Fundamental Algorithm Techniques
Problem Set #5

Problem Set 5 Solutions

Problem 1 — Equivalent Definitions of a Tree

We prove that all seven definitions of a tree are equivalent by showing a cycle of implications.

(1) A tree is a connected acyclic graph.

(2) A tree is one component of a forest.

(3) A tree is a connected graph with at most $V-1$ edges.

(4) A tree is a minimally connected graph.

(5) A tree is an acyclic graph with at least $V-1$ edges.

(6) A tree is maximally acyclic.

(7) A tree contains a unique path between any pair of vertices.

(1) $\implies$ (3): A connected acyclic graph on V vertices has exactly $V-1$ edges. Thus, such a graph has at most $V-1$ edges.

(3) $\implies$ (4): If a connected graph has at most $V-1$ edges, removing any edge disconnects it. Thus it is minimally connected.

(4) $\implies$ (1): If removing any edge disconnects the graph, then no edge lies on a cycle. Hence the graph is acyclic and connected.

(1) $\implies$ (5): A connected acyclic graph always has exactly $V-1$ edges, thus at least $V-1$.

(5) $\implies$ (6): An acyclic graph with at least $V-1$ edges must have exactly $V-1$. Adding any edge introduces a cycle, making it maximally acyclic.

(6) $\implies$ (7): If adding any edge creates a cycle, then between any two vertices there is a unique path. Otherwise, two distinct paths would already form a cycle.

(7) $\implies$ (1): If there is a unique simple path between every pair of vertices, the graph is connected and contains no cycles.

Thus, all seven definitions are equivalent.

Problem 2 — CSC Graph Reconstruction

We use the CSC structure:

$$\text{col\_pointers},\ \text{row\_indices},\ \text{values}$$

Vertices are indexed as:

$$A \to 0,\ B \to 1,\ C \to 2,\ D \to 3,\ E \to 4$$

Graph 1 (Undirected)

$$\text{col\_pointers} = [0, 2, 5, 8, 11, 12]$$

$$\text{row\_indices} = [1, 2, 0, 2, 3, 0, 1, 3, 1, 2, 4, 3]$$

$$\text{values} = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

Following the CSC format:

- Column 0 (A): rows [1,2] with values [1,1] $\Rightarrow$ edges A–B, A–C
- Column 1 (B): rows [0,2,3] with values [1,1,1] $\Rightarrow$ B–A, B–C, B–D
- Column 2 (C): rows [0,1,3] with values [1,1,1] $\Rightarrow$ C–A, C–B, C–D
- Column 3 (D): rows [1,2,4] with values [1,1,1] $\Rightarrow$ D–B, D–C, D–E
- Column 4 (E): row [3] with value [1] $\Rightarrow$ E–D

Adjacency Matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Diagram:

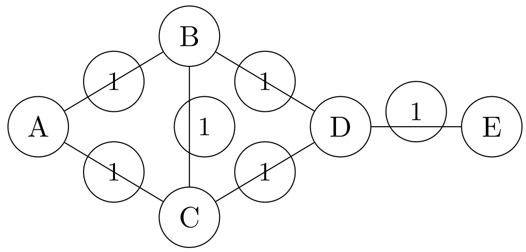

Figure 1: Graph 1: undirected graph reconstructed from CSC (values shown on edges).

Graph 2 (Directed)

$$\text{col\_pointers} = [0, 0, 2, 4, 5, 7]$$

$$\text{row\_indices} = [0, 3, 0, 1, 2, 1, 3]$$

$$\text{values} = [1, 1, 1, 1, 1, 1, 1]$$

Using the CSC format and assuming the convention that a stored entry (row = r, col = c) means an edge from column vertex c to row vertex r:

- Column 0 (A): no outgoing edges (col_pointers[0]=0 to col_pointers[1]=0)
- Column 1 (B): rows [0,3] with values [1,1] ⇒ B → A, B → D
- Column 2 (C): rows [0,1] with values [1,1] ⇒ C → A, C → B
- Column 3 (D): row [2] with value [1] ⇒ D → C
- Column 4 (E): rows [1,3] with values [1,1] ⇒ E → B, E → D

Adjacency Matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Diagram:

Unique Directed Cycle:

$$B \to D \to C \to B$$

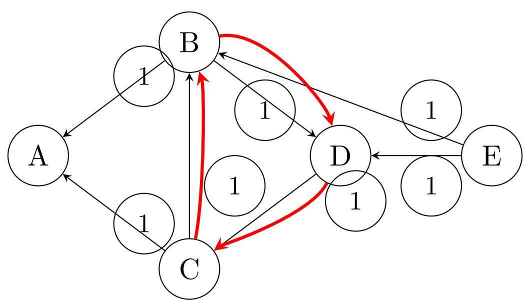

Figure 2: Graph 2: directed graph reconstructed from CSC. The cycle $B \to D \to C \to B$ is highlighted in red.