Transformation of Random Variables

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Contents

0.1 Gaussian to Other **1**

0.2 Conditional Probability **2**

0.1 Gaussian to Other

0.1.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{0.1.1.1}$$

Solution: The CDF and PDF of V are plotted in Fig. 0.1.1.1 and Fig. 0.1.1.2 respectively using the below code

```
codes/ch4_squares.py
```

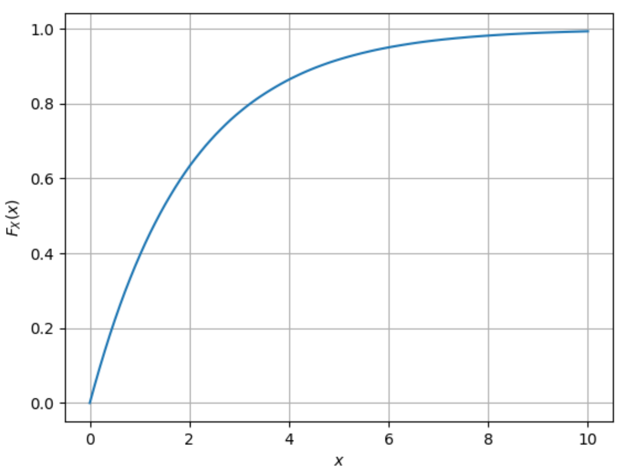

Figure 0.1.1.1: CDF of V

0.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{0.1.2.1}$$

find α.
Solution: Let $Z = X^2$ where $X \sim \mathcal{N}(0,1)$. Defining

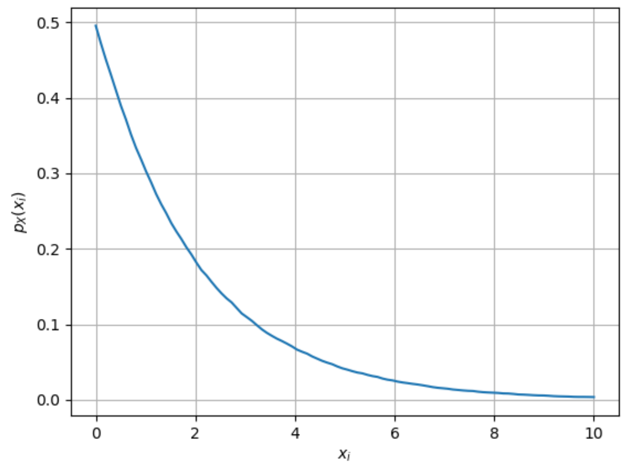

Figure 0.1.1.2: PDF of V

the CDF for Z,

$$\begin{aligned} P_Z(z) &= \Pr(Z < z) \\ &= \Pr\left(X^2 < z\right) \\ &= \Pr\left(-\sqrt{z} < X < \sqrt{z}\right) \\ &= \int_{-\sqrt{z}}^{\sqrt{z}} p_X(x)\, dx \end{aligned}$$

Using (**??**), the PDF of Z is given by

$$\frac{d}{dz} P_Z(z) = p_Z(z)$$

$$= \frac{p_X(\sqrt{z}) + p_X(-\sqrt{z})}{2\sqrt{z}} \text{ (Using Lebniz's rule)} \tag{0.1.2.2}$$

Substituting the standard gaussian density function $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ in (0.1.2.2),

$$p_Z(z) = \begin{cases} \frac{1}{\sqrt{2\pi z}} e^{-\frac{z}{2}} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{0.1.2.3}$$

The PDF of X_1^2 and X_2^2 are given by (0.1.2.3). Since V

is the sum of two independant random variables,

$$
\begin{aligned}
p_V(v) &= p_{X_1^2}(x_1) * p_{X_2^2}(x_2) \\
&= \frac{1}{2\pi} \int_0^v \frac{e^{-\frac{x}{2}}}{\sqrt{x}} \frac{e^{-\frac{v-x}{2}}}{\sqrt{v-x}} \, dx \\
&= \frac{e^{-\frac{v}{2}}}{2\pi} \int_0^v \frac{1}{\sqrt{x(v-x)}} \, dx \\
&= \frac{e^{-\frac{v}{2}}}{2\pi} \left[-\arcsin\left(\frac{v-2x}{v}\right) \right]_0^v \\
&= \frac{e^{-\frac{v}{2}}}{2\pi} \pi \\
&= \frac{e^{-\frac{v}{2}}}{2} \text{ for } v \geq 0
\end{aligned}
$$

$F_V(v)$ can be obtained from $p_V(v)$ using (**??**)

$$
\begin{aligned}
F_V(v) &= \frac{1}{2} \int_0^v \exp\left(-\frac{v}{2}\right) \\
&= 1 - \exp\left(-\frac{v}{2}\right) \text{ for } v \geq 0 \qquad (0.1.2.4)
\end{aligned}
$$

Comparing (0.1.2.4) with (0.1.2.1), $\alpha = \frac{1}{2}$

0.1.3 Plot the CDF and PDF of

$$A = \sqrt{V} \qquad (0.1.3.1)$$

Solution: The CDF and PDF of A are plotted in

$$Fig.\ 0.1.3.1$$

and

$$Fig.\ 0.1.3.2$$

respectively using the below code

```
codes/ch4_sqrt.py
```

The CDF of A is given by,

$$
\begin{aligned}
F_A(a) &= \Pr(A < a) && (0.1.3.2) \\
&= \Pr\left(\sqrt{V} < a\right) && (0.1.3.3) \\
&= \Pr\left(V < a^2\right) && (0.1.3.4) \\
&= F_V\left(a^2\right) && (0.1.3.5) \\
&= 1 - \exp\left(-\frac{a^2}{2}\right) && (0.1.3.6)
\end{aligned}
$$

Using (**??**), the PDF is found to be

$$p_A(a) = a \exp\left(-\frac{a^2}{2}\right) \qquad (0.1.3.7)$$

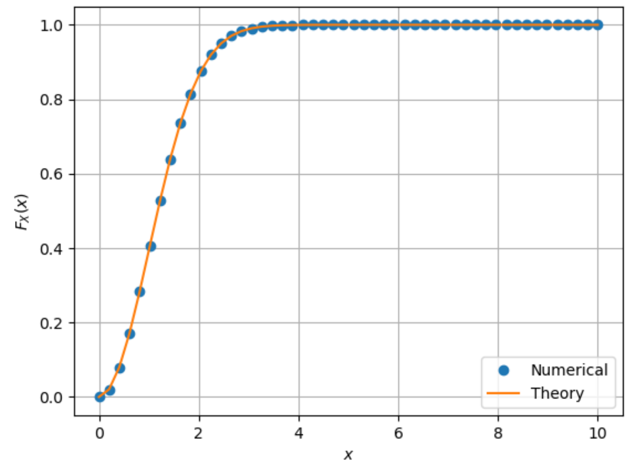

Figure 0.1.3.1: CDF of A

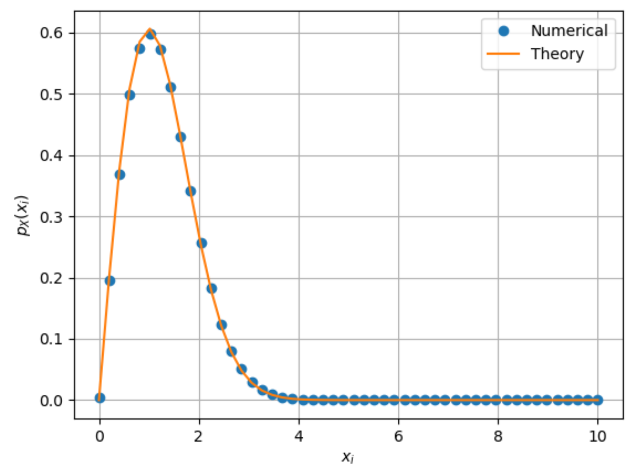

Figure 0.1.3.2: PDF of A

0.2 Conditional Probability

0.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \qquad (0.2.1.1)$$

for

$$Y = AX + N, \qquad (0.2.1.2)$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.
Solution: The blue dots in Fig. 0.2.4.1 is the required plot. The below code is used to generate the plot,

```
codes/ch4/ch4_err.py
```

0.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$
Solution: Assuming the decision rule in (**??**), when N

is constant, P_e is given by

$$
\begin{aligned}
P_e &= \Pr\left(\hat{X} = -1 | X = 1\right) \\
&= \Pr\left(Y < 0 | X = 1\right) \\
&= \Pr\left(AX + N < 0 | X = 1\right) \\
&= \Pr\left(A + N < 0\right) && (0.2.2.1) \\
&= \Pr\left(A < -N\right) \\
&= \begin{cases} F_A(-N) & N \geq 0 \\ 0 & N < 0 \end{cases} && (0.2.2.2)
\end{aligned}
$$

For a Rayleigh random variable X with $E\left[X^2\right] = \gamma$, the PDF and CDF are given by

$$
p_X(x) = \frac{2x}{\gamma} \exp\left(-\frac{x^2}{\gamma}\right) \quad \text{for } x \geq 0 \tag{0.2.2.3}
$$

$$
F_X(X) = 1 - \exp\left(-\frac{x^2}{\gamma}\right) \quad \text{for } x \geq 0 \tag{0.2.2.4}
$$

Substituting (0.2.2.4) in (0.2.2.2),

$$
P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{\gamma}\right) & N \geq 0 \\ 0 & N < 0 \end{cases} \tag{0.2.2.5}
$$

0.2.3 For a function g,

$$
E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{0.2.3.1}
$$

Find $P_e = E\left[P_e(N)\right]$.
Solution: Using $P_e(N)$ from (0.2.2.5),

$$
\begin{aligned}
P_e &= \int_{-\infty}^{\infty} P_e(x) p_N(x)\, dx \\
&= \int_{0}^{\infty} \left(1 - e^{-\frac{x^2}{\gamma}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx
\end{aligned}
$$

$$
P_e = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{x^2}{2}}\, dx - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \exp\left(-x^2\left(\frac{1}{\gamma} + \frac{1}{2}\right)\right) dx
$$

$$
P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}}
$$

0.2.4 Plot P_e in problems 0.2.1 and 0.2.3 on the same graph w.r.t γ. Comment.
Solution: P_e plotted in same graph in Fig. 0.2.4.1. The value of P_e is much higher when the channel gain A is Rayleigh distributed than the case where A is a constant (compare with Fig. **??**).

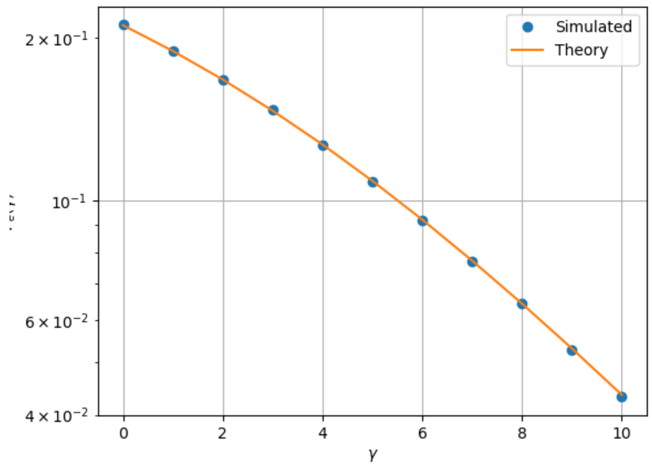

Figure 0.2.4.1: P_e versus γ

From (0.2.2.1), P_e is given by

$$
P_e = \Pr\left(A + N < 0\right) \tag{0.2.4.1}
$$

One method of computing (0.2.2.1) is by finding the PDF of $Z = A + N$ (as the convolution of the individual PDFs of A and N) and then integrating $p_Z(z)$ from $-\infty$ to 0. The other method is by first computing P_e for constant N and then finding the expectation of $P_e(N)$. Both provide the same result but the computation of integrals is simpler when using the latter method.