# CATEGORY THEORY

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## 1. Introduction

In each area of mathematics (e.g., sets, groups, topological spaces) there are available many definitions and constructions. It turns out, however, that there are a number of notions (e.g. that of a product) that occur naturally in various areas of mathematics, with only slight changes from one area to another. It is convenient to take advantage of this observation. Category theory can be described as that branch of mathematics in which one studies certain definitions in a broader context - without reference to the particular area to which the definition might be applied.

## 2. Categories

We begin by defining what we mean by a 'Category'.

**Definition 2.1** (Category)**.** A *category* $\mathcal{K}$ consists of the following three mathematical entities:

(1) A *class* $\mathrm{Ob}(\mathcal{K})$ of objects
(2) A class $\mathrm{Hom}(A, B)$ of *morphisms*, from $A \longrightarrow B$ such that $A, B \in \mathrm{Ob}(\mathcal{K})$.
e.g. $f : A \to B$ to mean $f \in \mathrm{Hom}(A, B)$.

*Remark.* The class of *all* morphisms of $\mathcal{K}$ is denoted $\mathrm{Hom}(\mathcal{K})$.

(3) Given $A, B, C \in \mathrm{Ob}(\mathcal{K})$, a binary operation $\circ : Hom(B, C) \times Hom(A, B) \to Hom(A, C)$ called *composition*, satisfying:
  (a) *(associativity)* Given $f : A \to B$, $g : B \to C$ and $h : C \to D$ we have $h \circ (g \circ f) = (h \circ g) \circ f$.

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[ur, "f"] \arrow[r, "h"'] & C \end{tikzcd}$$

  (b) *(identity)* For any object X there is an identity morphism $1_X : X \to X$ such that for any $f : A \to B$ we have $1_B \circ f = f = f \circ 1_A$.

$$\begin{tikzcd} X \arrow[loop above, "1_X"] \end{tikzcd}$$

It is also worth noting about what we mean by 'small' and 'large' categories.

**Definition 2.2** (Small Category)**.** A category $\mathcal{K}$ is called small if both $\mathrm{Ob}(\mathcal{K})$ and $\mathrm{Hom}(\mathcal{K})$ are sets. If $\mathcal{K}$ is not small, then it is called large. $\mathcal{K}$ is called locally small if $\mathrm{Hom}(A, B)$ is a set for all $A, B \in \mathrm{Ob}(\mathcal{K})$.

*Remark.* Most important categories in mathematics are not small however, are locally small.

## 3. Functors

A category is itself a type of mathematical structure, so we can look for "processes" which preserve this structure in some sense; such a process is called a functor. A functor associates to every object of one category an object of another category, and to every morphism in the first category a morphism in the second. In fact, what we have done is define a category of categories and functors  the objects are categories, and the morphisms (between categories) are functors. By studying categories and functors, we are not just studying a class of mathematical structures and the morphisms between them; we are studying the relationships between various classes of mathematical structures.

**Definition 3.1** (Functor)**.** Let $\mathcal{C}$ and $\mathcal{K}$ be categories. A *functor* $F$ from $\mathcal{C}$ to $\mathcal{K}$ is a mapping that:

(1) associates to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{K}$
(2) associates to each morphism $f : X \to Y \in \mathcal{C}$ a morphism $F(f) : F(X) \to F(Y) \in \mathcal{K}$ satisfying:
  (a) $F(id_X) = id_{F(X)}$ for every object $X \in \mathcal{C}$
  (b) $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$

  *Remark.* That is, functors must preserve identity morphisms and composition of morphisms.

## 4. Natural Transformations

In category theory, a branch of mathematics, a natural transformation provides a way of transforming one functor into another while respecting the internal structure (i.e. the composition of morphisms) of the categories involved. Hence, a natural transformation can be considered to be a "morphism of functors". Indeed this intuition can be formalized to define so-called functor categories. Natural transformations are, after categories and functors, one of the most basic notions of category theory and consequently appear in the majority of its applications.

**Definition 4.1** (Natural Transformation)**.** Let $F$ and $G$ be functors between the categories $\mathcal{C}$ and $\mathcal{K}$, then a *natural transformation* $\eta : F \to G$ associates to every object $X \in \mathcal{C}$ a morphism $\eta_X : F(X) \to G(X)$ between objects of $\mathcal{K}$, called the *component* of $\eta$ at $X$, such that for every morphism $f : X \to Y \in \mathcal{C}$ we have:
$\eta_Y \circ F(f) = G(g) \circ \eta_X$

Constructions are often "naturally related"  a vague notion, at first sight. This leads to the clarifying concept of natural transformation, a way to "map" one functor to another. Many important constructions in mathematics can be studied in this context. "Naturality" is a principle, like general covariance in physics, that cuts deeper than is initially apparent.

## 5. Monoids

**Definition 5.1** (Monoid)**.** A category with exactly one object is called a *monoid*.