TOPOLOGY I

EDWARD O'CALLAGHAN

Contents

1. Metric space

1.1. Metrics.

Definition 1.1 (Metric space). A *metric space* is a order pair (X, d) where X is a set and d is some function $d : X \times X \to X$ that satisfies, for all $x, y, z \in X$,

- $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$;
- $d(x, y) = d(y, x)$ (symmetric);
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

We call d a metric on X.

Problem 1.2 (Discrete metric). *Suppose*

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Prove $d(x, y)$ defines a metric.

Example 1.3 (Euclicean metric). Consider the set of real n-tuples $M = \mathbb{R}^n$.

For points $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, \ldots, y_n\}$ in $\mathbb{R}^n$ we set

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

Definition 1.4 (Continuity). Let (X, d_X) and (Y, d_Y) be metric spaces. We say that the mapping $f : X \to Y$ is *continuous at a point* $x_0 \in X$, if

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in X : d_X(x, x_0) < \delta \implies d_Y(y, y_0) < \epsilon.$$

The mapping $f : X \to Y$ is said to be *continuous* if f is continuous *at every point* $x_0 \in X$.

1.2. Topology of a metric space.

A metric space provides sufficient structure to study the notions of convergence and thus continuity. A closer study of continuity of mappings in the setting of metric spaces revels that a metric need not be of a specific type. Rather, a class of subsets defined by the metric lead to the concept of the underlying *topology* in a metric space that is decisive for continuity to make sense.

Definition 1.5 (Open set). A subset U of a metric space $M = (X, d)$ is said to be *open* if;

$$\forall x \in U \, \exists \epsilon > 0 : d(x, y) < \epsilon \, \forall y \in X \implies y \in U.$$

Alternatively, we may consider defining the notion of a *open ball* $B_\epsilon(x)$ and using this equivalently to redefine a *open set*.

Definition 1.6 (Open ball)**.** Let $M = (X, d)$ be an arbitrary metric space and let some point $x_0 \in X$ with $\epsilon \in \mathbb{R}^+$. Then an open ball with center x_0 and radius ϵ is defined as:

$$B_\epsilon(x_0) = \{x \in X : d(x_0, x) < \epsilon\}$$

Remark. A closed ball may be defined in a similar way, that is,

$$\bar{B}_\epsilon(x_0) = \{x \in X : d(x_0, x) \leq \epsilon\}$$

Hence we have the alternative definition in the following way.

Definition 1.7 (Open set - alternative)**.** For some arbitrary metric space $M = (X, d)$ and open ball $B_\epsilon(x)$ where $\epsilon > 0$. A subset $U \subset X$ is said to be a *open set* if,

$$\forall x \in U \, \exists \epsilon > 0 \, : B_\epsilon(x) \subseteq U.$$

Theorem 1.8. *The family of open sets $\mathcal{T}$ in an arbitrary metric space (X, d) has the following properties:*

- $\emptyset, X \in \mathcal{T}$*;*
- *Given some arbitrary finite set* $\{U_i\}_{i=1}^n \subseteq \mathcal{T}$ *where each* $U_i \subset X$*, then* $\displaystyle\bigcap_{i=1}^n U_i \in \mathcal{T}$*;*
- *Given some arbitrary subsets* $U_\lambda \in \mathcal{T} : \lambda \in \Lambda$ *in* X*, then the union* $\displaystyle\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$*.*

Hence, a arbitrary metric space $M = (X, d)$ with some family of open sets $\mathcal{T}$ in M is a **topology** $\mathcal{T}$ of M. The pair $(M, \mathcal{T})$ is called a **topological space**.

Proof. Since there are no points $x_0 \in \emptyset$, then the open ball $B_\epsilon(x)$ exists for all $x \in \emptyset$ for any $\epsilon > 0$.

Conversely, since every point $x \in X$ exists then the open ball $B_\epsilon(x)$ exists for all $x_0 \in X$. Hence both $\emptyset$ and X are open.

If $x_0 \in \displaystyle\bigcap_{i=1}^n U_i$ then $x_0 \in U_i$ for each U_i. Now, since every U_i is open, we have $B_{\epsilon_i}(x_0) \subseteq U_i$. By setting $\epsilon = \displaystyle\min_{1 \leq i \leq n} \epsilon_i$ we have $B_\epsilon(x) \subseteq U_i$ for all i. Thus $B_\epsilon(x) \subseteq \displaystyle\bigcap_{i=1}^n U_i$ and so $\displaystyle\bigcap_{i=1}^n U_i$ is open.

If $x_0 \in \displaystyle\bigcup_{\lambda \in \Lambda} U_\lambda$, then we may find some $\lambda_0 \in \Lambda$ so that $x_0 \in U_{\lambda_0}$. Since U_{λ_0} is open we have $B_\epsilon(x) \subseteq U_{\lambda_0}$ for some $\epsilon > 0$. Now since

$$U_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$$

then

$$B_\epsilon(x) \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$$

and so $\displaystyle\bigcup_{\lambda \in \Lambda} U_\lambda$ is open. $\square$

Lemma 1.9. *Any arbitrary metric space $M = (X, d)$ induces a topology $\mathcal{T}$ in M.*

Another piece of terminology that is often seen in topology is that of a neighbourhood which we define here for completeness.

Definition 1.10 (Neighbourhood)**.** Suppose a arbitrary metric space $M = (X, d)$. A *neighborhood* of some point $x \in X$ is a subset $V \subset X$ such that $B_\epsilon(x) \subseteq V$.

In this case, we call the open set V the ϵ-neighborhood of the point x in the set X.

1.3. **Continuous maps and homeomorphisms.** By the abstraction of open sets we may describe continuity of a mapping by way that is independent of the metric.

Theorem 1.11. *Let $f : X \to Y$ be a mapping between metric spaces (X, d_X) and (Y, d_Y). Then f is continuous if and only if for every open set V in Y, the set $f^{-1}(V)$ is an open set in X.*

Proof. Consider the mapping $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y). Now, suppose that $V \subset Y$ so that $\forall y \in V\, \exists \delta > 0 : B_\delta(y) \subseteq V$ is an open set V in Y and if $x \in f^{-1}(V)$, then there exists some $y \in V$ such that $y = f(x)$.

Now, if f is continuous and $U \subset X$ with $U = f^{-1}(V)$, we have

$$\begin{gathered}\forall x \in f^{-1}(V)\, \exists \epsilon > 0 : B_\epsilon(x) \subseteq f^{-1}(V) \\ \implies \forall x \in U\, \exists \epsilon > 0 : B_\epsilon(x) \subseteq U\end{gathered}$$

and so U is a open set in X. We now show the converse.

Suppose that $U = f^{-1}(V)$ is open in X whenever $V = f(U)$ is open in Y. Then,

$$\begin{gathered}\forall x \in U\, \exists \epsilon > 0 : B_\epsilon(x) \subseteq U \\ \forall x \in f^{-1}(V)\, \exists \epsilon > 0 : B_\epsilon(x) \subseteq f^{-1}(V) \\ \forall y \in f \cdot f^{-1}(V)\, \exists \delta > 0 : B_\delta(y) \subseteq f \cdot f^{-1}(V) \\ \implies \forall y \in V\, \exists \delta > 0 : B_\delta(y) \subseteq V.\end{gathered}$$

Thus f is continuous. ☐

Definition 1.12 (Homeomorphism)**.** A map $f : X \to Y$ is a *homeomorphism* if f is a continuous bijection.

Remark. A homeomorphism is the structure preserving mapping between topological spaces. As we shall see later, a isomorphism is insufficient to preserve the structure of a topology since a isomorphism need not be continuous.

1.4. **Closed sets.**

Definition 1.13. Consider a *sequence* x_n in a metric space (X, d). If $x \in X$ and given some fixed $\epsilon > 0$ we may find a corresponding integer $N \geq 1$ such that

$$d(x_n, x) < \epsilon \, \forall n \geq N.$$

Alternatively, in terms of an ϵ-neighbourhood we have,

$$\lim_{n \to \infty} B_\epsilon(x_n) \subseteq B_\epsilon(x) \text{ for some } \epsilon > 0.$$

In particular, we say that $x_n \to x$ as $n \to \infty$ and that x is the *limit* of the sequence x_n.

Example 1.14. Consider two metric spaces (X, d_X) and (Y, d_Y). Show that a function $f : X \to Y$ is continuous if and only if, whenever $x_n \in X$ and $x_n \to x$ as $n \to \infty$, we have $f(x_n) \to f(x)$.

We need only show that, for every open set V in Y, the set $f^{-1}(V)$ is open in X. Hence, given some $U \subset X$ with $U = f^{-1}(V)$, we have

$$\forall x_n \in U \, \exists \epsilon > 0 : B_\epsilon(x_n) \subseteq U$$

since,

$$\begin{aligned} &\lim_{n \to \infty} B_\epsilon(x_n) \subseteq B_\epsilon(x) \text{ for some } \epsilon > 0 \\ \implies &\forall x \in U \, \exists \epsilon > 0 : B_\epsilon(x) \subseteq U \end{aligned}$$

and so U is a open set in X. Therefore

$$\begin{aligned} &\forall y_n \in V \, \exists \delta > 0 : B_\delta(y_n) \subseteq V \\ \implies &\forall y \in V \, \exists \delta > 0 : B_\delta(y) \subseteq V \end{aligned}$$

and so V is open in Y whenever U is open in X.

2. Topological Space

Topological spaces allow for the formal definition of *convergence, continuity and connectedness*. These are central concepts found typically in analysis. However topology, the study of topological spaces, is more general and allows for study of the mechanisms behind analysis in this light. A mathematical hierarchy of generality can be view in this way;

Normed vector spaces $\subset$ Metric spaces $\subset$ Topological spaces.

In truth, topology captures the main aspects of geometry however dispenses with some notions, in particular, that of an angle. So topology is geometry with no such notion of angle and thus more general. Here is a light example:

Example 2.1.

$$\heartsuit = 0 = \square.$$

However,

$$\angle \neq \square.$$

Definition 2.2 (Topological space)**.** A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subset 2^X$ such that:

- $\emptyset, X \in \mathcal{T}$;
- For $\{U_i\}_{i=1}^n \subseteq \mathcal{T}$ we have $\displaystyle\bigcap_{i=1}^{n} U_i \in \mathcal{T}$;
- For $U_\lambda \in \mathcal{T} : \lambda \in \Lambda$ give some an arbitrary indexing set Λ we have $\displaystyle\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$.

In particular $\mathcal{T}$ is **closed under** *finite* intersections and *arbitrary*, possibly uncountably infinite, unions. We may denote the pair $(X, \mathcal{T})$ by $\mathcal{T}_X$.

If $(X, \mathcal{T})$ is a topological space, we call $\mathcal{T}$ a topology on X. A set $U \in \mathcal{T}$ is called the *open set* of the topology $\mathcal{T}$.

Problem 2.3. *Pick any arbitrary metric space $M = (X, d)$ and show this induces a topology $\mathcal{T}_X$. In particular, the metric topological space $(M, \mathcal{T}_M)$.*

$V \subseteq X$ is *closed* if its complement is *open*. The topology could be defined equivalently by the collection of closed sets, which enjoys finite unions and arbitrary intersection. If $Z \subset X$, the *closure* of Z, denoted $\bar{Z}$, is the intersection of all closed sets containing Z. By the arbitrary intersection property of closed sets, $\bar{Z}$ is *closed*. A *neighbourhood* of a point $x \in X$ is any open subset $V \subset X$ containing x.

Definition 2.4 (Discrete topology)**.** A *discrete topology* has every subset open.

Definition 2.5 (Indiscrete topology)**.** A *indiscrete topology* has no open sets except $\emptyset$ and X itself.

Definition 2.6 (Closure operator)**.** A *closure operator* is a function

$$\mathrm{cl} : 2^X \to 2^X$$

that satisfies, for $A, B \in 2^X$,

- $A \subset \mathrm{cl}(A)$ (extensively);
- $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ (idempotent);
- $\mathrm{cl}(\emptyset) = \emptyset$ (preserves nullary unions);
- $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ (preserves binary unions).

Note here that 2^X denotes the *power set* of X.

We may redefine a topological space equivalently in terms of a closure operator in the following way:

Definition 2.7 (Topological space - alternative)**.** A *topological space* (X, cl) is a set X endowed with closure operator from the power set of X to itself $\mathrm{cl} : 2^X \to 2^X$.

Notice that we may recover the topological definitions in terms of the closure operator. By considering a function $f : (X, \mathrm{cl}) \to (Y, \mathrm{cl}')$ between topological spaces (X, cl) and (Y, cl'). See that f is then said to be continuous if, for all $A \subset X$, we have $f(\mathrm{cl}(A)) \subset \mathrm{cl}'(f(A))$. A point $x \in A$ is closed in (X, cl) if $x \in \mathrm{cl}(A)$ and the set A is closed in (X, cl) if $A = \mathrm{cl}(A)$.