Problem 1

Proof of Equivalence:

We denote *"n"* as the number of vertices *"V"* and *"m"* as the number of edges.

- (1) → (2):

  A tree is a connected acyclic graph. A forest is defined as an acyclic graph. Since a tree is connected, it consists of a single connected component. Thus, a tree is one component of a forest.

- (2) → (3):

  If a graph is a component of a forest (acyclic), it is connected and acyclic. By induction on the number of vertices, any connected acyclic graph with "n" vertices has exactly "n-1" edges.

- (3) → (4):

  We have a connected graph with "n-1" edges. If we remove any edge, the number of edges becomes "n-2". A connected graph with "n" vertices requires at least "n-1" edges. Therefore, with "n-2" edges, the graph must be disconnected. This fits the definition of minimal connectivity.

- (4) → (5):

  A minimally connected graph contains no cycles (if there were a cycle, removing an edge from it would not disconnect the graph). Since it is connected and acyclic, it is a tree, and as shown in (2→3), it must have "n-1" edges. Since it's minimally connected, it cannot have more. Thus, it is acyclic with "n-1" edges.

- (5) → (6):

  An acyclic graph with "n-1" edges is a tree (standard theorem). If we add an edge between any two vertices "u" and "v", since there was already a unique path between them (connectedness), the new edge creates a cycle. Thus, it is maximally acyclic.

- (6) → (7):

  Being maximally acyclic implies the graph is connected (if not, adding an edge between components wouldn't create a cycle). Since it is connected and acyclic, there is exactly one path between any pair of nodes.

- (7) → (1):

  If there is a unique path between any pair, the graph is connected. Uniqueness implies no cycles (two paths would form a cycle). Thus, it is a connected acyclic graph.

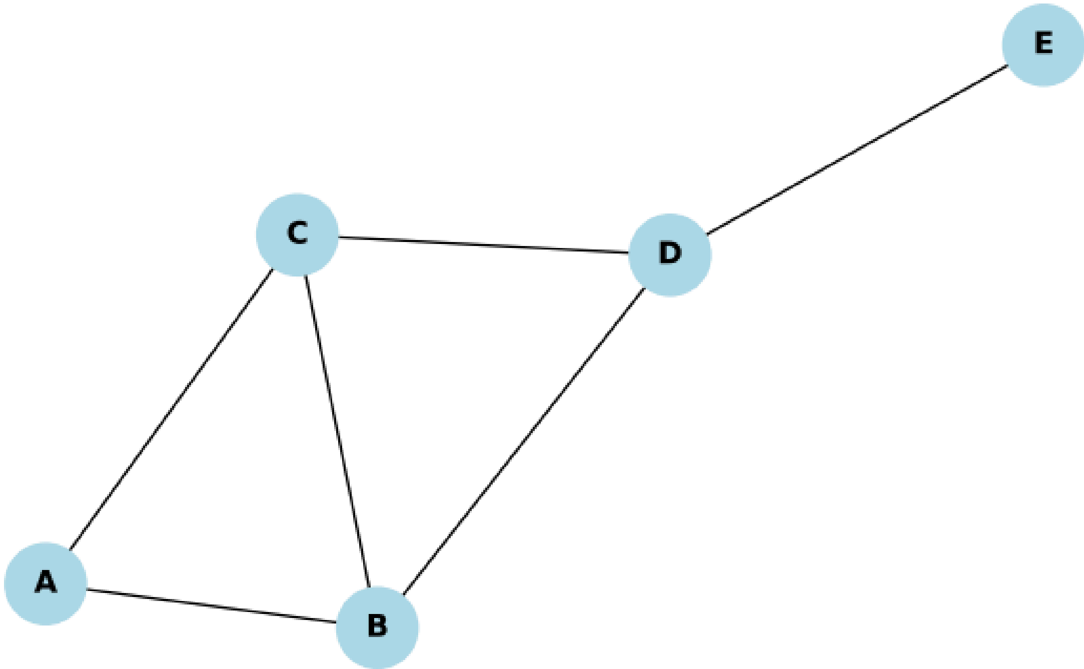
Problem 2

Graph 1 - undirected

1. Adjacency Matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

2. Diagram (Graph 1):



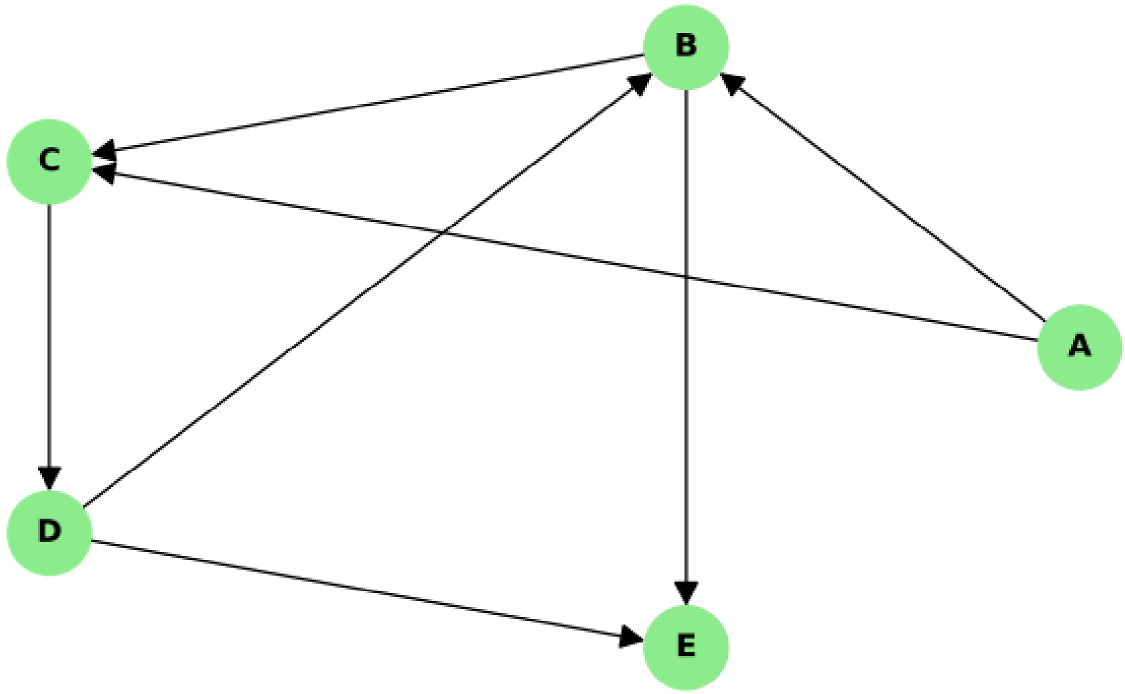
3. Unique cycle (Graph 1): Graph 1 has cycles (e.g., A-B-C-A), but the question asks about the unique cycle in the *directed* graph below. For Graph 1, it is connected but not a tree (it has cycles).

   Graph 2 - directed

1. Adjacency Matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Diagram:



3. **Unique cycle:** The unique cycle is: B → C → D → B.