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Fundamental Algorithmic Techniques VII

Outline
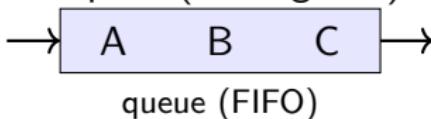
Graph Traversals: BFS vs DFS

Breadth First Search

Queue (FIFO)

Level-order

Shortest path (unweighted)

$\longrightarrow$ | A   B   C | $\longrightarrow$

queue (FIFO)
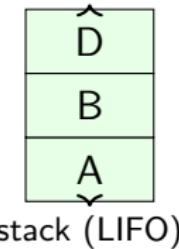
Depth First Search

Stack (LIFO)

Deep-first, backtrack
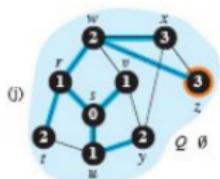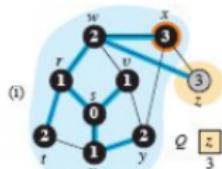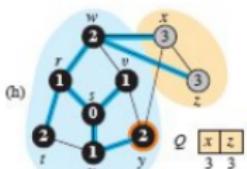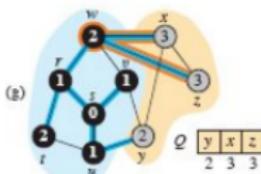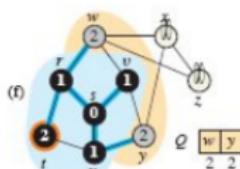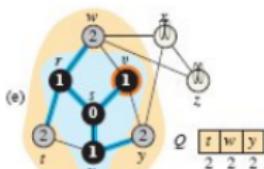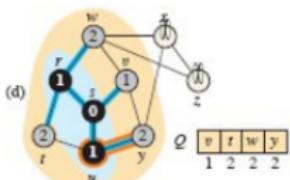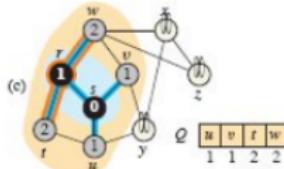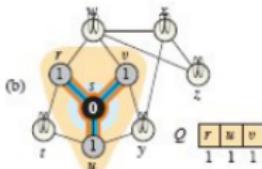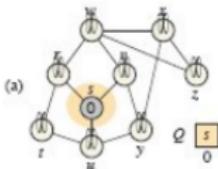
Discovery/finish times

SCCs = Strongly Connected Components

| D |
| B |
| A |

stack (LIFO)

Both solve reachability — BFS: wide, DFS: deep

Breadth First Search in Practice

Analysis of Search

Search on graph: $\mathcal{G} = (V, E)$,

- Each edge uv in the component traversed twice
  $\implies 2E + 1$
- Search in sparse! Adjacency matrix $\mathcal{O}(V)$,
  $\mathcal{O}(V^2)$ if not sparse!

Time complexity: $\mathcal{O}(V + ET)$

Cliques

A **clique** is a subset of vertices in an undirected graph such that:
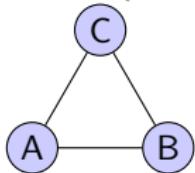
- Every two distinct vertices are **adjacent**
- The induced subgraph is **complete**

Every two distinct vertices are **adjacent**
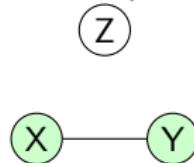
The induced subgraph is **complete**

Examples:

3-clique (size 3)

2-clique (size 2)

Note:

- Any single edge is a clique of size 2. The largest clique in a graph is the *maximum clique* (NP-hard to compute).
- Bron-Kerbosch algorithm for finding maximum clique

Minimum Spanning Tree (MST)

A **spanning tree** of a connected, undirected graph $G = (V, E)$ is:
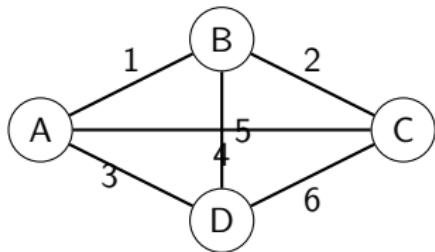
A subgraph that is a **tree**

Includes **all vertices** ($|V|$ nodes)
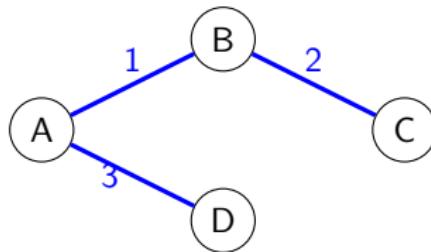
Has exactly $|V| - 1$ edges (no cycles)

A **minimum spanning tree** (MST) is a spanning tree with the **smallest possible total edge weight**.

Example:
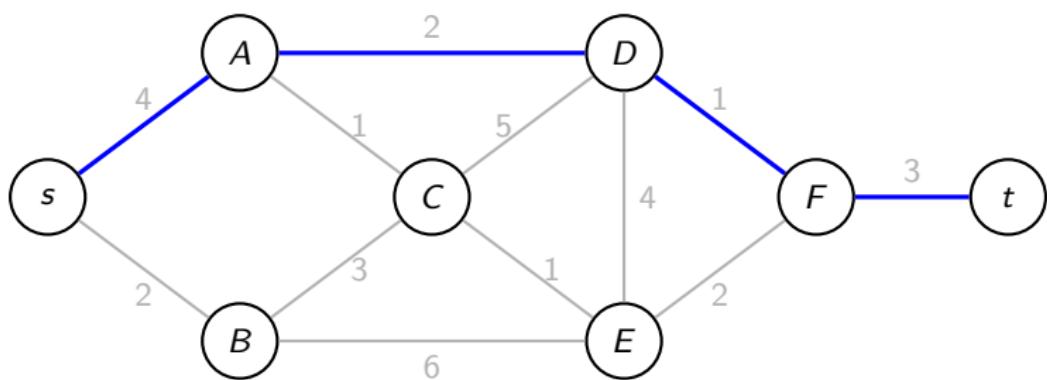
Original graph

MST (total weight $= 6$)
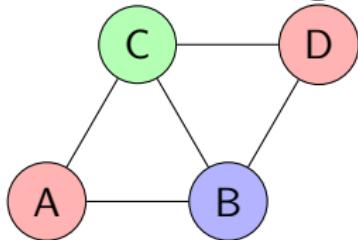
Shortest Path

The **shortest path** from s to t minimizes the sum of edge weights.
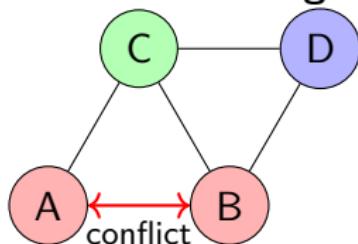
Graph Coloring

A **proper coloring** assigns colors to vertices so that **no two adjacent vertices share the same color**.

Valid 3-coloring

Invalid coloring
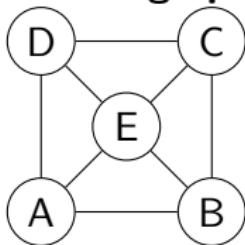
The smallest number of colors needed is the **chromatic number**.
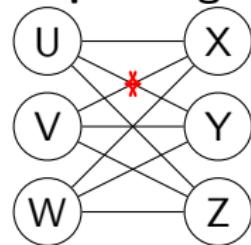
Planar Graphs

A graph is **planar** if it can be drawn in the plane **without edge crossings** (except at vertices).

Planar graph

Drawing without crossings $\to$ planar.

Non-planar graph

$K_{3,3}$ (complete bipartite) is non-planar.

Strongly Connected Components (SCCs)
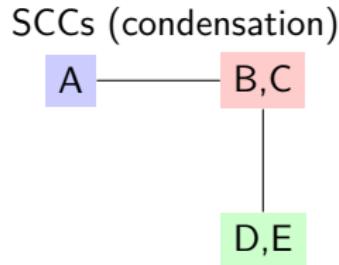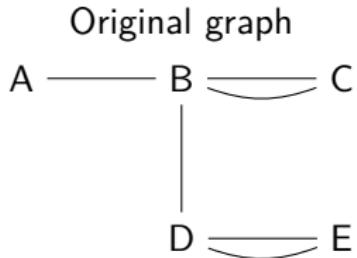
Definition: In a directed graph $G = (V, E)$, a strongly connected component is a maximal subset $C \subseteq V$ such that for every pair $u, v \in C$, there is a directed path from u to v **and** from v to u.

Key ideas:

Every vertex belongs to exactly one SCC.

SCCs partition the vertex set.
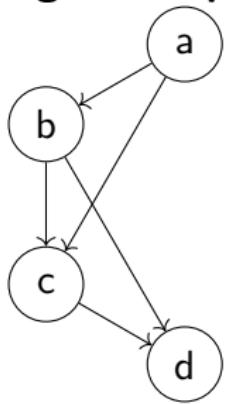
The *condensation* of G (contracting each SCC to a node) is a DAG.

Original graph

A —— B ⥮ C

B | D ⥮ E

SCCs (condensation)

A —— B,C

D,E
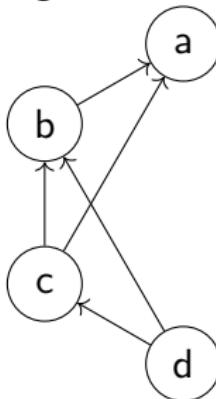
Algorithms: Kosaraju's, Tarjan's, or Gabow's (all linear time: $O(|V| + |E|)$).

Graph Transformations
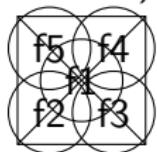
Original Graph

Transpose Graph
(Edges reversed)

Dual Graph
(Faces become
vertices)

Inverse Graph: If $(u, v) \in E$, then $(u, v) \notin E_{inv}$ (complement)