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Expected utility with uncertain probabilities theory

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ABSTRACT

This paper introduces a model of decision making under ambiguity by extending the Bayesian approach to *uncertain probabilities*. In this model, preferences for ambiguity pertain directly to probabilities such that attitude toward ambiguity is defined as attitude toward mean-preserving spreads in probabilities—analogous to the Rothschild–Stiglitz risk attitude toward mean-preserving spreads in outcomes. The model refines the separations between tastes and beliefs, and between risk and ambiguity. These separations are crucial for the measurement of the degree of ambiguity and for the elicitation and characterization of attitudes toward ambiguity, thereby providing an empirically and experimentally applicable framework.

1. Introduction

The separation that expected utility theory has attained between *risk* and individuals' *tastes* for risk has been a critical step toward the (empirical) measurement of risk independently of tastes and toward the (experimental) exploration of tastes for risk independently of risk. This can be considered a breakthrough that has led to the incorporation of the concept of risk into numerous studies in economics and finance. These fields are still pursuing such a breakthrough with respect to the notion of *ambiguity*—the uncertainty about probabilities.[1] As Knight (1921, page 19) stressed

> *"Uncertainty must be taken in a sense radically distinct from the familiar notion of Risk, from which it has never been properly separated."*

The search for models of decision making under ambiguity has been evolving toward the ultimate separations between tastes and beliefs, and between risk and ambiguity. This search can be viewed as having been started by Knight (1921) and enhanced with the *Choquet expected utility* (CEU) of Schmeidler (1989) and the *max–min expected utility* (MEU) of Gilboa and Schmeidler (1989) that distinguish ambiguity from risk. It has been followed by the *cumulative prospect theory* (CPT) of Tversky and Kahneman (1992) that refines preferences concerning losses and gains. This search has evolved further with the α-MEU of Ghirardato et al. (1998) that introduces a separation between tastes and beliefs with respect to ambiguity, and the *smooth model of ambiguity* of Klibanoff et al. (2005) that improves this separation, but still retains some relation between risk and ambiguity. Although these models, among others, have made significant contributions to our understanding of the mental and analytical processes of decision making under ambiguity, the search for an applicable model is still ongoing. Some fundamental questions such as "How can the degree of ambiguity be measured?" or "What is the relationship between risk and ambiguity (preferences)?" remain unanswered. Building upon previous literature, this paper furthers the development of a model that can be used in empirical and behavioral studies by refining the separations between tastes and beliefs, and between risk and ambiguity.

This paper provides a theoretical framework for measuring tastes for ambiguity independently of beliefs and of tastes for risk. To this end, it introduces a decision-making model, underpinned by a new theoretical concept. This concept proposes that preferences concerning ambiguity are applied directly to probabilities such that aversion to ambiguity is defined as aversion to *mean-preserving spreads in probabilities*—analogous to the Rothschild and Stiglitz (1970) aversion to mean-preserving spreads in outcomes (risk). Thereby, the degree of ambiguity can be measured by the volatility of probabilities, just as the degree of risk can be measured by the volatility of outcomes (Izhakian, 2016).[2]

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[1] *Risk* is defined as a condition in which the event to be realized is *a priori* unknown, but the odds of all possible events are perfectly known. *Ambiguity* (*Knightian uncertainty*) refers to conditions in which not only is the event to be realized *a priori* unknown, but the odds of events are also either not uniquely assigned or are unknown.

[2] Measuring risk by the volatility of outcomes is admissible under some conditions; the same is true for measuring ambiguity by the volatility of probabilities.

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0304-4068/

The resulting *risk-independent* measure of ambiguity can be employed to study the implications of ambiguity empirically.[3] The key feature of the proposed model is that it completely distinguishes tastes from beliefs and risk from ambiguity. It identifies the sources of uncertainty (consequences versus probabilities), allowing attitudes toward ambiguity to be elicited and characterized explicitly.

The proposed model, referred to as *expected utility with uncertain probabilities* (henceforth EUUP), aims to capture the multi-dimensional nature of uncertainty when probabilities of events are uncertain. It contains two tiers of uncertainty, one with respect to consequences (outcomes) and the other with respect to the probabilities of these consequences. This model assumes two differentiated phases of the decision process, each refers to one of these tiers. In the first phase – the *probability formation* phase – the decision maker (DM) forms a representation of her perceived probabilities (capacities) for all the events which are relevant to her decision.[4] Then, in the second phase—the *valuation* phase, she assesses the value of each alternative using her perceived probabilities and chooses accordingly. Ambiguity – the uncertainty about probabilities – plays a role in the probability formation phase, while risk – the uncertainty about consequences – plays a role in the valuation phase. This structure introduces a complete distinction of the sources of uncertainty, thereby of risk from ambiguity with regard to both beliefs and tastes. The degree of ambiguity and attitudes toward it are measured with respect to one tier, while risk and risk attitudes are measured with respect to the other tier.

The main idea of EUUP is that, in the probability formation phase, perceived probabilities are formed by the "*certainty equivalent probabilities*" of uncertain probabilities. That is to say, an uncertain probability is modeled explicitly in a state space that is subject to a prior probability, and the *perceived probability* is the unique certain probability value that the DM is willing to accept in exchange for the uncertain probability of a given event. As a consequence of probabilistic sensitivity, i.e., the nonlinear ways in which individuals may interpret probabilities, perceived probabilities are nonadditive. That is, the sum of the probabilities can be either smaller or greater than 1. Aversion to ambiguity results in a subadditive probability measure, while love for ambiguity results in a superadditive measure.

Preferences for risk and ambiguity in EUUP are shown to be represented by the sign-dependent (reference-dependent) Choquet expectation

$$
\begin{aligned}
&\mathrm{V}(f)\\
&= \int_{z\le 0}\left[\Gamma\left(\Upsilon^{-1}\left(\int_{\mathcal{P}}\Upsilon(\mathrm{P}(\{s\in\mathcal{S}\mid \mathrm{U}(f(s))\ge z\}))d\xi\right)\right)-1\right]dz\\
&\quad+\int_{z\ge 0}\Gamma\left(\Upsilon^{-1}\left(\int_{\mathcal{P}}\Upsilon(\mathrm{P}(\{s\in\mathcal{S}\mid \mathrm{U}(f(s))\ge z\}))d\xi\right)\right)dz, \qquad (1)
\end{aligned}
$$

where f is an act on a state space $\mathcal{S}$; U is a (von Neumann–Morgenstern) utility function; $\mathcal{P}$ is a set of possible probability measures P on a σ-algebra of subsets of $\mathcal{S}$; ξ is a subjective second-order belief (a probability measure on an algebra of subsets of $\mathcal{P}$); Υ captures attitude toward ambiguity; and Γ captures the impact of events on the desirability of the act ("decision weight"). EUUP allows U, Υ and Γ concerning unfavorable consequences to be different from those concerning favorable consequences.[5] In particular, a concave Υ characterizes ambiguity-averse behavior, and a convex Υ characterizes ambiguity-loving behavior. Following Kahneman and Tversky (1979, page 280), who stress that "*Decision weights measure the impact of events on the desirability of prospects, and not merely the perceived likelihood of these events*," EUUP distinguishes between perceived probabilities and the transformation (decision weights) applied to these probabilities.

To observe the structure of perceived probabilities, it might be helpful to write the *dual* representation of V in the special case of a DM whose preferences for risk and ambiguity are sign-independent and who does not transform (distort) her perceived probabilities. In this case, the CEU (Schmeidler, 1989) defined by the Choquet integral

$$\mathrm{W}(f) = \int_{\mathcal{S}} \mathrm{U}(f(s))\, d\mathrm{Q}, \qquad (2)$$

with respect to a particular capacity

$$\mathrm{Q}(E) = \Upsilon^{-1}\left(\int_{\mathcal{P}} \Upsilon(\mathrm{P}(E)\, d\xi)\right) \qquad (3)$$

where $E = \{t \in \mathcal{S} \mid f(t) \ge f(s)\}$, is obtained.

The concept of modeling attitudes toward ambiguity by nonreductive first-order and second-order probabilities, suggested by Segal (1987), inspires EUUP. This concept also inspires other models: Klibanoff et al.'s (2005) smooth model, its generalization to include intertemporal substitution, proposed by Hayashi and Miao (2011) and Ju and Miao (2012), and the second-order beliefs of Nau (2006), Chew and Sagi (2008), Ergin and Gul (2009) and Seo (2009). However, unlike EUUP, these models define attitude toward ambiguity as attitude toward mean-preserving spreads in certainty equivalent utilities, which are subject to risk and preferences for risk. By applying preferences for ambiguity directly and solely to probabilities, rather than to certainty equivalent utilities, EUUP achieves the separations between risk and ambiguity and between tastes and beliefs. Thereby, it enables the measurement of ambiguity attitudes independently of risk and risk attitudes.

EUUP employs the axiomatic foundations proposed by Kopylov (2010) and Wakker (2010) for modeling ambiguity and risk preferences. It relies upon the CEU of Schmeidler (1989), whose axiomatic foundation paved the way for modeling decision making under ambiguity. Schmeidler (1989), in his pioneering study, introduces the idea that, in the presence of ambiguity, the probabilities that reflect the DM's willingness to bet may not be additive. EUUP combines the concept of nonadditive probabilities with the idea of *reference-dependent beliefs*. Reference dependence is applied to differentiate between the perceived probabilities of *unfavorable* events and the perceived probabilities of *favorable* events.[6] This broader definition allows preferences for ambiguity, which pertain to the probabilities of these events, to be different for unfavorable events and for favorable events. Unlike EUUP, Tversky and Kahneman's (1992) CPT focuses on reference-dependent tastes to model a DM who has different risk attitudes for losses and for gains and has asymmetric capacities with different arbitrary weighting schemes for losses and for gains.[7] EUUP, on the other hand, shows that capacities are not arbitrary and may well be explained by the presence of ambiguity and associated tastes.[8]

3 See, for example, Brenner and Izhakian (2016), and Izhakian and Yermack (2017).

4 In this paper, the terms perceived probabilities, subjective probabilities and capacities are used interchangeably. The term probability is used in a broad sense, i.e., it can be nonadditive and either subjective or objective.

5 Consequences are classified as unfavorable or favorable relative to a reference point.

6 Losses and gains, for example, can be considered unfavorable and favorable events, respectively.

7 Using the perception of rank dependence and cumulative functionals proposed by Weymark (1981), Quiggin (1982), Yaari (1987) and Schmeidler (1989), CPT generalizes the original prospect theory (Kahneman and Tversky, 1979) from risk to uncertainty. It modifies the probability weighting functionals of the original prospect theory, such that it always satisfies stochastic dominance and supports an infinite state space.

8 To explain capacities, EUUP does not assume asymmetric risk attitude, different ambiguity attitudes for losses and for gains, or loss aversion.

Moreover, EUUP maintains the descriptively important (empirical) components of CPT including reference dependence, different risk attitude for gains and losses, and loss aversion, which are not supported by CEU.

EUUP can be interpreted as a model of robustness in the presence of *model uncertainty*. This class of models assumes an uncertainty about the true probability law governing the realization of states, and a DM, with her concerns about misclassification, looks for a robust decision-making process; e.g., Hansen et al. (1999), and Hansen and Sargent (2001). Ambiguity in this line of models is formulated by the deviation of probability distributions from a reference distribution (reference model). A related variational preferences model, suggested by Maccheroni et al. (2006) and generalized by Grant and Polak (2007), extends this line of models. EUUP is also related to Siniscalchi's (2009) *vector expected utility*, which assumes a baseline probability and different sources of ambiguity with respect to expected utility. Other models that consider reference expected utility include those of Roberts (1980) and Quiggin and Chambers (2004), for example; or consider a reference prior: Einhorn and Hogarth (1986), and Gajdos et al. (2008), for example. Kopylov's (2006) ϵ-contamination suggests the addition of an element of confidence (around a reference prior) to generate a set of priors. Chateauneuf et al. (2007) suggest new capacities (neo-additives) obtained from a set of priors generated by ϵ-contamination. Izhakian and Izhakian (2015) suggest that real probabilities are projections of uncertain probabilities in a multidimensional probability space. These models require the identification of a reference prior whereas in EUUP it is not required.

Prelec (1998), which specializes CPT to the domain of risk, considers a more structured representation of the probability weighting functions to derive their observed properties from preference axioms. The current model furthers this approach by considering a more structured representation of probability weighting functions in the domain of ambiguity. It aims to derive the observed properties of attitudes toward ambiguity and of second-order beliefs from preference axioms over probabilities. EUUP provides a framework that aspires to accommodate ambiguity and make its applied use easier. Its unique approach has two main characteristics. First, EUUP can be employed in empirical studies; it allows for the derivation of an applicable risk-independent measure of ambiguity. Second, EUUP can be employed in behavioral studies; it provides a framework in which attitudes toward ambiguity can be elicited and characterized explicitly. This framework enables the study of the relationship between the extent of ambiguity and risk, and the relationship between risk and ambiguity attitudes; for example, by allowing the comparison of the attitudes toward ambiguity of two DMs with different risk attitudes. It helps in answering questions about the nature of attitudes toward ambiguity such as whether individuals are typified by a constant relative ambiguity aversion, a constant absolute ambiguity aversion or perhaps a decreasing (increasing) relative (absolute) ambiguity aversion. That is to say, EUUP can also be used to study the relationship between the degree of ambiguity and the DM's attitude toward this ambiguity. In finance, for example, EUUP is used to study the risk–ambiguity–return relationship (Brenner and Izhakian, 2016), and the role ambiguity plays in employees' decisions of whether to exercise their vested options (Izhakian and Yermack, 2017).

The rest of the paper is organized as follows. Section 2 establishes the setup. Section 3 constructs perceived probabilities and uses them to introduce the decision making model. Section 4 discusses the characterization of DMs' attitudes toward ambiguity. Section 5 extends the model to asymmetric risk and ambiguity preferences. Section 6 discusses applications of the model, and Section 7 concludes. All proofs are provided in the Appendix.

2. The setup

2.1. Illustration

To illustrate the EUUP idea about the nature of ambiguity and the way individuals may perceive it, consider the following example. An Ellsberg urn contains 30 balls which are either black or yellow, in an *unknown* proportion. If you draw a yellow ball (Y), you win \$1; otherwise, if you draw a black ball (B), you win nothing. Say you were offered a similar bet on a second 30-ball urn with a *known* proportion of balls. What is the known proportion of balls in the second urn that you are willing to accept in exchange for the first urn with the unknown proportion of balls? In other words, what is the certain probability that you are willing to accept in exchange for the uncertain probability? Where do you perceive the break-even probability to be? These questions imply that decision making involves a process of perceiving probabilities built upon ambiguity and attitude toward ambiguity. The question of how uncertain probabilities are perceived by DMs underlies the main idea of EUUP.

In terms of EUUP, this example is formulated as follows. The primary space is defined by the states of drawing different balls, i.e., $\{B, Y\}$, and the secondary space is defined by the set of priors, determined by the possible compositions of the urn. The probabilities of drawing different balls are thus defined by the realization of one of these possible compositions. The probability of B can be one of the 31 possible values $\frac{0}{30}, \frac{1}{30}, \frac{2}{30}, \ldots, \frac{30}{30}$, and the same for the probability of Y. The precise probability of B can be thought of as determined by a second-order latent event. Such an event can be, for example, "The experimenter puts 20 black balls and 10 yellow balls in the urn". The DM in this example does not have any information indicating which of the possible urn compositions (probabilities) are more likely, and thus she acts as if she assigns an equal weight to each possibility, i.e., ξ in Eq. (3) is uniform.

Assume a risk- and ambiguity-averse DM who does not distort perceived probabilities, i.e., $\Gamma(x) = x$ in Eq. (1), and whose attitude toward ambiguity is represented by the concave function

$$\Upsilon(\mathrm{P}(E)) = \sqrt{\mathrm{P}(E)},$$

and her attitude toward risk is represented by the concave function

$$\mathrm{U}(x) = 1 - e^{-x},$$

where $\mathrm{P}(E)$ stands for the probability of event E and x stands for consumption. While making decisions, the DM first forms her perceived probabilities as the certain probabilities she is willing to accept in exchange for the uncertain probabilities, i.e., certainty equivalent probabilities. For example, the certainty equivalent probability $\mathrm{Q}(B)$ of B, derived from the 31 subjectively equally likely probabilities $\frac{0}{30}, \frac{1}{30}, \frac{2}{30}, \ldots, \frac{30}{30}$, satisfies

$$\sqrt{\mathrm{Q}(B)} = \frac{1}{31}\sum_{i=0}^{30}\sqrt{\frac{i}{30}}.$$

Applying Eq. (3), the perceived probability of B would be

$$\mathrm{Q}(B) = \left(\frac{1}{31}\sum_{i=0}^{30}\sqrt{\frac{i}{30}}\right)^2 = 0.44.$$

Similarly, the perceived probability of the complementary event, Y, would be $\mathrm{Q}(Y) = 0.44$. Given these probabilities, which are nonadditive, the DM then assesses the value (in terms of expected utility) of the urn using a two-sided Choquet expected utility for unfavorable outcomes and for favorable outcomes. Assuming a DM who considers strictly positive outcomes to be favorable, and otherwise to be unfavorable, by Eq. (1), the utility from this bet would be

$$\begin{aligned}
\mathrm{V} &= \underbrace{[1-\mathrm{Q}(Y)]\left(1-e^{-0}\right)}_{\text{Unfavorable}} + \underbrace{\mathrm{Q}(Y)\left(1-e^{-1}\right)}_{\text{Favorable}} \\
&= \left[1-\left(\frac{1}{31}\sum_{i=0}^{30}\sqrt{\frac{i}{30}}\right)^2\right]\left(1-e^{-0}\right) \\
&\quad + \left(\frac{1}{31}\sum_{i=0}^{30}\sqrt{\frac{i}{30}}\right)^2\left(1-e^{-1}\right) = 0.28.
\end{aligned}$$

This example demonstrates the separation EUUP attains between risk and ambiguity and between tastes and beliefs. As opposed to MEU and CEU, the effect of the DM's tastes for ambiguity is distinguished from the effect of her beliefs. It demonstrates that comparative statics questions about the distinct effects of a change in beliefs (change in the number of balls in the urn), change in tastes for ambiguity (change in Υ), or change in tastes for risk (change in U) can be answered.

2.2. The primary space

In EUUP, the two tiers of uncertainty (one with respect to consequences and the other with respect to the probabilities of these consequences) are modeled by two separate state spaces. Uncertainty with respect to consequences is modeled in a primary outcome space, while uncertainty with respect to probabilities is modeled in a secondary probability space. Events and their consequences are defined by the primary space, while the (nonadditive) probabilities of these events are defined later by the secondary space.

Let $\mathcal{S}$ be an infinite state space, called the *primary space*, endowed with a σ-algebra, $\mathcal{E}$, of subsets of $\mathcal{S}$. Generic elements of this σ-algebra are called *events* and are denoted by E. A λ-system $\mathcal{H} \subset \mathcal{E}$ should be thought of as containing events with an unambiguous probability.[9] In other words, these events can be thought of as events with a known, objective probability, agreed upon by all possible measures.[10]

Assumption 1. For every $c \in [0, 1]$ there exists an event $E \in \mathcal{H}$ with the probability $\mathrm{P}(E) = c$.[11]

Define $\mathcal{X} \subseteq \mathbb{R}$ to be a convex set of *consequences* that contains the interval $[0, 1]$, where, since this paper mostly deals with monetary outcomes, consequences are confined to real numbers. Let a *primary (first-order) act* $f : \mathcal{S} \to \mathcal{X}$ be a bounded $\mathcal{E}$-measurable function from states into consequences, and denote the set of all these (Savage) acts by $\mathcal{F}$. A *simple* primary act can be represented as a sequence of pairs,

$$f = (E_1 : x_1, \ldots, E_n : x_n),$$

where $(E_1, \ldots, E_n)$ is a generic partition of the state space $\mathcal{S}$; x_j is the consequence if event E_j occurs; and the consequences $x_1, \ldots, x_n$ are listed in a non-decreasing order. The set of all simple measurable primary acts is denoted $\mathcal{F}_0$. A primary act yielding the same consequence for any state $s \in \mathcal{S}$ is called a *constant act* and is designated by its resulting consequence $x \in \mathcal{X}$. A *positive (primary) indicator act*

$$\delta_E = \left(E^C : 0, E : 1\right)$$

assigns the outcome 1 to event $E \in \mathcal{E}$ and the outcome 0 to its complementary event $E^C \in \mathcal{E}$. A *negative indicator* act

$$-\delta_E = \left(E : -1, E^C : 0\right)$$

assigns the outcome -1 to event $E \in \mathcal{E}$ and the outcome 0 to its complementary event $E^C \in \mathcal{E}$. A primary act $l \in \mathcal{F}_0$ is said to be an *unambiguous act* (a *lottery*) if it is $\mathcal{H}$-measurable, i.e., $l^{-1}(x) \in \mathcal{H}$ for each $x \in \mathcal{X}$. An unambiguous (positive) indicator act is denoted $\mathbf{1}_E$.

A consequence $x \in \mathcal{X}$ is considered to be *unfavorable* if $x \le 0$ and *favorable* if $0 < x$, where $k = 0$ is the *reference point.*[12] An event $E \in \mathcal{E}$ is considered to be unfavorable under act f if $f(s) \le 0$ for any $s \in E$ and favorable if $0 < f(s)$ for any $s \in E$. A *capacity* G is a function from events $\mathcal{E}$ to $[0, 1]$, satisfying $\mathrm{G}(\emptyset) = 0$, $\mathrm{G}(\mathcal{S}) = 1$, and set monotonicity with respect to set-inclusion, i.e., if $E \subset F$ then $\mathrm{G}(E) \le \mathrm{G}(F)$.

Denote by $\succsim^1$ the DM's preference relation over the set of primary acts $\mathcal{F}_0$, and let the relations $\precsim^1$, $\prec^1$, $\succ^1$ and $\sim^1$ be defined as usual.

Assumption 2. The preference relation $\succsim^1$ over the set of primary acts $\mathcal{F}_0$ satisfies Wakker's (2010, Theorem 12.3.5) axioms.

Assumption 3. For any $E, F \in \mathcal{E}$,

$$\delta_E \succsim^1 \delta_F \iff -\delta_{E^C} \succsim^1 -\delta_{F^C}.$$

Assumption 2 is equivalent to the existence of a function $\mathrm{v} : \mathcal{F}_0 \to \mathbb{R}$ such that

$$f \succsim^1 g \iff \mathrm{v}(f) \ge \mathrm{v}(g),$$

for any $f, g \in \mathcal{F}_0$, where

$$\begin{aligned}
\mathrm{v}(f) &= \int_{z\le 0}\left[\mathrm{G}^-(\{s \in \mathcal{S} \mid \mathrm{U}(f(s)) \ge z\}) - 1\right]dz \\
&\quad + \int_{z\ge 0}\mathrm{G}^+(\{s \in \mathcal{S} \mid \mathrm{U}(f(s)) \ge z\})dz; \qquad (4)
\end{aligned}$$

$\mathrm{U} : \mathcal{X} \to \mathbb{R}$ is a strictly increasing continuous bounded utility function, satisfying $\mathrm{U}(k = 0) = 0$; and G^- and G^+ are capacities. By Wakker and Tversky (1993, Equation 6.2), the dual representation of Eq. (4) is

$$\mathrm{w}(f) = \int_{\mathcal{S}}\mathrm{U}\left(f^-(s)\right)d\hat{\mathrm{G}} + \int_{\mathcal{S}}\mathrm{U}\left(f^+(s)\right)d\mathrm{G}^+,$$

where $f^-(s) = f(s)$ if $f(s) \le 0$ and $f^-(s) = 0$ if $f(s) > 0$; $f^+(s) = f(s)$ if $f(s) \ge 0$ and $f^+(s) = 0$ if $f(s) < 0$; and $\hat{\mathrm{G}}(E) = \left[1 - \mathrm{G}^-\left(E^C\right)\right]$. Both v and w deliver a rank-dependent representation in the presence of ambiguity à la Quiggin (1982) and Schmeidler (1989). Assumption 3 implies that $\mathrm{G}^-(E) = \mathrm{G}^+(E)$ for every $E \in \mathcal{E}$. This is dropped in Section 5.

The function v applies a sign-dependent Choquet integration to unfavorable consequences and to favorable consequences (relative to the reference point k), where the utility function U is unique up to a positive scaling and the capacities G^- and G^+ are uniquely

9 The set $\mathcal{H}$ may not necessarily be an algebra as it may not be closed with respect to intersections. Yet, a probability measure can be defined on $\mathcal{H}$ (Epstein and Zhang, 2001).

10 These settings apply to any case where there is a sub σ-algebra isomorphic to the Borel sets on [0, 1] endowed with Lebesgue measure. In this case, the extended state space can be defined by $\mathcal{S} \times [0, 1]$; e.g., Sarin and Wakker (1992).

11 Note that this assumption is less restrictive than an alternative assumption of a state space $\mathcal{S} \times (0, 1]$, endowed with a product σ-algebra of $\mathcal{E}$ and a Borel σ-algebra of (0, 1].

12 The reference point, k, also referred to as the status quo, is subjectively defined by each DM and may be affected by the formulation of the offered prospects; e.g., Kahneman and Tversky (1979). For simplicity, as in Tversky and Kahneman (1992), Wakker and Tversky (1993, Equation 6.2), and Prelec (1998), it is assumed that $k = 0$.

determined.[13] This CPT representation generalizes the CEU representation by adding reference-dependent tastes, allowing attitude toward risk concerning unfavorable consequences to be different from attitude toward risk concerning favorable consequences, where attitudes toward risk are characterized by U.[14] Note that smaller values of a capacity imply lower values of acts (in terms of expected utility), regardless of whether the capacity is of unfavorable events or of favorable events.[15] CPT and CEU do not attempt to delve into the sources that shape capacities, which are determined by an arbitrary weighting scheme. In particular, they do not provide a distinction between the DM's beliefs and tastes concerning probabilities (ambiguity and attitude toward it), which determine the way she may perceive the likelihoods of events occurring.

By Assumptions 1 and 2, the following observation is immediate.

Observation 1. *For every event $E \in \mathcal{E}$ there exists an event $F \in \mathcal{H}$ with an unambiguous probability such that $\delta_E \sim^1 \mathbf{1}_F$.*

Preferences over primary acts thus generate preferences over consequences. Thus, by the convexity of $\mathcal{X}$ and the fact that $\mathcal{F}_0$ consists of all simple acts, the following observation is delivered.

Observation 2. *For every primary act $f \in \mathcal{F}_0$ there exists a certainty equivalent (constant act) $x \in \mathcal{F}_0$ satisfying $f \sim^1 x$.*

2.3. The secondary space

To explicitly model the uncertainty of probabilities in the primary space and formally identify the DM's beliefs and tastes concerning this uncertainty, one can consider preferences over secondary acts in a secondary space.

Probabilities of events $\mathcal{E}$ occurring in the primary space are determined in a nonempty *secondary space*, defined by a set $\mathcal{P}$ of all *additive* probability measures over the primary space $\mathcal{S}$ that agree on $\mathcal{H}$. Namely, $\mathcal{P}$ is viewed as a sort of "secondary state space" consisting of all possible probability measures over $\mathcal{S}$, and a *first-order probability* measure $\mathrm{P} \in \mathcal{P}$ is viewed as a state of nature in this secondary space. In this view, the state space $\mathcal{P}$ is assumed to be endowed with an algebra, $\Pi \subset 2^{\mathcal{P}}$, of subsets of $\mathcal{P}$. Π is assumed to satisfy the structure required by Kopylov (2010).

A *secondary (second-order) act*, $\widehat{f} : \mathcal{P} \to \mathcal{X}$, is a bounded Π-measurable function from the secondary space $\mathcal{P}$ into the set of consequences $\mathcal{X}$. The set of *all* secondary acts is denoted $\widehat{\mathcal{F}}$. A secondary act $\widehat{f}$ that describes the resulting expected outcome of a primary act $f \in \mathcal{F}$ contingent upon a prior $\mathrm{P} \in \mathcal{P}$ (on $\mathcal{S}$), is denoted $\hat{f}$; that is, $\hat{f} : \mathcal{P} \to \mathcal{X}$ satisfies

$$\hat{f}(\mathrm{P}) = \int_{\mathcal{S}} f d\mathrm{P}.$$

The set of *all* these secondary acts $\hat{f} \in \widehat{\mathcal{F}}$ is denoted $\hat{\mathcal{F}}$. In this framework, an unambiguous primary act $l \in \mathcal{F}$ is associated with a constant secondary act $\hat{l} \in \widehat{\mathcal{F}}$, assigning every $\mathrm{P} \in \mathcal{P}$ with the same $x \in \mathcal{X}$.

The class of secondary acts in $\widehat{\mathcal{F}}$ that are associated with (primary) indicator acts is our main interest. A positive secondary act $\hat{\delta}_E : \mathcal{P} \to [0, 1]$ in this class, associated with a positive indicator act $\delta_E \in \mathcal{F}_0$, is given by

$$\hat{\delta}_E(\mathrm{P}) = \mathrm{P}(E)$$

for every $\mathrm{P} \in \mathcal{P}$. Similarly, a negative secondary act $-\hat{\delta}_E : \mathcal{P} \to [-1, 0]$ in this class, associated a negative indicator act $-\delta_E \in \mathcal{F}_0$, is given by

$$-\hat{\delta}_E(\mathrm{P}) = -\mathrm{P}(E)$$

for every $\mathrm{P} \in \mathcal{P}$. A secondary act $\hat{\delta}_E$ can, therefore, be viewed as a function that assigns the event $E \in \mathcal{E}$ with its possible probabilities. In this view, $\hat{\delta}_E$ can be interpreted as an uncertain variable describing the probability $\mathrm{P}(E)$ of event E.

Observation 3. *For every $E \in \mathcal{E}$ there exists $\hat{\delta}_E \in \widehat{\mathcal{F}}$.*

This observation is immediate by the construction of $\widehat{\mathcal{F}}$ from $\mathcal{F}$, which consists of all bounded $\mathcal{E}$-measurable functions $f : \mathcal{S} \to \mathcal{X}$. Notice that $\hat{\delta}_E$ can be a constant secondary act if and only if $E \in \mathcal{H}$. The subset of all secondary acts $\hat{\delta}_E$ in $\widehat{\mathcal{F}}$ is denoted $\hat{\Delta}$, and the subset of all secondary acts $-\hat{\delta}_E$ in $\widehat{\mathcal{F}}$ is denoted $-\hat{\Delta}$.

Assumption 4. The preference relation $\succsim^2$ over the set of secondary acts $\widehat{\mathcal{F}}$ satisfies the extended Savage axioms of Kopylov (2010) and

$$x > y \iff x \succ^2 y,$$

for all $x, y \in \mathcal{X}$.

This assumption implies a unique Υ up to a positive linear transformation and a unique countably-additive measure ξ on Π such that

$$\widehat{f} \succsim^2 \widehat{g} \iff \int_{\mathcal{P}} \Upsilon\left(\widehat{f}(\mathrm{P})\right) d\xi \geq \int_{\mathcal{P}} \Upsilon\left(\widehat{g}(\mathrm{P})\right) d\xi.$$

The second-order countably-additive probability measure ξ on Π assigns each subset $A \in \Pi$ of first-order probability measures in $\mathcal{P}$ with a probability $\xi(A)$. The probability measure ξ may be thought of as the DM's *subjective* assessment of the likelihood that the probability measure P is in the subset A of $\mathcal{P}$. Kopylov's (2010) axioms are assumed instead of the Savage axioms since secondary acts do not generally have a finite range and evaluating their expected utility would generally require a countably additive measure.

In the view of $\hat{\delta}_E : \mathcal{P} \to [0, 1]$ as describing the (uncertain) probability $\mathrm{P}(E)$ of event E, the preference relation $\succsim^2$ over $\hat{\Delta}$ may well be referred to as a preference over probabilities. To understand this interpretation, consider the two secondary acts $\hat{\delta}_E, \hat{\delta}_F \in \hat{\Delta}$, associated with the indicator acts $\delta_E, \delta_F \in \mathcal{F}$ (whose outcomes are the same), and assume a DM who prefers $\hat{\delta}_E$ to $\hat{\delta}_F$. This means that she prefers to get the good outcome with the (uncertain) probability $\{\mathrm{P}(E)\}_{\mathrm{P} \in \mathcal{P}}$ than with the (uncertain) probability $\{\mathrm{P}(F)\}_{\mathrm{P} \in \mathcal{P}}$.

EUUP proposes that $\succsim^2$ is utilized solely to elicit the DM's certainty equivalent probability of each event. That is, given a secondary act $\hat{\delta}_E$, the preference $\succsim^2$ is used only to elicit a constant secondary act $p_E \in \hat{\Delta}$ that satisfies $\hat{\delta}_E \sim^2 p_E$. The second-order preference relation $\succsim^2$ over $\hat{\Delta}$ can, therefore, be interpreted as applied directly to probabilities, and each p_E, formed separately for each event, may well be referred to as the *perceived probability* of event E. The existence of $p_E \in \hat{\Delta}$, for any $p_E \in [0, 1]$, is delivered by Assumption 1. This, together with Kopylov's axioms, implies the follows.

Observation 4. *For every $\hat{\delta}_E \in \hat{\Delta}$ there exists a certainty equivalent (constant act) $p_E \in \hat{\Delta}$, satisfying $\hat{\delta}_E \sim^2 p_E$.*

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13 For more on the uniqueness of U, G^- and G^+, see also Köbberling and Wakker (2003, Observation 13).

14 The CPT representation, defined by v, differs from the Choquet integral in that the latter satisfies a "shift" axiom (i.e., $\int_{\mathcal{S}} (f + c)\, d\mathrm{G} = \int_{\mathcal{S}} f d\mathrm{G} + c$ for every f and c), whereas the former is sensitive to shifts of outcomes crossing the reference point.

15 Bommier (2014) suggests that this CPT representation is more suitable for discussing aversion to ambiguity concerning unfavorable events.

The use of secondary acts might be challenged by the claim that these acts are imaginary objects and that a DM cannot bet on a set of probability measures, as the realized probabilities are not verifiable. However, Klibanoff et al. (2005, page 1854), who also use the concept of secondary acts, stress that *"second order acts are not as strange or unfamiliar as they might first appear. (...) In a parametric portfolio investment example, these could be bets about the parameter values that characterize the asset returns, e.g., means, variances, and covariances. Similarly, in model uncertainty applications, second order acts are bets about the values of the relevant parameters in the underlying model. Closer to decision theory, for an Ellsberg urn, second order acts may be viewed as bets on the composition of the urn."* They further stress that "*Even when verifiability is an issue, preference axioms provide a useful conceptual underpinning to choice criteria. For example, economists often apply the subjective expected utility model to a variety of situations characterized by limited verifiability. For instance, consider an investors portfolio choice problem. The relevant states of the world for a particular stock may include events that take place inside the firm and in the wider market. It cannot be claimed that it is easy to verify, if at all, which of these relevant states actually obtain.*" Implicitly or explicitly, many other studies assume second-order acts or second-order beliefs. For example, Ju and Miao (2012), Hayashi and Miao (2011) and Seo (2009), which are underpinned by the smooth model of ambiguity, or Nau (2006), and Chew and Sagi (2008), to name a few. Some earlier formal decision theory studies also rely on similarly unobservable acts. For instance, the analytical framework modeling state-dependent preferences postulates the existence of a preference relation on hypothetical lotteries whose prizes are outcome-state pairs; e.g., Karni and Schmeidler (1981), Karni et al. (1983), and Grant and Karni (2004).

Indeed, technically, in most cases (secondary) acts are not observable and beliefs are not verifiable, even in the frameworks of classical decision-making models that do not consider ambiguity. For example, to elicit beliefs in Savage's framework the DM has to be presented with infinitely many choices. In MEU (Gilboa and Schmeidler, 1989), for example, the DM acts *as if* she chooses from among sets of priors, while the realized prior is not verifiable. The question, however, is whether when considering a DM who acts *as if* she chooses from among secondary acts (or from among sets of priors), beliefs have to be verifiable. To answer this question, one has to find circumstances in which a DM acts as if she prefers one set of priors over the other. This is the case in the Ellsberg urn choices, since the DM bets on the composition of the urn (probabilities). In this example, however, beliefs can be verified by counting the number of different colored balls in the urn.

In contrast, consider the point-spread wagering on football games in the National Football League (NFL), which is designed as follows. The oddsmaker establishes a point-spread line, k, by the difference in units of game points between the score of the winning team and the score of the losing team. A negative k means that the favorite team is expected to win the game in k points or more. Given the point-spread line k, a gambler chooses whether to wager that the favorite team will win the game by k points or more, or alternatively that the underdog team will win the game or lose by fewer than k points.[16] The bookmaker establishes a point-spread line such that the expected probabilities of winning a bet against and in favor of the favorite team are the same. In this circumstance, the decision of whether to wager on game A or game B can be viewed as betting on a set of possible probabilities, since the prizes and the expected probabilities associated with A and B are the same (and therefore also the risk). A gambler, who decides to wager on game A rather than on B, can be viewed as if she prefers the set of probabilities associated with game A over the set associated with game B. Clearly, in this case, the realized probabilities are not verifiable.

Previous behavioral studies also indicate that DMs act as if they have second-order preferences; e.g., Ellsberg (1961), Becker and Brownson (1964), Halevy (2007), and Hao and Houser (2012). Furthermore, it has been found that neural responses to second-order uncertainty (ambiguity) are associated with brain areas (e.g., Huettel et al., 2006; Bach et al., 2009 and Bach et al., 2011) that are distinct from those supporting first-order uncertainty (e.g., Dreher et al., 2006; Preuschoff et al., 2006 and Tobler et al., 2007). All decision situations produced in a laboratory setting can actually be designed to create and observe the resolution of secondary acts by allowing subjects to view (ex-post) the mechanism that generated the uncertainty about probabilities. For example, the number of balls in an Ellsberg urn can be determined by a randomizing device, such as a roulette wheel, whose resolution is (ex-post) observable to the subjects.

3. The decision-making model

The main idea of EUUP is that risk and preference for risk, formed by $\succsim^1$, apply to consequences in the primary space, whereas ambiguity and preference for ambiguity, formed by $\succsim^2$, apply to probabilities in the secondary space.

3.1. Consistency of preferences

The first-order preference $\succsim^1$ on the set of primary acts $\mathcal{F}_0$ satisfies Wakker (2010, Theorem 12.3.5) axioms (or, alternatively, other CPT axiomatization). The second-order preference $\succsim^2$ on a set of secondary acts $\widehat{\mathcal{F}}$ satisfies Kopylov (2010) axioms (or, alternatively, some other suitable SEU axiomatization). EUUP postulates an additional axiom to maintain the consistency of $\succsim^1$ with $\succsim^2$. In particular, this consistency axiom applies to the relation between indicator acts and between their associated secondary acts.

Axiom 1 (*Certainty Equivalent Probabilistic Consistency*)**.** Let $\mathbf{1}_F \in \mathcal{F}_0$ be an unambiguous indicator act whose associated (constant) secondary act pays some $c \in [0, 1]$. For every event $E \in \mathcal{E}$,

$$\delta_E \sim^1 \mathbf{1}_F \iff \hat{\delta}_E \sim^2 c.$$

In the view of $\hat{\delta}_E$ as the uncertain probability of event E, Axiom 1 suggests that certainty equivalent probabilities can be elicited by the preference over ambiguous and unambiguous (primary) indicator acts. The *certainty equivalent probability*, $c \sim^2 \hat{\delta}_E$, of a secondary act $\hat{\delta}_E$ is the certain probability that the DM is willing to accept in exchange for the uncertain probability induced by $\hat{\delta}_E$. It can, therefore, be interpreted as the *perceived probability* of event E. In this view, Axiom 1 provides a measurement tool for the DM's perceived probabilities.

It is important to note that the second-order preference relation $\succsim^2$ applies to the entire set of secondary acts $\widehat{\mathcal{F}}$ only for the technical reasons dictated by the Savage framework. Since $\succsim^2$ is used solely to identify the DM's certainty equivalent probability (perceived probability) of each event $E \in \mathcal{E}$ separately, the consistency axiom (Axiom 1) applies only to indicator acts and their associated secondary acts and not to all primary acts in $\mathcal{F}_0$ and their associated secondary acts in $\widehat{\mathcal{F}}$. Later, in Theorem 1, it is shown that only secondary acts that are associated with indicator acts (i.e., $\hat{\delta}_E \in \hat{\Delta}$) play a role in the functional representation of the DM's preferences.

[16] Standard bets on NFL games are made on the "11 for 10" rule: a gambler must lay out \$11 to win \$10.

A particular case of Axiom 1 is when the DM is risk neutral. The next observation refers to the relationship between first-order risk-neutral preference $\succsim^1$ over indicator acts and second-order preference $\succsim^2$ over their associated secondary acts. It shows that, for a risk-neutral DM, the certainty equivalent of an indicator act is identical to the certainty equivalent of its related secondary act. It followed from defining risk neutrality as $\mathbf{1}_F \sim^1 c$, where $c = \mathrm{P}(F)$ for every $\mathrm{P} \in \mathscr{P}$.

Observation 5. *Let $\mathbf{1}_F \in \mathcal{F}_0$ be an unambiguous indicator act whose associated (constant) secondary act pays some constant $c \in [0, 1]$. For a risk-neutral DM,*

$$\delta_E \sim^1 c \iff \hat{\delta}_E \sim^2 c,$$

for every event $E \in \mathcal{E}$.

Axiom 1 establishes a link between $\succsim^1$ and $\succsim^2$, i.e., between preference over consequences and preference over their probabilities. It provides the axiomatic basis for ordering events by their (uncertain) probabilities. This ordering is crucial for structuring perceived probabilities, based upon the DM's second-order belief and second-order taste (taste for ambiguity), from her observable choices. Using Axiom 1, the next proposition implies an ordering of events by their (uncertain) probabilities.

Proposition 1. *Given Assumptions 1–4, for all events $E, F \in \mathcal{E}$,*

$$\delta_E \succsim^1 \delta_F \iff \hat{\delta}_E \succsim^2 \hat{\delta}_F.$$

This proposition demonstrates that first-order preference over indicator acts is determined solely by the second-order preference over their associated secondary acts. It means that the DM prefers to get the better outcome if the event with the preferred probabilities occurs (with 0 being the alternative); that is, if the event with the higher perceived probability occurs. Proposition 1, therefore, provides a measurement tool for ordering events by their (uncertain) probabilities.

To avoid the relation between risk and ambiguity retained in the smooth model (Klibanoff et al., 2005) and refine the separation between risk and ambiguity, Axiom 1 weakens the consistency of first-order preferences $\succsim^1$ with second-order preferences $\succsim^2$. In the smooth model, Klibanoff et al. (2005, Assumption 3), first-order preferences satisfy $f \succsim^1 g \iff f^2 \succsim^2 g^2$ over all acts, where $f^2(\mathrm{P})$ is the certainty equivalent of f computed using $\mathrm{P} \in \mathscr{P}$. Thus, the source of the preference relation $f^2 \succsim^2 g^2$ cannot be determined explicitly, since f^2 is subject to risk and tastes for risk. On the other hand, by Axiom 1 (and Proposition 1), first-order preferences are required to satisfy $\delta_E \succsim^1 \delta_F \iff \hat{\delta}_E \succsim^2 \hat{\delta}_F$ over only indicator acts. This weakened consistency allows preferences for ambiguity, $\succsim^2$, to apply directly and solely to probabilities (rather than to certainty equivalents, which are a function of risk and risk preferences) and be defined as preferences for mean-preserving spreads in probabilities.

3.2. Perceived probabilities

The DM's subjective perception of likelihoods, resulting from aversion to or love for ambiguity, plays an important role in her decision process. Perceived probabilities are determined by attitudes toward ambiguity, defined by preferences over secondary acts $\hat{\Delta}$. These preferences may well be viewed as preferences over uncertain probabilities, where uncertain probabilities take the form of a set of probability measures and are viewed as states of nature in a secondary space that is subject to a prior probability. This concept extends the Bayesian approach, which asserts that everything that is not known should be modeled explicitly in a state space and be subject to a prior probability. Based upon this concept, the next proposition ties preferences over uncertain probabilities, i.e., preferences concerning ambiguity, to a functional representation.

Proposition 2. *Under Assumption 4, there exists a function $\mathrm{Q} : \hat{\Delta} \to [0, 1]$ such that*

$$\hat{\delta}_E \succsim^2 \hat{\delta}_F \iff \mathrm{Q}(\hat{\delta}_E) \geq \mathrm{Q}(\hat{\delta}_F)$$

for every $\hat{\delta}_E, \hat{\delta}_F \in \hat{\Delta}$, where

$$\mathrm{Q}(\hat{\delta}_E) = \Upsilon^{-1}\left(\int_{\mathscr{P}} \Upsilon\left(\hat{\delta}_E(\mathrm{P})\right) d\xi\right); \tag{5}$$

$\Upsilon : [0, 1] \to \mathbb{R}$ is a non-constant bounded function; and ξ is a non-atomic countably-additive probability measure. Furthermore, ξ is unique, and Υ is unique up to a positive linear transformation.

This proposition provides a theoretical foundation for extracting second-order belief, formed by ξ, and second-order taste (taste for ambiguity), formed by the *probability-outlook function* Υ, from the DM's preferences over secondary acts. It suggests a construct of perceived probabilities based upon the nonlinear way that individuals may see probabilities. Abusing notation, the *perceived probability* $\mathrm{Q}(E) = \mathrm{Q}(\hat{\delta}_E)$ of event E is determined by the DM's attitude toward ambiguity, defined by her preference over uncertain probabilities, i.e., $\succsim^2$ over secondary acts in $\hat{\Delta}$. These probabilities can be elicited, for example, by a simple variation of the Ellsberg two-color experiment. Subjects can be asked to propose an urn with a known proportion of balls which they are willing to accept in exchange for an urn with an unknown proportion of balls, where the prizes attached to each color are the same in both urns.

For the moment, assume that, as with risk attitudes, there are three types of attitudes toward ambiguity: ambiguity aversion, ambiguity loving and ambiguity neutrality. Ambiguity neutrality takes the form of a linear Υ; ambiguity aversion takes the form of a concave Υ; and ambiguity loving takes the form of a convex Υ. Later, Section 4 derives these functional forms of attitudes toward ambiguity from second-order preferences $\succsim^2$ and characterizes them.

An ambiguity-neutral DM can be considered a DM who reduces two-stage lotteries to compound lotteries in the usual way.[17] The perceived probabilities of an ambiguity-neutral DM are additive and equal to the expected probabilities. That is, $\mathrm{Q}(E) = \mathrm{E}[\mathrm{P}(E)]$, where the expected probability[18]

$$\mathrm{E}[\mathrm{P}(E)] = \int_{\mathscr{P}} \mathrm{P}(E)\, d\xi$$

is taken with respect to the second-order probabilities ξ. The perceived probabilities of an ambiguity-averse DM are lower than the expected probabilities, i.e., $\mathrm{Q}(E) < \mathrm{E}[\mathrm{P}(E)]$, and result in subadditive probabilities. The perceived probabilities of an ambiguity-loving DM are greater than the expected probabilities, i.e., $\mathrm{Q}(E) > \mathrm{E}[\mathrm{P}(E)]$, and result in superadditive probabilities. Consider for example an ambiguity-averse DM, Eq. (5) implies that a higher aversion to ambiguity (a more concave Υ function) or higher ambiguity (higher dispersions of probabilities), both result in lower perceived probabilities which in turn result in a lower value of the act (in terms of expected utility).[19] Perceived probabilities are uniquely

17 For the implications of this type of preferences see, for example, Halevy (2007).

18 Notice that the operator $\mathrm{E}[\cdot]$ (in straight font followed by square parentheses) stands for expectation, while the variable E (in slanted italic font) stands for an event in $\mathcal{E}$.

19 This result coincides with Ghirardato and Marinacci (2002) who show that a higher aversion to ambiguity implies lower capacities.

determined because Υ is unique up to a positive linear transformation and ξ is unique. The uniqueness of Υ and ξ is obtained since $\succsim^2$ satisfies Kopylov (2010) axioms. Clearly, the perceived probabilities of events with an unambiguous probability, i.e., any event $E \in \mathcal{H}$, are not affected by attitudes toward ambiguity.

3.3. The representation

The reduction of the family of priors $\mathcal{P}$ to a perceived (nonadditive) probability measure Q, is carried out using $\succsim^2$ over secondary acts in $\hat{\Delta}$. Using Q, the assessment of the values of primary acts is carried out using v in Eq. (4). The next theorem knits $\succsim^1$ with $\succsim^2$ to introduce a functional representation of these preferences.

Theorem 1. *Suppose that $\succsim^1$ and $\succsim^2$ satisfy Assumptions 1–4 and Axiom 1. Then, there exists a function $\mathrm{V} : \mathcal{F}_0 \to \mathbb{R}$ such that*

$$f \succsim^1 g \iff \mathrm{V}(f) \geq \mathrm{V}(g),$$

for every $f, g \in \mathcal{F}_0$, where

$$\begin{aligned}\mathrm{V}(f) &= \int_{z\leq 0} \left[\Gamma\left(\Upsilon^{-1}\left(\int_{\mathcal{P}} \Upsilon\left(\hat{\delta}_{\{s\in\mathcal{S}\mid \mathrm{U}(f(s))\geq z\}}(\mathrm{P})\right) d\xi\right)\right) - 1\right] dz \\ &\quad + \int_{z\geq 0} \Gamma\left(\Upsilon^{-1}\left(\int_{\mathcal{P}} \Upsilon\left(\hat{\delta}_{\{s\in\mathcal{S}\mid \mathrm{U}(f(s))\geq z\}}(\mathrm{P})\right) d\xi\right)\right) dz; \qquad (6)\end{aligned}$$

$\mathrm{U} : \mathcal{X} \to \mathbb{R}$ is a continuous strictly increasing bounded function, normalized such that $\mathrm{U}(0) = 0$[20]; $\Upsilon : [0, 1] \to \mathbb{R}$ is a continuous non-constant bounded function; $\Gamma : [0, 1] \to [0, 1]$ is a continuous strictly-increasing function; ξ is a non-atomic countably-additive probability measure; and every $\mathrm{P} \in \mathcal{P}$ is finitely-additive probability measure. Furthermore, ξ is unique, Υ is unique up to a positive linear transformation, U is unique up to a positive scaling, and Γ is unique.

The functional representation of the DM's aggregate preferences, established in Theorem 1, makes a complete distinction between beliefs and tastes, and between risk and ambiguity. First-order beliefs are formed by the uncertain probability measure P; second-order beliefs (or information) are formed by the probability measure ξ; risk attitudes are formed by the utility function U; ambiguity attitudes are formed by the outlook function Υ; and subjective weightings of perceived probabilities are formed by Γ. The following observation pertains to an important property of this functional representation.

Observation 6. *Suppose that the assumptions of Theorem 1 hold. Then, the perceived probabilities in the functional representation introduced in Eq. (6) are uniquely determined.*

The uniqueness of perceived probabilities is immediately obtained from the uniqueness of ξ and the uniqueness of Υ up to a positive linear transformation. Apart from providing what appears to be a new clarifying perspective on perceived probabilities and their construction, the model in Eq. (6) is particularly useful for measuring the degree of ambiguity, independently of risk. If ξ is objective, as it is arguably in the classical Ellsberg choice problem, then the concept underpinning the construction of perceived probabilities defines aversion to ambiguity as aversion to objective mean-preserving spreads in probabilities; otherwise it defines aversion to ambiguity as aversion to subjective mean-preserving spreads in probabilities. As such, Rothschild and Stiglitz (1970) approach, which is typically applied to outcomes when measuring the extent risk, can be applied to probabilities when measuring the extent of ambiguity (Izhakian, 2016).[21] The model in Eq. (6) is also useful in economic modeling that aims to answer comparative statics questions that involve ambiguity such as how the equilibrium would change when the extent of ambiguity changes, or how the equilibrium would change if the aversion to ambiguity changes.

Theorem 1 assumes that $\succsim^1$ satisfies Wakker's (2010) axioms to support reference-dependent preferences for risk and ambiguity. However, alternative sets of CPT axioms can be adopted; e.g., Tversky and Kahneman (1992), Wakker and Tversky (1993), Hong and Wakker (1996), and Kothiyal et al. (2011). When considering a DM whose preferences are reference-independent, the CEU axioms of Schmeidler (1989) and the mixture-space axioms of Ghirardato et al. (2003), for example, can also be adopted.[22] Similarly, although $\succsim^2$ is assumed to satisfy Kopylov's (2010) extension of Savage's axioms, alternative sets of axioms can be adopted, as long as they provide the uniqueness of Υ up to a positive linear transformation and a suitable and unique ξ; e.g., Wakker (1993) and Kopylov (2007).

To simplify the functional representation proposed in Theorem 1, secondary acts can be replaced by their resulting probabilities to obtain

$$\begin{aligned}\mathrm{V}(f) &= \int_{z\leq 0} \left[\Gamma\left(\Upsilon^{-1}\left(\int_{\mathcal{P}} \Upsilon\left(\mathrm{P}\left(\{s\in\mathcal{S}\mid \mathrm{U}(f(s))\geq z\}\right)\right) d\xi\right)\right) - 1\right] dz \\ &\quad + \int_{z\geq 0} \Gamma\left(\Upsilon^{-1}\left(\int_{\mathcal{P}} \Upsilon\left(\mathrm{P}\left(\{s\in\mathcal{S}\mid \mathrm{U}(f(s))\geq z\}\right)\right) d\xi\right)\right) dz, \qquad (7)\end{aligned}$$

where, as in Eq. (6), $\mathrm{U} : \mathcal{X} \to \mathbb{R}$ is a continuous strictly increasing bounded function, normalized around a reference point $k = 0$ such that $\mathrm{U}(0) = 0$; $\Upsilon : [0, 1] \to \mathbb{R}$ is a continuous non-constant bounded function; $\Gamma : [0, 1] \to [0, 1]$ is a continuous strictly increasing function; ξ is a non-atomic countably-additive probability measure over $\mathcal{P}$; and every $\mathrm{P} \in \mathcal{P}$ is a finitely-additive probability measure. Recall that U captures attitudes toward risk, Υ captures attitudes toward ambiguity, ξ captures beliefs, and Γ captures probability weightings. Notice that in the particular case of ambiguity neutrality (i.e., perceived probabilities are additive, and no distortion of perceived probabilities) the functional representation in Eq. (7) collapses to the classical expected utility representation.

In classical (subjective) expected utility theory, it is commonly suggested that a given DM displays the same risk attitude across settings in which she might hold different (subjective) beliefs. Theorem 1 maintains this property. In EUUP, risk attitude toward unambiguous primary acts (lotteries) and toward ambiguous primary acts is the same and independent of the DM's beliefs and attitude toward ambiguity. On the other hand, ambiguity attitude is independent of the DM's beliefs and attitude toward risk.

It may be useful to consider a case where objective probabilities are provided to the DM. Suppose that the DM is informed about the set of possible priors $\mathcal{P}$ and that each $\mathrm{P} \in \mathcal{P}$ is obtained with an objective probability $\xi(\{\mathrm{P}\})$. This can be generated by a randomizing device, such as a roulette wheel, that determines the number of balls in an Ellsberg urn. It is plausible that the DM would then view the objective probability distributions over consequences as her subjective set of priors and would adopt the roulette's probabilities as her second-order belief. In fact, Halevy (2007) finds that typically, while facing two-stage objective lotteries, individuals violate reduction of compound (objective) lotteries and exhibit ambiguity aversion behavior. Under EUUP,

20 Recall that the reference point (the "status quo") k is assumed to be 0.

21 Section 6 further elaborates on the applicability of EUUP for measuring the degree of ambiguity.

22 Ghirardato et al. (2003) simplify the CEU axiomatization (Schmeidler, 1989) by not using Anscombe and Aumann (1963) framework with objective lotteries.

such behavior can be explained by a nonlinear compounding of (objective) probabilities.

At first glance, the model proposed in Eq. (7) resembles the smooth model (Klibanoff et al., 2005). However, although both models rely upon the idea of second-order preferences over secondary acts, they are *conceptually* different. In the smooth model, second-order preferences are applied to certainty equivalent outcomes, which are subject to risk and preferences for risk, and aversion to ambiguity is defined by aversion to mean-preserving spreads in the related certainty equivalent utilities. As such, these preferences are captured by a function over expected utilities, where each expected utility corresponds to a $\mathrm{P} \in \mathcal{P}$, i.e., an "expected utility over expected utilities".[23] Unlike the smooth model, preferences for ambiguity in EUUP are applied directly to probabilities and aversion to ambiguity is defined as aversion to (subjective) mean-preserving spreads in probabilities. This important quality allows EUUP to facilitate a risk-independent measure of ambiguity.

4. Attitudes toward ambiguity

Attitudes toward ambiguity can be defined by the DMs' preferences over uncertain probabilities of event and their expected probabilities, i.e., preferences over secondary acts and constant secondary acts in $\hat{\Delta}$, respectively. An ambiguity-averse DM prefers the expectations of the uncertain probabilities over the uncertain probabilities themselves. An ambiguity-loving DM prefers the uncertain probabilities over their expectations, and an ambiguity-neutral DM is indifferent between the two. Such preferences define *absolute ambiguity attitude*. The next definition establishes this idea formally.

Definition 1. Let $p_E \in \hat{\Delta}$ be the constant secondary that pays $\mathrm{E}[\mathrm{P}(E)]$. As regards event $E \in \mathcal{E}$, ambiguity aversion is defined by

$$p_E \succsim^2 \hat{\delta}_E;$$

ambiguity loving is defined by

$$p_E \precsim^2 \hat{\delta}_E;$$

and ambiguity neutrality is defined by

$$p_E \sim^2 \hat{\delta}_E.$$

Note that Assumption 1 ensures that p_E always exists. If any of the three attitudes toward ambiguity in Definition 1 holds uniformly across ξ, then one can derive the shape of Υ.

Theorem 2. *Suppose that Assumptions 1 and 4 hold. For every $E \in \mathcal{E}$, let the constant secondary act $p_E \in \hat{\Delta}$ pay $\mathrm{E}[\mathrm{P}(E)]$. The preference relation $\succsim^2$ over secondary acts $\hat{\Delta}$ can then be characterized by one of the three types of global ambiguity attitudes:*

(i) *Ambiguity aversion: $\forall E \in \mathcal{E}$, $p_E \succsim^2 \hat{\delta}_E \iff p_E \geq \mathrm{Q}(\hat{\delta}_E)$ whenever Υ is concave;*
(ii) *Ambiguity loving: $\forall E \in \mathcal{E}$, $p_E \precsim^2 \hat{\delta}_E \iff p_E \leq \mathrm{Q}(\hat{\delta}_E)$ whenever Υ is convex;*
(iii) *Ambiguity neutrality: $\forall E \in \mathcal{E}$, $p_E \sim^2 \hat{\delta}_E \iff p_E = \mathrm{Q}(\hat{\delta}_E)$ whenever Υ is linear.*

[23] Formally, in the smooth model the value of a primary act is assessed by $\mathrm{V}(f) = \int_{\mathcal{P}} \phi\left(\int_{\mathcal{S}} \mathrm{U}(f(s))\, d\mathrm{P}\right) d\xi$, where the function ϕ over expected utilities captures preferences for ambiguity. The concept that preferences for ambiguity apply to certainty equivalent outcomes and represented by a function over expected utilities can lead to counterintuitive results. For example, Izhakian and Benninga (2011) show that in such settings, a higher aversion to risk can lead to a higher value of an act.

By this theorem, ambiguity neutrality implies a special case of the functional representation of preferences in Eq. (7).

Observation 7. *An ambiguity-neutral DM, who does not distort perceived probabilities, assesses the value of an act by reducing two-stage probabilities to compound (additive) probabilities that are equal to the expected probabilities of events. That is,*

$$\mathrm{V}(f) = \int_{\mathcal{S}} \mathrm{U}(f)\, d\mathrm{Q},$$

where $\mathrm{Q}(E) = \mathrm{E}[\mathrm{P}(E)]$.

In general, when the DM is not necessarily ambiguity neutral, Theorem 2 proposes that while making decisions, the DM, who views uncertain probabilities as a set of priors, aggregates these probabilities in a nonlinear way to form her perceived probabilities. Two special types of ambiguity attitudes can be defined: constant absolute and constant relative.

Definition 2. A DM is said to exhibit a constant absolute ambiguity attitude (CAAA) if for any events $E, F \in \mathcal{E}$

$$\hat{\delta}_E \succsim^2 \hat{\delta}_F \iff \hat{\delta}_E + c \succsim^2 \hat{\delta}_F + c,$$

where $c \in \mathbb{R}$ is a constant.

The intuition of CAAA is that ambiguity attitude is not sensitive to shifts in probabilities. This means that the preference concerning ambiguity remains unchanged when the probability ranges of the considered events are shifted linearly. It can be shown that CAAA is characterized by the exponential functional form

$$\Upsilon(\mathrm{P}(E)) = \frac{1 - e^{-\eta \mathrm{P}(E)}}{\eta},$$

where η is the coefficient of absolute ambiguity aversion.

Definition 3. A DM is said to exhibit a constant relative ambiguity attitude (CRAA) if for all events $E, F \in \mathcal{E}$

$$\hat{\delta}_E \succsim^2 \hat{\delta}_F \iff c\hat{\delta}_E \succsim^2 c\hat{\delta}_F,$$

where $c \in \mathbb{R}$ is a strictly positive constant.

The expression $c\hat{\delta}_E$ corresponds to the second order act that pays $cP(E)$ for any $\mathrm{P} \in \mathcal{P}$. Therefore, the intuition of CRAA is that ambiguity attitude is not sensitive to a positive probability scaling. This implies that, given an event, aversion to or love for ambiguity increases linearly with its probabilities. It can be shown that CRAA is characterized by the functional form

$$\Upsilon(\mathrm{P}(E)) = \frac{(\mathrm{P}(E))^{1-\eta}}{1-\eta}.$$

When the outlook function Υ is twice continuously differentiable, similarly to Arrow–Pratt's risk theory, the *coefficient* of absolute ambiguity aversion can be defined by $-\frac{\Upsilon''(\mathrm{P}(E))}{\Upsilon'(\mathrm{P}(E))}$, and the coefficient of relative ambiguity aversion by $-\frac{\Upsilon''(\mathrm{P}(E))}{\Upsilon'(\mathrm{P}(E))}\mathrm{P}(E)$.

The notion of ambiguity aversion can be used to define comparative ambiguity aversion, so that DMs can be ranked by their level of aversion to ambiguity. To conduct this kind of comparative static, it is assumed that the DMs under consideration share the same beliefs, i.e., they have the same set of priors $\mathcal{P}$ and the same measure ξ over $\mathcal{P}$.

Definition 4. Let $\succsim_A^2$ and $\succsim_B^2$ be, respectively, the preferences of two DMs A and B who share the same beliefs. As regards event $E \in \mathcal{E}$, DM A is at least as ambiguity averse as DM B if

$$\hat{\delta}_E \succsim_A^2 p_F \Longrightarrow \hat{\delta}_E \succsim_B^2 p_F,$$

where $p_F \in \hat{\Delta}$ is some constant secondary act.

The idea of this definition is that if DM A is more ambiguity averse whenever she prefers the uncertain probability over a certain probability, then DM B, who is less ambiguity averse, will follow suit. This implies that the set of $\hat{\delta}_E$ that DM B prefers to a given certain probability is larger than DM A's preferred set of $\hat{\delta}_E$. DM A is said to be at least as ambiguity averse as DM B if Definition 4 holds for every $E \in \mathcal{E}$.

A similar notion of comparative preferences is widely used with respect to risk; e.g., Yaari (1969). A closer definition has been proposed with respect to bet imprecision; e.g., Gajdos et al. (2008). The use of this notion has also been adopted to ambiguity; e.g., Epstein (1999), Ghirardato and Marinacci (2002), and Jewitt and Mukerji (2011). Formally, they define DM A to be at least as ambiguity averse as DM B if $f \succsim_A^1 l \implies f \succsim_B^1 l$, for every $l, f \in \mathcal{F}_0$, where l is an unambiguous (primary) act. Their definition relies upon comparative preferences over (primary) acts relative to unambiguous but risky acts. EUUP isolates the effect of ambiguity from risk, as preferences for ambiguity are applied to secondary acts in $\hat{\Delta}$ (i.e., to probabilities). Klibanoff et al. (2005) also suggest a comparison of ambiguity attitudes. Their comparison, however, is restricted to DMs who have the same risk preferences and the same beliefs. Whereas here the restriction on risk preferences is not needed, since ambiguity preferences are independent of risk preferences. Yet, the DMs' beliefs must coincide for this comparison to be meaningful.

Theorem 3. *Suppose that DMs A and B share the same beliefs and have strictly-increasing continuous outlook functions Υ_A and Υ_B. DM A is at least as ambiguity averse as DM B whenever $\Upsilon_A = h \circ \Upsilon_B$ for some increasing and concave function $h : \mathcal{G} \to \mathbb{R}$, where $\mathcal{G} = \Upsilon_B([0, 1])$.*

This theorem implies that whether a DM is ambiguity averse, ambiguity loving or ambiguity neutral, can be inferred purely by the shape of the outlook function Υ irrespective of the utility function U and the beliefs ξ involved. Applying Arrow–Pratt's risk theory to probabilities, the following corollary is an immediate derivation of Theorem 3.

Corollary 1. *Suppose that DMs A and B share the same beliefs. If Υ_A and Υ_B are twice continuously differentiable, then A is at least as ambiguity averse as B whenever*

$$-\frac{\Upsilon_A''(x)}{\Upsilon_A'(x)} \geq -\frac{\Upsilon_B''(x)}{\Upsilon_B'(x)},$$

for any $x \in [0, 1]$.

5. Asymmetric preferences

Prospect theory and the behavioral studies that followed it suggest that preferences for risk and for ambiguity might be different concerning unfavorable consequences (losses) and concerning favorable consequences (gains). Some studies find risk loving for losses and risk aversion for gains (e.g., Abdellaoui et al., 2010), and some find *loss aversion* (e.g., Barberis and Huang, 2001). Focusing on preferences for ambiguity, most studies document ambiguity loving for losses and ambiguity aversion for gains (e.g., Bier and Connell, 1994). To grant EUUP the flexibility to support different attitudes toward ambiguity and toward risk for different consequences, Theorem 1 is extended to asymmetric (reference-dependent) preferences.

The extension of Theorem 1 to asymmetric risk preferences is obtained immediately from Wakker (2010, Theorem 12.3.5). The extension to asymmetric ambiguity preferences, however, requires the redefinition of Axiom 1. To this end, the set of consequences $\mathcal{X} \subseteq \mathbb{R}$ is assumed to be a convex set that contains the interval $[-1, 1]$. The following observation is then delivered by Assumptions 1 and 2.

Observation 8. *For every event $E \in \mathcal{E}$ there exists an event $F \in \mathcal{H}$ with unambiguous probability such that $-\delta_E \sim^1 -\mathbf{1}_F$.*

The set of secondary acts $\widehat{\mathcal{F}}$ is built upon $\mathcal{F}$, which consists of all bounded $\mathcal{E}$-measurable functions $f : \mathcal{S} \to \mathcal{X}$. Therefore, the following observation is immediate.

Observation 9. *For every $E \in \mathcal{E}$ there exists $-\hat{\delta}_E \in \widehat{\mathcal{F}}$.*

Axiom 2 (*Asymmetric Certainty Equivalent Probabilistic Consistency*). Let $\mathbf{1}_F \in \mathcal{F}_0$ be an unambiguous positive indicator act whose associated (constant) secondary act pays c, and let $-\mathbf{1}_F \in \mathcal{F}_0$ be an unambiguous negative indicator act whose associated (constant) secondary act pays $-d$, for some constants $c, d \in [0, 1]$. For every event $E \in \mathcal{E}$

$$\delta_E \sim^1 \mathbf{1}_F \iff \hat{\delta}_E \sim^2 c$$

and

$$-\delta_E \sim^1 -\mathbf{1}_F \iff -\hat{\delta}_E \sim^2 -d.$$

Axiom 2 distinguishes between second-order preferences over positive secondary acts $\hat{\Delta}$ and over negative secondary acts $-\hat{\Delta}$. Thereby, it distinguishes between first-order preferences over positive (primary) indicator acts and over negative indicator acts. Since consequences lower than 0 are considered unfavorable, $-\hat{\delta}_E \in -\hat{\Delta}$ can be thought of as assigning an unfavorable (under some act) event $E \in \mathcal{E}$ with its possible (negative) probabilities, whereas $\hat{\delta}_E \in \hat{\Delta}$ can be thought of as assigning a favorable event $E \in \mathcal{E}$ with its possible (positive) probabilities. In this view, ambiguity loving (aversion) for unfavorable events can be defined along with ambiguity aversion (loving) for favorable events.

Definition 5. Let $p_E \in \hat{\Delta}$ be a positive constant secondary act that pays $\mathrm{E}[\mathrm{P}(E)]$, and let $-p_E \in \hat{\Delta}$ be a negative constant secondary act that pays $-\mathrm{E}[\mathrm{P}(E)]$. As regards a favorable (under some act) event $E \in \mathcal{E}$,

ambiguity aversion is defined by $\quad p_E \succsim^2 \hat{\delta}_E$;
ambiguity loving is defined by $\quad p_E \precsim^2 \hat{\delta}_E$;
and ambiguity neutrality is defined by $\quad p_E \sim^2 \hat{\delta}_E$.

As regards an unfavorable (under some act) event $E \in \mathcal{E}$,

ambiguity aversion is defined by $\quad -p_E \succsim^2 -\hat{\delta}_E$;
ambiguity loving is defined by $\quad -p_E \precsim^2 -\hat{\delta}_E$;
and ambiguity neutrality is defined by $\quad -p_E \sim^2 -\hat{\delta}_E$.

With this notion of distinguished preferences for ambiguity concerning favorable and unfavorable events, the outlook function can be defined by

$$\Upsilon(x) = \begin{cases} \Upsilon_{FV}(x), & x > 0 \\ \Upsilon_{UF}(-x), & x \leq 0, \end{cases}$$

for all $x \in \mathcal{X}$, where Υ_{FV} and Υ_{UF} capture attitudes toward ambiguity concerning favorable and unfavorable events, respectively.[24] Continuity of Υ implies $\Upsilon_{FV}(0) = \Upsilon_{UF}(0)$, which without loss of generality can be assumed to be zero. Proposition 2 can then be extended as follows.

24 A similar idea is suggested by Tversky and Kahneman (1992) for the utility of losses and gains.

Proposition 3. *Under Assumption 4, there exists a function* $\mathrm{Q} : -\hat{\Delta} \cup \hat{\Delta} \to [0,1]$ *such that*

$$\hat{\delta}_E \succsim^2 \hat{\delta}_F \iff \mathrm{Q}(\hat{\delta}_E) \geq \mathrm{Q}(\hat{\delta}_F)$$

for every $\hat{\delta}_E, \hat{\delta}_F \in \hat{\Delta}$ *and*

$$-\hat{\delta}_E \succsim^2 -\hat{\delta}_F \iff \mathrm{Q}(-\hat{\delta}_E) \geq \mathrm{Q}(-\hat{\delta}_F)$$

for every $-\hat{\delta}_E, -\hat{\delta}_F \in -\hat{\Delta}$, *where*

$$\mathrm{Q}(\hat{\delta}_E) = \Upsilon_{FV}^{-1}\left(\int_{\mathcal{P}} \Upsilon_{FV}\left(\hat{\delta}_E(\mathrm{P})\right) d\xi\right) \quad \text{and}$$

$$\mathrm{Q}(-\hat{\delta}_E) = \Upsilon_{UF}^{-1}\left(\int_{\mathcal{P}} \Upsilon_{UF}\left(\hat{\delta}_E(\mathrm{P})\right) d\xi\right);$$

$\Upsilon_{FV} : [0,1] \to \mathbb{R}$ *and* $\Upsilon_{UF} : [0,1] \to \mathbb{R}$ *are non-constant bounded functions; and* ξ *is a non-atomic countably-additive probability measure. Furthermore,* ξ *is unique, and* Υ_{FV} *and* Υ_{UF} *are unique up to a positive linear transformation.*

This proposition suggests that the perceived probabilities of unfavorable events may be different from those of favorable events, contingent upon the DM's attitudes toward ambiguity formed by Υ_{UF} and Υ_{FV}. Based upon Proposition 3, Theorem 1 can be generalized as follows.

Theorem 4. *Suppose that* $\succsim^1$ *and* $\succsim^2$ *satisfy Assumptions 1, 2 and 4, and Axiom 2. Then, there exists a function* $\mathrm{V} : \mathcal{F}_0 \to \mathbb{R}$ *such that*

$$f \succsim^1 g \iff \mathrm{V}(f) \geq \mathrm{V}(g),$$

for every $f, g \in \mathcal{F}_0$, *where*

$$\begin{aligned}\mathrm{V}(f) &= \int_{z\leq 0}\left[\Gamma_{UF}\left(\Upsilon_{UF}^{-1}\left(\int_{\mathcal{P}} \Upsilon_{UF}\left(\hat{\delta}_{\{s\in\mathcal{S}|\mathrm{U}_{UF}(f(s))\geq z\}}(\mathrm{P})\right)d\xi\right)\right)-1\right]dz \\ &\quad + \int_{z\geq 0}\Gamma_{FV}\left(\Upsilon_{FV}^{-1}\left(\int_{\mathcal{P}} \Upsilon_{FV}\left(\hat{\delta}_{\{s\in\mathcal{S}|\mathrm{U}_{FV}(f(s))\geq z\}}(\mathrm{P})\right)d\xi\right)\right)dz; \qquad (8)\end{aligned}$$

$\mathrm{U}_{FV} : \mathcal{X} \to \mathbb{R}$ *and* $\mathrm{U}_{UF} : \mathcal{X} \to \mathbb{R}$ *are continuous strictly increasing bounded functions, normalized such that* $\mathrm{U}_{FV}(0) = \mathrm{U}_{UF}(0) = 0$; $\Upsilon_{FV} : [0,1] \to \mathbb{R}$ *and* $\Upsilon_{UF} : [0,1] \to \mathbb{R}$ *are continuous non-constant bounded functions that vanish at* 0; $\Gamma_{FV} : [0,1] \to [0,1]$ *and* $\Gamma_{UF} : [0,1] \to [0,1]$ *are continuous strictly increasing functions;* ξ *is a non-atomic countably-additive probability measure; and every* $\mathrm{P} \in \mathcal{P}$ *is finitely-additive probability measure. Furthermore,* ξ *is unique,* Υ_{FV} *and* Υ_{UF} *are unique up to a (joint) positive scaling,* U_{FV} *and* U_{UF} *are unique up to a (joint) positive scaling; and* Γ_{FV} *and* Γ_{UF} *are unique.*

6. Applications

Consciously or subconsciously, ambiguity is embedded in nearly every real-life decision process. A critical step toward the quantification of the extent of ambiguity, independently of risk and independently of related tastes, is an ultimate separation of ambiguity from risk and from tastes. Based upon the EUUP concept that aversion to ambiguity is defined as aversion to mean-preserving spreads in probabilities, Izhakian (2016) introduces an empirically applicable, risk-independent *measure* of ambiguity. This study applies the Rothschild and Stiglitz (1970) approach to probabilities to show that the degree of ambiguity can be measured by the volatility of probabilities, just as the degree of risk can be measured by the volatility of outcomes. In particular, it shows that the degree of ambiguity can be measured by the expected variance of probabilities across the relevant events. Formally, this measure is given by

$$\mho^2[f] = \int_{\mathcal{X}} \mathrm{E}\left[\varphi_f(x)\right] \mathrm{Var}\left[\varphi_f(x)\right] dx,$$

where $\varphi_f(\cdot)$ is an uncertain probability density function under act f; and the expectation $\mathrm{E}[\cdot]$ and the variance $\mathrm{Var}[\cdot]$ are taken with respect to second-order probabilities ξ.[25] When ξ is objective, $\mho^2$ can be considered as a measure of objective ambiguity, otherwise it can be considered as a measure of subjective ambiguity. The proposed measure, $\mho^2$ can serve for introducing ambiguity into models that attempt to explain observable phenomena such as financial anomalies.[26]

It should be noted that other decision-making models do not permit a natural derivation of a risk-independent measure of ambiguity. For example, in MEU (Gilboa and Schmeidler, 1989) and α-MEU (Ghirardato et al., 1998) the set of priors captures both tastes and beliefs. Since in this class of models beliefs are not distinguished from ambiguity attitudes, they do not permit ambiguity to be measured. The same holds for the subjective nonadditive probabilities (Gilboa, 1987), the CEU (Schmeidler, 1989) and the CPT (Tversky and Kahneman, 1992) in which capacities reflect both tastes and beliefs. In the smooth model (Klibanoff et al., 2005) preferences for ambiguity apply to certainty equivalent outcomes. Hence, within this framework, ambiguity cannot be distinguished from risk in a way that a risk-independent measure of ambiguity can be derived.[27]

EUUP equipped with its derived measure of ambiguity can be used to introduce ambiguity into financial theories and study the importance of ambiguity and attitudes toward it in financial decisions (including pricing). For example, using EUUP, Izhakian's (2017) study constructs a well-defined ambiguity premium, solely attributed to ambiguity and attitude toward ambiguity, completely distinguished from risk and attitude toward risk, and which can be computed from the data and tested empirically. Brenner and Izhakian (2016) employ the EUUP framework to study the risk–ambiguity–return relationship in the stock market. To do so, they assume that each subset of stock returns is generated by the choices of a single representative DM conditional upon a different prior P within her set of priors $\mathcal{P}$. Assuming some structure on second-order beliefs, they compute ambiguity from the data and investigate its effect on stock returns. Their study elicits DMs' attitude toward ambiguity and finds that aversion to ambiguity increases with the expected probability of favorable returns (gains), and love for ambiguity increases with the expected probability of unfavorable returns (losses). The tractable framework of EUUP is also demonstrated in other studies of the effect of ambiguity on financial decisions. For example, Augustin and Izhakian (2016) employ this framework to study the effect of ambiguity on pricing credit default swaps, and Izhakian and Yermack (2017) employ it to study the role ambiguity plays in employees' decisions of whether to exercise their vested options.

25 In case of a finite state space, this measure takes the form $\mho^2[f] = \sum_j \mathrm{E}\left[\varphi_f(x_j)\right] \mathrm{Var}\left[\varphi_f(x_j)\right]$, where $\varphi_f(\cdot)$ is an uncertain probability mass function.

26 Notable examples in finance include the fact that individuals tend to hold very small portfolios, 3–4 stocks (Goetzmann and Kumar, 2008), the equity premium puzzle (Mehra and Prescott, 1985), the risk-free rate puzzle (Weil, 1989), the phenomenon of the observed equity volatility being too high to be justified by changes in the fundamental (Shiller, 1981), and the home bias puzzle (Coval and Moskowitz, 1999).

27 For example, based upon the smooth model and assuming that the variance of outcomes is known, Maccheroni et al. (2013) suggest to measure ambiguity by the variation of the mean (which is risk-dependent). Such a measure ignores the importance of ambiguous variance (Epstein and Ji, 2013), and in some scenarios has serious issues. On the other hand, the measure $\mho^2$ encompasses both, ambiguous variance and ambiguous mean, as well as the ambiguity of all higher moments of the probability distribution (i.e., skewness, kurtosis, etc.) through the uncertainty of probabilities.

7. Conclusion

The importance of ambiguity for understanding economic and financial decision processes has been recognized in the literature for the past century. Relevant studies have acknowledged that attempts to portray a realistic picture of observable phenomena (and anomalies) should consider also the dimension of uncertainty with respect to probabilities. This has led to the search for an *applicable* framework of modeling decision making under ambiguity.

The model of decision making under ambiguity introduced in this paper aims to address this need. It provides a framework that can be used for the (empirical) measurement of the degree of ambiguity, independently of tastes and of risk. This characteristic is of primary importance for introducing ambiguity into theoretical, behavioral and empirical studies. Thereby, this framework aims to pave the way not only for the introduction of ambiguity into empirical studies, but also for the expansion of theoretical and behavioral studies regarding the nature of ambiguity and related preferences.

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Appendix

Proof of Proposition 1. Let the unambiguous indicator acts $\mathbf{1}_{E'}, \mathbf{1}_{F'} \in \mathcal{F}_0$ be, respectively, associated with the secondary (constant) acts in $\hat{\Delta}$ paying $c_1 \geq 0$ and $c_2 \geq 0$ and such that $\delta_E \sim^1 \mathbf{1}_{E'}$ and $\delta_F \sim^1 \mathbf{1}_{F'}$. Then, by transitivity and monotonicity of $\succsim^1$ and $\succsim^2$, and by Axiom 1,

$$\delta_E \succsim^1 \delta_F \iff \mathbf{1}_{E'} \succsim^1 \mathbf{1}_{F'} \iff c_1 \geq c_2$$
$$\iff c_1 \succsim^2 c_2 \iff \hat{\delta}_E \succsim^2 \hat{\delta}_F. \quad \square$$

Proof of Proposition 2. Since $\hat{\Delta} \subset \widehat{\mathcal{F}} \subset \widehat{\mathcal{F}}$ and $\succsim^2$ over $\widehat{\mathcal{F}}$ satisfies Kopylov (2010) axioms,

$$\hat{\delta}_E \sim^2 c \iff \Upsilon(c) = \int_{\mathcal{P}} \Upsilon(\hat{\delta}_E(\mathrm{P}))d\xi$$

for some constant secondary act $c \in \hat{\Delta}$. The function Υ^{-1} is well defined over the entire range of Υ since, by outcome-monotonicity of $\succsim^2$, Υ is monotonic. Therefore,

$$\hat{\delta}_E \sim^2 c \iff c = \Upsilon^{-1}\left(\int_{\mathcal{P}} \Upsilon(\hat{\delta}_E(\mathrm{P}))d\xi\right) = \mathrm{Q}(\hat{\delta}_E).$$

Thus,

$$\hat{\delta}_E \succsim^2 \hat{\delta}_F \iff \mathrm{Q}(\hat{\delta}_E) \geq \mathrm{Q}(\hat{\delta}_F).$$

The uniqueness and countable-additivity of ξ and the uniqueness of Υ up to a positive linear transformation follow from Kopylov's (2010). $\square$

Proof of Theorem 1. Consider $E, F \in \mathcal{E}$. Since $\succsim^1$ over $\mathcal{F}_0$ satisfies Wakker's (2010, Theorem 12.3.5) axioms and $\mathrm{U}(0) = 0$, by Eq. (4),

$$\mathrm{v}(\delta_E) = \mathrm{G}(E)\mathrm{U}(1) \quad \text{and} \quad \mathrm{v}(\delta_F) = \mathrm{G}(F)\mathrm{U}(1). \tag{9}$$

By Propositions 1 and 2,

$$\mathrm{G}(E) \geq \mathrm{G}(F) \iff \delta_E \succsim^1 \delta_F \iff \hat{\delta}_E \succsim^2 \hat{\delta}_F \iff \mathrm{Q}(\hat{\delta}_E) \geq \mathrm{Q}(\hat{\delta}_F),$$

where

$$\mathrm{Q}(\hat{\delta}_E) = \Upsilon^{-1}\left(\int_{\mathcal{P}} \Upsilon\left(\hat{\delta}_E(\mathrm{P})\right)d\xi\right).$$

Thus, $\mathrm{G}(E)$ is a representation of $\succsim^2$ restricted to $\hat{\Delta}$ and must be equivalent to Q up to a monotonic transformation:

$$\mathrm{G}(E) = \Gamma\left(\Upsilon^{-1}\left(\int_{\mathcal{P}} \Upsilon\left(\hat{\delta}_E(\mathrm{P})\right)d\xi\right)\right),$$

where $\Gamma : [0, 1] \to [0, 1]$ is continuous, increasing and satisfies $\Gamma(0) = 0$ and $\Gamma(1) = 1$. The proof is then completed from Wakker (2010, Theorem 12.3.5) since, by Wakker (2010, Observation 12.3.5'), capacities are uniquely determined. $\square$

Proof of Theorem 2. The proof follows directly from Jensen's inequality. $\square$

Proof of Theorem 3. Standard result (e.g., Theorem 1 in Nielsen, 1988). $\square$

Proof of Proposition 3. The proof for positive secondary acts $\hat{\delta}_E, \hat{\delta}_F \in \hat{\Delta}$ is identical to that of Proposition 2. The proof for negative secondary acts follows similarly. $\square$

Proof of Theorem 4. The proof consists of separately applying the arguments in the proof of Theorem 1 to non-negative and non-positive secondary acts. $\square$

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