Problem Set #5
Fundamental Algorithm Techniques

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Problem 1: Equivalence of Tree Definitions

We show that the seven given characterizations of a tree are all equivalent. Let $G = (V, E)$ be a graph. The statements are:

1. G is connected and acyclic.
2. G is one component of a forest.
3. G is connected and has $|V| - 1$ edges.
4. G is minimally connected: removing any edge disconnects the graph.
5. G is acyclic and has at least $|V| - 1$ edges.
6. G is maximally acyclic: adding any edge between two vertices creates a cycle.
7. Between any two vertices there exists a unique path.

We prove the equivalence by showing

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (1).$$

(1) ⇒ (2)

A forest is a graph with no cycles. Since G is acyclic by (1), it is a forest. Because G is also connected, it consists of exactly one component of this forest. Hence (2) holds.

(2) ⇒ (3)

A fundamental property of forests is that a forest with k connected components has exactly $|V| - k$ edges. If G is one component of a forest, then $k = 1$, and thus $|E| = |V| - 1$. Since that component is connected, (3) follows.

(3) $\Rightarrow$ (4)

Suppose G is connected with $|V|-1$ edges. If some edge could be removed without disconnecting G, the resulting graph would still be connected, implying the original graph had at least $|V|$ edges. This contradicts the assumption. Thus G is minimally connected, proving (4).

(4) $\Rightarrow$ (5)

Assume G is minimally connected. If G contained a cycle, we could remove an edge from that cycle and the graph would remain connected, contradicting minimality. Thus G is acyclic. Since G is connected, it must have at least $|V|-1$ edges. Thus (5) holds.

(5) $\Rightarrow$ (6)

Let G be acyclic and have at least $|V|-1$ edges. An acyclic graph on $|V|$ vertices has at most $|V|-1$ edges, so G must have exactly $|V|-1$. Adding any new edge creates a cycle, so G is maximally acyclic. Thus (6) is true.

(6) $\Rightarrow$ (7)

If there were two distinct simple paths between vertices u and v, then adding an edge between u and v would not create a new cycle (one already exists), contradicting maximal acyclicity. Hence there must be exactly one simple path between any two vertices, establishing (7).

(7) $\Rightarrow$ (1)

If there is a unique path between any two vertices:

- G is connected (a path exists between every pair).
- G is acyclic (a cycle would create two distinct paths between some vertices).

Thus (1) holds.

Conclusion

All seven statements are equivalent, and therefore each one is a valid definition of a tree.

Problem 2: Sparse Representation of Graphs

We are given two graphs on vertices $\{A, B, C, D, E\}$ (indexed as $A \to 0$, $B \to 1$, $C \to 2$, $D \to 3$, $E \to 4$).

Graph 1 (Undirected)

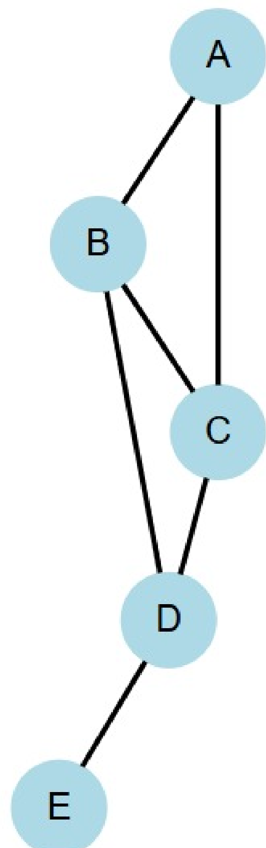
CSC Representation:

$$\begin{aligned} \text{col\_pointers} &= [0, 2, 5, 8, 11, 12] \\ \text{row\_indices} &= [1, 2, 0, 2, 3, 0, 1, 3, 1, 2, 4, 3] \\ \text{values} &= [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] \end{aligned}$$

Adjacency Matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Edges: A–B, A–C, B–C, B–D, C–D, D–E



Graph 2 (Directed)

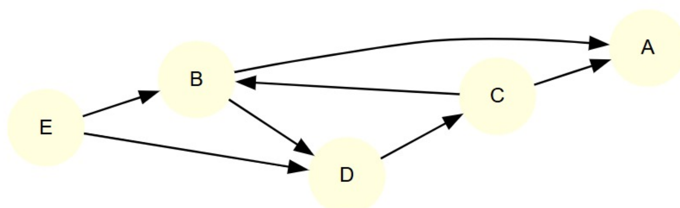
CSC Representation:

$$\begin{aligned} \text{col\_pointers} &= [0, 0, 2, 4, 5, 7] \\ \text{row\_indices} &= [0, 3, 0, 1, 2, 1, 3] \\ \text{values} &= [1, 1, 1, 1, 1, 1, 1] \end{aligned}$$

Adjacency Matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Edges: B→A, B→D, C→A, C→B, D→C, E→B, E→D



Unique cycle: B→D→C→B