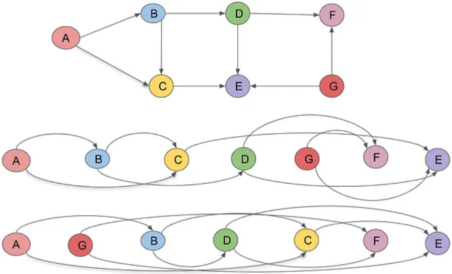
Fundamental Algorithmic Techniques VIII

November 22, 2025

Outline

Topological Sort: DFS Approach

- Works only on **Directed Acyclic Graphs** (DAGs)
- Detects cycles (if any node is visited twice)
- Result is **not unique** in general
- All trees have a topological order
- Course prerequisites, Build/makefile dependencies, Class loading in Java

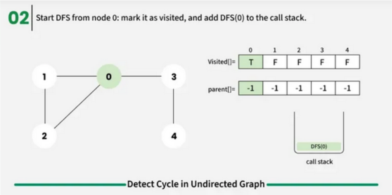


Topological Sort

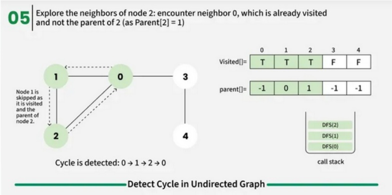
```
procedure TopologicalSort(G)
    L ← ∅
    visited ← all false
    for each vertex v do
        if not visited[v] then
            (v)
        end if
    end for
    return reverse(L)
end procedure

procedure DFS(v)
    visited[v] ← true
    for each neighbor w of v do
        if not visited[w] then
            (w)
        end if
    end for
    L.append(v)                ▷ ← finish time
end procedure
```

Cycle Detection: DFS vs BFS — Complexity



Depth-First Search step 0



Depth-First Search step 3

Both detect the cycle when exploring the back edge (e.g., $D \to A$): since the target node is already visited and not the immediate parent (in undirected) or is on the recursion stack (in directed).

Complexity:

- **Time:** $O(V + E)$ for both
  Every vertex and edge is processed at most once.
- **Space:** $O(V)$ for both
  - **DFS**: Call stack depth V (worst-case path).
  - **BFS**: Queue may hold up to $O(V)$ nodes (e.g., wide level).

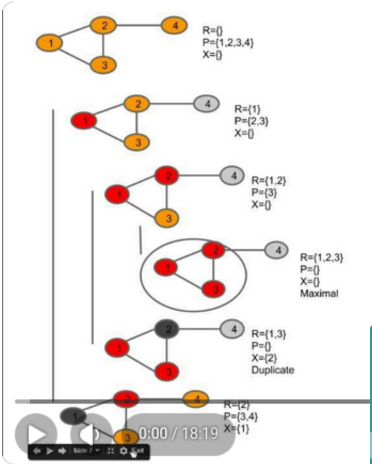
Bron–Kerbosch Algorithm: Maximal Clique Enumeration

Undirected graph $G = (V, E)$,
$N(v)$ = neighbors of v in G,

Initial call: BronKerbosch1(∅, V, ∅)

Pseudocode:

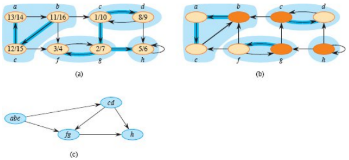
```
algorithm BronKerbosch1(R, P, X) is
    if P and X are both empty then
        report R as a maximal
clique
    for each vertex v in P do
        BronKerbosch1(R ∪ {v}, P ∩
N(v), X ∩ N(v))
        P := P \ {v}
        X := X ∪ {v}
```



Kosaraju's Algorithm - Strongly Connected Components

Kosaraju's Algorithm

1. **DFS on Original Graph**: Record finish times
2. **Transpose the Graph**: Reverse all edges
3. **DFS on Transposed Graph**: Process nodes in order of decreasing finish times to find SCCs

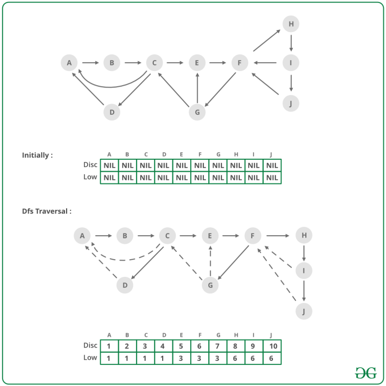


Two-pass DFS to find SCCs

Time Complexity: Depth First Search: $O(V + E)$

Space Complexity: Stack: $O(V)$

Tarjan's Algorithm for SCCs



Index: Discovery order in DFS

Low-link: Smallest index reachable via DFS (including back edges)

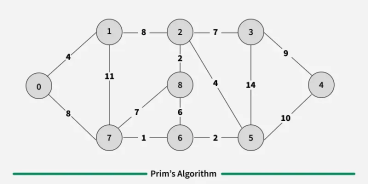
Problem: Random DFS order $\rightarrow$ ambiguous SCC boundaries

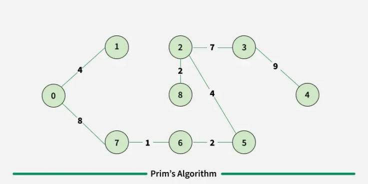
Solution: Use a stack to track active nodes

- When $\texttt{low}[u] = \texttt{index}[u]$: pop stack until u — those form one SCC

Complexity: $O(V + E)$ node/edge visited once

Jarník's (Prim's) Algorithm





Prim's Algorithm: initial graph (top) and MST

Steps

1. Start from arbitrary vertex for MST.
2. Till there are fringe vertices:
3. Find edges connecting tree & fringe vertices
4. Find the minimum among these edges
5. Add the chosen edge to the MST
6. Return the MST

Complexity

- Time: $\mathcal{O}(E \cdot \log V)$ with binary heap
- Space: $\mathcal{O}(V)$

Cuts and the Cut Property

Definition (Cut)

A *cut* $(S, V \setminus S)$ of an undirected graph $G = (V, E)$ is a partition of V into two non-empty disjoint subsets S and $V \setminus S$.

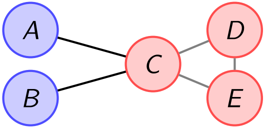
Cut-Crossing Edge

An edge $(u, v) \in E$ *crosses* the cut if exactly one of u or v is in S.

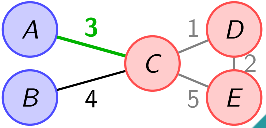
Cut Property / Theorem

For any cut $(S, V \setminus S)$, **if edge e's weight is the minimum among all edges crossing the cut, then e belongs to *some* MST of G.**

Cut Illustration



Cut Property in Action



Note: The green edge (weight 3) is the lightest crossing edge and must appear in *some* MST.

MST Building: Proof of Correctness

Key Invariant

At each step, the tree T maintained by the algorithm is a subset of some MST.

Proof by Induction

- **Base Case:** $T = \{v_0\}$ is trivially part of an MST.
- **Inductive Step:** Assume T is part of MST T^*. Let $e = (u, v)$ be the minimum-weight edge crossing the cut $(T, V \setminus T)$.
  - If $e \in T^*$: $T \cup \{e\}$ still part of T^*.
  - If $e \notin T^*$: $\exists e' \in T^*$ crossing same cut. Then $T^* - \{e'\} \cup \{e\}$ is also an MST (by cut property), since $w(e) \leq w(e')$.

Conclusion

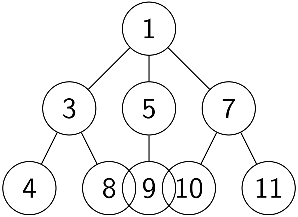
By induction, T is always part of an MST.

Final T is a spanning tree $\Rightarrow$ it is an MST.

Jarník's Algorithm: Binary Heap Implementation

```
procedure JarnikMST(G = (V, E))
    Q ← empty min-heap                              ▷ Vertices with key values
    for v ∈ V do
        key[v] ← ∞
        Q.insert(v, key[v])
    end for
    key[0] ← 0                                      ▷ Start from vertex 0
    while Q is not empty do
        u ← Q.extractMin()
        for v ∈ Adj[u] and v ∈ Q do
            if w(u, v) < key[v] then
                parent[v] ← u
                key[v] ← w(u, v)
            end if
        end for
    end while
end procedure
```

Complexity: $O(E \log V)$ if using min heap (Faster alternative with Fibonacci)



Min-heap example with 3 children for root

Kruskal's Algorithm - Greedy MST Construction

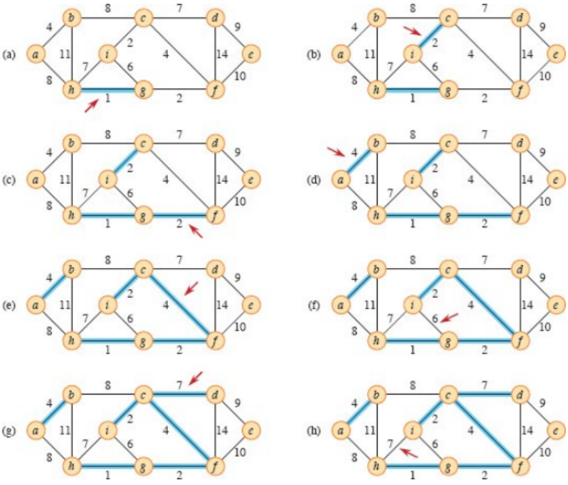
Kruskal's Algorithm Steps

1. **Initialize DSU**: Each vertex in its own component
2. **Sort edges**: By weight (ascending order)
3. **For each edge** (u, v) in sorted order:
4. **Check for cycle**: If `find(u)` $\neq$ `find(v)`
5. **Add to MST**: Include edge if no cycle
6. **Union**: Merge components using `union(u,v)`
7. **Skip**: If same component (cycle detected)

Greedy Strategy

Always pick the smallest available edge that doesn't create a cycle

Kruskal Algorithm: Execution



Stepwise execution of Kruskal Algorithm

Graph Coloring – Map and Schedule Applications

Problem: Assign as **few colors as possible** to vertices so that no two adjacent vertices share the same color.

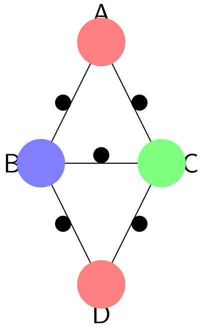
Example:

Vertices: Regions on a map or tasks needing resources

Edges: Conflicts
Chromatic Number: minimum colors needed: $\chi(G) = 3$ (NP Hard)

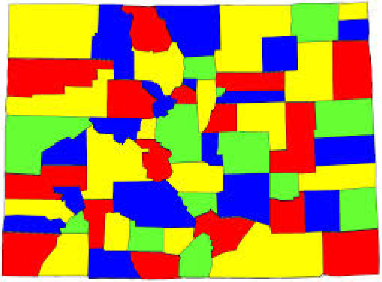
Real-world use cases:

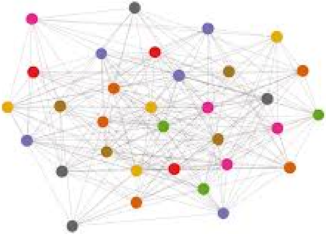
- Scheduling exams
- Register allocation in compilers
- Frequency assignment in wireless networks



A 3-coloring: A,D=red; B=blue; C=green

Nice examples of graph colouring problems





Bipartite Graphs & Graph Coloring

Bipartite Graph

A graph whose vertices can be divided into two disjoint sets U and V, such that every edge connects a vertex in U to one in V.

Equivalently: 2-colorable — no two adjacent vertices share the same color.

When is a graph bipartite?

$\iff$ No odd-length cycles.

Example: Trees, even cycles, hypercubes.

Beyond Two Colors: Four Color Theorem

Every planar graph (or map) is 4-colorable — no two adjacent regions need more than four colors.

First major theorem proven with computer assistance (1976, Appel & Haken)

Graph Coloring Algorithm: Greedy Coloring

Algorithm (Greedy Coloring):

1. Order vertices: $v_1, v_2, \ldots, v_n$
2. For each v_i in order:

   Assign the smallest color not used by its already-colored neighbors.

Key Properties:

Time complexity: $O(V + E)$

Not optimal — may use $> \chi(G)$ colors

Performance depends on vertex ordering

Worst case: $\chi(G) + 1$ colors

Heuristics: DSATUR, Largest First, Smallest Last

Vertex	Neighbors' Colors	Color Assign
v_1	—	1
v_2	$\{1\}$	2
v_3	$\{1,2\}$	3
v_4	$\{2,3\}$	1

Example run of greedy coloring

Flood Fill: More Than Just a Paint Tool

It's Graph Traversal on a Grid

Each pixel is a node; edges connect to 4 neighbors.
Flood fill = find connected component of same color.

Why Queue? Avoid Stack Overflow

Recursive DFS crashes on large regions (e.g., 1M pixels).
Queue → iterative BFS → safe, predictable memory use.

BFS vs DFS: Shape Matters!

BFS (Queue): Circular, even fill — *used in Photoshop*

DFS (Stack): Jagged, spiky fill — *faster on small areas*

Applications

- Medical imaging: Segment tumors or organs
- Computer vision: Object detection via region growing
- Game engines: Territory control, pathfinding

Dijkstra's Algorithm

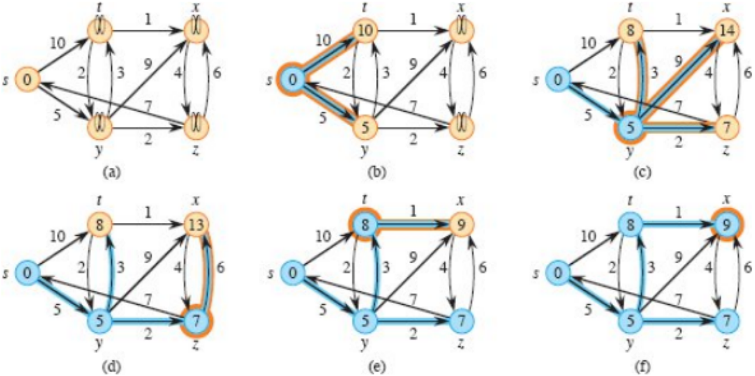
Goal: Find shortest paths from a source node to all other nodes in a weighted graph (non-negative weights).

Simple Steps:

1. Set distance to source $= 0$. Set all other distances to ∞. Mark all nodes unvisited.
2. While there are unvisited nodes:
3. Choose the unvisited node with the smallest known distance.
4. For each neighbor of that node:
   - Add the edge weight to the current node's distance.
   - If this gives a shorter path to the neighbor, update its distance.
   - Mark the current node as visited.

Key idea: Greedily expand the closest unvisited node — guarantees optimal paths.

Dijkstra's Algorithm: Step-by-Step Execution



Dijkstra's algorithm: shortest path tree built step by step

Bellman-Ford: Shortest Paths with Negative Weights

Why Bellman-Ford?

- Handles **negative edge weights** (unlike Dijkstra)
- Detects **negative cycles** — paths with $-\infty$ weight
- Works on graphs where Dijkstra fails due to negative edges

Key Algorithmic Ideas

- Initialize all distances to ∞, except source to 0
- **Relaxation**: If $\text{dist}[u] + \text{wt} < \text{dist}[v]$, update $\text{dist}[v]$
- Repeat relaxation for **at most** $V - 1$ **iterations** (tree property and so not sensitive to neg. paths)
- Run V-th iteration to detect negative cycles

Complexity

1. **Time**: $O(VE)$ — $V - 1$ passes, E edges each
2. **Space**: $O(V)$ — distance array only
3. **vs Dijkstra**: $O(E \log V)$, but no negative edges allowed

Pseudocode

```
for i = 1 to V − 1 :
  for each edge (u, v) with weight w :
    if dist[u] + w < dist[v] :
      dist[v] = dist[u] + w
for each edge (u, v) with weight w :    // Negative cycle check
  if dist[u] + w < dist[v] :  return "Negative cycle detected"
```

Flow Network

A **flow network** is a directed graph $G = (V, E)$ with:

A **source** $s \in V$ and a **sink** $t \in V$

A **capacity function** $c : E \to \mathbb{R}_{\geq 0}$

A **flow function** $f : E \to \mathbb{R}_{\geq 0}$ satisfying:

1. **Capacity constraint:** $0 \leq f(u, v) \leq c(u, v)$
2. **Flow conservation:** $\sum_w f(w, u) = \sum_w f(u, w)$ for all $u \neq s, t$

Value of flow: $|f| = \sum_v f(s, v) - \sum_v f(v, s)$

Goal: Find a flow of **maximum value** from s to t.

Max-Flow Min-Cut Theorem & Duality

Cut: A partition (S, T) of V with $s \in S$, $t \in T$. **Capacity of cut:** $c(S, T) = \sum_{u \in S, v \in T} c(u, v)$

Max-Flow Min-Cut Theorem

$$\max_f |f| = \min_{(S,T)} c(S, T)$$

The maximum flow value equals the minimum cut capacity.

LP Duality Perspective

Max-flow is a linear program.

The dual of the max-flow LP corresponds to a fractional min-cut.

Strong duality $\Rightarrow$ integral optimal solutions coincide.

$$\text{Primal (Max-Flow)} \quad \leftrightarrow \quad \text{Dual (Min-Cut)}$$