Ha Noi University of Science and Technology

2D LAPLACE EQUATION

WITH SUCCESSIVE OVER RELAXATION METHOD

Subject code
IT4110E

Class code
147831

Teacher
Vu Van Thieu

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MEMBER LIST

SCIENTIFIC COMPUTING

01 Nguyễn Đình Dương
20225966

02 Nguyễn Mỹ Duyên
20225967

03 Nguyễn Trọng Minh Phương
20225992

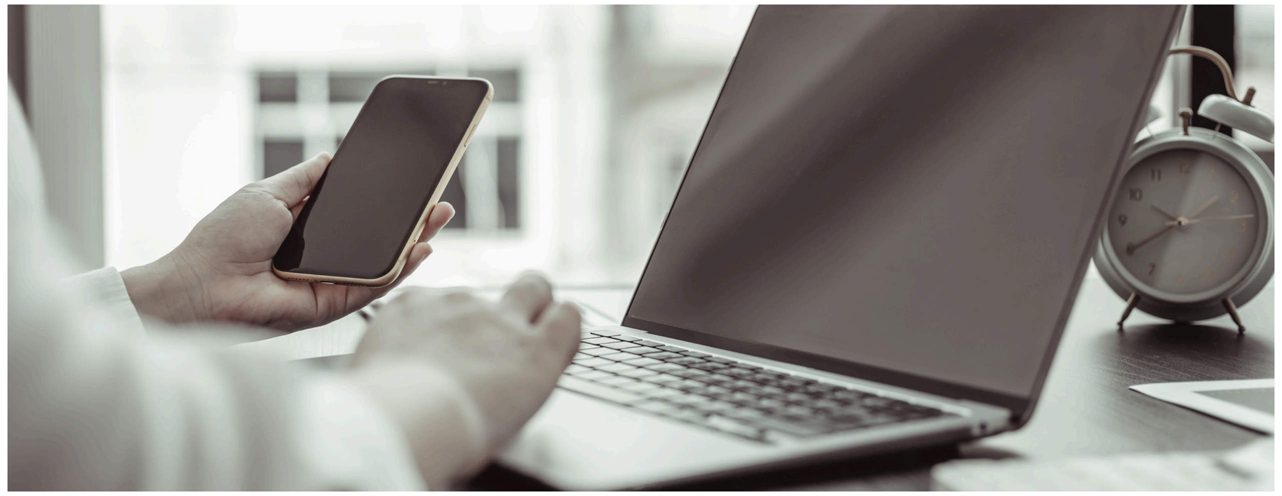
04 Nguyễn Lan Nhi
20225991

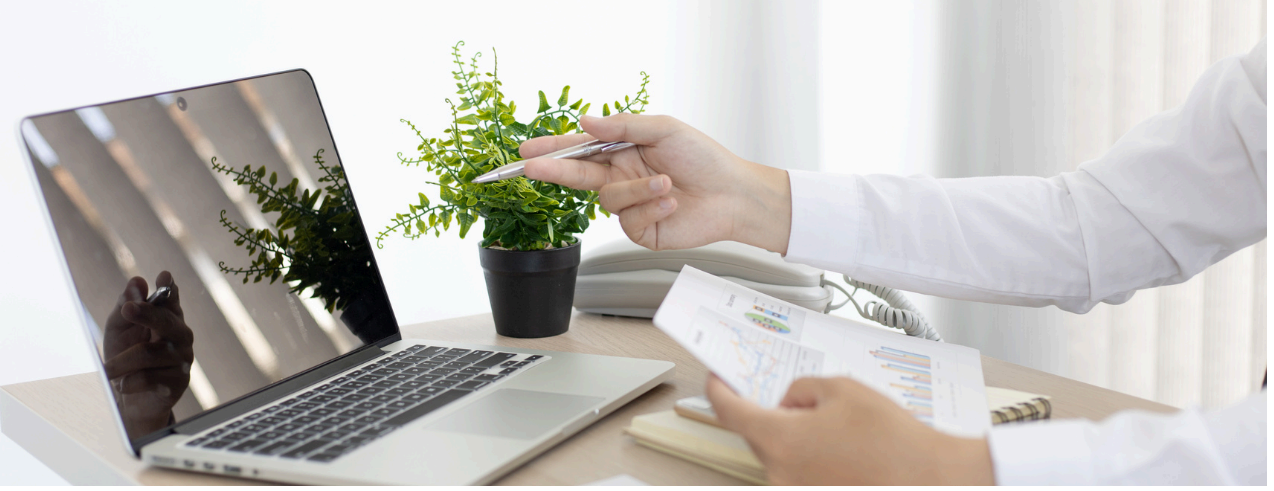
05 Nguyễn Phương Anh
20226011

06 Hồ Bảo Thư
20226003

APPROACH

TO SOLVE THE PROBLEM





01 Discretization

Convert the continuous PDE into a discrete system using finite difference methods

02 Iterative Solution

Use an iterative method, such as the SOR method, to solve the resulting system of linear equations

03 Boundary Conditions

Apply the given boundary conditions to the grid points to ensure that the solution conforms to the physical constraints of the problem

INTRODUCTION TO PROBLEM

Our study focuses on the numerical solution of the Steady-State Diffusion problem by transforming it into the 2D Laplace equation, which is then solved using the Successive OverRelaxation (SOR) method

01

STEADY STATE DIFFUSION

02

DIFFUSION COEFFICIENT

03

NON STEADY STATE DIFFUSION

STEADY-STATE DIFFUSION

Diffusion refers to the **net movement** of a species down a concentration gradient from an area of high concentration to an area of low concentration, takes place in any phase of matter including in solids

Two types: (a) steady state and (b) non steady state.

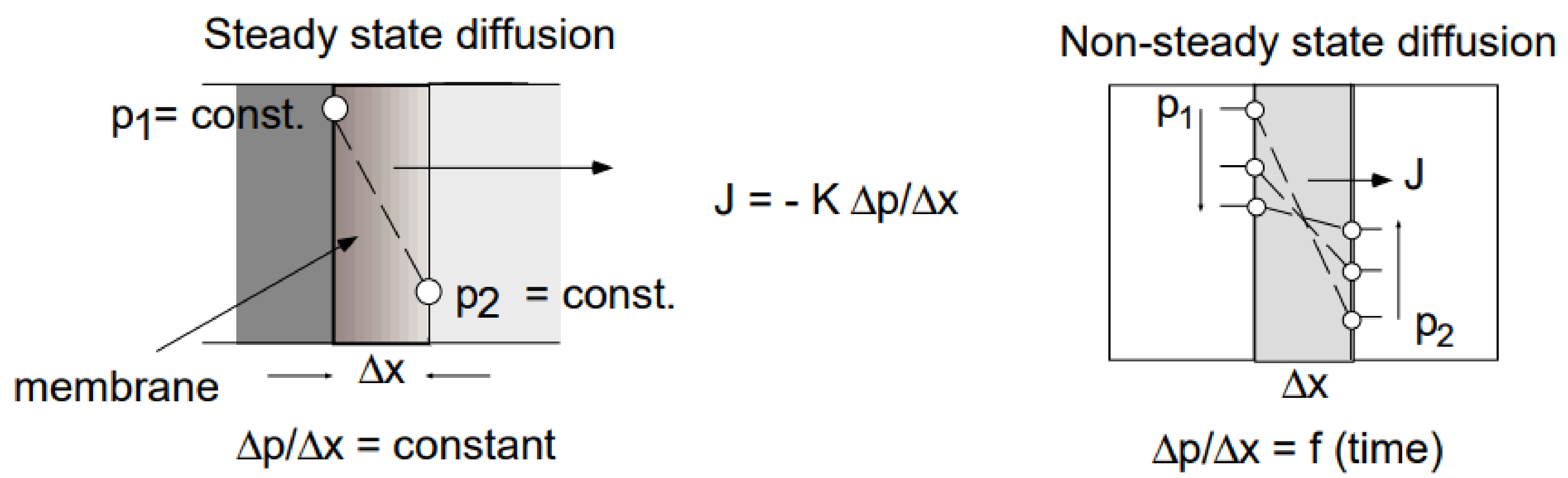


Figure 4 Steady state and Non-steady state diffusion

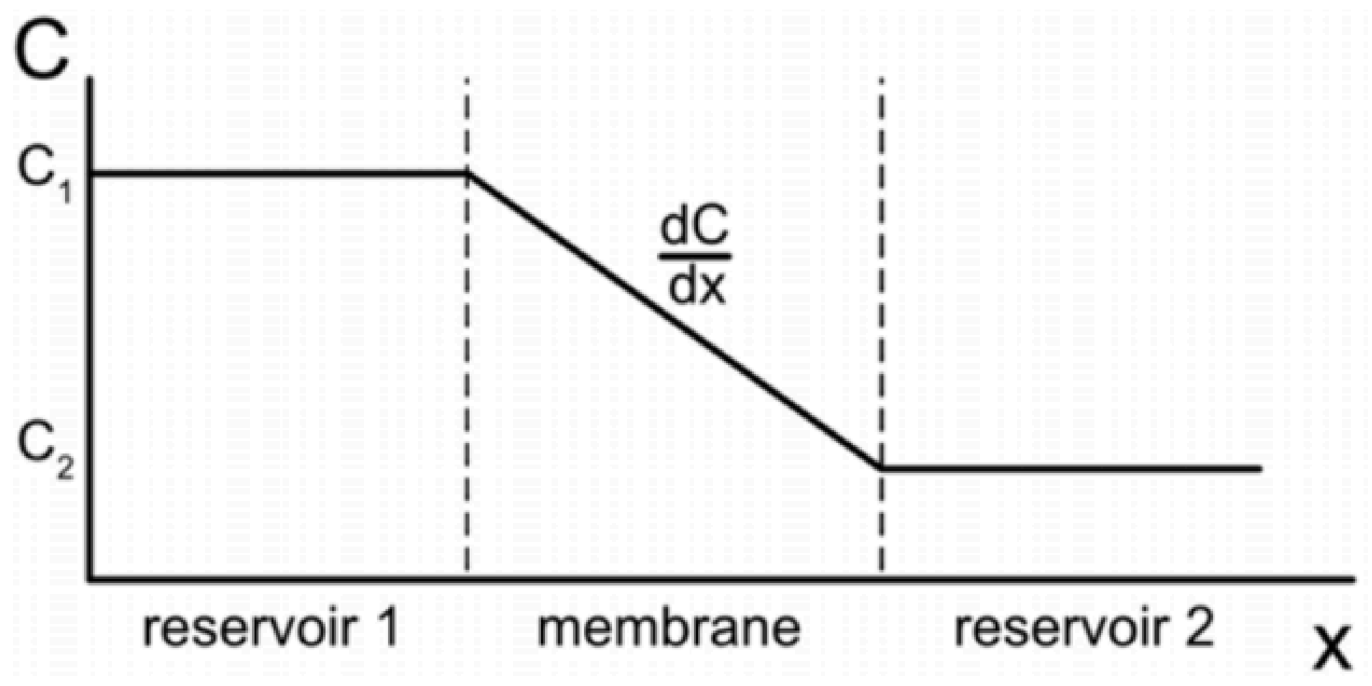


Figure 1: Concentration Profile Across a Membrane Between Two Reservoirs

Steady state diffusion

the number of atoms crossing a given interface is constant with time.

By Fick's first law:

$$J = -D\frac{dC}{dx}$$

DIFFUSION COEFFICIENT

The diffusion coefficient (D) is a measure of how easily molecules or atoms can move through a medium.

Units: square meters per second

Depends on: the type of diffusing species, the medium through which diffusion occurs, and the temperature.

Diffusing Substance	*Solvent*	*T (°C)*	*D (cm²·s⁻¹)*
Au	Cu	400	5×10^{-13}
Cu (Self-Diffusing)	(Cu)	650	3.2×10^{-12}
C	Fe (FCC)	950	10^{-7}
Methanol	H_2O	18	1.4×10^{-5}
O_2	Air	0	0.178
H_2	Air	0	0.611

Mathematically, it is used to describe **the rate** at which particles spread.

The energy required for diffusion to occur can be thought of as **an activation energy**.

The diffusion coefficient is:

$$D = D_0 e^{-\frac{E_a}{k_B T}}$$

NONSTEADY STATE DIFFUSION

If the concentration profile varies with respect to time, the steady-state assumption no longer holds and instead Fick's second law is used

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

Solutions to Fick's second law are of the form:

$$\frac{C(x,t) - C_0}{C_s - C_0} = 1 - \text{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$$

TRANSITION TO 2D LAPLACE EQUATION

FICK'S SECOND LAW

Use the Laplacian which generate the second derivative in the equation.

$$\frac{\partial C}{\partial t} = D\nabla^2 C$$

STEADY-STATE CONDITION

The concentration does not change over time

$$\frac{\partial C}{\partial t} = 0$$

Substituting:

$$D\nabla^2 C = 0$$

LAPLACE EQUATION

The resulting equation

$$\nabla^2 C = 0$$

is a second-order partial differential equation that describes systems in which the quantity of interest is in a state of equilibrium with no sources or sinks.

EXTENDING

Adding derivatives with respect to the additional spatial dimensions

$$\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} = 0$$

describes the steady-state distribution of concentration in a 2D space

SOLVING THE EQUATION

WITH SOR

The Successive Over-Relaxation (SOR)
an iterative technique used to solve linear systems of equations, particularly those arising from discretized partial differential equations like the Laplace equation.

It is an extension of the Gauss-Seidel method, incorporatinga relaxation parameter to accelerate convergence.

INITIALIZATION

Choose an initial guess for the concentration C at each grid point.

Set the relaxation parameter ω (typically between 1 and 2) and choose a convergence tolerance. (ω = 1.9 is the best for most cases)

ITERATION

$$C_{i,j}^{(k+1)} = (1-\omega)C_{i,j}^{(k)} + \frac{\omega}{4}\left(C_{i+1,j}^{(k)} + C_{i-1,j}^{(k+1)} + C_{i,j+1}^{(k)} + C_{i,j-1}^{(k+1)}\right)$$

where:

- $C_{i,j}^{(k)}$ is the concentration at grid point (i,j) in the k-th iteration.
- $C_{i+1,j}^{(k)}$, $C_{i-1,j}^{(k)}$, $C_{i,j+1}^{(k)}$, and $C_{i,j-1}^{(k)}$ are the concentrations at the neighboring grid points.
- ω is the relaxation parameter.

Iterate through each grid point

excluding the boundary points

Update the concentration value

at each grid point using the SOR formula

SOLVING THE EQUATION

WITH SOR

Convergence check

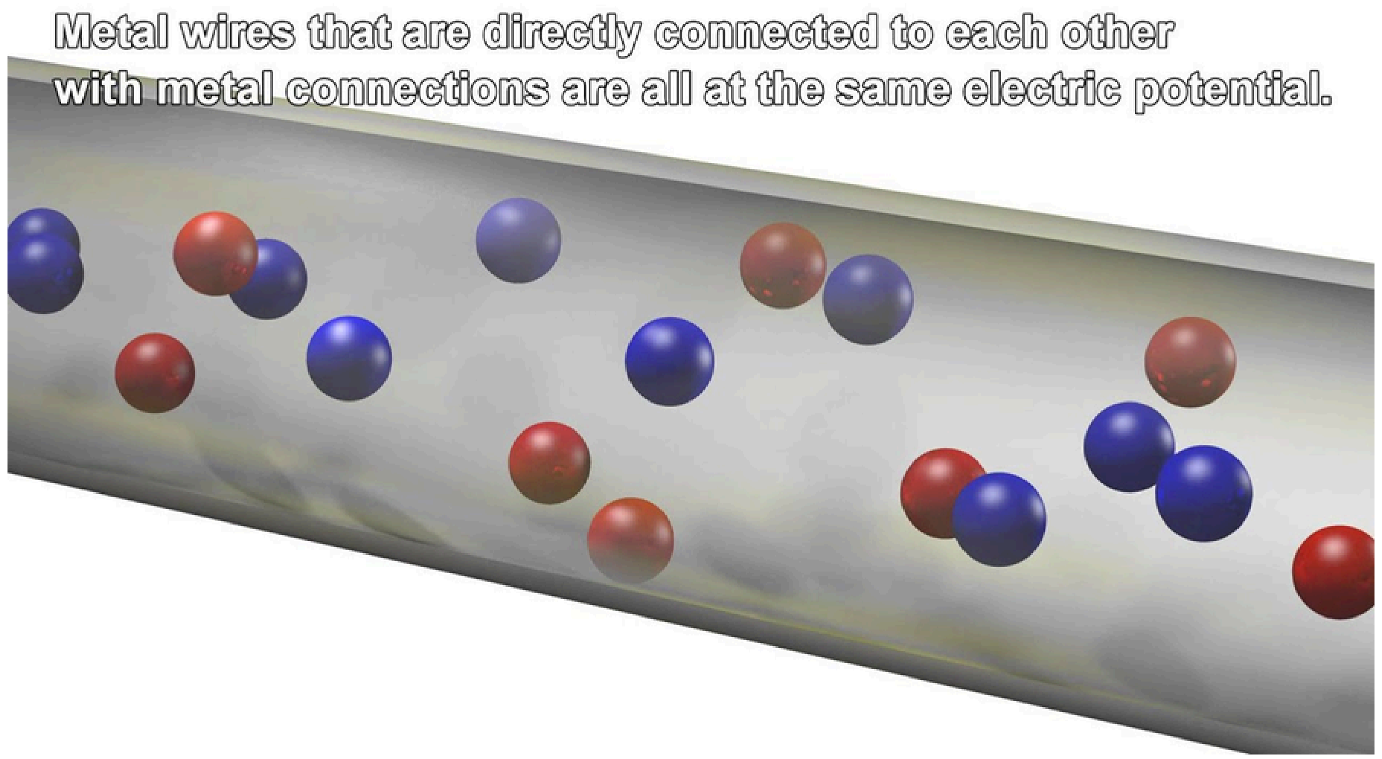
Check if the maximum change falls below the specified tolerance level. If so, stop iterating; otherwise, continue

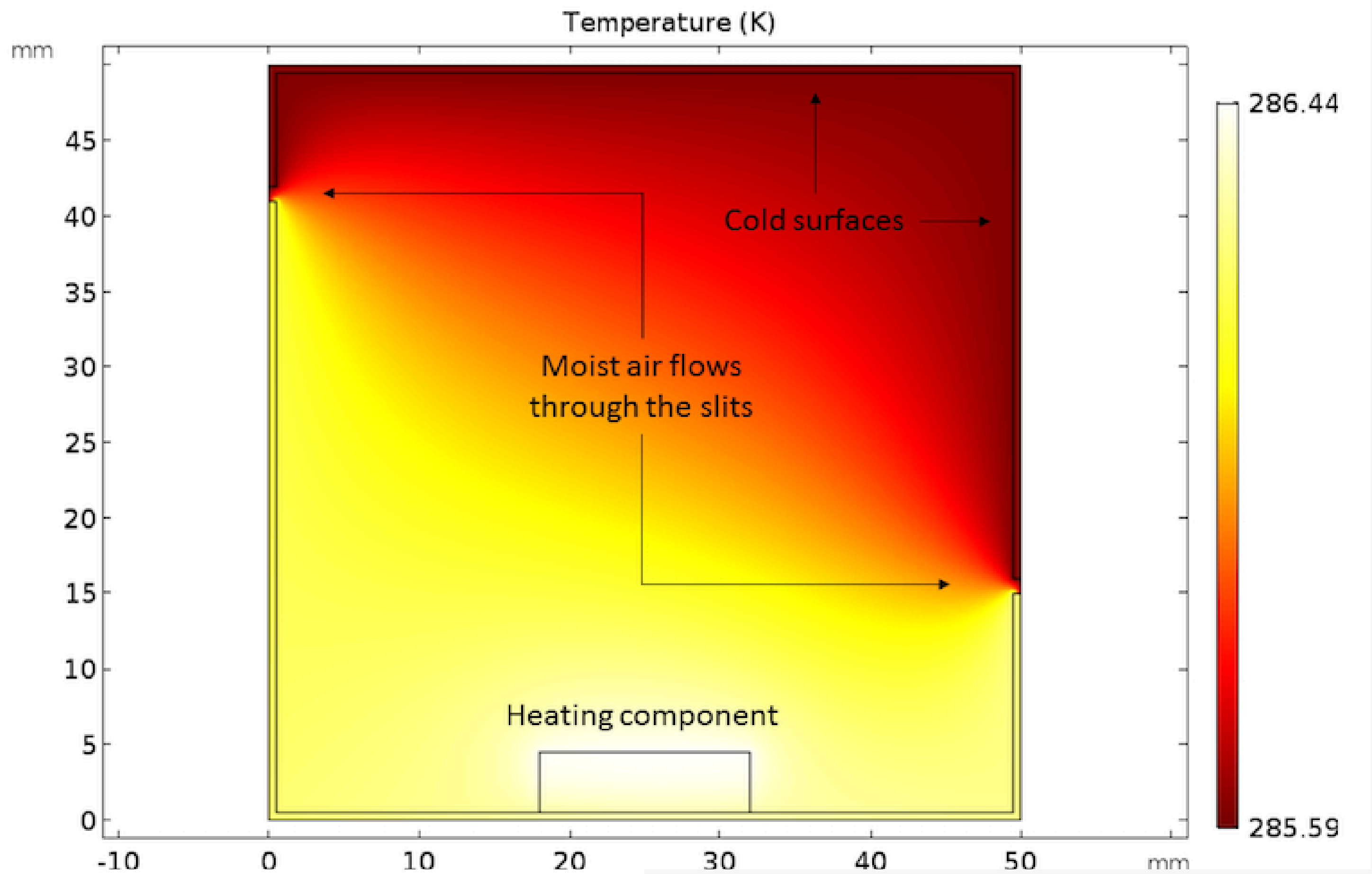
Solution extraction

After convergence is achieved, the concentration values at each grid point represent the numerical solution to the Laplace equation.

APPLICATION

IN OUR PROJECT





The 2D Laplace equation

a fundamental differential equation in physics and engineering, describes the distribution of a scalar field in a homogeneous and steady state

Objective

to determine the stable distribution of this field within a rectangular domain given specific boundary values

APPLICATION

Problem Characteristics

Computational Domain
A two-dimensional grid L × L with L is entered by the user.

Boundary Conditions
The potential φ is set to 100 at the vertical boundaries and 0 at the horizontal boundaries

Problem Challenges

01 Computational Efficiency

02 Algorithm Stability

03 Algorithm Stability

SPECIFIC PROBLEM STATEMENT

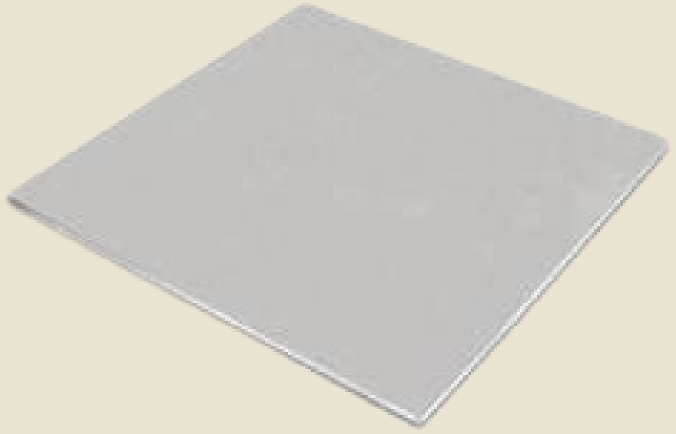
Given a rectangular metal plate with dimensions $L_x \times L_y$, we aim to determine the steady-state temperature distribution across the plate using the SOR method

The problem involves setting specific boundary conditions, handling random holes, incorporating singular temperature points, and adding dynamic heat sources

SPECIFIC PROBLEM

Computational Domain

Rectangular Metal Plate
The computational grid is initialized with zero temperatures



Boundary Conditions

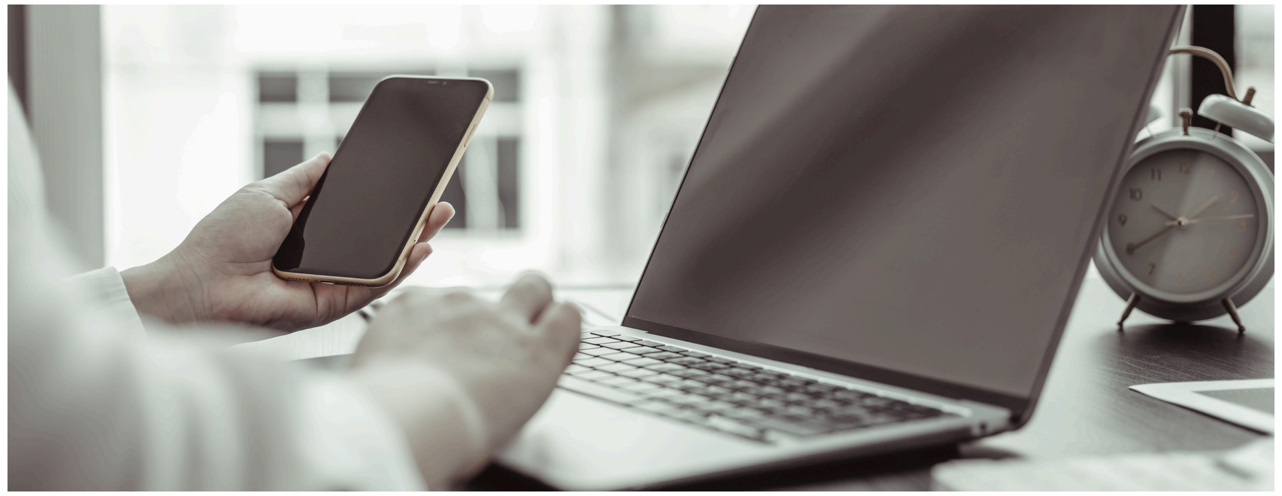
The temperature at the boundaries of the plate is specified by the user:

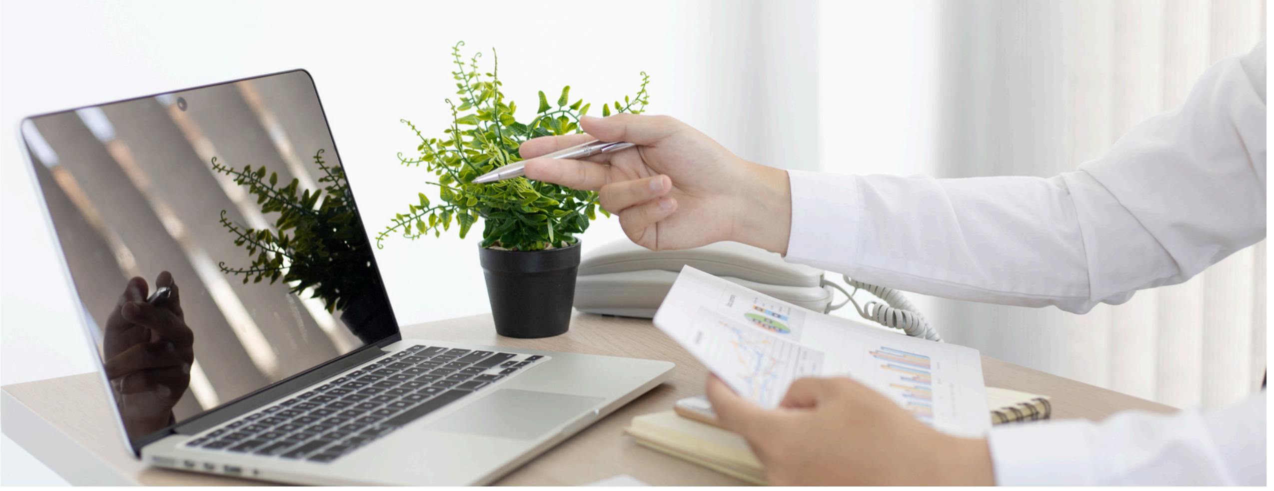
Left boundary
Right boundary
Bottom boundary
Top boundary

Singular Temperature Points

User can specify certain points within the plate where the temperature is fixed at specific values

SPECIFIC PROBLEM





04 Objective

Determine the Steady-State Temperature Distribution

Analyze Convergence

Visualize the Temperature Distribution

05 Challenges

Handling Boundary Conditions: Accurately implementing the user-defined boundary conditions

Stability of SOR Method: does not lead to divergence

06 Results

Contour Plot

Surface Plot

EXAMPLE

WITH USER INPUT DATA

Grid size: 20 × 20

Boundary temperatures:

– Left boundary (bottom to top): 400 to 600

– Right boundary (bottom to top): 400 to 600

– Bottom boundary (left to right): 400 to 600

– Top boundary (left to right): 4000 to 600

Singular temperature point:

– Coordinate (10, 10) with temperature value: 500

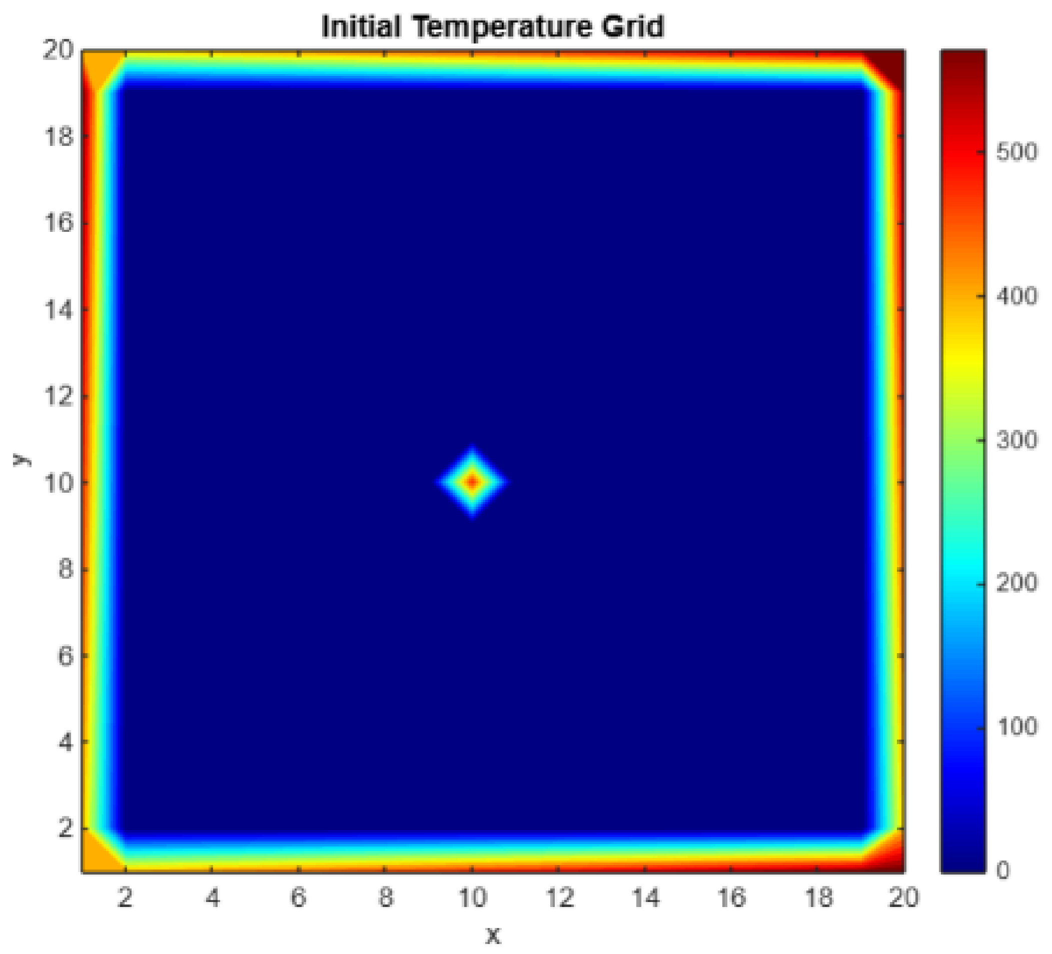


Figure 2: Initial Temperature Grid

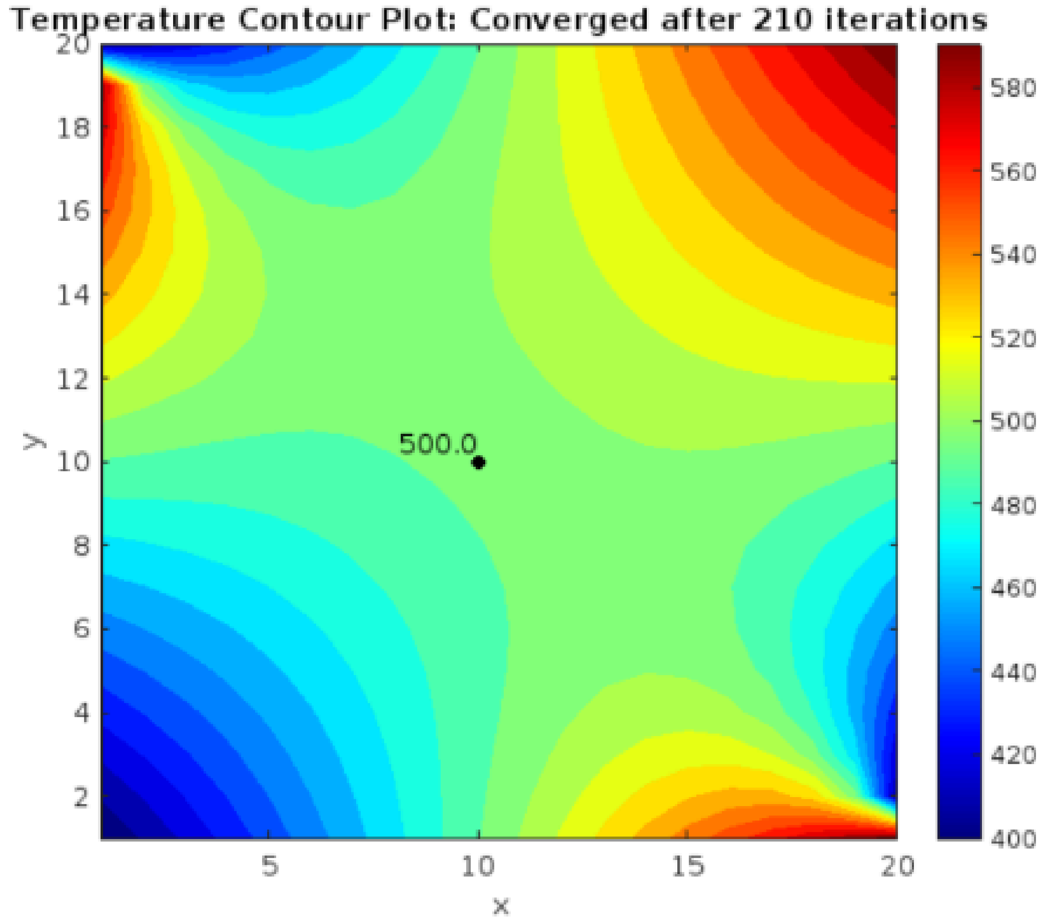


Figure 3: Temperature Contour Plot

CONCLUSION

Through the meticulous application of the SOR method, the Laplace equation governing steady-state diffusion can be effectively solved, offering valuable insights into concentration dynamics within the system. This iterative approach facilitates computational efficiency and enhances the understanding of diffusive phenomena in diverse scientific and engineering contexts

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