ELECTROMAGNETIC INVERSE PROBLEMS IN CHIRAL MEDIA

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1. Introduction

We study the problem of time-harmonic electromagnetic waves scattering by a bounded perfectly conducting obstacle embedded in a homogeneous chiral environment in $\mathbb{R}^3$.

The macroscopic Maxwell equations of electromagnetism are

$$\begin{aligned} \operatorname{div}\mathcal{D} &= \rho \qquad & \operatorname{curl}\mathcal{H} - \frac{\partial \mathcal{D}}{\partial t} &= \mathcal{J} \\ \operatorname{div}\mathcal{B} &= 0 \qquad & \operatorname{curl}\mathcal{E} + \frac{\partial \mathcal{B}}{\partial t} &= 0 \end{aligned}$$

where $\mathcal{D}$ denotes the electric displacement, $\mathcal{H}$ the magnetic field, $\mathcal{B}$ the magnetic induction, $\mathcal{E}$ the electric field, ρ the charge density and $\mathcal{J}$ the current density. The connections between derived fields $\mathcal{D}$, $\mathcal{H}$ and the fundamental terms $\mathcal{E}$, $\mathcal{B}$ are known as the constitutive relations.

In presence of the conducting obstacle the current density $\mathcal{J}$ satisfies Ohm's law

$$\mathcal{J} = \sigma\mathcal{E},$$

where σ is the electric conductivity. The homogeneous chiral medium obeys the Drude-Born-Fedorov constitutive relations

$$\begin{aligned} \mathcal{D} &= \varepsilon(\mathcal{E} + \beta \operatorname{curl}\mathcal{E}) \\ \mathcal{B} &= \mu(\mathcal{H} + \beta \operatorname{curl}\mathcal{H}) \end{aligned}$$

where ε denotes the electric permittivity, μ the magnetic permeability and β the chirality measure. For time-harmonic electromagnetic wave of the form

$$\begin{aligned} \mathcal{E}(x,t) &= \left(\varepsilon + \tfrac{i\sigma}{\omega}\right)^{-\frac{1}{2}} E(x) e^{-i\omega t} \\ \mathcal{H}(x,t) &= \mu^{-\frac{1}{2}} H(x) e^{-i\omega t} \end{aligned}$$

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we have the following reduced Maxwell equations for complex-valued spatial part $E(x)$, $H(x)$:

$$\begin{aligned} \operatorname{curl} E - ik(H + \beta \operatorname{curl} H) &= 0 \\ \operatorname{curl} H + ik(E + \beta \operatorname{curl} E) &= 0 \end{aligned} \tag{1.1}$$

where

$$k^2 = \omega\mu(\varepsilon\omega + i\sigma) \text{ and } \operatorname{Im} k \geqslant 0. \tag{1.2}$$

If we set

$$\begin{aligned} Q_{\mathrm{L}} &= E + iH \\ Q_{\mathrm{R}} &= E - iH, \end{aligned} \tag{1.3}$$

we can transform reduced Maxwell equations (1.1) into

$$\begin{aligned} \operatorname{curl} Q_{\mathrm{L}} &= \gamma_{\mathrm{L}} Q_{\mathrm{L}} \\ \operatorname{curl} Q_{\mathrm{R}} &= -\gamma_{\mathrm{R}} Q_{\mathrm{R}} \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} \gamma_{\mathrm{L}} &= \frac{k}{1 - k\beta} \\ \gamma_{\mathrm{R}} &= \frac{k}{1 + k\beta}. \end{aligned} \tag{1.5}$$

We say that the tuple (E, H, k) solves Maxwell equations if

$$\begin{aligned} \operatorname{curl} E - ikH &= 0 \qquad & \operatorname{div} E &= 0 \\ \operatorname{curl} H + ikE &= 0 \qquad & \operatorname{div} H &= 0 \end{aligned}$$

It is clear that $(Q_{\mathrm{L}}, -iQ_{\mathrm{L}}, \gamma_{\mathrm{L}})$ and $(Q_{\mathrm{R}}, iQ_{\mathrm{R}}, \gamma_{\mathrm{R}})$ solves Maxwell equations and this fact allows us to reuse the representation theorem (c.f. proposition 2) in achiral cases.

2. Notations, Definitions and Known Results

Here we collect relevant notations, definitions and propositions and refer to [1], [2], [3], [4] for proofs and details.

Definition 1 (Lipschitz domain)**.** An open set Ω is called a Lipschitz domain if for each point $x \in \Gamma$, the boundary of Ω, there exists a rectangular coordinate system (u, v), where $u \in \mathbb{R}^{n-1}$ and $v \in \mathbb{R}$, a neighborhood $U \equiv U(x)$ and a function $\phi \equiv \phi(x) : \mathbb{R}^{n-1} \to \mathbb{R}$ which satisfies

(1) $|\phi(s) - \phi(t)| \leqslant C(x)|s - t|$ for $s, t \in \mathbb{R}^{n-1}$ and $0 < C(x) < \infty$.
(2) $\Omega \cap U = \{(u, v) \mid v > \phi(u)\} \cap U$.

We assume that the Lipschitz domain Ω is bounded and connected. Define $\Omega_- := \Omega$ and $\Omega_+ := \mathbb{R}^3 \setminus \overline{\Omega}$.

Definition 2 (coordinate cylinder)**.** $Z \equiv Z(x,r)$, $x \in \Gamma$ is called a coordinate cylinder if

(1) $Z(x,r)$ is an open, right circular, doubly truncated cylinder centered at x with radius r.
(2) There exists a rectangular coordinate system (u,v) where $u \in \mathbb{R}^{n-1}$, $v \in \mathbb{R}$ such that the axis of Z is in the $v-$direction.
(3) There exists a function $\phi \equiv \phi(Z) : \mathbb{R}^{n-1} \to \mathbb{R}$ such that
   (a) $|\phi(s)-\phi(t)| \leqslant C(Z)|s-t|$ for s, $t \in \mathbb{R}^{n-1}$ and $0 < C(Z) < \infty$.
   (b) $\Omega \cap Z = \{(u,v) \mid v > \phi(u)\} \cap Z$.
   (c) $p = (0, \phi(0))$.
   (Z,ϕ) is called a coordinate pair.

Let $\mu Z := \mu Z(x,r)$, $\mu > 0$ be the dilation of Z about x by a factor of μ, i.e. $\mu Z = \{y \in \mathbb{R}^n \mid x + \frac{(y-x)}{\mu} \in Z\}$.

Definition 3 (regular family of cones)**.** Let $\zeta(x)$ with $x \in \Gamma$ be the open circular, doubly truncated cone centered at x with two nonempty, convex components in Ω_+ (denoted by $\zeta_+(x)$) and Ω_- (denoted by $\zeta_-(x)$) respectively.

$\{\zeta\} := \{\zeta(x) \mid \forall x \in \Gamma\}$ is called a regular family of cones if there exists a finite covering of Γ of coordinate cylinders such that for each coordinate pair $(Z(x,r),\phi)$ there exist three cones ζ_1, ζ_2, and ζ_3, all with vertex at 0, parallel to the axis of Z and satisfy

(1) $\zeta_1 \subset \overline{\zeta_2} \setminus \{0\} \subset \zeta_3$,
(2) For all $x \in \frac{4}{5}Z \cap \Gamma$ and $x \equiv (s, \phi(s))$
   (a) $\zeta_1 + x \subset \zeta(x) \subset \zeta_2 + x$,
   (b) $\zeta_3 + x \subset Z$,
   (c) $\{\frac{4}{5}Z\}$ covers Γ.

Proposition 1. Given a bounded Lipschitz domain Ω, the followings hold:

(1) There exists a regular family of cones $\{\zeta\}$.
(2) There exists a sequence of C^∞ domains $\Omega_i \subset \Omega$ and corresponding homeomorphisms $\Lambda_j : \Gamma \to \Gamma_i$ such that $\sup_{x\in\Gamma} |\Lambda_j(x) - x| \to 0$ as $j \to \infty$ and for all j and all $x \in \Gamma$, $\Lambda_j(x) \in \zeta(x)$.
(3) There exist positive functions $\omega_j : \Gamma \to \mathbb{R}^+$ bounded away from zero and infinity uniformly in j such that

(a) For any measurable set $V \subset \Gamma$

$$\int_V \omega_j \, d\sigma = \int_{\Lambda_j(V)} d\sigma_j.$$

(b) $\omega_j(x) \to 1$ pointwise a.e. for $x \in \Gamma$.

(4) $\nu(\Lambda_j(x)) \to \nu(x)$ pointwise a.e. for $x \in \Gamma$.

(5) There exists a real-valued C^∞ vector field h such that for all j and $x \in \Gamma$, $\nu(\Lambda_j(x)) \cdot h(\Lambda_j(x)) \geqslant \kappa > 0$, where κ depends on the Lipschitz character of Ω.

The approximation scheme described above is denoted by $\Omega_i \uparrow \Omega$. An analogous approximation scheme with C^∞ domains $\widetilde{\Omega}_i \supset \Omega$ exists and is denoted by $\widetilde{\Omega}_i \downarrow \Omega$.

Definition 4 (nontangential maximal function)**.** Given a function f in Ω and a regular family of cones $\{\zeta\}$, the nontangential maximal function f^* is defined by

$$f^*(x) = \sup\{|f(y)| \mid y \in \zeta(x), x \in \Gamma\}$$

Definition 5 (nontangential convergence)**.** We say that f converges nontangentially a.e. to u if for any regular family of cones $\{\zeta\}$

$$\lim_{\substack{y \to x \\ y \in \zeta(x)}} f(y) = u(x) \quad x \in \Gamma \text{ a.e.}$$

Let ν stands for the unit normal vector of Γ. Given a vector field E, the normal component $E_{\mathrm{n}} := (E \cdot \nu)\nu$, the tangential component $E_{\mathrm{t}} := E - E_{\mathrm{n}}$. Let ∇_{t} denotes the tangential differentiation

$$\nabla_{\mathrm{t}} := \nu \times (\nu \times \nabla).$$

A vector field a defined on Γ is called tangential if $a \cdot \nu = 0$ a.e. on Γ. The collection of complex-valued L_2-integrable tangential vector fields is denoted by $L_{2,\mathrm{t}}(\Gamma)$. The surface divergence $\operatorname{div}_\Gamma a$ of a given vector field a is defined as

$$\int_\Gamma \phi \operatorname{div}_\Gamma a \, d\sigma = -\int_\Gamma \nabla_{\mathrm{t}} \phi \cdot a \, d\sigma$$

for any $\phi \in C^\infty(\mathbb{R}^3)$. The function space $L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}$ is defined as

$$L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma} = \{a \in L_{2,\mathrm{t}}(\Gamma) \mid \operatorname{div}_\Gamma a \in L_2(\Gamma)\}.$$

Endowed with the norm

$$\|a\|_{L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}} := \|a\|_{L_2(\Gamma)} + \|\operatorname{div}_\Gamma a\|_{L_2(\Gamma)}$$

$L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}$ becomes a Banach space.

The notation $F \lesssim G$ means that, if there exists $C > 0$ such that for variables F, G, the inequality $F \leqslant CG$ holds uniformly. The notation $F \approx G$ means $F \lesssim G$ and $G \lesssim F$. The notation $\mathsf{K}(a)$ denotes a generic compact operator acting on a. Notations (small) and (large) stand for the positive constants which may be sufficiently small and large respectively. Note that the contant C appears in the inequalities generally depends on the underlying regular family of cones $\{\zeta\}$ (c.f. proposition 1, item (1)), the complex number k, $\kappa > 0$ and the L_∞ norms of h and ∇h in proposition 1, item (5).

Let $\Phi(x, y)$ denotes the fundamental solution of the Helmholtz operator $\triangle + k^2$ in $\mathbb{R}^3$:

$$\Phi(x, y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x - y|}.$$

Proposition 2 (Stratton-Chu formula)**.** Let E, H be vector fields defined in Ω such that E, $\operatorname{div} E$, $\operatorname{curl} E$ and H, $\operatorname{div} H$, $\operatorname{curl} H$ are in $L_p(\Omega)$ for a given p with $1 < p < \infty$, then the following identity holds for $x \in \Omega$ a.e.

$$\begin{aligned} E(x) = &- \operatorname{curl} \int_\Gamma \nu(y) \times E(y)\Phi(x, y)\, d\sigma(y) \\ &+ \nabla \int_\Gamma \nu(y) \cdot E(y)\Phi(x, y)\, d\sigma(y) \\ &- ik \int_\Gamma \nu(y) \times H(y)\Phi(x, y)\, d\sigma(y) \\ &+ \operatorname{curl} \int_\Omega \big\{\operatorname{curl} E(y) - ikH(y)\big\}\Phi(x, y)\, dV(y) \\ &- \nabla \int_\Omega \operatorname{div} E(y)\Phi(x, y)\, dV(y) \\ &+ ik \int_\Omega \big\{\operatorname{curl} H(y) + ikE(y)\big\}\Phi(x, y)\, dV(y). \end{aligned} \tag{2.1}$$

If E, H satisfy the above assumptions of Stratton-Chu formula and, in addition, Maxwell equations in Ω, then for $x \in \Omega$ we have

$$\begin{aligned} E(x) = &- \operatorname{curl} \int_\Gamma \nu(x) \times E(y)\Phi(x, y)\, d\sigma(y) \\ &- \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times H(y)\Phi(x, y)\, d\sigma(y) \end{aligned} \tag{2.2}$$

$$H(x) = -\operatorname{curl}\int_\Gamma \nu(x)\times H(y)\Phi(x,y)\,d\sigma(y) + \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times E(y)\Phi(x,y)\,d\sigma(y) \tag{2.3}$$

Definition 6 (Silver-Müller radiation condition)**.** Solutions E, H of Maxwell equations satisfy the Silver-Müller radiation condition if one of the following holds:

$$\lim_{|x|\to\infty}(x\times H + |x|E) = 0 \tag{2.4}$$

$$\lim_{|x|\to\infty}(x\times E - |x|H) = 0 \tag{2.5}$$

In each case the limit is hold uniformly in all directions $x/|x|$.

If E, H are defined in Ω_+ such that E, $\operatorname{div} E$, $\operatorname{curl} E$ and H, $\operatorname{div} H$, $\operatorname{curl} H$ are in $L_p(\Omega_+)$ for a given p with $1 < p < \infty$, satisfy Maxwell equations in Ω_+ and the Silver-Müller radiation condition, we have

$$E(x) = \operatorname{curl}\int_\Gamma \nu(x)\times E(y)\Phi(x,y)\,d\sigma(y) + \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times H(y)\Phi(x,y)\,d\sigma(y) \tag{2.6}$$

$$H(x) = \operatorname{curl}\int_\Gamma \nu(x)\times H(y)\Phi(x,y)\,d\sigma(y) - \frac{i}{k}\operatorname{curl}\operatorname{curl}\int_\Gamma \nu(y)\times E(y)\Phi(x,y)\,d\sigma(y). \tag{2.7}$$

Define the single layer potential $\mathcal{S}$ which acts on a scalar function $f \in L_2(\Gamma)$ for $x \in \mathbb{R}^3 \setminus \Gamma$ as

$$\mathcal{S}f(x) = \int_\Gamma \Phi(x,y)f(y)\,d\sigma(y).$$

In the sequel a stands for a $L_2(\Gamma)$ vector field.

Proposition 3.

$$\lim_{\substack{y\to x\\ y\in\zeta_+(x)}} \mathcal{S}f(y) = \lim_{\substack{y\to x\\ y\in\zeta_-(x)}} \mathcal{S}f(y) = \int_\Gamma \Phi(x,y)f(y)\,d\sigma(y) =: Sf(x)$$

Proposition 4.

$$\lim_{\substack{y\to x\\ y\in\zeta_\pm(x)}} \nabla\mathcal{S}f(y)\cdot\nu(x) = \big(\mp\frac{1}{2}I + K^*\big)f(x)$$

where K^* is the formal transpose of the bounded operator K defined as

$$Kf(x) := \frac{1}{4\pi}\lim_{\varepsilon\downarrow 0}\int_{|x-y|\geqslant\varepsilon}\frac{(x-y)\cdot\nu(y)}{|x-y|^3}e^{ik|x-y|}(1-ik|x-y|)f(y)\,d\sigma(y)$$

Proposition 5.

$$\|(\nabla Sf)^*\| \lesssim \|f\|$$

Proposition 6.

$$\lim_{\substack{y\to x\\ y\in\zeta_\pm(x)}} \operatorname{div}\mathcal{S}a(y) = \mp\frac{1}{2}\nu(x)\cdot a(x) + \operatorname{pv}\int_\Gamma \operatorname{div}_x\{\Phi(x,y)a(y)\}\,d\sigma(y)$$

$$\lim_{\substack{y\to x\\ y\in\zeta_\pm(x)}} \operatorname{curl}\mathcal{S}a(y) = \mp\frac{1}{2}\nu(x)\times a(x) + \operatorname{pv}\int_\Gamma \operatorname{curl}_x\{\Phi(x,y)a(y)\}\,d\sigma(y)$$

Proposition 7.

$$\lim_{\substack{y\to x\\ y\in\zeta_\pm(x)}} \nu(x)\times\operatorname{curl}\mathcal{S}a(y) = \pm\frac{1}{2}a(x) + \operatorname{pv}\int_\Gamma \nu(x)\times\operatorname{curl}_x\{\Phi(x,y)a(y)\}\,d\sigma(y)$$

$$\lim_{\substack{y_\pm\to x\\ y_\pm\in\zeta_\pm(x)}} \nu(x)\times\big(\operatorname{curl}\operatorname{curl}\mathcal{S}a(y_+) - \operatorname{curl}\operatorname{curl}\mathcal{S}a(y_-)\big) = 0$$

Proposition 7 motivates the definition of the magnetic dipole operator M and the electric dipole operator N for $x\in\Gamma$:

$$Ma(x) = \nu(x)\times\operatorname{pv}\int_\Gamma \operatorname{curl}_x\{\Phi(x,y)a(y)\}\,d\sigma(y)$$
$$Na(x) = \nu(x)\times\operatorname{pv}\int_\Gamma \operatorname{curl}\operatorname{curl}_x\{\Phi(x,y)a(y)\}\,d\sigma(y)$$

In the sequel we sometimes adopt the subscript convention, e.g. Φ_k, S_k, M_k, N_k, etc. to emphasize the dependence of k.

Proposition 8. For arbitrary γ_{L}, $\gamma_{\mathrm{R}}\in\mathbb{C}$,

(1) $M_{\gamma_{\mathrm{L}}} : L_{2,\mathrm{t}}(\Gamma)\to L_{2,\mathrm{t}}(\Gamma)$ is bounded.
(2) $M_{\gamma_{\mathrm{L}}} : L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}\to L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}$ is bounded.
(3) $N_{\gamma_{\mathrm{L}}} : L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}\to L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}$ is bounded.
(4) $M_{\gamma_{\mathrm{L}}} - M_{\gamma_{\mathrm{R}}} : L_{2,\mathrm{t}}(\Gamma)\to L_{2,\mathrm{t}}(\Gamma)$ is compact.
(5) $M_{\gamma_{\mathrm{L}}} - M_{\gamma_{\mathrm{R}}} : L_{2,\mathrm{t}}(\Gamma)\to L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}$ is compact.
(6) $N_{\gamma_{\mathrm{L}}} - N_{\gamma_{\mathrm{R}}} : L_{2,\mathrm{t}}(\Gamma)\to L_{2,\mathrm{t}}(\Gamma)$ is compact.
(7) $N_{\gamma_{\mathrm{L}}} - N_{\gamma_{\mathrm{R}}} : L_{2,\mathrm{t}}(\Gamma)\to L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}$ is bounded.

3. Spectral Theory of the Magnetic Dipole Operator M

In this section we set the following restriction on k:

$$k \in \mathbb{C} \setminus \{0\} \text{ and } \operatorname{Im} k \geqslant |\operatorname{Re} k|, \tag{3.1}$$

unless otherwise stated.

Lemma 1 (Rellich identity)**.** For a complex-valued $C^\infty(\overline{\Omega})$ vector field E and a real-valued $C^\infty(\mathbb{R}^3)$ vector field h

$$\begin{aligned}\int_\Gamma \Big\{\frac{1}{2}|E|^2(h\cdot\nu) - \operatorname{Re}\big((\overline{E}\cdot h)(E\cdot\nu)\big)\Big\}\, d\sigma \\ = \int_\Omega \operatorname{Re}\Big\{\frac{1}{2}|E|^2 \operatorname{div} h - (\overline{E}\cdot h)\operatorname{div} E - \overline{E}\cdot(\nabla h)E + (h\times\overline{E})\cdot\operatorname{curl} E\Big\}\, dV,\end{aligned} \tag{3.2}$$

where $\overline{E}\cdot(\nabla h)E$ denotes the quadratic form $\Sigma_{i,j}(D_i h_j)E_i\overline{E_j}$.

Proof. It is evident from

$$\begin{aligned}&\operatorname{div}\Big\{\frac{1}{2}|E|^2 h - \operatorname{Re}\big((\overline{E}\cdot h)E\big)\Big\} \\ &= \operatorname{Re}\Big\{\frac{1}{2}|E|^2 \operatorname{div} h - (\overline{E}\cdot h)\operatorname{div} E - \overline{E}\cdot(\nabla h)E + (h\times\overline{E})\cdot\operatorname{curl} E\Big\}\end{aligned}$$

and Divergence theorem. □

Lemma 2. For a complex-valued $C^\infty(\overline{\Omega})$ vector field E

$$\int_\Gamma |E|^2\, d\sigma \lesssim \int_\Gamma |E_{\mathrm{n}}|^2\, d\sigma + \int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2\, dV \tag{3.3}$$

$$\int_\Gamma |E|^2\, d\sigma \lesssim \int_\Gamma |E_{\mathrm{t}}|^2\, d\sigma + \int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2\, dV. \tag{3.4}$$

If $E \in C^\infty(\overline{\Omega_+})$ and decays at infinity then the above hold with Ω replaced by Ω_+.

Proof. Let h be the real-valued vector field which satisfies proposition 1, item (5), i.e. $h\cdot\nu \geqslant \kappa > 0$ on Γ. Decomposing E, h into mutually orthogonal parts $E = E_{\mathrm{t}} + E_{\mathrm{n}}$, $h = h_{\mathrm{t}} + h_{\mathrm{n}}$, we have

$$\begin{aligned}&\frac{1}{2}|E|^2(h\cdot\nu) - \operatorname{Re}\big((\overline{E}\cdot h)(E\cdot\nu)\big) \\ &\qquad = \frac{1}{2}|E_{\mathrm{t}}|^2(h\cdot\nu) - \frac{1}{2}|E_{\mathrm{n}}|^2(h\cdot\nu) - \operatorname{Re}\big((\overline{E_{\mathrm{t}}}\cdot h_{\mathrm{t}})(E_{\mathrm{n}}\cdot\nu)\big),\end{aligned}$$

thus the Rellich identity (3.2) is rewritten as

$$\int_\Gamma \frac{1}{2}|E_{\mathrm{t}}|^2(h\cdot\nu)\, d\sigma = \int_\Gamma \frac{1}{2}|E_{\mathrm{n}}|^2(h\cdot\nu)\, d\sigma + \Theta_1 + \Theta_2, \tag{3.5}$$

where

$$\Theta_1 := \int_\Gamma \operatorname{Re}\big((\overline{E_{\mathrm{t}}}\cdot h_{\mathrm{t}})(E_{\mathrm{n}}\cdot\nu)\big)\,d\sigma,$$

$$\Theta_2 := \int_\Omega \operatorname{Re}\Big\{\frac{1}{2}|E|^2 \operatorname{div} h - (\overline{E}\cdot h)\operatorname{div} E - \overline{E}\cdot(\nabla h)E + (h\times\overline{E})\cdot\operatorname{curl} E\Big\}\,dV$$

In view of (3.5) and $h\cdot\nu \geqslant \kappa > 0$ we have

$$\frac{1}{2}\kappa\int_\Gamma |E_{\mathrm{t}}|^2\,d\sigma \leqslant \frac{1}{2}\int_\Gamma |E_{\mathrm{n}}|^2\,d\sigma + \Theta_1 + \Theta_2. \tag{3.6}$$

By Young's inequality

$$ab \leqslant \varepsilon a^2 + \frac{1}{\varepsilon}b^2 \quad \forall\varepsilon > 0$$

(3.6) becomes

$$\int_\Gamma |E|^2\,d\sigma \lesssim \int_\Gamma |E_{\mathrm{n}}|^2\,d\sigma + \int_\Omega |E|^2 + |E||\operatorname{curl} E| + |E||\operatorname{div} E|\,dV \tag{3.7}$$

Similarly, from (3.5) and $h\cdot\nu \geqslant \kappa > 0$ we have

$$\begin{aligned}\frac{1}{2}\kappa\int_\Gamma |E_{\mathrm{n}}|^2\,d\sigma &\leqslant \frac{1}{2}\int_\Gamma |E_{\mathrm{t}}|^2\,d\sigma - \Theta_1 - \Theta_2\\ &\leqslant \frac{1}{2}\int_\Gamma |E_{\mathrm{t}}|^2\,d\sigma + |\Theta_1| + |\Theta_2|,\end{aligned} \tag{3.8}$$

hence by Young's inequality (3.8) becomes

$$\int_\Gamma |E|^2\,d\sigma \lesssim \int_\Gamma |E_{\mathrm{t}}|^2\,d\sigma + \int_\Omega |E|^2 + |E||\operatorname{curl} E| + |E||\operatorname{div} E|\,dV. \tag{3.9}$$

Once by Young's inequality

$$\int_\Omega |E|^2 + |E||\operatorname{curl} E| + |E||\operatorname{div} E|\,dV \lesssim \int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2\,dV,$$

and we may rewrite (3.7), (3.9) into (3.3), (3.4) respectively. □

Lemma 3. For the complex-valued $C^\infty(\overline{\Omega})$ vector field E which satisfies $(\triangle + k^2)E = 0$ in Ω,

$$\int_\Gamma |\nu\times E|^2\,d\sigma \lesssim \int_\Gamma |E\cdot\nu|^2 + |\operatorname{div} E|^2\,d\sigma + \Big|\int_\Gamma (\nu\times\overline{E})\cdot\operatorname{curl} E\,d\sigma\Big|. \tag{3.10}$$

Proof. Vector Green's theorem for vector fields a, b on Ω reads

$$\int_\Omega a\triangle b + \operatorname{curl} a\cdot\operatorname{curl} b + \operatorname{div} a\cdot\operatorname{div} b\,dV = \int_\Gamma (\nu\times a)\cdot\operatorname{curl} b + (\nu\cdot a)\operatorname{div} b\,d\sigma$$

Setting $a = \overline{E}$ and $b = E$ in vector Green's theorem we have

$$\int_\Gamma (\nu\times\overline{E})\cdot\operatorname{curl} E + (\overline{E}\cdot\nu)\operatorname{div} E\,d\sigma = \int_\Omega |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 - k^2|E|^2\,dV.$$

In view of the restriction on k (3.1), the above identity becomes

$$\begin{aligned}&\int_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2\,dV\\ &\qquad\lesssim \left|\int_\Gamma (\nu\times\overline{E})\cdot\operatorname{curl} E\,d\sigma\right| + \int_\Gamma |E\cdot\nu||\operatorname{div} E|\,d\sigma.\end{aligned}$$

Once by $|E\cdot\nu|\leqslant|E|$ and Young's inequality

$$\int_\Gamma |E\cdot\nu||\operatorname{div} E|\,d\sigma \leqslant (\text{small})\int_\Gamma |E|^2\,d\sigma + (\text{large})\int_\Gamma |\operatorname{div} E|^2\,d\sigma,$$

which turns (3.3) into (3.10). □

Theorem 1. Let X, Y, Z be Banach spaces and $A : X\to Y$ be a closed operator with dense domain. Then the followings are equivalent:

(1) A compact operator $T : X\to Z$ exists such that

$$\|x\|\leqslant C(\|Ax\| + \|Tx\|)\qquad \forall x\in\operatorname{dom} A. \tag{3.11}$$

(2) $\dim\ker A$ is finite, and $\operatorname{img} A$ is closed.

Proof of (1) *implies* (2). If $x\in\ker A$, by (3.11) we have $\|x\|\leqslant C\|Tx\|$, which implies that T^{-1} exists and is bounded. It follows that $I = T^{-1}T : \ker A\to\ker A$ is compact, so $\dim\ker A$ is finite.

Decompose X into direct sum

$$X = \widetilde{X}\oplus\ker A. \tag{3.12}$$

Extract a sequence $\{x_{\mathrm{n}}\}$ from $\widetilde{X}$ and let $Ax_{\mathrm{n}}\to y$. We claim that there exists $M>0$ such that $\|x_{\mathrm{n}}\|\leqslant M$ for all n. Assume the contrary; $\|x_{\mathrm{n}}\|\to\infty$. Let $x'_{\mathrm{n}} = \frac{x_{\mathrm{n}}}{\|x_{\mathrm{n}}\|}$, then $\|x'_{\mathrm{n}}\| = 1$, $Ax'_{\mathrm{n}}\to 0$. Since $\|x'_{\mathrm{n}}\| = 1$, there exists a subsequence $\{x'_{n_k}\}$ of $\{x'_{\mathrm{n}}\}$ such that $\{Tx'_{n_k}\}$ is a Cauchy sequence. Hence $\{x'_{n_k}\}$ has a limit $x\in\widetilde{X}$ and $\|x\| = 1$. Since $x\in\operatorname{dom} A$ and A is closed, $Ax = 0$. But $\widetilde{X}\cap\ker A = \{0\}$ implies $x = 0$, a contradiction.

From compactness of T and (3.11) we can extract a convergent subsequence $\{x_{n_k}\}$ from $\{x_{\mathrm{n}}\}$. If $x_{n_k}\to x$ and $Ax_{n_k}\to y$, then $x\in\operatorname{dom} A$ and $Ax = y$, which implies $\operatorname{img} A$ is closed. □

Proof of (2) *implies* (1). Decompose X into direct sum (3.12), denote by P the projection of X onto $\ker A$ parallel to $\widetilde{X}$, and define $\widetilde{A}$ as the restriction of the operator A to $\widetilde{X}$. Then $\ker \widetilde{A} = \{0\}$ and $\operatorname{img} \widetilde{A} = \operatorname{img} A$. $\widetilde{A}^{-1}$ is a closed operator on the Banach space $\operatorname{img} A$ and therefore is bounded; a constant $C' > 0$ exists such that $\forall \widetilde{x} \in \widetilde{X}$

$$\|\widetilde{x}\| \leqslant C' \|\widetilde{A}\widetilde{x}\|.$$

Given an arbitrary $x \in D(A)$, by (3.12) we have $x = x_1 + x_2$ with $x_1 = (I - P)x \in \widetilde{X}$ and $x_2 = Px \in \ker A$. The operator $T = P$ has finite rank and therefore is compact. Since $Ax = \widetilde{A}\widetilde{x}$, we have

$$\|x\| \leqslant \|x_1\| + \|x_2\| \leqslant C(\|Ax\| + \|Tx\|)$$

with $C = \max(1, C')$. □

Theorem 2. $\pm\frac{1}{2}I + M : L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma} \to L_{2,\mathrm{t}}(\Gamma)$ is injective with closed range.

Proof. From lemma 2, we have

$$\begin{aligned} \|E\|_{L_2(\Gamma)} &\lesssim \|E_{\mathrm{n}}\|_{L_2(\Gamma)} + \|(\operatorname{curl} E)_{\mathrm{t}}\|_{L_2(\Gamma)} + \|\operatorname{div} E\|_{L_2(\Gamma)}, \\ \|E\|_{L_2(\Gamma)} &\lesssim \|E_{\mathrm{t}}\|_{L_2(\Gamma)} + \|(\operatorname{curl} E)_{\mathrm{t}}\|_{L_2(\Gamma)} + \|\operatorname{div} E\|_{L_2(\Gamma)}. \end{aligned} \tag{3.13}$$

Suppose $(\triangle + k^2)E = 0$ and additionally $\operatorname{div} E = 0$. By writing $H = \frac{1}{ik}\operatorname{curl} E$, (3.13) becomes

$$\|E\|_{L_2(\Gamma)} \lesssim \|E_{\mathrm{n}}\|_{L_2(\Gamma)} + \|H_{\mathrm{t}}\|_{L_2(\Gamma)}, \tag{3.14}$$

$$\|E\|_{L_2(\Gamma)} \lesssim \|E_{\mathrm{t}}\|_{L_2(\Gamma)} + \|H_{\mathrm{t}}\|_{L_2(\Gamma)}. \tag{3.15}$$

From $\operatorname{curl}\operatorname{curl} E = -\triangle E + \nabla \operatorname{div} E$ we are free to permute E and H in (3.14), (3.15) and obtain

$$\|H\|_{L_2(\Gamma)} \lesssim \|H_{\mathrm{n}}\|_{L_2(\Gamma)} + \|E_{\mathrm{t}}\|_{L_2(\Gamma)}, \tag{3.16}$$

$$\|H\|_{L_2(\Gamma)} \lesssim \|H_{\mathrm{t}}\|_{L_2(\Gamma)} + \|E_{\mathrm{t}}\|_{L_2(\Gamma)}. \tag{3.17}$$

By (3.15) and (3.16),

$$\begin{aligned} \|E\|_{L_2(\Gamma)} &\lesssim \|E_{\mathrm{t}}\|_{L_2(\Gamma)} + \|H_{\mathrm{t}}\|_{L_2(\Gamma)} \\ &\lesssim \|E_{\mathrm{t}}\|_{L_2(\Gamma)} + \|H_{\mathrm{t}}\|_{L_2(\Gamma)} + \|H_{\mathrm{n}}\|_{L_2(\Gamma)} \\ &\lesssim \|E_{\mathrm{t}}\|_{L_2(\Gamma)} + \|H\|_{L_2(\Gamma)} \\ &\lesssim \|E_{\mathrm{t}}\|_{L_2(\Gamma)} + \|H_{\mathrm{n}}\|_{L_2(\Gamma)} + \|E_{\mathrm{t}}\|_{L_2(\Gamma)} \\ &\lesssim \|H_{\mathrm{n}}\|_{L_2(\Gamma)} + \|E_{\mathrm{t}}\|_{L_2(\Gamma)}. \end{aligned} \tag{3.18}$$

From (3.18), (3.16) and $\|E_{\mathrm{t}}\|_{L_2(\Gamma)} + \|H_{\mathrm{n}}\|_{L_2(\Gamma)} \lesssim \|E\|_{L_2(\Gamma)} + \|H\|_{L_2(\Gamma)}$, we have

$$\|E\|_{L_2(\Gamma)} + \|H\|_{L_2(\Gamma)} \approx \|E_{\mathrm{t}}\|_{L_2(\Gamma)} + \|H_{\mathrm{n}}\|_{L_2(\Gamma)}. \tag{3.19}$$

Once by permutting E and H in (3.19) we have

$$\|H\|_{L_2(\Gamma)} + \|E\|_{L_2(\Gamma)} \approx \|H_{\mathrm{t}}\|_{L_2(\Gamma)} + \|E_{\mathrm{n}}\|_{L_2(\Gamma)}, \tag{3.20}$$

hence (3.19), (3.20) amount to

$$\|E_{\mathrm{t}}\|_{L_2(\Gamma)} + \|H_{\mathrm{n}}\|_{L_2(\Gamma)} \approx \|E_{\mathrm{n}}\|_{L_2(\Gamma)} + \|H_{\mathrm{t}}\|_{L_2(\Gamma)}. \tag{3.21}$$

By $\|\cdot\|_{L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}} \equiv \|\cdot\|_{L_2(\Gamma)} + \|\operatorname{div}_\Gamma(\cdot)\|_{L_2(\Gamma)}$ and $\operatorname{div}_\Gamma(\nu \times E) = -\nu \cdot \operatorname{curl} E$, (3.21) is written as

$$\|\nu \times E\|_{L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}} \approx \|\nu \times \operatorname{curl} E\|_{L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}}$$

Now set $E := \operatorname{curl} \mathcal{S}a$ in $\mathbb{R}^3 \setminus \Gamma$ and apply proposition 7, we have

$$\|(\tfrac{1}{2}I + M)a\|_{L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}} \approx \|Na\|_{L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}} \approx \|(-\tfrac{1}{2}I + M)a\|_{L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}}.$$

Together with $a = (\frac{1}{2}I + M)a - (-\frac{1}{2}I + M)a$ we obtain

$$\|(\pm\tfrac{1}{2}I + M)a\|_{L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}} \approx \|a\|_{L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}},$$

which completes the proof. □

Lemma 4. For $f \in L_2(\Gamma)$ and $\lambda \in \mathbb{R}$, $|\lambda| > \frac{1}{2}$,

$$\|f\|_{L_2(\Gamma)} \leqslant C_\lambda\big(\|(\lambda I - K^*)f\|_{L_2(\Gamma)} + \|\,\mathsf{K}(f)\|_{L_2(\Gamma)}\big). \tag{3.22}$$

Proof. It suffices to prove the case $k = 0$ and the general validity follows from the compactness of $K^* - K_0^*$. Set $E = \nabla \mathcal{S}_0 f$ in Ω and let $T := \lambda I - K^*$, from $\operatorname{div} E = \operatorname{curl} E = 0$ and $|E \cdot \nu| \leqslant |E|$ on Γ, the Rellich identity (3.2) becomes

(3.23)

$$\begin{aligned}
&\int_\Gamma \frac{1}{2}|E\cdot\nu|^2 (h\cdot\nu)\, d\sigma \leqslant \int_\Gamma \frac{1}{2}|E|^2 (h\cdot\nu)\, d\sigma \\
&= \int_\Gamma \operatorname{Re}\big((\overline{E}\cdot h)(E\cdot\nu)\big)\, d\sigma + \int_\Omega \operatorname{Re}\Big\{\frac{1}{2}|E|^2 \operatorname{div} h - \overline{E}\cdot(\nabla h)E\Big\}\, dV
\end{aligned}$$

Note that on Γ

$$\begin{aligned}
E\cdot\nu &= (\lambda - \tfrac{1}{2})f - T(f) \\
E\cdot h &= -\frac{1}{2}(h\cdot\nu)f + \widetilde{K}f
\end{aligned}$$

where

$$\widetilde{K}f(x) = \int_\Gamma \frac{\partial \Phi_0(x,y)}{\partial h(x)} f(y)\, d\sigma(y).$$

From Green's theorem

$$\int_\Omega |E|^2\, dV = \int_\Gamma S_0(f)\cdot \overline{\frac{\partial S_0 f}{\partial \nu}}\, d\sigma = \int_\Gamma S_0(f)\cdot \overline{\big((\lambda - \tfrac{1}{2})f - T(f)\big)}\, d\sigma,$$

(3.23) becomes

$$
\begin{aligned}
\frac{1}{2}\int_\Gamma &(h\cdot\nu)|\lambda-\tfrac{1}{2}|^2|f|^2\,d\sigma \\
&\leqslant \int_\Gamma \mathrm{Re}\Big\{\overline{\big(\frac{1}{2}(h\cdot\nu)f+\widetilde{K}(f)\big)}\big((\lambda-\tfrac{1}{2})f-T(f)\big)\Big\}\,d\sigma \\
&\qquad + C_1\|f\|(\|S(f)\|+\|T(f)\|)+C_2\|S(f)\|\|T(f)\|,
\end{aligned}
$$

which is further simplified to

$$
\begin{aligned}
\frac{1}{2}\big(|\lambda|^2-\frac{1}{4}\big)&\int_\Gamma(h\cdot\nu)|f|^2\,d\sigma \\
&\leqslant \int_\Gamma \mathrm{Re}\Big\{\overline{\widetilde{K}(f)}\cdot(\lambda-\tfrac{1}{2})f\Big\}\,d\sigma \\
&\quad + C_1\|f\|(\|S(f)\|+\|T(f)\|)+C_2\|S(f)\|\|T(f)\|.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_\Gamma \mathrm{Re}\Big\{\overline{\widetilde{K}(f)}\cdot(\lambda-\frac{1}{2})f\Big\}\,d\sigma \\
&= \mathrm{Re}(\lambda-\tfrac{1}{2})\int_\Gamma \mathrm{Re}\Big\{\overline{\widetilde{K}(f)}\cdot f\Big\}\,d\sigma - \mathrm{Im}(\lambda-\frac{1}{2})\int_\Gamma \mathrm{Im}\Big\{\overline{\widetilde{K}(f)}\cdot f\Big\}\,d\sigma
\end{aligned}
$$

and

$$
\int_\Gamma \mathrm{Re}\Big\{\overline{\widetilde{K}(f)}\cdot f\Big\}\,d\sigma = \frac{1}{2}\int_\Gamma fQ(\overline{f})\,d\sigma,
$$

where $Q := \widetilde{K}+\widetilde{K}^*$. Q is an operator with weakly singular kernel, hence is compact. □

Theorem 3. For $a\in L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}$ and $\lambda\in\mathbb{R}$, $|\lambda|>\frac{1}{2}$,

$$
\begin{aligned}
\|a\|_{L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}} \leqslant C_\lambda\big(&\|(\lambda I+M)a\|_{L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}}+\|\,\mathsf{K}(\mathrm{div}_\Gamma a)\|_{L_2(\Gamma)} \\
&+\|\,\mathsf{K}(a)\|_{L_2(\Gamma)}\big).
\end{aligned} \tag{3.24}
$$

Proof. From

$$
\mathrm{div}_\Gamma Ma = -K^*\,\mathrm{div}_\Gamma a - k^2\nu\cdot Sa
$$

for $a\in L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}$, we have

$$
(\lambda I-K^*)\,\mathrm{div}_\Gamma a = \mathrm{div}_\Gamma(\lambda I+M)a + k^2\nu\cdot Sa.
$$

Applying lemma 4 and the compactness of S we obtain

$$
\begin{aligned}
\|\,\mathrm{div}_\Gamma a\|_{L_2(\Gamma)} \lesssim \|\,&\mathrm{div}_\Gamma(\lambda I+M)a\|_{L_2(\Gamma)} \\
&+\|\,\mathsf{K}(a)\|_{L_2(\Gamma)}+\|\,\mathsf{K}(\mathrm{div}_\Gamma a)\|_{L_2(\Gamma)}.
\end{aligned} \tag{3.25}
$$

Set $E = \operatorname{curl} \mathcal{S}a$ in Ω, we have

$$
\begin{aligned}
|\lambda + \frac{1}{2}| \|a\|_{L_2(\Gamma)} &\leqslant \|(\lambda I + M)a\|_{L_2(\Gamma)} + \|(-\frac{1}{2}I + M)a\|_{L_2(\Gamma)} \\
&= \|(\lambda I + M)a\|_{L_2(\Gamma)} + \|\nu \times E\|_{L_2(\Gamma)}.
\end{aligned}
$$

In view of lemma 3,

$$
\begin{aligned}
\|a\|_{L_2(\Gamma)} \lesssim \|(\lambda I + M)a\|_{L_2(\Gamma)} &+ \|E \cdot \nu\|_{L_2(\Gamma)} \\
&+ \Big| \int_\Gamma (\nu \times \overline{E}) \cdot \operatorname{curl} E \, d\sigma \Big|
\end{aligned}
\tag{3.26}
$$

From Stratton-Chu formula (2.1),

$$
K^*(E \cdot \nu) = -\nu \cdot \operatorname{curl} S(Ma) + \mathsf{K}(A) + \mathsf{K}(\operatorname{div}_\Gamma a),
$$

then

$$
(\lambda I - K^*)(E \cdot \nu) = \nu \cdot \operatorname{curl} S\big((\lambda I + M)a\big) + \mathsf{K}(A) + \mathsf{K}(\operatorname{div}_\Gamma a).
$$

Once by lemma 4 we have

$$
\begin{aligned}
\|E \cdot \nu\|_{L_2(\Gamma)} \lesssim \|(\lambda I + M)a\|_{L_2(\Gamma)} & \\
&+ \| \mathsf{K}(a)\|_{L_2(\Gamma)} + \| \mathsf{K}(\operatorname{div}_\Gamma a)\|_{L_2(\Gamma)}.
\end{aligned}
\tag{3.27}
$$

On Γ_-, $\nu \times E = (-\frac{1}{2}I + M)a = (-\frac{1}{2} - \lambda)a + (\lambda I + M)a$ is tangential; the tangential component of $\operatorname{curl} E$ is $k^2 Sa + \nabla S(\operatorname{div}_\Gamma a)$, hence

$$
\begin{aligned}
&\Big| \int_\Gamma (\nu \times \overline{E}) \cdot \operatorname{curl} E \, d\sigma \Big| \\
&\lesssim \Big| \int_\Gamma \overline{a} \cdot Sa \, d\sigma \Big| + \Big| \int_\Gamma \overline{a} \cdot \nabla S(\operatorname{div}_\Gamma a) \, d\sigma \Big| + \Big| \int_\Gamma \overline{(\lambda I + M)a} \cdot Sa \, d\sigma \Big| \\
&+ \Big| \int_\Gamma \overline{(\lambda I + M)a} \cdot \nabla S(\operatorname{div}_\Gamma a) \, d\sigma \Big| \\
&:= I + II + III + IV
\end{aligned}
$$

Applying Young's inequality,

$$
\begin{aligned}
I &\leqslant (\text{small})\|a\|^2_{L_2(\Gamma)} + (\text{large})\|Sa\|^2_{L_2(\Gamma)}, \\
II &= \Big| \int_\Gamma \overline{a} \cdot \nabla S(\operatorname{div}_\Gamma a) \, d\sigma \Big| = \Big| \int_\Gamma (\operatorname{div}_\Gamma \overline{a}) S(\operatorname{div}_\Gamma a) \, d\sigma \Big| \\
&\leqslant (\text{small})\| \operatorname{div}_\Gamma a\|^2_{L_2(\Gamma)} + (\text{large})\|S(\operatorname{div}_\Gamma a)\|^2_{L_2(\Gamma)}, \\
III &\leqslant (\text{small})\|a\|^2_{L_2(\Gamma)} + (\text{large})\|(\lambda I + M)a\|^2_{L_2(\Gamma)}, \\
IV &\leqslant (\text{small})\| \operatorname{div}_\Gamma a\|^2_{L_2(\Gamma)} + (\text{large})\|(\lambda I + M)a\|^2_{L_2(\Gamma)}
\end{aligned}
$$

By (3.27) and above results, (3.26) becomes

$$\begin{aligned}\|a\|_{L_2(\Gamma)} \lesssim \|(\lambda I + M)a\|_{L_2(\Gamma)} + \|\,\mathsf{K}(a)\|_{L_2(\Gamma)} + \|\,\mathsf{K}(\operatorname{div}_\Gamma a)\|_{L_2(\Gamma)} \\ + (\text{small})\|a\|_{L_2(\Gamma)} + (\text{small})\|\operatorname{div}_\Gamma a\|_{L_2(\Gamma)}\end{aligned}$$

Together with (3.25) the result follows. □

Theorem 4. $\lambda I + M : L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma} \to L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}$ is injective if $\lambda \in \mathbb{C}$ and $|\lambda| > 1/2$.

Proof. Let $\lambda \in \mathbb{C}$ be the eigenvalue of M and $a \in L_{2,\mathrm{t}}^{\operatorname{div}_\Gamma}$ be the corresponding eigenvector. Set $E = \operatorname{curl} \mathcal{S}a$ on $\mathbb{R}^3 \setminus \Gamma$, we have

$$\pm \int_{\Gamma_\pm} (\nu \times \overline{E}) \cdot \operatorname{curl} E \, d\sigma = \int_{\Omega_\pm} |\operatorname{curl} E|^2 - k^2 |E|^2 \, dV.$$

On $\Gamma_\pm$, $\nu \times \overline{E} = (\pm\frac{1}{2} + M)a = (\lambda \pm \frac{1}{2})a$ and hence is tangential; together with the fact that $(\operatorname{curl} E)_{\mathrm{t}}$ is continuous across $\Gamma_\pm$ and set

$$\mu_\pm := \int_{\Omega_\pm} |\operatorname{curl} E|^2 - k^2 |E|^2 \, dV$$

we get

$$|\lambda| = \frac{1}{2}\left|\frac{\mu_+ - \mu_-}{\mu_+ + \mu_-}\right|.$$

The restriction of k (3.1) implies that $\mu_\pm$ are in the same quadrant of $\mathbb{C}$, therefore $|\lambda| \leqslant \frac{1}{2}$. □

Theorem 5. Let W be a Hilbert space, X be a connected topological space and $T_\lambda : X \to \mathcal{L}(W)$, a continuous function from X to the set of all bounded operators on W. Assume that T_λ is injective with closed range for each $\lambda \in X$. If for some $\lambda_0 \in X$ the operator T_{λ_0} is actually an isomorphism of W, then T_λ is an isomorphism of W for any $\lambda \in X$.

Proof. Let $O := \{\lambda \in X \mid T_\lambda \text{ is invertible}\}$; O is nonempty because $\lambda_0 \in O$. To establish that $O = X$ we have to show that O is closed. Suppose $\lambda_j \in O$ such that $\lambda_j \to \lambda$. Let $u \in W$ and take $x_j \in W$ such that $T_{\lambda_j} x_j = u$ for each j. We claim that $\sup \|x_j\| < \infty$. From assumption and the open-mapping theorem, there exists an positive constant C_λ with $\|x\| \leqslant C_\lambda \|T_\lambda x\|$ for all $x \in W$. We have

$$\|x_j\| \leqslant C_\lambda \|T_\lambda x_j\| \leqslant C_\lambda \|T_{\lambda_j} x_j\| + C_\lambda \|(T_{\lambda_j} - T_\lambda) x_j\|$$

As $\|(T_{\lambda_j} - T_\lambda)x_j\| \leqslant \|T_{\lambda_j} - T_\lambda\| \|x_j\|$, the coefficient of $\|x_j\|$ becomes, for large j, small enough to be absorbed into the left-hand side, hence the claim is established.

From $\sup \|x_j\| < \infty$ we can find a subsequence of $\{x_j\}$ weakly convergent to some $x \in W$ and deduce that $T_\lambda x = u$, i.e. T_λ is invertible. □

Theorem 6. $\pm\frac{1}{2}I + M : L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma} \to L_{2,\mathrm{t}}(\Gamma)$ is invertible.

Proof. If k satisfies (3.1), from theorems 1, 2, 3 and 4, $\lambda I + M_k : L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma} \to L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}$ with $\lambda \in \mathbb{R}$, $|\lambda| \leqslant \frac{1}{2}$ is injective with closed range. Moreover, $\lambda I + M_k$ is invertible on $L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}$ for sufficiently large $|\lambda|$. By theorem 4, $\lambda I + M_k$ with $\lambda \in \mathbb{R}$, $|\lambda| \leqslant \frac{1}{2}$ is invertible. For general k the same conclusion follows from proposition 8, item (5) the decomposition

$$\lambda I + M_k = \lambda I + M_{k_0} + (M_k - M_{k_0}),$$

where k_0 satisfies (3.1). In particular, $\pm\frac{1}{2}I + M : L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma} \to L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}$ is invertible. □

4. Statement of the Direct Scattering Problem

Find the function E which satisfies

$$\begin{cases} \operatorname{curl}\operatorname{curl} E - 2\beta\gamma^2 \operatorname{curl} E - \gamma^2 E = 0 & \text{on } \Omega_+ \\ \nu \times E = 0 & \text{on } \Gamma \\ \frac{x}{|x|} \times H + E = o(|x|^{-1}) & |x| \to \infty \\ E^* \in L_2(\Gamma) \end{cases} \tag{4.1}$$

where $\gamma^2 = \frac{k^2}{1-k^2\beta^2}$ and k, H are defined by (1.2), (1.1) respectively.

Introducing the transformation (1.4), the direct scattering problem (4.1) is transformed to the following: Find Q_{L}, Q_{R} which satisfy

$$\begin{cases} \operatorname{curl} Q_{\mathrm{L}} = \gamma_{\mathrm{L}} Q_{\mathrm{L}} & \text{on } \Omega_+ \\ \operatorname{curl} Q_{\mathrm{R}} = -\gamma_{\mathrm{R}} Q_{\mathrm{R}} & \text{on } \Omega_+ \\ \nu \times (Q_{\mathrm{L}} + Q_{\mathrm{R}}) = f \in L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma} & \text{on } \Gamma_+ \\ \frac{x}{|x|} \times Q_{\mathrm{L}} + iQ_{\mathrm{L}} = o(|x|^{-1}) & |x| \to \infty \\ \frac{x}{|x|} \times Q_{\mathrm{R}} - iQ_{\mathrm{R}} = o(|x|^{-1}) & |x| \to \infty \\ Q_{\mathrm{L}}^*, Q_{\mathrm{R}}^* \in L_2(\Gamma) \end{cases} \tag{4.2}$$

where γ_{L}, γ_{R} are defined as in (1.5).

Inspired by Stratton-Chu formula, we propose the following ansatz

$$\begin{aligned} Q_{\mathrm{L}} &= \gamma_{\mathrm{L}} \operatorname{curl} \mathcal{S}_{\gamma_{\mathrm{L}}} a + \operatorname{curl}\operatorname{curl} \mathcal{S}_{\gamma_{\mathrm{L}}} a \\ Q_{\mathrm{R}} &= \gamma_{\mathrm{R}} \operatorname{curl} \mathcal{S}_{\gamma_{\mathrm{R}}} a - \operatorname{curl}\operatorname{curl} \mathcal{S}_{\gamma_{\mathrm{R}}} a, \end{aligned} \tag{4.3}$$

where $a \in L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}$ to be determined. Then (4.2) is reduced to the solution of the following boundary integral equation of unknown a:

$$\frac{1}{2}\gamma_{\mathrm{L}} a + \frac{1}{2}\gamma_{\mathrm{L}} M_{\gamma_{\mathrm{L}}} a + N_{\gamma_{\mathrm{L}}} a + \frac{1}{2}\gamma_{\mathrm{R}} a + \frac{1}{2}\gamma_{\mathrm{R}} M_{\gamma_{\mathrm{R}}} a - N_{\gamma_{\mathrm{R}}} a = f \tag{4.4}$$

5. Solvability of the Direct Scattering Problem

Theorem 7. The boundary integral equation (4.4) has an unique solution.

Proof of Existence. Let $T := (\gamma_{\mathrm{L}} + \gamma_{\mathrm{R}})\left(\frac{1}{2}I + M_{\gamma_{\mathrm{L}}}\right) + \gamma_{\mathrm{R}}(M_{\gamma_{\mathrm{R}}} - M_{\gamma_{\mathrm{L}}}) + (N_{\gamma_{\mathrm{L}}} - N_{\gamma_{\mathrm{R}}})$, then (4.4) is rearranged as $Ta = f$. We claim that, if $a \in L_{2,\mathrm{t}}(\Gamma)$ and $(\frac{1}{2}I + M)a \in L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}$, then $a \in L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}$. This is seen from

$$(-\tfrac{1}{2}I + K^*)(\mathrm{div}_\Gamma\, a) = -k^2\nu \cdot Sa - \mathrm{div}_\Gamma\big((\tfrac{1}{2}I + M)a\big)$$

and the invertibility of $(-\frac{1}{2}I + K^*)$. From this claim and the fact that the operator T is Fredholm with index zero on $L_{2,\mathrm{t}}(\Gamma)$, T is Fredholm with index zero on $L_{2,\mathrm{t}}^{\mathrm{div}_\Gamma}$. □

Proof of Uniqueness. We first show that the only solution for the homogeneous problem, i.e. (4.2) with $f = 0$, is $Q_{\mathrm{L}} = Q_{\mathrm{R}} = 0$. Let B_r be an open ball centered at 0 with radius r such that $\Omega \subset B_r$. From Silver-Müller radiation condition of Q_{L}, we have

$$\lim_{r\to\infty} \int_{\partial B_r} |\nu \times Q_{\mathrm{L}}|^2 + |Q_{\mathrm{L}}|^2 + 2\ \mathrm{Im}\Big\{\big(\nu \times Q_{\mathrm{L}}\big) \cdot \overline{Q_{\mathrm{L}}}\Big\}\, d\sigma = 0 \tag{5.1}$$

By Gauss divergence theorem and vector identities

$$(a \times b) \cdot c = (b \times c) \cdot a, \qquad \mathrm{div}(a \times b) = b \cdot \mathrm{curl}\, a - a \cdot \mathrm{curl}\, b,$$

we have

$$\begin{aligned}
\int_{\partial B_r} \big(\nu \times Q_{\mathrm{L}}\big) \cdot \overline{Q_{\mathrm{L}}}\, d\sigma &= \int_{\partial B_r} \big(Q_{\mathrm{L}} \times \overline{Q_{\mathrm{L}}}\big) \cdot \nu\, d\sigma \\
&= \int_{B_r\setminus\overline{\Omega}} \mathrm{div}\big(Q_{\mathrm{L}} \times \overline{Q_{\mathrm{L}}}\big)\, dV + \int_\Gamma \big(Q_{\mathrm{L}} \times \overline{Q_{\mathrm{L}}}\big) \cdot \nu\, d\sigma \\
&= \int_{B_r\setminus\overline{\Omega}} \mathrm{div}\big(Q_{\mathrm{L}} \times \overline{Q_{\mathrm{L}}}\big)\, dV + \int_\Gamma \big(Q_{\mathrm{R}} \times \overline{Q_{\mathrm{R}}}\big) \cdot \nu\, d\sigma \\
&= \int_{\partial B_r} \big(Q_{\mathrm{R}} \times \overline{Q_{\mathrm{R}}}\big) \cdot \nu\, d\sigma + 2i\, \mathrm{Im}\, \gamma_{\mathrm{R}} \int_{B_r\setminus\overline{\Omega}} |Q_{\mathrm{R}}|^2\, dV \\
&\quad + 2i\, \mathrm{Im}\, \gamma_{\mathrm{L}} \int_{B_r\setminus\overline{\Omega}} |Q_{\mathrm{L}}|^2\, dV
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\partial B_r} \mathrm{Im}\Big\{\big(\nu \times Q_{\mathrm{L}}\big) \cdot \overline{Q_{\mathrm{L}}} - \big(\nu \times Q_{\mathrm{R}}\big) \cdot \overline{Q_{\mathrm{R}}}\Big\}\, d\sigma \\
&\qquad = 2\, \mathrm{Im}\, \gamma_{\mathrm{R}} \int_{B_r\setminus\overline{\Omega}} |Q_{\mathrm{R}}|^2\, dV + 2\, \mathrm{Im}\, \gamma_{\mathrm{L}} \int_{B_r\setminus\overline{\Omega}} |Q_{\mathrm{L}}|^2\, dV.
\end{aligned} \tag{5.2}$$

From Silver-Müller radiation condition of Q_{R}, we have

$$\lim_{r\to\infty}\int_{\partial B_r} |\nu\times Q_{\mathrm{R}}|^2 + |Q_{\mathrm{R}}|^2 - 2\ \mathrm{Im}\Big\{\big(\nu\times Q_{\mathrm{R}}\big)\cdot\overline{Q_{\mathrm{R}}}\Big\}\,d\sigma = 0 \tag{5.3}$$

Add (5.3) and (5.1), by (5.2) we have

$$\begin{aligned}\lim_{r\to\infty}\bigg\{\int_{\partial B_r} &|\nu\times Q_{\mathrm{R}}|^2 + |Q_{\mathrm{R}}|^2 + |\nu\times Q_{\mathrm{R}}|^2 + |Q_{\mathrm{L}}|^2\,d\sigma \\ &+ 4\,\mathrm{Im}\,\gamma_{\mathrm{R}}\int_{B_r\setminus\overline{\Omega}}|Q_{\mathrm{R}}|^2\,dV + 4\,\mathrm{Im}\,\gamma_{\mathrm{L}}\int_{B_r\setminus\overline{\Omega}}|Q_{\mathrm{L}}|^2\,dV\bigg\} = 0\end{aligned}$$

By assumption $\mathrm{Im}\,\gamma_{\mathrm{L}}$, $\mathrm{Im}\,\gamma_{\mathrm{R}}$ are nonnegative, so we can deduce that

$$\lim_{r\to\infty}\int_{\partial B_r}|Q_{\mathrm{R}}|^2\,dV = 0$$
$$\lim_{r\to\infty}\int_{\partial B_r}|Q_{\mathrm{L}}|^2\,dV = 0$$

Note that Q_{L}, Q_{R} satisfy the Helmholtz equation; by Rellich's lemma $Q_{\mathrm{L}} = Q_{\mathrm{R}} = 0$ in Ω_+, hence

$$\nu\times Q_{\mathrm{L}} = \gamma_{\mathrm{L}}M_{\gamma_{\mathrm{L}}}a + \tfrac{1}{2}\gamma_{\mathrm{L}}a + N_{\gamma_{\mathrm{L}}}a = 0 \tag{5.4}$$
$$\nu\times Q_{\mathrm{R}} = \gamma_{\mathrm{R}}M_{\gamma_{\mathrm{R}}} + \tfrac{1}{2}\gamma_{\mathrm{R}}a - N_{\gamma_{\mathrm{R}}}a = 0 \tag{5.5}$$

Define

$$Q_1 = -\gamma_{\mathrm{L}}\,\mathrm{curl}\,\mathcal{S}_{\gamma_{\mathrm{L}}}a - \mathrm{curl}\,\mathrm{curl}\,\mathcal{S}_{\gamma_{\mathrm{L}}}a$$
$$Q_2 = -\gamma_{\mathrm{R}}\,\mathrm{curl}\,\mathcal{S}_{\gamma_{\mathrm{R}}}a + \mathrm{curl}\,\mathrm{curl}\,\mathcal{S}_{\gamma_{\mathrm{R}}}a$$

in Ω. From (5.4), (5.5) and the expressions

$$\nu\times Q_1 = -\gamma_{\mathrm{L}}M_{\gamma_{\mathrm{L}}}a + \tfrac{1}{2}\gamma_{\mathrm{L}}a - N_{\gamma_{\mathrm{L}}}a$$
$$\nu\times Q_2 = -\gamma_{\mathrm{R}}M_{\gamma_{\mathrm{R}}}a + \tfrac{1}{2}\gamma_{\mathrm{R}}a + N_{\gamma_{\mathrm{R}}}a$$

we have

$$\nu\times Q_1 = \alpha\nu\times Q_2$$

on Γ, where $\alpha = \frac{\gamma_{\mathrm{L}}}{\gamma_{\mathrm{R}}}$. We claim $Q_1 = Q_2 = 0$ in Ω. This can be seen from

$$
\begin{aligned}
2i \operatorname{Im} \gamma_{\mathrm{L}} \int_\Omega |Q_1|^2 \, dV &= \int_\Omega \overline{Q_1} \cdot \operatorname{curl} Q_1 - Q_1 \cdot \operatorname{curl} \overline{Q_1} \, dV \\
&= \int_\Omega \operatorname{div}\big(Q_1 \times \overline{Q_1}\big) \, dV \\
&= \int_\Gamma \big(Q_1 \times \overline{Q_1}\big) \cdot \nu \, d\sigma \\
&= \int_\Gamma \big(\nu \times Q_1\big) \cdot \overline{Q_1} \, d\sigma \\
&= \alpha \int_\Gamma \big(\nu \times Q_2\big) \cdot \overline{Q_1} \, d\sigma \\
&= -\alpha \int_\Gamma \big(\nu \times \overline{Q_1}\big) \cdot Q_2 \, d\sigma \\
&= -|\alpha|^2 \int_\Gamma \big(\nu \times \overline{Q_2}\big) \cdot Q_2 \, d\sigma \\
&= -|\alpha|^2 \int_\Gamma \big(\overline{Q_2} \times Q_2\big) \cdot \nu \, d\sigma \\
&= -|\alpha|^2 \int_\Omega \operatorname{div}\big(\overline{Q_2} \times Q_2\big) \, dV \\
&= -|\alpha|^2 \int_\Omega Q_2 \cdot \operatorname{curl} \overline{Q_2} - \overline{Q_2} \cdot \operatorname{curl} Q_2 \, dV \\
&= -2i|\alpha|^2 \operatorname{Im} \gamma_{\mathrm{R}} \int_\Omega Q_2|^2 \, dV
\end{aligned}
$$

So we have

$$
\operatorname{Im} \gamma_{\mathrm{L}} \int_\Omega |Q_1|^2 \, dV + |\alpha|^2 \operatorname{Im} \gamma_{\mathrm{R}} \int_\Omega |Q_2|^2 \, dV = 0,
$$

$Q_1 = Q_2 = 0$ in Ω. In particular,

$$
\nu \times Q_1 = -\gamma_{\mathrm{L}} M_{\gamma_{\mathrm{L}}} a + \tfrac{1}{2} \gamma_{\mathrm{L}} a - N_{\gamma_{\mathrm{L}}} a = 0 \tag{5.6}
$$

Add this with (5.4) we have $\gamma_{\mathrm{L}} a = 0$ which implies $a = 0$, as required. $\square$

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