# CATEGORY THEORY

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## 1. Prelude

First a word on how to study category theory. Category theory is extremely heavy on new terminology at first sight. It is typically misconceived as abstractly and arbitrarily renaming things to no real gain and a generally poorly understood topic in undergraduate mathematics. To get over the initial shock of the many perhaps new terms and to consolidate simple concepts. One should consider taking the time to highlight these notes in assortment of coloured pens, with each new term given a corresponding colour. Also note that almost everything, at least at this level in category theory, may be represented as a diagram. I highly recommend you copy each new term with a corresponding diagram to anecdote the idea at hand. If all else fails, be sure to remember to break the term down into prefix and suffix and reason it. Typically terms are fairly simple to reason out (e.g., mono = "one", auto = "self", morphism = "to shape").

*Note* (Class). A *class* is a collection of mathematical objects (e.g., sets) which can be unambiguously defined by a property that all its members share.

*Note* (Morphism). A *morphism*, from Greek, is an abstraction derived from structure-preserving mappings between two mathematical structures. The notion of morphism recurs in much of contemporary mathematics.

## 2. Introduction

In each area of mathematics (e.g., sets, groups, topological spaces) there are available many definitions and constructions. It turns out, however, that there are a number of notions (e.g. that of a product) that occur naturally in various areas of mathematics, with only slight changes from one area to another. It is convenient to take advantage of this observation. Category theory can be described as that branch of mathematics in which one studies certain definitions in a broader context - without reference to the particular area to which the definition might be applied.

## 3. Morphisms

A morphism is a map between two objects in an abstract category, more on this later. A general morphism is called a homomorphism. A homomorphism is a term typically used in category theory. The term itself derives from the Greek $o\mu o$ (omo) "alike" and $\mu o\rho\varphi\omega\sigma\iota\varsigma$ (morphosis), "to form" or "to shape".

*Remark.* The similarity in meaning and form of the words "homomorphism" and "homeomorphism" is unfortunate and typically a source of common confusion.

Morphisms can have any of the following properties.

**Definition 3.1** (Monomorphism)**.** A morphism $\varphi : X \to Y$ is a *monomorphism* if $\varphi \circ \psi = \varphi \circ \phi \implies \psi = \phi$ for all morphisms $\psi, \phi : X \to Y$.

**Definition 3.2** (Epimorphism)**.** A morphism $\varphi : X \to Y$ is a *epimorphism* if $\psi \circ \varphi = \phi \circ \varphi \implies \psi = \phi$ for all morphisms $\psi, \phi : Y \to X$.

**Definition 3.3** (Bimorphism)**.** A morphism $\varphi : X \to Y$ is a *bimorphism* if $\varphi$ is both a monomorphism and epimorphism.

**Definition 3.4** (Isomorphism)**.** A morphism $\varphi : X \to Y$ is a *isomorphism* if there exists a morphism $\psi : Y \to X : \varphi \circ \psi = id_Y$ and $\psi \circ \varphi = id_X$.

**Definition 3.5** (Endomorphism)**.** A morphism $\varphi : X \to Y$ is a *endomorphism* if $X = Y$. The class of endomorphisms of a is denoted by $end(a)$.

**Definition 3.6** (Automorphism)**.** A morphism $\varphi : X \to Y$ is a *automorphism* if $\varphi$ is both an endomorphism and an isomorphism. The class of automorphisms of a is denoted by $aut(a)$.

**Definition 3.7** (Retraction)**.** A morphism $\varphi : X \to Y$ has a *retraction* if there exists a morphism $\psi : Y \to X$ with $\varphi \circ \psi = id_Y$ - "if a right inverse of $\varphi$ exists".

**Definition 3.8** (Section)**.** A morphism $\varphi : X \to Y$ has a *section* if there exists a morphism $\psi : Y \to X$ with $\psi \circ \varphi = id_X$ - "if a left inverse of $\varphi$ exists".

**Lemma 3.9** (Retractions and Sections)**.** *Every retraction is an epimorphism and every section is a monomorphism. We have the following three equivalent statements:*

- $\varphi$ *is a monomorphism and a retraction;*
- $\varphi$ *is an epimorphism and a section;*
- $\varphi$ *is an isomorphism.*

## 4. Categories

A category consists of three things: a collection of objects, for each pair of objects a collection of morphisms from one to another, and a binary operation defined on compatible pairs of morphisms called composition. The category must satisfy an identity axiom and an associative axiom which is analogous to the monoid axioms, seen later. We require that morphisms preserve the mathematical structure of the objects (i.e., for vector spaces as objects one would choose linear maps as morphisms, and so on).

**Definition 4.1** (Category)**.** A *category* $\mathcal{K}$ consists of the following three mathematical entities:

(1) A *class* $\mathrm{Ob}(\mathcal{K})$ of objects
(2) A class $\mathrm{Hom}(A, B)$ of *morphisms*, from $A \longrightarrow B$ such that $A, B \in \mathrm{Ob}(\mathcal{K})$.
e.g. $\varphi : A \to B$ to mean $\varphi \in \mathrm{Hom}(A, B)$.

*Remark.* The class of *all* morphisms of $\mathcal{K}$ is denoted $\mathrm{Hom}(\mathcal{K})$.

(3) Given $A, B, C \in \mathrm{Ob}(\mathcal{K})$, a binary operation $\circ : Hom(A,B) \times Hom(B,C) \to Hom(A,C)$ called *composition*, satisfying:
   (a) *(closed associativity)* Given $\varphi : A \to B$, $\psi : B \to C$ and $\phi : A \to C$ we have $\phi \circ (\psi \circ \varphi) = (\phi \circ \psi) \circ \varphi$.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\phi"'] & B \arrow[d, "\psi"] \\ & C \end{tikzcd}$$

   (b) *(identity)* For any object X there is an identity morphism $id_X : X \to X$ such that for any $\varphi : A \to B$ we have $id_B \circ \varphi = \varphi = \varphi \circ id_A$.

$$\begin{tikzcd} X \arrow[loop above, "id_X"] \end{tikzcd}$$

It is also worth noting about what we mean by 'small' and 'large' categories.

**Definition 4.2** (Small Category)**.** A category $\mathcal{K}$ is called small if both $\mathrm{Ob}(\mathcal{K})$ and $\mathrm{Hom}(\mathcal{K})$ are sets. If $\mathcal{K}$ is not small, then it is called large. $\mathcal{K}$ is called locally small if $\mathrm{Hom}(A, B)$ is a set for all $A, B \in \mathrm{Ob}(\mathcal{K})$.

*Remark.* Most important categories in mathematics are not small however, are locally small.

Most basic categories have as objects certain mathematical structures, and the structure-preserving functions as morphisms. A common list is given;

- *Pos* is the category of partially ordered sets and monotonic functions.
- *Top* is the category of topological spaces and continuous functions.
- *Rng* is the category of rings and ring homomorphisms.
- *Grp* is the category of groups and group homomorphisms.
- *Grph* is the category of graphs and graph homomorphisms.

4.1. **Dual Category.** *Duality* is a correpsondence between properties of a category $\mathcal{K}$ and the notion of *dual properties* of the "opposite category" $\mathcal{K}^{op}$. Suppose some proposition regarding a category $\mathcal{K}$, by interchanging the domain and codomain of each morphism as well as the order of composition, a corresponding dual proposition is obtained of $\mathcal{K}^{op}$.

Duality, as such, is the assertion that truth is invariant under this operation on propositions.

**Lemma 4.3.** $(\mathcal{K}^{op})^{op} = \mathcal{K}$.

**Example 4.4.** Suppose a monomorphism in a category $\mathcal{K}$ then an epimorphism is the categorial dual in the dual category $\mathcal{K}^{op}$. Prove this by diagram.

## 5. Functors

A category is itself a type of mathematical structure and so, one can generalise the notion of a morphism thus preserve this structure by the notion of a functor. A functor associates to every object of one category an object of another category, and to every morphism in the first category a morphism in the second. Hence, functors are structure-preserving maps between categories and can be thought of as morphisms in the category of all (small) categories.

In particular, what we have done is define a category of categories and functors - the objects are categories, and the morphisms (between categories) are functors. By studying categories and functors, we are not merely studying a class of mathematical structures and the morphisms between them, we are studying the relationships between various classes of mathematical structures.

**Definition 5.1** (Functor)**.** Let $\mathcal{C}$ and $\mathcal{K}$ be categories. A *functor* $F$ from $\mathcal{C}$ to $\mathcal{K}$ is a mapping that:

(1) associates to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{K}$
(2) associates to each morphism $\varphi : X \to Y \in \mathcal{C}$ a morphism $F(\varphi) : F(X) \to F(Y) \in \mathcal{K}$ satisfying:
    (a) $F(id_X) = id_{F(X)}$ for every object $X \in \mathcal{C}$
    (b) $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$ for all morphisms $\varphi : X \to Y$ and $\psi : Y \to Z$

    *Remark.* That is, functors must preserve identity morphisms and composition of morphisms.

5.1. **Types of Functors.** Like many things in category theory, functors come in a kind of "dual" type in concepts; that of the *contravariant*- and *covariant*- functors, defined as follows:

**Definition 5.2** (Covariant Functor)**.** A *covariant* functor $F$ from a category $\mathcal{C}$ to a category $\mathcal{K}$, $F : \mathcal{C} \to \mathcal{K}$, consists of:

- for each object $X \in \mathcal{C}$, an object $F(X) \in \mathcal{K}$
- for each morphism $\varphi : X \to Y \in \mathcal{C}$, a morphism $F(\varphi) : F(X) \to F(Y)$

provided the following two properties hold:

- For every object $X \in \mathcal{C}$, $F(id_X) = id_{F(X)}$
- For all morphisms $\varphi : X \to Y$ and $\psi : Y \to Z$, $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$

**Definition 5.3** (Contravariant Functor)**.** A *contravariant* functor $F : \mathcal{C} \to \mathcal{K}$, is a 'reversed' covariant functor. In particular, for every morphism $\varphi : X \to Y \in \mathcal{C}$ must be assigned to a morphism $F(\varphi) : F(Y) \to F(X) \in \mathcal{K}$. Alternatively, contravariant functors act as covariant functors from the opposite category $\mathcal{C}^{op} \to \mathcal{K}$.

Functors have particular properties that essentially are the same as morphisms. This may seem obvious given that functors are a generalisation of the notion of a morphism. So, the following definitions should mostly be obvious if you reflect back to the corresponding morphism definitions above or simply break the word down grammatically.

**Definition 5.4** (Endofunctor)**.** A *endofunctor* is a functor that maps a category to its self, sometimes called a *identity functor*.

**Definition 5.5** (Bifunctor)**.** A *bifunctor* is a functor in *two* arguments. More formally, a bifunctor is a functor whose domain is a *product category*. The $Hom$ functor is a natural example; it is contravariant in one argument while covariant in the other. Hence, the $Hom$ functor is of the type $\mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$.

**Definition 5.6** (Multifunctor)**.** A *multifunctor* is a generalisation of the functor concept to n variables, (e.g., A *bifunctor* is when $n = 2$)

5.2. **Properties of Functors.** However, some properties are more particular to functors and we review them here to families ourselves. A functor $F : \mathcal{A} \to \mathcal{B}$ is

(1) *faithful* if for every parallel pair of morphisms $f, g : A \rightrightarrows A' \in \mathcal{A}$, one has $f = g$ whenever $F(f) = F(g)$
(2) *full* if for every morphism $b : F(A) \to F(A') \in \mathcal{B}$, there exists a morphism $a : A \to A' \in \mathcal{A}$ such that $F(a) = b$
(3) essentially *surjective* if for every object $B \in \mathcal{B}$, there exists an object $A \in \mathcal{A}$ with $B$ isomorphic to $F(A)$
(4) an *equivalence* if there exists a functor $F' : \mathcal{B} \to \mathcal{A}$ such that both $F \circ F'$ and $F' \circ F$ are naturally isomorphic to the identity functors; such a functor $F'$ is called a *quasi-inverse* of $F$
(5) an *isomorhpism* if there exists a functor $F' : \mathcal{B} \to \mathcal{A}$ such that both $F \circ F'$ and $F' \circ F$ are equal to the identity functors
(6) *conserative* if it reflects isomorphisms; that is, $a : A \to A'$ is an isomorphism whenever $F(a) : F(A) \to F(A')$ is

**Definition 5.7** (Functor categories and Yoneda embedding)**.**
(1) Given a category $\mathcal{A}$ and a small category $\mathcal{C}$, we denote by $\mathcal{A}^{\mathcal{C}}$ the category of functors from $\mathcal{C} \to \mathcal{A}$ and natural transformations.
(2) In case $\mathcal{A} = Set$, we have the *Yoneda embedding*:
$Y_{\mathcal{C}} : \mathcal{C}^{op} \to Set^{\mathcal{C}}$ $Y_{\mathcal{C}}(X) = \mathcal{C}(X, -)$,
which is full and faithful.

## 6. Monoids

**Definition 6.1** (Monoid)**.** A category with exactly one object is called a *monoid*.

Thus, expanding out the definition of a category we have;

**Corollary** (Monoid)**.** *A* monoid *is a set $X$ together with a law of composition $\circ$ which associates and has identity $id \in X$.*

**Example 6.2.** The following commutative diagram illustrates the monoidal structure of a group:

$$\begin{tikzcd}
(A \otimes A) \otimes A \otimes A \arrow[rr, leftrightarrow, "\alpha"] \arrow[d, "\mu \otimes id"] & & A \otimes (A \otimes A) \arrow[d, "id \otimes \mu"] \\
A \otimes A \arrow[dr, "\mu"] & & A \otimes A \arrow[dl, "\mu \otimes id"] \\
& A &
\end{tikzcd}$$

## 7. Natural Transformations

A natural transformation provides a way of transforming one functor into another while respecting the internal structure (i.e. the composition of morphisms) of the categories involved. Hence, a natural transformation can be considered to be a “morphism of functors”. Indeed this intuition can be formalized to define so-called functor categories. Natural transformations are, after categories and functors, one of the most basic notions of category theory and consequently appear in the majority of its applications.

**Definition 7.1** (Natural Transformation)**.** Let $F$ and $G$ be functors both from the category $\mathcal{C}$ to $\mathcal{K}$. Then a *natural transformation* $\eta : F \to G$ associates to every object $X \in \mathcal{C}$ a morphism $\eta_X : F(X) \to G(X) \in \mathcal{K}$, called the *component* of $\eta$ at $X$. The component morphism $\eta_X$ is subject to the condition that the diagram commutes for every morphism $\varphi : X \to Y \in \mathcal{C}$; that is, $\eta_Y \circ F(\varphi) = G(\varphi) \circ \eta_X$.

We draw the commutative diagram in the category $\mathcal{K}$:

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(\varphi)"] & G(X) \arrow[d, "G(\varphi)"] \\
F(Y) \arrow[r, "\eta_Y"] & G(Y)
\end{tikzcd}$$

Constructions are often ”naturally related” - a vague notion, at first sight. This leads to the clarifying concept of natural transformation, a way to ”map” one functor to another. Many important constructions in mathematics can be studied in this context. ”Naturality” is a principle, like general covariance in physics, that cuts deeper than is initially apparent.

## 8. Universal constructions

Categorial objects are both abstract and have atomic type. Thus we are face with the issue of defining objects without referring to the internal structure. Thus we characterise these objects in terms of their relations to other objects, as given by the morphisms of the respective categories.

Therefore we must find universal properties that uniquely determine the objects of interest. Indeed, it turns out that numerous important constructions can be described purely in this way. The central concept which is needed for this purpose is called categorical limit, and can be dualized to yield the notion of a colimit.

Suppose that $U : \mathcal{K} \to \mathcal{C}$ is a functor from a category $\mathcal{K}$ to a category $\mathcal{C}$, and let $X \in \mathrm{Ob}(\mathcal{C})$. Consider the following dual (opposite) notions:

**Definition 8.1** (Initial morphism)**.** An *initial morphism* from $X \to U$ is an initial object in the category $(X \downarrow U)$ of morphisms from $X \to U$. In other words, it consists of a pair $(A, \varphi)$ where $A \in \mathrm{Ob}(\mathcal{K})$ and $\varphi : X \to U(A) \in \mathrm{Hom}(\mathcal{C})$, such that the following initial property is satisfied:
Whenever $Y \in \mathrm{Ob}(\mathcal{K})$ and $f : X \to U(Y) \in \mathrm{Hom}(\mathcal{C})$, then there exists a unique morphism $g : A \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "f"'] & U(A) \arrow[d, dashed, "U(g)"] \\ & U(Y) \end{tikzcd}$$

**Definition 8.2** (Terminal morphism)**.** A *terminal morphism* from $U \to X$ is a terminal object in the *comma category (i.e., morphisms become objects)* $(U \downarrow X)$ of morphisms from $U \to X$. In other words, it consists of a pair $(A, \varphi)$ where $A \in \mathrm{Ob}(\mathcal{K})$ and $\varphi : U(A) \to X \in \mathrm{Hom}(\mathcal{C})$, such that the following terminal property is satisfied:
Whenever $Y \in \mathrm{Ob}(\mathcal{K})$ and $f : U(Y) \to X \in \mathrm{Hom}(\mathcal{C})$, then there exists a unique morphism $g : Y \to A$ such that the following diagram commutes:

$$\begin{tikzcd} U(Y) \arrow[d, dashed, "U(g)"'] \arrow[dr, "f"] & \\ U(A) \arrow[r, "\phi"'] & X \end{tikzcd}$$

8.1. **Limit.** TODO..

8.2. **Colimit.** TODO..

## 9. Monads

A *Monad* is an endofunctor, coupled with two natural transformations. The notion of ”algebras for a monad” generalizes classical notions from universal algebra, and in this sense, monads can be thought of as ”theories”. The computer science interpretation is that of a unit of computation.

**Definition 9.1** (Monad)**.** Given a category $\mathcal{K}$, a *monad* on $\mathcal{K}$ is the endofunctor $T : \mathcal{K} \to \mathcal{K}$ coupled with two natural transformations, $\eta : id_{\mathcal{C}} \to T$ and $\mu : (T \circ T) \to T$ provided they statisfy the so-called *coherence conditions*:

- $\mu \circ T\mu = \mu \circ \mu T$,
- $\mu \circ T\eta = \mu \circ \eta T = id_T$.

The coherence conditions may be represented by the following commutative diagrams: TODO..

9.1. **Algebras for Monads.** TODO..