Fundamental Algorithm Techniques
Problem Set #7

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Problem 1. Graph Play

This problem is devoted to understanding basic graph transformations and structural properties. The aim is not only to give examples, but also to build an intuition for how different graph concepts are related and why they are important in graph theory and algorithms.

1. Directed graphs and their transposed graphs

A directed graph $G = (V, E)$ consists of a set of vertices V and a set of directed edges $E \subseteq V \times V$. Each edge (u, v) is interpreted as a one-way connection from u to v.

The transpose of a directed graph, denoted by G^T, is obtained by reversing the direction of every edge:

$$(u, v) \in E(G) \iff (v, u) \in E(G^T).$$

This transformation preserves the vertex set and connectivity pattern but completely reverses the direction of information flow.

Figure 1 illustrates several directed graphs together with their transposes. In the simplest case, a single edge $A \to B$ becomes $B \to A$. In more complex examples, all edges are reversed simultaneously. A directed cycle remains a cycle after transposition, but its orientation is inverted.

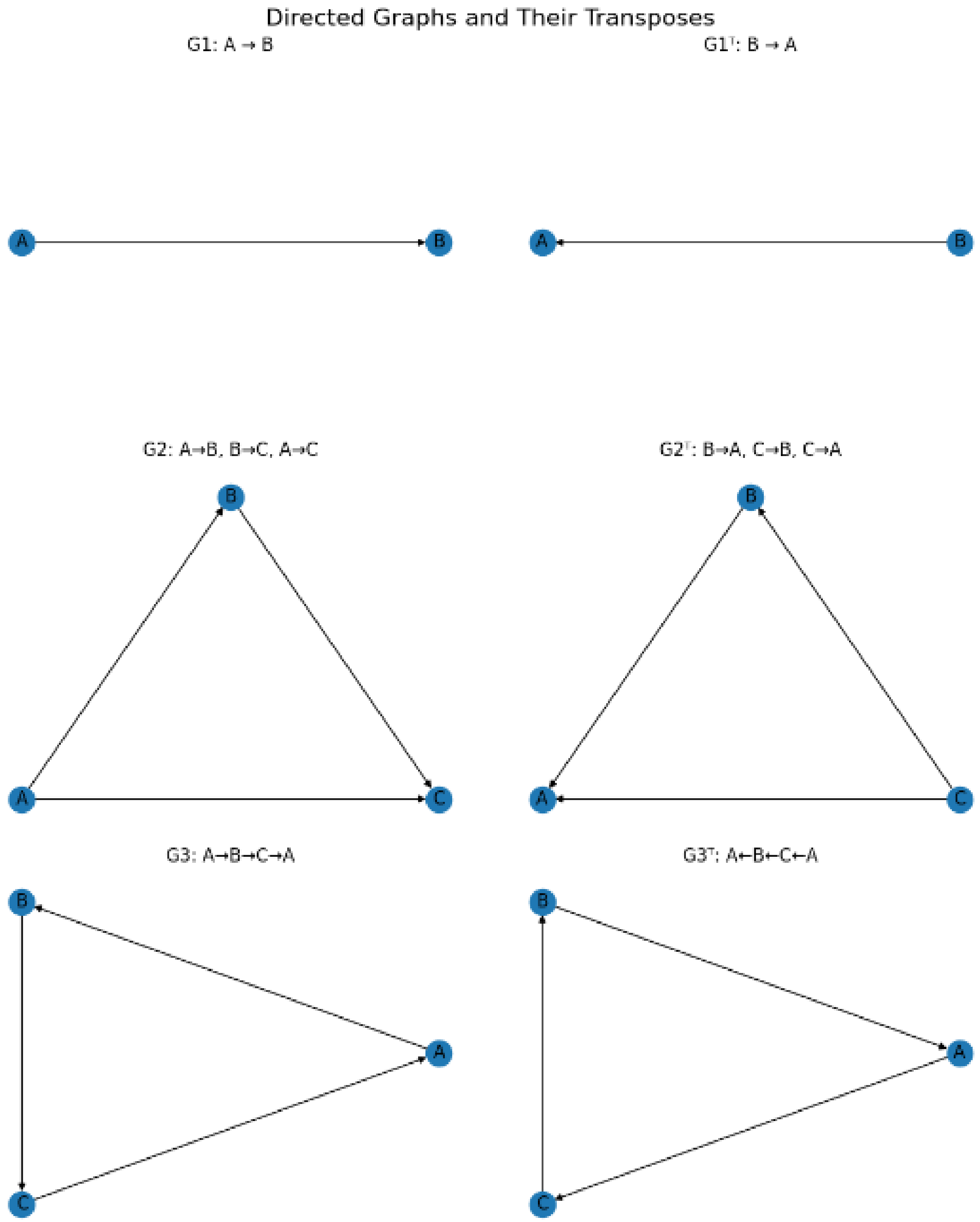

Figure 1: Directed graphs and their transposed graphs.

The notion of transposition is fundamental in algorithm design. For example, algorithms for finding strongly connected components explicitly rely on both a graph and its transpose, highlighting forward and backward reachability.

2. Undirected graphs and their inverse (complement) graphs

An undirected simple graph $H = (V, E)$ consists of vertices connected by unordered edges, with no self-loops or multiple edges. The inverse or complement graph $\overline{H}$ has the same vertex set, but edges represent missing connections from the original graph:

$$\{u, v\} \in E(\overline{H}) \iff \{u, v\} \notin E(H), \quad u \neq v.$$

In other words, every absent edge in H appears in $\overline{H}$, and every present edge is removed. This corresponds to flipping all off-diagonal entries of the adjacency matrix.

Figure 2 shows two examples. In the first graph, the only missing edge becomes the only edge in the complement. In the second example, a square graph is transformed into a graph connecting non-adjacent vertices.

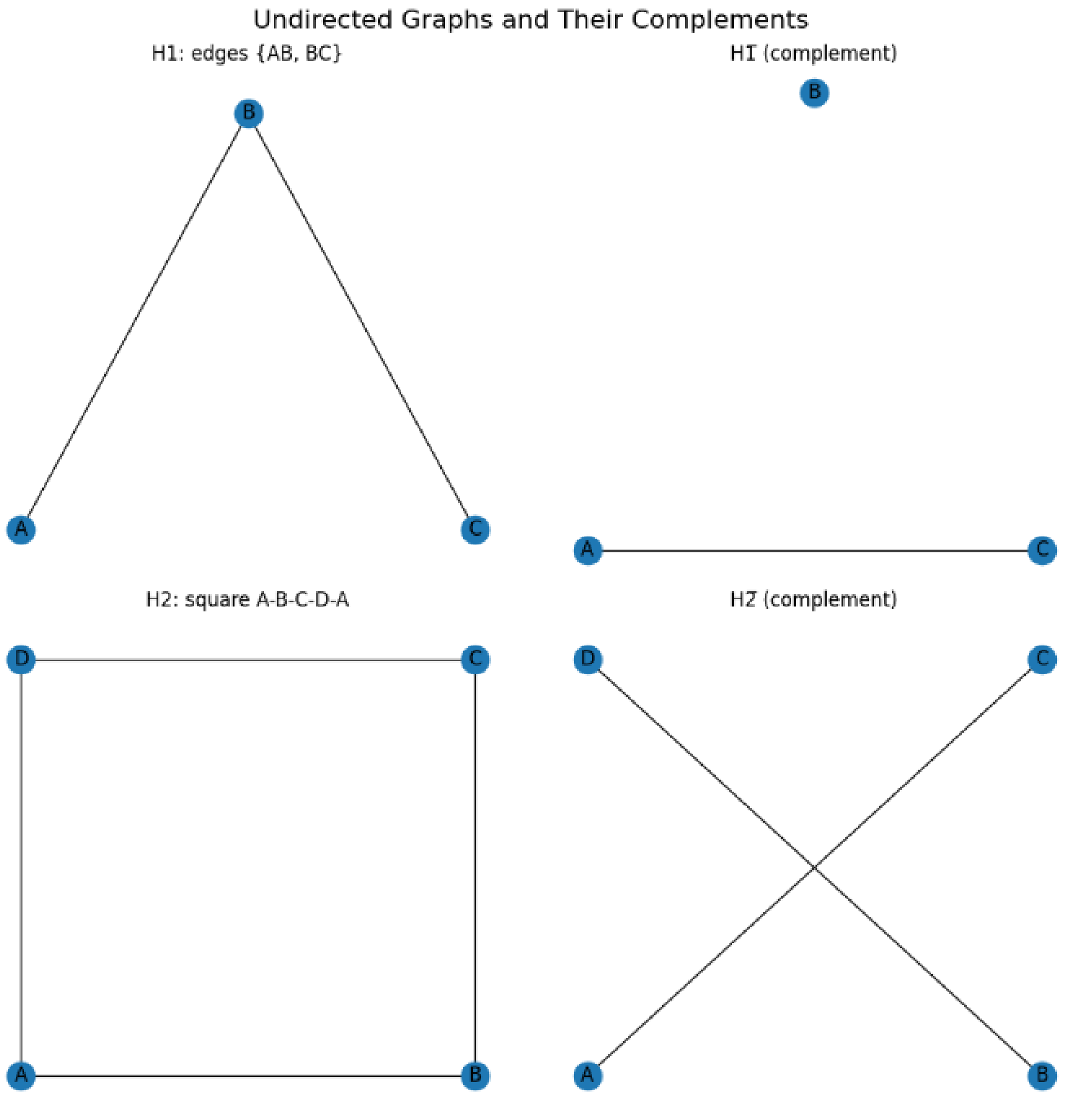

Figure 2: Undirected graphs and their complements.

The complement operation shifts attention from existing relationships to missing ones, which is useful in many theoretical and practical contexts.

3. Effect of density on the inverse graph

The density of a graph measures how close it is to being complete. For a graph with n vertices, the maximum number of undirected edges is $\binom{n}{2}$. Since the complement contains exactly the edges that are not present in the original graph, their densities satisfy

$$\text{density}(\overline{H}) = 1 - \text{density}(H).$$

Thus, a dense graph has a sparse complement, and a sparse graph has a dense complement. This inverse relationship is clearly visible in the examples presented in Figure 2.

4. Simple planar graphs and their dual graphs

A graph is called planar if it can be embedded in the plane without edge crossings. Such an embedding divides the plane into regions called faces, including one outer (unbounded) face.

For a connected planar graph, the dual graph is constructed by assigning one vertex to each face of the original graph and connecting two vertices whenever their corresponding faces share a common boundary edge.

Figure 3 presents a simple planar graph in the form of a square. This embedding has exactly two faces: an inner face and an outer face. As a result, the dual graph consists of two vertices connected by an edge.

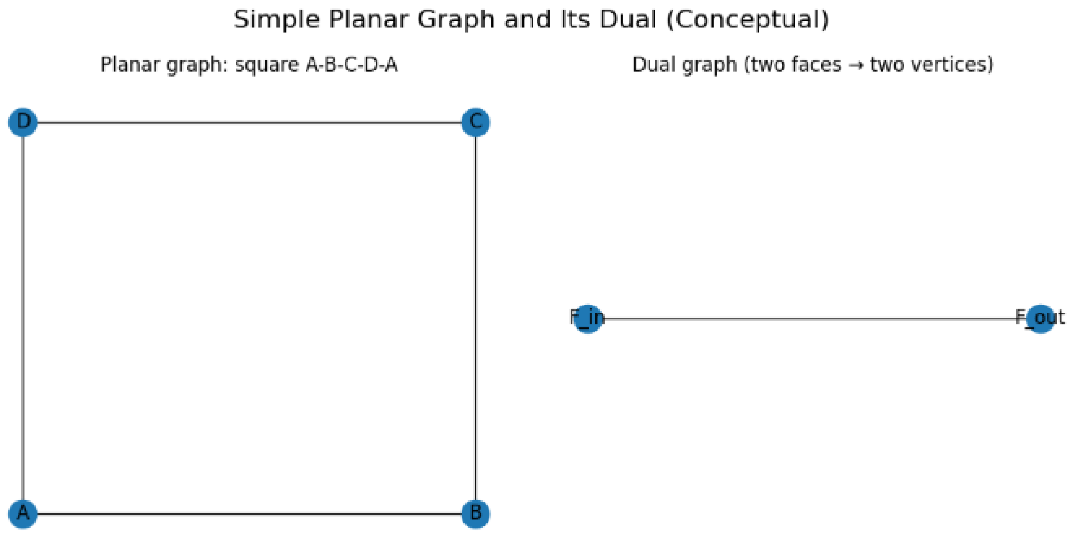

Figure 3: A simple planar graph and its dual graph.

Dual graphs provide a way to study planar graphs from a different perspective, focusing on relationships between faces rather than vertices.

5. Why the dual is only well-defined for planar graphs

The definition of a dual graph relies on the existence of well-defined faces, which only occur in planar embeddings. Non-planar graphs cannot be drawn in the plane without crossings, and different drawings may lead to inconsistent or ambiguous regions.

A standard example of a non-planar graph is the complete graph K_5 on five vertices. As shown in Figure 4, it is impossible to embed K_5 in the plane without edge crossings. Since no planar embedding exists, the faces required to construct a dual graph are undefined, and therefore K_5 has no dual graph.

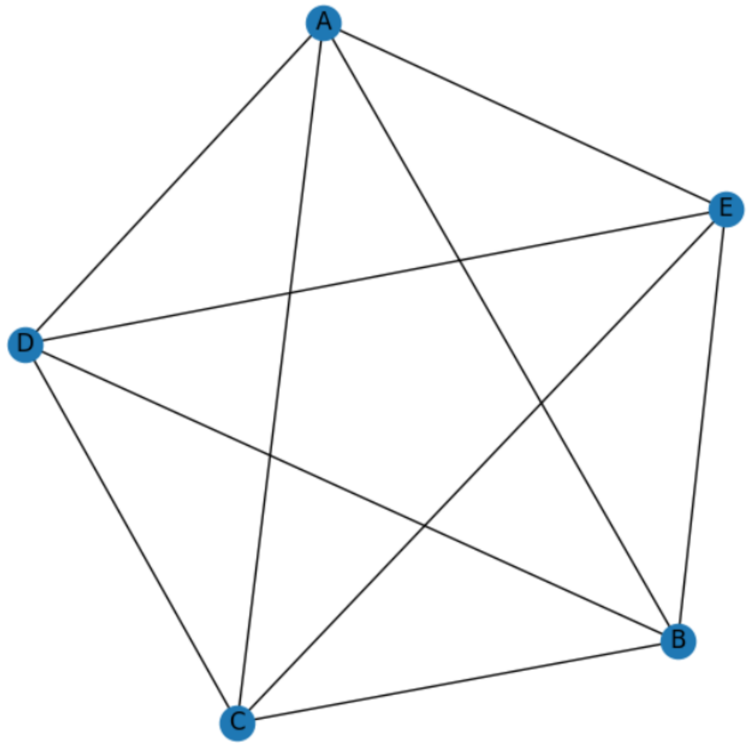

Figure 4: The complete graph K_5 as an example of a non-planar graph.