CATEGORY THEORY

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1. Introduction

We begin by defining what we mean by a 'Category'.

Definition 1.1 (Category)**.** A *category* $\mathcal{K}$ consists of the following three mathematical entities:

(1) A *class* $\mathrm{Ob}(\mathcal{K})$ of objects
(2) A class $\mathrm{Hom}(A, B)$ of *morphisms*, from $A \longrightarrow B$ such that $A, B \in \mathrm{Ob}(\mathcal{K})$.
e.g. $f : A \to B$ to mean $f \in \mathrm{Hom}(A, B)$.

Remark. The class of *all* morphisms of $\mathcal{K}$ is denoted $\mathrm{Hom}(\mathcal{K})$.

(3) Given $A, B, C \in \mathrm{Ob}(\mathcal{K})$, a binary operation $\circ : Hom(B, C) \times Hom(A, B) \to Hom(A, C)$ called *composition*, satisfying:
(a) *(associativity)* Given $f : A \to B$, $g : B \to C$ and $h : C \to D$
we have $h \circ (g \circ f) = (h \circ g) \circ f$.

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[ur, "f"] \arrow[r, "h"'] & C \end{tikzcd}$$

(b) *(identity)* For any object X there is an identity morphism $1_X : X \to X$ such that for any $f : A \to B$ we have $1_B \circ f = f = f \circ 1_A$.

$$\begin{tikzcd} X \arrow[loop above, "1_X"] \end{tikzcd}$$

It is also worth noting about what we mean by 'small' and 'large' categories.

Definition 1.2 (Small Category)**.** A category $\mathcal{K}$ is called small if both $\mathrm{Ob}(\mathcal{K})$ and $\mathrm{Hom}(\mathcal{K})$ are sets. If $\mathcal{K}$ is not small, then it is called large. $\mathcal{K}$ is called locally small if $\mathrm{Hom}(A, B)$ is a set for all $A, B \in \mathrm{Ob}(\mathcal{K})$.

Remark. Most important categories in mathematics are not small however, are locally small.