Problem Set #5

Problem 1. $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, e.g. $n=5$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^5 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$ $n=6$ $n=5$

1) to compute A^n, let's use:

if n is <u>even</u>: $A^n = (A^2)^{n/2}$

if n is <u>odd</u>: $A^n = A^1 \cdot A^{n-1}$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\vec{F}_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$

$\vec{F}_{k+1} = A \vec{F}_k \longrightarrow \vec{F}_n = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

e.g. $k=0$, then $F_1 = 1$, $F_0 = 0$, and

$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} F_2 \\ F_1 \end{pmatrix}$ $\quad \begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix} = A \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} \Rightarrow$

$\Rightarrow \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\boxed{A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}}$$

2) Algebraically equivalent:

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{n/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ — Squaring matrix and dividing n by 2 (sq root)

e.g. $A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

that's why:

if <u>even</u> n: $\boxed{A^n = (A^2)^{n/2}}$ and for <u>odd</u> n:

$$A^n = A(A^2)^{(n-1)/2}$$

Why time complexity is $O(\log n)$?

Each step divides indicator by 2. Then number of multiplications — $\lfloor \log_2 n \rfloor$;

Calculating it can be done by square-and-multiply, so the base multiplications equals to number of „1" in binary record of n:

Let's calculate f_{10}:

1) binary exponent: $10_{(10)} = 1010_{(2)}$

2) Take A^1, A^2, A^4, A^8 and calculate multiplication of $\underbrace{A^8 \text{ and } A^2}_{A^{10}}$. Number of multiplications: $2\lfloor \log_2 10 \rfloor = 6$

$\lfloor \log_2 n \rfloor$ squaring + $\lfloor \log_2 n \rfloor$ multiplications $\to \Theta(\log n)$

Problem 2.

1) Idea of Greedy approach:
taking items by value/weight, assuming that is optimal for best selection. But it's not true for 0/1 version of the problem.

Counter-example

Item	Weight	Value	V/W
1	3	4	1,33
2	4	5	1,25
3	5	6	1,2

$W = 8$

Greedy selects 1 and 2 (value = 9),
but optimal – 2 and 3 (value = 11)

Dynamic Programming is better for property of problem:
Optimal structures and overlapping subproblems

2) $W=8$, items (i, w_i, v_i): $1:(2,3)$, $2:(3,4)$, $3:(4,5)$, $4:(5,6)$.

i/W	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	3	3	3	3	3	3	3
2	0	0	3	4	4	7	7	7	7
3	0	0	3	4	5	7	8	9	9
4	0	0	3	4	5	7	8	9	10 - answer

$$ks[i][W] = \begin{cases} ks[i-1][W], & \text{if } w_i > W \\ Max(ks[i-1][W],\ ks[i-1][W-w_i]+v_i). & \text{if } w_i \le W \end{cases}$$