REAL ANALYSIS I

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1. Convergence

Definition 1.1 (Limit)**.** Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of vectors in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say that $\{x_k\}_{k=1}^{\infty}$ *converges* to a *limit* x if,

$$\lim_{k\to\infty} d(x_k, x) = \lim_{k\to\infty} \|x_k - x\| \to 0 \in \mathbb{R}$$

written $x_k \to x$.

2. Compactness

Let I denote any, possibly infinite, indexing set.

Definition 2.1 (Open Cover)**.** A **open cover** of a set $S \subseteq \mathbb{R}^n$ is a collection $\{V_i\}_{i\in I}$ of open sets of $\mathbb{R}^n$ such that $S \subset \bigcup_{i\in I} V_i$. A *subcover* is a subcollection which also covers S.

Definition 2.2 (Compact)**.** A subset $S \subseteq \mathbb{R}^n$ is said to be **compact** if from every open cover of S we may find a *finite* subcover of S.

3. Limits

Definition 3.1. Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \bar{\Omega}$. A point $y \in \mathbb{R}^m$ is said to be the *limit* of f as $x \to \infty$, written as

$$\lim_{x\to\infty} f(x) = y,$$

if for every neighborhood $B_\epsilon(y)$ of $y \in \mathbb{R}^m$ there exists a neighborhood $B_\delta(a)$ of $a \in \Omega$ such that whenever $x \in \Omega \cap B_\delta(a)$, this implies $f(x) \in B_\epsilon(y)$. That is,

$$f(\Omega \cap B_\delta(a)) \subseteq B_\epsilon(y).$$

Proposition 3.2. *For $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ we have*

$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x).$$

If, $g(x) \neq 0$ everywhere *on Ω, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

Problem 3.3. *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{a} \in \Omega$. Then,*

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{y}$$

if and only if,

$$\lim_{\vec{x}\to\vec{a}} f_i(\vec{x}) = y_i$$

for each i in $1 \leq i \leq m$.

Example 3.4. Show that,

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0.$$

Let $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$ and fix some arbitrarily small $\epsilon > 0$. Then we must find a $\delta > 0$ such that $|f(x,y) - 0| < \epsilon$ whenever $0 < \|(x,y) - (0,0)\| < \delta$.

Now, note that $|x^3| \leq (x^2+y^2)^{3/2}$ and so by the triangle inequality, we have

$$\begin{aligned} |f(x,y)| &= \left|\frac{x^3+y^3}{x^2+y^2}\right| \\ &\leq \frac{2(x^2+y^2)^{3/2}}{x^2+y^2} \\ &= 2\sqrt{x^2+y^2}. \end{aligned}$$

whence, choosing $\delta = \epsilon/2$, we get $|f(x,y)| < \epsilon$ provided that $\sqrt{x^2+y^2} < \delta$.

Thus, $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$.

Example 3.5. Show that,

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1.$$

Let $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ and take $t = x^2+y^2$ so that we have,

$$\begin{aligned} &\lim_{(x,y)\to(0,0)} f(x,y) \\ &= \lim_{t\to 0} \frac{\sin(t)}{t} = 1. \end{aligned}$$

4. Differentiable

Definition 4.1 (Differentiable)**.** Let $f : \Omega \to \mathbb{R}^m$ be a function defined on an open set $\Omega \in \mathbb{R}^n$ and point $\vec{a} \in \Omega$. The function f is said to be *differentiable* at $\vec{a}$ if there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that,

$$\lim_{\mathbf{x}\to\vec{a}} \frac{\|f(\mathbf{x}) - f(\vec{a}) - T(\mathbf{x}-\vec{a})\|_m}{\|\mathbf{x}-\vec{a}\|_n} = 0.$$

If f is differentiable at every point $\vec{a} \in \Omega$ then f is differentiable *on* Ω.

Theorem 4.2 (Linear Approximation)**.** *The function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at point $\vec{a}$ if and only if there is a function $\epsilon(\mathbf{x})$ so that for $\mathbf{x} \in \Omega$ we have,*

$$f(\mathbf{x}) = f(\vec{a}) + T(\mathbf{x}-\vec{a}) + \epsilon(\mathbf{x})\|\mathbf{x}-\vec{a}\|$$

with $\epsilon(\mathbf{x}) \to 0$ as $\mathbf{x} \to \vec{a}$.

Proof.

Set

$$\epsilon(\mathbf{x}) = \frac{f(\mathbf{x}) - f(\vec{a}) - T(\mathbf{x} - \vec{a})}{\|\mathbf{x} - \vec{a}\|} : \mathbf{x} \neq \vec{a}.$$

Now, if f is differentiable at $\vec{a}$, then $\lim_{\mathbf{x} \to \vec{a}} \epsilon(\mathbf{x}) = 0$.

Conversely, suppose

$$f(\mathbf{x}) = f(\vec{a}) + T(\mathbf{x} - \vec{a}) + \epsilon(\mathbf{x})\|\mathbf{x} - \vec{a}\|$$

holds, and since $\mathbf{x} \neq \vec{a}$, we have

$$\frac{f(\mathbf{x}) - f(\vec{a}) - T(\mathbf{x} - \vec{a})}{\|\mathbf{x} - \vec{a}\|} = \epsilon(\mathbf{x}) \to 0$$

as $\mathbf{x} \to \vec{a}$ and so f is differentiable at the point $\vec{a}$.

□

Theorem 4.3 (Chain Rule)**.** *Let Ω be a open set in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$ and $g : U \to \mathbb{R}^p$, where U is a open set in $\mathbb{R}^m$ with $f(\Omega) \subseteq U$. If f is differentiable at $\vec{a} \in \Omega$ and g is differentiable at $f(\vec{a})$, then $g \circ f$ is differentiable at $\vec{a}$ and*

$$D_{(g \circ f)}(\vec{a}) = D_g(f(\vec{a}))D_f(\vec{a}).$$

Proof. TODO □

4.1. **Partial derivatives.**

Definition 4.4 (Directional Derivative)**.** The directional derivative of f at $\vec{a}$ in the direction of a non-zero vector $\vec{u} \in \mathbb{R}^n$, denoted by $D_{\vec{u}}f(\vec{a})$ is defined by,

$$D_{\vec{u}}f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t},$$

whenever the limit exists.

Theorem 4.5. *If $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{a} \in \Omega$, then for any direction non-zero $\vec{u} \neq 0$, $\vec{u} \in \mathbb{R}^n$, $D_{\vec{u}}f(\vec{a})$ exists and*

$$D_{\vec{u}}f(\vec{a}) = \langle \nabla f(\vec{a}), \vec{u} \rangle.$$

Proof. TODO □

5. Mean Value Theorem

Theorem 5.1. *Suppose $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is some differentiable function on the open* convex set Ω. *Let $\vec{a}, \vec{b} \in \Omega$ and $\gamma(\lambda) = \vec{a} + \lambda(\vec{b} - \vec{a})$ be the line segment joining points $\vec{a}$ and $\vec{b}$. Then there exists some point $\vec{c}$ on the line $\gamma(\lambda)$ such that,*

$$f(\vec{b}) - f(\vec{a}) = \langle \nabla f(\vec{c}), (\vec{b} - \vec{a}) \rangle.$$

Proof. TODO □

Corollary. *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on a convex subset $K \subset \Omega$. If $\|\nabla f(x)\| \leq M \, \forall x \in K$, then*

$$|f(x) - f(y)| \leq M \, \|x - y\|$$

for all $x, y \in K$.

Corollary. *Let f be a differentiable function on an open convex set $\Omega \subseteq \mathbb{R}^n$. If $\nabla f(x) = 0 \forall x \in \Omega$, then f is constant on Ω.*

Proof. Let $x, y \in \Omega : x \neq y$. Then we have $|f(x) - f(y)| = 0$. That is, $f(x) = f(y)$. □

6. Taylor Series

Theorem 6.1 (Taylor Series)**.** *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be $f \in \mathcal{C}^2(\Omega)$ on the open convex set Ω and $a, x \in \Omega$. Then there exists a point c on the line segment joining a and x, such that*

$$f(x) = f(a) + \langle \nabla f(a), (x - a) \rangle + \frac{1}{2!} \langle H_f(c)(x - a), (x - a) \rangle.$$

7. Constrain extrema - Lagrange multipliers

Theorem 7.1 (Lagrange multiplier)**.** *Let $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1(\Omega)$ in the open set Ω. Let $S = \{x \in \Omega : g(x) = 0\}$ and $s \in S$ with $\nabla g(s) \neq 0$. If the restriction $f|_S$ of f to S takes on an extreme value at s, then there exists some $\lambda \in \mathbb{R}$ such that*

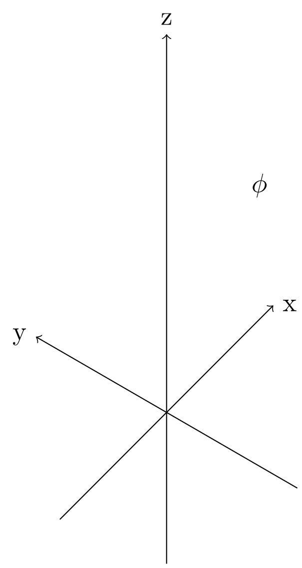
$$\nabla f(s) = \lambda \nabla g(s).$$

Example 7.2. ..

8. Cylindrical Coordinates

Definition 8.1 (Cylindrical Coordinates)**.** The *cylindrical coordinates* are defined to be the polar map with z-coordinate. That is,

$$\phi(r, \theta, z) = \Big(r\cos(\theta), \quad r\sin(\theta), \quad z \Big)$$



9. Green's theorem

Green's theorem is a two dimensional analog of the Fundamental Theorem of Calculus.

Theorem 9.1. *Let $\mathbf{F}(x,y) = (F(x,y), G(x,y))$ be, $\mathcal{C}^1(\bar{\Omega})$, a continuous vector field on an open set containing domain $\Omega \subseteq \mathbb{R}^2$ whose boundary curve, $C = \partial\Omega$, is closed and piecewise smooth. Then, by considering $\partial\Omega$ to have positive, or counterclockwise, orientation, we have:*

$$\iint_\Omega \left(\frac{\partial G(x,y)}{\partial x} - \frac{\partial F(x,y)}{\partial y} \right) \mathrm{d}y \, \mathrm{d}x = \int_{\partial\Omega} \left(F(x,y)\mathrm{d}x + G(x,y)\mathrm{d}y \right).$$

Proof.

First suppose Ω is an elementary region of the form:

$$\Omega = \{(x,y) : a \le x \le b, \phi_1(x) \le y \le \psi_1(x)\} \qquad (x\text{-simple})$$

where $\phi_1, \psi_1 \in \mathcal{C}([a,b])$ and

$$\Omega = \{(x,y) : c \le y \le d, \phi_2(y) \le x \le \psi_2(y)\} \qquad (y\text{-simple})$$

where $\phi_2, \psi_2 \in \mathcal{C}([c,d])$.

Since $\mathbf{F}(x,y) \in \mathcal{C}^1(\bar{\Omega})$ then $F(x,y) \in \mathcal{C}^1(\bar{\Omega})$. Hence we first show,

$$-\iint_\Omega \frac{\partial F}{\partial y} \mathrm{d}y \, \mathrm{d}x = \int_{\partial\Omega} F(x,y)\mathrm{d}x.$$

By writing Ω as x-simple we see that,

$$\int_{\partial\Omega} F(x,y)\mathrm{d}x = \int_{C_1} F\mathrm{d}x + \underbrace{\int_{C_2} F\mathrm{d}x}_{\dagger} + \int_{C_3} F\mathrm{d}x + \underbrace{\int_{C_4} F\mathrm{d}x}_{\dagger}.$$

Observe† that the curves C_2, C_4 are the vertical line portions $x = a$ and $x = b$ respectively. So any parametrisation $x'(t) = 0$ of constant terms gives us $\mathrm{d}x = 0$ and so the sums are zero measure.

Consider also the parameterisations $\gamma_1(x) = (x, \phi_1(x))$ and $\gamma_3(x) = (x, \psi_1(x))$ for the curves C_1 and C_3 respectively with $x \in [a, b]$. Since C_3 has negative orientation, we have

$$\int_{\partial\Omega} F(x,y)\mathrm{d}x = \int_a^b \left(F(x, \phi_1(x)) - F(x, \psi_1(x))\right)\mathrm{d}x.$$

Holding x fixed we have, by the Fundamental Theorem of Calculus, the following

$$\iint_{\Omega} \frac{\partial F(x,y)}{\partial y}\mathrm{d}y\,\mathrm{d}x = \int_a^b \left(\int_{\phi_1(x)}^{\psi_1(x)} \frac{\partial F(x,y)}{\partial y}\mathrm{d}y\right)\mathrm{d}x$$

and so,

$$\iint_{\Omega} \frac{\partial F(x,y)}{\partial y}\mathrm{d}y\,\mathrm{d}x = -\int_{\partial\Omega} F(x,y)\mathrm{d}x. \tag{1}$$

Similarly, expressing Ω as a y-simple region, we obtain,

$$\iint_{\Omega} \frac{\partial G(x,y)}{\partial x}\mathrm{d}y\,\mathrm{d}x = \int_{\partial\Omega} G(x,y)\mathrm{d}y. \tag{2}$$

By adding (1) and (2) together we have the required result in this simple case.

□

10. Fourier Transform

Definition 10.1 (Fourier transform)**.** Let $f : \mathbb{R} \to \mathbb{C}$ be an absolutely integrable function on $\mathbb{R}$. The *Fourier transform* $\mathcal{F}(f) \equiv \hat{f}$ of f is defined by the integral

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\omega x}\mathrm{d}x.$$

Definition 10.2 (Characteristic function)**.**

$$\chi_{(-a,a)}(x) \doteq \begin{cases} 1 & \text{if } |x| < a, \\ 0 & \text{if } |x| \geq a. \end{cases}$$

Example 10.3. Suppose $f(x) = \chi_{(-1,1)}(x)$. Find the Fourier transform $\hat{f}(\omega)$.

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{(-1,1)}(x) e^{-i\omega x} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{-i\omega x}}{-i\omega} \right\}_{-1}^{1} \\ &= \frac{2}{\sqrt{2\pi}} \left\{ \frac{e^{-i\omega x}}{2\omega i} \right\}_{1}^{-1} \\ &= \sqrt{\frac{2}{\pi}} \left(\frac{\sin(\omega)}{\omega} \right). \end{aligned}$$