Problem Set 7.

Problem 1:

1. Directed Graphs and their transposes

For a directed graph $G=(V,E)$, the transpose G^T is formed by reversing the direction of every edge.

Formally, if $(u,v)\in E$, then $(v,u)\in E^T$

Example 1. Original graph G — Transpose G^T

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[d] \\ & C \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dr] \\ C \arrow[ur] \arrow[rr] & & A \end{tikzcd}$$

Example 2. Original H — Transpose H^T

$$\begin{tikzcd} X \arrow[r] & Y \arrow[dl] \\ Z \end{tikzcd} \qquad \begin{tikzcd} X & Y \arrow[l] \\ Z \arrow[ur] \end{tikzcd}$$

2. Undirected graphs and their inverse graphs

The inverse graph of an undirected graph has an edge exactly where the original does not.

Example 1.

Original graph G on $\{A,B,C\}$, $E=\{AB, BC\}$

$$\begin{tikzcd} A \arrow[r, dash] & B \arrow[dl, dash] \\ C \end{tikzcd}$$

Inverse G^{-1} edges $= AC$

Example 2:

H on $\{1,2,3,4\}$ Edges $-$ 12, 23, 34

Inverse H^{-1} $=$ 13, 14, 24

3. What happens if the original graph is dense for the inverse?

If the original graph is dense, meaning close to complete, its inverse becomes: sparse; possibly even empty, if the original is a complete graph.

Example: If G is K_4, a complete graph on 4 vertices $\to$ 6 edges

Then the inverse has 0 edges.

4. Simple examples of undirected graph and their dual graph

Dual graph is formed for planar embeddings: Each face becomes a vertex, and edges represent adjacency between faces:

Example: Square

Original planar graph:

$$\begin{tikzcd} A \arrow[r, dash] \arrow[d, dash] & B \arrow[d, dash] \\ D \arrow[r, dash] & C \end{tikzcd}$$

Example: Triangle

Original: A
\ B
/ C

Faces: f_in = inside triangle
f_out = outside

5. Why is the dual only well-defined for planar graphs

Dual graphs require:

- A clear definition of faces
- Non-planar graphs cannot be drawn without crossing edges.

Example: Non-planar graph $K_{3,3}$

Nodes on left: $\{a, b, c\}$

Nodes on right: $\{x, y, z\}$

Edges: all bipartite connections.

Problem 2

Graph: $V = \{A, B, C, D\}$ $E = AB, AC, BC, CD$

1. Initial Call

$R = \emptyset$

$P = \{A, B, C, D\}$

$X = \emptyset$

2. Trace first two recursive calls leading to a maximal clique

$R = \{A\}$ $P = \{B, C\}$ $X = \emptyset$ - Choose A

$R = \{A, B\}$ $P = \{C\}$ $X = \emptyset$ - Choose B

$R = \{A, B, C\}$ $P = \emptyset$ $X = \emptyset$ - Choose C.

This the first maximal clique

3. List all maximal cliques & identify maximum

All maximal cliques

1. $\{A, B, C\}$
2. $\{C, D\}$

Maximum clique size = 3