Fundamental Algorithm Techniques
Problem Set #7

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Abstract

This report is devoted to the rigorous study of fundamental graph transformations and their structural properties. The objective of this work is not only to formally define key graph operations such as transposition, complementation, and duality, but also to analyze their mathematical meaning and their role in modern algorithm design. Special attention is paid to the relationship between graph structure and computational properties.

1 Problem 1. Graph Play

Graph transformations play a central role in graph algorithms, including reachability analysis, network flow problems, connectivity testing, planar graph embeddings, and structural optimization tasks. Understanding these transformations provides a solid theoretical foundation for advanced studies in algorithmic graph theory.

In the following sections we consider directed graphs and their transposes, undirected graphs and their complements, the effect of density on the complement graph, planar graphs and their duals, and the limitations of duality for non–planar graphs.

2 Directed Graphs and Their Transposed Graphs

A directed graph (digraph) is formally defined as an ordered pair

$$G = (V, E),$$

where:

- V is a finite, non-empty set of vertices (nodes),
- $E \subseteq V \times V$ is a set of ordered pairs called directed edges (arcs).

Each directed edge $(u, v) \in E$ represents a one-directional connection from vertex u to vertex v. The direction of edges introduces asymmetry into the structure of the graph and determines the direction of information flow, control flow, or dependencies in modeled systems.

Definition of the Transpose Graph

The transpose of a directed graph G, denoted by G^T, is defined as the graph obtained by reversing all directed edges:

$$(u, v) \in E(G) \iff (v, u) \in E(G^T).$$

Thus, the vertex set remains unchanged:

$$V(G^T) = V(G),$$

while the edge set is completely inverted with respect to direction.

Structural Properties

- The transposition operator preserves the number of vertices and the number of edges.
- Strong connectivity is preserved: a digraph G is strongly connected if and only if G^T is strongly connected.
- The in-degree and out-degree of every vertex are swapped in the transposed graph.

Algorithmic Significance

Graph transposition is a fundamental operation in multiple classical algorithms. The most notable example is Kosaraju's algorithm for finding strongly connected components (SCCs). This algorithm performs:

1. Depth-first search (DFS) on the original graph.
2. DFS on the transposed graph in a specific order determined by the first pass.

The correctness of such algorithms critically depends on the theoretical properties of the transposition operation.

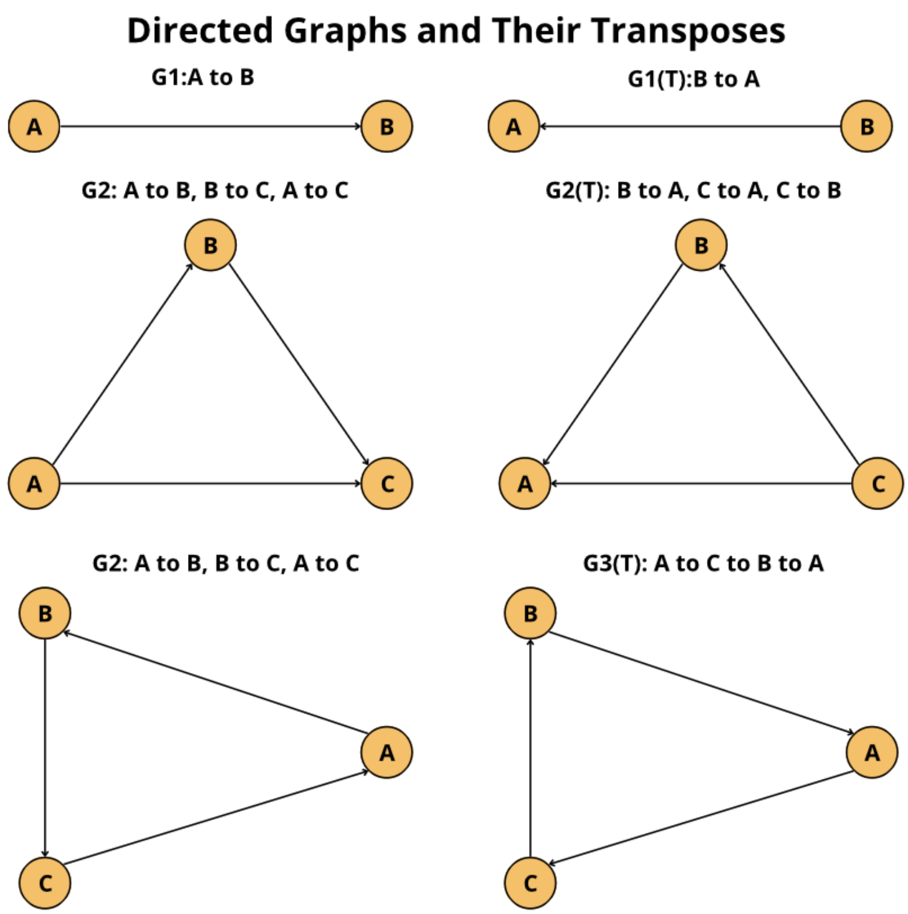

Figure 1: Directed graphs and their transposed graphs.

3 Undirected Graphs and Their Complement Graphs

An undirected simple graph is defined as

$$H = (V, E),$$

where:

- V is a finite set of vertices,
- $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$.

The graph contains no self-loops and no multiple edges. Each edge represents a symmetric relationship between its endpoints.

Definition of the Complement Graph

The complement of an undirected graph H, denoted by $\overline{H}$, is defined as:

$$\{u, v\} \in E(\overline{H}) \iff \{u, v\} \notin E(H), \quad u \neq v.$$

Thus, $\overline{H}$ contains exactly those edges that are missing in the original graph. The vertex set remains unchanged.

Matrix Interpretation

If $A(H)$ is the adjacency matrix of graph H, then the adjacency matrix of its complement graph is given by:

$$A(\overline{H}) = J - I - A(H),$$

where:

- J is the all-ones matrix,
- I is the identity matrix.

This shows that complementation corresponds to flipping all off-diagonal entries of the adjacency matrix.

Theoretical Importance

Complement graphs are widely used in:

- Ramsey theory,
- extremal graph theory,
- network modeling and communication systems,
- optimization problems where absent connections are as important as existing ones.

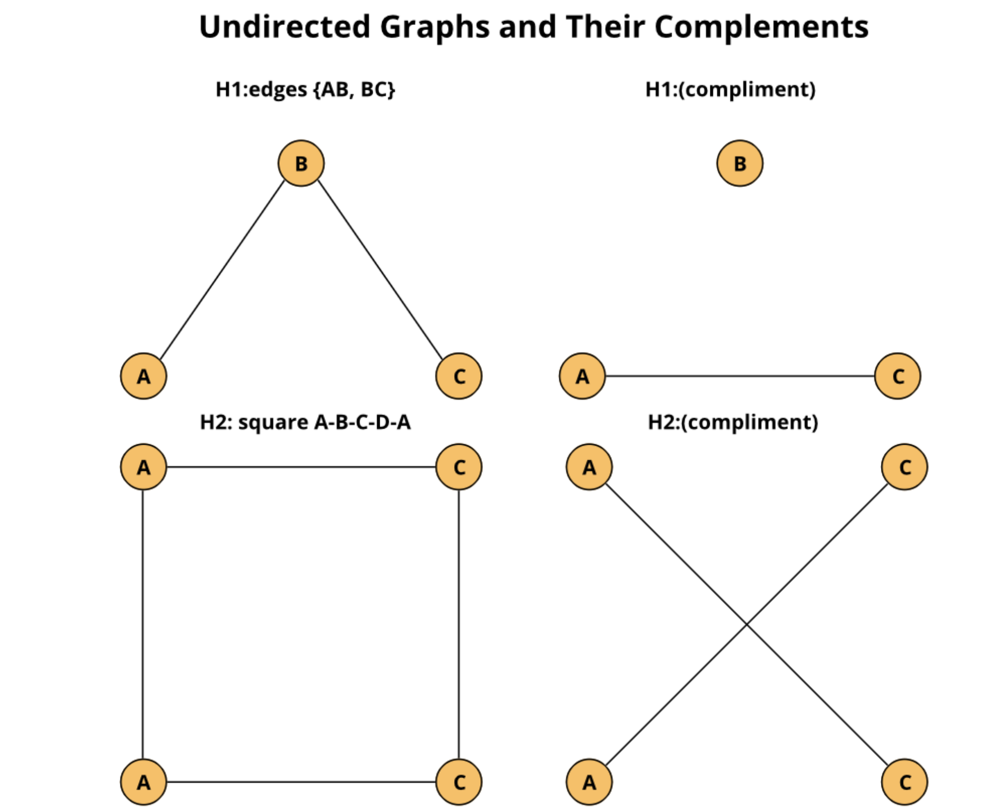

Figure 2: Undirected graphs and their complement graphs.

4 Effect of Density on the Complement Graph

Let H be an undirected graph with n vertices. The maximum possible number of edges in H is

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

The density of the graph is defined as

$$\text{density}(H) = \frac{|E(H)|}{\binom{n}{2}}.$$

Since the complement graph $\overline{H}$ contains all edges that are not present in H, its density satisfies

$$\text{density}(\overline{H}) = 1 - \text{density}(H).$$

Consequences

- High-density graphs have low-density complements.
- Sparse graphs generate dense complements.
- This duality is crucial in extremal graph problems, where bounds are often derived using complements.

5 Simple Planar Graphs and Their Dual Graphs

A graph is said to be planar if there exists an embedding of the graph in the Euclidean plane such that no two edges intersect except at their endpoints.

Such an embedding divides the plane into regions called faces. One of these faces is unbounded and is called the outer face.

Definition of the Dual Graph

Let G be a connected planar graph with a fixed planar embedding. The dual graph G^* is constructed as follows:

- Each face of G corresponds to a vertex in G^*.
- Two vertices in G^* are connected by an edge if and only if their corresponding faces in G share a common boundary edge.

The dual graph represents how regions of the plane interact rather than how vertices interact. This transformation is fundamental in planar graph algorithms, electrical network analysis, computational geometry, and finite element methods.

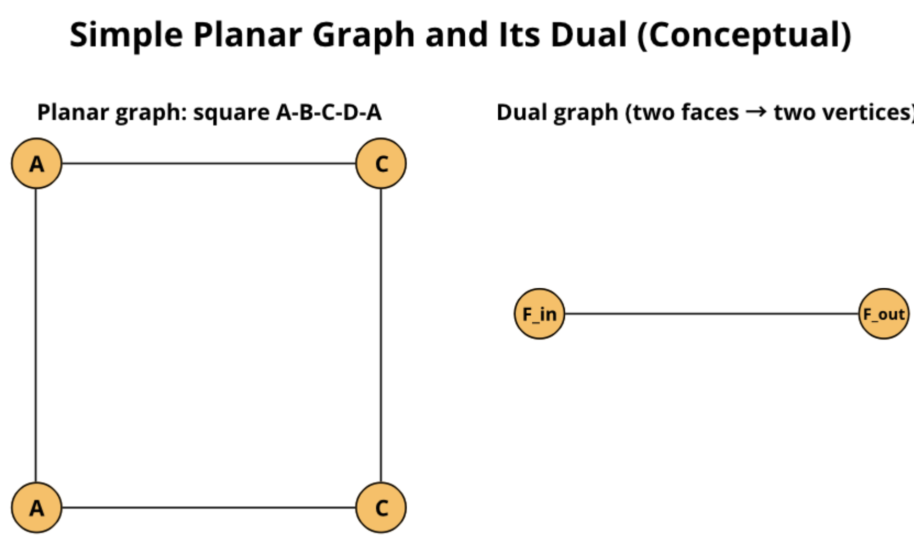

Figure 3: Simple planar graph and its dual graph.

6 Why the Dual Graph Is Only Well-Defined for Planar Graphs

The existence of a dual graph depends entirely on the existence of well-defined faces. However, faces only exist when a graph admits a planar embedding.

A classic counterexample is the complete graph K_5 on five vertices. This graph is known to be non-planar, meaning that it cannot be embedded in the plane without edge crossings. Since no consistent planar embedding exists, the concept of distinct faces becomes undefined. As a result, a dual graph cannot be constructed for K_5.

This limitation is formalized in Kuratowski's Theorem, which characterizes non-planar graphs through subdivisions of K_5 and $K_{3,3}$.

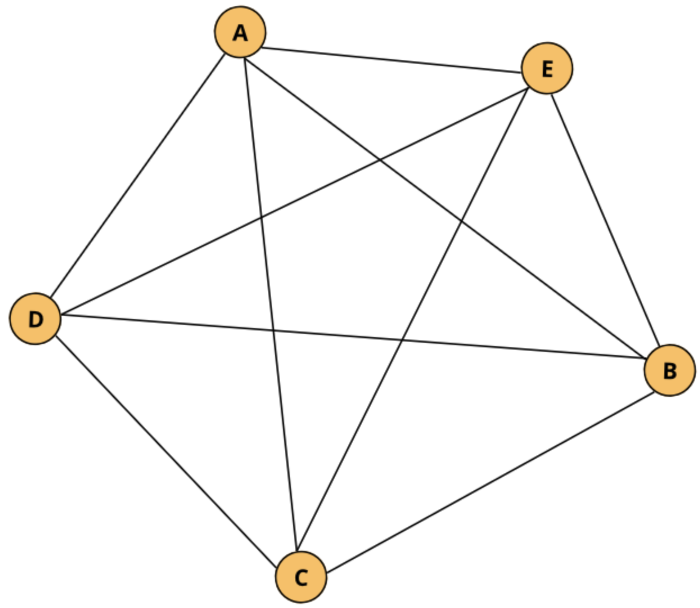

Figure 4: The complete graph K_5 as an example of a non-planar graph.

7 Conclusion

This work has provided a detailed and rigorous study of the most fundamental graph transformations used in theoretical and applied algorithmics:

- the transpose of directed graphs, which inverts all arc directions and is crucial for reachability analysis and SCC algorithms;
- the complement of undirected graphs, which reveals missing connections and is central to extremal graph theory;
- the inverse relationship between graph density and complement density;
- the concept of dual graphs, which transforms planar embeddings into face-adjacency structures;
- the strict limitation of dual graphs to planar graphs only, due to the necessity of well-defined faces.

Together, these transformations form a powerful toolkit for understanding the deep structural properties of graphs and serve as a foundation for advanced algorithmic techniques in computer science, engineering, and applied mathematics.