Problem 2. Bron-Kerbosch Algorithm

In this problem we apply the Bron-Kerbosch algorithm without pivoting in order to find all maximal cliques of a given undirected graph. The goal is to trace the algorithm step by step and understand the role of the sets R, P, and X.

Given graph

The undirected graph G has the vertex set

$$V = \{A, B, C, D\}$$

and the edge set

$$E = \{AB, AC, BC, CD\}.$$

Thus, vertices A, B, and C form a triangle, and vertex D is connected only to C.

The neighbourhoods of the vertices are:

$$\begin{aligned} N(A) &= \{B, C\}, \\ N(B) &= \{A, C\}, \\ N(C) &= \{A, B, D\}, \\ N(D) &= \{C\}. \end{aligned}$$

Cliques

A clique is a set of vertices such that every two vertices in the set are connected by an edge.

A clique is called maximal if no additional vertex can be added to it while preserving the clique property.

The task of the Bron-Kerbosch algorithm is to enumerate all maximal cliques of a graph.

Bron-Kerbosch algorithm

The algorithm maintains three sets:

- R – the current clique,
- P – candidate vertices that can be added to R,
- X – vertices already processed in this branch.

If both P and X are empty, then R is reported as a maximal clique.

Otherwise, for each vertex v in P, the algorithm calls itself recursively with updated sets:

$$(R \cup \{v\},\ P \cap N(v),\ X \cap N(v)).$$

Initial call

At the beginning, the algorithm starts with:

$$R = \emptyset, \quad P = \{A, B, C, D\}, \quad X = \emptyset.$$

First maximal clique

The algorithm first selects vertex A.

After adding A:

$$R = \{A\}, \quad P = \{B, C\}, \quad X = \emptyset.$$

Next, vertex B is added:

$$R = \{A, B\}, \quad P = \{C\}, \quad X = \emptyset.$$

Then vertex C is added:

$$R = \{A, B, C\}, \quad P = \emptyset, \quad X = \emptyset.$$

Since both P and X are empty, the algorithm reports the clique

$$\{A, B, C\}$$

as a maximal clique.

Second maximal clique

After all branches starting with A are processed, the algorithm selects vertex C at the top level.

At this point:

$$R = \emptyset, \quad P = \{C, D\}, \quad X = \{A, B\}.$$

After adding C:

$$R = \{C\}, \quad P = \{D\}, \quad X = \{A, B\}.$$

Then vertex D is added:

$$R = \{C, D\}, \quad P = \emptyset, \quad X = \emptyset.$$

Thus, the algorithm reports the clique

$$\{C, D\}$$

as a maximal clique.

Final result

The maximal cliques found by the algorithm are:

$$\{A, B, C\}, \quad \{C, D\}.$$

The sizes of these cliques are:

$$|\{A, B, C\}| = 3, \quad |\{C, D\}| = 2.$$

Therefore, the clique {A, B, C} is the maximum clique of the graph.