Problem Set #5: Graph Theory and Sparse Representations

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Problem 1. Equivalence of Tree Definitions

We prove that all seven definitions of a tree are equivalent by showing a chain of logical implications.

(1) $\Rightarrow$ (7)

If a graph is connected and acyclic, then between any two vertices there is exactly one path. Connectivity guarantees at least one path, and acyclicity guarantees that two different paths cannot exist (otherwise a cycle would appear). Thus the path is unique.

(7) $\Rightarrow$ (6)

If there is a unique path between any two vertices, adding an extra edge between them will create a second path. Therefore a cycle must form, and the graph is maximally acyclic.

(6) $\Rightarrow$ (5)

A maximally acyclic graph is one in which adding any edge produces a cycle. The only way this can happen is if the graph already has at least $V - 1$ edges.

(5) $\Rightarrow$ (3)

Any acyclic graph with at least $V - 1$ edges must have exactly $V - 1$ edges. If it had fewer, it would have more than one component. Thus it is a connected graph with at most (in fact exactly) $V - 1$ edges.

(3) $\Rightarrow$ (4)

A connected graph with exactly $V - 1$ edges is minimally connected. Removing any edge reduces the number of edges below the minimum needed for connectivity, so the graph becomes disconnected.

(4) $\Rightarrow$ (1)

If removing any edge disconnects the graph, then the graph is connected. It also cannot contain cycles, because removing an edge from a cycle would not disconnect the graph. Thus the graph is connected and acyclic.

(1) $\Rightarrow$ (2)

A connected acyclic graph is exactly one connected component of a forest. Therefore it satisfies the definition of a tree as a component of a forest.

(2) $\Rightarrow$ (1)

A component of a forest is connected and acyclic by definition. Hence it is a tree.

Conclusion

All seven definitions are logically equivalent and describe the same concept: a tree.

Problem 2. Sparse Representation of Graphs

Vertices are labeled as: A $\to$ 0, B $\to$ 1, C $\to$ 2, D $\to$ 3, E $\to$ 4.

Graph 1 (Undirected)

CSC representation:

- `col_pointers = [0, 2, 5, 8, 11, 12]`
- `row_indices = [1, 2, 0, 2, 3, 0, 1, 3, 1, 2, 4, 3]`

(a) Adjacency Matrix

	A	B	C	D	E
A	0	1	1	0	0
B	1	0	1	1	0
C	1	1	0	1	0
D	0	1	1	0	1
E	0	0	0	1	0

(b) Graph Diagram

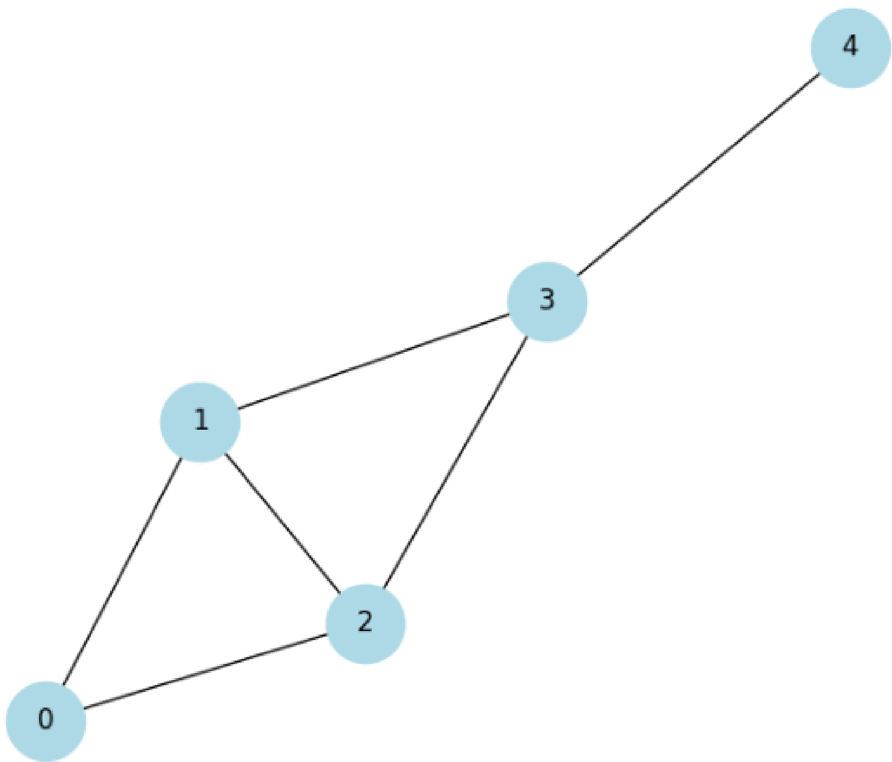

Figure 1: Graph 1 reconstructed from CSC format (undirected).

Graph 2 (Directed)

CSC representation:

- `col_pointers = [0, 0, 2, 4, 5, 7]`
- `row_indices = [0, 3, 0, 1, 2, 1, 3]`

(a) Adjacency Matrix

	A	B	C	D	E
A	0	0	0	0	0
B	1	0	1	1	0
C	1	1	0	0	0
D	0	0	1	0	0
E	0	1	0	1	0

(b) Directed Graph Diagram

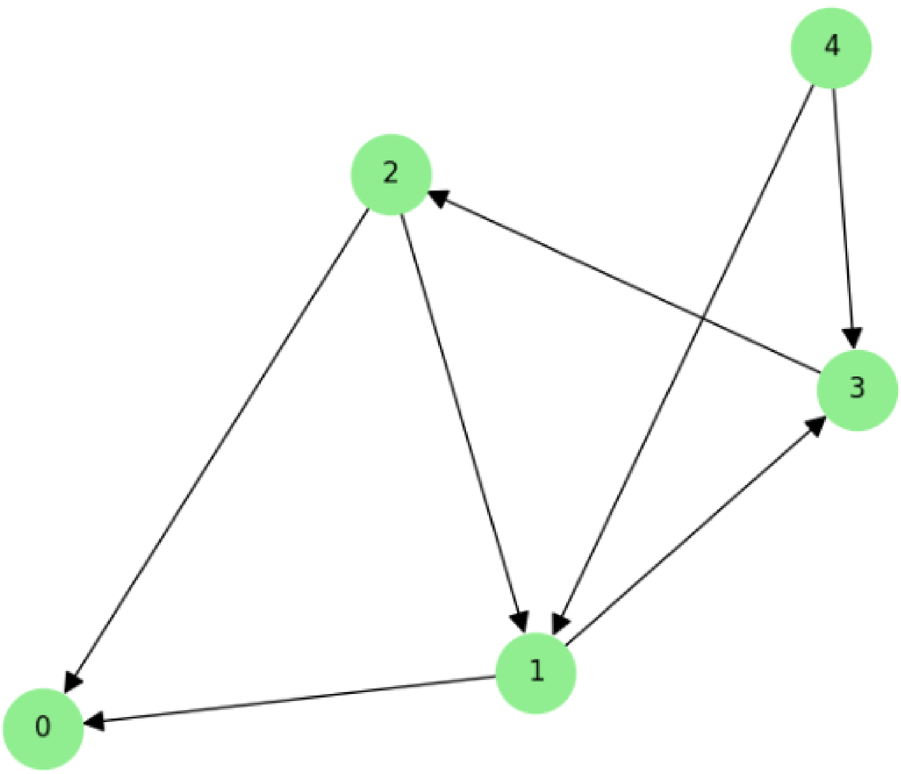

Figure 2: Graph 2 reconstructed from CSC format (directed).

(c) Unique Directed Cycle

The directed cycle in the graph is:

$$B \rightarrow D \rightarrow C \rightarrow B$$

Thus, the graph contains exactly one directed cycle.