Lecture 07 – Solving Linear Systems (Part 1)

Kyle Vaughn & Prof. Brendan Kochunas

NERS/ENGR 570 - Methods and Practice of Scientific Computing (F22)

COLLEGE OF ENGINEERING
NUCLEAR ENGINEERING & RADIOLOGICAL SCIENCES
UNIVERSITY OF MICHIGAN

Outline

- Review Lab 2 Assignment
- Homework 2 Overview
- Recap: Basic Linear Algebra Operations
- Overview of Solution of Linear Systems
- Direct Methods and Matrix Factorizations
- Deep-dive into LU decomposition

Lab 2 Assignment Review

- Ex. 1: Parametric Calc. & Function Approximation
  - No common errors
  - Produced the data + close to correct coefficients = full points
- Ex. 2: Searching & Parsing
  - Most common error: includes and excludes when searching for jackpot

```
grep -r Jackpot --include=*.{doc,xml} --exclude='jackpot_locations.txt'
```

- Other errors
  - File naming
    - -1 point per file with incorrect name.
  - Compression with tar
    - Make sure you submit just the required files!
    - A few people practically gave me their whole desktop (-5 points)

Homework 2 Overview

Homework 2 Overview

- Will try to get solutions written/Autograder up by Friday
- vecfilein/vecfileout will be N lines with a single value on each line
- Timing table may be subject to change
  - Benchmarking is hard
    - Are you measuring I/O or the mat-vec multiplication?
  - Matrices vary greatly in size
    - One N may be simultaneously too small and too large
    - Don’t want to wait 11 minutes for dense matrix multiplication
  - nmult = floor(C / N^2)?

Learning Objectives: By the end of Today's Lecture you should be able to

- *(Knowledge)* list several methods for factorizing a matrix
- *(Knowledge)* have a high level view of various ways to solve linear systems
- (*Skill*) implement LU factorization (without pivoting)

Basic Linear Algebra Operations

Residual and Norms of Vectors

$\mathbf{r} = \mathbf{Ax} - \mathbf{b}$ residual

$\|\mathbf{r}\|_1 = \sum_i |r_i|$ 1-norm ("total absolute error")

$\|\mathbf{r}\|_2 = \sqrt{\sum_i r_i^2}$ 2-norm ("average error")

$\|\mathbf{r}\|_\infty = \max_i \left(|r_i|\right)$ ∞-norm ("max local error")

$\|\mathbf{r}\|_p = \left(\sum_i |r_i|^p\right)^{1/p}$ *p*-norm

Inner/Dot Product (vector-vector multiply)

$$\mathbf{u}^T \cdot \mathbf{v} = \sum_i u_i v_i$$

Matrix-vector Multiply

$$\mathbf{Ax} = \mathbf{b} \rightarrow b_i = \sum_j a_{i,j} x_j$$

Matrix-Matrix Multiply

$$\mathbf{AB} = \mathbf{C} \rightarrow c_{i,j} = \sum_k a_{i,k} b_{k,j}$$

Solving Linear Systems

How do we solve linear systems?

Types of Matrices

- Structure
  - Diagonal
  - Triangular
  - Symmetric (Hermitian)
- Properties
  - Orthonormal (Unitary)

Direct Solution Methods

aka Matrix Factorizations

LU Factorization

- L is lower triangular
- U is upper triangular

$$A = LU$$

QR Factorization

- Q is orthonormal
- R is upper triangular

$$A = QR$$

Cholesky Factorization

- L is lower triangular
- A must be symmetric

$$A = LL^*$$

Singular Value Decomposition

- U is unitary
- Σ is diagonal
- V is unitary

$$A = U\Sigma V^*$$

Eigendecomposition

- Q is orthonormal
- Λ is diagonal
- A must be square and diagonalizable

$$A = Q\Lambda Q^{-1}$$

LU Factorization

LU Factorization

LU: Forward Elimination

LU: Backward Substitution

Fast LU Factorizations