ALGEBRA I

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Contents

1. Metric space

1.1. Metrics.

Definition 1.1 (Metric space)**.** A *metric space* is a order pair (X, d) where X is a set and d is some function $d : X \times X \to X$ that satisfies, for all $x, y, z \in X$,

- $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$;
- $d(x, y) = d(y, x)$ (symmetric);
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

We call d a metric on X.

Problem 1.2 (Discrete metric)**.** *Suppose*

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Prove $d(x, y)$ defines a metric.

Example 1.3 (Eucliean metric)**.** Consider the set of real n-tuples $M = \mathbb{R}^n$.

For points $\mathbf{x} = \{x_1, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, \ldots, y_n\}$ in $\mathbb{R}^n$ we set

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

Definition 1.4 (Continuity)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. We say that the mapping $f : X \to Y$ is *continuous at a point* $x_0 \in X$, if

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in X : d_X(x, x_0) < \delta \implies d_Y(y, y_0) < \epsilon.$$

The mapping $f : X \to Y$ is said to be *continuous* if f is continuous *at every point* $x_0 \in X$.

1.2. Topology of a metric space.

A metric space provides sufficient structure to study the notions of convergence and thus continuity. A closer study of continuity of mappings in the setting of metric spaces revels that a metric need not be of a specific type. Rather, a class of subsets defined by the metric lead to the concept of the underlying *topology* in a metric space that is decisive for continuity to make sense.

Definition 1.5 (Open set)**.** A subset U of a metric space $M = (X, d)$ is said to be *open* if;

$$\forall x \in U \, \exists \epsilon > 0 : d(x, y) < \epsilon \, \forall y \in X \implies y \in U.$$

Alternatively, we may consider defining the notion of a *open ball* $B_\epsilon(x)$ and using this equivalently to redefine a *open set*.

Definition 1.6 (Open ball). Let $M = (X, d)$ be an arbitrary metric space and let some point $x_0 \in X$ with $\epsilon \in \mathbb{R}^+$. Then an open ball with center x_0 and radius ϵ is defined as:

$$B_\epsilon(x_0) = \{x \in X : d(x_0, x) < \epsilon\}$$

Remark. A closed ball may be defined in a similar way, that is,

$$B_\epsilon(x_0) = \{x \in X : d(x_0, x) \leq \epsilon\}$$

Hence we have the alternative definition in the following way.

Definition 1.7 (Open set - alternative). For some arbitrary metric space $M = (X, d)$ and open ball $B_\epsilon(x)$ where $\epsilon > 0$. A subset $U \subset X$ is said to be a *open set* if,

$$\forall x \in U \, \exists \epsilon > 0 \, : B_\epsilon(x) \subseteq U.$$

Another piece of terminology that is often seen in topology is that of a neighbourhood which we define here for completeness.

Definition 1.8 (Neighbourhood). Suppose a arbitrary metric space $M = (X, d)$. A *neighborhood* of some point $x \in X$ is a subset $V \subset X$ such that $B_\epsilon(x) \subseteq V$.

In this case, we call the open set V the ϵ-neighborhood of the point x in the set X.

By the abstraction of open sets we may describe continuity of a mapping by way that is independent of the metric.

Theorem 1.9. *Let $f : X \to Y$ be a mapping between metric spaces (X, d_X) and (Y, d_Y). Then f is continuous if and only if for every open set V in Y, the set $f^{-1}(V)$ is an open set in X.*

Proof. Consider the mapping $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y). Now, suppose that $V \subset Y$ so that $\forall y \in V \, \exists \epsilon > 0 : B_\epsilon(y) \subseteq V$ is an open set V in Y.

Then, if f is continuous and $U \subset X$ with $X = f^{-1}(V)$, we have

$$\begin{aligned} &\forall x \in f^{-1}(V) \, \exists \delta > 0 : B_\delta(x) \subseteq f^{-1}(V) \\ \implies &\forall x \in U \, \exists \delta > 0 : B_\delta(x) \subseteq U \end{aligned}$$

and so U is a open set in X. Since f^{-1} exists the converse is trivially so. □

2. Topological Space

Definition 2.1 (Topological space). A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subset 2^X$ such that:

- $\emptyset, X \in \mathcal{T}$;
- For $\{U_i\}_{i=1}^n \subseteq \mathcal{T}$ we have $\displaystyle\bigcap_{i=1}^n U_i \in \mathcal{T}$;

- For $U_\lambda \in \mathcal{T} : \lambda \in \Lambda$ give some an arbitrary indexing set Λ we have $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$.

In particular $\mathcal{T}$ is **closed under** *finite* intersections and *arbitrary*, possibly uncountably infinite, unions. We may denote the pair $(X, \mathcal{T})$ by $\mathcal{T}_X$.

If $(X, \mathcal{T})$ is a topological space, we call $\mathcal{T}$ a topology on X. A set $U \in \mathcal{T}$ is called the *open set* of the topology $\mathcal{T}$.

Example 2.2. TODO..

$V \subseteq X$ is *closed* if its complement is *open*. The topology could be defined equivalently by the collection of closed sets, which enjoys finite unions and arbitrary intersection. If $Z \subset X$, the *closure* of Z, denoted $\bar{Z}$, is the intersection of all closed sets containing Z. By the arbitrary intersection property of closed sets, $\bar{Z}$ is *closed*. A *neighbourhood* of a point $x \in X$ is any open subset $V \subset X$ containing x.

Definition 2.3 (Discrete topology)**.** A *discrete topology* has every subset open.

Definition 2.4 (Indiscrete topology)**.** A *indiscrete topology* has no open sets except $\emptyset$ and X itself.