{curl} U'. \tag{1.9}$$

From $(U')' = U$, we have

$$\begin{aligned} \operatorname{curl} U' &= k(U')' + k\beta \operatorname{curl}(U')' \\ &= kU + k\beta \operatorname{curl} U. \end{aligned} \tag{1.10}$$

Eliminate $\operatorname{curl} U'$ from (1.9), (1.10), we have

$$\operatorname{curl} U = \gamma^2 \beta U + \frac{\gamma^2}{k} U' \tag{1.11}$$

where

$$\gamma^2 = \frac{k^2}{1 - k^2\beta^2} \tag{1.12}$$

By taking the curl of (1.11) and using (1.10), we have

$$\operatorname{curl}\operatorname{curl} U - 2\gamma^2\beta \operatorname{curl} U - \gamma^2 U = 0 \tag{1.13}$$

With the scaling (1.6) performed in each region, the transmission boundary condition becomes

$$\begin{aligned} \nu \times (E_{\mathrm{e}} + E_{\mathrm{o}}) &= \delta\, \nu \times E_{\mathrm{i}} \\ \nu \times (H_{\mathrm{e}} + H_{\mathrm{o}}) &= \rho\, \nu \times H_{\mathrm{i}} \end{aligned} \tag{1.14}$$

where

$$\delta = \sqrt{\frac{\mu_{\mathrm{i}}}{\mu_{\mathrm{e}}}}, \quad \rho = \sqrt{\frac{\varepsilon_{\mathrm{i}}}{\varepsilon_{\mathrm{e}}}}. \tag{1.15}$$

Bohren's Transformation

By expanding (1.11), we have

$$\left(1 - k^2\beta^2\right) \operatorname{curl} E = ikH + k^2\beta E \tag{1.16}$$

and

$$\left(1 - k^2\beta^2\right) \operatorname{curl} H = -ikE + k^2\beta H \tag{1.17}$$

The above two equations can be rewritten into matrix form:

$$\begin{pmatrix}\operatorname{curl} E\\ \operatorname{curl} H\end{pmatrix} = \frac{1}{1-k^2\beta^2}\begin{pmatrix}k^2\beta & ik\\ -ik & k^2\beta\end{pmatrix}\begin{pmatrix}E\\ H\end{pmatrix} := A\begin{pmatrix}E\\ H\end{pmatrix}$$

Computing the roots of the equation

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix}\frac{k^2\beta}{1-k^2\beta^2}-\lambda & \frac{ik}{1-k^2\beta^2}\\ \frac{-ik}{1-k^2\beta^2} & \frac{k^2\beta}{1-k^2\beta^2}-\lambda\end{vmatrix}\\
&= \left(\lambda-\frac{k^2\beta}{1-k^2\beta^2}\right)^2-\frac{k^2}{(1-k^2\beta^2)^2}\\
&= \left(\lambda-\frac{k^2\beta}{1-k^2\beta^2}\right)^2-\left(\frac{k}{1-k^2\beta^2}\right)^2\\
&= \left(\lambda-\frac{k^2\beta+k}{1-k^2\beta^2}\right)\left(\lambda-\frac{k^2\beta-k}{1-k^2\beta^2}\right)\\
&= \left(\lambda-\frac{k(k\beta+1)}{(1+k\beta)(1-k\beta)}\right)\left(\lambda-\frac{k(k\beta-1)}{(1+k\beta)(1-k\beta)}\right)\\
&= \left(\lambda-\frac{k}{1-k\beta}\right)\left(\lambda+\frac{k}{1+k\beta}\right)\\
&= 0,
\end{aligned}$$

the eigenvalues of A are $\frac{k}{1-k\beta}$ and $-\frac{k}{1+k\beta}$ with corresponding orthonormal eigenvectors $\frac{1}{\sqrt{2}}(1,-i)^\top$ and $\frac{1}{\sqrt{2}}(1,i)^\top$. Let P be the matrix formed by the eigenvectors as columns,

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}$$

then

$$P^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}$$

is the basis transformation matrix. Hence if we set

$$\begin{aligned} Q_{\mathrm{l}} &= E + iH\\ Q_{\mathrm{r}} &= E - iH \end{aligned} \tag{1.18}$$

the Maxwell equations (1.16), (1.17) are transformed into the diagonalized form

$$\operatorname{curl} Q_{\mathrm{l}} = \gamma_{\mathrm{l}}\, Q_{\mathrm{l}} \tag{1.19}$$
$$\operatorname{curl} Q_{\mathrm{r}} = -\gamma_{\mathrm{r}}\, Q_{\mathrm{r}} \tag{1.20}$$

where

$$\begin{aligned} \gamma_{\mathrm{l}} &= \frac{k}{1-k\beta},\\ \gamma_{\mathrm{r}} &= \frac{k}{1+k\beta} \end{aligned} \tag{1.21}$$

and this can be directly verified. From (1.18), we have

$$\begin{aligned} E &= \frac{1}{2}\left(Q_{\mathrm{r}} + Q_{\mathrm{l}}\right) \\ H &= \frac{i}{2}\left(Q_{\mathrm{r}} - Q_{\mathrm{l}}\right) \end{aligned} \tag{1.22}$$

Substituting (1.22) into (1.10), we have

$$\begin{aligned} \operatorname{curl}\left(\frac{1}{2}\left(Q_{\mathrm{r}} + Q_{\mathrm{l}}\right)\right) - ik\left\{\frac{i}{2}\left(Q_{\mathrm{r}} - Q_{\mathrm{l}}\right) + \beta \operatorname{curl}\left(\frac{i}{2}\left(Q_{\mathrm{r}} - Q_{\mathrm{l}}\right)\right)\right\} &= 0 \\ \operatorname{curl}\left(\frac{i}{2}\left(Q_{\mathrm{r}} - Q_{\mathrm{l}}\right)\right) + ik\left\{\frac{1}{2}\left(Q_{\mathrm{r}} + Q_{\mathrm{l}}\right) + \beta \operatorname{curl}\left(\frac{1}{2}\left(Q_{\mathrm{r}} + Q_{\mathrm{l}}\right)\right)\right\} &= 0 \end{aligned}$$

which can be simplified as

$$\operatorname{curl}\left(Q_{\mathrm{r}} + Q_{\mathrm{l}}\right) + k\left(Q_{\mathrm{r}} - Q_{\mathrm{l}}\right) + k\beta \operatorname{curl}\left(Q_{\mathrm{r}} - Q_{\mathrm{l}}\right) = 0 \tag{1.23}$$

$$\operatorname{curl}\left(Q_{\mathrm{r}} - Q_{\mathrm{l}}\right) + k\left(Q_{\mathrm{r}} + Q_{\mathrm{l}}\right) + k\beta \operatorname{curl}\left(Q_{\mathrm{r}} + Q_{\mathrm{l}}\right) = 0 \tag{1.24}$$

By performing (1.23)+(1.24), (1.23)−(1.24) we recover (1.19).

Hereafter each of (1.18), (1.22) denotes “Bohren’s Transformation”.

Reduction to Two Dimension

Starting from

$$\left(1 - k^2\beta^2\right) \operatorname{curl} E = ikH + k^2\beta E \tag{1.25}$$

$$\left(1 - k^2\beta^2\right) \operatorname{curl} H = -ikE + k^2\beta H \tag{1.26}$$

Note that all ∂_3 derivatives vanish, thus the expression of curl becomes

$$\operatorname{curl} U = \begin{pmatrix} \partial_2 U_3 - \partial_3 U_2 \\ \partial_3 U_1 - \partial_1 U_3 \\ \partial_1 U_2 - \partial_2 U_1 \end{pmatrix} = \begin{pmatrix} \partial_2 U_3 \\ -\partial_1 U_3 \\ \partial_1 U_2 - \partial_2 U_1 \end{pmatrix}$$

Expanding (1.25), we have

$$\left(1 - k^2\beta^2\right) \begin{pmatrix} \partial_2 E_3 \\ -\partial_1 E_3 \\ \partial_1 E_2 - \partial_2 E_1 \end{pmatrix} = ik \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} + k^2\beta \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

The first row reads

$$\left(1 - k^2\beta^2\right) \partial_2 E_3 = ikH_1 + k^2\beta E_1$$

Differentiate with x_2,

$$\left(1 - k^2\beta^2\right) \partial_2^2 E_3 = ik\, \partial_2 H_1 + k^2\beta\, \partial_2 E_1 \tag{1.27}$$

The second row reads

$$-\left(1 - k^2\beta^2\right) \partial_1 E_3 = ikH_2 + k^2\beta E_2$$

Differentiate with x_1,

$$-\left(1-k^2\beta^2\right)\partial_1^2 E_3 = ik\,\partial_1 H_2 + k^2\beta\,\partial_1 E_2 \tag{1.28}$$

Combining (1.27), (1.28), we obtain the equation

$$\begin{aligned}
&\left(1-k^2\beta^2\right)\left(\partial_1^2+\partial_2^2\right)E_3\\
&= ik\left(\partial_2 H_1 - \partial_1 H_2\right) + k^2\beta\left(\partial_2 E_1 - \partial_1 E_2\right)\\
&= \frac{1}{1-k^2\beta^2}\left\{ik\left(ikE_3 - k^2\beta H_3\right) - k^2\beta\left(ikH_3 + k^2\beta E_3\right)\right\}\\
&= \frac{1}{1-k^2\beta^2}\left\{-\left(k^2+k^4\beta^2\right)E_3 - 2ik^3\beta H_3\right\}
\end{aligned} \tag{1.29}$$

Similarly, by expanding (1.26), we have

$$\left(1-k^2\beta^2\right)\begin{pmatrix}\partial_2 H_3\\ -\partial_1 H_3\\ \partial_1 H_2 - \partial_2 H_1\end{pmatrix} = -ik\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix} + k^2\beta\begin{pmatrix}H_1\\H_2\\H_3\end{pmatrix}$$

The first row reads

$$\left(1-k^2\beta^2\right)\partial_2 H_3 = -ikE_1 + k^2\beta H_1$$

Differentiate with x_2,

$$\left(1-k^2\beta^2\right)\partial_2^2 H_3 = -ik\,\partial_2 E_1 + k^2\beta\,\partial_2 H_1 \tag{1.30}$$

The second row reads

$$-\left(1-k^2\beta^2\right)\partial_1 H_3 = -ikE_2 + k^2\beta H_2$$

Differentiate with x_1,

$$-\left(1-k^2\beta^2\right)\partial_1^2 H_3 = -ik\,\partial_1 E_2 + k^2\beta\,\partial_1 H_2 \tag{1.31}$$

Combining (1.30), (1.31), we obtain the equation

$$\begin{aligned}
&\left(1-k^2\beta^2\right)\left(\partial_1^2+\partial_2^2\right)H_3\\
&= -ik\left(\partial_2 E_1 - \partial_1 E_2\right) + k^2\beta\left(\partial_2 H_1 - \partial_1 H_2\right)\\
&= \frac{1}{1-k^2\beta^2}\left\{ik\left(ikH_3 + k^2\beta E_3\right) + k^2\beta\left(ikE_3 - k^2\beta H_3\right)\right\}\\
&= \frac{1}{1-k^2\beta^2}\left\{-\left(k^2+k^4\beta^2\right)H_3 + 2ik^3\beta E_3\right\}
\end{aligned} \tag{1.32}$$

Now we derive the relations between E_1, H_1, E_2, H_2 and E_3, H_3. Starting from the first two rows of (1.25), (1.26),

$$\begin{aligned}
\left(1-k^2\beta^2\right)\partial_2 E_3 &= ikH_1 + k^2\beta E_1\\
-\left(1-k^2\beta^2\right)\partial_1 E_3 &= ikH_2 + k^2\beta E_2\\
\left(1-k^2\beta^2\right)\partial_2 H_3 &= -ikE_1 + k^2\beta H_1\\
-\left(1-k^2\beta^2\right)\partial_1 H_3 &= -ikE_2 + k^2\beta H_2
\end{aligned}$$

Rewriting in matrix form:

$$\begin{pmatrix} k^2\beta & 0 & ik & 0 \\ 0 & k^2\beta & 0 & ik \\ -ik & 0 & k^2\beta & 0 \\ 0 & -ik & 0 & k^2\beta \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{pmatrix} = \left(1 - k^2\beta^2\right) \begin{pmatrix} \partial_2 E_3 \\ -\partial_1 E_3 \\ \partial_2 H_3 \\ -\partial_1 H_3 \end{pmatrix}$$

The inverse matrix of the left hand side is

$$\begin{pmatrix} k^2\beta & 0 & ik & 0 \\ 0 & k^2\beta & 0 & ik \\ -ik & 0 & k^2\beta & 0 \\ 0 & -ik & 0 & k^2\beta \end{pmatrix}^{-1} = \frac{1}{1-k^2\beta^2} \begin{pmatrix} -\beta & 0 & \frac{i}{k} & 0 \\ 0 & -\beta & 0 & \frac{i}{k} \\ -\frac{i}{k} & 0 & -\beta & 0 \\ 0 & -\frac{i}{k} & 0 & -\beta \end{pmatrix}$$

Hence

$$\begin{aligned} \begin{pmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{pmatrix} &= \begin{pmatrix} -\beta & 0 & \frac{i}{k} & 0 \\ 0 & -\beta & 0 & \frac{i}{k} \\ -\frac{i}{k} & 0 & -\beta & 0 \\ 0 & -\frac{i}{k} & 0 & -\beta \end{pmatrix} \begin{pmatrix} \partial_2 E_3 \\ -\partial_1 E_3 \\ \partial_2 H_3 \\ -\partial_1 H_3 \end{pmatrix} \\ &= \begin{pmatrix} -\beta\partial_2 E_3 + \frac{i}{k}\partial_2 H_3 \\ \beta\partial_1 E_3 - \frac{i}{k}\partial_1 H_3 \\ -\beta\partial_2 H_3 - \frac{i}{k}\partial_2 E_3 \\ \beta\partial_1 H_3 + \frac{i}{k}\partial_1 E_3 \end{pmatrix} \end{aligned}$$

We have

$$\begin{aligned} E_1 &= -\beta\partial_2 E_3 + \frac{i}{k}\partial_2 H_3 \\ E_2 &= \beta\partial_1 E_3 - \frac{i}{k}\partial_1 H_3 \\ H_1 &= -\beta\partial_2 H_3 - \frac{i}{k}\partial_2 E_3 \\ H_2 &= \beta\partial_1 H_3 + \frac{i}{k}\partial_1 E_3 \end{aligned} \tag{1.33}$$

Now we derive terms used in formulating boundary conditions. In two dimensional case, the third component of ν vanishes. For any field U we have

$$\nu \times U = \begin{pmatrix} \nu_1 \\ \nu_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} \nu_2 U_3 \\ -\nu_1 U_3 \\ \nu_1 U_2 - \nu_2 U_1 \end{pmatrix} \tag{1.34}$$

The term that would need further deduction is $\nu_1 U_2 - \nu_2 U_1$. Combining (1.33), we have

$$\begin{aligned} \nu_1 E_2 - \nu_2 E_1 &= \nu_1 \left(\beta\partial_1 E_3 - \frac{i}{k}\partial_1 H_3\right) - \nu_2 \left(-\beta\partial_2 E_3 + \frac{i}{k}\partial_2 H_3\right) \\ &= \beta\nu \cdot \nabla E_3 - \frac{i}{k}\nu \cdot \nabla H_3 \\ &= \beta\frac{\partial E_3}{\partial\nu} - \frac{i}{k}\frac{\partial H_3}{\partial\nu} \end{aligned} \tag{1.35}$$

and

$$\begin{aligned}\nu_1 H_2 - \nu_2 H_1 &= \nu_1\left(\beta\partial_1 H_3 + \frac{i}{k}\partial_1 E_3\right) - \nu_2\left(-\beta\partial_2 H_3 - \frac{i}{k}\partial_2 E_3\right)\\ &= \beta\nu\cdot\nabla H_3 + \frac{i}{k}\nu\cdot\nabla E_3 \\ &= \beta\frac{\partial H_3}{\partial\nu} + \frac{i}{k}\frac{\partial E_3}{\partial\nu}\end{aligned} \tag{1.36}$$

The boundary conditions are

$$\nu\times E_{\mathrm{o}} = \delta\,\nu\times E_{\mathrm{i}} - \nu\times E_{\mathrm{e}} \tag{1.37}$$

$$\nu\times H_{\mathrm{o}} = \rho\,\nu\times H_{\mathrm{i}} - \nu\times H_{\mathrm{e}} \tag{1.38}$$

Expanding the first two rows of (1.37), we have

$$\begin{aligned}\nu_2 E_{\mathrm{o}3} &= \delta\,\nu_2 E_{\mathrm{i}3} - \nu_2 E_{\mathrm{e}3}\\ -\nu_1 E_{\mathrm{o}3} &= -\delta\,\nu_1 E_{\mathrm{i}3} + \nu_1 E_{\mathrm{e}3}\end{aligned}$$

Here the term $E_{\mathrm{o}3}$ denotes the third component of E_{o}; the meaning of similar terms $E_{\mathrm{i}3}$, $E_{\mathrm{e}3}$, $E_{\mathrm{e}1}$, etc. should be clear. The two equations are in fact only one, namely

$$E_{\mathrm{o}3} = \delta\,E_{\mathrm{i}3} - E_{\mathrm{e}3} \tag{1.39}$$

Similarly, expanding the first two rows of (1.38), we have

$$\begin{aligned}\nu_2 H_{\mathrm{o}3} &= \rho\,\nu_2 H_{\mathrm{i}3} - \nu_2 H_{\mathrm{e}3}\\ -\nu_1 H_{\mathrm{o}3} &= -\rho\,\nu_1 H_{\mathrm{i}3} + \nu_1 H_{\mathrm{e}3}\end{aligned}$$

which can be summarized as

$$H_{\mathrm{o}3} = \rho\,H_{\mathrm{i}3} - H_{\mathrm{e}3} \tag{1.40}$$

With the expressions (1.35), (1.36), the remaining two boundary conditions are

$$\nu_1 E_{\mathrm{o}2} - \nu_2 E_{\mathrm{o}1} = \delta\left(\beta_{\mathrm{i}}\frac{\partial E_{\mathrm{i}3}}{\partial\nu} - \frac{i}{k_{\mathrm{i}}}\frac{\partial H_{\mathrm{i}3}}{\partial\nu}\right) - \left(\beta_{\mathrm{e}}\frac{\partial E_{\mathrm{e}3}}{\partial\nu} - \frac{i}{k_{\mathrm{e}}}\frac{\partial H_{\mathrm{e}3}}{\partial\nu}\right) \tag{1.41}$$

and

$$\nu_1 H_{\mathrm{o}2} - \nu_2 H_{\mathrm{o}1} = \rho\left(\beta_{\mathrm{i}}\frac{\partial H_{\mathrm{i}3}}{\partial\nu} + \frac{i}{k_{\mathrm{i}}}\frac{\partial E_{\mathrm{i}3}}{\partial\nu}\right) - \left(\beta_{\mathrm{e}}\frac{\partial H_{\mathrm{e}3}}{\partial\nu} + \frac{i}{k_{\mathrm{e}}}\frac{\partial E_{\mathrm{e}3}}{\partial\nu}\right) \tag{1.42}$$

Recall the Bohren's transformation in outer environment and inner obstacle:

$$\begin{aligned}E_{\mathrm{e}3} &= \frac{1}{2}\left(Q_{\mathrm{er}} + Q_{\mathrm{el}}\right)\\ H_{\mathrm{e}3} &= \frac{i}{2}\left(Q_{\mathrm{er}} - Q_{\mathrm{el}}\right)\\ E_{\mathrm{i}3} &= \frac{1}{2}\left(Q_{\mathrm{ir}} + Q_{\mathrm{il}}\right)\\ H_{\mathrm{i}3} &= \frac{i}{2}\left(Q_{\mathrm{ir}} - Q_{\mathrm{il}}\right)\end{aligned} \tag{1.43}$$

Apply (1.43) in (1.39),

$$E_{o3} = \frac{\delta}{2}(Q_{ir} + Q_{il}) - \frac{1}{2}(Q_{er} + Q_{el})$$

Apply (1.43) in (1.40),

$$H_{o3} = \frac{i\rho}{2}(Q_{ir} - Q_{il}) - \frac{i}{2}(Q_{er} - Q_{el})$$

Apply (1.43) in (1.41),

$$\begin{aligned}
-\nu_1 E_{o2} + \nu_2 E_{o1} &= \delta \left\{ \beta_i \frac{\partial}{\partial \nu} \left(\frac{1}{2}(Q_{ir} + Q_{il}) \right) - \frac{i}{k_i} \frac{\partial}{\partial \nu} \left(\frac{i}{2} (Q_{ir} - Q_{il}) \right) \right\} \\
&\quad - \left\{ \beta_e \frac{\partial}{\partial \nu} \left(\frac{1}{2}(Q_{er} + Q_{el}) \right) - \frac{i}{k_e} \frac{\partial}{\partial \nu} \left(\frac{i}{2} (Q_{er} - Q_{el}) \right) \right\} \\
&= \frac{\delta}{2} \left(\frac{1 + k_i\beta_i}{k_i} \frac{\partial Q_{ir}}{\partial \nu} - \frac{1 - k_i\beta_i}{k_i} \frac{\partial Q_{il}}{\partial \nu} \right) \\
&\quad - \frac{1}{2} \left(\frac{1 + k_e\beta_e}{k_e} \frac{\partial Q_{er}}{\partial \nu} - \frac{1 - k_e\beta_e}{k_e} \frac{\partial Q_{el}}{\partial \nu} \right) \\
&= \frac{\delta}{2} \left(\frac{1}{\gamma_{ir}} \frac{\partial Q_{ir}}{\partial \nu} - \frac{1}{\gamma_{il}} \frac{\partial Q_{il}}{\partial \nu} \right) - \frac{1}{2} \left(\frac{1}{\gamma_{er}} \frac{\partial Q_{er}}{\partial \nu} - \frac{1}{\gamma_{el}} \frac{\partial Q_{el}}{\partial \nu} \right)
\end{aligned}$$

Apply (1.43) in (1.42),

$$\begin{aligned}
-\nu_1 H_{o2} + \nu_2 H_{o1} &= \rho \left\{ \beta_i \frac{\partial}{\partial \nu} \left(\frac{i}{2}(Q_{ir} - Q_{il}) \right) + \frac{i}{k_i} \frac{\partial}{\partial \nu} \left(\frac{1}{2} (Q_{ir} + Q_{il}) \right) \right\} \\
&\quad - \left\{ \beta_e \frac{\partial}{\partial \nu} \left(\frac{i}{2}(Q_{er} - Q_{el}) \right) + \frac{i}{k_e} \frac{\partial}{\partial \nu} \left(\frac{1}{2} (Q_{er} + Q_{el}) \right) \right\} \\
&= \frac{i\rho}{2} \left(\frac{1 + k_i\beta_i}{k_i} \frac{\partial Q_{ir}}{\partial \nu} + \frac{1 - k_i\beta_i}{k_i} \frac{\partial Q_{il}}{\partial \nu} \right) \\
&\quad - \frac{i}{2} \left(\frac{1 + k_e\beta_e}{k_e} \frac{\partial Q_{er}}{\partial \nu} + \frac{1 - k_e\beta_e}{k_e} \frac{\partial Q_{el}}{\partial \nu} \right) \\
&= \frac{i\rho}{2} \left(\frac{1}{\gamma_{ir}} \frac{\partial Q_{ir}}{\partial \nu} + \frac{1}{\gamma_{il}} \frac{\partial Q_{il}}{\partial \nu} \right) - \frac{i}{2} \left(\frac{1}{\gamma_{er}} \frac{\partial Q_{er}}{\partial \nu} + \frac{1}{\gamma_{el}} \frac{\partial Q_{el}}{\partial \nu} \right)
\end{aligned}$$

By writing

$$\begin{aligned}
v_0 &:= E_{o3} \\
w_0 &:= H_{o3} \\
v_1 &:= -\nu_1 E_{o2} + \nu_2 E_{o1} \\
w_1 &:= -\nu_1 H_{o2} + \nu_2 H_{o1}
\end{aligned} \tag{1.44}$$

we collect the previous four equations as the "master equations":

$$\begin{aligned}
v_0 &= \frac{\delta}{2}(Q_{\mathrm{ir}} + Q_{\mathrm{il}}) - \frac{1}{2}(Q_{\mathrm{er}} + Q_{\mathrm{el}}) \\
w_0 &= \frac{i\rho}{2}(Q_{\mathrm{ir}} - Q_{\mathrm{il}}) - \frac{i}{2}(Q_{\mathrm{er}} - Q_{\mathrm{el}}) \\
v_1 &= \frac{\delta}{2}\left(\frac{1}{\gamma_{\mathrm{ir}}}\frac{\partial Q_{\mathrm{ir}}}{\partial \nu} - \frac{1}{\gamma_{\mathrm{il}}}\frac{\partial Q_{\mathrm{il}}}{\partial \nu}\right) - \frac{1}{2}\left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} - \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right) \\
w_1 &= \frac{i\rho}{2}\left(\frac{1}{\gamma_{\mathrm{ir}}}\frac{\partial Q_{\mathrm{ir}}}{\partial \nu} + \frac{1}{\gamma_{\mathrm{il}}}\frac{\partial Q_{\mathrm{il}}}{\partial \nu}\right) - \frac{i}{2}\left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} + \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right)
\end{aligned} \tag{1.45}$$

1.3 Notations, Definitions and Prerequisites

Definition 1.1 (Boundary)**.** (**?** , p. 5]) Let Ω be an open subset in $\mathbb{R}^n$. The boundary $\Gamma = \partial\Omega$ is $C^{k,1}$ (resp. Lipchitz) if for $x \in \Gamma$ there exists a neighborhood V of x and new orthogonal coordinates $\{y_1, y_2, \ldots, y_n\}$ such that

1. V is an hypercube in the new coordinates:

$$V = \{(y_1, y_2, \ldots, y_n) | -a_j < y_j < a_j, 1 \leqslant j \leqslant n\}$$

2. There exists a $C^{k,1}$ (resp. Lipschitz) function φ, defined in

$$V' = \{(y_1, y_2, \ldots, y_{n-1}) | -a_j < y_j < a_j, 1 \leqslant j \leqslant n-1\}$$

such that

$$\begin{aligned}
|\varphi(y')| &\leqslant \frac{a_n}{2} \quad \forall y' = (y_1, y_2, \ldots, y_{n-1}) \in V' \\
\Omega \cap V &= \{y = (y', y_n) \in V |\, y_n < \varphi(y')\} \\
\Gamma \cap V &= \{y = (y', y_n) \in V |\, y_n = \varphi(y')\}
\end{aligned}$$

Proposition 1.1 (Vector Green Formula)**.**

$$\begin{aligned}
&\int_\Omega (E \cdot \Delta H - H \cdot \Delta E)\,\mathrm{d}V \\
&\qquad = \int_\Gamma (E \times \operatorname{curl} H + E \operatorname{div} H - H \times \operatorname{curl} E - H \operatorname{div} E) \cdot \nu \,\mathrm{d}\sigma
\end{aligned}$$

If $\operatorname{div} E = \operatorname{div} H = 0$, then

$$\begin{aligned}
\int_\Omega E \cdot \operatorname{curl}\operatorname{curl} H - H \cdot \operatorname{curl}\operatorname{curl} E \,\mathrm{d}V &= \int_\Gamma (E \times \operatorname{curl} H - H \times \operatorname{curl} E) \cdot \nu \,\mathrm{d}\sigma \\
&= \int_\Gamma (\nu \times E) \cdot \operatorname{curl} H - (\nu \times H) \cdot \operatorname{curl} E \,\mathrm{d}\sigma
\end{aligned} \tag{1.46}$$

Proposition 1.2 (Fundamental Theorem of Vector Analysis).

$$
\begin{aligned}
E(x) = &- \operatorname{curl} \int_\Gamma \nu(y) \times E(y)\Phi(x,y)\,\mathrm{d}\sigma(y) + \nabla \int_\Gamma \nu(y) \cdot E(y)\Phi(x,y)\,\mathrm{d}\sigma(y) \\
&- ik \int_\Gamma \nu(y) \times H(y)\Phi(x,y)\,\mathrm{d}\sigma(y) + \operatorname{curl} \int_\Omega \big\{\operatorname{curl} E(y) - ikH(y)\big\}\Phi(x,y)\,\mathrm{d}V(y) \\
&- \nabla \int_\Omega \operatorname{div} E(y)\Phi(x,y)\,\mathrm{d}V(y) + ik \int_\Omega \big\{\operatorname{curl} H(y) + ikE(y)\big\}\Phi(x,y)\,\mathrm{d}V(y).
\end{aligned}
$$

Proposition 1.3 (Stratton-Chu Representation Formula). If $E, H \in C^1(\Omega_+) \cap C(\Omega_+ \cup \Gamma)$ satisfy Maxwell equations in Ω_+ and the Silver-Müller radiation condition, then for $x \in \Omega_+$

$$
E(x) = \operatorname{curl} \int_\Gamma \nu(x) \times E(y)\Phi(x,y)\,\mathrm{d}\sigma(y) + \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times H(y)\Phi(x,y)\,\mathrm{d}\sigma(y)
$$

$$
H(x) = \operatorname{curl} \int_\Gamma \nu(x) \times H(y)\Phi(x,y)\,\mathrm{d}\sigma(y) - \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times E(y)\Phi(x,y)\,\mathrm{d}\sigma(y).
$$

For $x \in \Omega_-$:

$$
E(x) = - \operatorname{curl} \int_\Gamma \nu(y) \times E(y)\Phi(x,y)\,\mathrm{d}\sigma(y) - \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times H(y)\Phi(x,y)\,\mathrm{d}\sigma(y)
$$

$$
H(x) = - \operatorname{curl} \int_\Gamma \nu(y) \times H(y)\Phi(x,y)\,\mathrm{d}\sigma(y) + \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times E(y)\Phi(x,y)\,\mathrm{d}\sigma(y)
$$

Proposition 1.4 (Far Field Patterns).

$$
\begin{aligned}
E^\infty(\hat{x}) &= ik\,\hat{x} \times \int_\Gamma \{\nu(y) \times E(y) + (\nu(y) \times H(y)) \times \hat{x}\}\, e^{-ik\hat{x}\cdot y}\,\mathrm{d}\sigma(y) \\
H^\infty(\hat{x}) &= ik\,\hat{x} \times \int_\Gamma \{\nu(y) \times H(y) - (\nu(y) \times E(y)) \times \hat{x}\}\, e^{-ik\hat{x}\cdot y}\,\mathrm{d}\sigma(y)
\end{aligned}
$$

Proposition 1.5 (Rellich Lemma). If $E, H \in C^1(\Omega_+)$ is a radiating solution of Maxwell equations such that the electric far field pattern vanishes identically, then $E = H = 0$ in Ω_+.

Definition 1.2.

1. Γ: The regular (Lipschitzian) boundary of the open bounded set Ω_i in $\mathbb{R}^3$.

2. The tangential differentiation ∇_t is defined by

$$
\nabla_\mathrm{t} := \nu \times (\nu \times \nabla).
$$

3. Given a tangential vector field a, the surface divergence $\operatorname{div}_\Gamma a$ is defined as

$$
\int_\Gamma \phi \operatorname{div}_\Gamma a\,\mathrm{d}\sigma = -\int_\Gamma \nabla_\mathrm{t}\phi \cdot a\,\mathrm{d}\sigma, \qquad \forall \phi \in C^\infty(\mathbb{R}^3)
$$

4. $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma) = \{v \,|\, v \in L_2(\Gamma)^3,\ \nu \cdot v = 0,\ \operatorname{div}_\Gamma v \in L_2(\Gamma)\}$.

5. $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma) = \{v \mid v \in L_2(\Gamma)^3,\ \nu \cdot v = 0,\ \mathrm{curl}_\Gamma v \in L_2(\Gamma)\}$.

Proposition 1.6 (cf. Cessenat [6] section 2.4, corollary 2)**.** $v \to \nu \times v$ is an isomorphism from $L_{2,\mathrm{t}}^{\mathrm{curl}_\Gamma}$ to $L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}$ with inverse $w \to -\nu \times w$, and we have

$$\begin{aligned}\mathrm{curl}_\Gamma\, v &= -\,\mathrm{div}_\Gamma(\nu \times v)\\ \mathrm{div}_\Gamma\, w &= \mathrm{curl}_\Gamma(\nu \times w)\end{aligned}$$

for $v \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma), w \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$.

Definition 1.3. The Maxwell problem is to find a pair of radiating solution (E, H) to the Maxwell equations

$$\begin{aligned}\mathrm{curl}\, E - ikH &= 0\\ \mathrm{curl}\, H + ikE &= 0\end{aligned}$$

in $\mathbb{R}^3 \setminus \Omega$ with the boundary condition

$$\nu \times E = f$$

where $f \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$. The data-to-pattern operator $G : \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma) \to \mathbf{L}^2_{\mathrm{t}}(\mathbb{S}^2)$ is defined by

$$Gf = E^\infty$$

where E^∞ denotes the far field pattern of the radiating solution E of the Maxwell problem.

$$f^*(x) = \sup\{|f(y)| \mid y \in \Gamma(x), x \in \Gamma\}$$

$$\lim_{\substack{y \to x\\ y \in \Gamma(x)}} f(y) = u(x) \quad x \in \Gamma \text{ a.e.}$$

$$\begin{aligned}E_{\mathrm{n}} &:= (E \cdot \nu)\nu\\ E_{\mathrm{t}} &:= E - E_{\mathrm{n}}\\ \nabla_{\mathrm{t}} &:= \nu \times (\nu \times \nabla)\end{aligned}$$

Definition 1.4 (Silver-Müller radiation condition)**.**

$$\begin{aligned}\lim_{|x| \to \infty} (x \times H + |x| E) &= 0\\ \lim_{|x| \to \infty} (x \times E - |x| H) &= 0\end{aligned}$$

$$\mathcal{S}f(x) = \int_\Gamma \Phi(x, y) f(y)\, \mathrm{d}\sigma(y), \quad f \in L_2(\Gamma),\ x \in \mathbb{R}^3 \setminus \Gamma$$

$$\lim_{\substack{y\to x\\ y\in\Gamma_+(x)}} \mathcal{S}f(y) = \lim_{\substack{y\to x\\ y\in\Gamma_-(x)}} \mathcal{S}f(y) = \int_\Gamma \Phi(x,y)f(y)\,\mathrm{d}\sigma(y) =: Sf(x)$$

$$Kf(x) := \frac{1}{4\pi}\lim_{\varepsilon\downarrow 0}\int_{|x-y|\geqslant\varepsilon} \frac{(x-y)\cdot\nu(y)}{|x-y|^3} e^{ik|x-y|}(1-ik|x-y|)f(y)\,\mathrm{d}\sigma(y)$$

$$\lim_{\substack{y\to x\\ y\in\Gamma_\pm(x)}} \nabla\mathcal{S}f(y)\cdot\nu(x) = \big(\mp\frac{1}{2}I + K^*\big)f(x)$$

$$\|(\nabla Sf)^*\| \lesssim \|f\|$$

$$\lim_{\substack{y\to x\\ y\in\Gamma_\pm(x)}} \operatorname{div}\mathcal{S}a(y) = \mp\frac{1}{2}\nu(x)\cdot a(x) + \mathrm{pv}\int_\Gamma \operatorname{div}_x\{\Phi(x,y)a(y)\}\,\mathrm{d}\sigma(y)$$
$$\lim_{\substack{y\to x\\ y\in\Gamma_\pm(x)}} \operatorname{curl}\mathcal{S}a(y) = \mp\frac{1}{2}\nu(x)\times a(x) + \mathrm{pv}\int_\Gamma \operatorname{curl}_x\{\Phi(x,y)a(y)\}\,\mathrm{d}\sigma(y)$$

$$\lim_{\substack{y\to x\\ y\in\Gamma_\pm(x)}} \nu(x)\times\operatorname{curl}\mathcal{S}a(y) = \pm\frac{1}{2}a(x) + \mathrm{pv}\int_\Gamma \nu(x)\times\operatorname{curl}_x\{\Phi(x,y)a(y)\}\,\mathrm{d}\sigma(y)$$

$$\lim_{\substack{y_\pm\to x\\ y_\pm\in\Gamma_\pm(x)}} \nu(x)\times\big(\operatorname{curl}\operatorname{curl}\mathcal{S}a(y_+) - \operatorname{curl}\operatorname{curl}\mathcal{S}a(y_-)\big) = 0$$

$$Ma(x) = \nu(x)\times\mathrm{pv}\int_\Gamma \operatorname{curl}_x\{\Phi(x,y)a(y)\}\,\mathrm{d}\sigma(y)$$
$$Na(x) = \nu(x)\times\mathrm{pv}\int_\Gamma \operatorname{curl}\operatorname{curl}_x\{\Phi(x,y)a(y)\}\,\mathrm{d}\sigma(y)$$

$$\begin{aligned}
\nabla_y\times qe^{-ikx\cdot y} &= ik(q\times x)e^{-ikx\cdot y}\\
\nabla_y\times(\nabla_y\times qe^{-ikx\cdot y}) &= ik\cdot ik((q\times x)\times x)e^{-ikx\cdot y}\\
&= k^2 x\times(q\times x)e^{-ikx\cdot y}
\end{aligned}$$

$$\operatorname{curl}_x\left\{a(y)\frac{e^{ik|x-y|}}{|x-y|}\right\} = ik\frac{e^{ik|x|}}{|x|}\left\{e^{-ik\hat{x}\cdot y}(\hat{x}\times a) + O\left(\frac{|a|}{|x|}\right)\right\}$$
$$\operatorname{curl}\operatorname{curl}_x\left\{a(y)\frac{e^{ik|x-y|}}{|x-y|}\right\} = k^2\frac{e^{ik|x|}}{|x|}\left\{e^{-ik\hat{x}\cdot y}\,\hat{x}\times(\hat{x}\times a) + O\left(\frac{|a|}{|x|}\right)\right\}$$

$$\operatorname{div}_\Gamma Ma = -k^2\nu\cdot Sa - K^*(\operatorname{div}_\Gamma a) \quad \text{for tangential } a$$
$$\operatorname{div}_\Gamma(\nu\times E) = -\nu\cdot\operatorname{curl}E$$

1.3.1 Potentials and Boundary Integral Operators

$$
\begin{aligned}
\nu \times M_k^{\mathrm{i}}\, \varphi &= (M_k - I)\, \varphi \\
\nu \times M_k^{\mathrm{e}}\, \varphi &= (M_k + I)\, \varphi \\
\nu \times N_k^{\mathrm{i}}\, \varphi &= N_k\, \varphi \\
\nu \times N_k^{\mathrm{e}}\, \varphi &= N_k\, \varphi
\end{aligned}
$$

In two dimension, the fundamental solution of Helmholtz equation is

$$
\Phi_k(x,y) = \frac{i}{4} H_0^1(k|x-y|), \quad x \neq y
$$

Given k, for $x \in \Omega_{\mathrm{e}}$, define the potentials $S_k^{\mathrm{e}}, K_k^{\mathrm{e}}$

$$
\begin{aligned}
(S_k^{\mathrm{e}}\psi)(x) &:= 2 \int_{\Omega_{\mathrm{e}}} \Phi_k(x,y)\psi(y)\, \mathrm{d}V(y) \\
(K_k^{\mathrm{e}}\psi)(x) &:= 2 \int_{\Omega_{\mathrm{e}}} \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\psi(y)\, \mathrm{d}V(y)
\end{aligned}
$$

For $x \in \Omega_{\mathrm{i}}$, define the potentials $S_k^{\mathrm{i}}, K_k^{\mathrm{i}}$

$$
\begin{aligned}
(S_k^{\mathrm{i}}\psi)(x) &:= 2 \int_{\Omega_{\mathrm{i}}} \Phi_k(x,y)\psi(y)\, \mathrm{d}V(y) \\
(K_k^{\mathrm{i}}\psi)(x) &:= 2 \int_{\Omega_{\mathrm{i}}} \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\psi(y)\, \mathrm{d}V(y)
\end{aligned}
$$

For $x \in \Gamma$, define the boundary integral operators S_k, K_k, K_k' and T_k

$$
\begin{aligned}
(S_k\psi)(x) &:= 2 \int_{\Gamma} \Phi_k(x,y)\psi(y)\, \mathrm{d}\sigma(y) \\
(K_k\psi)(x) &:= 2 \int_{\Gamma} \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\psi(y)\, \mathrm{d}\sigma(y) \\
(K_k'\psi)(x) &:= 2 \int_{\Gamma} \frac{\partial \Phi_k(x,y)}{\partial \nu(x)}\psi(y)\, \mathrm{d}\sigma(y) \\
(T_k\psi)(x) &:= 2 \frac{\partial}{\partial \nu(x)} \int_{\Gamma} \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\psi(y)\, \mathrm{d}\sigma(y)
\end{aligned}
$$

1.4 Problem Statements

1.4.1 Direct Problem

Transmission Problem

Find electric fields $E_{\mathrm{e}}, E_{\mathrm{i}}$ and magnetic fields $H_{\mathrm{e}}, H_{\mathrm{i}}$ which satisfy the following equations

$$
\begin{aligned}
\operatorname{curl} E_{\mathrm{i}} &= \gamma_{\mathrm{i}}^2\, \beta_{\mathrm{i}}\, E_{\mathrm{i}} + i k_{\mathrm{i}} \left(\frac{\gamma_{\mathrm{i}}}{k_{\mathrm{i}}}\right)^2 H_{\mathrm{i}} \\
\operatorname{curl} H_{\mathrm{i}} &= \gamma_{\mathrm{i}}^2\, \beta_{\mathrm{i}}\, H_{\mathrm{i}} - i k_{\mathrm{i}} \left(\frac{\gamma_{\mathrm{i}}}{k_{\mathrm{i}}}\right)^2 E_{\mathrm{i}}
\end{aligned}
\tag{1.47}
$$

Table 1.1: All Possible Media Combinations

outer environment	inner obstacle	case no.
achiral	perfect conductor	I
achiral	achiral	II
achiral	chiral	III
chiral	perfect conductor	IV
chiral	achiral	V
chiral	chiral	VI
perfect conductor	perfect conductor	VII
perfect conductor	achiral	VIII
perfect conductor	chiral	IX

in Ω_i and

$$\begin{aligned} \operatorname{curl} E_\mathrm{e} &= \gamma_\mathrm{e}^2\,\beta_\mathrm{e}\,E_\mathrm{e} + ik_\mathrm{e}\left(\frac{\gamma_\mathrm{e}}{k_\mathrm{e}}\right)^2 H_\mathrm{e} \\ \operatorname{curl} H_\mathrm{e} &= \gamma_\mathrm{e}^2\,\beta_\mathrm{e}\,H_\mathrm{e} - ik_\mathrm{e}\left(\frac{\gamma_\mathrm{e}}{k_\mathrm{e}}\right)^2 E_\mathrm{e} \end{aligned} \tag{1.48}$$

in Ω_e, with boundary conditions

$$\begin{aligned} \nu \times E_\mathrm{o} &= \delta\,\nu \times E_\mathrm{i} - \nu \times E_\mathrm{e} \\ \nu \times H_\mathrm{o} &= \rho\,\nu \times H_\mathrm{i} - \nu \times H_\mathrm{e} \end{aligned} \tag{1.49}$$

and one of the following two Silver-Müller conditions

$$\hat{x} \times H_\mathrm{e}(x) + E_\mathrm{e}(x) = \mathrm{o}\left(\frac{1}{|x|}\right) \tag{1.50}$$

$$\hat{x} \times E_\mathrm{e}(x) - H_\mathrm{e}(x) = \mathrm{o}\left(\frac{1}{|x|}\right) \tag{1.51}$$

Here $E_\mathrm{o}, H_\mathrm{o}$ are the given incident fields in Ω_e.

Beltrami Transmission Problem

Given

$$\gamma_\mathrm{il} = \frac{k_\mathrm{i}}{1 - k_\mathrm{i}\,\beta_\mathrm{i}} \qquad \gamma_\mathrm{ir} = \frac{k_\mathrm{i}}{1 + k_\mathrm{i}\,\beta_\mathrm{i}}, \qquad \gamma_\mathrm{el} = \frac{k_\mathrm{e}}{1 - k_\mathrm{e}\,\beta_\mathrm{e}}, \qquad \gamma_\mathrm{er} = \frac{k_\mathrm{e}}{1 + k_\mathrm{e}\,\beta_\mathrm{e}} \tag{1.52}$$

find the fields $Q_\mathrm{il}, Q_\mathrm{ir}$ and $Q_\mathrm{el}, Q_\mathrm{er}$ which satisfy the following equations

$$\begin{aligned} \operatorname{curl} Q_\mathrm{il} &= \gamma_\mathrm{il}\,Q_\mathrm{il} \\ \operatorname{curl} Q_\mathrm{ir} &= -\gamma_\mathrm{ir}\,Q_\mathrm{ir} \end{aligned} \tag{1.53}$$

in Ω_i and

$$\begin{aligned} \operatorname{curl} Q_\mathrm{el} &= \gamma_\mathrm{el}\,Q_\mathrm{el} \\ \operatorname{curl} Q_\mathrm{er} &= -\gamma_\mathrm{er}\,Q_\mathrm{er} \end{aligned} \tag{1.54}$$

in Ω_{e}, with boundary conditions

$$\begin{aligned} \nu \times E_{\mathrm{o}} &= \delta\, \nu \times \frac{1}{2}\left(Q_{\mathrm{ir}} + Q_{\mathrm{il}}\right) - \nu \times \frac{1}{2}\left(Q_{\mathrm{er}} + Q_{\mathrm{el}}\right) \\ \nu \times H_{\mathrm{o}} &= \rho\, \nu \times \frac{i}{2}\left(Q_{\mathrm{ir}} - Q_{\mathrm{il}}\right) - \nu \times \frac{i}{2}\left(Q_{\mathrm{er}} - Q_{\mathrm{el}}\right) \end{aligned} \tag{1.55}$$

and Silver-Müller conditions

$$\begin{aligned} \hat{x} \times Q_{\mathrm{el}}(x) + i\, Q_{\mathrm{el}}(x) &= \mathrm{o}\left(\frac{1}{|x|}\right) \\ \hat{x} \times Q_{\mathrm{er}}(x) - i\, Q_{\mathrm{er}}(x) &= \mathrm{o}\left(\frac{1}{|x|}\right) \end{aligned} \tag{1.56}$$

Here $E_{\mathrm{o}}, H_{\mathrm{o}}$ are the given incident fields in Ω_{e}.

2D Transmission Problem

Find electric fields $E_{\mathrm{i}}, E_{\mathrm{e}}$ and magnetic fields $H_{\mathrm{i}}, H_{\mathrm{e}}$ which satisfy the following equations

$$\begin{aligned} \Delta E_{\mathrm{i}} &= \frac{1}{(1-k_{\mathrm{i}}^2\beta_{\mathrm{i}}^2)^2}\left\{-\left(k_{\mathrm{i}}^2 + k_{\mathrm{i}}^4\beta_{\mathrm{i}}^2\right)E_{\mathrm{i}} - 2ik_{\mathrm{i}}^3\beta_{\mathrm{i}}H_{\mathrm{i}}\right\} \\ \Delta H_{\mathrm{i}} &= \frac{1}{(1-k_{\mathrm{i}}^2\beta_{\mathrm{i}}^2)^2}\left\{-\left(k_{\mathrm{i}}^2 + k_{\mathrm{i}}^4\beta_{\mathrm{i}}^2\right)H_{\mathrm{i}} + 2ik_{\mathrm{i}}^3\beta_{\mathrm{i}}E_{\mathrm{i}}\right\} \end{aligned} \tag{1.57}$$

in Ω_{i} and

$$\begin{aligned} \Delta E_{\mathrm{e}} &= \frac{1}{(1-k_{\mathrm{e}}^2\beta_{\mathrm{e}}^2)^2}\left\{-\left(k_{\mathrm{e}}^2 + k_{\mathrm{e}}^4\beta_{\mathrm{e}}^2\right)E_{\mathrm{e}} - 2ik_{\mathrm{e}}^3\beta_{\mathrm{e}}H_{\mathrm{e}}\right\} \\ \Delta H_{\mathrm{e}} &= \frac{1}{(1-k_{\mathrm{e}}^2\beta_{\mathrm{e}}^2)^2}\left\{-\left(k_{\mathrm{e}}^2 + k_{\mathrm{e}}^4\beta_{\mathrm{e}}^2\right)H_{\mathrm{e}} + 2ik_{\mathrm{e}}^3\beta_{\mathrm{e}}E_{\mathrm{e}}\right\} \end{aligned} \tag{1.58}$$

in Ω_{e}, with boundary conditions

$$\begin{aligned} v_0 &= \delta\, E_{\mathrm{i}} - E_{\mathrm{e}} \\ w_0 &= \rho\, H_{\mathrm{i}} - H_{\mathrm{e}} \\ v_1 &= \delta\left(\beta_{\mathrm{i}}\frac{\partial E_{\mathrm{i}}}{\partial \nu} - \frac{i}{k_{\mathrm{i}}}\frac{\partial H_{\mathrm{i}}}{\partial \nu}\right) - \left(\beta_{\mathrm{e}}\frac{\partial E_{\mathrm{e}}}{\partial \nu} - \frac{i}{k_{\mathrm{e}}}\frac{\partial H_{\mathrm{e}}}{\partial \nu}\right) \\ w_1 &= \rho\left(\beta_{\mathrm{i}}\frac{\partial H_{\mathrm{i}}}{\partial \nu} + \frac{i}{k_{\mathrm{i}}}\frac{\partial E_{\mathrm{i}}}{\partial \nu}\right) - \left(\beta_{\mathrm{e}}\frac{\partial H_{\mathrm{e}}}{\partial \nu} + \frac{i}{k_{\mathrm{e}}}\frac{\partial E_{\mathrm{e}}}{\partial \nu}\right) \end{aligned} \tag{1.59}$$

and Sommerfeld radiation conditions

$$\frac{\partial E_{\mathrm{e}}}{\partial r} - ik_{\mathrm{e}}E_{\mathrm{e}} = \mathrm{o}\left(\frac{1}{\sqrt{r}}\right) \tag{1.60}$$

$$\frac{\partial H_{\mathrm{e}}}{\partial r} + ik_{\mathrm{e}}H_{\mathrm{e}} = \mathrm{o}\left(\frac{1}{\sqrt{r}}\right) \tag{1.61}$$

as $r \to \infty$. Here v_0, w_0, v_1, w_1 are the given incident fields in Ω_{e}.

2D Beltrami Transmission Problem

Find fields $Q_{\mathrm{il}}, Q_{\mathrm{ir}}$ and $Q_{\mathrm{el}}, Q_{\mathrm{er}}$ which satisfy the following equations

$$\begin{aligned}(\Delta + \gamma_{\mathrm{il}}^2)\, Q_{\mathrm{il}} &= 0 \\ (\Delta + \gamma_{\mathrm{ir}}^2)\, Q_{\mathrm{ir}} &= 0\end{aligned} \tag{1.62}$$

in Ω_{i} and

$$\begin{aligned}(\Delta + \gamma_{\mathrm{el}}^2)\, Q_{\mathrm{el}} &= 0 \\ (\Delta + \gamma_{\mathrm{er}}^2)\, Q_{\mathrm{er}} &= 0\end{aligned} \tag{1.63}$$

in Ω_{e}, with boundary conditions

$$\begin{aligned}
v_0 &= \frac{\delta}{2}(Q_{\mathrm{ir}} + Q_{\mathrm{il}}) - \frac{1}{2}(Q_{\mathrm{er}} + Q_{\mathrm{el}}) \\
w_0 &= \frac{i\rho}{2}(Q_{\mathrm{ir}} - Q_{\mathrm{il}}) - \frac{i}{2}(Q_{\mathrm{er}} - Q_{\mathrm{el}}) \\
v_1 &= \frac{\delta}{2}\left(\frac{1}{\gamma_{\mathrm{ir}}}\frac{\partial Q_{\mathrm{ir}}}{\partial \nu} - \frac{1}{\gamma_{\mathrm{il}}}\frac{\partial Q_{\mathrm{il}}}{\partial \nu}\right) - \frac{1}{2}\left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} - \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right) \\
w_1 &= \frac{i\rho}{2}\left(\frac{1}{\gamma_{\mathrm{ir}}}\frac{\partial Q_{\mathrm{ir}}}{\partial \nu} + \frac{1}{\gamma_{\mathrm{il}}}\frac{\partial Q_{\mathrm{il}}}{\partial \nu}\right) - \frac{i}{2}\left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} + \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right)
\end{aligned} \tag{1.64}$$

and Sommerfeld radiation conditions

$$\frac{\partial Q_{\mathrm{el}}}{\partial r} + ik_{\mathrm{e}} Q_{\mathrm{el}} = o\left(\frac{1}{\sqrt{r}}\right) \tag{1.65}$$

$$\frac{\partial Q_{\mathrm{er}}}{\partial r} - ik_{\mathrm{e}} Q_{\mathrm{er}} = o\left(\frac{1}{\sqrt{r}}\right) \tag{1.66}$$

as $r \to \infty$. Here v_0, w_0, v_1, w_1 are the given incident fields in Ω_{e}.

1.4.2 Inverse Problem

$\mathbf{L}^2_{\mathrm{t}}$, $\mathbf{H}^1(\Omega)$

Chapter 2

Direct Problems

2.1 Three Dimensional Cases

2.1.1 Uniqueness

Lemma 2.1. For $Q_1 \in H^1_{\mathrm{div}_\Gamma}(\Omega_\mathrm{i})$ and $Q_2 \in H^1_{\mathrm{div}_\Gamma,\mathrm{loc}}(\Omega_\mathrm{e})$ satisfy

$$\begin{aligned} \operatorname{curl} Q_1 &= \lambda_1 Q_1 \\ \operatorname{curl} Q_2 &= \lambda_2 Q_2 \end{aligned}$$

and the radiation condition

$$\hat{x} \times Q_2(x) + i\, Q_2(x) = \mathrm{o}\left(\frac{1}{|x|}\right), \qquad |x| \to \infty$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\Im\lambda_1, \Im\lambda_2 \geqslant 0$, if

$$\nu \times Q_2 = \alpha\, \nu \times Q_1 \qquad \text{on } \Gamma \tag{2.1}$$

for $\alpha \in \mathbb{C} \setminus \{0\}$, then

$$\begin{aligned} Q_1 &= 0 \qquad \text{on } \Omega_\mathrm{i} \\ Q_2 &= 0 \qquad \text{on } \Omega_\mathrm{e} \end{aligned}$$

Proof. From (2.1) we have

$$s$$

□

2.1.2 Existence

Chiral-Chiral

We propose the following ansatz

$$\begin{aligned} Q_{\mathrm{il}} &= \left(\gamma_{\mathrm{il}}\, M^{\mathrm{i}}_{\gamma_{\mathrm{il}}} + N^{\mathrm{i}}_{\gamma_{\mathrm{il}}}\right) \psi_1 \\ Q_{\mathrm{ir}} &= \left(-\gamma_{\mathrm{ir}}\, M^{\mathrm{i}}_{\gamma_{\mathrm{ir}}} + N^{\mathrm{i}}_{\gamma_{\mathrm{ir}}}\right) \psi_2 \\ Q_{\mathrm{el}} &= \left(\gamma_{\mathrm{el}}\, M^{\mathrm{e}}_{\gamma_{\mathrm{el}}} + N^{\mathrm{e}}_{\gamma_{\mathrm{el}}}\right) (\zeta_{11}\, \psi_1 + \zeta_{12}\, \psi_2) \\ Q_{\mathrm{er}} &= \left(-\gamma_{\mathrm{er}}\, M^{\mathrm{e}}_{\gamma_{\mathrm{er}}} + N^{\mathrm{e}}_{\gamma_{\mathrm{er}}}\right) (\zeta_{21}\, \psi_1 + \zeta_{22}\, \psi_2) \end{aligned}$$

where ψ_j's are unknowns and ζ_{ij}'s are constants to be determined later. The tangential boundary traces are

$$\begin{aligned} \nu \times Q_{\mathrm{il}} &= \big(\gamma_{\mathrm{il}}(M_{\gamma_{\mathrm{il}}} - I) + N_{\gamma_{\mathrm{il}}}\big)\psi_1 \\ \nu \times Q_{\mathrm{ir}} &= \big(-\gamma_{\mathrm{ir}}(M_{\gamma_{\mathrm{ir}}} - I) + N_{\gamma_{\mathrm{ir}}}\big)\psi_2 \\ \nu \times Q_{\mathrm{el}} &= \big(\gamma_{\mathrm{el}}(M_{\gamma_{\mathrm{el}}} + I) + N_{\gamma_{\mathrm{el}}}\big)(\zeta_{11}\, \psi_1 + \zeta_{12}\, \psi_2) \\ \nu \times Q_{\mathrm{er}} &= \big(-\gamma_{\mathrm{er}}(M_{\gamma_{\mathrm{er}}} + I) + N_{\gamma_{\mathrm{er}}}\big)(\zeta_{21}\, \psi_1 + \zeta_{22}\, \psi_2) \end{aligned}$$

Substituting into the "master equations", we have

$$\begin{aligned} v =& \frac{\delta}{2}(-\gamma_{\mathrm{ir}}(M_{\gamma_{\mathrm{ir}}} - I)\psi_2 + \gamma_{\mathrm{il}}(M_{\gamma_{\mathrm{il}}} - I)\psi_1 + N_{\gamma_{\mathrm{ir}}}\psi_2 + N_{\gamma_{\mathrm{il}}}\psi_1) \\ &- \frac{1}{2}\big((N_{\gamma_{\mathrm{er}}} - \gamma_{\mathrm{er}}(I + M_{\gamma_{\mathrm{er}}}))(\psi_2\zeta_{22} + \psi_1\zeta_{21}) + (\gamma_{\mathrm{el}}(I + M_{\gamma_{\mathrm{el}}}) + N_{\gamma_{\mathrm{el}}})(\psi_2\zeta_{12} + \psi_1\zeta_{11})\big) \end{aligned}$$

and

$$\begin{aligned} w =& \frac{i\rho}{2}\big(-\gamma_{\mathrm{ir}}(M_{\gamma_{\mathrm{ir}}} - I)\psi_2 - \gamma_{\mathrm{il}}(M_{\gamma_{\mathrm{il}}} - I)\psi_1 + N_{\gamma_{\mathrm{ir}}}\psi_2 - N_{\gamma_{\mathrm{il}}}\psi_1\big) \\ &- \frac{i}{2}\big((N_{\gamma_{\mathrm{er}}} - \gamma_{\mathrm{er}}(I + M_{\gamma_{\mathrm{er}}}))(\psi_2\zeta_{22} + \psi_1\zeta_{21}) - (\gamma_{\mathrm{el}}(I + M_{\gamma_{\mathrm{el}}}) + N_{\gamma_{\mathrm{el}}})(\psi_2\zeta_{12} + \psi_1\zeta_{11})\big) \end{aligned}$$

Put the previous two equations into matrix form, we have

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix}$$

where

$$
\begin{aligned}
c_{11} &= \left(\frac{\gamma_{\mathrm{er}}\zeta_{21}}{2} - \frac{\gamma_{\mathrm{el}}\zeta_{11}}{2} - \frac{\delta\gamma_{\mathrm{il}}}{2}\right) I - N_{\gamma_{\mathrm{er}}}\frac{\zeta_{21}}{2} + \gamma_{\mathrm{er}} M_{\gamma_{\mathrm{er}}}\frac{\zeta_{21}}{2} \\
&\qquad - N_{\gamma_{\mathrm{el}}}\frac{\zeta_{11}}{2} - \gamma_{\mathrm{el}} M_{\gamma_{\mathrm{el}}}\frac{\zeta_{11}}{2} + N_{\gamma_{\mathrm{il}}}\frac{\delta}{2} + \frac{\delta\gamma_{\mathrm{il}}}{2} M_{\gamma_{\mathrm{il}}} \\
c_{12} &= \left(\frac{\gamma_{\mathrm{er}}\zeta_{22}}{2} - \frac{\gamma_{\mathrm{el}}\zeta_{12}}{2} + \frac{\delta\gamma_{\mathrm{ir}}}{2}\right) I - N_{\gamma_{\mathrm{er}}}\frac{\zeta_{22}}{2} + M_{\gamma_{\mathrm{er}}}\frac{\gamma_{\mathrm{er}}\zeta_{22}}{2} \\
&\qquad - N_{\gamma_{\mathrm{el}}}\frac{\zeta_{12}}{2} - M_{\gamma_{\mathrm{el}}}\frac{\gamma_{\mathrm{el}}\zeta_{12}}{2} + N_{\gamma_{\mathrm{ir}}}\frac{\delta}{2} - \frac{\delta\gamma_{\mathrm{ir}}}{2} M_{\gamma_{\mathrm{ir}}} \\
c_{21} &= \left(\frac{i\gamma_{\mathrm{er}}\zeta_{21}}{2} + \frac{i\gamma_{\mathrm{el}}\zeta_{11}}{2} + \frac{i\gamma_{\mathrm{il}}\rho}{2}\right) I - iN_{\gamma_{\mathrm{er}}}\frac{\zeta_{21}}{2} + i\gamma_{\mathrm{er}} M_{\gamma_{\mathrm{er}}}\frac{\zeta_{21}}{2} \\
&\qquad + iN_{\gamma_{\mathrm{el}}}\frac{\zeta_{11}}{2} + i\gamma_{\mathrm{el}} M_{\gamma_{\mathrm{el}}}\frac{\zeta_{11}}{2} - iN_{\gamma_{\mathrm{il}}}\frac{\rho}{2} - i\gamma_{\mathrm{il}} M_{\gamma_{\mathrm{il}}}\frac{\rho}{2} \\
c_{22} &= \left(\frac{i\gamma_{\mathrm{er}}\zeta_{22}}{2} + \frac{i\gamma_{\mathrm{el}}\zeta_{12}}{2} + \frac{i\gamma_{\mathrm{ir}}\rho}{2}\right) I - iN_{\gamma_{\mathrm{er}}}\frac{\zeta_{22}}{2} + i\gamma_{\mathrm{er}} M_{\gamma_{\mathrm{er}}}\frac{\zeta_{22}}{2} \\
&\qquad + iN_{\gamma_{\mathrm{el}}}\frac{\zeta_{12}}{2} + i\gamma_{\mathrm{el}} M_{\gamma_{\mathrm{el}}}\frac{\zeta_{12}}{2} + iN_{\gamma_{\mathrm{ir}}}\frac{\rho}{2} - i\gamma_{\mathrm{ir}} M_{\gamma_{\mathrm{ir}}}\frac{\rho}{2}
\end{aligned}
$$

We wish to make the appearance of hypersingular operators N_k's in $c_{11}, c_{12}, c_{21}, c_{22}$ to be in the form of the linear combinations of $N_{k_1} - N_{k_2}$, hence

$$
\begin{aligned}
-\frac{\zeta_{21}}{2} - \frac{\zeta_{11}}{2} + \frac{\delta}{2} &= 0 \\
-\frac{\zeta_{22}}{2} - \frac{\zeta_{12}}{2} + \frac{\delta}{2} &= 0 \\
-\frac{i\zeta_{21}}{2} + i\frac{\zeta_{11}}{2} - i\frac{\rho}{2} &= 0 \\
-i\frac{\zeta_{22}}{2} + i\frac{\zeta_{12}}{2} + i\frac{\rho}{2} &= 0
\end{aligned}
$$

Solving the above, we have

$$
\zeta_{11} = \frac{\delta+\rho}{2} \qquad \zeta_{12} = \frac{\delta-\rho}{2} \qquad \zeta_{21} = \frac{\delta-\rho}{2} \qquad \zeta_{22} = \frac{\delta+\rho}{2}
$$

Hence

$$\begin{aligned}
c_{11} = &\left(\frac{-\gamma_{\mathrm{el}}(\rho+\delta)}{4} + \frac{\gamma_{\mathrm{er}}(\delta-\rho)}{4} - \frac{\delta\gamma_{\mathrm{il}}}{2}\right) I - N_{\gamma_{\mathrm{el}}}\frac{\rho+\delta}{4} - \gamma_{\mathrm{el}} M_{\gamma_{\mathrm{el}}}\frac{\rho+\delta}{4} \\
&+ N_{\gamma_{\mathrm{er}}}\frac{\rho-\delta}{4} - \gamma_{\mathrm{er}} M_{\gamma_{\mathrm{er}}}\frac{\rho-\delta}{4} + N_{\gamma_{\mathrm{il}}}\frac{\delta}{2} + M_{\gamma_{\mathrm{il}}}\frac{\delta\gamma_{\mathrm{il}}}{2} \\
c_{12} = &\left(\frac{\gamma_{\mathrm{er}}(\rho+\delta)}{4} - \frac{\gamma_{\mathrm{el}}(\delta-\rho)}{4} + \frac{\delta\gamma_{\mathrm{ir}}}{2}\right) I - N_{\gamma_{\mathrm{er}}}\frac{\rho+\delta}{4} + \gamma_{\mathrm{er}} M_{\gamma_{\mathrm{er}}}\frac{\rho+\delta}{4} \\
&+ N_{\gamma_{\mathrm{el}}}\frac{\rho-\delta}{4} + \gamma_{\mathrm{el}} M_{\gamma_{\mathrm{el}}}\frac{\rho-\delta}{4} + N_{\gamma_{\mathrm{ir}}}\frac{\delta}{2} - M_{\gamma_{\mathrm{ir}}}\frac{\delta\gamma_{\mathrm{ir}}}{2} \\
c_{21} = &\left(\frac{i\gamma_{\mathrm{el}}(\rho+\delta)}{4} + \frac{i\gamma_{\mathrm{il}}\rho}{2} + \frac{i\gamma_{\mathrm{er}}(\delta-\rho)}{4}\right) I + iN_{\gamma_{\mathrm{el}}}\frac{\rho+\delta}{4} + i\gamma_{\mathrm{el}} M_{\gamma_{\mathrm{el}}}\frac{\rho+\delta}{4} \\
&+ iN_{\gamma_{\mathrm{er}}}\frac{\rho-\delta}{4} - i\gamma_{\mathrm{er}} M_{\gamma_{\mathrm{er}}}\frac{\rho-\delta}{4} - iN_{\gamma_{\mathrm{il}}}\frac{\rho}{2} - i\gamma_{\mathrm{il}} M_{\gamma_{\mathrm{il}}}\frac{\rho}{2} \\
c_{22} = &\left(\frac{i\gamma_{\mathrm{er}}(\rho+\delta)}{4} + \frac{i\gamma_{\mathrm{ir}}\rho}{2} + \frac{i\gamma_{\mathrm{el}}(\delta-\rho)}{4}\right) I - iN_{\gamma_{\mathrm{er}}}\frac{\rho+\delta}{4} + i\gamma_{\mathrm{er}} M_{\gamma_{\mathrm{er}}}\frac{\rho+\delta}{4} \\
&- iN_{\gamma_{\mathrm{el}}}\frac{\rho-\delta}{4} - i\gamma_{\mathrm{el}} M_{\gamma_{\mathrm{el}}}\frac{\rho-\delta}{4} + iN_{\gamma_{\mathrm{ir}}}\frac{\rho}{2} - i\gamma_{\mathrm{ir}} M_{\gamma_{\mathrm{ir}}}\frac{\rho}{2}
\end{aligned}$$

Decomposing $c_{ij} = e_{ij} + a_{ij}$ where e_{ij} only involves the identity transform I and $a_{ij} = c_{ij} - e_{ij}$, we have

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with

$$\begin{aligned}
e_{11} &= \left(\frac{-\gamma_{\mathrm{el}}(\rho+\delta)}{4} + \frac{\gamma_{\mathrm{er}}(\delta-\rho)}{4} - \frac{\delta\gamma_{\mathrm{il}}}{2}\right) I \\
e_{12} &= \left(\frac{\gamma_{\mathrm{er}}(\rho+\delta)}{4} - \frac{\gamma_{\mathrm{el}}(\delta-\rho)}{4} + \frac{\delta\gamma_{\mathrm{ir}}}{2}\right) I \\
e_{21} &= \left(\frac{i\gamma_{\mathrm{el}}(\rho+\delta)}{4} + \frac{i\gamma_{\mathrm{il}}\rho}{2} + \frac{i\gamma_{\mathrm{er}}(\delta-\rho)}{4}\right) I \\
e_{22} &= \left(\frac{i\gamma_{\mathrm{er}}(\rho+\delta)}{4} + \frac{i\gamma_{\mathrm{ir}}\rho}{2} + \frac{i\gamma_{\mathrm{el}}(\delta-\rho)}{4}\right) I
\end{aligned}$$

and

$$\begin{aligned}
a_{11} &= -N_{\gamma_{\mathrm{el}}}\frac{\rho+\delta}{4} - \gamma_{\mathrm{el}} M_{\gamma_{\mathrm{el}}}\frac{\rho+\delta}{4} + N_{\gamma_{\mathrm{er}}}\frac{\rho-\delta}{4} - \gamma_{\mathrm{er}} M_{\gamma_{\mathrm{er}}}\frac{\rho-\delta}{4} + N_{\gamma_{\mathrm{il}}}\frac{\delta}{2} + M_{\gamma_{\mathrm{il}}}\frac{\delta\gamma_{\mathrm{il}}}{2} \\
a_{12} &= -N_{\gamma_{\mathrm{er}}}\frac{\rho+\delta}{4} + \gamma_{\mathrm{er}} M_{\gamma_{\mathrm{er}}}\frac{\rho+\delta}{4} + N_{\gamma_{\mathrm{el}}}\frac{\rho-\delta}{4} + \gamma_{\mathrm{el}} M_{\gamma_{\mathrm{el}}}\frac{\rho-\delta}{4} + N_{\gamma_{\mathrm{ir}}}\frac{\delta}{2} - M_{\gamma_{\mathrm{ir}}}\frac{\delta\gamma_{\mathrm{ir}}}{2} \\
a_{21} &= +iN_{\gamma_{\mathrm{el}}}\frac{\rho+\delta}{4} + i\gamma_{\mathrm{el}} M_{\gamma_{\mathrm{el}}}\frac{\rho+\delta}{4} + iN_{\gamma_{\mathrm{er}}}\frac{\rho-\delta}{4} - i\gamma_{\mathrm{er}} M_{\gamma_{\mathrm{er}}}\frac{\rho-\delta}{4} - iN_{\gamma_{\mathrm{il}}}\frac{\rho}{2} - i\gamma_{\mathrm{il}} M_{\gamma_{\mathrm{il}}}\frac{\rho}{2} \\
a_{22} &= -iN_{\gamma_{\mathrm{er}}}\frac{\rho+\delta}{4} + i\gamma_{\mathrm{er}} M_{\gamma_{\mathrm{er}}}\frac{\rho+\delta}{4} - iN_{\gamma_{\mathrm{el}}}\frac{\rho-\delta}{4} - i\gamma_{\mathrm{el}} M_{\gamma_{\mathrm{el}}}\frac{\rho-\delta}{4} + iN_{\gamma_{\mathrm{ir}}}\frac{\rho}{2} - i\gamma_{\mathrm{ir}} M_{\gamma_{\mathrm{ir}}}\frac{\rho}{2}
\end{aligned}$$

The determinant of $\{e_{ij}\}$ is

$$-\frac{i}{8}\big(\gamma_{\mathrm{er}}\gamma_{\mathrm{ir}}\rho^2+\gamma_{\mathrm{el}}\gamma_{\mathrm{ir}}\rho^2+\gamma_{\mathrm{er}}\gamma_{\mathrm{il}}\rho^2+\gamma_{\mathrm{el}}\gamma_{\mathrm{il}}\rho^2 \\ +4\delta\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}\rho-2\delta\gamma_{\mathrm{er}}\gamma_{\mathrm{ir}}\rho+2\delta\gamma_{\mathrm{el}}\gamma_{\mathrm{ir}}\rho+2\delta\gamma_{\mathrm{er}}\gamma_{\mathrm{il}}\rho-2\delta\gamma_{\mathrm{el}}\gamma_{\mathrm{il}}\rho+4\delta\gamma_{\mathrm{el}}\gamma_{\mathrm{er}}\rho \\ +\delta^2\gamma_{\mathrm{er}}\gamma_{\mathrm{ir}}+\delta^2\gamma_{\mathrm{el}}\gamma_{\mathrm{ir}}+\delta^2\gamma_{\mathrm{er}}\gamma_{\mathrm{il}}+\delta^2\gamma_{\mathrm{el}}\gamma_{\mathrm{il}}\big)$$

Chiral-Perfect Conductor

$$\frac{i(\gamma_{\mathrm{er}}+\gamma_{\mathrm{el}})}{2}I+\frac{i}{2}N_{\gamma_{\mathrm{el}}}-\frac{i}{2}N_{\gamma_{\mathrm{er}}}+\frac{i\gamma_{\mathrm{er}}}{2}M_{\gamma_{\mathrm{er}}}+\frac{i\gamma_{\mathrm{el}}}{2}M_{\gamma_{\mathrm{el}}}$$

Theorem 2.1 (Analytic Fredholm Theorem)**.** Let D be an open connected subset of $\mathbb{C}$, and $f : D \to \mathcal{L}(H)$ be an analytic operator-valued function such that $f(z)$ is compact for $z \in D$. Then one of the following cases holds:

1. $(1-f(z))^{-1}$ exists for no $z \in D$.

2. $(1-f(z))^{-1}$ exists for $z \in D \setminus S$, where S has no limit points in D.

In case (2), $(1-f(z))^{-1}$ is meromorphic in D, analytic in $D \setminus S$, the residues at the poles are finite rank operators and if $z \in S$ then $f(z)\psi = \psi$ has a nonzero solution in H.

2.2 Two Dimensional Cases

2.2.1 Uniqueness

2.2.2 Existence

Chiral-Chiral

We propose the following ansatz

$$\begin{aligned}
Q_{\mathrm{il}} &= K^{\mathrm{i}}_{\gamma_{\mathrm{il}}}(\zeta_{11}\psi_1+\zeta_{12}\psi_2)+S^{\mathrm{i}}_{\gamma_{\mathrm{il}}}(\zeta_{13}\psi_3+\zeta_{14}\psi_4)\\
Q_{\mathrm{ir}} &= K^{\mathrm{i}}_{\gamma_{\mathrm{ir}}}(\zeta_{21}\psi_1+\zeta_{22}\psi_2)+S^{\mathrm{i}}_{\gamma_{\mathrm{ir}}}(\zeta_{23}\psi_3+\zeta_{24}\psi_4)\\
Q_{\mathrm{el}} &= K^{\mathrm{e}}_{\gamma_{\mathrm{el}}}(\zeta_{31}\psi_1+\zeta_{32}\psi_2)+S^{\mathrm{e}}_{\gamma_{\mathrm{el}}}(\zeta_{33}\psi_3+\zeta_{34}\psi_4)\\
Q_{\mathrm{er}} &= K^{\mathrm{e}}_{\gamma_{\mathrm{er}}}(\zeta_{41}\psi_1+\zeta_{42}\psi_2)+S^{\mathrm{e}}_{\gamma_{\mathrm{er}}}(\zeta_{43}\psi_3+\zeta_{44}\psi_4)
\end{aligned}$$

where ψ_j's are unknowns and ζ_{ij}'s are constants to be determined later. The boundary traces are

$$\begin{aligned}
Q_{\text{il}} &= (K_{\gamma_{\text{il}}} - I)(\zeta_{11}\psi_1 + \zeta_{12}\psi_2) + S_{\gamma_{\text{il}}}(\zeta_{13}\psi_3 + \zeta_{14}\psi_4)\\
Q_{\text{ir}} &= (K_{\gamma_{\text{ir}}} - I)(\zeta_{21}\psi_1 + \zeta_{22}\psi_2) + S_{\gamma_{\text{ir}}}(\zeta_{23}\psi_3 + \zeta_{24}\psi_4)\\
Q_{\text{el}} &= (K_{\gamma_{\text{el}}} + I)(\zeta_{31}\psi_1 + \zeta_{32}\psi_2) + S_{\gamma_{\text{el}}}(\zeta_{33}\psi_3 + \zeta_{34}\psi_4)\\
Q_{\text{er}} &= (K_{\gamma_{\text{er}}} + I)(\zeta_{41}\psi_1 + \zeta_{42}\psi_2) + S_{\gamma_{\text{er}}}(\zeta_{43}\psi_3 + \zeta_{44}\psi_4)\\
\frac{\partial Q_{\text{il}}}{\partial \nu} &= T_{\gamma_{\text{il}}}(\zeta_{11}\psi_1 + \zeta_{12}\psi_2) + (K'_{\gamma_{\text{il}}} + I)(\zeta_{13}\psi_3 + \zeta_{14}\psi_4)\\
\frac{\partial Q_{\text{ir}}}{\partial \nu} &= T_{\gamma_{\text{ir}}}(\zeta_{21}\psi_1 + \zeta_{22}\psi_2) + (K'_{\gamma_{\text{ir}}} + I)(\zeta_{23}\psi_3 + \zeta_{24}\psi_4)\\
\frac{\partial Q_{\text{el}}}{\partial \nu} &= T_{\gamma_{\text{el}}}(\zeta_{31}\psi_1 + \zeta_{32}\psi_2) + (K'_{\gamma_{\text{el}}} - I)(\zeta_{33}\psi_3 + \zeta_{34}\psi_4)\\
\frac{\partial Q_{\text{er}}}{\partial \nu} &= T_{\gamma_{\text{er}}}(\zeta_{41}\psi_1 + \zeta_{42}\psi_2) + (K'_{\gamma_{\text{er}}} - I)(\zeta_{43}\psi_3 + \zeta_{44}\psi_4)
\end{aligned}$$

Substituting into the "master equations" (1.45), we have

$$\begin{aligned}
v_0 = \frac{\delta}{2}\big(&(K_{\gamma_{\text{ir}}} - I)(\zeta_{21}\psi_1 + \zeta_{22}\psi_2) + (K_{\gamma_{\text{il}}} - I)(\zeta_{11}\psi_1 + \zeta_{12}\psi_2)\\
&+S_{\gamma_{\text{ir}}}(\zeta_{23}\psi_3 + \zeta_{24}\psi_4) + S_{\gamma_{\text{il}}}(\zeta_{13}\psi_3 + \zeta_{14}\psi_4)\big)\\
&- \frac{1}{2}\big((K_{\gamma_{\text{er}}} + I)(\zeta_{41}\psi_1 + \zeta_{42}\psi_2) + (K_{\gamma_{\text{el}}} + I)(\zeta_{31}\psi_1 + \zeta_{32}\psi_2)\\
&+ S_{\gamma_{\text{er}}}(\zeta_{43}\psi_3 + \zeta_{44}\psi_4) + S_{\gamma_{\text{el}}}(\zeta_{33}\psi_3 + \zeta_{34}\psi_4)\big)
\end{aligned}$$

and

$$\begin{aligned}
w_0 = \frac{i\rho}{2}\big(&(K_{\gamma_{\text{ir}}} - I)(\psi_2\zeta_{22} + \psi_1\zeta_{21}) - (K_{\gamma_{\text{il}}} - I)(\psi_2\zeta_{12} + \psi_1\zeta_{11})\\
&+S_{\gamma_{\text{ir}}}(\psi_4\zeta_{24} + \psi_3\zeta_{23}) - S_{\gamma_{\text{il}}}(\psi_4\zeta_{14} + \psi_3\zeta_{13})\big)\\
&- \frac{i}{2}\big((K_{\gamma_{\text{er}}} + I)(\psi_2\zeta_{42} + \psi_1\zeta_{41}) - (K_{\gamma_{\text{el}}} + I)(\psi_2\zeta_{32} + \psi_1\zeta_{31})\\
&+S_{\gamma_{\text{er}}}(\psi_4\zeta_{44} + \psi_3\zeta_{43}) - S_{\gamma_{\text{el}}}(\psi_4\zeta_{34} + \psi_3\zeta_{33})\big)
\end{aligned}$$

and

$$\begin{aligned}
v_1 = \frac{\delta}{2}\Bigg(&\frac{1}{\gamma_{\text{ir}}}\big((K'_{\gamma_{\text{ir}}} + I)(\psi_4\zeta_{24} + \psi_3\zeta_{23}) + T_{\gamma_{\text{ir}}}(\psi_2\zeta_{22} + \psi_1\zeta_{21})\big)\\
&-\frac{1}{\gamma_{\text{il}}}\big((K'_{\gamma_{\text{il}}} + I)(\psi_4\zeta_{14} + \psi_3\zeta_{13}) + T_{\gamma_{\text{il}}}(\psi_2\zeta_{12} + \psi_1\zeta_{11})\big)\Bigg)\\
&- \frac{1}{2}\Bigg(\frac{1}{\gamma_{\text{er}}}\big((K'_{\gamma_{\text{er}}} - I)(\psi_4\zeta_{44} + \psi_3\zeta_{43}) + T_{\gamma_{\text{er}}}(\psi_2\zeta_{42} + \psi_1\zeta_{41})\big)\\
&-\frac{1}{\gamma_{\text{el}}}\big((K'_{\gamma_{\text{el}}} - I)(\psi_4\zeta_{34} + \psi_3\zeta_{33}) + T_{\gamma_{\text{el}}}(\psi_2\zeta_{32} + \psi_1\zeta_{31})\big)\Bigg)
\end{aligned}$$

and

$$
\begin{aligned}
w_1 = \frac{i\rho}{2}\Bigg(&\frac{1}{\gamma_{\rm ir}}\left((K'_{\gamma_{\rm ir}}+I)(\psi_4\zeta_{24}+\psi_3\zeta_{23})+T_{\gamma_{\rm ir}}(\psi_2\zeta_{22}+\psi_1\zeta_{21})\right)\\
&\qquad +\frac{1}{\gamma_{\rm il}}\left((K'_{\gamma_{\rm il}}+I)(\psi_4\zeta_{14}+\psi_3\zeta_{13})+T_{\gamma_{\rm il}}(\psi_2\zeta_{12}+\psi_1\zeta_{11})\right)\Bigg)\\
&\quad -\frac{i}{2}\Bigg(\frac{1}{\gamma_{\rm er}}\left((K'_{\gamma_{\rm er}}-I)(\psi_4\zeta_{44}+\psi_3\zeta_{43})+T_{\gamma_{\rm er}}(\psi_2\zeta_{42}+\psi_1\zeta_{41})\right)\\
&\qquad\qquad +\frac{1}{\gamma_{\rm el}}\left((K'_{\gamma_{\rm el}}-I)(\psi_4\zeta_{34}+\psi_3\zeta_{33})+T_{\gamma_{\rm el}}(\psi_2\zeta_{32}+\psi_1\zeta_{31})\right)\Bigg)
\end{aligned}
$$

Put the previous four equations into matrix form, we have

$$
\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14}\\ c_{21} & c_{22} & c_{23} & c_{24}\\ c_{31} & c_{32} & c_{33} & c_{34}\\ c_{41} & c_{42} & c_{43} & c_{44}\end{pmatrix}
\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} =
\begin{pmatrix} v_0\\ w_0\\ v_1\\ w_1\end{pmatrix}
$$

where

$$
\begin{aligned}
c_{11} &= -\frac{\zeta_{41}+\zeta_{31}+\delta\zeta_{21}+\delta\zeta_{11}}{2}I - K_{\gamma_{\rm er}}\frac{\zeta_{41}}{2} - K_{\gamma_{\rm el}}\frac{\zeta_{31}}{2} + K_{\gamma_{\rm ir}}\frac{\delta\zeta_{21}}{2} + K_{\gamma_{\rm il}}\frac{\delta\zeta_{11}}{2}\\
c_{12} &= -\frac{\zeta_{42}+\zeta_{32}+\delta\zeta_{22}+\delta\zeta_{12}}{2}I - K_{\gamma_{\rm er}}\frac{\zeta_{42}}{2} - K_{\gamma_{\rm el}}\frac{\zeta_{32}}{2} + K_{\gamma_{\rm ir}}\frac{\delta\zeta_{22}}{2} + K_{\gamma_{\rm il}}\frac{\delta\zeta_{12}}{2}\\
c_{13} &= -S_{\gamma_{\rm er}}\frac{\zeta_{43}}{2} - S_{\gamma_{\rm el}}\frac{\zeta_{33}}{2} + S_{\gamma_{\rm ir}}\frac{\delta\zeta_{23}}{2} + S_{\gamma_{\rm il}}\frac{\delta\zeta_{13}}{2}\\
c_{14} &= -S_{\gamma_{\rm er}}\frac{\zeta_{44}}{2} - S_{\gamma_{\rm el}}\frac{\zeta_{34}}{2} + \delta S_{\gamma_{\rm ir}}\frac{\zeta_{24}}{2} + \delta S_{\gamma_{\rm il}}\frac{\zeta_{14}}{2}\\
c_{21} &= \frac{i(-\zeta_{41}+\zeta_{31}-\rho\zeta_{21}+\rho\zeta_{11})}{2}I - iK_{\gamma_{\rm er}}\frac{\zeta_{41}}{2} + iK_{\gamma_{\rm el}}\frac{\zeta_{31}}{2} + iK_{\gamma_{\rm ir}}\frac{\rho\zeta_{21}}{2} - iK_{\gamma_{\rm il}}\frac{\rho\zeta_{11}}{2}\\
c_{22} &= \frac{i(-\zeta_{42}+\zeta_{32}-\rho\zeta_{22}+\rho\zeta_{12})}{2}I - iK_{\gamma_{\rm er}}\frac{\zeta_{42}}{2} + iK_{\gamma_{\rm el}}\frac{\zeta_{32}}{2} + iK_{\gamma_{\rm ir}}\frac{\rho\zeta_{22}}{2} - iK_{\gamma_{\rm il}}\frac{\rho\zeta_{12}}{2}\\
c_{23} &= -iS_{\gamma_{\rm er}}\frac{\zeta_{43}}{2} + iS_{\gamma_{\rm el}}\frac{\zeta_{33}}{2} + i\rho S_{\gamma_{\rm ir}}\frac{\zeta_{23}}{2} - i\rho S_{\gamma_{\rm il}}\frac{\zeta_{13}}{2}\\
c_{24} &= -iS_{\gamma_{\rm er}}\frac{\zeta_{44}}{2} + iS_{\gamma_{\rm el}}\frac{\zeta_{34}}{2} + i\rho S_{\gamma_{\rm ir}}\frac{\zeta_{24}}{2} - i\rho S_{\gamma_{\rm il}}\frac{\zeta_{14}}{2}\\
c_{31} &= -T_{\gamma_{\rm er}}\frac{\zeta_{41}}{2\gamma_{\rm er}} + T_{\gamma_{\rm el}}\frac{\zeta_{31}}{2\gamma_{\rm el}} + \delta T_{\gamma_{\rm ir}}\frac{\zeta_{21}}{2\gamma_{\rm ir}} - \delta T_{\gamma_{\rm il}}\frac{\zeta_{11}}{2\gamma_{\rm il}}\\
c_{32} &= -T_{\gamma_{\rm er}}\frac{\zeta_{42}}{2\gamma_{\rm er}} + T_{\gamma_{\rm el}}\frac{\zeta_{32}}{2\gamma_{\rm el}} + \delta T_{\gamma_{\rm ir}}\frac{\zeta_{22}}{2\gamma_{\rm ir}} - \delta T_{\gamma_{\rm il}}\frac{\zeta_{12}}{2\gamma_{\rm il}}
\end{aligned}
$$

$$c_{33} = \left(\frac{\zeta_{43}}{2\gamma_{\mathrm{er}}} - \frac{\zeta_{33}}{2\gamma_{\mathrm{el}}} + \frac{\delta\zeta_{23}}{2\gamma_{\mathrm{ir}}} - \frac{\delta\zeta_{13}}{2\gamma_{\mathrm{il}}}\right) I - K'_{\gamma_{\mathrm{er}}}\frac{\zeta_{43}}{2\gamma_{\mathrm{er}}} + K'_{\gamma_{\mathrm{el}}}\frac{\zeta_{33}}{2\gamma_{\mathrm{el}}} + \delta K'_{\gamma_{\mathrm{ir}}}\frac{\zeta_{23}}{2\gamma_{\mathrm{ir}}} - \delta K'_{\gamma_{\mathrm{il}}}\frac{\zeta_{13}}{2\gamma_{\mathrm{il}}}$$

$$c_{34} = \left(\frac{\zeta_{44}}{2\gamma_{\mathrm{er}}} - \frac{\zeta_{34}}{2\gamma_{\mathrm{el}}} + \frac{\delta\zeta_{24}}{2\gamma_{\mathrm{ir}}} - \frac{\delta\zeta_{14}}{2\gamma_{\mathrm{il}}}\right) I - K'_{\gamma_{\mathrm{er}}}\frac{\zeta_{44}}{2\gamma_{\mathrm{er}}} + K'_{\gamma_{\mathrm{el}}}\frac{\zeta_{34}}{2\gamma_{\mathrm{el}}} + \delta K'_{\gamma_{\mathrm{ir}}}\frac{\zeta_{24}}{2\gamma_{\mathrm{ir}}} - \delta K'_{\gamma_{\mathrm{il}}}\frac{\zeta_{14}}{2\gamma_{\mathrm{il}}}$$

$$c_{41} = -iT_{\gamma_{\mathrm{er}}}\frac{\zeta_{41}}{2\gamma_{\mathrm{er}}} - iT_{\gamma_{\mathrm{el}}}\frac{\zeta_{31}}{2\gamma_{\mathrm{el}}} + i\rho T_{\gamma_{\mathrm{ir}}}\frac{\zeta_{21}}{2\gamma_{\mathrm{ir}}} + i\rho T_{\gamma_{\mathrm{il}}}\frac{\zeta_{11}}{2\gamma_{\mathrm{il}}}$$

$$c_{42} = -iT_{\gamma_{\mathrm{er}}}\frac{\zeta_{42}}{2\gamma_{\mathrm{er}}} - iT_{\gamma_{\mathrm{el}}}\frac{\zeta_{32}}{2\gamma_{\mathrm{el}}} + i\rho T_{\gamma_{\mathrm{ir}}}\frac{\zeta_{22}}{2\gamma_{\mathrm{ir}}} + i\rho T_{\gamma_{\mathrm{il}}}\frac{\zeta_{12}}{2\gamma_{\mathrm{il}}}$$

$$c_{43} = i\left(\frac{\zeta_{43}}{2\gamma_{\mathrm{er}}} + \frac{\zeta_{33}}{2\gamma_{\mathrm{el}}} + \frac{\rho\zeta_{23}}{2\gamma_{\mathrm{ir}}} + \frac{\rho\zeta_{13}}{2\gamma_{\mathrm{il}}}\right) I - iK'_{\gamma_{\mathrm{er}}}\frac{\zeta_{43}}{2\gamma_{\mathrm{er}}} - iK'_{\gamma_{\mathrm{el}}}\frac{\zeta_{33}}{2\gamma_{\mathrm{el}}} + iK'_{\gamma_{\mathrm{ir}}}\frac{\rho\zeta_{23}}{2\gamma_{\mathrm{ir}}} + iK'_{\gamma_{\mathrm{il}}}\frac{\rho\zeta_{13}}{2\gamma_{\mathrm{il}}}$$

$$c_{44} = i\left(\frac{\zeta_{44}}{2\gamma_{\mathrm{er}}} + \frac{\zeta_{34}}{2\gamma_{\mathrm{el}}} + \frac{\rho\zeta_{24}}{2\gamma_{\mathrm{ir}}} + \frac{\rho\zeta_{14}}{2\gamma_{\mathrm{il}}}\right) I - iK'_{\gamma_{\mathrm{er}}}\frac{\zeta_{44}}{2\gamma_{\mathrm{er}}} - iK'_{\gamma_{\mathrm{el}}}\frac{\zeta_{34}}{2\gamma_{\mathrm{el}}} + iK'_{\gamma_{\mathrm{ir}}}\frac{\rho\zeta_{24}}{2\gamma_{\mathrm{ir}}} + iK'_{\gamma_{\mathrm{il}}}\frac{\rho\zeta_{14}}{2\gamma_{\mathrm{il}}}$$

We wish to make the appearance of hypersingular operators T_k's in $c_{31}, c_{32}, c_{41}, c_{42}$ to be in the form of the linear combinations of $T_{k_1} - T_{k_2}$, hence

$$0 = -\frac{\zeta_{41}}{2\gamma_{\mathrm{er}}} + \frac{\zeta_{31}}{2\gamma_{\mathrm{el}}} + \delta\frac{\zeta_{21}}{2\gamma_{\mathrm{ir}}} - \delta\frac{\zeta_{11}}{2\gamma_{\mathrm{il}}}$$

$$0 = -\frac{\zeta_{42}}{2\gamma_{\mathrm{er}}} + \frac{\zeta_{32}}{2\gamma_{\mathrm{el}}} + \delta\frac{\zeta_{22}}{2\gamma_{\mathrm{ir}}} - \delta\frac{\zeta_{12}}{2\gamma_{\mathrm{il}}}$$

$$0 = -i\frac{\zeta_{41}}{2\gamma_{\mathrm{er}}} - i\frac{\zeta_{31}}{2\gamma_{\mathrm{el}}} + i\rho\frac{\zeta_{21}}{2\gamma_{\mathrm{ir}}} + i\rho\frac{\zeta_{11}}{2\gamma_{\mathrm{il}}}$$

$$0 = -i\frac{\zeta_{42}}{2\gamma_{\mathrm{er}}} - i\frac{\zeta_{32}}{2\gamma_{\mathrm{el}}} + i\rho\frac{\zeta_{22}}{2\gamma_{\mathrm{ir}}} + i\rho\frac{\zeta_{12}}{2\gamma_{\mathrm{il}}}$$

Solving the above, we have

$$\zeta_{31} = \frac{\gamma_{\mathrm{el}}\gamma_{\mathrm{ir}}\zeta_{11}(\rho+\delta) + \gamma_{\mathrm{el}}\gamma_{\mathrm{il}}\zeta_{21}(\rho-\delta)}{2\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}}$$

$$\zeta_{32} = \frac{\gamma_{\mathrm{el}}\gamma_{\mathrm{ir}}\zeta_{12}(\rho+\delta) + \gamma_{\mathrm{el}}\gamma_{\mathrm{il}}\zeta_{22}(\rho-\delta)}{2\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}}$$

$$\zeta_{41} = \frac{-\gamma_{\mathrm{er}}\gamma_{\mathrm{ir}}\zeta_{11}(\delta-\rho) + \gamma_{\mathrm{er}}\gamma_{\mathrm{il}}\zeta_{21}(\rho+\delta)}{2\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}}$$

$$\zeta_{42} = \frac{-\gamma_{\mathrm{er}}\gamma_{\mathrm{ir}}\zeta_{12}(\delta-\rho) + \gamma_{\mathrm{er}}\gamma_{\mathrm{il}}\zeta_{22}(\rho+\delta)}{2\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}}$$

All ζ_{ij}'s but $\zeta_{31}, \zeta_{32}, \zeta_{41}, \zeta_{42}$ are independent constants; we have the following selection

$$\begin{array}{cccc} \zeta_{11} = 2\gamma_{\mathrm{il}} & \zeta_{12} = 0 & \zeta_{13} = 1 & \zeta_{14} = 0 \\ \zeta_{21} = 0 & \zeta_{22} = 2\gamma_{\mathrm{ir}} & \zeta_{23} = 0 & \zeta_{24} = 1 \\ \zeta_{31} = \gamma_{\mathrm{el}}(\rho+\delta) & \zeta_{32} = \gamma_{\mathrm{el}}(\rho-\delta) & \zeta_{33} = 1 & \zeta_{34} = 0 \\ \zeta_{41} = \gamma_{\mathrm{er}}(\rho-\delta) & \zeta_{42} = \gamma_{\mathrm{er}}(\rho+\delta) & \zeta_{43} = 0 & \zeta_{44} = 1 \end{array}$$

With this selection we have

$$\begin{aligned}
Q_{\mathrm{il}} &= 2\gamma_{\mathrm{il}}\, K^{\mathrm{i}}_{\gamma_{\mathrm{il}}}\psi_1 + S^{\mathrm{i}}_{\gamma_{\mathrm{il}}}\psi_3\\
Q_{\mathrm{ir}} &= 2\gamma_{\mathrm{ir}}\, K^{\mathrm{i}}_{\gamma_{\mathrm{ir}}}\psi_2 + S^{\mathrm{i}}_{\gamma_{\mathrm{ir}}}\psi_4\\
Q_{\mathrm{el}} &= K^{\mathrm{e}}_{\gamma_{\mathrm{el}}}(\gamma_{\mathrm{el}}(\rho+\delta)\psi_1 + \gamma_{\mathrm{el}}(\rho-\delta)\psi_2) + S^{\mathrm{e}}_{\gamma_{\mathrm{el}}}\psi_3\\
Q_{\mathrm{er}} &= K^{\mathrm{e}}_{\gamma_{\mathrm{er}}}(\gamma_{\mathrm{er}}(\rho-\delta)\psi_1 + \gamma_{\mathrm{er}}(\rho+\delta)\psi_2) + S^{\mathrm{e}}_{\gamma_{\mathrm{er}}}\psi_4
\end{aligned}$$

and

$$\begin{aligned}
c_{11} &= -\frac{(2\gamma_{\mathrm{il}}+\gamma_{\mathrm{el}}-\gamma_{\mathrm{er}})\delta+(\gamma_{\mathrm{er}}+\gamma_{\mathrm{el}})\rho}{2}I - \frac{\gamma_{\mathrm{el}}(\rho+\delta)}{2}K_{\gamma_{\mathrm{el}}} - \frac{\gamma_{\mathrm{er}}(\rho-\delta)}{2}K_{\gamma_{\mathrm{er}}} + \delta\gamma_{\mathrm{il}}K_{\gamma_{\mathrm{il}}}\\
c_{12} &= -\frac{(2\gamma_{\mathrm{ir}}+\gamma_{\mathrm{er}}-\gamma_{\mathrm{el}})\delta+(\gamma_{\mathrm{er}}+\gamma_{\mathrm{el}})\rho}{2}I - \frac{\gamma_{\mathrm{er}}(\rho+\delta)}{2}K_{\gamma_{\mathrm{er}}} - \frac{\gamma_{\mathrm{el}}(\rho-\delta)}{2}K_{\gamma_{\mathrm{el}}} + \delta\gamma_{\mathrm{ir}}K_{\gamma_{\mathrm{ir}}}\\
c_{13} &= \frac{\delta}{2}S_{\gamma_{\mathrm{il}}} - \frac{1}{2}S_{\gamma_{\mathrm{el}}}\\
c_{14} &= \frac{\delta}{2}S_{\gamma_{\mathrm{ir}}} - \frac{1}{2}S_{\gamma_{\mathrm{er}}}\\
c_{21} &= i\frac{(2\gamma_{\mathrm{il}}-\gamma_{\mathrm{er}}+\gamma_{\mathrm{el}})\rho+(\gamma_{\mathrm{er}}+\gamma_{\mathrm{el}})\delta}{2}I + \frac{i\gamma_{\mathrm{el}}(\rho+\delta)}{2}K_{\gamma_{\mathrm{el}}} - \frac{i\gamma_{\mathrm{er}}(\rho-\delta)}{2}K_{\gamma_{\mathrm{er}}} - i\gamma_{\mathrm{il}}\rho K_{\gamma_{\mathrm{il}}}\\
c_{22} &= -i\frac{(2\gamma_{\mathrm{ir}}+\gamma_{\mathrm{er}}-\gamma_{\mathrm{el}})\rho+(\gamma_{\mathrm{er}}+\gamma_{\mathrm{el}})\delta}{2}I - \frac{i\gamma_{\mathrm{er}}(\rho+\delta)}{2}K_{\gamma_{\mathrm{er}}} + \frac{i\gamma_{\mathrm{el}}(\rho-\delta)}{2}K_{\gamma_{\mathrm{el}}} + i\gamma_{\mathrm{ir}}\rho K_{\gamma_{\mathrm{ir}}}\\
c_{23} &= \frac{i}{2}S_{\gamma_{\mathrm{el}}} - \frac{i\rho}{2}S_{\gamma_{\mathrm{il}}}\\
c_{24} &= \frac{i\rho}{2}S_{\gamma_{\mathrm{ir}}} - \frac{i}{2}S_{\gamma_{\mathrm{er}}}\\
c_{31} &= -\delta T_{\gamma_{\mathrm{il}}} - \frac{\rho-\delta}{2}T_{\gamma_{\mathrm{er}}} + \frac{\rho+\delta}{2}T_{\gamma_{\mathrm{el}}}\\
c_{32} &= \delta T_{\gamma_{\mathrm{ir}}} - \frac{\rho+\delta}{2}T_{\gamma_{\mathrm{er}}} + \frac{\rho-\delta}{2}T_{\gamma_{\mathrm{el}}}\\
c_{33} &= -\frac{\gamma_{\mathrm{il}}+\delta\gamma_{\mathrm{el}}}{2\gamma_{\mathrm{il}}\gamma_{\mathrm{el}}}I - \frac{\delta}{2\gamma_{\mathrm{il}}}K'_{\gamma_{\mathrm{il}}} + \frac{1}{2\gamma_{\mathrm{el}}}K'_{\gamma_{\mathrm{el}}}\\
c_{34} &= \frac{\gamma_{\mathrm{ir}}+\delta\gamma_{\mathrm{er}}}{2\gamma_{\mathrm{ir}}\gamma_{\mathrm{er}}}I + \frac{\delta}{2\gamma_{\mathrm{ir}}}K'_{\gamma_{\mathrm{ir}}} - \frac{1}{2\gamma_{\mathrm{er}}}K'_{\gamma_{\mathrm{er}}}\\
c_{41} &= i\rho T_{\gamma_{\mathrm{il}}} - \frac{i(\rho-\delta)}{2}T_{\gamma_{\mathrm{er}}} - \frac{i(\rho+\delta)}{2}T_{\gamma_{\mathrm{el}}}\\
c_{42} &= i\rho T_{\gamma_{\mathrm{ir}}} - \frac{i(\rho+\delta)}{2}T_{\gamma_{\mathrm{er}}} - \frac{i(\rho-\delta)}{2}T_{\gamma_{\mathrm{el}}}\\
c_{43} &= \frac{i(\gamma_{\mathrm{il}}+\rho\gamma_{\mathrm{el}})}{2\gamma_{\mathrm{il}}\gamma_{\mathrm{el}}}I + \frac{i\rho}{2\gamma_{\mathrm{il}}}K'_{\gamma_{\mathrm{il}}} - \frac{i}{2\gamma_{\mathrm{el}}}K'_{\gamma_{\mathrm{el}}}\\
c_{44} &= \frac{i(\gamma_{\mathrm{ir}}+\rho\gamma_{\mathrm{er}})}{2\gamma_{\mathrm{ir}}\gamma_{\mathrm{er}}}I + \frac{i\rho}{2\gamma_{\mathrm{ir}}}K'_{\gamma_{\mathrm{ir}}} - \frac{i}{2\gamma_{\mathrm{er}}}K'_{\gamma_{\mathrm{er}}}
\end{aligned}$$

Decomposing $c_{ij} = e_{ij} + a_{ij}$ where e_{ij} only involves the identity transform I and $a_{ij} = c_{ij} - e_{ij}$, we have

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14}\\ c_{21} & c_{22} & c_{23} & c_{24}\\ c_{31} & c_{32} & c_{33} & c_{34}\\ c_{41} & c_{42} & c_{43} & c_{44}\end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14}\\ e_{21} & e_{22} & e_{23} & e_{24}\\ e_{31} & e_{32} & e_{33} & e_{34}\\ e_{41} & e_{42} & e_{43} & e_{44}\end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}$$

with

$$
\begin{aligned}
e_{11} &= -\frac{(2\gamma_{\text{il}} + \gamma_{\text{el}} - \gamma_{\text{er}})\delta + (\gamma_{\text{er}} + \gamma_{\text{el}})\rho}{2} I \\
e_{12} &= -\frac{(2\gamma_{\text{ir}} + \gamma_{\text{er}} - \gamma_{\text{el}})\delta + (\gamma_{\text{er}} + \gamma_{\text{el}})\rho}{2} I \\
e_{13} &= 0 \\
e_{14} &= 0 \\
e_{21} &= i\frac{(2\gamma_{\text{il}} - \gamma_{\text{er}} + \gamma_{\text{el}})\rho + (\gamma_{\text{er}} + \gamma_{\text{el}})\delta}{2} I \\
e_{22} &= -i\frac{(2\gamma_{\text{ir}} + \gamma_{\text{er}} - \gamma_{\text{el}})\rho + (\gamma_{\text{er}} + \gamma_{\text{el}})\delta}{2} I \\
e_{23} &= 0 \\
e_{24} &= 0 \\
e_{31} &= 0 \\
e_{32} &= 0 \\
e_{33} &= -\frac{\gamma_{\text{il}} + \delta\gamma_{\text{el}}}{2\gamma_{\text{il}}\gamma_{\text{el}}} I \\
e_{34} &= \frac{\gamma_{\text{ir}} + \delta\gamma_{\text{er}}}{2\gamma_{\text{ir}}\gamma_{\text{er}}} I \\
e_{41} &= 0 \\
e_{42} &= 0 \\
e_{43} &= \frac{i(\gamma_{\text{il}} + \rho\gamma_{\text{el}})}{2\gamma_{\text{il}}\gamma_{\text{el}}} I \\
e_{44} &= \frac{i(\gamma_{\text{ir}} + \rho\gamma_{\text{er}})}{2\gamma_{\text{ir}}\gamma_{\text{er}}} I
\end{aligned}
$$

and

$$
\begin{aligned}
a_{11} &= -\frac{\gamma_{\text{el}}(\rho + \delta)}{2} K_{\gamma_{\text{el}}} - \frac{\gamma_{\text{er}}(\rho - \delta)}{2} K_{\gamma_{\text{er}}} + \delta\gamma_{\text{il}} K_{\gamma_{\text{il}}} \\
a_{12} &= -\frac{\gamma_{\text{er}}(\rho + \delta)}{2} K_{\gamma_{\text{er}}} - \frac{\gamma_{\text{el}}(\rho - \delta)}{2} K_{\gamma_{\text{el}}} + \delta\gamma_{\text{ir}} K_{\gamma_{\text{ir}}} \\
a_{13} &= 0 \\
a_{14} &= 0 \\
a_{21} &= \frac{i\gamma_{\text{el}}(\rho + \delta)}{2} K_{\gamma_{\text{el}}} - \frac{i\gamma_{\text{er}}(\rho - \delta)}{2} K_{\gamma_{\text{er}}} - i\gamma_{\text{il}}\rho K_{\gamma_{\text{il}}} \\
a_{22} &= -\frac{i\gamma_{\text{er}}(\rho + \delta)}{2} K_{\gamma_{\text{er}}} + \frac{i\gamma_{\text{el}}(\rho - \delta)}{2} K_{\gamma_{\text{el}}} + i\gamma_{\text{ir}}\rho K_{\gamma_{\text{ir}}} \\
a_{23} &= \frac{i}{2} S_{\gamma_{\text{el}}} - \frac{i\rho}{2} S_{\gamma_{\text{il}}} \\
a_{24} &= \frac{i\rho}{2} S_{\gamma_{\text{ir}}} - \frac{i}{2} S_{\gamma_{\text{er}}}
\end{aligned}
$$

$$
\begin{aligned}
a_{31} &= -\delta T_{\gamma_{\mathrm{il}}} - \frac{\rho-\delta}{2} T_{\gamma_{\mathrm{er}}} + \frac{\rho+\delta}{2} T_{\gamma_{\mathrm{el}}} \\
a_{32} &= \delta T_{\gamma_{\mathrm{ir}}} - \frac{\rho+\delta}{2} T_{\gamma_{\mathrm{er}}} + \frac{\rho-\delta}{2} T_{\gamma_{\mathrm{el}}} \\
a_{33} &= -\frac{\delta}{2\gamma_{\mathrm{il}}} K'_{\gamma_{\mathrm{il}}} + \frac{1}{2\gamma_{\mathrm{el}}} K'_{\gamma_{\mathrm{el}}} \\
a_{34} &= \frac{\delta}{2\gamma_{\mathrm{ir}}} K'_{\gamma_{\mathrm{ir}}} - \frac{1}{2\gamma_{\mathrm{er}}} K'_{\gamma_{\mathrm{er}}} \\
a_{41} &= i\rho T_{\gamma_{\mathrm{il}}} - \frac{i(\rho-\delta)}{2} T_{\gamma_{\mathrm{er}}} - \frac{i(\rho+\delta)}{2} T_{\gamma_{\mathrm{el}}} \\
a_{42} &= i\rho T_{\gamma_{\mathrm{ir}}} - \frac{i(\rho+\delta)}{2} T_{\gamma_{\mathrm{er}}} - \frac{i(\rho-\delta)}{2} T_{\gamma_{\mathrm{el}}} \\
a_{43} &= \frac{i\rho}{2\gamma_{\mathrm{il}}} K'_{\gamma_{\mathrm{il}}} - \frac{i}{2\gamma_{\mathrm{el}}} K'_{\gamma_{\mathrm{el}}} \\
a_{44} &= \frac{i\rho}{2\gamma_{\mathrm{ir}}} K'_{\gamma_{\mathrm{ir}}} - \frac{i}{2\gamma_{\mathrm{er}}} K'_{\gamma_{\mathrm{er}}}
\end{aligned}
$$

The determinant of $\{e_{ij}\}$ is

$$
\begin{aligned}
&\frac{\gamma_{\mathrm{el}}\gamma_{\mathrm{ir}}\rho + \gamma_{\mathrm{er}}\gamma_{\mathrm{il}}\rho + 2\delta\gamma_{\mathrm{el}}\gamma_{\mathrm{er}}\rho + 2\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}} + \delta\gamma_{\mathrm{el}}\gamma_{\mathrm{ir}} + \delta\gamma_{\mathrm{er}}\gamma_{\mathrm{il}}}{8\gamma_{\mathrm{el}}\gamma_{\mathrm{er}}\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}} \\
&\qquad \times \big((\gamma_{\mathrm{er}}\gamma_{\mathrm{ir}} + \gamma_{\mathrm{el}}\gamma_{\mathrm{ir}} + \gamma_{\mathrm{er}}\gamma_{\mathrm{il}} + \gamma_{\mathrm{el}}\gamma_{\mathrm{il}})\rho^2 \\
&\qquad + (4\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}} - 2\gamma_{\mathrm{er}}\gamma_{\mathrm{ir}} + 2\gamma_{\mathrm{el}}\gamma_{\mathrm{ir}} + 2\gamma_{\mathrm{er}}\gamma_{\mathrm{il}} - 2\gamma_{\mathrm{el}}\gamma_{\mathrm{il}} + 4\gamma_{\mathrm{el}}\gamma_{\mathrm{er}})\delta\rho \\
&\qquad + (\gamma_{\mathrm{er}}\gamma_{\mathrm{ir}} + \gamma_{\mathrm{el}}\gamma_{\mathrm{ir}} + \gamma_{\mathrm{er}}\gamma_{\mathrm{il}} + \gamma_{\mathrm{el}}\gamma_{\mathrm{il}})\delta^2\big) ,
\end{aligned}
$$

which can be simplified as

$$
\begin{aligned}
&\frac{(\gamma_{\mathrm{el}}\gamma_{\mathrm{ir}} + \gamma_{\mathrm{er}}\gamma_{\mathrm{il}})\rho + (\gamma_{\mathrm{el}}\gamma_{\mathrm{ir}} + \gamma_{\mathrm{er}}\gamma_{\mathrm{il}})\delta + 2\delta\gamma_{\mathrm{el}}\gamma_{\mathrm{er}}\rho + 2\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}}{8\gamma_{\mathrm{el}}\gamma_{\mathrm{er}}\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}} \\
&\quad \times \big((\gamma_{\mathrm{er}}\gamma_{\mathrm{ir}} + \gamma_{\mathrm{el}}\gamma_{\mathrm{il}})(\rho-\delta)^2 + (\gamma_{\mathrm{el}}\gamma_{\mathrm{ir}} + \gamma_{\mathrm{er}}\gamma_{\mathrm{il}})(\rho+\delta)^2 + 4(\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}} + \gamma_{\mathrm{el}}\gamma_{\mathrm{er}})\delta\rho\big) .
\end{aligned}
$$

Chiral-Achiral

The "master equations" are

$$
\begin{aligned}
v_0 &= \frac{\delta}{2}(Q_{\mathrm{ir}} + Q_{\mathrm{il}}) - \frac{1}{2}(Q_{\mathrm{er}} + Q_{\mathrm{el}}) \\
w_0 &= \frac{i\rho}{2}(Q_{\mathrm{ir}} - Q_{\mathrm{il}}) - \frac{i}{2}(Q_{\mathrm{er}} - Q_{\mathrm{el}}) \\
v_1 &= \frac{\delta}{2}\left(\frac{1}{k_{\mathrm{i}}}\frac{\partial Q_{\mathrm{ir}}}{\partial \nu} - \frac{1}{k_{\mathrm{i}}}\frac{\partial Q_{\mathrm{il}}}{\partial \nu}\right) - \frac{1}{2}\left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} - \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right) \\
w_1 &= \frac{i\rho}{2}\left(\frac{1}{k_{\mathrm{i}}}\frac{\partial Q_{\mathrm{ir}}}{\partial \nu} + \frac{1}{k_{\mathrm{i}}}\frac{\partial Q_{\mathrm{il}}}{\partial \nu}\right) - \frac{i}{2}\left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} + \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right)
\end{aligned}
$$

We propose the following ansatz

$$\begin{aligned}
Q_{\mathrm{il}} &= K^{\mathrm{i}}_{k_{\mathrm{i}}}(\zeta_{11}\psi_1 + \zeta_{12}\psi_2) + S^{\mathrm{i}}_{k_{\mathrm{i}}}(\zeta_{13}\psi_3 + \zeta_{14}\psi_4)\\
Q_{\mathrm{ir}} &= K^{\mathrm{i}}_{k_{\mathrm{i}}}(\zeta_{21}\psi_1 + \zeta_{22}\psi_2) + S^{\mathrm{i}}_{k_{\mathrm{i}}}(\zeta_{23}\psi_3 + \zeta_{24}\psi_4)\\
Q_{\mathrm{el}} &= K^{\mathrm{e}}_{\gamma_{\mathrm{el}}}(\zeta_{31}\psi_1 + \zeta_{32}\psi_2) + S^{\mathrm{e}}_{\gamma_{\mathrm{il}}}(\zeta_{33}\psi_3 + \zeta_{34}\psi_4)\\
Q_{\mathrm{er}} &= K^{\mathrm{e}}_{\gamma_{\mathrm{er}}}(\zeta_{41}\psi_1 + \zeta_{42}\psi_2) + S^{\mathrm{e}}_{\gamma_{\mathrm{er}}}(\zeta_{43}\psi_3 + \zeta_{44}\psi_4)
\end{aligned}$$

where ψ_j's are unknowns and ζ_{ij}'s are constants to be determined later. The boundary traces are

$$\begin{aligned}
Q_{\mathrm{il}} &= (K_{k_{\mathrm{i}}} - I)(\zeta_{11}\psi_1 + \zeta_{12}\psi_2) + S_{k_{\mathrm{i}}}(\zeta_{13}\psi_3 + \zeta_{14}\psi_4)\\
Q_{\mathrm{ir}} &= (K_{k_{\mathrm{i}}} - I)(\zeta_{21}\psi_1 + \zeta_{22}\psi_2) + S_{k_{\mathrm{i}}}(\zeta_{23}\psi_3 + \zeta_{24}\psi_4)\\
Q_{\mathrm{el}} &= (K_{\gamma_{\mathrm{el}}} + I)(\zeta_{31}\psi_1 + \zeta_{32}\psi_2) + S_{\gamma_{\mathrm{el}}}(\zeta_{33}\psi_3 + \zeta_{34}\psi_4)\\
Q_{\mathrm{er}} &= (K_{\gamma_{\mathrm{er}}} + I)(\zeta_{41}\psi_1 + \zeta_{42}\psi_2) + S_{\gamma_{\mathrm{er}}}(\zeta_{43}\psi_3 + \zeta_{44}\psi_4)\\
\frac{\partial Q_{\mathrm{il}}}{\partial \nu} &= T_{k_{\mathrm{i}}}(\zeta_{11}\psi_1 + \zeta_{12}\psi_2) + (K'_{k_{\mathrm{i}}} + I)(\zeta_{13}\psi_3 + \zeta_{14}\psi_4)\\
\frac{\partial Q_{\mathrm{ir}}}{\partial \nu} &= T_{k_{\mathrm{i}}}(\zeta_{21}\psi_1 + \zeta_{22}\psi_2) + (K'_{k_{\mathrm{i}}} + I)(\zeta_{23}\psi_3 + \zeta_{24}\psi_4)\\
\frac{\partial Q_{\mathrm{el}}}{\partial \nu} &= T_{\gamma_{\mathrm{el}}}(\zeta_{31}\psi_1 + \zeta_{32}\psi_2) + (K'_{\gamma_{\mathrm{el}}} - I)(\zeta_{33}\psi_3 + \zeta_{34}\psi_4)\\
\frac{\partial Q_{\mathrm{er}}}{\partial \nu} &= T_{\gamma_{\mathrm{er}}}(\zeta_{41}\psi_1 + \zeta_{42}\psi_2) + (K'_{\gamma_{\mathrm{er}}} - I)(\zeta_{43}\psi_3 + \zeta_{44}\psi_4)
\end{aligned}$$

Substituting into "master equations" we have

$$\begin{aligned}
v_0 = \frac{\delta}{2}\big(&(K_{k_{\mathrm{i}}} - I)(\psi_2\zeta_{22} + \psi_1\zeta_{21}) + (K_{k_{\mathrm{i}}} - I)(\psi_2\zeta_{12} + \psi_1\zeta_{11})\\
&+S_{k_{\mathrm{i}}}(\psi_4\zeta_{24} + \psi_3\zeta_{23}) + S_{k_{\mathrm{i}}}(\psi_4\zeta_{14} + \psi_3\zeta_{13})\big)\\
&- \frac{1}{2}\big((K_{\gamma_{\mathrm{er}}} + I)(\psi_2\zeta_{42} + \psi_1\zeta_{41}) + (K_{\gamma_{\mathrm{el}}} + I)(\psi_2\zeta_{32} + \psi_1\zeta_{31})\\
&+S_{\gamma_{\mathrm{er}}}(\psi_4\zeta_{44} + \psi_3\zeta_{43}) + S_{\gamma_{\mathrm{el}}}(\psi_4\zeta_{34} + \psi_3\zeta_{33})\big)
\end{aligned}$$

and

$$\begin{aligned}
w_0 = \frac{i\rho}{2}\big(&(K_{k_{\mathrm{i}}} - I)(\psi_2\zeta_{22} + \psi_1\zeta_{21}) - (K_{k_{\mathrm{i}}} - I)(\psi_2\zeta_{12} + \psi_1\zeta_{11})\\
&+S_{k_{\mathrm{i}}}(\psi_4\zeta_{24} + \psi_3\zeta_{23}) - S_{k_{\mathrm{i}}}(\psi_4\zeta_{14} + \psi_3\zeta_{13})\big)\\
&- \frac{i}{2}\big((K_{\gamma_{\mathrm{er}}} + I)(\psi_2\zeta_{42} + \psi_1\zeta_{41}) - (K_{\gamma_{\mathrm{el}}} + I)(\psi_2\zeta_{32} + \psi_1\zeta_{31})\\
&+S_{\gamma_{\mathrm{er}}}(\psi_4\zeta_{44} + \psi_3\zeta_{43}) - S_{\gamma_{\mathrm{el}}}(\psi_4\zeta_{34} + \psi_3\zeta_{33})\big)
\end{aligned}$$

and

$$
\begin{aligned}
v_1 = \frac{\delta}{2} \Bigg(& \frac{1}{k_{\mathrm{i}}} \big((K'_{k_{\mathrm{i}}} + I)(\psi_4 \zeta_{24} + \psi_3 \zeta_{23}) + T_{k_{\mathrm{i}}}(\psi_2 \zeta_{22} + \psi_1 \zeta_{21}) \big) \\
& - \frac{1}{k_{\mathrm{i}}} \big((K'_{k_{\mathrm{i}}} + I)(\psi_4 \zeta_{14} + \psi_3 \zeta_{13}) + T_{k_{\mathrm{i}}}(\psi_2 \zeta_{12} + \psi_1 \zeta_{11}) \big) \Bigg) \\
& - \frac{1}{2} \Bigg(\frac{1}{\gamma_{\mathrm{er}}} \big((K'_{\gamma_{\mathrm{er}}} - I)(\psi_4 \zeta_{44} + \psi_3 \zeta_{43}) + T_{\gamma_{\mathrm{er}}}(\psi_2 \zeta_{42} + \psi_1 \zeta_{41}) \big) \\
& - \frac{1}{\gamma_{\mathrm{el}}} \big((K'_{\gamma_{\mathrm{el}}} - I)(\psi_4 \zeta_{34} + \psi_3 \zeta_{33}) + T_{\gamma_{\mathrm{el}}}(\psi_2 \zeta_{32} + \psi_1 \zeta_{31}) \big) \Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
w_1 = \frac{i\rho}{2} \Bigg(& \frac{1}{k_{\mathrm{i}}} \big((K'_{k_{\mathrm{i}}} + I)(\psi_4 \zeta_{24} + \psi_3 \zeta_{23}) + T_{k_{\mathrm{i}}}(\psi_2 \zeta_{22} + \psi_1 \zeta_{21}) \big) \\
& + \frac{1}{k_{\mathrm{i}}} \big((K'_{k_{\mathrm{i}}} + I)(\psi_4 \zeta_{14} + \psi_3 \zeta_{13}) + T_{k_{\mathrm{i}}}(\psi_2 \zeta_{12} + \psi_1 \zeta_{11}) \big) \Bigg) \\
& - \frac{i}{2} \Bigg(\frac{1}{\gamma_{\mathrm{er}}} \big((K'_{\gamma_{\mathrm{er}}} - I)(\psi_4 \zeta_{44} + \psi_3 \zeta_{43}) + T_{\gamma_{\mathrm{er}}}(\psi_2 \zeta_{42} + \psi_1 \zeta_{41}) \big) \\
& + \frac{1}{\gamma_{\mathrm{el}}} \big((K'_{\gamma_{\mathrm{el}}} - I)(\psi_4 \zeta_{34} + \psi_3 \zeta_{33}) + T_{\gamma_{\mathrm{el}}}(\psi_2 \zeta_{32} + \psi_1 \zeta_{31}) \big) \Bigg)
\end{aligned}
$$

Put the previous four equations into matrix form, we have

$$
\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}
\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}
=
\begin{pmatrix} v_0 \\ w_0 \\ v_1 \\ w_1 \end{pmatrix}
$$

where

$$
\begin{aligned}
c_{11} &= -\frac{\zeta_{41} + \zeta_{31} + \delta\zeta_{21} + \delta\zeta_{11}}{2} I - K_{\gamma_{\mathrm{er}}} \frac{\zeta_{41}}{2} - K_{\gamma_{\mathrm{el}}} \frac{\zeta_{31}}{2} + K_{k_{\mathrm{i}}} \frac{\delta(\zeta_{21} + \zeta_{11})}{2} \\
c_{12} &= -\frac{\zeta_{42} + \zeta_{32} + \delta\zeta_{22} + \delta\zeta_{12}}{2} I - K_{\gamma_{\mathrm{er}}} \frac{\zeta_{42}}{2} - K_{\gamma_{\mathrm{el}}} \frac{\zeta_{32}}{2} + K_{k_{\mathrm{i}}} \frac{\delta(\zeta_{22} + \zeta_{12})}{2} \\
c_{13} &= -S_{\gamma_{\mathrm{er}}} \frac{\zeta_{43}}{2} - S_{\gamma_{\mathrm{el}}} \frac{\zeta_{33}}{2} + S_{k_{\mathrm{i}}} \frac{\delta(\zeta_{23} + \zeta_{13})}{2} \\
c_{14} &= -S_{\gamma_{\mathrm{er}}} \frac{\zeta_{44}}{2} - S_{\gamma_{\mathrm{el}}} \frac{\zeta_{34}}{2} + S_{k_{\mathrm{i}}} \frac{\delta(\zeta_{24} + \zeta_{14})}{2} \\
c_{21} &= -\frac{i(\zeta_{41} - \zeta_{31} + \rho\zeta_{21} - \rho\zeta_{11})}{2} I - iK_{\gamma_{\mathrm{er}}} \frac{\zeta_{41}}{2} + iK_{\gamma_{\mathrm{el}}} \frac{\zeta_{31}}{2} + K_{k_{\mathrm{i}}} \frac{i\rho(\zeta_{21} - \zeta_{11})}{2} \\
c_{22} &= -\frac{i(\zeta_{42} - \zeta_{32} + \rho\zeta_{22} - \rho\zeta_{12})}{2} I - iK_{\gamma_{\mathrm{er}}} \frac{\zeta_{42}}{2} + iK_{\gamma_{\mathrm{el}}} \frac{\zeta_{32}}{2} + K_{k_{\mathrm{i}}} \frac{i\rho(\zeta_{22} - \zeta_{12})}{2} \\
c_{23} &= -iS_{\gamma_{\mathrm{er}}} \frac{\zeta_{43}}{2} + iS_{\gamma_{\mathrm{el}}} \frac{\zeta_{33}}{2} + S_{k_{\mathrm{i}}} \frac{i\rho(\zeta_{23} - \zeta_{13})}{2} \\
c_{24} &= -iS_{\gamma_{\mathrm{er}}} \frac{\zeta_{44}}{2} + iS_{\gamma_{\mathrm{el}}} \frac{\zeta_{34}}{2} + S_{k_{\mathrm{i}}} \frac{i\rho(\zeta_{24} - \zeta_{14})}{2}
\end{aligned}
$$

$$
\begin{aligned}
c_{31} &= -T_{\gamma_{\rm er}}\frac{\zeta_{41}}{2\gamma_{\rm er}} + T_{\gamma_{\rm el}}\frac{\zeta_{31}}{2\gamma_{\rm el}} + T_{k_{\rm i}}\frac{\delta(\zeta_{21}-\zeta_{11})}{2k_{\rm i}}\\
c_{32} &= -T_{\gamma_{\rm er}}\frac{\zeta_{42}}{2\gamma_{\rm er}} + T_{\gamma_{\rm el}}\frac{\zeta_{32}}{2\gamma_{\rm el}} + T_{k_{\rm i}}\frac{\delta(\zeta_{22}-\zeta_{12})}{2k_{\rm i}}\\
c_{33} &= \frac{\gamma_{\rm el}k_{\rm i}\zeta_{43}-\gamma_{\rm er}k_{\rm i}\zeta_{33}+\delta\gamma_{\rm el}\gamma_{\rm er}\zeta_{23}-\delta\gamma_{\rm el}\gamma_{\rm er}\zeta_{13}}{2\gamma_{\rm el}\gamma_{\rm er}k_{\rm i}}I - K'_{\gamma_{\rm er}}\frac{\zeta_{43}}{2\gamma_{\rm er}} + K'_{\gamma_{\rm el}}\frac{\zeta_{33}}{2\gamma_{\rm el}} + K'_{k_{\rm i}}\frac{\delta(\zeta_{23}-\zeta_{13})}{2k_{\rm i}}\\
c_{34} &= \frac{\gamma_{\rm el}k_{\rm i}\zeta_{44}-\gamma_{\rm er}k_{\rm i}\zeta_{34}+\delta\gamma_{\rm el}\gamma_{\rm er}\zeta_{24}-\delta\gamma_{\rm el}\gamma_{\rm er}\zeta_{14}}{2\gamma_{\rm el}\gamma_{\rm er}k_{\rm i}}I - K'_{\gamma_{\rm er}}\frac{\zeta_{44}}{2\gamma_{\rm er}} + K'_{\gamma_{\rm el}}\frac{\zeta_{34}}{2\gamma_{\rm el}} + K'_{k_{\rm i}}\frac{\delta(\zeta_{24}-\zeta_{14})}{2k_{\rm i}}\\
c_{41} &= -iT_{\gamma_{\rm er}}\frac{\zeta_{41}}{2\gamma_{\rm er}} - iT_{\gamma_{\rm el}}\frac{\zeta_{31}}{2\gamma_{\rm el}} + T_{k_{\rm i}}\frac{i\rho(\zeta_{21}+\zeta_{11})}{2k_{\rm i}}\\
c_{42} &= -iT_{\gamma_{\rm er}}\frac{\zeta_{42}}{2\gamma_{\rm er}} - iT_{\gamma_{\rm el}}\frac{\zeta_{32}}{2\gamma_{\rm el}} + T_{k_{\rm i}}\frac{i\rho(\zeta_{22}+\zeta_{12})}{2k_{\rm i}}\\
c_{43} &= i\frac{\gamma_{\rm el}k_{\rm i}\zeta_{43}+\gamma_{\rm er}k_{\rm i}\zeta_{33}+\gamma_{\rm el}\gamma_{\rm er}\rho\zeta_{23}+\gamma_{\rm el}\gamma_{\rm er}\rho\zeta_{13}}{2\gamma_{\rm el}\gamma_{\rm er}k_{\rm i}}I - iK'_{\gamma_{\rm er}}\frac{\zeta_{43}}{2\gamma_{\rm er}} - iK'_{\gamma_{\rm el}}\frac{\zeta_{33}}{2\gamma_{\rm el}} + K'_{k_{\rm i}}\frac{i\rho(\zeta_{23}+\zeta_{13})}{2k_{\rm i}}\\
c_{44} &= i\frac{\gamma_{\rm el}k_{\rm i}\zeta_{44}+\gamma_{\rm er}k_{\rm i}\zeta_{34}+\gamma_{\rm el}\gamma_{\rm er}\rho\zeta_{24}+\gamma_{\rm el}\gamma_{\rm er}\rho\zeta_{14}}{2\gamma_{\rm el}\gamma_{\rm er}k_{\rm i}}I - iK'_{\gamma_{\rm er}}\frac{\zeta_{44}}{2\gamma_{\rm er}} - iK'_{\gamma_{\rm el}}\frac{\zeta_{34}}{2\gamma_{\rm el}} + K'_{k_{\rm i}}\frac{i\rho(\zeta_{24}+\zeta_{14})}{2k_{\rm i}}
\end{aligned}
$$

We wish to make the appearance of hypersingular operators T_k's in $c_{31}, c_{32}, c_{41}, c_{42}$ to be in the form of the linear combinations of $T_{k_1} - T_{k_2}$, hence

$$
\begin{aligned}
0 &= -\frac{\zeta_{41}}{2\gamma_{\rm er}} + \frac{\zeta_{31}}{2\gamma_{\rm el}} + \frac{\delta(\zeta_{21}-\zeta_{11})}{2k_{\rm i}}\\
0 &= -\frac{\zeta_{42}}{2\gamma_{\rm er}} + \frac{\zeta_{32}}{2\gamma_{\rm el}} + \frac{\delta(\zeta_{22}-\zeta_{12})}{2k_{\rm i}}\\
0 &= -i\frac{\zeta_{41}}{2\gamma_{\rm er}} - i\frac{\zeta_{31}}{2\gamma_{\rm el}} + \frac{i\rho(\zeta_{21}+\zeta_{11})}{2k_{\rm i}}\\
0 &= -i\frac{\zeta_{42}}{2\gamma_{\rm er}} - i\frac{\zeta_{32}}{2\gamma_{\rm el}} + \frac{i\rho(\zeta_{22}+\zeta_{12})}{2k_{\rm i}}
\end{aligned}
$$

Solving the above, we have

$$
\begin{aligned}
\zeta_{31} &= \frac{\gamma_{\rm el}\left(\zeta_{11}(\rho+\delta)+\zeta_{21}(\rho-\delta)\right)}{2k_{\rm i}}\\
\zeta_{32} &= \frac{\gamma_{\rm el}\left(\zeta_{12}(\rho+\delta)+\zeta_{22}(\rho-\delta)\right)}{2k_{\rm i}}\\
\zeta_{41} &= \frac{-\gamma_{\rm er}\left(\zeta_{11}(\delta-\rho)-\zeta_{21}(\rho+\delta)\right)}{2k_{\rm i}}\\
\zeta_{42} &= \frac{-\gamma_{\rm er}\left(\zeta_{12}(\delta-\rho)-\zeta_{22}(\rho+\delta)\right)}{2k_{\rm i}}
\end{aligned}
$$

All ζ_{ij}'s but $\zeta_{31}, \zeta_{32}, \zeta_{41}, \zeta_{42}$ are independent constants; we have the following selection

$$
\begin{array}{cccc}
\zeta_{11} = 2k_{\mathrm{i}} & \zeta_{12} = 0 & \zeta_{13} = 1 & \zeta_{14} = 0 \\
\zeta_{21} = 0 & \zeta_{22} = 2k_{\mathrm{i}} & \zeta_{23} = 0 & \zeta_{24} = 1 \\
\zeta_{31} = \gamma_{\mathrm{el}}(\rho+\delta) & \zeta_{32} = \gamma_{\mathrm{el}}(\rho-\delta) & \zeta_{33} = 1 & \zeta_{34} = 0 \\
\zeta_{41} = \gamma_{\mathrm{er}}(\rho-\delta) & \zeta_{42} = \gamma_{\mathrm{er}}(\rho+\delta) & \zeta_{43} = 0 & \zeta_{44} = 1
\end{array}
$$

With this selection we have

$$
\begin{aligned}
Q_{\mathrm{il}} &= 2k_{\mathrm{i}}\, K^{\mathrm{i}}_{k_{\mathrm{i}}}\psi_1 + S^{\mathrm{i}}_{k_{\mathrm{i}}}\psi_3 \\
Q_{\mathrm{ir}} &= 2k_{\mathrm{i}}\, K^{\mathrm{i}}_{k_{\mathrm{i}}}\psi_2 + S^{\mathrm{i}}_{k_{\mathrm{i}}}\psi_4 \\
Q_{\mathrm{el}} &= K^{\mathrm{e}}_{\gamma_{\mathrm{el}}}(\gamma_{\mathrm{el}}(\rho+\delta)\psi_1 + \gamma_{\mathrm{el}}(\rho-\delta)\psi_2) + S^{\mathrm{e}}_{\gamma_{\mathrm{el}}}\psi_3 \\
Q_{\mathrm{er}} &= K^{\mathrm{e}}_{\gamma_{\mathrm{er}}}(\gamma_{\mathrm{er}}(\rho-\delta)\psi_1 + \gamma_{\mathrm{er}}(\rho+\delta)\psi_2) + S^{\mathrm{e}}_{\gamma_{\mathrm{er}}}\psi_4
\end{aligned}
$$

and

$$
\begin{aligned}
c_{11} &= -\frac{(\gamma_{\mathrm{er}}+\gamma_{\mathrm{el}})\rho + (2k_{\mathrm{i}} - \gamma_{\mathrm{er}} + \gamma_{\mathrm{el}})\delta}{2} I - \frac{\gamma_{\mathrm{er}}(\rho-\delta)}{2} K_{\gamma_{\mathrm{er}}} - \frac{\gamma_{\mathrm{el}}(\rho+\delta)}{2} K_{\gamma_{\mathrm{el}}} + \delta k_{\mathrm{i}} K_{k_{\mathrm{i}}} \\
c_{12} &= -\frac{(\gamma_{\mathrm{er}}+\gamma_{\mathrm{el}})\rho + (2k_{\mathrm{i}} + \gamma_{\mathrm{er}} - \gamma_{\mathrm{el}})\delta}{2} I - \frac{\gamma_{\mathrm{er}}(\rho+\delta)}{2} K_{\gamma_{\mathrm{er}}} - \frac{\gamma_{\mathrm{el}}(\rho-\delta)}{2} K_{\gamma_{\mathrm{el}}} + \delta k_{\mathrm{i}} K_{k_{\mathrm{i}}} \\
c_{13} &= \frac{\delta}{2} S_{k_{\mathrm{i}}} - \frac{1}{2} S_{\gamma_{\mathrm{el}}} \\
c_{14} &= \frac{\delta}{2} S_{k_{\mathrm{i}}} - \frac{1}{2} S_{\gamma_{\mathrm{er}}} \\
c_{21} &= i\frac{(\gamma_{\mathrm{er}}+\gamma_{\mathrm{el}})\delta + (2k_{\mathrm{i}} - \gamma_{\mathrm{er}} + \gamma_{\mathrm{el}})\rho}{2} I + \frac{i\gamma_{\mathrm{el}}(\rho+\delta)}{2} K_{\gamma_{\mathrm{el}}} - \frac{i\gamma_{\mathrm{er}}(\rho-\delta)}{2} K_{\gamma_{\mathrm{er}}} - ik_{\mathrm{i}}\rho K_{k_{\mathrm{i}}} \\
c_{22} &= -i\frac{(\gamma_{\mathrm{er}}+\gamma_{\mathrm{el}})\delta + (2k_{\mathrm{i}} + \gamma_{\mathrm{er}} - \gamma_{\mathrm{el}})\rho}{2} I - \frac{i\gamma_{\mathrm{er}}(\rho+\delta)}{2} K_{\gamma_{\mathrm{er}}} + \frac{i\gamma_{\mathrm{el}}(\rho-\delta)}{2} K_{\gamma_{\mathrm{el}}} + ik_{\mathrm{i}}\rho K_{k_{\mathrm{i}}}
\end{aligned}
$$

$$
\begin{aligned}
c_{23} &= \frac{i}{2} S_{\gamma_{\mathrm{el}}} - \frac{i\rho}{2} S_{k_{\mathrm{i}}} \\
c_{24} &= \frac{i\rho}{2} S_{k_{\mathrm{i}}} - \frac{i}{2} S_{\gamma_{\mathrm{er}}} \\
c_{31} &= -\delta T_{k_{\mathrm{i}}} - \frac{\rho-\delta}{2} T_{\gamma_{\mathrm{er}}} + \frac{\rho+\delta}{2} T_{\gamma_{\mathrm{el}}} \\
c_{32} &= \delta T_{k_{\mathrm{i}}} - \frac{\rho+\delta}{2} T_{\gamma_{\mathrm{er}}} + \frac{\rho-\delta}{2} T_{\gamma_{\mathrm{el}}} \\
c_{33} &= -\frac{k_{\mathrm{i}} + \delta\gamma_{\mathrm{el}}}{2\gamma_{\mathrm{el}} k_{\mathrm{i}}} I - \frac{\delta}{2k_{\mathrm{i}}} K'_{k_{\mathrm{i}}} + \frac{1}{2\gamma_{\mathrm{el}}} K'_{\gamma_{\mathrm{el}}} \\
c_{34} &= \frac{k_{\mathrm{i}} + \delta\gamma_{\mathrm{er}}}{2\gamma_{\mathrm{er}} k_{\mathrm{i}}} I + \frac{\delta}{2k_{\mathrm{i}}} K'_{k_{\mathrm{i}}} - \frac{1}{2\gamma_{\mathrm{er}}} K'_{\gamma_{\mathrm{er}}} \\
c_{41} &= i\rho T_{k_{\mathrm{i}}} - \frac{i(\rho-\delta)}{2} T_{\gamma_{\mathrm{er}}} - \frac{i(\rho+\delta)}{2} T_{\gamma_{\mathrm{el}}} \\
c_{42} &= i\rho T_{k_{\mathrm{i}}} - \frac{i(\rho+\delta)}{2} T_{\gamma_{\mathrm{er}}} - \frac{i(\rho-\delta)}{2} T_{\gamma_{\mathrm{el}}} \\
c_{43} &= \frac{i(k_{\mathrm{i}} + \rho\gamma_{\mathrm{el}})}{2\gamma_{\mathrm{el}} k_{\mathrm{i}}} I + \frac{i\rho}{2k_{\mathrm{i}}} K'_{k_{\mathrm{i}}} - \frac{i}{2\gamma_{\mathrm{el}}} K'_{\gamma_{\mathrm{el}}} \\
c_{44} &= \frac{i(k_{\mathrm{i}} + \rho\gamma_{\mathrm{er}})}{2\gamma_{\mathrm{er}} k_{\mathrm{i}}} I + \frac{i\rho}{2k_{\mathrm{i}}} K'_{k_{\mathrm{i}}} - \frac{i}{2\gamma_{\mathrm{er}}} K'_{\gamma_{\mathrm{er}}}
\end{aligned}
$$

Decomposing $c_{ij} = e_{ij} + a_{ij}$ where e_{ij} only involves the identity transform I and $a_{ij} = c_{ij} - e_{ij}$, we have

$$
\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}
$$

with

$$
\begin{aligned}
e_{11} &= -\frac{(\gamma_{\mathrm{er}} + \gamma_{\mathrm{el}})\rho + (2k_{\mathrm{i}} - \gamma_{\mathrm{er}} + \gamma_{\mathrm{el}})\delta}{2} I \\
e_{12} &= -\frac{(\gamma_{\mathrm{er}} + \gamma_{\mathrm{el}})\rho + (2k_{\mathrm{i}} + \gamma_{\mathrm{er}} - \gamma_{\mathrm{el}})\delta}{2} I \\
e_{13} &= 0 \\
e_{14} &= 0 \\
e_{21} &= i\frac{(\gamma_{\mathrm{er}} + \gamma_{\mathrm{el}})\delta + (2k_{\mathrm{i}} - \gamma_{\mathrm{er}} + \gamma_{\mathrm{el}})\rho}{2} I \\
e_{22} &= -i\frac{(\gamma_{\mathrm{er}} + \gamma_{\mathrm{el}})\delta + (2k_{\mathrm{i}} + \gamma_{\mathrm{er}} - \gamma_{\mathrm{el}})\rho}{2} I \\
e_{23} &= 0 \\
e_{24} &= 0
\end{aligned}
$$

$$\begin{aligned}
e_{31} &= 0 \\
e_{32} &= 0 \\
e_{33} &= -\frac{k_\mathrm{i} + \delta\gamma_\mathrm{el}}{2\gamma_\mathrm{el} k_\mathrm{i}} I \\
e_{34} &= \frac{k_\mathrm{i} + \delta\gamma_\mathrm{er}}{2\gamma_\mathrm{er} k_\mathrm{i}} I \\
e_{41} &= 0 \\
e_{42} &= 0 \\
e_{43} &= \frac{i(k_\mathrm{i} + \rho\gamma_\mathrm{el})}{2\gamma_\mathrm{el} k_\mathrm{i}} I \\
e_{44} &= \frac{i(k_\mathrm{i} + \rho\gamma_\mathrm{er})}{2\gamma_\mathrm{er} k_\mathrm{i}} I
\end{aligned}$$

and

$$\begin{aligned}
a_{11} &= -\frac{\gamma_\mathrm{er}(\rho-\delta)}{2} K_{\gamma_\mathrm{er}} - \frac{\gamma_\mathrm{el}(\rho+\delta)}{2} K_{\gamma_\mathrm{el}} + \delta k_\mathrm{i} K_{k_\mathrm{i}} \\
a_{12} &= -\frac{\gamma_\mathrm{er}(\rho+\delta)}{2} K_{\gamma_\mathrm{er}} - \frac{\gamma_\mathrm{el}(\rho-\delta)}{2} K_{\gamma_\mathrm{el}} + \delta k_\mathrm{i} K_{k_\mathrm{i}} \\
a_{13} &= \frac{\delta}{2} S_{k_\mathrm{i}} - \frac{1}{2} S_{\gamma_\mathrm{el}} \\
a_{14} &= \frac{\delta}{2} S_{k_\mathrm{i}} - \frac{1}{2} S_{\gamma_\mathrm{er}} \\
a_{21} &= \frac{i\gamma_\mathrm{el}(\rho+\delta)}{2} K_{\gamma_\mathrm{el}} - \frac{i\gamma_\mathrm{er}(\rho-\delta)}{2} K_{\gamma_\mathrm{er}} - i k_\mathrm{i} \rho K_{k_\mathrm{i}} \\
a_{22} &= -\frac{i\gamma_\mathrm{er}(\rho+\delta)}{2} K_{\gamma_\mathrm{er}} + \frac{i\gamma_\mathrm{el}(\rho-\delta)}{2} K_{\gamma_\mathrm{el}} + i k_\mathrm{i} \rho K_{k_\mathrm{i}} \\
a_{23} &= \frac{i}{2} S_{\gamma_\mathrm{el}} - \frac{i\rho}{2} S_{k_\mathrm{i}} \\
a_{24} &= \frac{i\rho}{2} S_{k_\mathrm{i}} - \frac{i}{2} S_{\gamma_\mathrm{er}} \\
a_{31} &= -\delta T_{k_\mathrm{i}} - \frac{\rho-\delta}{2} T_{\gamma_\mathrm{er}} + \frac{\rho+\delta}{2} T_{\gamma_\mathrm{el}} \\
a_{32} &= \delta T_{k_\mathrm{i}} - \frac{\rho+\delta}{2} T_{\gamma_\mathrm{er}} + \frac{\rho-\delta}{2} T_{\gamma_\mathrm{el}} \\
a_{33} &= -\frac{\delta}{2k_\mathrm{i}} K'_{k_\mathrm{i}} + \frac{1}{2\gamma_\mathrm{el}} K'_{\gamma_\mathrm{el}} \\
a_{34} &= \frac{\delta}{2k_\mathrm{i}} K'_{k_\mathrm{i}} - \frac{1}{2\gamma_\mathrm{er}} K'_{\gamma_\mathrm{er}} \\
a_{41} &= i\rho T_{k_\mathrm{i}} - \frac{i(\rho-\delta)}{2} T_{\gamma_\mathrm{er}} - \frac{i(\rho+\delta)}{2} T_{\gamma_\mathrm{el}} \\
a_{42} &= i\rho T_{k_\mathrm{i}} - \frac{i(\rho+\delta)}{2} T_{\gamma_\mathrm{er}} - \frac{i(\rho-\delta)}{2} T_{\gamma_\mathrm{el}} \\
a_{43} &= \frac{i\rho}{2k_\mathrm{i}} K'_{k_\mathrm{i}} - \frac{i}{2\gamma_\mathrm{el}} K'_{\gamma_\mathrm{el}} \\
a_{44} &= \frac{i\rho}{2k_\mathrm{i}} K'_{k_\mathrm{i}} - \frac{i}{2\gamma_\mathrm{er}} K'_{\gamma_\mathrm{er}}
\end{aligned}$$

The determinant of $\{e_{ij}\}$ is

$$\frac{1}{4\gamma_{\text{el}}\gamma_{\text{er}}k_{\text{i}}^2}\left(\gamma_{\text{er}}k_{\text{i}}\rho+\gamma_{\text{el}}k_{\text{i}}\rho+2\delta\gamma_{\text{el}}\gamma_{\text{er}}\rho+2k_{\text{i}}^2+\delta\gamma_{\text{er}}k_{\text{i}}+\delta\gamma_{\text{el}}k_{\text{i}}\right)$$
$$\times\left(\gamma_{\text{er}}k_{\text{i}}\rho^2+\gamma_{\text{el}}k_{\text{i}}\rho^2+2\delta k_{\text{i}}^2\rho+2\delta\gamma_{\text{el}}\gamma_{\text{er}}\rho+\delta^2\gamma_{\text{er}}k_{\text{i}}+\delta^2\gamma_{\text{el}}k_{\text{i}}\right)$$

Achiral-Chiral

The "master equations" are

$$\begin{aligned}
v_0 &= \frac{\delta}{2}(Q_{\text{ir}}+Q_{\text{il}})-\frac{1}{2}(Q_{\text{er}}+Q_{\text{el}})\\
w_0 &= \frac{i\rho}{2}(Q_{\text{ir}}-Q_{\text{il}})-\frac{i}{2}(Q_{\text{er}}-Q_{\text{el}})\\
v_1 &= \frac{\delta}{2}\left(\frac{1}{\gamma_{\text{ir}}}\frac{\partial Q_{\text{ir}}}{\partial\nu}-\frac{1}{\gamma_{\text{il}}}\frac{\partial Q_{\text{il}}}{\partial\nu}\right)-\frac{1}{2}\left(\frac{1}{k_{\text{e}}}\frac{\partial Q_{\text{er}}}{\partial\nu}-\frac{1}{k_{\text{e}}}\frac{\partial Q_{\text{el}}}{\partial\nu}\right)\\
w_1 &= \frac{i\rho}{2}\left(\frac{1}{\gamma_{\text{ir}}}\frac{\partial Q_{\text{ir}}}{\partial\nu}+\frac{1}{\gamma_{\text{il}}}\frac{\partial Q_{\text{il}}}{\partial\nu}\right)-\frac{i}{2}\left(\frac{1}{k_{\text{e}}}\frac{\partial Q_{\text{er}}}{\partial\nu}+\frac{1}{k_{\text{e}}}\frac{\partial Q_{\text{el}}}{\partial\nu}\right)
\end{aligned}$$

We propose the following ansatz

$$\begin{aligned}
Q_{\text{il}} &= K^{\text{i}}_{\gamma_{\text{il}}}(\zeta_{11}\psi_1+\zeta_{12}\psi_2)+S^{\text{i}}_{\gamma_{\text{il}}}(\zeta_{13}\psi_3+\zeta_{14}\psi_4)\\
Q_{\text{ir}} &= K^{\text{i}}_{\gamma_{\text{ir}}}(\zeta_{21}\psi_1+\zeta_{22}\psi_2)+S^{\text{i}}_{\gamma_{\text{ir}}}(\zeta_{23}\psi_3+\zeta_{24}\psi_4)\\
Q_{\text{el}} &= K^{\text{e}}_{k_{\text{e}}}(\zeta_{31}\psi_1+\zeta_{32}\psi_2)+S^{\text{e}}_{k_{\text{e}}}(\zeta_{33}\psi_3+\zeta_{34}\psi_4)\\
Q_{\text{er}} &= K^{\text{e}}_{k_{\text{e}}}(\zeta_{41}\psi_1+\zeta_{42}\psi_2)+S^{\text{e}}_{k_{\text{e}}}(\zeta_{43}\psi_3+\zeta_{44}\psi_4)
\end{aligned}$$

where ψ_j's are unknowns and ζ_{ij}'s are constants to be determined later. The boundary traces are

$$\begin{aligned}
Q_{\text{il}} &= (K_{\gamma_{\text{il}}}-I)(\zeta_{11}\psi_1+\zeta_{12}\psi_2)+S_{\gamma_{\text{il}}}(\zeta_{13}\psi_3+\zeta_{14}\psi_4)\\
Q_{\text{ir}} &= (K_{\gamma_{\text{ir}}}-I)(\zeta_{21}\psi_1+\zeta_{22}\psi_2)+S_{\gamma_{\text{ir}}}(\zeta_{23}\psi_3+\zeta_{24}\psi_4)\\
Q_{\text{el}} &= (K_{k_{\text{e}}}+I)(\zeta_{31}\psi_1+\zeta_{32}\psi_2)+S_{k_{\text{e}}}(\zeta_{33}\psi_3+\zeta_{34}\psi_4)\\
Q_{\text{er}} &= (K_{k_{\text{e}}}+I)(\zeta_{41}\psi_1+\zeta_{42}\psi_2)+S_{k_{\text{e}}}(\zeta_{43}\psi_3+\zeta_{44}\psi_4)\\
\frac{\partial Q_{\text{il}}}{\partial\nu} &= T_{\gamma_{\text{il}}}(\zeta_{11}\psi_1+\zeta_{12}\psi_2)+(K'_{\gamma_{\text{il}}}+I)(\zeta_{13}\psi_3+\zeta_{14}\psi_4)\\
\frac{\partial Q_{\text{ir}}}{\partial\nu} &= T_{\gamma_{\text{ir}}}(\zeta_{21}\psi_1+\zeta_{22}\psi_2)+(K'_{\gamma_{\text{ir}}}+I)(\zeta_{23}\psi_3+\zeta_{24}\psi_4)\\
\frac{\partial Q_{\text{el}}}{\partial\nu} &= T_{k_{\text{e}}}(\zeta_{31}\psi_1+\zeta_{32}\psi_2)+(K'_{k_{\text{e}}}-I)(\zeta_{33}\psi_3+\zeta_{34}\psi_4)\\
\frac{\partial Q_{\text{er}}}{\partial\nu} &= T_{k_{\text{e}}}(\zeta_{41}\psi_1+\zeta_{42}\psi_2)+(K'_{k_{\text{e}}}-I)(\zeta_{43}\psi_3+\zeta_{44}\psi_4)
\end{aligned}$$

Substituting into "master equations", we have

$$
\begin{aligned}
v_0 = \frac{\delta}{2}\big((K_{\gamma_{\mathrm{ir}}} - I)(\psi_2\zeta_{22} + \psi_1\zeta_{21}) + (K_{\gamma_{\mathrm{il}}} - I)(\psi_2\zeta_{12} + \psi_1\zeta_{11}) \\
+S_{\gamma_{\mathrm{ir}}}(\psi_4\zeta_{24} + \psi_3\zeta_{23}) + S_{\gamma_{\mathrm{il}}}(\psi_4\zeta_{14} + \psi_3\zeta_{13})\big) \\
- \frac{1}{2}\big((K_{k_{\mathrm{e}}} + I)(\psi_2\zeta_{42} + \psi_1\zeta_{41}) + (K_{k_{\mathrm{e}}} + I)(\psi_2\zeta_{32} + \psi_1\zeta_{31}) \\
+S_{k_{\mathrm{e}}}(\psi_4\zeta_{44} + \psi_3\zeta_{43}) + S_{k_{\mathrm{e}}}(\psi_4\zeta_{34} + \psi_3\zeta_{33})\big)
\end{aligned}
$$

and

$$
\begin{aligned}
w_0 = \frac{i\rho}{2}\big((K_{\gamma_{\mathrm{ir}}} - I)(\psi_2\zeta_{22} + \psi_1\zeta_{21}) - (K_{\gamma_{\mathrm{il}}} - I)(\psi_2\zeta_{12} + \psi_1\zeta_{11}) \\
+S_{\gamma_{\mathrm{ir}}}(\psi_4\zeta_{24} + \psi_3\zeta_{23}) - S_{\gamma_{\mathrm{il}}}(\psi_4\zeta_{14} + \psi_3\zeta_{13})\big) \\
- \frac{i}{2}\big((K_{k_{\mathrm{e}}} + I)(\psi_2\zeta_{42} + \psi_1\zeta_{41}) - (K_{k_{\mathrm{e}}} + I)(\psi_2\zeta_{32} + \psi_1\zeta_{31}) \\
+S_{k_{\mathrm{e}}}(\psi_4\zeta_{44} + \psi_3\zeta_{43}) - S_{k_{\mathrm{e}}}(\psi_4\zeta_{34} + \psi_3\zeta_{33})\big)
\end{aligned}
$$

and

$$
\begin{aligned}
v_1 = \frac{\delta}{2}\Bigg(\frac{1}{\gamma_{\mathrm{ir}}}\big((K'_{\gamma_{\mathrm{ir}}} + I)(\psi_4\zeta_{24} + \psi_3\zeta_{23}) + T_{\gamma_{\mathrm{ir}}}(\psi_2\zeta_{22} + \psi_1\zeta_{21})\big) \\
-\frac{1}{\gamma_{\mathrm{il}}}\big((K'_{\gamma_{\mathrm{il}}} + I)(\psi_4\zeta_{14} + \psi_3\zeta_{13}) + T_{\gamma_{\mathrm{il}}}(\psi_2\zeta_{12} + \psi_1\zeta_{11})\big)\Bigg) \\
- \frac{1}{2}\Bigg(\frac{1}{k_{\mathrm{e}}}\big((K'_{k_{\mathrm{e}}} - I)(\psi_4\zeta_{44} + \psi_3\zeta_{43}) + T_{k_{\mathrm{e}}}(\psi_2\zeta_{42} + \psi_1\zeta_{41})\big) \\
-\frac{1}{k_{\mathrm{e}}}\big((K'_{k_{\mathrm{e}}} - I)(\psi_4\zeta_{34} + \psi_3\zeta_{33}) + T_{k_{\mathrm{e}}}(\psi_2\zeta_{32} + \psi_1\zeta_{31})\big)\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
w_1 = \frac{i\rho}{2}\Bigg(\frac{1}{\gamma_{\mathrm{ir}}}\big((K'_{\gamma_{\mathrm{ir}}} + I)(\psi_4\zeta_{24} + \psi_3\zeta_{23}) + T_{\gamma_{\mathrm{ir}}}(\psi_2\zeta_{22} + \psi_1\zeta_{21})\big) \\
+\frac{1}{\gamma_{\mathrm{il}}}\big((K'_{\gamma_{\mathrm{il}}} + I)(\psi_4\zeta_{14} + \psi_3\zeta_{13}) + T_{\gamma_{\mathrm{il}}}(\psi_2\zeta_{12} + \psi_1\zeta_{11})\big)\Bigg) \\
- \frac{i}{2}\Bigg(\frac{1}{k_{\mathrm{e}}}\big((K'_{k_{\mathrm{e}}} - I)(\psi_4\zeta_{44} + \psi_3\zeta_{43}) + T_{k_{\mathrm{e}}}(\psi_2\zeta_{42} + \psi_1\zeta_{41})\big) \\
+\frac{1}{k_{\mathrm{e}}}\big((K'_{k_{\mathrm{e}}} - I)(\psi_4\zeta_{34} + \psi_3\zeta_{33}) + T_{k_{\mathrm{e}}}(\psi_2\zeta_{32} + \psi_1\zeta_{31})\big)\Bigg)
\end{aligned}
$$

Put the previous four equations into matrix form, we have

$$
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} \\
c_{21} & c_{22} & c_{23} & c_{24} \\
c_{31} & c_{32} & c_{33} & c_{34} \\
c_{41} & c_{42} & c_{43} & c_{44}
\end{pmatrix}
\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}
=
\begin{pmatrix} v_0 \\ w_0 \\ v_1 \\ w_1 \end{pmatrix}
$$

where

$$
\begin{aligned}
c_{11} &= -\frac{\zeta_{41}+\zeta_{31}+\delta\zeta_{21}+\delta\zeta_{11}}{2}I - K_{k_e}\frac{\zeta_{41}+\zeta_{31}}{2} + K_{\gamma_{ir}}\frac{\delta\zeta_{21}}{2} + K_{\gamma_{il}}\frac{\delta\zeta_{11}}{2}\\
c_{12} &= -\frac{\zeta_{42}+\zeta_{32}+\delta\zeta_{22}+\delta\zeta_{12}}{2}I - K_{k_e}\frac{\zeta_{42}+\zeta_{32}}{2} + K_{\gamma_{ir}}\frac{\delta\zeta_{22}}{2} + K_{\gamma_{il}}\frac{\delta\zeta_{12}}{2}\\
c_{13} &= -S_{k_e}\frac{\zeta_{43}+\zeta_{33}}{2} + S_{\gamma_{ir}}\frac{\delta\zeta_{23}}{2} + S_{\gamma_{il}}\frac{\delta\zeta_{13}}{2}\\
c_{14} &= -S_{k_e}\frac{\zeta_{44}+\zeta_{34}}{2} + S_{\gamma_{ir}}\frac{\delta\zeta_{24}}{2} + S_{\gamma_{il}}\frac{\delta\zeta_{14}}{2}\\
c_{21} &= -\frac{i(\zeta_{41}-\zeta_{31}+\rho\zeta_{21}-\rho\zeta_{11})}{2}I - K_{k_e}\frac{i(\zeta_{41}-\zeta_{31})}{2} + K_{\gamma_{ir}}\frac{i\rho\zeta_{21}}{2} - K_{\gamma_{il}}\frac{i\rho\zeta_{11}}{2}\\
c_{22} &= -\frac{i(\zeta_{42}-\zeta_{32}+\rho\zeta_{22}-\rho\zeta_{12})}{2}I - K_{k_e}\frac{i(\zeta_{42}-\zeta_{32})}{2} + K_{\gamma_{ir}}\frac{i\rho\zeta_{22}}{2} - K_{\gamma_{il}}\frac{i\rho\zeta_{12}}{2}\\
c_{23} &= -S_{k_e}\frac{i(\zeta_{43}-\zeta_{33})}{2} + S_{\gamma_{ir}}\frac{i\rho\zeta_{23}}{2} - S_{\gamma_{il}}\frac{i\rho\zeta_{13}}{2}\\
c_{24} &= -S_{k_e}\frac{i(\zeta_{44}-\zeta_{34})}{2} + S_{\gamma_{ir}}\frac{i\rho\zeta_{24}}{2} - S_{\gamma_{il}}\frac{i\rho\zeta_{14}}{2}\\
c_{31} &= -T_{k_e}\frac{\zeta_{41}-\zeta_{31}}{2k_e} + T_{\gamma_{ir}}\frac{\delta\zeta_{21}}{2\gamma_{ir}} - T_{\gamma_{il}}\frac{\delta\zeta_{11}}{2\gamma_{il}}\\
c_{32} &= -T_{k_e}\frac{\zeta_{42}-\zeta_{32}}{2k_e} + T_{\gamma_{ir}}\frac{\delta\zeta_{22}}{2\gamma_{ir}} - T_{\gamma_{il}}\frac{\delta\zeta_{12}}{2\gamma_{il}}\\
c_{33} &= \frac{\gamma_{il}\gamma_{ir}\zeta_{43}-\gamma_{il}\gamma_{ir}\zeta_{33}+\delta\gamma_{il}k_e\zeta_{23}-\delta\gamma_{ir}k_e\zeta_{13}}{2\gamma_{il}\gamma_{ir}k_e}I - K'_{k_e}\frac{\zeta_{43}-\zeta_{33}}{2k_e} + K'_{\gamma_{ir}}\frac{\delta\zeta_{23}}{2\gamma_{ir}} - K'_{\gamma_{il}}\frac{\delta\zeta_{13}}{2\gamma_{il}}\\
c_{34} &= \frac{\gamma_{il}\gamma_{ir}\zeta_{44}-\gamma_{il}\gamma_{ir}\zeta_{34}+\delta\gamma_{il}k_e\zeta_{24}-\delta\gamma_{ir}k_e\zeta_{14}}{2\gamma_{il}\gamma_{ir}k_e}I - K'_{k_e}\frac{\zeta_{44}-\zeta_{34}}{2k_e} + K'_{\gamma_{ir}}\frac{\delta\zeta_{24}}{2\gamma_{ir}} - K'_{\gamma_{il}}\frac{\delta\zeta_{14}}{2\gamma_{il}}\\
c_{41} &= -T_{k_e}\frac{i(\zeta_{41}+\zeta_{31})}{2k_e} + T_{\gamma_{ir}}\frac{i\rho\zeta_{21}}{2\gamma_{ir}} + T_{\gamma_{il}}\frac{i\rho\zeta_{11}}{2\gamma_{il}}\\
c_{42} &= -T_{k_e}\frac{i(\zeta_{42}+\zeta_{32})}{2k_e} + T_{\gamma_{ir}}\frac{i\rho\zeta_{22}}{2\gamma_{ir}} + T_{\gamma_{il}}\frac{i\rho\zeta_{12}}{2\gamma_{il}}\\
c_{43} &= \frac{i(\gamma_{il}\gamma_{ir}\zeta_{43}+\gamma_{il}\gamma_{ir}\zeta_{33}+\gamma_{il}k_e\rho\zeta_{23}+\gamma_{ir}k_e\rho\zeta_{13})}{2\gamma_{il}\gamma_{ir}k_e}I - K'_{k_e}\frac{i(\zeta_{43}+\zeta_{33})}{2k_e} + K'_{\gamma_{ir}}\frac{i\rho\zeta_{23}}{2\gamma_{ir}} + K'_{\gamma_{il}}\frac{i\rho\zeta_{13}}{2\gamma_{il}}\\
c_{44} &= \frac{i(\gamma_{il}\gamma_{ir}\zeta_{44}+\gamma_{il}\gamma_{ir}\zeta_{34}+\gamma_{il}k_e\rho\zeta_{24}+\gamma_{ir}k_e\rho\zeta_{14})}{2\gamma_{il}\gamma_{ir}k_e}I - K'_{k_e}\frac{i(\zeta_{44}+\zeta_{34})}{2k_e} + K'_{\gamma_{ir}}\frac{i\rho\zeta_{24}}{2\gamma_{ir}} + K'_{\gamma_{il}}\frac{i\rho\zeta_{14}}{2\gamma_{il}}
\end{aligned}
$$

We wish to make the appearance of hypersingular operators T_k's in $c_{31}, c_{32}, c_{41}, c_{42}$ to be in the form of the linear combinations of $T_{k_1} - T_{k_2}$, hence

$$
\begin{aligned}
0 &= -\frac{\zeta_{41}-\zeta_{31}}{2k_e} + \frac{\delta\zeta_{21}}{2\gamma_{ir}} - \frac{\delta\zeta_{11}}{2\gamma_{il}}\\
0 &= -\frac{\zeta_{42}-\zeta_{32}}{2k_e} + \frac{\delta\zeta_{22}}{2\gamma_{ir}} - \frac{\delta\zeta_{12}}{2\gamma_{il}}\\
0 &= -\frac{i(\zeta_{41}+\zeta_{31})}{2k_e} + \frac{i\rho\zeta_{21}}{2\gamma_{ir}} + \frac{i\rho\zeta_{11}}{2\gamma_{il}}\\
0 &= -\frac{i(\zeta_{42}+\zeta_{32})}{2k_e} + \frac{i\rho\zeta_{22}}{2\gamma_{ir}} + \frac{i\rho\zeta_{12}}{2\gamma_{il}}
\end{aligned}
$$

Solving the above, we have

$$\begin{aligned}
\zeta_{31} &= \frac{k_{\mathrm{e}}\big(\gamma_{\mathrm{ir}}\zeta_{11}(\rho+\delta)+\gamma_{\mathrm{il}}\zeta_{21}(\rho-\delta)\big)}{2\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}}\\
\zeta_{32} &= \frac{k_{\mathrm{e}}\big(\gamma_{\mathrm{ir}}\zeta_{12}(\rho+\delta)+\gamma_{\mathrm{il}}\zeta_{22}(\rho-\delta)\big)}{2\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}}\\
\zeta_{41} &= \frac{-k_{\mathrm{e}}\big(\gamma_{\mathrm{ir}}\zeta_{11}(\delta-\rho)-\gamma_{\mathrm{il}}\zeta_{21}(\rho+\delta)\big)}{2\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}}\\
\zeta_{42} &= \frac{-k_{\mathrm{e}}\big(\gamma_{\mathrm{ir}}\zeta_{12}(\delta-\rho)-\gamma_{\mathrm{il}}\zeta_{22}(\rho+\delta)\big)}{2\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}}
\end{aligned}$$

All ζ_{ij}'s but $\zeta_{31}, \zeta_{32}, \zeta_{41}, \zeta_{42}$ are independent constants; we have the following selection

$$\begin{array}{cccc}
\zeta_{11} = 2\gamma_{\mathrm{il}} & \zeta_{12} = 0 & \zeta_{13} = 1 & \zeta_{14} = 0\\
\zeta_{21} = 0 & \zeta_{22} = 2\gamma_{\mathrm{ir}} & \zeta_{23} = 0 & \zeta_{24} = 1\\
\zeta_{31} = k_{\mathrm{e}}(\rho+\delta) & \zeta_{32} = k_{\mathrm{e}}(\rho-\delta) & \zeta_{33} = 1 & \zeta_{34} = 0\\
\zeta_{41} = k_{\mathrm{e}}(\rho-\delta) & \zeta_{42} = k_{\mathrm{e}}(\rho+\delta) & \zeta_{43} = 0 & \zeta_{44} = 1
\end{array}$$

With this selection we have

$$\begin{aligned}
Q_{\mathrm{il}} &= 2\gamma_{\mathrm{il}}\, K^{\mathrm{i}}_{\gamma_{\mathrm{il}}}\psi_1 + S^{\mathrm{i}}_{\gamma_{\mathrm{il}}}\psi_3\\
Q_{\mathrm{ir}} &= 2\gamma_{\mathrm{ir}}\, K^{\mathrm{i}}_{\gamma_{\mathrm{ir}}}\psi_2 + S^{\mathrm{i}}_{\gamma_{\mathrm{ir}}}\psi_4\\
Q_{\mathrm{el}} &= K^{\mathrm{e}}_{k_{\mathrm{e}}}(k_{\mathrm{e}}(\rho+\delta)\psi_1 + k_{\mathrm{e}}(\rho-\delta)\psi_2) + S^{\mathrm{e}}_{k_{\mathrm{e}}}\psi_3\\
Q_{\mathrm{er}} &= K^{\mathrm{e}}_{k_{\mathrm{e}}}(k_{\mathrm{e}}(\rho-\delta)\psi_1 + k_{\mathrm{e}}(\rho+\delta)\psi_2) + S^{\mathrm{e}}_{k_{\mathrm{e}}}\psi_4
\end{aligned}$$

and

$$
\begin{aligned}
c_{11} &= (-k_{\mathrm{e}}\rho - \delta\gamma_{\mathrm{il}})I - k_{\mathrm{e}}\rho K_{k_{\mathrm{e}}} + \delta\gamma_{\mathrm{il}}K_{\gamma_{\mathrm{il}}} \\
c_{12} &= (-k_{\mathrm{e}}\rho - \delta\gamma_{\mathrm{ir}})I - k_{\mathrm{e}}\rho K_{k_{\mathrm{e}}} + \delta\gamma_{\mathrm{ir}}K_{\gamma_{\mathrm{ir}}} \\
c_{13} &= \frac{\delta}{2}S_{\gamma_{\mathrm{il}}} - \frac{1}{2}S_{k_{\mathrm{e}}} \\
c_{14} &= \frac{\delta}{2}S_{\gamma_{\mathrm{ir}}} - \frac{1}{2}S_{k_{\mathrm{e}}} \\
c_{21} &= i(\gamma_{\mathrm{il}}\rho + \delta k_{\mathrm{e}})I + ik_{\mathrm{e}}\delta K_{k_{\mathrm{e}}} - i\gamma_{\mathrm{il}}\rho K_{\gamma_{\mathrm{il}}} \\
c_{22} &= -i(\gamma_{\mathrm{ir}}\rho + \delta k_{\mathrm{e}})I - ik_{\mathrm{e}}\delta K_{k_{\mathrm{e}}} + i\gamma_{\mathrm{ir}}\rho K_{\gamma_{\mathrm{ir}}} \\
c_{23} &= \frac{i}{2}S_{k_{\mathrm{e}}} - \frac{i\rho}{2}S_{\gamma_{\mathrm{il}}} \\
c_{24} &= \frac{i\rho}{2}S_{\gamma_{\mathrm{ir}}} - \frac{i}{2}S_{k_{\mathrm{e}}} \\
c_{31} &= -\delta T_{\gamma_{\mathrm{il}}} + \delta T_{k_{\mathrm{e}}} \\
c_{32} &= \delta T_{\gamma_{\mathrm{ir}}} - \delta T_{k_{\mathrm{e}}} \\
c_{33} &= -\frac{\delta k_{\mathrm{e}} + \gamma_{\mathrm{il}}}{2\gamma_{\mathrm{il}}k_{\mathrm{e}}}I - \frac{\delta}{2\gamma_{\mathrm{il}}}K'_{\gamma_{\mathrm{il}}} + \frac{1}{2k_{\mathrm{e}}}K'_{k_{\mathrm{e}}} \\
c_{34} &= \frac{\delta k_{\mathrm{e}} + \gamma_{\mathrm{ir}}}{2\gamma_{\mathrm{ir}}k_{\mathrm{e}}}I + \frac{\delta}{2\gamma_{\mathrm{ir}}}K'_{\gamma_{\mathrm{ir}}} - \frac{1}{2k_{\mathrm{e}}}K'_{k_{\mathrm{e}}} \\
c_{41} &= i\rho T_{\gamma_{\mathrm{il}}} - i\rho T_{k_{\mathrm{e}}} \\
c_{42} &= i\rho T_{\gamma_{\mathrm{ir}}} - i\rho T_{k_{\mathrm{e}}} \\
c_{43} &= \frac{i(k_{\mathrm{e}}\rho + \gamma_{\mathrm{il}})}{2\gamma_{\mathrm{il}}k_{\mathrm{e}}}I + \frac{i\rho}{2\gamma_{\mathrm{il}}}K'_{\gamma_{\mathrm{il}}} - \frac{i}{2k_{\mathrm{e}}}K'_{k_{\mathrm{e}}} \\
c_{44} &= \frac{i(k_{\mathrm{e}}\rho + \gamma_{\mathrm{ir}})}{2\gamma_{\mathrm{ir}}k_{\mathrm{e}}}I + \frac{i\rho}{2\gamma_{\mathrm{ir}}}K'_{\gamma_{\mathrm{ir}}} - \frac{i}{2k_{\mathrm{e}}}K'_{k_{\mathrm{e}}}
\end{aligned}
$$

Decomposing $c_{ij} = e_{ij} + a_{ij}$ where e_{ij} only involves the identity transform I and $a_{ij} = c_{ij} - e_{ij}$, we have

$$
\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}
$$

with

$$
\begin{aligned}
e_{11} &= (-k_{\mathrm{e}}\rho - \delta\gamma_{\mathrm{il}})\, I\\
e_{12} &= (-k_{\mathrm{e}}\rho - \delta\gamma_{\mathrm{ir}})\, I\\
e_{13} &= 0\\
e_{14} &= 0\\
e_{21} &= i\,(\gamma_{\mathrm{il}}\rho + \delta k_{\mathrm{e}})\, I\\
e_{22} &= -i(\gamma_{\mathrm{ir}}\rho + \delta k_{\mathrm{e}}) I\\
e_{23} &= 0\\
e_{24} &= 0\\
e_{31} &= 0\\
e_{32} &= 0\\
e_{33} &= -\frac{\delta k_{\mathrm{e}} + \gamma_{\mathrm{il}}}{2\gamma_{\mathrm{il}} k_{\mathrm{e}}} I\\
e_{34} &= \frac{\delta k_{\mathrm{e}} + \gamma_{\mathrm{ir}}}{2\gamma_{\mathrm{ir}} k_{\mathrm{e}}} I\\
e_{41} &= 0\\
e_{42} &= 0\\
e_{43} &= \frac{i(k_{\mathrm{e}}\rho + \gamma_{\mathrm{il}})}{2\gamma_{\mathrm{il}} k_{\mathrm{e}}} I\\
e_{44} &= \frac{i(k_{\mathrm{e}}\rho + \gamma_{\mathrm{ir}})}{2\gamma_{\mathrm{ir}} k_{\mathrm{e}}} I
\end{aligned}
$$

and

$$
\begin{aligned}
a_{11} &= -k_{\mathrm{e}}\rho K_{k_{\mathrm{e}}} + \delta\gamma_{\mathrm{il}} K_{\gamma_{\mathrm{il}}}\\
a_{12} &= -k_{\mathrm{e}}\rho K_{k_{\mathrm{e}}} + \delta\gamma_{\mathrm{ir}} K_{\gamma_{\mathrm{ir}}}\\
a_{13} &= \frac{\delta}{2} S_{\gamma_{\mathrm{il}}} - \frac{1}{2} S_{k_{\mathrm{e}}}\\
a_{14} &= \frac{\delta}{2} S_{\gamma_{\mathrm{ir}}} - \frac{1}{2} S_{k_{\mathrm{e}}}\\
a_{21} &= i k_{\mathrm{e}}\delta K_{k_{\mathrm{e}}} - i\gamma_{\mathrm{il}}\rho K_{\gamma_{\mathrm{il}}}\\
a_{22} &= -i k_{\mathrm{e}}\delta K_{k_{\mathrm{e}}} + i\gamma_{\mathrm{ir}}\rho K_{\gamma_{\mathrm{ir}}}\\
a_{23} &= \frac{i}{2} S_{k_{\mathrm{e}}} - \frac{i\rho}{2} S_{\gamma_{\mathrm{il}}}\\
a_{24} &= \frac{i\rho}{2} S_{\gamma_{\mathrm{ir}}} - \frac{i}{2} S_{k_{\mathrm{e}}}
\end{aligned}
$$

$$
\begin{aligned}
a_{31} &= -\delta T_{\gamma_{\mathrm{il}}} + \delta T_{k_{\mathrm{e}}} \\
a_{32} &= \delta T_{\gamma_{\mathrm{ir}}} - \delta T_{k_{\mathrm{e}}} \\
a_{33} &= -\frac{\delta}{2\gamma_{\mathrm{il}}} K'_{\gamma_{\mathrm{il}}} + \frac{1}{2k_{\mathrm{e}}} K'_{k_{\mathrm{e}}} \\
a_{34} &= \frac{\delta}{2\gamma_{\mathrm{ir}}} K'_{\gamma_{\mathrm{ir}}} - \frac{1}{2k_{\mathrm{e}}} K'_{k_{\mathrm{e}}} \\
a_{41} &= i\rho T_{\gamma_{\mathrm{il}}} - i\rho T_{k_{\mathrm{e}}} \\
a_{42} &= i\rho T_{\gamma_{\mathrm{ir}}} - i\rho T_{k_{\mathrm{e}}} \\
a_{43} &= \frac{i\rho}{2\gamma_{\mathrm{il}}} K'_{\gamma_{\mathrm{il}}} - \frac{i}{2k_{\mathrm{e}}} K'_{k_{\mathrm{e}}} \\
a_{44} &= \frac{i\rho}{2\gamma_{\mathrm{ir}}} K'_{\gamma_{\mathrm{ir}}} - \frac{i}{2k_{\mathrm{e}}} K'_{k_{\mathrm{e}}}
\end{aligned}
$$

The determinant of $\{e_{ij}\}$ is

$$
\begin{aligned}
\frac{1}{4\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}k_{\mathrm{e}}^2} &\left(2\delta k_{\mathrm{e}}^2\rho + \gamma_{\mathrm{ir}}k_{\mathrm{e}}\rho + \gamma_{\mathrm{il}}k_{\mathrm{e}}\rho + \delta\gamma_{\mathrm{ir}}k_{\mathrm{e}} + \delta\gamma_{\mathrm{il}}k_{\mathrm{e}} + 2\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}\right) \\
&\times \left(\gamma_{\mathrm{ir}}k_{\mathrm{e}}\rho^2 + \gamma_{\mathrm{il}}k_{\mathrm{e}}\rho^2 + 2\delta k_{\mathrm{e}}^2\rho + 2\delta\gamma_{\mathrm{il}}\gamma_{\mathrm{ir}}\rho + \delta^2\gamma_{\mathrm{ir}}k_{\mathrm{e}} + \delta^2\gamma_{\mathrm{il}}k_{\mathrm{e}}\right)
\end{aligned}
$$

Chiral-Perfect Conductor

The “master equations” are

$$
\begin{aligned}
w_0 &= -\frac{i}{2}(Q_{\mathrm{er}} - Q_{\mathrm{el}}) \\
w_1 &= -\frac{i}{2}\left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} + \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right)
\end{aligned}
$$

We propose the following ansatz

$$
\begin{aligned}
Q_{\mathrm{el}} &= K^{\mathrm{e}}_{\gamma_{\mathrm{el}}}\zeta_{11}\psi_1 + S^{\mathrm{e}}_{\gamma_{\mathrm{el}}}\zeta_{12}\psi_2 \\
Q_{\mathrm{er}} &= K^{\mathrm{e}}_{\gamma_{\mathrm{er}}}\zeta_{21}\psi_1 + S^{\mathrm{e}}_{\gamma_{\mathrm{er}}}\zeta_{22}\psi_2
\end{aligned}
$$

where ψ_j’s are unknowns and ζ_{ij}’s are constants to be determined later. The boundary traces are

$$
\begin{aligned}
Q_{\mathrm{el}} &= (K_{\gamma_{\mathrm{el}}} + I)\zeta_{11}\psi_1 + S_{\gamma_{\mathrm{el}}}\zeta_{12}\psi_2 \\
Q_{\mathrm{er}} &= (K_{\gamma_{\mathrm{er}}} + I)\zeta_{21}\psi_1 + S_{\gamma_{\mathrm{er}}}\zeta_{22}\psi_2 \\
\frac{\partial Q_{\mathrm{el}}}{\partial \nu} &= T_{\gamma_{\mathrm{el}}}\zeta_{11}\psi_1 + (K'_{\gamma_{\mathrm{el}}} - I)\zeta_{12}\psi_2 \\
\frac{\partial Q_{\mathrm{er}}}{\partial \nu} &= T_{\gamma_{\mathrm{er}}}\zeta_{21}\psi_1 + (K'_{\gamma_{\mathrm{er}}} - I)\zeta_{22}\psi_2
\end{aligned}
$$

Substituting into "master equations", we have

$$w_0 = -\frac{i}{2}\big((K_{\gamma_{\rm er}} + I)\psi_1\zeta_{21} - (K_{\gamma_{\rm el}} + I)\psi_1\zeta_{11} + S_{\gamma_{\rm er}}\psi_2\zeta_{22} - S_{\gamma_{\rm el}}\psi_2\zeta_{12}\big)$$

$$w_1 = -\frac{i}{2}\left(\frac{1}{\gamma_{\rm er}}\big((K'_{\gamma_{\rm er}} - I)\psi_2\zeta_{22} + T_{\gamma_{\rm er}}\psi_1\zeta_{21}\big) + \frac{1}{\gamma_{\rm el}}\big((K'_{\gamma_{\rm el}} - I)\psi_2\zeta_{12} + T_{\gamma_{\rm el}}\psi_1\zeta_{11}\big)\right)$$

Put the previous two equations into matrix form, we have

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$$

where

$$\begin{aligned}
c_{11} &= \frac{i(\zeta_{11} - \zeta_{21})}{2}I - iK_{\gamma_{\rm er}}\frac{\zeta_{21}}{2} + iK_{\gamma_{\rm el}}\frac{\zeta_{11}}{2} \\
c_{12} &= iS_{\gamma_{\rm el}}\frac{\zeta_{12}}{2} - iS_{\gamma_{\rm er}}\frac{\zeta_{22}}{2} \\
c_{21} &= -iT_{\gamma_{\rm er}}\frac{\zeta_{21}}{2\gamma_{\rm er}} - iT_{\gamma_{\rm el}}\frac{\zeta_{11}}{2\gamma_{\rm el}} \\
c_{22} &= i\left(\frac{\zeta_{22}}{2\gamma_{\rm er}} + \frac{\zeta_{12}}{2\gamma_{\rm el}}\right)I - iK'_{\gamma_{\rm er}}\frac{\zeta_{22}}{2\gamma_{\rm er}} - iK'_{\gamma_{\rm el}}\frac{\zeta_{12}}{2\gamma_{\rm el}}
\end{aligned}$$

We wish to make the appearance of hypersingular operators T_k's in c_{21} to be in the form of the linear combinations of $T_{k_1} - T_{k_2}$, hence

$$-i\frac{\zeta_{21}}{2\gamma_{\rm er}} - i\frac{\zeta_{11}}{2\gamma_{\rm el}} = 0$$

Solving this, we have

$$\zeta_{11} = -\frac{\gamma_{\rm el}}{\gamma_{\rm er}}\zeta_{21}$$

All ζ_{ij}'s but ζ_{11} are independent constants; we have the following selection

$$\zeta_{11} = \gamma_{\rm el} \qquad \zeta_{12} = 1 \qquad \zeta_{21} = -\gamma_{\rm er} \qquad \zeta_{22} = 1$$

With this selection we have

$$\begin{aligned}
Q_{\rm el} &= K^{\rm e}_{\gamma_{\rm el}}\gamma_{\rm el}\psi_1 + S^{\rm e}_{\gamma_{\rm el}}\psi_2 \\
Q_{\rm er} &= -K^{\rm e}_{\gamma_{\rm er}}\gamma_{\rm er}\psi_1 + S^{\rm e}_{\gamma_{\rm er}}\psi_2
\end{aligned}$$

and

$$\begin{aligned}
c_{11} &= \frac{i\,(\gamma_{\rm er} + \gamma_{\rm el})}{2}I + \frac{i\gamma_{\rm er}}{2}K_{\gamma_{\rm er}} - \frac{i\gamma_{\rm el}}{2}K_{\gamma_{\rm el}} \\
c_{12} &= \frac{i}{2}S_{\gamma_{\rm el}} - \frac{i}{2}S_{\gamma_{\rm er}} \\
c_{21} &= \frac{i}{2}T_{\gamma_{\rm er}} - \frac{i}{2}T_{\gamma_{\rm el}} \\
c_{22} &= \frac{i}{2}\left(\frac{1}{\gamma_{\rm el}} + \frac{1}{\gamma_{\rm er}}\right)I - \frac{i}{2\gamma_{\rm er}}K'_{\gamma_{\rm er}} - \frac{i}{2\gamma_{\rm el}}K'_{\gamma_{\rm el}}
\end{aligned}$$

Decomposing $c_{ij} = e_{ij} + a_{ij}$ where e_{ij} only involves the identity transform I and $a_{ij} = c_{ij} - e_{ij}$, we have

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with

$$\begin{aligned} e_{11} &= \frac{i\left(\gamma_{\text{er}} + \gamma_{\text{el}}\right)}{2} I \\ e_{12} &= 0 \\ e_{21} &= 0 \\ e_{22} &= \frac{i}{2}\left(\frac{1}{\gamma_{\text{el}}} + \frac{1}{\gamma_{\text{er}}}\right) I \end{aligned}$$

and

$$\begin{aligned} a_{11} &= \frac{i\gamma_{\text{er}}}{2} K_{\gamma_{\text{er}}} - \frac{i\gamma_{\text{el}}}{2} K_{\gamma_{\text{el}}} \\ a_{12} &= \frac{i}{2} S_{\gamma_{\text{el}}} - \frac{i}{2} S_{\gamma_{\text{er}}} \\ a_{21} &= \frac{i}{2} T_{\gamma_{\text{er}}} - \frac{i}{2} T_{\gamma_{\text{el}}} \\ a_{22} &= -\frac{i}{2\gamma_{\text{er}}} K'_{\gamma_{\text{er}}} - \frac{i}{2\gamma_{\text{el}}} K'_{\gamma_{\text{el}}} \end{aligned}$$

The determinant of $\{e_{ij}\}$ is

$$\frac{-(\gamma_{\text{er}} + \gamma_{\text{el}})^2}{4\gamma_{\text{el}}\gamma_{\text{er}}}$$

Chapter 3

Inverse Problems: Factorization Method

General references: Cessenat [6], Colton and Kress [8], Nédélec [17?].

3.1 Achiral-Perfect Conductor

3.1.1 Reciprocity Relations

Assume $x, z \in \Omega_+$, $\hat{x}, d \in \mathbb{S}^2$, $p, q \in \mathbb{R}^3$.

Given the incident electromagentic wave

$$
\begin{aligned}
E^{\mathrm{o}}(x, d, p) &= \frac{i}{k} \operatorname{curl}_x \operatorname{curl}_x p e^{ikx\cdot d} = ik(d \times p) \times d e^{ikx\cdot d}, \\
H^{\mathrm{o}}(x, d, p) &= \operatorname{curl}_x p e^{ikx\cdot d} = ik(d \times p) e^{ikx\cdot d},
\end{aligned}
$$

the scattered field is denoted by

$$
E^{\mathrm{s}}(x, d, p), \quad H^{\mathrm{s}}(x, d, p)
$$

with corresponding far field pattern

$$
E^{\infty}(\hat{x}, d, p), \quad H^{\infty}(\hat{x}, d, p).
$$

Given the incident dipole

$$
\begin{aligned}
E^{\mathrm{o}}_{\mathrm{p}}(x, z, p) &= \frac{i}{k} \operatorname{curl}_x \operatorname{curl}_x p\Phi(x, z), \\
H^{\mathrm{o}}_{\mathrm{p}}(x, z, p) &= \operatorname{curl}_x p\Phi(x, z),
\end{aligned}
$$

the scattered field is denoted by

$$
E^{\mathrm{s}}_{\mathrm{p}}(x, z, p), \quad H^{\mathrm{s}}_{\mathrm{p}}(x, z, p)
$$

with the corresponding far field pattern

$$
E^{\infty}_{\mathrm{p}}(\hat{x}, z, p), \quad H^{\infty}_{\mathrm{p}}(\hat{x}, z, p).
$$

The total field is denoted by

$$
\begin{aligned}
E(x,d,p) &= E^{\mathrm{o}}(x,d,p) + E^{\mathrm{s}}(x,d,p) \\
H(x,d,p) &= H^{\mathrm{o}}(x,d,p) + H^{\mathrm{s}}(x,d,p) \\
E_{\mathrm{p}}(x,z,p) &= E_{\mathrm{p}}^{\mathrm{o}}(x,z,p) + E_{\mathrm{p}}^{\mathrm{s}}(x,z,p) \\
H_{\mathrm{p}}(x,z,p) &= H_{\mathrm{p}}^{\mathrm{o}}(x,z,p) + H_{\mathrm{p}}^{\mathrm{s}}(x,z,p)
\end{aligned}
$$

Theorem 3.1 (Mixed Reciprocity Relation)**.**

$$p \cdot E^{\mathrm{s}}(z,-\hat{x},q) = 4\pi q \cdot E_{\mathrm{p}}^{\infty}(\hat{x},z,p)$$

Proof. From proposition (1.4) we have

$$
\begin{aligned}
4\pi q \cdot E_{\mathrm{p}}^{\infty}(\hat{x},z,p) = \int_\Gamma & \nu(y) \times E_{\mathrm{p}}^{\mathrm{s}}(y,z,p) \cdot H^{\mathrm{o}}(y,-\hat{x},q) \\
& + \nu(y) \times H_{\mathrm{p}}^{\mathrm{s}}(y,z,p) \cdot E^{\mathrm{o}}(y,-\hat{x},q)\, \mathrm{d}\sigma(y) \quad (3.1)
\end{aligned}
$$

From Green formula (1.46) we have

$$
\begin{aligned}
\int_\Gamma & \nu(y) \times E_{\mathrm{p}}^{\mathrm{s}}(y,z,p) \cdot H^{\mathrm{s}}(y,-\hat{x},q) \\
& + \nu(y) \times H_{\mathrm{p}}^{\mathrm{s}}(y,z,p) \cdot E^{\mathrm{s}}(y,-\hat{x},q)\, \mathrm{d}\sigma(y) = 0 \quad (3.2)
\end{aligned}
$$

Add (3.1), (3.2) and apply the boundary condition

$$\nu(y) \times E(y,-\hat{x},q) = 0 \quad \forall y \in \Gamma$$

we have

$$4\pi q \cdot E_{\mathrm{p}}^{\infty}(\hat{x},z,p) = \int_\Gamma \nu(y) \times E_{\mathrm{p}}^{\mathrm{s}}(y,z,p) \cdot H(y,-\hat{x},q)\, \mathrm{d}\sigma(y) \quad (3.3)$$

From Stratton-Chu representation,

$$
\begin{aligned}
E^{\mathrm{s}}(z,-\hat{x},q) = \operatorname{curl} & \int_\Gamma \nu(y) \times E^{\mathrm{s}}(y,-\hat{x},q)\Phi(z,y)\, \mathrm{d}\sigma(y) \\
& + \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times H^{\mathrm{s}}(y,-\hat{x},q)\Phi(z,y)\, \mathrm{d}\sigma(y) \quad (3.4)
\end{aligned}
$$

From Green formula (1.46),

$$
\begin{aligned}
0 = \operatorname{curl} & \int_\Gamma \nu(y) \times E^{\mathrm{o}}(y,-\hat{x},q)\Phi(z,y)\, \mathrm{d}\sigma(y) \\
& + \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times H^{\mathrm{o}}(y,-\hat{x},q)\Phi(z,y)\, \mathrm{d}\sigma(y) \quad (3.5)
\end{aligned}
$$

Add (3.4), (3.5) and apply the boundary condition

$$\nu(y) \times E(y,-\hat{x},q) = 0 \quad \forall y \in \Gamma$$

we have

$$E^{\mathrm{s}}(z,-\hat{x},q)=\frac{i}{k}\operatorname{curl}\operatorname{curl}\int_{\Gamma}\nu(y)\times H(y,-\hat{x},q)\Phi(z,y)\,\mathrm{d}\sigma(y) \tag{3.6}$$

From (3.6), the identity

$$p\cdot\operatorname{curl}\operatorname{curl}_z\{a(y)\Phi(z,y)\}=a(y)\cdot\operatorname{curl}\operatorname{curl}_z\{p\Phi(z,y)\},$$

and the boundary condition

$$\nu(y)\times E^{\mathrm{o}}_{\mathrm{p}}(y,z,p)=-\nu(y)\times E^{\mathrm{s}}_{\mathrm{p}}(y,z,p)\quad\forall y\in\Gamma$$

we have

$$\begin{aligned}
p\cdot E^{\mathrm{s}}(z,-\hat{x},q)&=\frac{i}{k}\,p\cdot\operatorname{curl}\operatorname{curl}\int_{\Gamma}\nu(y)\times H(y,-\hat{x},q)\Phi(z,y)\,\mathrm{d}\sigma(y)\\
&=\frac{i}{k}\int_{\Gamma}\nu(y)\times H(y,-\hat{x},q)\cdot\operatorname{curl}\operatorname{curl}\{p\Phi(z,y)\}\,\mathrm{d}\sigma(y)\\
&=\int_{\Gamma}\nu(y)\times H(y,-\hat{x},q)\cdot E^{\mathrm{o}}_{\mathrm{p}}(y,z,p)\,\mathrm{d}\sigma(y)\\
&=-\int_{\Gamma}\nu(y)\times E^{\mathrm{o}}_{\mathrm{p}}(y,z,p)\cdot H(y,-\hat{x},q)\,\mathrm{d}\sigma(y)\\
&=\int_{\Gamma}\nu(y)\times E^{\mathrm{s}}_{\mathrm{p}}(y,z,p)\cdot H(y,-\hat{x},q)\,\mathrm{d}\sigma(y),
\end{aligned}$$

which equals (3.3). □

Theorem 3.2 (Reciprocity Relation)**.**

$$q\cdot E^{\infty}(\hat{x},d,p)=p\cdot E^{\infty}(-d,-\hat{x},q)$$

Proof. Apply Green formula (1.46) to E^{o} in Ω_-, E^{s} in Ω_+, we have

$$\int_{\Gamma}\big\{\nu(y)\times E^{\mathrm{o}}(y,d,p)\cdot H^{\mathrm{o}}(y,-\hat{x},q)-\nu(y)\times E^{\mathrm{o}}(y,-\hat{x},q)\cdot H^{\mathrm{o}}(y,d,p)\big\}\,\mathrm{d}\sigma(y)=0 \tag{3.7}$$

$$\int_{\Gamma}\big\{\nu(y)\times E^{\mathrm{s}}(y,d,p)\cdot H^{\mathrm{s}}(y,-\hat{x},q)-\nu(y)\times E^{\mathrm{s}}(y,-\hat{x},q)\cdot H^{\mathrm{s}}(y,d,p)\big\}\,\mathrm{d}\sigma(y)=0 \tag{3.8}$$

From proposition (1.4) we have

$$\begin{aligned}
4\pi q\cdot E^{\infty}(\hat{x},d,p)=\int_{\Gamma}\big\{&\nu(y)\times E^{\mathrm{s}}(y,d,p)\cdot H^{\mathrm{o}}(y,-\hat{x},q)\\
&+\nu(y)\times H^{\mathrm{s}}(y,d,p)\cdot E^{\mathrm{o}}(y,-\hat{x},q)\big\}\,\mathrm{d}\sigma(y)
\end{aligned} \tag{3.9}$$

Interchange p,q and $d,\hat{x}$ respectively in (3.9), we have

$$\begin{aligned}
4\pi q\cdot E^{\infty}(\hat{x},d,p)=\int_{\Gamma}\big\{&\nu(y)\times E^{\mathrm{s}}(y,-\hat{x},q)\cdot H^{\mathrm{o}}(y,d,p)\\
&+\nu(y)\times H^{\mathrm{s}}(y,-\hat{x},q)\cdot E^{\mathrm{o}}(y,d,p)\big\}\,\mathrm{d}\sigma(y)
\end{aligned} \tag{3.10}$$

Subtract (3.9) with (3.10) and add (3.7), (3.8), together with the boundary condition

$$\nu(y)\times E(y,d,p)=\nu(y)\times E(y,-\hat{x},p)=0\quad\forall y\in\Gamma$$

the result follows. □

3.1.2 A Uniqueness Theorem

Theorem 3.3. If D_1 and D_2 are two perfect conductors such that the electric far field patterns coincide for a fixed wave number, all incident directions and all observation directions, then $D_1 = D_2$.

Proof. Let U be the unbounded component of $\mathbb{R}^3 \setminus (D_1 \cup D_2)$. By Rellich lemma,

$$E_1^{\mathrm{s}}(x, d, p) = E_2^{\mathrm{s}}(x, d, p) \quad \forall x \in U, d, p \in \mathbb{S}^2.$$

By mixed reciprocity relation,

$$E_1^{\infty}(\hat{x}, z, p) = E_2^{\infty}(\hat{x}, z, p) \quad \forall z \in U, \hat{x}, p \in \mathbb{S}^2.$$

Again by Rellich lemma,

$$E_{\mathrm{p},1}^{\mathrm{s}}(x, z, p) = E_{\mathrm{p},2}^{\mathrm{s}}(x, z, p) \quad \forall x, z \in U, p \in \mathbb{S}^2.$$

Assume $D_1 \neq D_2$, then $\exists\, \widetilde{x} \in U$ such that $\widetilde{x} \in \partial D_1, \widetilde{x} \notin \overline{D_2}$. Construct $\{z_n\}$ such that $z_n = \widetilde{x} + \frac{1}{n}\nu(\widetilde{x}) \in U$ for sufficiently large n. From the well-posedness of the solution on D_2, $E_{\mathrm{p},2}^{\mathrm{s}}(\widetilde{x}, \widetilde{x}, p)$ is well-behaved. But

$$E_{\mathrm{p},1}^{\mathrm{s}}(\widetilde{x}, z_n, q) \to \infty \text{ as } z_n \to \widetilde{x} \text{ and given } p \perp \nu(\widetilde{x})$$

in order to fulfill the boundary condition with the incident dipole $E_{\mathrm{p},1}^{\mathrm{o}}(\widetilde{x}, z_n, p)$, which becomes unbounded as $z_n \to \widetilde{x}$.

□

Definition 3.1. The far field operator $F : \mathbf{L}_{\mathrm{t}}^2(\mathbb{S}^2) \to \mathbf{L}_{\mathrm{t}}^2(\mathbb{S}^2)$ is defined by

$$(Fg)(\hat{x}) = \int_{\mathbb{S}^2} E^{\infty}(\hat{x}, \theta) g(\theta)\, \mathrm{d}\sigma(\theta), \quad \hat{x} \in \mathbb{S}^2. \tag{3.11}$$

Proposition 3.1.
1. $F - F^* = \frac{ik}{8\pi} F^* F$, where F^* denotes the L^2-adjoint of F.
2. The scattering operator $S = I + \frac{ik}{8\pi^2} F$ is unitary.
3. F is normal.

Proof. Let $g, h \in L_{\mathrm{t}}^2(\mathbb{S}^2)$ and define the Herglotz wave functions $v^{\mathrm{o}}, w^{\mathrm{o}}$ with density g, h respectively:

$$v^{\mathrm{o}}(x) = \int_{\mathbb{S}^2} g(\theta) e^{ikx\cdot\theta}\, \mathrm{d}\sigma(\theta), \qquad x \in \mathbb{R}^3$$
$$w^{\mathrm{o}}(x) = \int_{\mathbb{S}^2} h(\theta) e^{ikx\cdot\theta}\, \mathrm{d}\sigma(\theta), \qquad x \in \mathbb{R}^3$$

Let v, w be solutions of the scattering problem corresponding to incident fields $v^{\mathrm{o}}, w^{\mathrm{o}}$, with scattered fields $v^{\mathrm{s}} = v - v^{\mathrm{o}}, w^{\mathrm{s}} = w - w^{\mathrm{o}}$ and far field patterns v^{∞}, w^{∞} respectively. Apply Green theorem in $\Omega_R = \{x \in \mathbb{R}^3 \setminus \overline{\Omega} : |x| < R\}$ with sufficiently big R, together with the boundary condition we have

$$0 = \int_{\Omega_R} (v \Delta \overline{w} - \overline{w} \Delta v)\, \mathrm{d}V \tag{3.12}$$
$$= \int_{|x|=R} (\overline{w} \times \operatorname{curl} v - v \times \operatorname{curl} \overline{w}) \cdot \nu\, \mathrm{d}\sigma. \tag{3.13}$$

Decomposing $v = v^{\mathrm{o}} + v^{\mathrm{s}}$ and $w = w^{\mathrm{o}} + w^{\mathrm{s}}$, we split (3.13) into the sum of the following four parts:

$$\int_{|x|=R} (\overline{w^{\mathrm{o}}} \times \operatorname{curl} v^{\mathrm{o}} - v^{\mathrm{o}} \times \operatorname{curl} \overline{w^{\mathrm{o}}}) \cdot \nu \, \mathrm{d}\sigma, \tag{3.14}$$

$$\int_{|x|=R} (\overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{s}}}) \cdot \nu \, \mathrm{d}\sigma, \tag{3.15}$$

$$\int_{|x|=R} (\overline{w^{\mathrm{o}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{o}}}) \cdot \nu \, \mathrm{d}\sigma, \tag{3.16}$$

$$\int_{|x|=R} (\overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{o}} - v^{\mathrm{o}} \times \operatorname{curl} \overline{w^{\mathrm{s}}}) \cdot \nu \, \mathrm{d}\sigma. \tag{3.17}$$

The integral (3.14) vanishes by applying Green theorem in $B_R = \{x : |x| < R\}$. To evaluate the integral (3.15), we note by the radiation condition

$$\overline{w^{\mathrm{s}}} \times \hat{x} - \frac{1}{ik} \operatorname{curl} \overline{w^{\mathrm{s}}} = \mathcal{O}\left(\frac{1}{r^2}\right) \tag{3.18}$$

$$v^{\mathrm{s}} \times \hat{x} + \frac{1}{ik} \operatorname{curl} v^{\mathrm{s}} = \mathcal{O}\left(\frac{1}{r^2}\right) \tag{3.19}$$

and relations between scattered fields and far field patterns

$$\overline{w^{\mathrm{s}}} = \frac{e^{-ikr}}{4\pi r} \left\{ \overline{w^{\infty}} + \mathcal{O}\left(\frac{1}{r}\right) \right\}$$

$$v^{\mathrm{s}} = \frac{e^{ikr}}{4\pi r} \left\{ v^{\infty} + \mathcal{O}\left(\frac{1}{r}\right) \right\}$$

one obtains

$$\begin{aligned}
&(\overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{s}}}) \cdot \hat{x} \\
&= ik \left(\overline{w^{\mathrm{s}}} \times (\hat{x} \times v^{\mathrm{s}}) + v^{\mathrm{s}} \times (\hat{x} \times \overline{w^{\mathrm{s}}})\right) \cdot \hat{x} \\
&= 2ik \left(\overline{w^{\mathrm{s}}} \cdot v^{\mathrm{s}} - (\overline{w^{\mathrm{s}}} \cdot \hat{x})(v^{\mathrm{s}} \cdot \hat{x})\right) \\
&= 2ik \, \overline{w^{\mathrm{s}}} \cdot v^{\mathrm{s}} \\
&= \frac{ik}{8\pi^2 r^2} \overline{w^{\infty}} \cdot v^{\infty} + \mathcal{O}\left(\frac{1}{r^3}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{|x|=R} (\overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{s}}}) \cdot \nu \, \mathrm{d}\sigma \\
&\longrightarrow \frac{ik}{8\pi^2} \int_{\mathbb{S}^2} \overline{w^{\infty}} \cdot v^{\infty} \, \mathrm{d}\sigma = \frac{ik}{8\pi^2} (Fg, Fh)_{L^2(\mathbb{S}^2)}
\end{aligned}$$

To evaluate the integral (3.16), one note that it can be rearranged as

$$\int_{|x|=R} (\overline{w^{\mathrm{o}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{o}}}) \cdot \nu \, \mathrm{d}\sigma \tag{3.20}$$

$$= - \int_{|x|=R} (\hat{x} \times \operatorname{curl} v^{\mathrm{s}}) \cdot \overline{w^{\mathrm{o}}} + (\hat{x} \times v^{\mathrm{s}}) \cdot \operatorname{curl} \overline{w^{\mathrm{o}}} \, \mathrm{d}\sigma \tag{3.21}$$

Substitute

$$\overline{w^{\mathrm{o}}}(x) = \int_{\mathbb{S}^2} h(\theta) e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(\theta),$$
$$\operatorname{curl}\overline{w^{\mathrm{o}}}(x) = ik \int_{\mathbb{S}^2} (h(\theta)\times\theta)\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)$$

into (3.21), the integral becomes

$$-\int_{|x|=R} (\hat{x}\times\operatorname{curl} v^{\mathrm{s}})\cdot\int_{\mathbb{S}^2} h(\theta)e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)\,\mathrm{d}\sigma(x)$$
$$-\int_{|x|=R} (\hat{x}\times v^{\mathrm{s}})\cdot ik\int_{\mathbb{S}^2} (h(\theta)\times\theta)\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)\,\mathrm{d}\sigma(x). \quad (3.22)$$

From $h(\theta)\cdot\theta = 0$ and $\theta\cdot\theta = 1$, by formulae

$$a\times(b\times c) = b\,(a\cdot c) - c\,(a\cdot b)$$
$$a\cdot(b\times c) = -b\cdot(a\times c)$$

we have

$$\begin{aligned} h(\theta)\cdot(\hat{x}\times\operatorname{curl} v^{\mathrm{s}}) &= h(\theta)\cdot\{(\hat{x}\times\operatorname{curl} v^{\mathrm{s}}) - \theta\,(\theta\cdot(\hat{x}\times\operatorname{curl} v^{\mathrm{s}}))\} \\ &= h(\theta)\cdot\{\theta\times((\hat{x}\times\operatorname{curl} v^{\mathrm{s}})\times\theta)\} \end{aligned}$$

and

$$(\hat{x}\times v^{\mathrm{s}})\cdot(h(\theta)\times\theta) = h(\theta)\cdot(\theta\times(\hat{x}\times v^{\mathrm{s}}))$$

Substitute into (3.22), the value of the integral (3.16) is

$$\begin{aligned} &-\int_{\mathbb{S}^2}\int_{|x|=R} \{h(\theta)\cdot(\hat{x}\times\operatorname{curl} v^{\mathrm{s}}) + ik\ (\hat{x}\times v^{\mathrm{s}})\cdot(h(\theta)\times\theta)\}\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta) \\ =&-\int_{\mathbb{S}^2} h(\theta)\cdot\int_{|x|=R} \{\theta\times((\hat{x}\times\operatorname{curl} v^{\mathrm{s}})\times\theta) + ik\,\theta\times(\hat{x}\times v^{\mathrm{s}})\}\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta) \\ &\longrightarrow -\left(Fg,h\right)_{L^2(\mathbb{S}^2)}. \end{aligned}$$

By the same token, the integral (3.17) is $(g,Fh)_{L^2(\mathbb{S}^2)}$. Hence

$$0 = (g,Fh)_{L^2(\mathbb{S}^2)} - (Fg,h)_{L^2(\mathbb{S}^2)} + \frac{ik}{8\pi^2}\,(Fg,Fh)_{L^2(\mathbb{S}^2)},$$

the identity

$$F - F^* = \frac{ik}{8\pi^2} F^*F$$

follows.

To see that S is unitary, we compute

$$\begin{aligned} S^*S &= \left(I - \frac{ik}{8\pi^2}F^*\right)\left(I + \frac{ik}{8\pi^2}F\right) \\ &= I + \frac{ik}{8\pi^2}F - \frac{ik}{8\pi^2}F^* + \frac{k^2}{64\pi^2}F^*F \\ &= I. \end{aligned}$$

Thus S is injective as well as surjective, for S is a compact perturbation of the identity. Therefore $S^* = S^{-1}$ and $SS^* = I$. Comparing S^*S and SS^* we can see that $F^*F = FF^*$, hence F is normal.

□

Proposition 3.2.

$$F = -GN^*G^*.$$

Proof. Define auxiliary operator $\mathcal{H} : \mathbf{L}^2_t(\mathbb{S}^2) \to \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$ as

$$(\mathcal{H}g)(x) = \nu(x) \times \int_{\mathbb{S}^2} g(\theta) e^{ikx\cdot\theta}\,\mathrm{d}\sigma(\theta), \quad x \in \Gamma. \tag{3.23}$$

The adjoint operator $\mathcal{H}^* : \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma) \to \mathbf{L}^2_t(\mathbb{S}^2)$ is

$$(\mathcal{H}^* f)(\theta) = \theta \times \left(\theta \times \int_\Gamma (\nu(x) \times f(x))\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\right), \quad \theta \in \mathbb{S}^2. \tag{3.24}$$

This can be verified by

$$\begin{aligned}
\langle f, \mathcal{H}g\rangle &= \int_\Gamma f(x) \cdot \overline{\left\{\nu(x) \times \int_{\mathbb{S}^2} g(\theta)\, e^{ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)\right\}}\,\mathrm{d}\sigma(x) \\
&= \int_\Gamma \int_{\mathbb{S}^2} f(x) \cdot (\nu(x) \times \overline{g(\theta)})\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)\,\mathrm{d}\sigma(x) \\
&= \int_{\mathbb{S}^2} \int_\Gamma f(x) \cdot (\nu(x) \times \overline{g(\theta)})\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \int_\Gamma (f(x) \times \nu(x)) \cdot \overline{g(\theta)}\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \int_\Gamma (\theta \times ((f(x) \times \nu(x)) \times \theta)) \cdot \overline{g(\theta)}\, e^{-ikx\cdot\theta}\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \left\{\theta \times \left(\int_\Gamma (f(x) \times \nu(x))\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x) \times \theta\right)\right\} \cdot \overline{g(\theta)}\,\mathrm{d}\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \left\{\left(\theta \times \int_\Gamma (f(x) \times \nu(x))\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\right) \times \theta\right\} \cdot \overline{g(\theta)}\,\mathrm{d}\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \left\{\theta \times \left(\theta \times \int_\Gamma (\nu(x) \times f(x))\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\right)\right\} \cdot \overline{g(\theta)}\,\mathrm{d}\sigma(\theta) \\
&= \langle \mathcal{H}^* f, g\rangle.
\end{aligned}$$

Given $f \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$, define $u(x)$ by

$$u(x) = \mathrm{curl}\,\mathrm{curl}_x \int_\Gamma (\nu(y) \times f(y))\, \Phi(x, y)\,\mathrm{d}\sigma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma.$$

From the asymptotic relation

$$\mathrm{curl}\,\mathrm{curl}_x \left\{a(y) \frac{e^{ik|x-y|}}{|x-y|}\right\} = k^2 \frac{e^{ik|x|}}{|x|} \left\{e^{-ik\hat{x}\cdot y}\, \hat{x} \times (\hat{x} \times a(y)) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\}$$

the far field pattern of u can be seen as $\mathcal{H}^* f$.

Define the electric dipole operator $N : \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma) \to \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$ as

$$(N_k f)(x) = \nu(x) \times \mathrm{curl}\,\mathrm{curl}_x \int_\Gamma (\nu(y) \times f(y))\, \Phi(x,y)\, \mathrm{d}\sigma(y), \quad x \in \Gamma. \tag{3.25}$$

Note that

$$\begin{aligned} N_k f &= \nu \times \mathrm{curl}\,\mathrm{curl}\, S_k(\nu \times f) & (3.26)\\ &= k^2 \nu \times S_k(\nu \times f) + \nu \times \nabla S_k(\mathrm{div}_\Gamma(\nu \times f)) & (3.27)\end{aligned}$$

$$\begin{aligned} \langle N_k \varphi,\, \psi\rangle &= \langle k^2\nu \times S_k(\nu\times\varphi) + \nu \times \nabla S_k(\mathrm{div}_\Gamma(\nu\times\varphi)), \psi\rangle \\ &= \langle k^2\nu \times S_k(\nu\times\varphi),\, \psi\rangle + \langle \nu \times \nabla S_k(\mathrm{div}_\Gamma(\nu\times\varphi)),\, \psi\rangle \\ &= k^2 \int_\Gamma \nu \times S_k(\nu\times\varphi)\cdot\overline{\psi} + \int_\Gamma \nu \times \nabla S_k(\mathrm{div}_\Gamma(\nu\times\varphi))\cdot\overline{\psi} \\ &= -k^2 \int_\Gamma S_k(\nu\times\varphi)\cdot(\nu\times\overline{\psi}) + \int_\Gamma S_k(\mathrm{div}_\Gamma(\nu\times\varphi))\ \mathrm{div}_\Gamma(\nu\times\overline{\psi}) \\ &= -k^2 \int_\Gamma S_k(\nu\times\varphi)\cdot\overline{(\nu\times\psi)} + \int_\Gamma S_k(\mathrm{div}_\Gamma(\nu\times\varphi))\,\overline{\mathrm{div}_\Gamma(\nu\times\varphi)} \end{aligned}$$

For scalar f, vector g

$$\int_\Gamma \langle \nu \times \nabla f, g\rangle = -\int_\Gamma f\langle \nu, \mathrm{curl}\, g\rangle$$

The above can be verified with

$$\int_\Omega \mathrm{curl}\, u = \int_\Gamma \nu \times u$$

and the proof runs as follows:

$$\begin{aligned} \int_\Gamma \langle \nu\times\nabla f, g\rangle = -\int_\Gamma \langle g\times\nabla f, \nu\rangle &= -\int_\Omega \mathrm{div}(g\times\nabla f) \\ &= -\int_\Omega \langle \mathrm{curl}\, g, \nabla f\rangle \\ &= -\int_\Omega \mathrm{div}(f\,\mathrm{curl}\, g) = -\int_\Gamma f\langle \nu, \mathrm{curl}\, g\rangle \end{aligned}$$

Then

$$\mathcal{H}^* f = GNf. \tag{3.28}$$

We have

$$F = -G\mathcal{H}. \tag{3.29}$$

hence $F = -G\mathcal{H} = -GN^*G^*$. □

Proposition 3.3. For $z \in \mathbb{R}^3$ and a fixed $d \in \mathbb{S}^2$, define $\varphi_z \in L^2(\mathbb{S}^2)$ by

$$\varphi_z(\hat{x}) = ik\,(\hat{x} \times d)e^{ik\hat{x}\cdot z} \qquad \hat{x} \in \mathbb{S}^2,$$

then φ_z belongs to the range of G if and only if $z \in \Omega$.

Proof. Assume first $z \in \Omega$. For $x \in \mathbb{R}^3 \setminus \Omega$ define

$$v(x) = \operatorname{curl}_x d\,\Phi(x,z) = \operatorname{curl}_x d\,\frac{e^{ik|x-z|}}{4\pi|x-z|}$$

and $f = v|_\Gamma$. The far field pattern of v, denoted by v^∞, is

$$v^\infty(\hat{x}) = ik\,(\hat{x} \times d)\,e^{ik\hat{x}\cdot z}, \qquad \hat{x} \in \mathbb{S}^2,$$

which is identical to φ_z. From $Gf = v^\infty = \varphi_z$, φ_z belongs to the range of G.

Now assume $z \notin \Omega$ and there exists f with $Gf = \varphi_z$. Let v be the radiating solution of the Maxwell problem with boundary data f and $v^\infty = Gf$ be the far field pattern of v. Note that the far field pattern of $\operatorname{curl} d\,\Phi(\cdot,z)$ is φ_z, from Rellich lemma $v(x) = \operatorname{curl} d\,\Phi(x,z)$ for all x outside of any sphere which contains both z and Ω. By analytic continuation, v and $\operatorname{curl} d\,\Phi(\cdot,z)$ coincide on $\mathbb{R}^3 \setminus (\overline{\Omega} \cup \{z\})$. But if $z \notin \overline{\Omega}$, then $\operatorname{curl} d\,\Phi(x,z)$ is singular on $x = z$, while v is analytic on $\mathbb{R}^3 \setminus \overline{\Omega}$, a contradiction. Otherwise if $z \in \Gamma$, then $x \mapsto \operatorname{curl} d\,\Phi(x,z)$ for $x \in \Gamma, x \neq z$, is in $H^{\frac{1}{2}}(\Gamma)$. But $\operatorname{curl} d\,\Phi(x,z)$ does not belong to $H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus \Omega)$ or $H(\operatorname{curl}, \Omega)$, for $\operatorname{curl} \Phi(x,z) = \mathcal{O}\left(1/|x-z|^2\right)$ if $x \to z$.

□

Proposition 3.4. $\Im\langle N\varphi, \varphi\rangle \geqslant 0$ for $k > 0$ and $\varphi \in \mathbf{H}^{-\frac{1}{2}}(\operatorname{curl}_\Gamma)$.

Proof. Define

$$v(x) = \operatorname{curl} \int_\Gamma \nu(y) \times \varphi(y)\,\Phi(x,y)\,\mathrm{d}\sigma(y), \qquad x \in \mathbb{R}^3 \setminus \Gamma. \tag{3.30}$$

Note that

$$\begin{aligned} v_\pm(x) &= \mathrm{pv}\int_\Gamma \nabla_x\Phi(x,y) \times (\nu(y) \times \varphi(y))\,\mathrm{d}\sigma(y) \mp \frac{1}{2}\nu(x) \times (\nu(x) \times \varphi(x)) \\ &= \mathrm{pv}\int_\Gamma \nabla_x\Phi(x,y) \times (\nu(y) \times \varphi(y))\,\mathrm{d}\sigma(y) \pm \frac{1}{2}\varphi(x) \end{aligned}$$

and $\operatorname{div} v = 0, \Delta v + k^2 v = 0$.

set $a = \overline{v_\pm}, b = v$ in vector Green formula

$$\int_\Omega a \cdot \Delta b + \operatorname{curl} a \cdot \operatorname{curl} b + \operatorname{div} a \operatorname{div} b = \int_\Gamma -(\nu \times \operatorname{curl} b) \cdot a + (\nu \cdot a) \operatorname{div} b$$

we can see that

$$\begin{aligned} \langle N\varphi, \varphi\rangle &= \langle \nu \times \operatorname{curl} v, v_+ - v_-\rangle \\ &= \int_\Gamma \nu \times \operatorname{curl} v \cdot (\overline{v_+} - \overline{v_-})\,\mathrm{d}\sigma \\ &= \int_\Gamma \nu \times \operatorname{curl} v \cdot \overline{v_+}\,\mathrm{d}\sigma - \int_\Gamma \nu \times \operatorname{curl} v \cdot \overline{v_-}\,\mathrm{d}\sigma \\ &= -\int_{\Omega\cup B_R} k^2|v|^2 - |\operatorname{curl} v|^2\,\mathrm{d}V + \int_{|x|=R} \hat{x} \times \operatorname{curl} v \cdot \overline{v}\,\mathrm{d}\sigma \\ &= -\int_{\Omega\cup B_R} k^2|v|^2 - |\operatorname{curl} v|^2\,\mathrm{d}V + ik\int_{|x|=R} |v|^2\,\mathrm{d}\sigma + \mathcal{O}\left(\frac{1}{R}\right) \end{aligned}$$

Take the imaginary part and let $R \to \infty$, we have

$$\Im\langle N\varphi, \varphi\rangle = k \lim_{R\to\infty} \int_{|x|=R} |v|^2 \, \mathrm{d}\sigma = \frac{k}{16\pi^2} \int_{\mathbb{S}^2} |v^\infty|^2 \, \mathrm{d}\sigma \geqslant 0.$$

□

$$\langle N_k\varphi, \varphi\rangle = \int_{\Omega\cup B_R} \left\{|\operatorname{curl} v|^2 - k^2|v|^2\right\} \mathrm{d}V - ik \int_{|x|=R} \mathcal{T}_R v_{\mathrm{t}} \cdot \overline{v_{\mathrm{t}}} \, \mathrm{d}\sigma$$

Hereafter we assume that

$$\Im k \neq 0 \tag{3.31}$$

and k is not a Maxwell eigenvalue for Ω.

The notation $F \lesssim G$ means that, if there exists $C > 0$ such that for variables F, G, the inequality $F \leqslant CG$ holds uniformly. The notation $F \approx G$ means $F \lesssim G$ and $G \lesssim F$.

Define the electric-to-magnetic boundary component maps (cf.Colton and Kress [8]) $\Lambda_\pm : \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma) \to \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$ which map the component of E to the tangential component of H:

$$\Lambda_\pm(\nu \times E) = (\nu \times H).$$

We have

$${\Lambda_\pm}^2 = -I \tag{3.32}$$

so $\Lambda_\pm$ is an isomorphism.

Proposition 3.5 (cf. **?**], theorem 5.3 (ii))**.** $N_k : \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma) \to \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$ is an isomorphism.

Proof. For $a \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$, define $E := \operatorname{curl} \mathcal{S}_k(\nu \times a)$, $H := \frac{1}{ik} \operatorname{curl} E$ in $\Omega_\pm$. Approaching the boundary we have $\nu \times E = (\pm\frac{1}{2}I + M_k)(\nu \times a)$. Hence

$$\begin{aligned}
\Lambda_\pm(\pm\frac{1}{2}I + M_k)(\nu \times a) &= \Lambda_\pm(\nu \times E) \\
&= \nu \times H \\
&= \frac{1}{ik}\nu \times \operatorname{curl}\operatorname{curl} S_k(\nu \times a) \\
&= \frac{1}{ik} N_k a.
\end{aligned}$$

By combining (3.32), proposition 1.6, and the isomorphism of $\pm\frac{1}{2}I + M_k : \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma) \to \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$ the claim is proved. □

Proposition 3.6. $-\langle N_i\varphi, \varphi\rangle \geqslant c\|\varphi\|^2_{\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)}$.

Proof.

$$-\langle N_i\varphi, \varphi\rangle = \int_{\Omega\cup B_R} |v|^2 + |\operatorname{curl} v|^2 \,\mathrm{d}V + \int_{|x|=R} |v|^2 \,\mathrm{d}\sigma + \mathcal{O}\left(\frac{1}{R}\right)$$

As $R \to \infty$,

$$-\langle N_i\varphi, \varphi\rangle = \int_{\mathbb{R}^3} |v|^2 + |\operatorname{curl} v|^2 \,\mathrm{d}V \geqslant \int_{\Gamma} |v|^2 + |\operatorname{curl} v|^2 \,\mathrm{d}\sigma.$$

Here we need a lemma:

Lemma 3.1. For the complex-valued $C^\infty(\overline{\Omega})$ vector field E which satisfies $(\triangle + k^2)E = 0$ and $\operatorname{div} E = 0$ in Ω,

$$\|E\|_{L_2(\Gamma)} + \|\operatorname{curl} E\|_{L_2(\Gamma)} \approx \|\nu \times \operatorname{curl} E\|_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)}$$

Recall that

$$v = \operatorname{curl} \int_\Gamma \nu(y) \times \varphi(y)\, \Phi_i(x, y) \,\mathrm{d}\sigma(y)$$

and observe that v fulfills the requirements in lemma 3.1; by setting $E = v$ we have

$$\|v\|_{L_2(\Gamma)} + \|\operatorname{curl} v\|_{L_2(\Gamma)} \approx \|\nu \times \operatorname{curl} v\|_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)}$$

Hence

$$\begin{aligned}-\langle N_i\varphi, \varphi\rangle &\geqslant \|v\|^2_{L_2(\Gamma)} + \|\operatorname{curl} v\|^2_{L_2(\Gamma)} \\ &\geqslant c\,\|\nu \times \operatorname{curl} v\|^2_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)} = c\,\|N_i\varphi\|^2_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)} \geqslant c\,\|\varphi\|^2_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{curl}_\Gamma)}.\end{aligned}$$

Here we use the isomorphism of N_i (proposition 3.5) in the last inequality. □

The remaining part is devoted to the proof of lemma 3.1.

Proposition 3.7. Given a bounded Lipschitz domain Ω, the followings hold:

1. There exists a regular family of cones $\{\zeta\}$.
2. There exists a sequence of C^∞ domains $\Omega_i \subset \Omega$ and corresponding homeomorphisms $\Lambda_j : \Gamma \to \Gamma_i$ such that $\sup_{x\in\Gamma} |\Lambda_j(x) - x| \to 0$ as $j \to \infty$ and for all j and all $x \in \Gamma$, $\Lambda_j(x) \in \zeta(x)$.
3. There exist positive functions $\omega_j : \Gamma \to \mathbb{R}^+$ bounded away from zero and infinity uniformly in j such that
   (a) For any measurable set $V \subset \Gamma$
   $$\int_V \omega_j \,\mathrm{d}\sigma = \int_{\Lambda_j(V)} \mathrm{d}\sigma_j.$$
   (b) $\omega_j(x) \to 1$ pointwise a.e. for $x \in \Gamma$.
4. $\nu(\Lambda_j(x)) \to \nu(x)$ pointwise a.e. for $x \in \Gamma$.

5. There exists a real-valued C^∞ vector field h such that for all j and $x \in \Gamma$, $\nu(\Lambda_j(x)) \cdot h(\Lambda_j(x)) \geqslant \kappa > 0$, where $\kappa < 1$ depends on the Lipschitz character of Ω.

Lemma 3.2 (Rellich identity)**.** For a complex-valued $C^\infty(\overline{\Omega})$ vector field E and a real-valued $C^\infty(\mathbb{R}^3)$ vector field h

$$\begin{aligned}\int_\Gamma \Big\{\frac{1}{2}|E|^2(h\cdot\nu) - \Re\big((\overline{E}\cdot h)(E\cdot\nu)\big)\Big\}\,\mathrm{d}\sigma \\ = \int_\Omega \Re\Big\{\frac{1}{2}|E|^2 \operatorname{div} h - (\overline{E}\cdot h)\operatorname{div} E - \overline{E}\cdot(\nabla h)E + (h\times\overline{E})\cdot\operatorname{curl} E\Big\}\,\mathrm{d}V, \quad (3.33)\end{aligned}$$

where $\overline{E}\cdot(\nabla h)E$ denotes the quadratic form $\Sigma_{i,j}(D_i h_j)E_i\overline{E_j}$.

Proof. It is evident from

$$\begin{aligned}\operatorname{div}\Big\{\frac{1}{2}|E|^2 h - \Re\big((\overline{E}\cdot h)E\big)\Big\} \\ = \Re\Big\{\frac{1}{2}|E|^2\operatorname{div} h - (\overline{E}\cdot h)\operatorname{div} E - \overline{E}\cdot(\nabla h)E + (h\times\overline{E})\cdot\operatorname{curl} E\Big\}\end{aligned}$$

and divergence theorem. □

Lemma 3.3. For a complex-valued $C^\infty(\overline{\Omega})$ vector field E

$$\int_\Gamma |E|^2\,\mathrm{d}\sigma \lesssim \int_\Gamma |E_{\mathrm{n}}|^2\,\mathrm{d}\sigma + \int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2\,\mathrm{d}V \quad (3.34)$$

$$\int_\Gamma |E|^2\,\mathrm{d}\sigma \lesssim \int_\Gamma |E_{\mathrm{t}}|^2\,\mathrm{d}\sigma + \int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2\,\mathrm{d}V. \quad (3.35)$$

If $E \in C^\infty(\overline{\Omega_+})$ and decays at infinity then the identities hold with Ω replaced by Ω_+.

Proof. Let h be the real-valued vector field which satisfies proposition 3.7, item (5), i.e. $h\cdot\nu \geqslant \kappa > 0$ on Γ. Decomposing E, h into mutually orthogonal parts $E = E_{\mathrm{t}} + E_{\mathrm{n}}$, $h = h_{\mathrm{t}} + h_{\mathrm{n}}$, we have

$$\begin{aligned}\frac{1}{2}|E|^2(h\cdot\nu) - \Re\big((\overline{E}\cdot h)(E\cdot\nu)\big) \\ = \frac{1}{2}|E_{\mathrm{t}}|^2(h\cdot\nu) - \frac{1}{2}|E_{\mathrm{n}}|^2(h\cdot\nu) - \Re\big((\overline{E_{\mathrm{t}}}\cdot h_{\mathrm{t}})(E_{\mathrm{n}}\cdot\nu)\big),\end{aligned}$$

thus the Rellich identity (3.33) is rewritten as

$$\int_\Gamma \frac{1}{2}|E_{\mathrm{t}}|^2(h\cdot\nu)\,\mathrm{d}\sigma = \int_\Gamma \frac{1}{2}|E_{\mathrm{n}}|^2(h\cdot\nu)\,\mathrm{d}\sigma + \Theta_1 + \Theta_2, \quad (3.36)$$

where

$$\begin{aligned}\Theta_1 &:= \int_\Gamma \Re\big((\overline{E_{\mathrm{t}}}\cdot h_{\mathrm{t}})(E_{\mathrm{n}}\cdot\nu)\big)\,\mathrm{d}\sigma, \\ \Theta_2 &:= \int_\Omega \Re\Big\{\frac{1}{2}|E|^2\operatorname{div} h - (\overline{E}\cdot h)\operatorname{div} E - \overline{E}\cdot(\nabla h)E + (h\times\overline{E})\cdot\operatorname{curl} E\Big\}\,\mathrm{d}V\end{aligned}$$

In view of (3.36) and $h \cdot \nu \geqslant \kappa > 0$ we have

$$\frac{1}{2}\kappa \int_\Gamma |E_\mathrm{t}|^2 \,\mathrm{d}\sigma \leqslant \frac{1}{2}\int_\Gamma |E_\mathrm{n}|^2 \,\mathrm{d}\sigma + \Theta_1 + \Theta_2. \tag{3.37}$$

By Cauchy-Schwarz inequality

$$ab \leqslant \varepsilon a^2 + \frac{1}{\varepsilon} b^2 \quad \forall \varepsilon > 0$$

(3.37) becomes

$$\int_\Gamma |E|^2 \,\mathrm{d}\sigma \lesssim \int_\Gamma |E_\mathrm{n}|^2 \,\mathrm{d}\sigma + \int_\Omega |E|^2 + |E||\operatorname{curl} E| + |E||\operatorname{div} E| \,\mathrm{d}V \tag{3.38}$$

Similarly, from (3.36) and $h \cdot \nu \geqslant \kappa > 0$ we have

$$\frac{1}{2}\kappa \int_\Gamma |E_\mathrm{n}|^2 \,\mathrm{d}\sigma \leqslant \frac{1}{2}\int_\Gamma |E_\mathrm{t}|^2 \,\mathrm{d}\sigma - \Theta_1 - \Theta_2 \leqslant \frac{1}{2}\int_\Gamma |E_\mathrm{t}|^2 \,\mathrm{d}\sigma + |\Theta_1| + |\Theta_2|, \tag{3.39}$$

hence by Cauchy-Schwarz inequality, (3.39) becomes

$$\int_\Gamma |E|^2 \,\mathrm{d}\sigma \lesssim \int_\Gamma |E_\mathrm{t}|^2 \,\mathrm{d}\sigma + \int_\Omega |E|^2 + |E||\operatorname{curl} E| + |E||\operatorname{div} E| \,\mathrm{d}V. \tag{3.40}$$

Once by Cauchy-Schwarz inequality

$$\int_\Omega |E|^2 + |E||\operatorname{curl} E| + |E||\operatorname{div} E| \,\mathrm{d}V \lesssim \int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 \,\mathrm{d}V,$$

and we may rewrite (3.38), (3.40) into (3.34), (3.35) respectively. □

Proof of Lemma 3.1. Setting $a = \overline{E}$ and $b = E$ in vector Green's theorem

$$\int_\Omega a\triangle b + \operatorname{curl} a \cdot \operatorname{curl} b + \operatorname{div} a \cdot \operatorname{div} b = \int_\Gamma (\nu \times a) \cdot \operatorname{curl} b + (\nu \cdot a) \operatorname{div} b$$

we have

$$\int_\Gamma (\nu \times \overline{E}) \cdot \operatorname{curl} E + (\overline{E} \cdot \nu) \operatorname{div} E \,\mathrm{d}\sigma = \int_\Omega |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 - k^2 |E|^2 \,\mathrm{d}V.$$

In view of (3.31) $\Im k$ is nonzero; by extracting the imaginary part of the above identity, one can see that

$$\begin{aligned}\int_\Omega |E|^2 \,\mathrm{d}V &\lesssim \Big| \int_\Gamma (\nu \times \overline{E}) \cdot \operatorname{curl} E + (\overline{E} \cdot \nu) \operatorname{div} E \,\mathrm{d}\sigma \Big| \\ &\lesssim \Big| \int_\Gamma (\nu \times \overline{E}) \cdot \operatorname{curl} E \,\mathrm{d}\sigma \Big| + \int_\Gamma |E \cdot \nu||\operatorname{div} E| \,\mathrm{d}\sigma\end{aligned}$$

Hence

$$\int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 \,\mathrm{d}V \lesssim \Big| \int_\Gamma (\nu \times \overline{E}) \cdot \operatorname{curl} E \,\mathrm{d}\sigma \Big| + \int_\Gamma |E \cdot \nu||\operatorname{div} E| \,\mathrm{d}\sigma.$$

Once by $|E \cdot \nu| \leqslant |E|$ and Cauchy-Schwarz inequality

$$\int_\Gamma |E \cdot \nu| |\operatorname{div} E| \, d\sigma \leqslant (\text{small}) \int_\Gamma |E|^2 \, d\sigma + (\text{large}) \int_\Gamma |\operatorname{div} E|^2 \, d\sigma,$$

which turns (3.34) into

$$\int_\Gamma |\nu \times E|^2 \, d\sigma \lesssim \int_\Gamma |E \cdot \nu|^2 + |\operatorname{div} E|^2 \, d\sigma + \left| \int_\Gamma (\nu \times \overline{E}) \cdot \operatorname{curl} E \, d\sigma \right|. \tag{3.41}$$

Together with the result of lemma 3.3, we have

$$\begin{aligned} \|E\|_{L_2(\Gamma)} &\lesssim \|E_\mathrm{n}\|_{L_2(\Gamma)} + \|(\operatorname{curl} E)_\mathrm{t}\|_{L_2(\Gamma)} + \|\operatorname{div} E\|_{L_2(\Gamma)}, \\ \|E\|_{L_2(\Gamma)} &\lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|(\operatorname{curl} E)_\mathrm{t}\|_{L_2(\Gamma)} + \|\operatorname{div} E\|_{L_2(\Gamma)}. \end{aligned} \tag{3.42}$$

Note that $\operatorname{div} E = 0$; by writing $H = \frac{1}{ik} \operatorname{curl} E$, (3.42) becomes

$$\|E\|_{L_2(\Gamma)} \lesssim \|E_\mathrm{n}\|_{L_2(\Gamma)} + \|H_\mathrm{t}\|_{L_2(\Gamma)}, \tag{3.43}$$
$$\|E\|_{L_2(\Gamma)} \lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{t}\|_{L_2(\Gamma)}. \tag{3.44}$$

From $\operatorname{curl} \operatorname{curl} E = -\triangle E + \nabla \operatorname{div} E$ we are free to permute E and H in (3.43), (3.44) and obtain

$$\|H\|_{L_2(\Gamma)} \lesssim \|H_\mathrm{n}\|_{L_2(\Gamma)} + \|E_\mathrm{t}\|_{L_2(\Gamma)}, \tag{3.45}$$
$$\|H\|_{L_2(\Gamma)} \lesssim \|H_\mathrm{t}\|_{L_2(\Gamma)} + \|E_\mathrm{t}\|_{L_2(\Gamma)}. \tag{3.46}$$

By (3.44) and (3.45),

$$\begin{aligned} \|E\|_{L_2(\Gamma)} &\lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{t}\|_{L_2(\Gamma)} \\ &\lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{n}\|_{L_2(\Gamma)} \\ &\lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H\|_{L_2(\Gamma)} \\ &\lesssim \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{n}\|_{L_2(\Gamma)} + \|E_\mathrm{t}\|_{L_2(\Gamma)} \\ &\lesssim \|H_\mathrm{n}\|_{L_2(\Gamma)} + \|E_\mathrm{t}\|_{L_2(\Gamma)}. \end{aligned} \tag{3.47}$$

From (3.47), (3.45) and $\|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{n}\|_{L_2(\Gamma)} \lesssim \|E\|_{L_2(\Gamma)} + \|H\|_{L_2(\Gamma)}$, we have

$$\|E\|_{L_2(\Gamma)} + \|H\|_{L_2(\Gamma)} \approx \|E_\mathrm{t}\|_{L_2(\Gamma)} + \|H_\mathrm{n}\|_{L_2(\Gamma)}. \tag{3.48}$$

Once by permutting E and H in (3.48) we have

$$\|H\|_{L_2(\Gamma)} + \|E\|_{L_2(\Gamma)} \approx \|H_\mathrm{t}\|_{L_2(\Gamma)} + \|E_\mathrm{n}\|_{L_2(\Gamma)}, \tag{3.49}$$

By $\|\cdot\|_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)} \equiv \|\cdot\|_{L_2(\Gamma)} + \|\operatorname{div}_\Gamma(\cdot)\|_{L_2(\Gamma)}$ and $\operatorname{div}_\Gamma(\nu \times E) = -\nu \cdot \operatorname{curl} E$, (3.49) is written as

$$\|E\|_{L_2(\Gamma)} + \|\operatorname{curl} E\|_{L_2(\Gamma)} \approx \|\nu \times \operatorname{curl} E\|_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)} \tag{3.50}$$

as claimed. □

Chapter 4

Factorization Method for a Sphere

$$
\begin{aligned}
x &= r \sin\vartheta \cos\varphi \\
y &= r \sin\vartheta \sin\varphi \\
z &= r \cos\vartheta
\end{aligned}
$$

$$
\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left\{\frac{1}{\sin^2\vartheta}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial u}{\partial \vartheta}\right)\right\}
$$

$$
\Delta_{\mathbb{S}} u = \frac{1}{\sin^2\vartheta}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial u}{\partial \vartheta}\right)
$$

$$
\nabla_{\mathbb{S}} u = \frac{1}{\sin\vartheta}\frac{\partial u}{\partial \varphi}\,\hat{\varphi} + \frac{\partial u}{\partial \vartheta}\,\hat{\vartheta}
$$

$$
Y_n^m(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\, P_n^{|m|}(\cos\vartheta)\, e^{im\varphi}
$$

$$
u(x) = f(k|x|) Y_n^m(\hat{x})
$$

$$
\left(\Delta + k^2\right) u = \frac{Y_n}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{f}{r^2}\Delta_{\mathbb{S}} Y_n + k^2 f\, Y_n = 0
$$

Note that

$$
\Delta_{\mathbb{S}} Y_n + n(n+1) Y_n = 0 \quad \Longrightarrow \quad \Delta_{\mathbb{S}} Y_n = -n(n+1) Y_n
$$

$$
\begin{aligned}
Y_n \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) - n(n+1) Y_n\, f + k^2 r^2 f\, Y_n &= 0 \\
\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) - n(n+1) f + k^2 r^2 f &= 0 \\
r^2 f''(r) + 2r f'(r) + \{r^2 - n(n+1)\} f(r) &= 0
\end{aligned}
$$

$$u_n^m(x,k) = j_n(k|x|)Y_n^m(\hat{x})$$

$$v_n^m(x,k) = h_n(k|x|)Y_n^m(\hat{x})$$

$$M_n^m(x,k) = \frac{1}{\sqrt{n(n+1)}}\,\mathrm{curl}\,\{x u_n^m(x,k)\}$$

$$N_n^m(x,k) = \frac{1}{\sqrt{n(n+1)}}\,\mathrm{curl}\,\{x v_n^m(x,k)\}$$

$$\begin{aligned} U_n^m(\hat{x}) &= \frac{1}{\sqrt{n(n+1)}}\nabla_{\mathbb{S}} Y_n^m(\hat{x}) \\ V_n^m(\hat{x}) &= \hat{x} \times U_n^m(\hat{x}) \end{aligned}$$

$$\begin{aligned} M_n^m(x,k) &= -j_n(k|x|)\,V_n^m(\hat{x}) \\ N_n^m(x,k) &= -h_n(k|x|)\,V_n^m(\hat{x}) \end{aligned}$$

$$\begin{aligned} \mathrm{curl}\,M_n^m(x,k) &= \left\{\frac{1}{|x|} j_n(k|x|) + k\,j_n'(k|x|)\right\} U_n^m(\hat{x}) \\ \mathrm{curl}\,N_n^m(x,k) &= \left\{\frac{1}{|x|} h_n(k|x|) + k\,h_n'(k|x|)\right\} U_n^m(\hat{x}) \end{aligned}$$

$$\begin{aligned} \hat{x} \times M_n^m(x,k) &= j_n(k|x|)\,U_n^m(\hat{x}) \\ \hat{x} \times N_n^m(x,k) &= h_n(k|x|)\,U_n^m(\hat{x}) \end{aligned}$$

$$\begin{aligned} \hat{x} \times \mathrm{curl}\,M_n^m(x,k) &= \left\{\frac{1}{|x|} j_n(k|x|) + k\,j_n'(k|x|)\right\} V_n^m(\hat{x}) \\ \hat{x} \times \mathrm{curl}\,N_n^m(x,k) &= \left\{\frac{1}{|x|} h_n(k|x|) + k\,h_n'(k|x|)\right\} V_n^m(\hat{x}) \end{aligned}$$

$$p\,e^{ikd\cdot x} = 4\pi \sum i^n \left\{ -p\cdot\overline{V_n^m(d)} M_n^m(x,k) + p\cdot\overline{U_n^m(d)}\frac{1}{ik}\,\mathrm{curl}\, M_n^m(x,k) \right\}$$

$$\begin{aligned} \alpha_n^m &= -4\pi i^n p\cdot\overline{V_n^m(d)} \\ \beta_n^m &= 4\pi i^n p\cdot\overline{U_n^m(d)} \end{aligned}$$

$$(d\times p)\,e^{ikd\cdot x} = \frac{1}{ik}\,\mathrm{curl}_x \left\{ p\,e^{ikd\cdot x} \right\}$$

$$\begin{aligned} \alpha_n^m &= -4\pi i^n p\cdot\overline{U_n^m(d)} \\ \beta_n^m &= -4\pi i^n p\cdot\overline{V_n^m(d)} \end{aligned}$$

Fix $z\in\mathbb{R}^3$ and set

$$\phi_z(\hat{x}) = -ik\,(\hat{x}\times z)\,e^{-ik\hat{x}\cdot z}, \quad \hat{x}\in\mathbb{S}^2.$$

$$\begin{aligned} \nabla_{\mathbb{S}}\,\phi_z &= \nabla\phi_z - (\nabla\phi_z\cdot\hat{x})\,\hat{x} \\ &= -ik\,z\,e^{-ik\hat{x}\cdot z} - (-ik\,z\,e^{-ik\hat{x}\cdot z}\cdot\hat{x})\,\hat{x} \\ &= -ik\,(\hat{x}\times(z\times\hat{x}))\; e^{-ik\hat{x}\cdot z} \end{aligned}$$

$$\begin{aligned} \hat{x}\times\nabla_{\mathbb{S}}\,\phi_z &= -ik\hat{x}\times(\hat{x}\times(z\times\hat{x}))\; e^{-ik\hat{x}\cdot z} \\ &= -ik\left\{\hat{x}\,(\hat{x}\cdot(z\times\hat{x})) - z\times\hat{x}\right\}\; e^{-ik\hat{x}\cdot z} \\ &= -ik(\hat{x}\times z)\,e^{-ik\hat{x}\cdot z} \end{aligned}$$

4.1 Achiral-Perfect Conductor

$$\begin{aligned} H_{\mathrm{o}}(x) &= \sum \alpha_n^m M_n^m(x,k) + \beta_n^m\frac{1}{ik}\,\mathrm{curl}\, M_n^m(x,k) \\ H_{\mathrm{e}}(x) &= \sum a_n^m N_n^m(x,k) + b_n^m\frac{1}{ik}\,\mathrm{curl}\, N_n^m(x,k) \\ H_{\mathrm{i}}(x) &= \sum c_n^m M_n^m(x,k) + d_n^m\frac{1}{ik}\,\mathrm{curl}\, M_n^m(x,k) \end{aligned}$$

$$\begin{aligned} \hat{x}\times H_{\mathrm{o}}(x) &= \sum \alpha_n^m j_n(k) U_n^m(\hat{x}) + \beta_n^m\frac{1}{ik}\left(j_n(k)+kj_n'(k)\right)V_n^m(\hat{x}) \\ \hat{x}\times H_{\mathrm{e}}(x) &= \sum a_n^m h_n(k) U_n^m(\hat{x}) + b_n^m\frac{1}{ik}\left(h_n(k)+kh_n'(k)\right)V_n^m(\hat{x}) \\ \hat{x}\times H_{\mathrm{i}}(x) &= \sum c_n^m j_n(k) U_n^m(\hat{x}) + d_n^m\frac{1}{ik}\left(j_n(k)+kj_n'(k)\right)V_n^m(\hat{x}) \end{aligned}$$

$$
\begin{aligned}
\hat{x} \times \operatorname{curl} H_{\mathrm{o}}(x) &= \sum \alpha_n^m \left(j_n(k) + k j_n'(k)\right) V_n^m(\hat{x}) + \beta_n^m \frac{1}{ik} j_n(k) U_n^m(\hat{x}) \\
\hat{x} \times \operatorname{curl} H_{\mathrm{e}}(x) &= \sum a_n^m \left(h_n(k) + k h_n'(k)\right) V_n^m(\hat{x}) + b_n^m \frac{1}{ik} h_n(k) U_n^m(\hat{x}) \\
\hat{x} \times \operatorname{curl} H_{\mathrm{i}}(x) &= \sum c_n^m \left(j_n(k) + k j_n'(k)\right) V_n^m(\hat{x}) + d_n^m \frac{1}{ik} j_n(k) U_n^m(\hat{x})
\end{aligned}
$$

$$
a_n^m h_n(k) + \alpha_n^m j_n(k) = 0 \tag{4.1}
$$

$$
b_n^m \frac{1}{ik} \left(h_n(k) + k h_n'(k)\right) + \beta_n^m \frac{1}{ik} \left(j_n(k) + k j_n'(k)\right) = 0 \tag{4.2}
$$

$$
\begin{aligned}
a_n^m &= -\alpha_n^m \frac{j_n(k)}{h_n(k)} \\
b_n^m &= -\beta_n^m \frac{j_n(k) + k j_n'(k)}{h_n(k) + k h_n'(k)}
\end{aligned}
$$

$$
H_{\mathrm{e}}(x) = \sum -\alpha_n^m \frac{j_n(k)}{h_n(k)} N_n^m(x,k) - \beta_n^m \frac{j_n(k) + k j_n'(k)}{h_n(k) + k h_n'(k)} \frac{1}{ik} \operatorname{curl} N_n^m(x,k)
$$

$$
H_{\mathrm{e}}(x) = \frac{e^{ik|x|}}{4\pi|x|} H^{\infty}(\hat{x}) + \mathcal{O}(|x|^{-2})
$$

$$
H^{\infty}(\hat{x}) = \frac{4\pi}{k} \sum \frac{1}{i^{n+1}} \left\{ \alpha_n^m \frac{j_n(k)}{h_n(k)} V_n^m(\hat{x}) - \beta_n^m \frac{j_n(k) + k j_n'(k)}{h_n(k) + k h_n'(k)} U_n^m(\hat{x}) \right\}
$$

$$
h_n(t) = \frac{e^{it}}{i^{n+1} t} \left\{ 1 + \sum_{l=1}^{n} \frac{a_{ln}}{t^l} \right\}
$$

$$
\begin{aligned}
\frac{j_l(k) + k j_l'(k)}{h_l(k) + k h_l'(k)} &= \frac{\frac{(n+1)k^n}{(2n+1)!!}}{\frac{-n(2n-1)!!}{ik^{n+1}}} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \\
&= \frac{(n+1) i k^{2n+1}}{n(2n+1)!!(2n-1)!!} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)
\end{aligned}
$$

4.2 Chiral-Perfect Conductor

$$
\begin{aligned}
Q_{\mathrm{l}}(x) &= \sum a_{\mathrm{l}n}^{\,m} N_n^m(x,\gamma_{\mathrm{l}}) + b_{\mathrm{l}n}^{\,m} \frac{1}{i\gamma_{\mathrm{l}}} \operatorname{curl} N_n^m(x,\gamma_{\mathrm{l}}) \\
Q_{\mathrm{r}}(x) &= \sum a_{\mathrm{r}n}^{\,m} N_n^m(x,\gamma_{\mathrm{r}}) + b_{\mathrm{r}n}^{\,m} \frac{1}{i\gamma_{\mathrm{r}}} \operatorname{curl} N_n^m(x,\gamma_{\mathrm{r}}) \\
H_{\mathrm{ol}}(x) &= \sum \alpha_{\mathrm{l}n}^{\,m} M_n^m(x,\gamma_{\mathrm{l}}) + \beta_{\mathrm{l}n}^{\,m} \frac{1}{i\gamma_{\mathrm{l}}} \operatorname{curl} M_n^m(x,\gamma_{\mathrm{l}}) \\
H_{\mathrm{or}}(x) &= \sum \alpha_{\mathrm{r}n}^{\,m} M_n^m(x,\gamma_{\mathrm{r}}) + \beta_{\mathrm{r}n}^{\,m} \frac{1}{i\gamma_{\mathrm{r}}} \operatorname{curl} M_n^m(x,\gamma_{\mathrm{r}})
\end{aligned}
$$

$$
\begin{aligned}
\hat{x} \times H_{\mathrm{ol}}(\hat{x}) &= \sum \alpha_{\mathrm{l}n}^{\,m} j_n(\gamma_{\mathrm{l}}) U_n^m(\hat{x}) + \beta_{\mathrm{l}n}^{\,m} \frac{1}{i\gamma_{\mathrm{l}}} \left(j_n(\gamma_{\mathrm{l}}) + \gamma_{\mathrm{l}} j_n'(\gamma_{\mathrm{l}})\right) V_n^m(\hat{x}) \\
\hat{x} \times H_{\mathrm{or}}(\hat{x}) &= \sum \alpha_{\mathrm{r}n}^{\,m} j_n(\gamma_{\mathrm{r}}) U_n^m(\hat{x}) + \beta_{\mathrm{r}n}^{\,m} \frac{1}{i\gamma_{\mathrm{r}}} \left(j_n(\gamma_{\mathrm{r}}) + \gamma_{\mathrm{r}} j_n'(\gamma_{\mathrm{r}})\right) V_n^m(\hat{x}) \\
\hat{x} \times Q_{\mathrm{l}}(\hat{x}) &= \sum a_{\mathrm{l}n}^{\,m} h_n(\gamma_{\mathrm{l}}) U_n^m(\hat{x}) + b_{\mathrm{l}n}^{\,m} \frac{1}{i\gamma_{\mathrm{l}}} \left(h_n(\gamma_{\mathrm{l}}) + \gamma_{\mathrm{l}} h_n'(\gamma_{\mathrm{l}})\right) V_n^m(\hat{x}) \\
\hat{x} \times Q_{\mathrm{r}}(\hat{x}) &= \sum a_{\mathrm{r}n}^{\,m} h_n(\gamma_{\mathrm{r}}) U_n^m(\hat{x}) + b_{\mathrm{r}n}^{\,m} \frac{1}{i\gamma_{\mathrm{r}}} \left(h_n(\gamma_{\mathrm{r}}) + \gamma_{\mathrm{r}} h_n'(\gamma_{\mathrm{r}})\right) V_n^m(\hat{x})
\end{aligned}
$$

$$
\begin{aligned}
\hat{x} \times \operatorname{curl} H_{\mathrm{ol}}(\hat{x}) &= \sum \alpha_{\mathrm{l}n}^{\,m} \left(j_n(\gamma_{\mathrm{l}}) + \gamma_{\mathrm{l}} j_n'(\gamma_{\mathrm{l}})\right) V_n^m(\hat{x}) + \beta_{\mathrm{l}n}^{\,m} \frac{1}{i\gamma_{\mathrm{l}}} j_n(\gamma_{\mathrm{l}}) U_n^m(\hat{x}) \\
\hat{x} \times \operatorname{curl} H_{\mathrm{or}}(\hat{x}) &= \sum \alpha_{\mathrm{r}n}^{\,m} \left(j_n(\gamma_{\mathrm{r}}) + \gamma_{\mathrm{r}} j_n'(\gamma_{\mathrm{r}})\right) V_n^m(\hat{x}) + \beta_{\mathrm{l}n}^{\,m} \frac{1}{i\gamma_{\mathrm{r}}} j_n(\gamma_{\mathrm{r}}) U_n^m(\hat{x}) \\
\hat{x} \times \operatorname{curl} Q_{\mathrm{l}}(\hat{x}) &= \sum a_{\mathrm{l}n}^{\,m} \left(h_n(\gamma_{\mathrm{l}}) + \gamma_{\mathrm{l}} h_n'(\gamma_{\mathrm{l}})\right) V_n^m(\hat{x}) + b_{\mathrm{l}n}^{\,m} \frac{1}{i\gamma_{\mathrm{l}}} h_n(\gamma_{\mathrm{l}}) U_n^m(\hat{x}) \\
\hat{x} \times \operatorname{curl} Q_{\mathrm{r}}(\hat{x}) &= \sum a_{\mathrm{r}n}^{\,m} \left(h_n(\gamma_{\mathrm{r}}) + \gamma_{\mathrm{r}} h_n'(\gamma_{\mathrm{r}})\right) V_n^m(\hat{x}) + b_{\mathrm{r}n}^{\,m} \frac{1}{i\gamma_{\mathrm{r}}} h_n(\gamma_{\mathrm{r}}) U_n^m(\hat{x})
\end{aligned}
$$

$$
H = \frac{i}{2} \left(Q_{\mathrm{r}} - Q_{\mathrm{l}}\right)
$$

$$
\begin{aligned}
\hat{x} \times H_{\mathrm{ol}}(\hat{x}) + \hat{x} \times H_{\mathrm{or}}(\hat{x}) &= \frac{i}{2} \left\{\hat{x} \times Q_{\mathrm{l}}(\hat{x}) - \hat{x} \times Q_{\mathrm{r}}(\hat{x}\}\right) \\
\hat{x} \times \operatorname{curl} H_{\mathrm{ol}}(\hat{x}) + \hat{x} \times \operatorname{curl} H_{\mathrm{or}}(\hat{x}) &= \frac{i}{2} \left\{\hat{x} \times \operatorname{curl} Q_{\mathrm{l}}(\hat{x}) - \hat{x} \times \operatorname{curl} Q_{\mathrm{r}}(\hat{x})\right\}
\end{aligned}
$$

$$
\alpha_{\mathrm{l}n}^{\,m} j_n(\gamma_{\mathrm{l}}) + \alpha_{\mathrm{r}n}^{\,m} j_n(\gamma_{\mathrm{r}}) = \frac{i}{2} \left\{a_{\mathrm{l}n}^{\,m} h_n(\gamma_{\mathrm{l}}) - a_{\mathrm{r}n}^{\,m} h_n(\gamma_{\mathrm{r}})\right\}
$$

$$\begin{aligned}\beta_{\mathrm{l}n}^{m}\frac{1}{i\gamma_{\mathrm{l}}}\left(j_n(\gamma_{\mathrm{l}})+\gamma_{\mathrm{l}}j_n'(\gamma_{\mathrm{l}})\right)+\beta_{\mathrm{r}n}^{m}\frac{1}{i\gamma_{\mathrm{r}}}\left(j_n(\gamma_{\mathrm{r}})+\gamma_{\mathrm{r}}j_n'(\gamma_{\mathrm{r}})\right)&\\ =\frac{i}{2}\left\{b_{\mathrm{l}n}^{m}\frac{1}{i\gamma_{\mathrm{l}}}\left(h_n(\gamma_{\mathrm{l}})+\gamma_{\mathrm{l}}h_n'(\gamma_{\mathrm{l}})\right)-b_{\mathrm{r}n}^{m}\frac{1}{i\gamma_{\mathrm{r}}}\left(h_n(\gamma_{\mathrm{r}})+\gamma_{\mathrm{r}}h_n'(\gamma_{\mathrm{r}})\right)\right\}&\end{aligned}$$

$$\begin{aligned}\alpha_{\mathrm{l}n}^{m}\left(j_n(\gamma_{\mathrm{l}})+\gamma_{\mathrm{l}}j_n'(\gamma_{\mathrm{l}})\right)+\alpha_{\mathrm{r}n}^{m}\left(j_n(\gamma_{\mathrm{r}})+\gamma_{\mathrm{r}}j_n'(\gamma_{\mathrm{r}})\right)&\\ =\frac{i}{2}\left\{a_{\mathrm{l}n}^{m}\left(h_n(\gamma_{\mathrm{l}})+\gamma_{\mathrm{l}}h_n'(\gamma_{\mathrm{l}})\right)-a_{\mathrm{r}n}^{m}\left(h_n(\gamma_{\mathrm{r}})+\gamma_{\mathrm{r}}h_n'(\gamma_{\mathrm{r}})\right)\right\}&\end{aligned}$$

$$\beta_{\mathrm{l}n}^{m}\frac{1}{i\gamma_{\mathrm{l}}}j_n(\gamma_{\mathrm{l}})+\beta_{\mathrm{r}n}^{m}\frac{1}{i\gamma_{\mathrm{r}}}j_n(\gamma_{\mathrm{r}})=\frac{i}{2}\left\{b_{\mathrm{l}n}^{m}\frac{1}{i\gamma_{\mathrm{l}}}h_n(\gamma_{\mathrm{l}})-b_{\mathrm{r}n}^{m}\frac{1}{i\gamma_{\mathrm{r}}}h_n(\gamma_{\mathrm{r}})\right\}$$

$$\begin{aligned}\begin{pmatrix} j_n(\gamma_{\mathrm{l}}) & j_n(\gamma_{\mathrm{r}}) \\ j_n(\gamma_{\mathrm{l}})+\gamma_{\mathrm{l}}j_n'(\gamma_{\mathrm{l}}) & j_n(\gamma_{\mathrm{r}})+\gamma_{\mathrm{r}}j_n'(\gamma_{\mathrm{r}})\end{pmatrix}\begin{pmatrix}\alpha_{\mathrm{l}n}^{m}\\ \alpha_{\mathrm{r}n}^{m}\end{pmatrix}&\\ =\frac{i}{2}\begin{pmatrix} h_n(\gamma_{\mathrm{l}}) & -h_n(\gamma_{\mathrm{r}}) \\ h_n(\gamma_{\mathrm{l}})+\gamma_{\mathrm{l}}h_n'(\gamma_{\mathrm{l}}) & -\left(h_n(\gamma_{\mathrm{r}})+\gamma_{\mathrm{r}}h_n'(\gamma_{\mathrm{r}})\right)\end{pmatrix}\begin{pmatrix}a_{\mathrm{l}n}^{m}\\ a_{\mathrm{r}n}^{m}\end{pmatrix}&\end{aligned}$$

$$\begin{aligned}\begin{pmatrix} \frac{1}{\gamma_{\mathrm{l}}}\left(j_n(\gamma_{\mathrm{l}})+\gamma_{\mathrm{l}}j_n'(\gamma_{\mathrm{l}})\right) & \frac{1}{\gamma_{\mathrm{r}}}\left(j_n(\gamma_{\mathrm{r}})+\gamma_{\mathrm{r}}j_n'(\gamma_{\mathrm{r}})\right) \\ \frac{1}{\gamma_{\mathrm{l}}}j_n(\gamma_{\mathrm{l}}) & \frac{1}{\gamma_{\mathrm{r}}}j_n(\gamma_{\mathrm{r}})\end{pmatrix}\begin{pmatrix}\beta_{\mathrm{l}n}^{m}\\ \beta_{\mathrm{r}n}^{m}\end{pmatrix}&\\ =\frac{i}{2}\begin{pmatrix} \frac{1}{\gamma_{\mathrm{l}}}\left(h_n(\gamma_{\mathrm{l}})+\gamma_{\mathrm{l}}h_n'(\gamma_{\mathrm{l}})\right) & -\frac{1}{\gamma_{\mathrm{r}}}\left(h_n(\gamma_{\mathrm{r}})+\gamma_{\mathrm{r}}h_n'(\gamma_{\mathrm{r}})\right) \\ \frac{1}{\gamma_{\mathrm{l}}}h_n(\gamma_{\mathrm{l}}) & -\frac{1}{\gamma_{\mathrm{r}}}h_n(\gamma_{\mathrm{r}})\end{pmatrix}\begin{pmatrix}b_{\mathrm{l}n}^{m}\\ b_{\mathrm{r}n}^{m}\end{pmatrix}&\end{aligned}$$

$$\begin{aligned}Q_{\mathrm{l}}^{\infty}(\hat{x})&=\frac{4\pi}{\gamma_{\mathrm{l}}}\sum\frac{1}{i^{n+1}}\left\{-a_{\mathrm{l}n}^{m}V_n^m(\hat{x})+b_{\mathrm{l}n}^{m}U_n^m(\hat{x})\right\}\\ Q_{\mathrm{r}}^{\infty}(\hat{x})&=\frac{4\pi}{\gamma_{\mathrm{r}}}\sum\frac{1}{i^{n+1}}\left\{-a_{\mathrm{r}n}^{m}V_n^m(\hat{x})+b_{\mathrm{r}n}^{m}U_n^m(\hat{x})\right\}\end{aligned}$$

Chapter 5

Numerical Results for 2D Problems

5.1 Direct Problems

5.1.1 Discretization of Integral Equations

The boundary Γ is assumed to be of the 2π periodic parametric form

$$z(t) = (z_1(t), z_2(t)), \quad 0 \leqslant t \leqslant 2\pi \tag{5.1}$$

with

$$(z_1'(t))^2 + (z_2'(t))^2 > 0. \tag{5.2}$$

Note that

$$\frac{d\sigma(z(\tau))}{d\tau} = |z'(\tau)|, \tag{5.3a}$$

$$\nu(z(\tau)) = \frac{1}{|z'(\tau)|}(z_2'(\tau), -z_1'(\tau)). \tag{5.3b}$$

By means of the periodic boundary representation (5.1), boundary integral operators on Γ of the form

$$\int_\Gamma f(x,y)g(y)\,\mathrm{d}\sigma(y), \quad x \in \Gamma$$

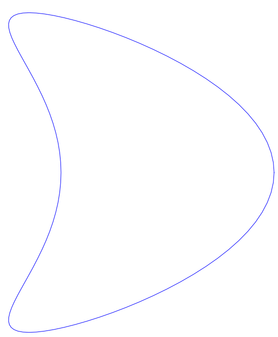

Figure 5.1: The "kite" domain with boundary $\Gamma = (\cos t + 0.65\cos(2t) - 0.65, 1.5\sin t)$, $t \in [0, 2\pi]$.

with kernel f and operand g will be transformed into

$$\int_0^{2\pi} f(z(t), z(\tau))g(z(\tau))|z'(\tau)|\,\mathrm{d}\tau, \quad 0 \leqslant t \leqslant 2\pi.$$

$$H_n^1(z) = J_n(z) + i\,Y_n(z) \tag{5.4}$$

$$\begin{aligned}
Y_0(z) &= \frac{2}{\pi}\left(\gamma + \ln\left(\frac{z}{2}\right)\right) J_0(z) - \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{j}J_{2j}(z) \\
Y_1(z) &= -\frac{2}{\pi z}J_0(z) + \frac{2}{\pi}\left(\ln\left(\frac{z}{2}\right) + \gamma - 1\right) J_1(z) - \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^j(2j+1)}{j(j+1)}J_{2j+1}(z)
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
\left(\widetilde{T}_k\varphi\right)(x) &= (T_k\varphi)(x) - (T_0\varphi)(x) \\
&= \frac{\partial}{\partial\nu(x)}\int_\Gamma \frac{\partial}{\partial\nu(y)}\left\{\frac{i}{2}H_0^1(k|x-y|) + \frac{1}{\pi}\ln|x-y|\right\}\varphi(y)\,\mathrm{d}\sigma(y)
\end{aligned} \tag{5.6}$$

$$S_k\varphi(t) = \int_0^{2\pi} \mathsf{M}_k(t,\tau)\varphi(\tau)\,d\tau \tag{5.7a}$$

$$K_k\varphi(t) = \int_0^{2\pi} \mathsf{L}_k(t,\tau)\varphi(\tau)\,d\tau \tag{5.7b}$$

$$K_k^*\varphi(t) = \int_0^{2\pi} \mathsf{L}_k^*(t,\tau)\varphi(\tau)\,d\tau \tag{5.7c}$$

$$\widetilde{T}_k\varphi(t) = \int_0^{2\pi} \mathsf{N}_k(t,\tau)\varphi(\tau)\,d\tau \tag{5.7d}$$

With

$$r(t,\tau) = \left\{(z_1(t) - z_1(\tau))^2 + (z_2(t) - z_2(\tau))^2\right\}^{\frac{1}{2}}, \tag{5.8}$$

we have

$$\mathsf{M}_k(t,\tau) = \frac{i}{2}H_0^1(kr(t,\tau))|z'(\tau)| \tag{5.9a}$$

$$\mathsf{L}_k(t,\tau) = \frac{ik}{2}\left\{z_2'(\tau)[z_1(t) - z_1(\tau)] - z_1'(\tau)[z_2(t) - z_2(\tau)]\right\}\frac{H_1^1(kr(t,\tau))}{r(t,\tau)} \tag{5.9b}$$

$$\mathsf{L}_k^*(t,\tau) = -\frac{ik}{2}\left\{z_2'(t)[z_1(t) - z_1(\tau)] - z_1'(t)[z_2(t) - z_2(\tau)]\right\}\frac{|z'(\tau)|}{|z'(t)|}\frac{H_1^1(kr(t,\tau))}{r(t,\tau)} \tag{5.9c}$$

$$= \frac{|z'(\tau)|}{|z'(t)|}\mathsf{L}_k(\tau,t) \tag{5.9d}$$

$$\begin{aligned}
\mathsf{N}_k(t,\tau) &= \left\{z_2'(t)[z_1(t) - z_1(\tau)] - z_1'(t)[z_2(t) - z_2(\tau)]\right\} \\
&\times \left\{z_2'(\tau)[z_1(t) - z_1(\tau)] - z_1'(\tau)[z_2(t) - z_2(\tau)]\right\} \\
&\times \frac{1}{|z'(t)|r(t,\tau)^4}\left\{\frac{ik^2}{2}H_0^1(kr(t,\tau))r(t,\tau)^2 - ikH_1^1(kr(t,\tau))r(t,\tau) + \frac{2}{\pi}\right\} \\
&+ \frac{z_1'(t)z_1'(\tau) + z_2'(t)z_2'(\tau)}{|z'(t)|r(t,\tau)^2}\left\{\frac{ik}{2}H_1^1(kr(t,\tau))r(t,\tau) - \frac{1}{\pi}\right\}
\end{aligned} \tag{5.9e}$$

$$H_0^1(z) = \frac{2i}{\pi} \ln\left(\frac{z}{2}\right) J_0(z) + \cdots \tag{5.10a}$$

$$H_1^1(z) = \frac{2i}{\pi} \ln\left(\frac{z}{2}\right) J_1(z) + \cdots \tag{5.10b}$$

$$\ln\frac{z}{2} = \ln\left(\frac{kr(t,\tau)}{4|\sin\frac{(t-\tau)}{2}|}\right) + \frac{1}{2}\ln\left(4\sin^2\frac{t-\tau}{2}\right) + \cdots \tag{5.10c}$$

$$\mathsf{M}_k(t,\tau) = \mathsf{M}_k^1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + \mathsf{M}_k^2(t,\tau) \tag{5.11a}$$

$$\mathsf{L}_k(t,\tau) = \mathsf{L}_k^1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + \mathsf{L}_k^2(t,\tau) \tag{5.11b}$$

$$\mathsf{L}_k^*(t,\tau) = \mathsf{L}_k^{*1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + \mathsf{L}_k^{*2}(t,\tau) \tag{5.11c}$$

$$\mathsf{N}_k(t,\tau) = \mathsf{N}_k^1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + \mathsf{N}_k^2(t,\tau) \tag{5.11d}$$

$$\mathsf{M}_k^1(t,\tau) = -\frac{1}{2\pi}J_0(kr(t,\tau))|z'(\tau)| \tag{5.12a}$$

$$\mathsf{M}_k^2(t,\tau) = \mathsf{M}_k(t,\tau) - \mathsf{M}_k^1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) \tag{5.12b}$$

$$\mathsf{L}_k^1(t,\tau) = -\frac{k}{2\pi}\left\{z_2'(\tau)[z_1(t)-z_1(\tau)] - z_1'(\tau)[z_2(t)-z_2(\tau)]\right\}\frac{J_1(kr(t,\tau))}{r(t,\tau)} \tag{5.12c}$$

$$\mathsf{L}_k^2(t,\tau) = \mathsf{L}_k(t,\tau) - \mathsf{L}_k^1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) \tag{5.12d}$$

$$\mathsf{L}_k^{*1}(t,\tau) = \frac{|z'(\tau)|}{|z'(t)|}L_k^1(t,\tau) \tag{5.12e}$$

$$\begin{aligned}\mathsf{L}_k^{*2}(t,\tau) &= \mathsf{L}_k^*(t,\tau) - \mathsf{L}_k^{*1}(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) \\ &= \frac{|z'(\tau)|}{|z'(t)|}\mathsf{L}_k^2(t,\tau)\end{aligned} \tag{5.12f}$$

$$\begin{aligned}\mathsf{N}_k^1(t,\tau) =& \left\{z_2'(t)[z_1(t)-z_1(\tau)] - z_1'(t)[z_2(t)-z_2(\tau)]\right\} \\ &\times \left\{z_2'(\tau)[z_1(t)-z_1(\tau)] - z_1'(\tau)[z_2(t)-z_2(\tau)]\right\} \\ &\times \frac{1}{|z'(t)|r(t,\tau)^4}\left\{-\frac{k^2}{2\pi}J_0(kr(t,\tau))r(t,\tau)^2 + \frac{k}{\pi}J_1(kr(t,\tau))r(t,\tau)\right\} \\ &- \frac{k}{2\pi}\frac{z_1'(t)z_1'(\tau)+z_2'(t)z_2'(\tau)}{|z'(t)|r(t,\tau)^2}J_1(kr(t,\tau)) \\ \mathsf{N}_k^2(t,\tau) =& \mathsf{N}_k(t,\tau) - \mathsf{N}_k^1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)\end{aligned} \tag{5.12g}$$

$$\mathsf{M}_k^1(t,t) = -\frac{1}{2\pi}|z'(\tau)| \tag{5.13a}$$

$$\mathsf{M}_k^2(t,t) = |z'(t)| \left\{ \frac{i}{2} - \frac{\gamma}{\pi} - \frac{1}{2\pi} \ln \frac{k^2|z'(t)|^2}{4} \right\} \tag{5.13b}$$

$$\mathsf{L}_k^1(t,t) = 0 \tag{5.13c}$$

$$\mathsf{L}_k^2(t,t) = -\frac{1}{2\pi} \frac{z_1'(t)z_2''(t) - z_2'(t)z_1''(t)}{|z'(t)^2|} \tag{5.13d}$$

$$\mathsf{L}_k^{*1}(t,t) = 0 \tag{5.13e}$$

$$\mathsf{L}_k^{*2}(t,t) = \mathsf{L}_k^2(t,t) \tag{5.13f}$$

$$\mathsf{N}_k^1(t,t) = -\frac{k^2}{4\pi}|z'(t)| \tag{5.13g}$$

$$\mathsf{N}_k^2(t,t) = \frac{k^2}{2} \left\{ \frac{i}{2} - \frac{\gamma}{\pi} - \frac{1}{2\pi} \ln \frac{k^2|z'(t)|^2}{4} + \frac{1}{2\pi} \right\} |z'(t)| \tag{5.13h}$$

$$\int_0^{2\pi} \ln \left(4 \sin^2 \frac{t-\tau}{2} \right) f(\tau)\, d\tau \approx \sum_{j=0}^{2n-1} R_j^{(n)}(t) f(t_j), \quad 0 \leqslant t \leqslant 2\pi \tag{5.14}$$

$$t_j = \frac{\pi j}{n}, \quad j = 0, \cdots, 2n-1$$

$$R_j^{(n)}(t) = -\frac{2\pi}{n} \sum_{l=1}^{n-1} \frac{1}{l} \cos\left(l(t-t_j)\right) - \frac{\pi}{n^2} \cos\left(n(t-t_j)\right), \quad j = 0, \cdots, 2n-1 \tag{5.15}$$

$$\int_0^{2\pi} f(\tau)\, d\tau \approx \frac{\pi}{n} \sum_{j=0}^{2n-1} f(t_j) \tag{5.16}$$

$$E\psi^{(n)}(t) + \sum_{j=0}^{2n-1} \left\{ R_j^{(n)}(t) A^1(t,t_j) + \frac{\pi}{n} A^2(t,t_j) \right\} \psi^{(n)}(t_j) = u(t), \quad 0 \leqslant t \leqslant 2\pi \tag{5.17}$$

$$\psi_i^{(n)} := \psi^{(n)}(t_i), \quad i = 0, \cdots, 2n-1 \tag{5.18}$$

$$R_{|i-j|}^{(n)} := R_j^{(n)}(t_i) \tag{5.19}$$

$$E\psi_i^{(n)} + \sum_{j=0}^{2n-1} \left\{ R_{|i-j|}^{(n)} A^1(t_i,t_j) + \frac{\pi}{n} A^2(t_i,t_j) \right\} \psi_j^{(n)} = u(t_i), \quad i = 0, \cdots, 2n-1 \tag{5.20}$$

$$R_l^{(n)} = R_l^{(n)}(0) = -\frac{2\pi}{n} \sum_{m=1}^{n-1} \frac{1}{m} \cos \frac{ml\pi}{n} - \frac{(-1)^l \pi}{n^2}, \quad j = 0, \cdots, 2n-1 \tag{5.21}$$

$$
\begin{aligned}
&Q_{\mathrm{er}}^{\infty}(\hat{x})\\
&= \frac{e^{-\frac{i\pi}{4}}}{\sqrt{8\pi\gamma_{\mathrm{er}}}} \int_0^{2\pi} \left\{\gamma_{\mathrm{er}}[\hat{x}_1 z_2'(\tau) - \hat{x}_2 z_1'(\tau)]\psi_3(\tau) + ic_1|z'(\tau)|\psi_1(\tau)\right\} e^{-i\gamma_{\mathrm{er}}(\hat{x_1}z_1(\tau)+\hat{x_2}z_2(\tau))}\, d\tau\\
&= \frac{e^{-\frac{i\pi}{4}}}{\sqrt{8\pi\gamma_{\mathrm{er}}}} \frac{\pi}{n} \sum_{j=0}^{2n-1} \left\{\gamma_{\mathrm{er}}[\hat{x}_1 z_2'(t_j) - \hat{x}_2 z_1'(t_j)]\psi_{j;3}^{(n)} + ic_1|z'(t_j)|\psi_{j;1}^{(n)}(t_j)\right\} e^{-i\gamma_{\mathrm{er}}(\hat{x_1}z_1(t_j)+\hat{x_2}z_2(t_j))}
\end{aligned} \tag{5.22}
$$

$$
\begin{aligned}
&Q_{\mathrm{el}}^{\infty}(\hat{x})\\
&= \frac{e^{-\frac{i\pi}{4}}}{\sqrt{8\pi\gamma_{\mathrm{el}}}} \int_0^{2\pi} \left\{\gamma_{\mathrm{el}}[\hat{x}_1 z_2'(\tau) - \hat{x}_2 z_1'(\tau)]\psi_4(\tau) + ic_1|z'(\tau)|\psi_2(\tau)\right\} e^{-i\gamma_{\mathrm{el}}(\hat{x_1}z_1(\tau)+\hat{x_2}z_2(\tau))}\, d\tau\\
&= \frac{e^{-\frac{i\pi}{4}}}{\sqrt{8\pi\gamma_{\mathrm{el}}}} \frac{\pi}{n} \sum_{j=0}^{2n-1} \left\{\gamma_{\mathrm{el}}[\hat{x}_1 z_2'(t_j) - \hat{x}_2 z_1'(t_j)]\psi_{j;4}^{(n)} + ic_1|z'(t_j)|\psi_{j;2}^{(n)}(t_j)\right\} e^{-i\gamma_{\mathrm{el}}(\hat{x_1}z_1(t_j)+\hat{x_2}z_2(t_j))}
\end{aligned} \tag{5.23}
$$

5.1.2 Calibration

Note that if

$$
\begin{aligned}
Q_1 &= c_1 J_0(k_1|x|)\\
Q_2 &= c_2 H_0^1(k_2|x|)
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{\partial Q_1}{\partial \nu} &= -k_1 c_1 J_1(k_1|x|)\, \nu(x) \cdot \frac{x}{|x|}\\
\frac{\partial Q_2}{\partial \nu} &= -k_2 c_2 H_1^1(k_2|x|)\, \nu(x) \cdot \frac{x}{|x|}
\end{aligned}
$$

Chiral-Chiral

The boundary terms according to “master equations” are

$$
\begin{aligned}
u_1 &= \delta(Q_{\mathrm{ir}} + Q_{\mathrm{il}}) - (Q_{\mathrm{er}} + Q_{\mathrm{el}})\\
u_2 &= i\rho(Q_{\mathrm{ir}} - Q_{\mathrm{il}}) - i(Q_{\mathrm{er}} - Q_{\mathrm{el}})\\
u_3 &= \delta\left(\frac{1}{\gamma_{\mathrm{ir}}}\frac{\partial Q_{\mathrm{ir}}}{\partial \nu} - \frac{1}{\gamma_{\mathrm{il}}}\frac{\partial Q_{\mathrm{il}}}{\partial \nu}\right) - \left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} - \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right)\\
u_4 &= i\rho\left(\frac{1}{\gamma_{\mathrm{ir}}}\frac{\partial Q_{\mathrm{ir}}}{\partial \nu} + \frac{1}{\gamma_{\mathrm{il}}}\frac{\partial Q_{\mathrm{il}}}{\partial \nu}\right) - i\left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} + \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right)
\end{aligned}
$$

Given the fields

$$
\begin{aligned}
Q_{\mathrm{er}} &= c_{\mathrm{er}} H_0^1(\gamma_{\mathrm{er}}\,|x|)\\
Q_{\mathrm{el}} &= c_{\mathrm{el}} H_0^1(\gamma_{\mathrm{el}}\,|x|)\\
Q_{\mathrm{ir}} &= c_{\mathrm{ir}} J_0(\gamma_{\mathrm{ir}}\,|x|)\\
Q_{\mathrm{il}} &= c_{\mathrm{il}} J_0(\gamma_{\mathrm{il}}\,|x|)
\end{aligned}
$$

where c_i's are constants, we have

$$
\begin{aligned}
u_1 &= c_{\mathrm{ir}}\delta J_0(\gamma_{\mathrm{ir}}|x|) + c_{\mathrm{il}}\delta J_0(\gamma_{\mathrm{il}}|x|) - c_{\mathrm{er}} H_0^1(\gamma_{\mathrm{er}}|x|) - c_{\mathrm{el}} H_0^1(\gamma_{\mathrm{el}}|x|)\\
u_2 &= i\big(c_{\mathrm{ir}} J_0(\gamma_{\mathrm{ir}}|x|)\rho - c_{\mathrm{il}} J_0(\gamma_{\mathrm{il}}|x|)\rho - c_{\mathrm{er}} H_0^1(\gamma_{\mathrm{er}}|x|) + c_{\mathrm{el}} H_0^1(\gamma_{\mathrm{el}}|x|)\big)\\
u_3 &= -\nu(x)\cdot\frac{x}{|x|}\big(c_{\mathrm{ir}}\delta J_1(\gamma_{\mathrm{ir}}|x|) - c_{\mathrm{il}}\delta J_1(\gamma_{\mathrm{il}}|x|) - c_{\mathrm{er}} H_1^1(\gamma_{\mathrm{er}}|x|) + c_{\mathrm{el}} H_1^1(\gamma_{\mathrm{el}}|x|)\big)\\
u_4 &= -i\nu(x)\cdot\frac{x}{|x|}\big(c_{\mathrm{ir}} J_1(\gamma_{\mathrm{ir}}|x|)\rho + c_{\mathrm{il}} J_1(\gamma_{\mathrm{il}}|x|)\rho - c_{\mathrm{er}} H_1^1(\gamma_{\mathrm{er}}|x|) - c_{\mathrm{el}} H_1^1(\gamma_{\mathrm{el}}|x|)\big)
\end{aligned}
$$

Achiral-Chiral

The boundary terms according to "master equations" are

$$
\begin{aligned}
u_1 &= \delta(Q_{\mathrm{ir}} + Q_{\mathrm{il}}) - (Q_{\mathrm{er}} + Q_{\mathrm{el}})\\
u_2 &= i\rho(Q_{\mathrm{ir}} - Q_{\mathrm{il}}) - i(Q_{\mathrm{er}} - Q_{\mathrm{el}})\\
u_3 &= \delta\left(\frac{1}{\gamma_{\mathrm{ir}}}\frac{\partial Q_{\mathrm{ir}}}{\partial\nu} - \frac{1}{\gamma_{\mathrm{il}}}\frac{\partial Q_{\mathrm{il}}}{\partial\nu}\right) - \left(\frac{1}{k_{\mathrm{e}}}\frac{\partial Q_{\mathrm{er}}}{\partial\nu} - \frac{1}{k_{\mathrm{e}}}\frac{\partial Q_{\mathrm{el}}}{\partial\nu}\right)\\
u_4 &= i\rho\left(\frac{1}{\gamma_{\mathrm{ir}}}\frac{\partial Q_{\mathrm{ir}}}{\partial\nu} + \frac{1}{\gamma_{\mathrm{il}}}\frac{\partial Q_{\mathrm{il}}}{\partial\nu}\right) - i\left(\frac{1}{k_{\mathrm{e}}}\frac{\partial Q_{\mathrm{er}}}{\partial\nu} + \frac{1}{k_{\mathrm{e}}}\frac{\partial Q_{\mathrm{el}}}{\partial\nu}\right)
\end{aligned}
$$

Given the fields

$$
\begin{aligned}
Q_{\mathrm{er}} &= c_{\mathrm{er}} H_0^1(k_{\mathrm{e}}|x|)\\
Q_{\mathrm{el}} &= c_{\mathrm{el}} H_0^1(k_{\mathrm{e}}|x|)\\
Q_{\mathrm{ir}} &= c_{\mathrm{ir}} J_0(\gamma_{\mathrm{ir}}|x|)\\
Q_{\mathrm{il}} &= c_{\mathrm{il}} J_0(\gamma_{\mathrm{il}}|x|)
\end{aligned}
$$

where c_i's are constants, we have

$$
\begin{aligned}
u_1 &= c_{\mathrm{ir}}\delta J_0(\gamma_{\mathrm{ir}}|x|) + c_{\mathrm{il}}\delta J_0(\gamma_{\mathrm{il}}|x|) - c_{\mathrm{er}} H_0^1(k_{\mathrm{e}}|x|) - c_{\mathrm{el}} H_0^1(k_{\mathrm{e}}|x|)\\
u_2 &= i\big(c_{\mathrm{ir}} J_0(\gamma_{\mathrm{ir}}|x|)\rho - c_{\mathrm{il}} J_0(\gamma_{\mathrm{il}}|x|)\rho - c_{\mathrm{er}} H_0^1(k_{\mathrm{e}}|x|) + c_{\mathrm{el}} H_0^1(k_{\mathrm{e}}|x|)\big)\\
u_3 &= -\nu(x)\cdot\frac{x}{|x|}\big(c_{\mathrm{ir}}\delta J_1(\gamma_{\mathrm{ir}}|x|) - c_{\mathrm{il}}\delta J_1(\gamma_{\mathrm{il}}|x|) - c_{\mathrm{er}} H_1^1(k_{\mathrm{e}}|x|) + c_{\mathrm{el}} H_1^1(k_{\mathrm{e}}|x|)\big)\\
u_4 &= -i\nu(x)\cdot\frac{x}{|x|}\big(c_{\mathrm{ir}} J_1(\gamma_{\mathrm{ir}}|x|)\rho + c_{\mathrm{il}} J_1(\gamma_{\mathrm{il}}|x|)\rho - c_{\mathrm{er}} H_1^1(k_{\mathrm{e}}|x|) - c_{\mathrm{el}} H_1^1(k_{\mathrm{e}}|x|)\big)
\end{aligned}
$$

Chiral-Achiral

The boundary terms according to "master equations" are

$$
\begin{aligned}
u_1 &= \delta(Q_{\mathrm{ir}} + Q_{\mathrm{il}}) - (Q_{\mathrm{er}} + Q_{\mathrm{el}}) \\
u_2 &= i\rho(Q_{\mathrm{ir}} - Q_{\mathrm{il}}) - i(Q_{\mathrm{er}} - Q_{\mathrm{el}}) \\
u_3 &= \delta\left(\frac{1}{k_{\mathrm{i}}}\frac{\partial Q_{\mathrm{ir}}}{\partial \nu} - \frac{1}{k_{\mathrm{i}}}\frac{\partial Q_{\mathrm{il}}}{\partial \nu}\right) - \left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} - \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right) \\
u_4 &= i\rho\left(\frac{1}{k_{\mathrm{i}}}\frac{\partial Q_{\mathrm{ir}}}{\partial \nu} + \frac{1}{k_{\mathrm{i}}}\frac{\partial Q_{\mathrm{il}}}{\partial \nu}\right) - i\left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} + \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right)
\end{aligned}
$$

Given the fields

$$
\begin{aligned}
Q_{\mathrm{er}} &= c_{\mathrm{er}} H_0^1(\gamma_{\mathrm{er}}|x|) \\
Q_{\mathrm{el}} &= c_{\mathrm{el}} H_0^1(\gamma_{\mathrm{el}}|x|) \\
Q_{\mathrm{ir}} &= c_{\mathrm{ir}} J_0(k_{\mathrm{i}}|x|) \\
Q_{\mathrm{il}} &= c_{\mathrm{il}} J_0(k_{\mathrm{i}}|x|)
\end{aligned}
$$

where c_i's are constants, we have

$$
\begin{aligned}
u_1 &= c_{\mathrm{ir}}\delta J_0(k_{\mathrm{i}}|x|) + c_{\mathrm{il}}\delta J_0(k_{\mathrm{i}}|x|) - c_{\mathrm{er}} H_0^1(\gamma_{\mathrm{er}}|x|) - c_{\mathrm{el}} H_0^1(\gamma_{\mathrm{el}}|x|) \\
u_2 &= i\big(c_{\mathrm{ir}} J_0(k_{\mathrm{i}}|x|)\rho - c_{\mathrm{il}} J_0(k_{\mathrm{i}}|x|)\rho - c_{\mathrm{er}} H_0^1(\gamma_{\mathrm{er}}|x|) + c_{\mathrm{el}} H_0^1(\gamma_{\mathrm{el}}|x|)\big) \\
u_3 &= -\nu(x)\cdot\frac{x}{|x|}\big(c_{\mathrm{ir}}\delta J_1(k_{\mathrm{i}}|x|) - c_{\mathrm{il}}\delta J_1(k_{\mathrm{i}}|x|) - c_{\mathrm{er}} H_1^1(\gamma_{\mathrm{er}}|x|) + c_{\mathrm{el}} H_1^1(\gamma_{\mathrm{el}}|x|)\big) \\
u_4 &= -i\nu(x)\cdot\frac{x}{|x|}\big(c_{\mathrm{ir}} J_1(k_{\mathrm{i}}|x|)\rho + c_{\mathrm{il}} J_1(k_{\mathrm{i}}|x|)\rho - c_{\mathrm{er}} H_1^1(\gamma_{\mathrm{er}}|x|) - c_{\mathrm{el}} H_1^1(\gamma_{\mathrm{el}}|x|)\big)
\end{aligned}
$$

Chiral-Perfect Conductor

The boundary terms according to "master equations" are

$$
\begin{aligned}
u_1 &= -i(Q_{\mathrm{er}} - Q_{\mathrm{el}}) \\
u_2 &= -i\left(\frac{1}{\gamma_{\mathrm{er}}}\frac{\partial Q_{\mathrm{er}}}{\partial \nu} + \frac{1}{\gamma_{\mathrm{el}}}\frac{\partial Q_{\mathrm{el}}}{\partial \nu}\right)
\end{aligned}
$$

Given the fields

$$
\begin{aligned}
Q_{\mathrm{er}} &= c_{\mathrm{er}} H_0^1(\gamma_{\mathrm{er}}|x|) \\
Q_{\mathrm{el}} &= c_{\mathrm{el}} H_0^1(\gamma_{\mathrm{el}}|x|)
\end{aligned}
$$

where c_i's are constants, we have

$$
\begin{aligned}
u_1 &= -i\big(c_{\mathrm{er}} H_0^1(\gamma_{\mathrm{er}}|x|) - c_{\mathrm{el}} H_0^1(\gamma_{\mathrm{el}}|x|)\big) \\
u_2 &= i\nu(x)\cdot\frac{x}{|x|}\big(c_{\mathrm{er}} H_1^1(\gamma_{\mathrm{er}}|x|) + c_{\mathrm{el}} H_1^1(\gamma_{\mathrm{el}}|x|)\big)
\end{aligned}
$$

5.1.3 Calibration Results

5.2 Inverse Problem

Table 5.1: Parameters Used in Calibration

parameter	chiral-chiral	chiral-achiral	achiral-chiral	chiral-perfect conductor
c_{il}	$1+i$	$1+i$	$1+i$	—
c_{ir}	$2i$	3	3	—
c_{el}	$1-0.5i$	1	1	1
c_{er}	2	2	2	2
ε_{i}	1.4	1.4	1.4	—
μ_{i}	1.2	1.2	1.2	—
β_{i}	0.1	0	0.1	—
ε_{e}	1.3	1.1	1	1.4
μ_{e}	1.25	1.15	1	1.2
β_{e}	0.05	0.05	0	0.1

Table 5.2: Q^{∞}_{el}, Chiral-Chiral, $\omega = 1$

n	$\Re Q^{\infty}_{\mathrm{el}}$	$\Im Q^{\infty}_{\mathrm{el}}$	error
8	0.243508249431	-0.724463645911	0.00252276843633
16	0.241757267708	-0.725273624079	7.57632109346e-07
32	0.241757794244	-0.725273382729	1.30143110535e-12
64	0.241757794243	-0.72527338273	7.48452457873e-16
128	0.241757794243	-0.72527338273	—

exact $\Re Q^{\infty}_{\mathrm{el}} = 0.241757794243$, $\Im Q^{\infty}_{\mathrm{el}} = -0.72527338273$

Table 5.3: Q^{∞}_{er}, Chiral-Chiral, $\omega = 1$

n	$\Re Q^{\infty}_{\mathrm{er}}$	$\Im Q^{\infty}_{\mathrm{er}}$	error
8	1.03242377734	-1.0282192268	0.00208410886337
16	1.03076319169	-1.03076453615	7.75447408805e-07
32	1.0307634315	-1.0307634315	1.74692244341e-12
64	1.0307634315	-1.0307634315	5.49209323305e-16
128	1.0307634315	-1.0307634315	—

exact $\Re Q^{\infty}_{\mathrm{er}} = 1.0307634315$, $\Im Q^{\infty}_{\mathrm{er}} = -1.0307634315$

Table 5.4: Q^{∞}_{el}, Chiral-Chiral, $\omega = 5$

n	$\Re Q^{\infty}_{\mathrm{el}}$	$\Im Q^{\infty}_{\mathrm{el}}$	error
8	-1.54250039	-0.590088131481	5.70708508424
16	-1.32334465864	-0.341629394985	4.85869482999
32	0.157231611887	-0.334208562175	0.29760446524
64	0.0922336179119	-0.27668319242	2.26074544743e-05
128	0.0922294278071	-0.276688283421	6.01956349633e-15

exact $\Re Q^{\infty}_{\mathrm{el}} = 0.0922294278071$, $\Im Q^{\infty}_{\mathrm{el}} = -0.276688283421$

Table 5.5: Q^{∞}_{er}, Chiral-Chiral, $\omega = 5$

n	$\Re Q^{\infty}_{\mathrm{er}}$	$\Im Q^{\infty}_{\mathrm{er}}$	error
8	-0.914018631255	-1.10274229685	2.12747483551
16	0.485271823154	-0.531252645336	0.0458354064657
32	0.514160036173	-0.514124508444	0.00174134854985
64	0.513248314271	-0.513249251167	9.25516228666e-07
128	0.513248703975	-0.513248703975	1.86706958735e-15

exact $\Re Q^{\infty}_{\mathrm{er}} = 0.513248703975$, $\Im Q^{\infty}_{\mathrm{er}} = -0.513248703975$

Table 5.6: Q^{∞}_{el}, Chiral-Achiral, $\omega = 1$

n	$\Re Q^{\infty}_{\mathrm{el}}$	$\Im Q^{\infty}_{\mathrm{el}}$	error
8	0.5177553169413	-0.515454523971	0.00211709931463
16	0.516813776585	-0.51681440272	8.11410019314e-07
32	0.516813810653	-0.516813810653	1.89097093567e-12
64	0.516813810652	-0.516813810652	3.03802341641e-16
128	0.516813810652	-0.516813810652	—

exact $\Re Q^{\infty}_{\mathrm{el}} = 0.516813810652$, $\Im Q^{\infty}_{\mathrm{el}} = -0.516813810652$

Table 5.7: Q^{∞}_{er}, Chiral-Achiral, $\omega = 1$

n	$\Re Q^{\infty}_{\mathrm{er}}$	$\Im Q^{\infty}_{\mathrm{er}}$	error
8	1.09460136497	-1.09072789032	0.00192359535948
16	1.09348494462	-1.09348581917	4.1512146724e-07
32	1.09348526007	-1.09348526007	2.27060990893e-12
64	1.09348526007	-1.09348526007	5.92020513263e-16
128	1.09348526007	-1.09348526007	—

exact $\Re Q^{\infty}_{\mathrm{er}} = 1.09348526007$, $\Im Q^{\infty}_{\mathrm{er}} = -1.09348526007$

Table 5.8: Q^{∞}_{el}, Chiral-Achiral, $\omega = 5$

n	$\Re Q^{\infty}_{\mathrm{el}}$	$\Im Q^{\infty}_{\mathrm{el}}$	error
8	0.30026870953	-0.712558815495	1.82383712303
16	0.200579979636	-0.410772136414	0.732891144812
32	0.20170499083	-0.20171835406	3.41968012876e-05
64	0.201709958923	-0.201709958923	3.21233736018e-15
128	0.201709958923	-0.201709958923	—

exact $\Re Q^{\infty}_{\mathrm{el}} = 0.201709958923$, $\Im Q^{\infty}_{\mathrm{el}} = -0.201709958923$

Table 5.9: Q^{∞}_{er}, Chiral-Achiral, $\omega = 5$

n	$\Re Q^{\infty}_{\mathrm{er}}$	$\Im Q^{\infty}_{\mathrm{er}}$	error
8	-0.20701351838	-0.109212287124	1.12960975313
16	0.547735625223	-0.536176317021	0.0124250415665
32	0.538582690729	-0.53858241491	7.65288725345e-07
64	0.538582941234	-0.538582941234	1.56989871106e-15
128	0.538582941234	-0.538582941234	—

exact $\Re Q^{\infty}_{\mathrm{er}} = 0.538582941234$, $\Im Q^{\infty}_{\mathrm{er}} = -0.538582941234$

Table 5.10: Q^{∞}_{el}, Achiral-Chiral, $\omega = 1$

n	$\Re Q^{\infty}_{\text{el}}$	$\Im Q^{\infty}_{\text{el}}$	error
8	0.564621897758	-0.562665174455	0.00198590726036
16	0.564189380256	-0.564189852447	4.22488162925e-07
32	0.564189583547	-0.564189583545	3.02452614006e-12
64	0.564189583548	-0.564189583548	1.12182947538e-15
128	0.564189583548	-0.564189583548	—

exact $\Re Q^{\infty}_{\text{el}} = 0.564189583548$, $\Im Q^{\infty}_{\text{el}} = -0.564189583548$

Table 5.11: Q^{∞}_{er}, Achiral-Chiral, $\omega = 1$

n	$\Re Q^{\infty}_{\text{er}}$	$\Im Q^{\infty}_{\text{er}}$	error
8	1.12939682663	-1.12573323433	0.00177650273147
16	1.12837893453	-1.12837995564	5.15193216419e-07
32	1.12837916709	-1.12837916709	2.01395207188e-12
64	1.1283791671	-1.1283791671	4.40017721993e-16
128	1.1283791671	-1.1283791671	—

exact $\Re Q^{\infty}_{\text{er}} = 1.1283791671$, $\Im Q^{\infty}_{\text{er}} = -1.1283791671$

Table 5.12: Q^{∞}_{el}, Achiral-Chiral, $\omega = 5$

n	$\Re Q^{\infty}_{\text{el}}$	$\Im Q^{\infty}_{\text{el}}$	error
8	-1.38080129148	-1.2059316398	5.29994165852
16	-0.176692628905	-0.350908885116	1.23363025131
32	0.2768771934	-0.182011193168	0.208701580971
64	0.252325246554	-0.252321699817	4.11143070506e-05
128	0.252313252202	-0.252313252202	3.45420918345e-15

exact $\Re Q^{\infty}_{\text{el}} = 0.252313252202$, $\Im Q^{\infty}_{\text{el}} = -0.252313252202$

Table 5.13: Q^∞_{er}, Achiral-Chiral, $\omega = 5$

n	$\Re Q^\infty_{\text{er}}$	$\Im Q^\infty_{\text{er}}$	error
8	-0.283950058348	-0.305333620019	1.13973283355
16	0.495725962103	-0.501971233041	0.0130150291233
32	0.503544847795	-0.505898206999	0.00233937408589
64	0.504625161441	-0.504626766172	1.91723918982e-06
128	0.504626504404	-0.504626504404	2.11597746912e-15

exact $\Re Q^\infty_{\text{er}} = 0.504626504404$, $\Im Q^\infty_{\text{er}} = -0.504626504404$

Table 5.14: Q^∞_{el}, Chiral-Perfect Conductor, $\omega = 1$

n	$\Re Q^\infty_{\text{el}}$	$\Im Q^\infty_{\text{el}}$	error
8	0.463559211052	-0.460881978899	0.00290528086333
16	0.462330209434	-0.462330386333	2.61720255786e-06
32	0.462331504632	-0.462331504679	5.35048489119e-11
64	0.462331504662	-0.462331504662	6.1222832642e-16
128	0.462331504662	-0.462331504662	—

exact $\Re Q^\infty_{\text{el}} = 0.462331504662$, $\Im Q^\infty_{\text{el}} = -0.462331504662$

Table 5.15: Q^∞_{er}, Chiral-Perfect Conductor, $\omega = 1$

n	$\Re Q^\infty_{\text{er}}$	$\Im Q^\infty_{\text{er}}$	error
8	1.05517487942	-1.05108924213	0.00195575830377
16	1.05339681853	-1.05340126998	2.11298904147e-06
32	1.05339906484	-1.05339906484	2.14225596584e-11
64	1.05339906482	-1.05339906482	4.71337831248e-16
128	1.05339906482	-1.05339906482	—

exact $\Re Q^\infty_{\text{er}} = 1.05339906482$, $\Im Q^\infty_{\text{er}} = -1.05339906482$

Table 5.16: Q^{∞}_{el}, Chiral-Perfect Conductor, $\omega = 5$

n	$\Re Q^{\infty}_{\text{el}}$	$\Im Q^{\infty}_{\text{el}}$	error
8	-1.3859292999	-2.36655602671	14.529533948
16	1.18976771222	-1.24398483508	8.25824844059
32	0.151186067614	0.157450832994	1.55754026122
64	0.132779683355	-0.132115233073	0.00782681251132
128	0.131473545422	-0.131473545422	9.09496118187e-15

$$\text{exact } \Re Q^{\infty}_{\text{el}} = 0.131473545422,\ \Im Q^{\infty}_{\text{el}} = -0.131473545422$$

Table 5.17: Q^{∞}_{er}, Chiral-Perfect Conductor, $\omega = 5$

n	$\Re Q^{\infty}_{\text{er}}$	$\Im Q^{\infty}_{\text{er}}$	error
8	-0.0113788027834	1.58425059547	2.77129864916
16	0.509670739085	-0.617842449859	0.0954999201488
32	0.572425660343	-0.579442075825	0.0136177506068
64	0.5690246566	-0.569024880861	2.56074083006e-07
128	0.569024675678	-0.569024675678	1.00438927556e-15

$$\text{exact } \Re Q^{\infty}_{\text{er}} = 0.569024675678,\ \Im Q^{\infty}_{\text{er}} = -0.569024675678$$

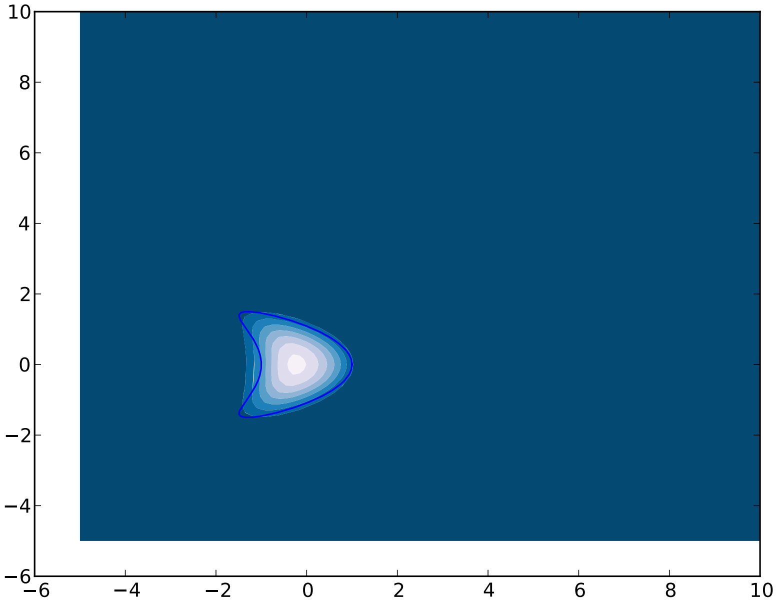

Figure 5.2: Achiral-Chiral Reconstruction

Appendix A

Symbolic Manipulation Procedures

In order to automate the equation derivation processes and to eliminate the inaccuracies in the numerical codes and the final TEX output, a few programs have been written.

The main program is written in the open-source computer algebra system **maxima**.

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