Inverse Electromagnetic Obstacle Scattering Problem for Perfect Conductors: A Personal Survey

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To the best of my knowledge, one of the earliest documented attempt of the factorization method to perfect conductors appears in Kress [22, Theorem 8.4, pp. 201]; see also Colton and Kress [12, pp. 260]. Modulo a constant, the far-field operator F can be factorized into the form

$$F = GN^*G^*. \tag{1}$$

Here G maps the electrical tangential components of radiating solution to the Maxwell equations onto the electric far-field patterns, and N is the electric dipole operator defined by

$$(Nf)(x) = \nu(x) \times \operatorname{curl}\operatorname{curl}_x \int_\Gamma \big(\nu(y) \times f(y)\big)\Phi_k(x, y)\,\mathrm{d}\sigma(y), \quad x \in \Gamma \tag{2}$$

and G^*, N^* denote the adjoint of G, N respectively.

The formula 1 resembles the corresponding factorization in impenetrable acoustic inverse scattering problem, but with a big caveat: the electric dipole operator N, usually operates on $\mathbf{H}^{-\frac{1}{2}}(\operatorname{curl}_\Gamma)$ with value in $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)$, is no longer coercive. To be exact, no constant $c > 0$ exists such that

$$|\langle N\varphi, \varphi\rangle| \geqslant c\,\|\varphi\|^2_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{curl}_\Gamma)}, \quad \forall \varphi \in \operatorname{dom} N$$

holds. This fact is sketched in a few papers, e.g. Hiptmair and Schwab [18], Buffa and Hiptmair [8]. The underlying function spaces $\mathbf{H}^{-\frac{1}{2}}(\operatorname{curl}_\Gamma), \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)$ which defined on a piecewise smooth domain are studied in Buffa and Ciarlet Jr. [6], [7]; on general Lipshitz domain Buffa et al. [9] remains the definitive treatment. In Buffa and Hiptmair [8], a heuristic energy argument is given regarding the noncoercive nature of N; in acoustics the potential energy is a compact perturbation of the kinetic energy, but in electromagnetism the electric and the magnetic energy are perfectly symmetric, neither one is a compact perturbation of the other. Furthermore, given the tangential nature of the electromagnetic far-field patterns, there is no hope of circumventing the appearances of N in other impenetrable cases (e.g. the impedance boundary condition); by direct computation this is verified. Finally, factorization with respect to $F^\#$ provides no help in this regard.

After these considerations and trials, I turned my attention to the improvement of the linear sampling method, mainly from the range inclusion identities viewpoint; this was inspired by Hanke [17]. For this purpose, the following papers (in chronological order) are of interest: Douglas [13], Fillmore and Williams [14], Anderson and Trapp [1], Barnes [4]. For if the factorization $F = GN^*G^*$ holds, then a priori $\operatorname{range}(F) \subseteq \operatorname{range}(G)$. If N is coercive, then range($(F^*F)^{\frac{1}{4}}$) = range(G); this is established by you and used in Arens [2], Arens and Lechleiter [3]. Apparently, alternative path has yet to be found.

Interest for persuing the aforementioned papers comes from the following fact (see Fillmore and Williams [14, Theorem 4.2], Anderson and Trapp [1, Theorem 11]):

Theorem 1. For positive operators A, B on a Hilbert space H, define the parallel sum $A : B = (A^{-1} + B^{-1})^{-1}$. Then the following holds:

1. range($A : B$) $\supset$ range(A) $\cap$ range(B).
2. range($(A : B)^{\frac{1}{2}}$) = range($A^{\frac{1}{2}}$) $\cap$ range($B^{\frac{1}{2}}$).

Note that $A : B = A(A+B)^{-1}B$; if we take $A = I, B = F^*F$, then $A : B = (I + F^*F)^{-1}F^*F$, which resembles the Tikhonov regularization. By item 2 of the above theorem, range($((I + F^*F)^{-1}F^*F)^{\frac{1}{2}}$) = range($(F^*F)^{\frac{1}{2}}$). In the perfect conductor case F is normal, so range($(F^*F)^{\frac{1}{2}}$) = range($(FF^*)^{\frac{1}{2}}$), which equals range(F) by Douglas [13, Theorem 1].

1 Notations, Definitions and Prerequisites

Definition 1 (Boundary)**.** Let Ω be an open subset in $\mathbb{R}^n$. The boundary $\Gamma = \partial\Omega$ is $C^{k,1}$ (resp. Lipchitz) if for $x \in \Gamma$ there exists a neighborhood V of x and new orthogonal coordinates $\{y_1, y_2, \dots, y_n\}$ such that

1. V is an hypercube in the new coordinates:
$$V = \{(y_1, y_2, \dots, y_n) | -a_j < y_j < a_j, 1 \leqslant j \leqslant n\}$$
2. There exists a $C^{k,1}$ (resp. Lipschitz) function φ, defined in
$$V' = \{(y_1, y_2, \dots, y_{n-1}) | -a_j < y_j < a_j, 1 \leqslant j \leqslant n-1\}$$
such that
$$\begin{aligned} |\varphi(y')| &\leqslant \frac{a_n}{2} \quad \forall y' = (y_1, y_2, \dots, y_{n-1}) \in V' \\ \Omega \cap V &= \{y = (y', y_n) \in V \,|\, y_n < \varphi(y')\} \\ \Gamma \cap V &= \{y = (y', y_n) \in V \,|\, y_n = \varphi(y')\} \end{aligned}$$

Proposition 1 (Vector Green Formula)**.**

$$\begin{aligned} &\int_\Omega (E \cdot \Delta H - H \cdot \Delta E) \,\mathrm{d}V \\ &\qquad = \int_\Gamma (E \times \operatorname{curl} H + E \operatorname{div} H - H \times \operatorname{curl} E - H \operatorname{div} E) \cdot \nu \,\mathrm{d}\sigma \end{aligned}$$

If $\operatorname{div} E = \operatorname{div} H = 0$, then

$$\begin{aligned} \int_\Omega E \cdot \operatorname{curl}\operatorname{curl} H - H \cdot \operatorname{curl}\operatorname{curl} E \,\mathrm{d}V &= \int_\Gamma (E \times \operatorname{curl} H - H \times \operatorname{curl} E) \cdot \nu \,\mathrm{d}\sigma \\ &= \int_\Gamma (\nu \times E) \cdot \operatorname{curl} H - (\nu \times H) \cdot \operatorname{curl} E \,\mathrm{d}\sigma \end{aligned} \tag{3}$$

Proposition 2 (Fundamental Theorem of Vector Analysis).

$$
\begin{aligned}
E(x) = &- \operatorname{curl} \int_\Gamma \nu(y) \times E(y)\, \Phi_k(x,y)\, \mathrm{d}\sigma(y) + \nabla \int_\Gamma \nu(y) \cdot E(y)\, \Phi_k(x,y)\, \mathrm{d}\sigma(y) \\
&- ik \int_\Gamma \nu(y) \times H(y)\, \Phi_k(x,y)\, \mathrm{d}\sigma(y) + \operatorname{curl} \int_\Omega \left\{ \operatorname{curl} E(y) - ikH(y) \right\} \Phi_k(x,y)\, \mathrm{d}V(y) \\
&- \nabla \int_\Omega \operatorname{div} E(y)\, \Phi_k(x,y)\, \mathrm{d}V(y) + ik \int_\Omega \left\{ \operatorname{curl} H(y) + ikE(y) \right\} \Phi_k(x,y)\, \mathrm{d}V(y).
\end{aligned}
$$

Proposition 3 (Stratton-Chu Representation Formula). If $E, H \in C^1(\Omega_+) \cap C(\Omega_+ \cup \Gamma)$ satisfy Maxwell equations in Ω_+ and the Silver-Müller radiation condition, then for $x \in \Omega_+$

$$
E(x) = \operatorname{curl} \int_\Gamma \nu(x) \times E(y)\, \Phi_k(x,y)\, \mathrm{d}\sigma(y) + \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times H(y)\, \Phi_k(x,y)\, \mathrm{d}\sigma(y)
$$

$$
H(x) = \operatorname{curl} \int_\Gamma \nu(x) \times H(y)\, \Phi_k(x,y)\, \mathrm{d}\sigma(y) - \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times E(y)\, \Phi_k(x,y)\, \mathrm{d}\sigma(y).
$$

For $x \in \Omega_-$:

$$
E(x) = -\operatorname{curl} \int_\Gamma \nu(y) \times E(y)\, \Phi_k(x,y)\, \mathrm{d}\sigma(y) - \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times H(y)\, \Phi_k(x,y)\, \mathrm{d}\sigma(y)
$$

$$
H(x) = -\operatorname{curl} \int_\Gamma \nu(y) \times H(y)\, \Phi_k(x,y)\, \mathrm{d}\sigma(y) + \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_\Gamma \nu(y) \times E(y)\, \Phi_k(x,y)\, \mathrm{d}\sigma(y)
$$

Proposition 4 (Far Field Patterns).

$$
\begin{aligned}
E^\infty(\hat{x}) &= ik\, \hat{x} \times \int_\Gamma \left\{ \nu(y) \times E(y) + (\nu(y) \times H(y)) \times \hat{x} \right\} e^{-ik\hat{x}\cdot y}\, \mathrm{d}\sigma(y) \\
H^\infty(\hat{x}) &= ik\, \hat{x} \times \int_\Gamma \left\{ \nu(y) \times H(y) - (\nu(y) \times E(y)) \times \hat{x} \right\} e^{-ik\hat{x}\cdot y}\, \mathrm{d}\sigma(y)
\end{aligned}
$$

Proposition 5 (Rellich Lemma). If $E, H \in C^1(\Omega_+)$ is a radiating solution of Maxwell equations such that the electric far field pattern vanishes identically, then $E = H = 0$ in Ω_+.

Definition 2.

1. Γ: The regular (Lipschitzian) boundary of the open bounded set Ω_i in $\mathbb{R}^3$.
2. The tangential differentiation ∇_t is defined by
$$\nabla_\mathrm{t} := \nu \times (\nu \times \nabla).$$
3. Given a tangential vector field a, the surface divergence $\operatorname{div}_\Gamma a$ is defined as
$$\int_\Gamma \phi \operatorname{div}_\Gamma a\, \mathrm{d}\sigma = -\int_\Gamma \nabla_\mathrm{t}\phi \cdot a\, \mathrm{d}\sigma, \quad \forall \phi \in C^\infty(\mathbb{R}^3)$$
4. $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma) = \{ v \mid v \in \mathbf{L}_2(\Gamma),\ \nu \cdot v = 0,\ \operatorname{div}_\Gamma v \in L_2(\Gamma) \}$.

5. $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma) = \{v \mid v \in \mathbf{L}_2(\Gamma),\ \nu \cdot v = 0,\ \mathbf{curl}_\Gamma v \in L_2(\Gamma)\}$.

Proposition 6. $v \to \nu \times v$ is an isomorphism from $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$ to $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$ with inverse $w \to -\nu \times w$, and we have

$$\begin{aligned}\mathrm{curl}_\Gamma v &= -\mathrm{div}_\Gamma(\nu \times v)\\ \mathrm{div}_\Gamma w &= \mathrm{curl}_\Gamma(\nu \times w)\end{aligned}$$

for $v \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$, $w \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$.

Definition 3 (The Maxwell Problem)**.** The Maxwell problem is to find a pair of solution (E, H) to the Maxwell equations

$$\begin{aligned}\mathrm{curl}\, E - ikH &= 0\\ \mathrm{curl}\, H + ikE &= 0\end{aligned}$$

in Ω_+, with the boundary condition

$$\nu \times E|_+ = f \tag{4}$$

on Γ where $f \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$, and (E, H) satisfies the Silver-Müller radiation condition

$$H \times \frac{x}{|x|} - E = \mathcal{O}\left(|x|^{-2}\right) \quad |x| \to \infty. \tag{5}$$

The data to far field pattern operator $G : \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma) \to \mathbf{L}^2_{\mathrm{t}}(\mathbb{S}^2)$ is defined as

$$Gf = E^\infty \tag{6}$$

where E^∞ denotes the far field pattern of the solution E of the Maxwell problem.

Definition 4 (The Tangential Maxwell Problem)**.** The tangential Maxwell problem is to find a pair of solution $(E_\perp, H_\perp)$ to the Maxwell equations

$$\begin{aligned}\mathrm{curl}\, E_\perp - ikH_\perp &= 0\\ \mathrm{curl}\, H_\perp + ikE_\perp &= 0\end{aligned}$$

in Ω_+, with the boundary condition

$$\left(\nu \times E_\perp\right) \times \nu|_+ = g \tag{7}$$

on Γ where $g \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$, and $(E_\perp, H_\perp)$ satisfies the Silver-Müller radiation condition

$$H_\perp \times \frac{x}{|x|} - E_\perp = \mathcal{O}\left(|x|^{-2}\right) \quad |x| \to \infty. \tag{8}$$

The tangential Maxwell data-to-pattern operator $G_\perp : \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma) \to \mathbf{L}^2_{\mathrm{t}}(\mathbb{S}^2)$ is defined as

$$G_\perp g = E_\perp^\infty \tag{9}$$

where $E_\perp^\infty$ denotes the far field pattern of the solution $E_\perp$ of the tangential Maxwell problem.

2 Reciprocity Relations

Assume $x, z \in \Omega_+$, $\hat{x}, d \in \mathbb{S}^2$, $p, q \in \mathbb{R}^3$.

Given the incident electromagentic wave

$$
\begin{aligned}
E^{\mathrm{i}}(x, d, p) &= \frac{i}{k} \operatorname{curl}_x \operatorname{curl}_x p e^{ikx\cdot d} = ik(d \times p) \times d e^{ikx\cdot d}, \\
H^{\mathrm{i}}(x, d, p) &= \operatorname{curl}_x p e^{ikx\cdot d} = ik(d \times p) e^{ikx\cdot d},
\end{aligned}
$$

the scattered field is denoted by

$$
E^{\mathrm{s}}(x, d, p), \quad H^{\mathrm{s}}(x, d, p)
$$

with corresponding far field pattern

$$
E^{\infty}(\hat{x}, d, p), \quad H^{\infty}(\hat{x}, d, p).
$$

Given the incident dipole

$$
\begin{aligned}
E^{\mathrm{i}}_{\mathrm{p}}(x, z, p) &= \frac{i}{k} \operatorname{curl}_x \operatorname{curl}_x p\, \Phi_k(x, z), \\
H^{\mathrm{i}}_{\mathrm{p}}(x, z, p) &= \operatorname{curl}_x p\, \Phi_k(x, z),
\end{aligned}
$$

the scattered field is denoted by

$$
E^{\mathrm{s}}_{\mathrm{p}}(x, z, p), \quad H^{\mathrm{s}}_{\mathrm{p}}(x, z, p)
$$

with the corresponding far field pattern

$$
E^{\infty}_{\mathrm{p}}(\hat{x}, z, p), \quad H^{\infty}_{\mathrm{p}}(\hat{x}, z, p).
$$

The total field is denoted by

$$
\begin{aligned}
E(x, d, p) &= E^{\mathrm{i}}(x, d, p) + E^{\mathrm{s}}(x, d, p) \\
H(x, d, p) &= H^{\mathrm{i}}(x, d, p) + H^{\mathrm{s}}(x, d, p) \\
E_{\mathrm{p}}(x, z, p) &= E^{\mathrm{i}}_{\mathrm{p}}(x, z, p) + E^{\mathrm{s}}_{\mathrm{p}}(x, z, p) \\
H_{\mathrm{p}}(x, z, p) &= H^{\mathrm{i}}_{\mathrm{p}}(x, z, p) + H^{\mathrm{s}}_{\mathrm{p}}(x, z, p)
\end{aligned}
$$

Theorem 2 (Mixed Reciprocity Relation)**.**

$$
p \cdot E^{\mathrm{s}}(z, -\hat{x}, q) = 4\pi q \cdot E^{\infty}_{\mathrm{p}}(\hat{x}, z, p)
$$

Proof.

$$
\begin{aligned}
4\pi q \cdot E^{\infty}_{\mathrm{p}}(\hat{x}, z, p) = \int_{\Gamma} & \nu(y) \times E^{\mathrm{s}}_{\mathrm{p}}(y, z, p) \cdot H^{\mathrm{i}}(y, -\hat{x}, q) \\
& + \nu(y) \times H^{\mathrm{s}}_{\mathrm{p}}(y, z, p) \cdot E^{\mathrm{i}}(y, -\hat{x}, q) \, \mathrm{d}\sigma(y)
\end{aligned} \tag{10}
$$

From Green formula

$$
\int_{\Gamma} \nu(y) \times E^{\mathrm{s}}_{\mathrm{p}}(y, z, p) \cdot H^{\mathrm{s}}(y, -\hat{x}, q) + \nu(y) \times H^{\mathrm{s}}_{\mathrm{p}}(y, z, p) \cdot E^{\mathrm{s}}(y, -\hat{x}, q) \, \mathrm{d}\sigma(y) = 0 \tag{11}
$$

Add (10), (11) and apply the boundary condition

$$\nu(y) \times E(y, -\hat{x}, q) = 0 \quad y \in \Gamma$$

we have

$$4\pi q \cdot E_{\mathrm{p}}^{\infty}(\hat{x}, z, p) = \int_{\Gamma} \nu(y) \times E_{\mathrm{p}}^{\mathrm{s}}(y, z, p) \cdot H(y, -\hat{x}, q)\, \mathrm{d}\sigma(y) \tag{12}$$

From Stratton-Chu representation,

$$\begin{aligned} E^{\mathrm{s}}(z, -\hat{x}, q) = \operatorname{curl} \int_{\Gamma} \nu(y) \times E^{\mathrm{s}}(y, -\hat{x}, q)\, \Phi_k(z, y)\, \mathrm{d}\sigma(y) \\ + \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_{\Gamma} \nu(y) \times H^{\mathrm{s}}(y, -\hat{x}, q)\, \Phi_k(z, y)\, \mathrm{d}\sigma(y) \end{aligned} \tag{13}$$

From Green formula

$$\begin{aligned} 0 = \operatorname{curl} \int_{\Gamma} \nu(y) \times E^{\mathrm{i}}(y, -\hat{x}, q)\, \Phi_k(z, y)\, \mathrm{d}\sigma(y) \\ + \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_{\Gamma} \nu(y) \times H^{\mathrm{i}}(y, -\hat{x}, q)\, \Phi_k(z, y)\, \mathrm{d}\sigma(y) \end{aligned} \tag{14}$$

Add (13), (14) and apply the boundary condition

$$\nu(y) \times E(y, -\hat{x}, q) = 0 \quad y \in \Gamma$$

we have

$$E^{\mathrm{s}}(z, -\hat{x}, q) = \frac{i}{k} \operatorname{curl}\operatorname{curl} \int_{\Gamma} \nu(y) \times H(y, -\hat{x}, q)\, \Phi_k(z, y)\, \mathrm{d}\sigma(y) \tag{15}$$

From (15), the identity

$$p \cdot \operatorname{curl}\operatorname{curl}_z \{a(y)\, \Phi_k(z, y)\} = a(y) \cdot \operatorname{curl}\operatorname{curl}_z \{p\, \Phi_k(z, y)\},$$

and the boundary condition

$$\nu(y) \times E_{\mathrm{p}}^{\mathrm{i}}(y, z, p) = -\nu(y) \times E_{\mathrm{p}}^{\mathrm{s}}(y, z, p) \quad y \in \Gamma$$

we have

$$\begin{aligned} p \cdot E^{\mathrm{s}}(z, -\hat{x}, q) &= \frac{i}{k}\, p \cdot \operatorname{curl}\operatorname{curl} \int_{\Gamma} \nu(y) \times H(y, -\hat{x}, q)\, \Phi_k(z, y)\, \mathrm{d}\sigma(y) \\ &= \frac{i}{k} \int_{\Gamma} \nu(y) \times H(y, -\hat{x}, q) \cdot \operatorname{curl}\operatorname{curl}\{p\, \Phi_k(z, y)\}\, \mathrm{d}\sigma(y) \\ &= \int_{\Gamma} \nu(y) \times H(y, -\hat{x}, q) \cdot E_{\mathrm{p}}^{\mathrm{i}}(y, z, p)\, \mathrm{d}\sigma(y) \\ &= -\int_{\Gamma} \nu(y) \times E_{\mathrm{p}}^{\mathrm{i}}(y, z, p) \cdot H(y, -\hat{x}, q)\, \mathrm{d}\sigma(y) \\ &= \int_{\Gamma} \nu(y) \times E_{\mathrm{p}}^{\mathrm{s}}(y, z, p) \cdot H(y, -\hat{x}, q)\, \mathrm{d}\sigma(y), \end{aligned}$$

which equals (12). $\square$

Theorem 3 (Reciprocity Relation)**.**

$$q \cdot E^{\infty}(\hat{x}, d, p) = p \cdot E^{\infty}(-d, -\hat{x}, q)$$

Proof. Apply Green formula to E^{i} in Ω_{-}, E^{s} in Ω_{+}, we have

$$\int_{\Gamma} \left\{ \nu(y) \times E^{\mathrm{i}}(y, d, p) \cdot H^{\mathrm{i}}(y, -\hat{x}, q) - \nu(y) \times E^{\mathrm{i}}(y, -\hat{x}, q) \cdot H^{\mathrm{i}}(y, d, p) \right\} \mathrm{d}\sigma(y) = 0 \quad (16)$$

$$\int_{\Gamma} \left\{ \nu(y) \times E^{\mathrm{s}}(y, d, p) \cdot H^{\mathrm{s}}(y, -\hat{x}, q) - \nu(y) \times E^{\mathrm{s}}(y, -\hat{x}, q) \cdot H^{\mathrm{s}}(y, d, p) \right\} \mathrm{d}\sigma(y) = 0 \quad (17)$$

$$\begin{aligned} 4\pi q \cdot E^{\infty}(\hat{x}, d, p) = \int_{\Gamma} \Big\{ & \nu(y) \times E^{\mathrm{s}}(y, d, p) \cdot H^{\mathrm{i}}(y, -\hat{x}, q) \\ & + \nu(y) \times H^{\mathrm{s}}(y, d, p) \cdot E^{\mathrm{i}}(y, -\hat{x}, q) \Big\} \mathrm{d}\sigma(y) \end{aligned} \quad (18)$$

Interchange p, q and $d, \hat{x}$ respectively in (18), we have

$$\begin{aligned} 4\pi q \cdot E^{\infty}(\hat{x}, d, p) = \int_{\Gamma} \Big\{ & \nu(y) \times E^{\mathrm{s}}(y, -\hat{x}, q) \cdot H^{\mathrm{i}}(y, d, p) \\ & + \nu(y) \times H^{\mathrm{s}}(y, -\hat{x}, q) \cdot E^{\mathrm{i}}(y, d, p) \Big\} \mathrm{d}\sigma(y) \end{aligned} \quad (19)$$

Subtract (18) with (19) and add (16), (17), together with the boundary condition

$$\nu(y) \times E(y, d, p) = \nu(y) \times E(y, -\hat{x}, p) = 0, \quad y \in \Gamma$$

the result follows. □

3 The Uniqueness Theorem

Theorem 4. If D_1 and D_2 are two perfect conductors such that the electric far field patterns coincide for a fixed wave number, all incident directions and all observation directions, then $D_1 = D_2$.

Proof. Let U be the unbounded component of $\mathbb{R}^3 \setminus (D_1 \cup D_2)$. By Rellich lemma,

$$E_1^{\mathrm{s}}(x, d, p) = E_2^{\mathrm{s}}(x, d, p) \quad \forall x \in U, d, p \in \mathbb{S}^2.$$

By mixed reciprocity relation,

$$E_1^{\infty}(\hat{x}, z, p) = E_2^{\infty}(\hat{x}, z, p) \quad \forall z \in U, \hat{x}, p \in \mathbb{S}^2.$$

Again by Rellich lemma,

$$E_{\mathrm{p},1}^{\mathrm{s}}(x, z, p) = E_{\mathrm{p},2}^{\mathrm{s}}(x, z, p) \quad \forall x, z \in U, p \in \mathbb{S}^2.$$

Assume $D_1 \neq D_2$, then $\exists\, \widetilde{x} \in U$ such that $\widetilde{x} \in \partial D_1, \widetilde{x} \notin \overline{D_2}$. Construct $\{z_n\}$ such that $z_n = \widetilde{x} + \frac{1}{n}\nu(\widetilde{x}) \in U$ for sufficiently large n. From the well-posedness of the solution on D_2, $E_{\mathrm{p},2}^{\mathrm{s}}(\widetilde{x}, \widetilde{x}, p)$ is well-behaved. But

$$E_{\mathrm{p},1}^{\mathrm{s}}(\widetilde{x}, z_n, q) \to \infty \text{ as } z_n \to \widetilde{x} \text{ and given } p \perp \nu(\widetilde{x})$$

in order to fulfill the boundary condition with the incident dipole $E_{\mathrm{p},1}^{\mathrm{i}}(\widetilde{x}, z_n, p)$, which becomes unbounded as $z_n \to \widetilde{x}$. □

4 The Factorization Method

Definition 5 (The Far Field Operator)**.** The far field operator $F : \mathbf{L}^2_{\mathrm{t}}(\mathbb{S}^2) \to \mathbf{L}^2_{\mathrm{t}}(\mathbb{S}^2)$ is

$$(Fg)(\hat{x}) = \int_{\mathbb{S}^2} E^\infty(\hat{x}, \theta, g(\theta))\, \mathrm{d}\sigma(\theta), \quad \hat{x} \in \mathbb{S}^2. \tag{20}$$

Proposition 7. The far field operator F is compact, injective with dense range.

Proof. □

Proposition 8. The far field operator F is normal, i.e. $F^*F = FF^*$.

Proof. Let $g, h \in \mathbf{L}^2_{\mathrm{t}}(\mathbb{S}^2)$ and define the Herglotz wave functions $v^{\mathrm{i}}, w^{\mathrm{i}}$ with density g, h respectively:

$$\begin{aligned} v^{\mathrm{i}}(x) &= \int_{\mathbb{S}^2} g(\theta)\, e^{ikx\cdot\theta}\, \mathrm{d}\sigma(\theta), \quad x \in \mathbb{R}^3 \\ w^{\mathrm{i}}(x) &= \int_{\mathbb{S}^2} h(\theta)\, e^{ikx\cdot\theta}\, \mathrm{d}\sigma(\theta). \quad x \in \mathbb{R}^3 \end{aligned}$$

Let v, w be solutions of the scattering problem corresponding to incident fields $v^{\mathrm{i}}, w^{\mathrm{i}}$ with scattered fields $v^{\mathrm{s}} = v - v^{\mathrm{i}}, w^{\mathrm{s}} = w - w^{\mathrm{i}}$ and far field patterns v^∞, w^∞ respectively. Apply Green theorem in $\Omega_R = \{x \in \mathbb{R}^3 \setminus \overline{\Omega} : |x| < R\}$ with sufficiently big R, together with the boundary condition we have

$$0 = \int_{\Omega_R} \left\{ v\, \Delta\overline{w} - \overline{w}\, \Delta v \right\}\, \mathrm{d}V \tag{21}$$

$$= \int_{\mathbb{S}^2} \left\{ \overline{w} \times \operatorname{curl} v - v \times \operatorname{curl} \overline{w} \right\} \cdot \nu\, \mathrm{d}\sigma. \tag{22}$$

Decomposing $v = v^{\mathrm{i}} + v^{\mathrm{s}}$ and $w = w^{\mathrm{i}} + w^{\mathrm{s}}$, we split (22) into the sum of the following four parts:

$$\int_{\mathbb{S}^2} \left\{ \overline{w^{\mathrm{i}}} \times \operatorname{curl} v^{\mathrm{i}} - v^{\mathrm{i}} \times \operatorname{curl} \overline{w^{\mathrm{i}}} \right\} \cdot \nu\, \mathrm{d}\sigma, \tag{23}$$

$$\int_{\mathbb{S}^2} \left\{ \overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{s}}} \right\} \cdot \nu\, \mathrm{d}\sigma, \tag{24}$$

$$\int_{\mathbb{S}^2} \left\{ \overline{w^{\mathrm{i}}} \times \operatorname{curl} v^{\mathrm{s}} - v^{\mathrm{s}} \times \operatorname{curl} \overline{w^{\mathrm{i}}} \right\} \cdot \nu\, \mathrm{d}\sigma, \tag{25}$$

$$\int_{\mathbb{S}^2} \left\{ \overline{w^{\mathrm{s}}} \times \operatorname{curl} v^{\mathrm{i}} - v^{\mathrm{i}} \times \operatorname{curl} \overline{w^{\mathrm{s}}} \right\} \cdot \nu\, \mathrm{d}\sigma. \tag{26}$$

The integral (23) vanishes by applying Green theorem in $B_R = \{x : |x| < R\}$. To evaluate the integral (24), we note by the radiation condition

$$\overline{w^{\mathrm{s}}} \times \hat{x} - \frac{1}{ik} \operatorname{curl} \overline{w^{\mathrm{s}}} = \mathcal{O}\left(r^{-2}\right) \tag{27}$$

$$v^{\mathrm{s}} \times \hat{x} + \frac{1}{ik} \operatorname{curl} v^{\mathrm{s}} = \mathcal{O}\left(r^{-2}\right) \tag{28}$$

and relations between scattered fields and far field patterns

$$\begin{aligned}\overline{w^{\mathrm{s}}} &= \frac{e^{-ikr}}{4\pi r}\left\{\overline{w^{\infty}}+\mathcal{O}\left(r^{-1}\right)\right\}\\ v^{\mathrm{s}} &= \frac{e^{ikr}}{4\pi r}\left\{v^{\infty}+\mathcal{O}\left(r^{-1}\right)\right\}\end{aligned}$$

one obtains

$$\begin{aligned}&\left\{\overline{w^{\mathrm{s}}}\times\operatorname{curl}v^{\mathrm{s}}-v^{\mathrm{s}}\times\operatorname{curl}\overline{w^{\mathrm{s}}}\right\}\cdot\hat{x}\\ &= ik\left\{\overline{w^{\mathrm{s}}}\times(\hat{x}\times v^{\mathrm{s}})+v^{\mathrm{s}}\times\left(\hat{x}\times\overline{w^{\mathrm{s}}}\right)\right\}\cdot\hat{x}\\ &= 2ik\left\{\overline{w^{\mathrm{s}}}\cdot v^{\mathrm{s}}-\left(\overline{w^{\mathrm{s}}}\cdot\hat{x}\right)(v^{\mathrm{s}}\cdot\hat{x})\right\}\\ &= 2ik\,\overline{w^{\mathrm{s}}}\cdot v^{\mathrm{s}}\\ &= \frac{ik}{8\pi^2r^2}\,\overline{w^{\infty}}\cdot v^{\infty}+\mathcal{O}\left(r^{-3}\right)\end{aligned}$$

Hence

$$\begin{aligned}&\int_{\mathbb{S}^2}\left\{\overline{w^{\mathrm{s}}}\times\operatorname{curl}v^{\mathrm{s}}-v^{\mathrm{s}}\times\operatorname{curl}\overline{w^{\mathrm{s}}}\right\}\cdot\nu\,\mathrm{d}\sigma\\ &\longrightarrow\frac{ik}{8\pi^2}\int_{\mathbb{S}^2}\overline{w^{\infty}}\cdot v^{\infty}\,\mathrm{d}\sigma=\frac{ik}{8\pi^2}\left(Fg,Fh\right)_{L^2(\mathbb{S}^2)}\end{aligned}$$

To evaluate the integral (25), one note that it can be rearranged as

$$\int_{\mathbb{S}^2}\left\{\overline{w^{\mathrm{i}}}\times\operatorname{curl}v^{\mathrm{s}}-v^{\mathrm{s}}\times\operatorname{curl}\overline{w^{\mathrm{i}}}\right\}\cdot\nu\,\mathrm{d}\sigma \tag{29}$$

$$=-\int_{\mathbb{S}^2}\left\{(\hat{x}\times\operatorname{curl}v^{\mathrm{s}})\cdot\overline{w^{\mathrm{i}}}+(\hat{x}\times v^{\mathrm{s}})\cdot\operatorname{curl}\overline{w^{\mathrm{i}}}\right\}\,\mathrm{d}\sigma \tag{30}$$

Substitute

$$\begin{aligned}\overline{w^{\mathrm{i}}}(x) &= \int_{\mathbb{S}^2}h(\theta)\,e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(\theta),\\ \operatorname{curl}\overline{w^{\mathrm{i}}}(x) &= ik\int_{\mathbb{S}^2}(h(\theta)\times\theta)\,e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)\end{aligned}$$

into (30), it becomes

$$\begin{aligned}&-\int_{|x|=r}(\hat{x}\times\operatorname{curl}v^{\mathrm{s}})\cdot\left\{\int_{\mathbb{S}^2}h(\theta)\,e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)\right\}\,\mathrm{d}\sigma(x)\\ &\qquad-\int_{|x|=r}(\hat{x}\times v^{\mathrm{s}})\cdot\left\{ik\int_{\mathbb{S}^2}(h(\theta)\times\theta)\;e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)\right\}\,\mathrm{d}\sigma(x).\end{aligned} \tag{31}$$

From $h(\theta)\cdot\theta=0$ and $\theta\cdot\theta=1$, by formulae

$$\begin{aligned}a\times(b\times c) &= b\;(a\cdot c)-c\;(a\cdot b)\\ a\cdot(b\times c) &= -b\cdot(a\times c)\end{aligned}$$

we have

$$\begin{aligned}h(\theta)\cdot(\hat{x}\times\operatorname{curl}v^{\mathrm{s}}) &= h(\theta)\cdot\{(\hat{x}\times\operatorname{curl}v^{\mathrm{s}})-\theta\,(\theta\cdot(\hat{x}\times\operatorname{curl}v^{\mathrm{s}}))\}\\ &= h(\theta)\cdot\{\theta\times((\hat{x}\times\operatorname{curl}v^{\mathrm{s}})\times\theta)\}\end{aligned}$$

and

$$(\hat{x} \times v^s) \cdot (h(\theta) \times \theta) = h(\theta) \cdot (\theta \times (\hat{x} \times v^s))$$

Substitute into (31), the integral (25) is

$$\begin{aligned}
&- \int_{\mathbb{S}^2} \int_{|x|=r} \left\{ h(\theta) \cdot (\hat{x} \times \operatorname{curl} v^s) + ik\ (\hat{x} \times v^s) \cdot (h(\theta) \times \theta) \right\} e^{-ikx\cdot\theta} \,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta) \\
= &- \int_{\mathbb{S}^2} h(\theta) \cdot \left\{ \int_{|x|=r} \left\{ \theta \times ((\hat{x} \times \operatorname{curl} v^s) \times \theta) + ik\, \theta \times (\hat{x} \times v^s) \right\} e^{-ikx\cdot\theta} \,\mathrm{d}\sigma(x) \right\} \mathrm{d}\sigma(\theta) \\
&\longrightarrow -(Fg, h)_{L^2(\mathbb{S}^2)} .
\end{aligned}$$

By the same token, the integral (26) is $(g, Fh)_{L^2(\mathbb{S}^2)}$. Hence

$$0 = (g, Fh)_{L^2(\mathbb{S}^2)} - (Fg, h)_{L^2(\mathbb{S}^2)} + \frac{ik}{8\pi^2} (Fg, Fh)_{L^2(\mathbb{S}^2)} ,$$

the identity

$$F - F^* = \frac{ik}{8\pi^2} F^* F$$

follows.

Now set $S = I + \frac{ik}{8\pi^2} F$, we have

$$\begin{aligned}
S^* S &= \left(I - \frac{ik}{8\pi^2} F^* \right) \left(I + \frac{ik}{8\pi^2} F \right) \\
&= I + \frac{ik}{8\pi^2} F - \frac{ik}{8\pi^2} F^* + \frac{k^2}{64\pi^2} F^* F \\
&= I.
\end{aligned}$$

If $Sg = 0$, then $g = S^* S g = 0$, hence S is injective. Note that S is a compact perturbation of the identity, from Fredholm theory S is an isomorphism. Therefore $S^* = S^{-1}$ and $SS^* = I$. Comparing $S^* S$ and SS^* we can see that $F^* F = F F^*$, i.e. F is normal. □

Proposition 9. The data to far field pattern operator G is compact, injective with dense range.

Proof. □

Proposition 10. For $z \in \mathbb{R}^3$ and a fixed $d \in \mathbb{S}^2$, define

$$\varphi_z(\hat{x}) = ik\, (\hat{x} \times d)\, e^{ik\hat{x}\cdot z}, \quad \hat{x} \in \mathbb{S}^2,$$

then φ_z belongs to the range of G if and only if $z \in \Omega$.

Proof. Assume first $z \in \Omega$. define

$$v(x) = \operatorname{curl}_x \left\{ \Phi_k(x, z)\, d \right\}, \quad \forall x \in \mathbb{R}^3 \setminus \Omega$$

and $f = v|_\Gamma$. The far field pattern of v, denoted by v^∞, is

$$v^\infty(\hat{x}) = ik\, (\hat{x} \times d)\, e^{ik\hat{x}\cdot z}, \quad \hat{x} \in \mathbb{S}^2,$$

which is identical to φ_z. From $Gf = v^\infty = \varphi_z$, φ_z belongs to the range of G.

Now assume $z \notin \Omega$ and there exists f with $Gf = \varphi_z$. Let v be the radiating solution of the Maxwell problem with boundary data f and $v^\infty = Gf$ be the far field pattern of v. Note that the far field pattern of $\operatorname{curl}\left\{\Phi_k(\cdot, z)\, d\right\}$ is φ_z, from Rellich lemma $v(x) = \operatorname{curl}\left\{\Phi_k(x, z)\, d\right\}$ for all x outside of any sphere which contains both z and Ω. By analytic continuation, v and $\operatorname{curl}\left\{\Phi_k(\cdot, z)\, d\right\}$ coincide on $\mathbb{R}^3 \setminus \left(\overline{\Omega} \cup \{z\}\right)$. But if $z \notin \overline{\Omega}$, then $\operatorname{curl}\left\{\Phi_k(x, z)\, d\right\}$ is singular on $x = z$, while v is analytic on $\mathbb{R}^3 \setminus \overline{\Omega}$, a contradiction. Otherwise if $z \in \Gamma$, then $x \mapsto \operatorname{curl}\left\{\Phi_k(x, z)\, d\right\}$ for $x \in \Gamma, x \neq z$, is in $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. But $\operatorname{curl}\{\Phi(x, z)\, d\}$ does not belong to $\mathbf{H}_{\text{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus \Omega)$ or $\mathbf{H}(\operatorname{curl}, \Omega)$, for $\operatorname{curl} \Phi_k(x, z) = \mathcal{O}\left(|x - z|^{-2}\right)$ if $x \to z$. □

Definition 6. The single layer operator $S_k : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ with density f is

$$(S_k f)(x) = \int_\Gamma f(y)\, \Phi_k(x, y)\, \mathrm{d}\sigma(y), \quad x \in \Gamma. \tag{32}$$

The vector single layer operator $S_k : \mathbf{H}^{-\frac{1}{2}}(\Gamma) \to \mathbf{H}^{\frac{1}{2}}(\Gamma)$ is formed with vector density g:

$$(S_k g)(x) = \int_\Gamma g(y)\, \Phi_k(x, y)\, \mathrm{d}\sigma(y), \quad x \in \Gamma. \tag{33}$$

The electric dipole operator $N_k : \mathbf{H}^{-\frac{1}{2}}(\operatorname{curl}_\Gamma) \to \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma)$ is

$$(N_k f)(x) = \nu(x) \times \operatorname{curl}\operatorname{curl}_x \int_\Gamma \big(\nu(y) \times f(y)\big) \Phi_k(x, y)\, \mathrm{d}\sigma(y), \quad x \in \Gamma. \tag{34}$$

By $\operatorname{curl}\operatorname{curl} \cdot = \nabla \operatorname{div} \cdot + \Delta \cdot$,

$$\begin{aligned} N_k f &= \nu \times \operatorname{curl}\operatorname{curl} S_k(\nu \times f) \\ &= k^2 \nu \times S_k(\nu \times f) + \nu \times \nabla S_k(\operatorname{div}_\Gamma(\nu \times f)) \end{aligned} \tag{35}$$

We note the following formula: for scalar f, vector g

$$\int_\Gamma \langle \nu \times \nabla f, g\rangle = -\int_\Gamma f \langle \nu, \operatorname{curl} g\rangle$$

This can be verified with

$$\int_\Omega \operatorname{curl} u = \int_\Gamma \nu \times u$$

and the proof runs as follows:

$$\begin{aligned} \int_\Gamma \langle \nu \times \nabla f, g\rangle &= -\int_\Gamma \langle g \times \nabla f, \nu\rangle = -\int_\Omega \operatorname{div}(g \times \nabla f) \\ &= -\int_\Omega \langle \operatorname{curl} g, \nabla f\rangle \\ &= -\int_\Omega \operatorname{div}(f \operatorname{curl} g) = -\int_\Gamma f \langle \nu, \operatorname{curl} g\rangle \end{aligned}$$

Set $f = S_k(\mathrm{div}_\Gamma(\nu \times \varphi))$, $g = \overline{\psi}$ and recall that $\mathrm{div}_\Gamma(\nu \times \overline{\psi}) = -\nu \cdot \mathrm{curl}\,\overline{\psi}$, we have

$$
\begin{aligned}
\langle N_k \varphi,\, \psi \rangle &= \langle k^2 \nu \times S_k(\nu \times \varphi) + \nu \times \nabla S_k(\mathrm{div}_\Gamma\, \nu \times \varphi),\, \psi \rangle \\
&= k^2 \int_\Gamma \big(\nu \times S_k(\nu \times \varphi)\big) \cdot \overline{\psi} + \int_\Gamma \big(\nu \times \nabla S_k(\mathrm{div}_\Gamma(\nu \times \varphi))\big) \cdot \overline{\psi} \\
&= -k^2 \int_\Gamma S_k(\nu \times \varphi) \cdot \big(\nu \times \overline{\psi}\big) + \int_\Gamma S_k(\mathrm{div}_\Gamma(\nu \times \varphi))\, \big(\nu \cdot \mathrm{curl}\,\overline{\psi}\big) \\
&= -k^2 \int_\Gamma S_k(\nu \times \varphi) \cdot \overline{(\nu \times \psi)} + \int_\Gamma S_k(\mathrm{div}_\Gamma(\nu \times \varphi))\, \overline{\mathrm{div}_\Gamma\, (\nu \times \psi)} \\
&= -k^2 \,\langle S_k(\nu \times \varphi),\, \nu \times \psi \rangle + \langle S_k(\mathrm{div}_\Gamma(\nu \times \varphi)),\, \mathrm{div}_\Gamma(\nu \times \psi) \rangle\,. \qquad (36)
\end{aligned}
$$

Proposition 11. The adjoint operator $N_k^* : \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma) \to \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$ is N_{-k}, i.e.

$$
(N_k^* f)(x) = \nu(x) \times \mathrm{curl}_x\, \mathrm{curl}_x \int_\Gamma (\nu(y) \times f(y))\, \Phi_{-k}(x, y)\, \mathrm{d}\sigma(y), \quad x \in \Gamma. \qquad (37)
$$

Proof. Note that

$$
\begin{aligned}
\nabla_x \cdot \nabla_y \Phi_{-k}(x, y) &= -\nabla_y \cdot \nabla_y \Phi_{-k}(x, y), \\
\Big(\Big(\nu(y) \times \overline{g(y)}\Big) \cdot \nabla_x\Big) \nabla_y \Phi_{-k}(x, y) &= -\Big(\Big(\nu(y) \times \overline{g(y)}\Big) \cdot \nabla_y\Big) \nabla_y \Phi_{-k}(x, y),
\end{aligned}
$$

which can be verified by straightforward differentiation. Then

$$
\begin{aligned}
\langle f, N_k g \rangle &= \int_\Gamma f(x) \cdot \overline{\left\{ \nu(x) \times \mathrm{curl}_x\, \mathrm{curl}_x \int_\Gamma (\nu(y) \times g(y))\ \Phi_k(x, y)\, \mathrm{d}\sigma(y) \right\}}\, \mathrm{d}\sigma(x) \\
&= \int_\Gamma \int_\Gamma f(x) \cdot \Big\{ \nu(x) \times \mathrm{curl}_x \Big(\nabla_x \Phi_{-k}(x, y) \times \Big(\nu(y) \times \overline{g(y)}\Big)\Big)\Big\}\ \mathrm{d}\sigma(y)\, \mathrm{d}\sigma(x) \\
&= \int_\Gamma \int_\Gamma f(x) \cdot \Big\{ \nu(x) \times \mathrm{curl}_x \Big(\nabla_x \Phi_{-k}(x, y) \times \Big(\nu(y) \times \overline{g(y)}\Big)\Big)\Big\}\ \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(y) \\
&= \int_\Gamma \int_\Gamma (f(x) \times \nu(x)) \cdot \mathrm{curl}_x \Big(\nabla_x \Phi_{-k}(x, y) \times \Big(\nu(y) \times \overline{g(y)}\Big)\Big)\ \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(y) \\
&= \int_\Gamma \int_\Gamma (\nu(x) \times f(x)) \cdot \mathrm{curl}_x \Big(\nabla_y \Phi_{-k}(x, y) \times \Big(\nu(y) \times \overline{g(y)}\Big)\Big)\ \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(y) \\
&= \int_\Gamma \int_\Gamma (\nu(x) \times f(x)) \cdot \Big\{ -\Big(\nu(y) \times \overline{g(y)}\Big) \big(\nabla_x \cdot \nabla_y \Phi_{-k}(x, y)\big) \\
&\qquad\qquad + \Big(\Big(\nu(y) \times \overline{g(y)}\Big) \cdot \nabla_x\Big) \nabla_y \Phi_{-k}(x, y) \Big\}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(y) \\
&= \int_\Gamma \int_\Gamma (\nu(x) \times f(x)) \cdot \Big\{ \Big(\nu(y) \times \overline{g(y)}\Big) \big(\nabla_y \cdot \nabla_y \Phi_{-k}(x, y)\big) \\
&\qquad\qquad - \Big(\Big(\nu(y) \times \overline{g(y)}\Big) \cdot \nabla_y\Big) \nabla_y \Phi_{-k}(x, y) \Big\}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(y) \\
&= \int_\Gamma \left\{ -\mathrm{curl}_y\, \mathrm{curl}_y \int_\Gamma (\nu(x) \times f(x))\, \Phi_{-k}(x, y)\, \mathrm{d}\sigma(x) \right\} \cdot \Big(\nu(y) \times \overline{g(y)}\Big)\ \mathrm{d}\sigma(y) \\
&= \int_\Gamma \left\{ \nu(y) \times \mathrm{curl}_y\, \mathrm{curl}_y \int_\Gamma (\nu(x) \times f(x))\, \Phi_{-k}(x, y)\, \mathrm{d}\sigma(x) \right\} \cdot \overline{g(y)}\, \mathrm{d}\sigma(y) \\
&= \langle N_k^* f, g \rangle.
\end{aligned}
$$

□

Proposition 12.

$$F = \frac{1}{k^2} G N_{-k} G^*.$$

Proof. Define auxiliary operator $\mathcal{H} : \mathbf{L}^2_t(\mathbb{S}^2) \to \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$ as

$$(\mathcal{H}g)(x) = \nu(x) \times \int_{\mathbb{S}^2} g(\theta)\, e^{ikx\cdot\theta}\, \mathrm{d}\sigma(\theta) \quad x \in \Gamma,$$

then the adjoint operator $\mathcal{H}^* : \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma) \to \mathbf{L}^2_t(\mathbb{S}^2)$ is

$$(\mathcal{H}^* f)(\theta) = \theta \times \left(\theta \times \int_\Gamma (\nu(x) \times f(x))\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\right), \quad \theta \in \mathbb{S}^2. \tag{38}$$

This can be verified by

$$\begin{aligned}
\langle f, \mathcal{H}g\rangle &= \int_\Gamma f(x) \cdot \overline{\left\{\nu(x) \times \int_{\mathbb{S}^2} g(\theta)\, e^{ikx\cdot\theta}\, \mathrm{d}\sigma(\theta)\right\}}\, \mathrm{d}\sigma(x) \\
&= \int_\Gamma \int_{\mathbb{S}^2} f(x) \cdot \left(\nu(x) \times \overline{g(\theta)}\right) e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(\theta)\, \mathrm{d}\sigma(x) \\
&= \int_{\mathbb{S}^2} \int_\Gamma f(x) \cdot \left(\nu(x) \times \overline{g(\theta)}\right) e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \int_\Gamma (f(x) \times \nu(x)) \cdot \overline{g(\theta)}\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \int_\Gamma (f(x) \times \nu(x)) \cdot \left(\left(\theta \times \overline{g(\theta)}\right) \times \theta\right) e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \int_\Gamma (\theta \times (f(x) \times \nu(x))) \cdot \left(\theta \times \overline{g(\theta)}\right) e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \int_\Gamma (\theta \times ((f(x) \times \nu(x)) \times \theta)) \cdot \overline{g(\theta)}\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \left\{\left(\theta \times \int_\Gamma (f(x) \times \nu(x))\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\right) \times \theta\right\} \cdot \overline{g(\theta)}\, \mathrm{d}\sigma(\theta) \\
&= \int_{\mathbb{S}^2} \left\{\theta \times \left(\theta \times \int_\Gamma (\nu(x) \times f(x))\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\right)\right\} \cdot \overline{g(\theta)}\, \mathrm{d}\sigma(\theta) \\
&= \langle \mathcal{H}^* f, g\rangle.
\end{aligned}$$

Given $f \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$, define $u(x)$ by

$$u(x) = \mathrm{curl}\,\mathrm{curl}_x \int_\Gamma (\nu(y) \times f(y))\, \Phi_k(x, y)\, \mathrm{d}\sigma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma.$$

From the asymptotic relation (c.f. Colton and Kress [12] (6.27))

$$\mathrm{curl}\,\mathrm{curl}_x \left\{a(y) \frac{e^{ik|x-y|}}{|x-y|}\right\} = -k^2 \frac{e^{ik|x|}}{|x|} \left\{\hat{x} \times \left(\hat{x} \times a(y)\, e^{-ik\hat{x}\cdot y}\right) + \mathcal{O}\left(|x|^{-1}\right)\right\}$$

the far field pattern of u can be seen as $-k^2\mathcal{H}^* f$; the trace $\nu(x) \times u(x) = N_k f$. Hence, $-k^2\mathcal{H}^* f = G N_k f \implies \mathcal{H}^* = -\frac{1}{k^2} G N_k$, so $\mathcal{H} = -\frac{1}{k^2} N_k^* G^* = -\frac{1}{k^2} N_{-k} G^*$. By definition $F = -G\mathcal{H}$, hence

$$F = -G\mathcal{H} = -G\left(-\frac{1}{k^2} N_{-k} G^*\right) = \frac{1}{k^2} G N_{-k} G^*. \tag{39}$$

□

$$\Lambda_k = -N_k R \left(\frac{I}{2} + M_k\right)^{-1} \frac{1}{ik}. \tag{40}$$

$$\begin{aligned}\Lambda_k^{-1} &= -(ik)\left(\frac{I}{2} + M_k\right)(-R)\, N_k^{-1} \\ &= ik\left(\frac{I}{2} + M_k\right) R N_k^{-1}. \end{aligned} \tag{41}$$

$$\begin{aligned}\left(\Lambda_k^{-1}\right)^* &= \left(N_k^{-1}\right)^* R^* \left(\frac{I}{2} + M_k\right)^* (ik)^* \\ &= \left(N_{-k}\right)^{-1}(-R)\left(\frac{I}{2} + M_k\right)^*(-ik) \\ &= ik\left(N_{-k}\right)^{-1} R\left(\frac{I}{2} + M_k\right)^*. \end{aligned} \tag{42}$$

$$\begin{aligned}\Lambda_k^{-1} N_{-k}\left(\Lambda_k^{-1}\right)^* &= \left\{ik\left(\frac{I}{2} + M_k\right) R N_k^{-1}\right\} N_{-k}\left\{ik\left(N_{-k}\right)^{-1} R\left(\frac{I}{2} + M_k\right)^*\right\} \\ &= -k^2\left(\frac{I}{2} + M_k\right) R N_k^{-1} R\left(\frac{I}{2} + M_k\right)^*. \end{aligned} \tag{43}$$

Proposition 13.

$$F_0 = \frac{1}{k^2} G \Lambda_k^{-1} N_{-k}\left(\Lambda_k^{-1}\right)^* G^*.$$

Proof. Define auxiliary operator $\mathcal{H}_0 : \mathbf{L}^2_{\mathrm{t}}(\mathbb{S}^2) \to \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$ as

$$(\mathcal{H}_0 g)(x) = \nu(x) \times \int_{\mathbb{S}^2} (\theta \times g(\theta))\, e^{ikx\cdot\theta}\, \mathrm{d}\sigma(\theta), \quad x \in \Gamma. \tag{44}$$

The adjoint operator $\mathcal{H}_0^* : \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma) \to \mathbf{L}^2_{\mathrm{t}}(\mathbb{S}^2)$ is

$$(\mathcal{H}_0^* f)(\theta) = \theta \times \int_\Gamma (\nu(x) \times f(x))\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x), \quad \theta \in \mathbb{S}^2. \tag{45}$$

This can be verified by

$$\begin{aligned}\langle f, \mathcal{H}_0 g\rangle &= \int_\Gamma f(x) \cdot \overline{\left\{\nu(x) \times \int_{\mathbb{S}^2} (\theta \times g(\theta))\, e^{ikx\cdot\theta}\, \mathrm{d}\sigma(\theta)\right\}}\, \mathrm{d}\sigma(x) \\ &= \int_\Gamma \int_{\mathbb{S}^2} f(x) \cdot \left(\nu(x) \times \left(\theta \times \overline{g(\theta)}\right)\right) e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(\theta)\, \mathrm{d}\sigma(x) \\ &= \int_{\mathbb{S}^2} \int_\Gamma f(x) \cdot \left(\nu(x) \times \left(\theta \times \overline{g(\theta)}\right)\right) e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(\theta) \\ &= \int_{\mathbb{S}^2} \int_\Gamma (f(x) \times \nu(x)) \cdot \left(\theta \times \overline{g(\theta)}\right) e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(\theta) \\ &= \int_{\mathbb{S}^2} \int_\Gamma (\theta \times (\nu(x) \times f(x))) \cdot \overline{g(\theta)}\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\, \mathrm{d}\sigma(\theta) \\ &= \int_{\mathbb{S}^2} \left\{\theta \times \int_\Gamma (\nu(x) \times f(x))\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x)\right\} \cdot \overline{g(\theta)}\, \mathrm{d}\sigma(\theta) \\ &= \langle \mathcal{H}_0^* f, g\rangle. \end{aligned}$$

Given $f \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$, define $u_0(x)$ by

$$u_0(x) = \mathrm{curl}_x \int_\Gamma (\nu(y) \times f(y))\, \Phi_k(x,y)\, \mathrm{d}\sigma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma.$$

From the asymptotic relation

$$\mathrm{curl}_x \left\{ a(y) \frac{e^{ik|x-y|}}{4\pi|x-y|} \right\} = ik \frac{e^{ik|x|}}{4\pi|x|} \left\{ \hat{x} \times a(y)\, e^{-ik\hat{x}\cdot y} + \mathcal{O}\left(|x|^{-1}\right) \right\}$$

the far field pattern of u_0 can be seen as $ik\mathcal{H}_0^* f$; the trace $\nu(x) \times \frac{1}{ik} \mathrm{curl}_x\, u_0(x)|_+ = \frac{1}{ik} N_k f$. Hence, $ik\mathcal{H}_0^* f = G_0 \frac{1}{ik} N_k f \implies \mathcal{H}_0^* = -\frac{1}{k^2} G_0 N_k$, so $\mathcal{H}_0 = -\frac{1}{k^2} N_k^* G_0^* = -\frac{1}{k^2} N_{-k} G_0^*$. By definition $F_0 = -G_0 \mathcal{H}_0$, then

$$F_0 = -G_0 \left(-\frac{1}{k^2} N_{-k} G_0^* \right) = \frac{1}{k^2} G_0 N_{-k} G_0^*. \tag{46}$$

Also $G_0 = G\Lambda_k^{-1}$, $G_0^* = \left(\Lambda_k^{-1}\right)^* G^*$. Hence finally

$$F_0 = \frac{1}{k^2} G_0 N_{-k} G_0^* = \frac{1}{k^2} G \Lambda_k^{-1} N_{-k} \left(\Lambda_k^{-1}\right)^* G^*. \tag{47}$$

□

Proposition 14.

$$F_\perp = -\frac{1}{k^2} G_\perp R N_{-k} G_\perp^*.$$

Proof. Define auxiliary operator $\mathcal{H}_1 : \mathbf{L}_\mathrm{t}^2(\mathbb{S}^2) \to \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$ as

$$(\mathcal{H}_1 g)(x) = \left(\nu(x) \times \int_{\mathbb{S}^2} g(\theta)\, e^{ikx\cdot\theta}\, \mathrm{d}\sigma(\theta) \right) \times \nu(x), \quad \lambda \in \mathbb{R},\ x \in \Gamma. \tag{48}$$

The adjoint operator $\mathcal{H}_1^* : \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma) \to \mathbf{L}_\mathrm{t}^2(\mathbb{S}^2)$ is

$$(\mathcal{H}_1^* f)(\theta) = -\theta \times \left(\theta \times \int_\Gamma (\nu(x) \times f(x)) \times \nu(x)\, e^{-ikx\cdot\theta}\, \mathrm{d}\sigma(x) \right), \quad \theta \in \mathbb{S}^2. \tag{49}$$

This can be verified by

$$
\begin{aligned}
\langle f, \mathcal{H}_1 g\rangle &= \int_\Gamma f(x)\cdot \overline{\left\{\left(\nu(x)\times\int_{\mathbb{S}^2} g(\theta)\, e^{ikx\cdot\theta}\,\mathrm{d}\sigma(\theta)\right)\times\nu(x)\right\}}\,\mathrm{d}\sigma(x)\\
&= \int_\Gamma\int_{\mathbb{S}^2} f(x)\cdot\left(\left(\nu(x)\times\overline{g(\theta)}\, e^{-ikx\cdot\theta}\right)\times\nu(x)\right)\,\mathrm{d}\sigma(\theta)\,\mathrm{d}\sigma(x)\\
&= \int_{\mathbb{S}^2}\int_\Gamma f(x)\cdot\left(\left(\nu(x)\times\overline{g(\theta)}\, e^{-ikx\cdot\theta}\right)\times\nu(x)\right)\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta)\\
&= \int_{\mathbb{S}^2}\int_\Gamma (\nu(x)\times f(x))\cdot\left(\nu(x)\times\overline{g(\theta)}\, e^{-ikx\cdot\theta}\right)\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta)\\
&= \int_{\mathbb{S}^2}\int_\Gamma \left((\nu(x)\times f(x))\times\nu(x)\, e^{-ikx\cdot\theta}\right)\cdot\overline{g(\theta)}\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta)\\
&= \int_{\mathbb{S}^2}\int_\Gamma \left((\nu(x)\times f(x))\times\nu(x)\, e^{-ikx\cdot\theta}\right)\cdot\left(\left(\theta\times\overline{g(\theta)}\right)\times\theta\right)\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta)\\
&= \int_{\mathbb{S}^2}\int_\Gamma \left\{\theta\times(\nu(x)\times f(x))\times\nu(x)\, e^{-ikx\cdot\theta}\right\}\cdot\left(\theta\times\overline{g(\theta)}\right)\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(\theta)\\
&= \int_{\mathbb{S}^2}\left\{-\theta\times\left(\theta\times\int_\Gamma(\nu(x)\times f(x))\times\nu(x)\, e^{-ikx\cdot\theta}\,\mathrm{d}\sigma(x)\right)\right\}\cdot\overline{g(\theta)}\,\mathrm{d}\sigma(\theta)\\
&= \langle\mathcal{H}_1^* f, g\rangle.
\end{aligned}
$$

Given $f\in\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)$, define $u_1(x)$ by

$$
u_1(x) = \operatorname{curl}\operatorname{curl}_x\int_\Gamma(\nu(y)\times f(y))\times\nu(y)\,\Phi_k(x,y)\,\mathrm{d}\sigma(y),\quad x\in\mathbb{R}^3\setminus\Gamma.
$$

From the asymptotic relation

$$
\operatorname{curl}\operatorname{curl}_x\left\{a(y)\frac{e^{ik|x-y|}}{|x-y|}\right\} = -k^2\frac{e^{ik|x|}}{|x|}\left\{\hat{x}\times\left(\hat{x}\times a(y)\, e^{-ik\hat{x}\cdot y}\right)+\mathcal{O}\left(|x|^{-1}\right)\right\}
$$

the far field pattern of u_1 can be seen as $k^2\mathcal{H}_1^* f$; the trace $\left(\nu(x)\times u_1(x)\right)\times\nu(x) = -RN_k(-R)f = RN_kRf$. Hence, $k^2\mathcal{H}_1^* f = G_\perp RN_kRf \implies \mathcal{H}_1^* = \frac{1}{k^2}G_\perp RN_kR$, so $\mathcal{H}_1 = \frac{1}{k^2}R^*N_k^*R^*G_\perp^* = \frac{1}{k^2}(-R)N_{-k}(-R)G_\perp^* = \frac{1}{k^2}RN_{-k}RG_\perp^*$. By definition $F_\perp = -G_\perp\mathcal{H}_1$, hence

$$
F_\perp = -G_\perp\mathcal{H}_1 = -G_\perp\left(\frac{1}{k^2}RN_{-k}RG_\perp^*\right) = -\frac{1}{k^2}G_\perp RN_{-k}RG_\perp^*. \tag{50}
$$

□

Proposition 15. 1. The data-to-pattern operators G and $G_\perp$ satisfy

$$
G = -G_\perp R. \tag{51}
$$

In particular, the ranges of G and $G_\perp$ coincide.

2. $G_\perp$ is compact and injective.

Proof. 1. Consider the Maxwell problem with $\nu\times E = f$ on Γ, then E also solves the tangential Maxwell problem with $g = (\nu\times E)\times\nu = -Rf$. So for the far field pattern E^∞ of E,

$$
E^\infty = Gf = G_\perp g = -G_\perp Rf, \tag{52}
$$

which implies (51).

2. The compactness and the injectivity of $G_\perp$ follows from (51) and the compactness and injectivity of G.

$\square$

Right multiply both sides of (51) by R, we have $G_\perp = GR$; so $G_\perp^* = (GR)^* = R^*G^* = -RG^*$. Hence $F_\perp = -\frac{1}{k^2}G_\perp R N_{-k} R G_\perp^* = -\frac{1}{k^2}(GR)RN_{-k}R(-RG^*) = \frac{1}{k^2}GN_{-k}G^*$.

Proposition 16. $\Im\langle S_{-k}\varphi,\, \varphi\rangle < 0$ for $\varphi \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and $\varphi \neq 0$.

Proof. Given $\varphi \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$, define

$$v(x) = \int_\Gamma \varphi(y)\, \Phi_{-k}(x,y)\, \mathrm{d}\sigma(y), \quad x \in \mathbb{R}^3 \setminus \Gamma. \tag{53}$$

Note that $\Delta v + k^2 v = 0$ for $x \in \mathbb{R}^3 \setminus \Gamma$,

$$\frac{\partial v_\pm}{\partial \nu} = \int_\Gamma \varphi(y) \left(\nabla_x \Phi_{-k}(x,y)\cdot \nu(x)\right)\, \mathrm{d}\sigma(y) \mp \frac{1}{2}\varphi(x),$$

and v satisfies the radiation condition

$$\frac{\partial v(x)}{\partial \nu} + ikv(x) = \mathcal{O}\left(|x|^{-2}\right), \quad |x| \to \infty. \tag{54}$$

Then

$$\begin{aligned}
\langle S_{-k}\varphi,\, \varphi\rangle &= \left\langle v, \frac{\partial v_-}{\partial \nu} - \frac{\partial v_+}{\partial \nu}\right\rangle \\
&= \int_\Gamma v\cdot \frac{\partial \overline{v}_-}{\partial \nu}\,\mathrm{d}\sigma - \int_\Gamma v\cdot\frac{\partial \overline{v}_+}{\partial \nu}\,\mathrm{d}\sigma \\
&= \int_{B_R\cup\Omega_-} \left\{|\nabla v|^2 - k^2|v|^2\right\}\, \mathrm{d}V - \int_{\mathbb{S}^2} v\cdot\frac{\partial \overline{v}}{\partial \nu}\,\mathrm{d}\sigma \qquad (55)\\
&= \int_{B_R\cup\Omega_-} \left\{|\nabla v|^2 - k^2|v|^2\right\}\, \mathrm{d}V - ik\int_{\mathbb{S}^2} |v|^2\,\mathrm{d}\sigma + \mathcal{O}\left(|x|^{-1}\right) \qquad (56)
\end{aligned}$$

where we use the radiation condition (54) into the second integral of (55). Now take the imaginary part and let $R \to \infty$,

$$\Im\langle S_{-k}\varphi,\, \varphi\rangle = -k\lim_{R\to\infty}\int_{\mathbb{S}^2} |v|^2\,\mathrm{d}\sigma = -\frac{k}{16\pi^2}\int_{\mathbb{S}^2} \mathrm{d}\sigma(\theta)|v^\infty|^2 \leqslant 0.$$

Let $\Im\langle S_{-k}\varphi,\, \varphi\rangle = 0$ for some $\varphi \in \mathbf{H}^{-\frac{1}{2}}(\Gamma)$, then by (56) $v^\infty = 0$; via Rellich's lemma and unique continuation $v = 0$ in Ω_+, hence $S_{-k}\varphi = 0 \implies \varphi = 0$, for S_{-k} is an isomorphism. $\square$

Proposition 17. $\Im\langle\varphi, N_k\varphi\rangle > 0$ for $k > 0$ and $\varphi \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$.

Proof. Given $\varphi \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)$, define

$$v(x) = \mathrm{curl}\int_\Gamma \nu(y)\times\varphi(y)\,\Phi(x,y)\,\mathrm{d}\sigma(y), \quad x \in \mathbb{R}^3\setminus\Gamma. \tag{57}$$

Note that $\operatorname{div} v \equiv 0$ for $x \in \mathbb{R}^3$, $\Delta v + k^2 v = 0$ for $x \in \mathbb{R}^3 \setminus \Gamma$,

$$\begin{aligned} v_\pm(x) &= \int_\Gamma \nabla_x \Phi(x,y) \times \big(\nu(y) \times \varphi(y)\big)\, \mathrm{d}\sigma(y) \mp \frac{1}{2} \nu(x) \times \big(\nu(x) \times \varphi(x)\big) \\ &= \int_\Gamma \nabla_x \Phi(x,y) \times \big(\nu(y) \times \varphi(y)\big)\, \mathrm{d}\sigma(y) \pm \frac{1}{2} \varphi(x) \end{aligned}$$

(c.f. Colton and Kress [12] Theorem 6.13), and the radiation condition

$$\operatorname{curl} v(x) \times \frac{x}{|x|} - ikv(x) = \mathcal{O}\left(|x|^{-2}\right), \quad |x| \to \infty. \tag{58}$$

By vector Green formula

$$\int_\Omega a \cdot \Delta b + \operatorname{curl} a \cdot \operatorname{curl} b + \operatorname{div} a \operatorname{div} b = \int_\Gamma -a \cdot (\nu \times \operatorname{curl} b) + (\nu \cdot a) \operatorname{div} b$$

with $a = v_\pm$, $b = \overline{v}$, we have

$$\begin{aligned} \langle \varphi, N_k \varphi \rangle &= \big\langle v_+ - v_-,\, \nu \times \operatorname{curl} v \big\rangle \\ &= \int_\Gamma \big(v_+ - v_-\big) \cdot \big(\nu \times \operatorname{curl} \overline{v}\big)\, \mathrm{d}\sigma \\ &= \int_\Gamma v_+ \cdot \big(\nu \times \operatorname{curl} \overline{v}\big)\, \mathrm{d}\sigma - \int_\Gamma v_- \cdot \big(\nu \times \operatorname{curl} \overline{v}\big)\, \mathrm{d}\sigma \\ &= \int_{B_R \cup \Omega_-} \left\{ |\operatorname{curl} v|^2 - k^2 |v|^2 \right\} \mathrm{d}V + \int_{\mathbb{S}^2} v \cdot \big(\hat{x} \times \operatorname{curl} \overline{v}\big)\, \mathrm{d}\sigma \qquad (59) \\ &= \int_{B_R \cup \Omega_-} \left\{ |\operatorname{curl} v|^2 - k^2 |v|^2 \right\} \mathrm{d}V + ik \int_{\mathbb{S}^2} |v|^2\, \mathrm{d}\sigma + \mathcal{O}\left(|x|^{-1}\right) \qquad (60) \end{aligned}$$

where we use the radiation condition (58) into the second integral of (59). Now take the imaginary part and let $R \to \infty$,

$$\Im \langle \varphi, N_k \varphi \rangle = k \lim_{R \to \infty} \int_{\mathbb{S}^2} |v|^2\, \mathrm{d}\sigma = \frac{k}{16\pi^2} \int_{\mathbb{S}^2} \mathrm{d}\sigma(\theta) |v^\infty|^2 \geqslant 0.$$

Let $\Im \langle \varphi, N_k \varphi \rangle = 0$ for some $\varphi \in \mathbf{H}^{-\frac{1}{2}}(\operatorname{curl}_\Gamma)$, then by (60) $v^\infty = 0$; via Rellich's lemma and unique continuation $v = 0$ in Ω_+, hence $N_k \varphi = 0 \Longrightarrow \varphi = 0$, for N_k is an isomorphism. □

5 Range Inclusion Identities

Throughout this section X, Y, Z are reflexive Banach spaces. The collection of all bounded linear opertors from X into Y is denoted by $\mathcal{L}(X, Y)$. For $S \in \mathcal{L}(X, Y)$, let $\mathcal{R}(S)$ and $\mathcal{N}(S)$ be the range and the null space of S, respectively. The dual space of X is denoted by X^*. The duality pairing in $X^* \times X$ is denoted by $\langle \cdot, \cdot \rangle$. The adjoint of $S \in \mathcal{L}(X, Y)$ is denoted by $S^* \in \mathcal{L}(Y^*, X^*)$. An operator $U \in \mathcal{L}(X, Y)$ is an isomorphism if U is a bijection mapping. By the corollary of Open Mapping Theorem (Rudin [27], 2.12 Corollaries (b), pp. 49) $U^{-1} \in \mathcal{L}(Y, X)$. Note that $U^* \in \mathcal{L}(Y^*, X^*)$ is an isomorphism if $U \in \mathcal{L}(X, Y)$ is, and $(U^{-1})^* = (U^*)^{-1}$.

Let $M \subseteq X, N \subseteq X^*$. The annihilators $M^\perp, {}^\perp N$ are defined as

$$M^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0 \quad \forall x \in M\},$$
$$N^\perp = \{x \in X : \langle x^*, x\rangle = 0 \quad \forall x^* \in N\}.$$

For $S \in \mathcal{L}(X, Y)$, the following identities hold (Rudin [27], 4.12 Theorem, pp.99):

$$\mathcal{N}(S) = \mathcal{R}(S^*)^\perp, \quad \mathcal{N}(S^*) = \mathcal{R}(S)^\perp. \tag{61}$$

Definition 7. (Barnes [4] Definition 1) Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(X, Z)$. We say that T majorize S if there exists $c > 0$ such that

$$\|Sx\| \leqslant c\|Tx\| \quad \forall x \in X.$$

Proposition 18. (Barnes [4] Proposition 3, Theorem 7(1)) Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(X, Z)$.

1. T majorize S if and only if there exists $V \in \mathcal{L}(\overline{\mathcal{R}(T)}, Z)$ with $S = VT$.
2. If T majorize S, then $\mathcal{R}(S^*) \subseteq \mathcal{R}(T^*)$.

Proof.

1. ($\Longrightarrow$) Define $V : \mathcal{R}(T) \to Z$ by $V(Tx) = Sx$; V is well defined since $\mathcal{N}(T) \subseteq \mathcal{N}(S)$. By definition $\|V(Tx)\| = \|Sx\| \leqslant c\|T\|\,\|x\|$, hence V has a bounded extension on $\overline{\mathcal{R}(T)}$, which we still denote as V. ($\Longleftarrow$) $\|Sx\| = \|VTx\| \leqslant \|V\|\,\|Tx\|$ for all $x \in X$.
2. From (1) $\exists V \in \mathcal{L}(\overline{\mathcal{R}(T)}, Z)$ with $S = VT$. Naturally $S^* \in \mathcal{L}(Z^*, X^*), T^* \in \mathcal{L}(Y^*, X^*)$. Let $\alpha \in Z^*$; for $x \in X$

$$\langle x, S^*\alpha\rangle = \langle Sx, \alpha\rangle = \langle VTx, \alpha\rangle = \langle Tx, V^*\alpha\rangle$$

where $V^*\alpha$ is a continuous linear function on $\overline{\mathcal{R}(T)}$. By Hahn-Banach theorem there exists an extension of $V^*\alpha$ to Y^*, say β. Then for $x \in X$

$$\langle x, S^*\alpha\rangle = \langle Tx, \beta\rangle = \langle x, T^*\beta\rangle,$$

$S^*\alpha = T^*\beta$. Hence $\mathcal{R}(S^*) \subseteq \mathcal{R}(T^*)$.

□

Proposition 19. Let $U \in \mathcal{L}(Y, Z)$ be an isomorphism. For $T \in \mathcal{L}(X, Y)$,

$$\mathcal{R}(T^*) = \mathcal{R}(T^*U^*).$$

Proof. Set $W = UT \in \mathcal{L}(X, Z)$. Evidently T majorize W, by proposition 18 $\mathcal{R}(W^*) \subseteq \mathcal{R}(T^*)$. Also we have $T = U^{-1}W \Longrightarrow W$ majorize T; $\mathcal{R}(T^*) \subseteq \mathcal{R}(W^*)$. Hence $\mathcal{R}(T^*) = \mathcal{R}(W^*) = \mathcal{R}(T^*U^*)$. □

Proposition 20. Let $U \in \mathcal{L}(X, Y)$ be an isomorphism and $D \subseteq X$ be dense in X. Then $U(D)$ is dense in Y.

Proof. If $U(D)$ is not dense in Y, then there exists a nonempty open set $O \subseteq (Y \setminus U(D))$. Set $Q = U^{-1}(O)$; Q is nonempty for U is an isomorphism; Q is open for U is continuous. By construction $Q \not\subseteq D$, which contradicts that D is dense in X. □

Proposition 21. Let $A \in \mathcal{L}(X, Y)$, $B \in \mathcal{L}(X^*, X)$ with A injective and $U \in \mathcal{L}(X^*, X^*)$ be an isomorphism. Then

$$\mathcal{R}(ABUA^*) = \mathcal{R}(ABA^*). \tag{62}$$

Proof. For injective A, by (61) $\overline{\mathcal{R}(A^*)} = X^*$. By Proposition 20 $\overline{U(\mathcal{R}(A^*))} = X^*$. Consider

$$\begin{aligned} P_1 &= AB : \mathcal{R}(A^*) \to Y, \\ P_2 &= AB : U(\mathcal{R}(A^*)) \to Y. \end{aligned}$$

For Y is complete and $\overline{\mathcal{R}(A^*)} = \overline{U(\mathcal{R}(A^*))} = X^*$, by BLT theorem (Reed and Simon [26] Theorem I.7, pp.9) P_1, P_2 have the same extension $P : X^* \to Y$ with the same norm. Redefine AB as P, we see that (61) holds. □

6 Factorization Method Revisited

Let V, W be complex Hilbert spaces with corresponding inner products $(\cdot,\cdot)_V, (\cdot,\cdot)_W$; the induced norms are denoted as $\|\cdot\|_V, \|\cdot\|_W$. The duality pairings are denoted as $\langle\cdot,\cdot\rangle_{V^*,V}, \langle\cdot,\cdot\rangle_{W^*,W}$; subscripts would be suppressed should no confusion arise.

Proposition 22. Let $T \in \mathcal{L}(V, W^*)$ be an isomporphism, then there exists an isomorphism $U \in \mathcal{L}(V, W)$ such that

$$\exists c > 0, \quad \left|\langle v, T^* U v\rangle\right| \geqslant c\|v\|_V^2 \quad \forall v \in V. \tag{63}$$

Proof. By Riesz Representation Theorem there exists an isomorphism $\Lambda \in \mathcal{L}(W^*, W)$ such that

$$(\Lambda w^*, w)_W = \langle w^*, w\rangle, \quad \forall (w^*, w) \in W^* \times W.$$

Set $U = \Lambda \circ T \in \mathcal{L}(V, W)$. U is an isomorphism with continuous inverse $U^{-1} \in \mathcal{L}(W, V)$; hence for $v \in V$

$$\|v\|_V = \|U^{-1} U v\|_V \leqslant \|U^{-1}\|_V \|Uv\|_W$$

and

$$\left|\langle Tv, Uv\rangle\right| = \left|(\Lambda \circ Tv, Uv)_W\right| = \|Uv\|_W^2 \geqslant \frac{1}{\|U^{-1}\|_V^2}\|v\|_V^2.$$

□

Proposition 23. Let V, W be complex Hilbert spaces and $A \in \mathcal{L}(V^*, W)$, $B \in \mathcal{L}(V, V^*)$. Let $F = AB^*A^* \in \mathcal{L}(W^*, W)$ and B be an isomorphism. For $\varphi \in W, \varphi \neq 0$, define

$$\begin{aligned} W_1 &= \{\psi \in W^* : \|\psi\|_{W^*} = 1\}, \\ W_\varphi &= \{\psi \in W_1 : |\langle \psi, \varphi\rangle| \neq 0\}. \end{aligned}$$

1. If $\varphi \in \mathcal{R}(A)$, then for $\psi \in W_\varphi$, there exists $\psi_0 \in W_1$ such that $|\langle \psi, F\psi_0 \rangle| > 0$.
2. If $\varphi \notin \mathcal{R}(A)$, then $\inf_{\psi \in W_\varphi} |\langle \psi, F\psi \rangle| = 0$.

Proof. 1. By Proposition 22 there exists an isomorphism $U \in \mathcal{L}(V, V)$ such that

$$\exists c > 0, \quad \left|\langle v, B^* U v \rangle\right| \geqslant c\|v\|_V^2 \quad \forall v \in V. \tag{64}$$

Set $\widetilde{F} = AB^*UA^*$. For $\psi \in W^*$, we have

$$\left|\langle \psi, \widetilde{F}\psi \rangle\right| = \left|\langle \psi, AB^*UA^*\psi \rangle\right| = \left|\langle A^*\psi, B^*UA^*\psi \rangle\right| \geqslant c\|A^*\psi\|^2. \tag{65}$$

For $\varphi \in \mathcal{R}(A)$, there exists $\phi_0 \in V^*$ such that $\varphi = A\phi_0$. For $\psi \in W_\varphi$

$$\begin{aligned} \left|\langle \psi, \widetilde{F}\psi \rangle\right| \geqslant c\|A^*\psi\|^2 &= \frac{c}{\|\phi_0\|^2}\|A^*\psi\|^2\,\|\phi_0\|^2 \\ &\geqslant \frac{c}{\|\phi_0\|^2}\left|\langle A^*\psi, \phi_0 \rangle\right|^2 \\ &= \frac{c}{\|\phi_0\|^2}\left|\langle \psi, A\phi_0 \rangle\right|^2 > 0 \end{aligned}$$

By Proposition 21 we have $\mathcal{R}(\widetilde{F}) = \mathcal{R}(F)$, thus

$$\forall \psi \in W^*, \exists \xi \in W^* \text{ such that } \widetilde{F}\psi = F\xi. \tag{66}$$

So

$$\left|\langle \psi, \widetilde{F}\psi \rangle\right| = \left|\langle \psi, F\xi \rangle\right| = \|\xi\|\left|\langle \psi, F\frac{\xi}{\|\xi\|} \rangle\right| > 0.$$

2. From (Nachman et al. [25] Lemma 2.1) we have

$$\varphi \notin \mathcal{R}(A) \quad \Longrightarrow \quad \exists\{\psi_n\} \in W^* \text{ such that } |\langle \psi_n, \varphi \rangle| = 1, \|A^*\psi_n\| \to 0.$$

Note that for $\psi \in W^*$,

$$|\langle \psi, F\psi \rangle| = \left|\langle \psi, AB^*A^*\psi \rangle\right| = \left|\langle A^*\psi, B^*A^*\psi \rangle\right| \leqslant \|B^*\|\,\|A^*\psi\|^2,$$

so $\left|\langle \psi_n, F\psi_n \rangle\right| \leqslant \|B^*\|\,\|A^*\psi_n\|^2$. □

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