Sum of Independent Random Variables

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CONTENTS

1 Two dice

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

1) *The Uniform Distribution:* Let $X_i \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2$, be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{1.0.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{1.0.2}$$

$$\implies X \in \{1, 2, \dots, 12\} \tag{1.0.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{1.0.4}$$

2) *Convolution:* From (1.0.2),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{1.0.5}$$

$$= \sum_k \Pr(X_1 = n - k | X_2 = k)\, p_{X_2}(k) \tag{1.0.6}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr(X_1 = n - k | X_2 = k) = \Pr(X_1 = n - k) = p_{X_1}(n - k) \tag{1.0.7}$$

From (1.0.6) and (1.0.7),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{1.0.8}$$

where $*$ denotes the convolution operation. Substituting from (1.0.1) in (1.0.8),

$$p_X(n) = \frac{1}{6} \sum_{k=1}^{6} p_{X_1}(n - k) = \frac{1}{6} \sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{1.0.9}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{1.0.10}$$

From (1.0.9),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6} \sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n - 1 \le 6 \\ \frac{1}{6} \sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n - 6 \le 6 \\ 0 & n > 12 \end{cases} \tag{1.0.11}$$

Substituting from (1.0.1) in (1.0.11),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{1.0.12}$$

satisfying (1.0.4).

3) *The Z-transform:* The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{1.0.13}$$

From (1.0.1) and (1.0.13),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{1.0.14}$$

$$= \frac{z^{-1}\left(1-z^{-6}\right)}{6\left(1-z^{-1}\right)}, \quad |z| > 1 \tag{1.0.15}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{1.0.16}$$

$$P_X(z) = P_{X_1}(z)P_{X_2}(z) \tag{1.0.17}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (1.0.15) and (1.0.17),

$$P_X(z) = \left\{\frac{z^{-1}\left(1-z^{-6}\right)}{6\left(1-z^{-1}\right)}\right\}^2 \tag{1.0.18}$$

$$= \frac{1}{36}\frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{\left(1-z^{-1}\right)^2} \tag{1.0.19}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} ZP_X(z)z^{-k}, \tag{1.0.20}$$

$$nu(n) \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{z^{-1}}{\left(1-z^{-1}\right)^2} \tag{1.0.21}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{1}{36}\frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{\left(1-z^{-1}\right)^2} \tag{1.0.22}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{1.0.23}$$

From (1.0.13), (1.0.19) and (1.0.22)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{1.0.24}$$

which is the same as (1.0.12). Note that (1.0.12) can be obtained from (1.0.22) using contour integration as well.

4) The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 0. The theoretical pmf obtained in (1.0.12) is plotted for comparison.

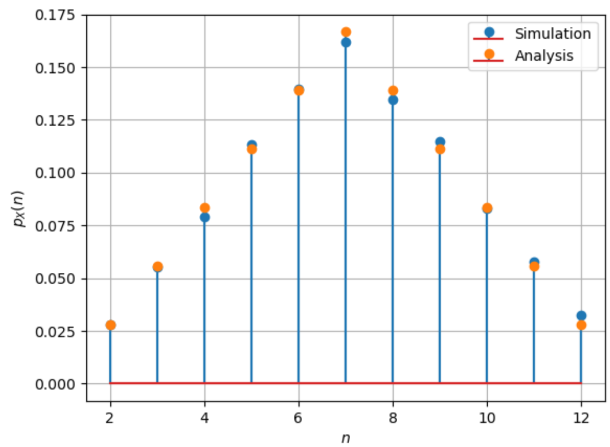

Fig. 0: Plot of $p_X(n)$. Simulations are close to the analysis.

5) The python code is available in

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/codes/ch1.py
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