ALGEBRA I

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Contents

1. Prelude

TODO: Fix notation here...

2. Introduction

In this course we builds up the rudiments of some important notions of algebraic structures. That is, a algebraic structure of an arbitrary set, or carrier set, coupled with various finitary operations defined on it. ..

3. Groups

Definition 3.1 (Binary operation). A **binary operation** on a set $\mathcal{G}$ is a map $\circ : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$. **N.B.** that the binary operation is *closed*.

Definition 3.2 (Magma). A **magma** is a set $\mathcal{M}$ equipped with a binary operation $\circ$. We denote the magma as the tuple pair $(\mathcal{M}, \circ)$.

Definition 3.3 (Semi-group). A **semi-group** is a set $\mathcal{G}$ equipped with binary operation that is *associative*. Hence, a semi-group is a magma where the operation is *associative*; That is, given any $x, y, z \in \mathcal{G}$ then $x \circ (y \circ z) = (x \circ y) \circ z \in \mathcal{G}$. We denote the semi-group as the tuple pair $(\mathcal{G}, \circ)$, not to be confused with a magma from context.

Definition 3.4 (Monoid). A **semi-group with idenitity** or, **monoid** for short, is a semi-group $(\mathcal{G}, \circ)$ with a unquie idenitity elememt $e \in \mathcal{G}$ such that $x \circ e = x = e \circ x \, \forall x \in \mathcal{G}$

Example 3.5. Given $\mathcal{G} = \mathbb{Z}$ with the binary law of composition $\circ$ to be defined as arithmetic addition $+$. Then, $(\mathbb{Z}, +)$ forms a semi-group with idenitity 0. Verify the axioms.

Definition 3.6 (Group). A **group** is a monoid where every element has an inverse. A abelian group is a group that is commutative.

3.1. Non-commutative groups.

A common class of non-commutative groups are transformation groups. Note:

Definition 3.7 (Transformation). A bijective map $\varphi : X \to X$ is called a **transformation** of X.

Note. The most trivial case is the *idenitity map* id_X by $id_X(x) = x$, $\forall x \in X$.

Hence, there exists a inverse φ^{-1} of φ such that $\varphi^{-1} \circ \varphi = id_X = \varphi \circ \varphi^{-1}$. Now, take two transformations of X, φ and ψ, and let the product $\varphi \circ \psi$ be well defined. Then the set of all transformations of X form the group **Transf(X)**. Since, given $\varphi, \psi, \phi \in Transf(X)$ then we have associatity, $\varphi \circ (\psi \circ \phi) = (\varphi \circ \psi) \circ \phi$. We have idenitity $e = id_X \in Transf(X)$ and so, inverses $\forall \varphi \in Transf(X) \exists! \varphi^{-1} : \varphi \circ \varphi^{-1} = e$. Closure follows from the composition of two transformations φ and ψ, since $(\varphi \circ \psi)^{-1} = \psi^{-1} \circ \varphi^{-1}$.

A transformation group is a type of group action which describes symmetries of objects. More abstractly, since a group $\mathcal{G}$ is a category with a single object in which every morphism is bijective. Then, a group action is a *forgetful functor* $\mathcal{F}$ from the group $\mathcal{G}$ in the category **Grp** to the set category **Set** that is, $\mathcal{F} : \mathcal{G} \to \textbf{Set}$.

That is, for a group $\mathcal{G}$ and set X, a group action is defined as a group homomorphism φ from $\mathcal{G}$ to the symmetric group of X. The action assigns a permutation of X to each element of the group in such a way that the permutation of X assigned to:

- The identity element $e \in \mathcal{G}$ is the identity transformation of X, that is, id_X;
- A product $\varphi \circ \psi \in \mathcal{G}$ is the composite of the permutations assigned to φ and ψ.

Given that each element of $\mathcal{G}$ is represented as a permutation. Then a group action can also be consider as a permutation representation.

3.2. **Permutations.** Now take a finite set X with $|X| = n$, then the transformations of X are called **permutations** of the elements of X. In particular, the group of permutatons of $X = \{1, 2, \cdots, n\}$ is a **symmetric group** of *order* n, denoted S_n with **order** $|S_n| = n!$. Thus, by taking any subgroup of S_n we have a **permutation group**. Also note that, for finite sets, *permutations* and *bijective maps* refer to the same operation, namely rearrangement of elements of X.

A permutaton $\sigma \in S_n$ can be notated by,

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \text{ where } a_1 = \sigma(1), a_2 = \sigma(2), \cdots .$$

The idenitity permutation $id_n \in S_n$ is simply,

$$id_n = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

.

Since $|S_n| = n!$ then the total number of ways n elements maybe permuted is $n!$.

Take any two permutations $\sigma, \pi \in S_n$ then composition is well defined as **functional composition** as follows.

Given,

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \textit{ and } \pi = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

then,

$$\begin{aligned} \pi \circ \sigma &= \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(a_1) & \pi(a_2) & \cdots & \pi(a_n) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \cdots & n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} \end{aligned}$$

.

A inverse of any permutation $\sigma \in S_n$ is given by,

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}^{-1} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & 2 & \cdots & n \end{pmatrix}$$