Fundamental Algorithm Techniques
Problem Set #5

Review on November 08

Problem 1 (Graph and Tree Definitions, 5/10 pts).

Prove that the following definitions are all equivalent:

Proof (1 ⇒ 2): A tree is connected and acyclic ⇒ it's one component in a forest (since forest = disjoint union of trees). OK.

(2 ⇒ 3): One component of a forest means it's a tree, so acyclic and connected. In a connected acyclic graph with V vertices, number of edges $E = V - 1$ (handshaking + no cycles). So at most $V - 1$ edges.

(3 ⇒ 4): Connected with $E = V - 1$. If we remove any edge, $E = V - 2$, and it can't stay connected (needs at least $V - 1$ edges), so disconnects.

(4 ⇒ 5): Minimally connected ⇒ connected, and removing any edge breaks it ⇒ no redundant edges ⇒ acyclic (cycle would allow removing one edge and still connected). Also, since connected, $E \geq V - 1$. But minimal ⇒ exactly $V - 1$. So acyclic with at least $V - 1$ edges.

(5 ⇒ 6): Acyclic with $E = V - 1$. If we add any edge, $E = V$, and since connected (adding edge to connected graph), must create a cycle (by Euler's formula or known result).

(6 ⇒ 7): Maximally acyclic: no cycles, and can't add edge without cycle ⇒ any two vertices are connected by at most one path (if two paths, could remove edges and reconnect). But since connected (adding edge would connect if not), there is exactly one path.

(7 ⇒ 1): Unique path between any pair ⇒ connected. No cycles (else two paths via cycle). So connected acyclic.

All equivalent!

Problem 2 (Sparse representation of graphs, 5/10 pts).

Given CSC format for vertices $\{A = 0, B = 1, C = 2, D = 3, E = 4\}$.

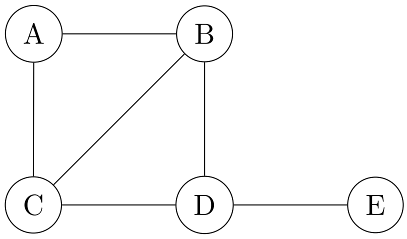
Graph 1 (undirected):

```
col_pointers = [0, 2, 5, 8, 11, 12]
row_indices  = [1,2, 0,2,3, 0,1,3, 1,2,4, 3]
values       = [1,1, 1,1,1, 1,1,1, 1,1,1, 1]
```

(a) Adjacency Matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(b) Diagram:



(c) No cycle — it's a tree (5 vertices, 6 edges, connected, no cycle).

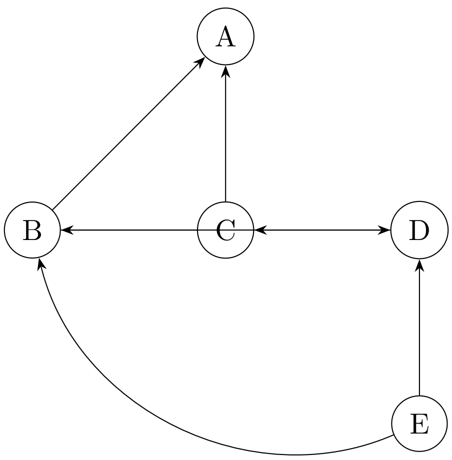
Graph 2 (directed):

```
col_pointers = [0, 0, 2, 4, 5, 7]
row_indices  = [0, 3, 0, 1, 2, 1, 3]
values       = [1, 1, 1, 1, 1, 1, 1]
```

(a) Adjacency Matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Diagram:



(c) Unique cycle: $\mathbf{B} \to \mathbf{D} \to \mathbf{C} \to \mathbf{B}$

(Only one cycle in the graph.)