REAL ANALYSIS I

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Contents

1. Set Theory

Here we fix some notation.

If $p()$ is some *predicate* that is either true or false for every element of some set X, then the notation $\{x \in X : p(x)\}$ will be used to denote the subset of X consisting of all those elements of X for which $p(x)$ is true.

If A and B are sets, we write $A \backslash B$ for the *difference*, that is, $A \backslash B \doteq \{x \in A : x \notin B\}$.

Proposition 1.1. *For $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ we have*

$$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x).$$

If, $g(x) \neq 0$ everywhere *on Ω, then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}.$$

2. Metric Spaces

Definition 2.1 (Metric)**.** A *metric* d_X on set X is a function $d_X : X \times X \to \mathbb{R}$ such that, for any $x, y, z \in X$;

i.) $d_X(x, y) \geq 0$ and $d_X(x, y) = 0$ iff $x = y$ (semi-positive definate),
ii.) $d_X(x, y) = d_X(y, x)$ (symmetric),
iii.) $d_X(x, z) \leq d_X(x, y) + d_X(y, z)$ (triangle inequality).

Definition 2.2 (Metric Space)**.** A *metric space* is the pair (X, d_X) where X is a set and d_X is the metric defined on the set X.

Definition 2.3 (Open Ball)**.** An *open ball* in a metric space (X, d_X) with center $x_0 \in X$ and ϵ-neighborhood with $\epsilon > 0$ is defined as the set;

$$\mathcal{B}_\epsilon(x_0) \doteq \{x \in X : d_X(x_0, x) < \epsilon\}.$$

Definition 2.4 (Closed Ball)**.** A *closed ball* in X with center $x_0 \in X$ and ϵ-neighborhood with $\epsilon > 0$ is defined as the set;

$$\overline{\mathcal{B}_\epsilon(x_0)} \doteq \{x \in X : d_X(x_0, x) \leq \epsilon\}.$$

Definition 2.5 (Open set)**.** Let $\Omega \subseteq X$ of a metric space (X, d_X). The set Ω is called an *open set* in X if; for each $x \in \Omega$ there exists some $\delta > 0$ such that $\mathcal{B}_\delta(x) \subseteq \Omega$.

Lemma 2.6. *The whole space X and the empty set $\emptyset$ are trivially open.*

Definition 2.7 (Closed set)**.** A *closed set* is the complement, denoted Ω^c, of the open set Ω.

Lemma 2.8. *The whole space X and the empty set $\emptyset$ are trivially closed.*

Definition 2.9 (Clopen set)**.** A set that is both open and closed is said to be an *clopen set.*

Definition 2.10 (Boundary)**.** The *boundary* of a set $\Omega \subseteq X$ of a metric space (X, d_X), denoted by $\delta\Omega$, is defined as;

$$\delta\Omega \doteq \{x \in X : \mathcal{B}_\epsilon(x) \cap \Omega \neq \emptyset, \mathcal{B}_\epsilon(x) \cap \Omega^c \neq \emptyset\}.$$

Lemma 2.11. *We have that, $\delta\Omega = \delta(\Omega^c)$, is trivially so.*

Proposition 2.12. *Any open set does not contain any of its boundary points.*

Proof. Let $\Omega \subseteq X$ be open so that Ω^c is closed. It remains that $\delta\Omega^c \subseteq \Omega^c$. $\square$

Proposition 2.13. *A set S is* closed *if and only if it contains all its boundary $\delta S \subseteq S$.*

Proof. Suppose that S is closed so that S^c is open. It follows that $S^c \cap \delta S = \emptyset$ and hence $\delta S \subseteq S$. Conversely, now consider some $x \in S^c$. Since $\delta S \subseteq S$, it follows that $x \notin \delta S$. It remains by definition of boundary point, we have some $\epsilon > 0$ such that $\mathcal{B}_\epsilon(x) \cap S = \emptyset$. Thus, $\mathcal{B}_\epsilon(x) \subseteq S$ and so S^c is open. $\square$

Definition 2.14 (Interior (Point))**.** Let $\Omega \subseteq X$ be some region of a metric space (X, d_X). A point $x \in \Omega$ is said to be an *interior point* if there exists some $\delta > 0$ such that $\mathcal{B}_\delta(x) \subseteq \Omega$. The set of all interior points, denoted Ω°, is called the *interior* of the region Ω.

Definition 2.15 (Closure)**.** Let $S \subseteq X$. We define the *closure* of S, denoted $\overline{S}$, by the set $\overline{S} = S \cup \delta S$.

Definition 2.16 (Accumulation Point)**.** Let $\Omega \subseteq X$ and fix some point $x \in X$. We call x an *accumulation*, or *limit*, *point* of Ω if every open ball around x contains atleast one distinct point $y \in \Omega$. In particular, for every $\epsilon > 0$ we have that, $(\mathcal{B}_\epsilon(x) - y) \cap \Omega \neq \emptyset$.

3. Convergence

Definition 3.1 (Limit)**.** Let $\{x_k\}_{k=1}^\infty$ be a sequence of vectors in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say that $\{x_k\}_{k=1}^\infty$ *converges* to a *limit* x if,

$$\lim_{k\to\infty} d(x_k, x) = \lim_{k\to\infty} \|x_k - x\| \to 0 \in \mathbb{R}$$

written $x_k \to x$.

4. Compactness

Let I denote any, possibly infinite, indexing set.

Definition 4.1 (Open Cover)**.** A **open cover** of a set $S \subseteq \mathbb{R}^n$ is a collection $\{V_i\}_{i\in I}$ of open sets of $\mathbb{R}^n$ such that $S \subset \bigcup_{i\in I} V_i$. A *subcover* is a subcollection which also covers S.

Definition 4.2 (Compact)**.** A subset $S \subseteq \mathbb{R}^n$ is said to be **compact** if from every open cover of S we may find a *finite* subcover of S.

5. Limits

Definition 5.1. Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \bar{\Omega}$. A point $y \in \mathbb{R}^m$ is said to be the *limit* of f as $x \to \infty$, written as

$$\lim_{x \to \infty} f(x) = y,$$

if for every neighborhood $B_\epsilon(y)$ of $y \in \mathbb{R}^m$ there exists a neighborhood $B_\delta(a)$ of $a \in \Omega$ such that whenever $x \in \Omega \cap B_\delta(a)$, this implies $f(x) \in B_\epsilon(y)$. That is,

$$f(\Omega \cap B_\delta(a)) \subseteq B_\epsilon(y).$$

Proposition 5.2. *For $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ we have*

$$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x).$$

If, $g(x) \neq 0$ everywhere *on Ω, then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}.$$

Problem 5.3. *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{a} \in \Omega$. Then,*

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \vec{y}$$

if and only if,

$$\lim_{\vec{x} \to \vec{a}} f_i(\vec{x}) = y_i$$

for each i in $1 \leq i \leq m$.

Example 5.4. Show that,

$$\lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = 0.$$

Let $f(x, y) = \frac{x^3+y^3}{x^2+y^2}$ and fix some arbitrarily small $\epsilon > 0$. Then we must find a $\delta > 0$ such that $|f(x, y) - 0| < \epsilon$ whenever $0 < \|(x, y) - (0, 0)\| < \delta$.

Now, note that $|x^3| \leq (x^2 + y^2)^{3/2}$ and so by the triangle inequality, we have

$$\begin{aligned} |f(x,y)| &= \left| \frac{x^3 + y^3}{x^2 + y^2} \right| \\ &\leq \frac{2(x^2 + y^2)^{3/2}}{x^2 + y^2} \\ &= 2\sqrt{x^2 + y^2}. \end{aligned}$$

whence, choosing $\delta = \epsilon/2$, we get $|f(x, y)| < \epsilon$ provided that $\sqrt{x^2 + y^2} < \delta$.

Thus, $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$.

Example 5.5. Show that,

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1.$$

Let $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ and take $t = x^2+y^2$ so that we have,

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

$$= \lim_{t\to 0} \frac{\sin(t)}{t} = 1.$$

6. Differentiable

Definition 6.1 (Differentiable)**.** Let $f : \Omega \to \mathbb{R}^m$ be a function defined on an open set $\Omega \in \mathbb{R}^n$ and point $\vec{a} \in \Omega$. The function f is said to be *differentiable* at $\vec{a}$ if there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that,

$$\lim_{\mathbf{x}\to\vec{a}} \frac{\|f(\mathbf{x}) - f(\vec{a}) - T(\mathbf{x}-\vec{a})\|_m}{\|\mathbf{x}-\vec{a}\|_n} = 0.$$

If f is differentiable at every point $\vec{a} \in \Omega$ then f is differentiable *on* Ω.

Theorem 6.2 (Linear Approximation)**.** *The function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at point $\vec{a}$ if and only if there is a function $\epsilon(\mathbf{x})$ so that for $\mathbf{x} \in \Omega$ we have,*

$$f(\mathbf{x}) = f(\vec{a}) + T(\mathbf{x}-\vec{a}) + \epsilon(\mathbf{x})\|\mathbf{x}-\vec{a}\|$$

with $\epsilon(\mathbf{x}) \to 0$ as $\mathbf{x} \to \vec{a}$.

Proof.

Set

$$\epsilon(\mathbf{x}) = \frac{f(\mathbf{x}) - f(\vec{a}) - T(\mathbf{x}-\vec{a})}{\|\mathbf{x}-\vec{a}\|} : \mathbf{x} \neq \vec{a}.$$

Now, if f is differentiable at $\vec{a}$, then $\lim_{\mathbf{x}\to\vec{a}} \epsilon(\mathbf{x}) = 0$.

Conversely, suppose

$$f(\mathbf{x}) = f(\vec{a}) + T(\mathbf{x}-\vec{a}) + \epsilon(\mathbf{x})\|\mathbf{x}-\vec{a}\|$$

holds, and since $\mathbf{x} \neq \vec{a}$, we have

$$\frac{f(\mathbf{x}) - f(\vec{a}) - T(\mathbf{x}-\vec{a})}{\|\mathbf{x}-\vec{a}\|} = \epsilon(\mathbf{x}) \to 0$$

as $\mathbf{x} \to \vec{a}$ and so f is differentiable at the point $\vec{a}$.

$\square$

Theorem 6.3 (Chain Rule)**.** *Let Ω be a open set in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$ and $g : U \to \mathbb{R}^p$, where U is a open set in $\mathbb{R}^m$ with $f(\Omega) \subseteq U$. If f is differentiable at $\vec{a} \in \Omega$ and g is differentiable at $f(\vec{a})$, then $g \circ f$ is differentiable at $\vec{a}$ and*

$$D_{(g\circ f)}(\vec{a}) = D_g(f(\vec{a}))D_f(\vec{a}).$$

Proof. TODO $\square$

6.1. **Partial derivatives.**

Definition 6.4 (Directional Derivative)**.** The directional derivative of f at $\vec{a}$ in the direction of a non-zero vector $\vec{u} \in \mathbb{R}^n$, denoted by $D_{\vec{u}}f(\vec{a})$ is defined by,

$$D_{\vec{u}}f(\vec{a}) = \lim_{t\to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t},$$

whenever the limit exists.

Theorem 6.5. *If $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a} \in \Omega$, then for any direction non-zero $\vec{u} \neq 0$, $\vec{u} \in \mathbb{R}^n$, $D_{\vec{u}}f(\vec{a})$ exists and*

$$D_{\vec{u}}f(\vec{a}) = \langle \nabla f(\vec{a}), \vec{u}\rangle.$$

Proof. TODO $\square$

7. Leibniz Rule

We begin by generalising the product rule in the following ways and building a general result about integration by parts for vector valued functions called the *Leibniz Rule*.

Proposition 7.1. *Suppose that $f, g \in \mathcal{C}^n(\mathbb{R})$ are n-times differentiable. Then the n^{th} derivative of the product $f \cdot g$ is given by,*

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

Proof. We prove by mathematical induction.

For the base case, let $n = 1$,

$$\begin{aligned}\sum_{k=0}^{n=1} \binom{n}{k} f^k g^{(n-k)} &= \binom{1}{0} f^{(0)} g^{(1)} + \binom{1}{1} f^{(1)} g^{(0)} \\ &= f\frac{dg}{dx} + g\frac{df}{dx}.\end{aligned}$$

Now assume that,

$$(f \cdot g)^{(t)} = \sum_{k=0}^{t} \binom{t}{k} f^{(k)} g^{(t-k)}.$$

Hence for $t+1$ we have,

$$(f \cdot g)^{(t+1)} = \sum_{k=0}^{t+1} \binom{t+1}{k} f^{(k)} g^{(t+1-k)}$$

...

□

Even more generally we can show that the product rule in this way holds for vector valued functions by the multinomial theorem.

Proposition 7.2. *Suppose that the vector valued functions $\vec{f}, \vec{g} \in \mathcal{C}^k(\mathbb{R}^n)$. Then the k^{th} partial derivative of the product $\vec{f} \cdot \vec{g}$ is given by,*

$$\partial^\alpha(\vec{f} \cdot \vec{g}) = \sum_{\beta : \beta \leq \alpha} \binom{\alpha}{\beta} (\partial^{(\alpha-\beta)} \vec{f})(\partial^\beta \vec{g})$$

where we make use of the multi-index *notation that, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ are both n-tuples i.e., $\alpha, \beta \in \mathbb{N}_0^n$.*

Proof. TODO.. □

Theorem 7.3 (Leibniz Rule of Integration)**.** *Let $\vec{f} = f(x,t) \in \mathcal{C}^1(\mathbb{R}^2)$ and $a(t), b(t) \in \mathcal{C}^1(\mathbb{R})$ exist. Then,*

$$\frac{d}{dt} \int_{a(t)}^{b(t)} \vec{f} \mathrm{d}x = \int_{a(t)}^{b(t)} \frac{\partial \vec{f}}{\partial t} \mathrm{d}x + \frac{\partial b(t)}{\partial t} f(b(t), t) - \frac{\partial a(t)}{\partial t} f(a(t), t).$$

Proof. TODO.. □

8. Mean Value Theorem

Theorem 8.1. *Suppose $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is some differentiable function on the open* convex set *Ω. Let $\vec{a}, \vec{b} \in \Omega$ and $\gamma(\lambda) = \vec{a} + \lambda(\vec{b} - \vec{a})$ be the line segment joining points $\vec{a}$ and $\vec{b}$. Then there exists some point $\vec{c}$ on the line $\gamma(\lambda)$ such that,*

$$f(\vec{b}) - f(\vec{a}) = \langle \nabla f(\vec{c}), (\vec{b} - \vec{a}) \rangle.$$

Proof. TODO □

Corollary. *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on a convex subset $K \subset \Omega$. If $\|\nabla f(x)\| \leq M \, \forall x \in K$, then*

$$|f(x) - f(y)| \leq M \|x - y\|$$

for all $x, y \in K$.

Corollary. *Let f be a differentiable function on an open convex set $\Omega \subseteq \mathbb{R}^n$. If $\nabla f(x) = 0 \forall x \in \Omega$, then f is constant on Ω.*

Proof. Let $x, y \in \Omega : x \neq y$. Then we have $|f(x) - f(y)| = 0$. That is, $f(x) = f(y)$. □

9. Taylor Series

Theorem 9.1 (Taylor Series)**.** *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be $f \in \mathcal{C}^2(\Omega)$ on the open convex set Ω and $a, x \in \Omega$. Then there exists a point c on the line segment joining a and x, such that*

$$f(x) = f(a) + \langle \nabla f(a), (x-a) \rangle + \frac{1}{2!} \langle H_f(c)(x-a), (x-a) \rangle.$$

10. Constrain extrema - Lagrange multipliers

Theorem 10.1 (Lagrange multiplier)**.** *Let $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1(\Omega)$ in the open set Ω. Let $S = \{x \in \Omega : g(x) = 0\}$ and $s \in S$ with $\nabla g(s) \neq 0$. If the restriction $f|_S$ of f to S takes on an extreme value at s, then there exists some $\lambda \in \mathbb{R}$ such that*

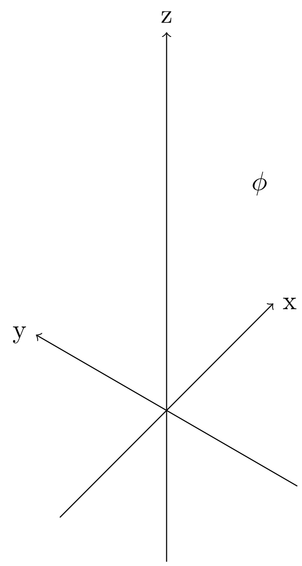
$$\nabla f(s) = \lambda \nabla g(s).$$

Example 10.2. ..

11. Cylindrical Coordinates

Definition 11.1 (Cylindrical Coordinates)**.** The *cylindrical coordinates* are defined to be the polar map with z-coordinate. That is,

$$\phi(r, \theta, z) = \Big(r\cos(\theta), \quad r\sin(\theta), \quad z \Big)$$



12. Green's theorem

Green's theorem is a two dimensional analog of the Fundamental Theorem of Calculus.

Theorem 12.1. *Let $\mathbf{F}(x,y) = (F(x,y), G(x,y))$ be, $\mathcal{C}^1(\bar{\Omega})$, a continuous vector field on an open set containing domain $\Omega \subseteq \mathbb{R}^2$ whose boundary curve, $C = \partial\Omega$, is closed and piecewise smooth. Then, by considering $\partial\Omega$ to have positive, or counterclockwise, orientation, we have:*

$$\iint_\Omega \left(\frac{\partial G(x,y)}{\partial x} - \frac{\partial F(x,y)}{\partial y} \right) \mathrm{d}y \, \mathrm{d}x = \int_{\partial\Omega} \left(F(x,y)\mathrm{d}x + G(x,y)\mathrm{d}y \right).$$

Proof.

First suppose Ω is an elementary region of the form:

$$\Omega = \{(x,y) : a \le x \le b, \phi_1(x) \le y \le \psi_1(x)\} \qquad (x\text{-simple})$$

where $\phi_1, \psi_1 \in \mathcal{C}([a,b])$ and

$$\Omega = \{(x,y) : c \le y \le d, \phi_2(y) \le x \le \psi_2(y)\} \qquad (y\text{-simple})$$

where $\phi_2, \psi_2 \in \mathcal{C}([c,d])$.

Since $\mathbf{F}(x,y) \in \mathcal{C}^1(\bar{\Omega})$ then $F(x,y) \in \mathcal{C}^1(\bar{\Omega})$. Hence we first show,

$$-\iint_\Omega \frac{\partial F}{\partial y} \mathrm{d}y \, \mathrm{d}x = \int_{\partial\Omega} F(x,y)\mathrm{d}x.$$

By writing Ω as x-simple we see that,

$$\int_{\partial\Omega} F(x,y)\mathrm{d}x = \int_{C_1} F\mathrm{d}x + \underbrace{\int_{C_2} F\mathrm{d}x}_{\dagger} + \int_{C_3} F\mathrm{d}x + \underbrace{\int_{C_4} F\mathrm{d}x}_{\dagger}.$$

Observe† that the curves C_2, C_4 are the vertical line portions $x = a$ and $x = b$ respectively. So any parametrisation $x'(t) = 0$ of constant terms gives us $\mathrm{d}x = 0$ and so the sums are zero measure.

Consider also the parameterisations $\gamma_1(x) = (x, \phi_1(x))$ and $\gamma_3(x) = (x, \psi_1(x))$ for the curves C_1 and C_3 respectively with $x \in [a,b]$. Since C_3 has negative orientation, we have

$$\int_{\partial\Omega} F(x,y)\mathrm{d}x = \int_a^b \left(F(x,\phi_1(x)) - F(x,\psi_1(x))\right)\mathrm{d}x.$$

Holding x fixed we have, by the Fundamental Theorem of Calculus, the following

$$\iint_{\Omega} \frac{\partial F(x,y)}{\partial y}\mathrm{d}y\,\mathrm{d}x = \int_a^b \left(\int_{\phi_1(x)}^{\psi_1(x)} \frac{\partial F(x,y)}{\partial y}\mathrm{d}y\right)\mathrm{d}x$$

and so,

$$\iint_{\Omega} \frac{\partial F(x,y)}{\partial y}\mathrm{d}y\,\mathrm{d}x = -\int_{\partial\Omega} F(x,y)\mathrm{d}x. \tag{1}$$

Similarly, expressing Ω as a y-simple region, we obtain,

$$\iint_{\Omega} \frac{\partial G(x,y)}{\partial x}\mathrm{d}y\,\mathrm{d}x = \int_{\partial\Omega} G(x,y)\mathrm{d}y. \tag{2}$$

By adding (1) and (2) together we have the required result in this simple case.

□

13. Fourier Transform

Definition 13.1 (Fourier transform)**.** Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function on $\mathbb{R}$. The *Fourier transform* $\mathcal{F}(f) \equiv \hat{f}$ of f is defined by the integral

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\omega x}\mathrm{d}x.$$

Definition 13.2 (Characteristic function)**.**

$$\chi_{(-a,a)}(x) \doteq \begin{cases} 1 & \text{if } |x| < a, \\ 0 & \text{if } |x| \geq a. \end{cases}$$

Example 13.3. Suppose $f(x) = \chi_{(-1,1)}(x)$. Find the Fourier transform $\hat{f}(\omega)$.

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{(-1,1)}(x) e^{-i\omega x} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{-i\omega x}}{-i\omega} \right\}_{-1}^{1} \\ &= \frac{2}{\sqrt{2\pi}} \left\{ \frac{e^{-i\omega x}}{2\omega i} \right\}_{1}^{-1} \\ &= \sqrt{\frac{2}{\pi}} \left(\frac{\sin(\omega)}{\omega} \right). \end{aligned}$$