DIGITAL COMMUNICATION

Through Simulations

By- Sampath Govardhan

Contents

Introduction

This book introduces digital communication through probability.

Chapter 1

Sum of Independent Random Variables

1.1. Two dice

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

1. The Uniform Distribution: Let $X_i \in \{1,2,3,4,5,6\}, i = 1,2,$ be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr\left(X_i = n\right) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{1.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{1.2}$$

$$\implies X \in \{1, 2, \ldots, 12\} \tag{1.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{1.4}$$

2. Convolution: From (1.2),

$$p_X(n) = \Pr\left(X_1 + X_2 = n\right) = \Pr\left(X_1 = n - X_2\right) \tag{1.5}$$

$$= \sum_k \Pr\left(X_1 = n - k | X_2 = k\right) p_{X_2}(k) \tag{1.6}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr\left(X_1 = n - k | X_2 = k\right)$$
$$= \Pr\left(X_1 = n - k\right) = p_{X_1}(n - k) \tag{1.7}$$

From (1.6) and (1.7),

$$p_X(n) = \sum_k p_{X_1}(n - k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{1.8}$$

where $*$ denotes the convolution operation.

Substituting from (1.1) in (1.8),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{1.9}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{1.10}$$

From (1.9),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{1.11}$$

Substituting from (1.1) in (1.11),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{1.12}$$

satisfying (1.4).

3. <u>The Z-transform:</u> The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{1.13}$$

From (1.1) and (1.13),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{1.14}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{1.15}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{1.16}$$

$$P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{1.17}$$

The above property follows from Fourier analysis and is fundamental to signal processing.

From (1.15) and (1.17),

$$P_X(z) = \left\{\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right\}^2 \tag{1.18}$$

$$= \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{1.19}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{1.20}$$

$$nu(n) \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1-z^{-1}\right)^2} \tag{1.21}$$

after some algebra, it can be shown that

$$\begin{aligned} &\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7)\right. \\ &\qquad\qquad \left. + (n-13)\,u(n-13)\right] \\ &\qquad\qquad\qquad \overset{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36} \frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{\left(1-z^{-1}\right)^2} \end{aligned} \tag{1.22}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{1.23}$$

From (1.13), (1.19) and (1.22)

$$\begin{aligned} p_X(n) = &\frac{1}{36}\left[(n-1)\,u(n-1)\right. \\ &\left. -2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \end{aligned} \tag{1.24}$$

which is the same as (1.12). Note that (1.12) can be obtained from (1.22) using contour integration as well.

4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 1.1. The theoretical pmf obtained in (1.12) is plotted for comparison.

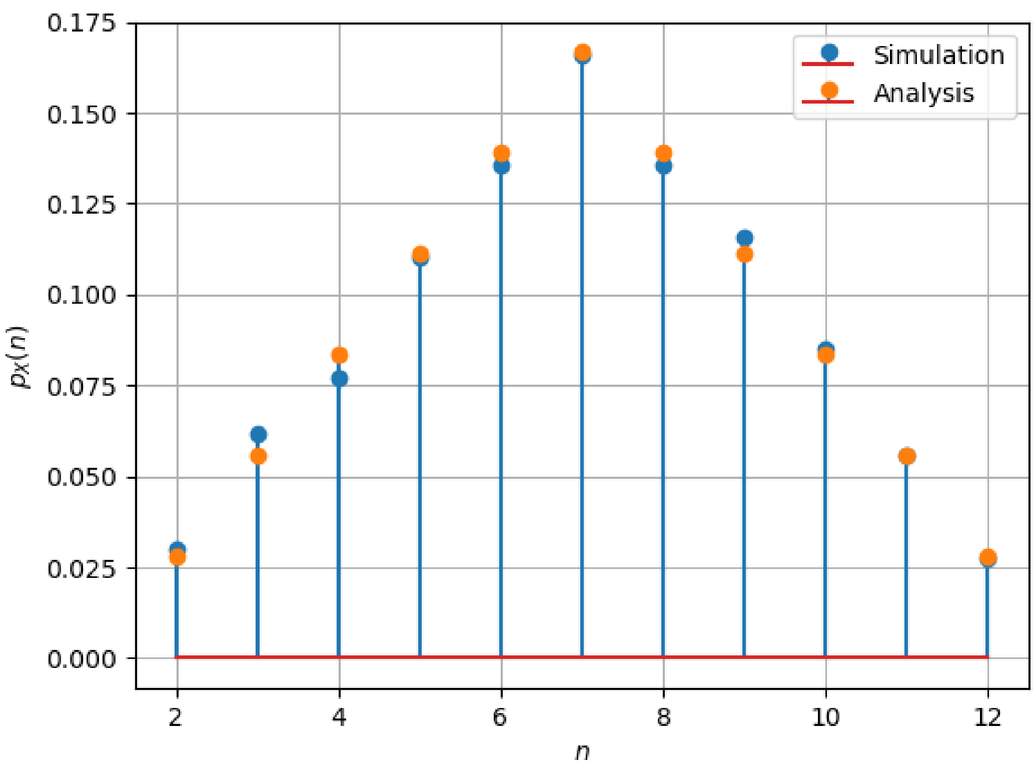

Figure 1.1: Plot of $p_X(n)$. Simulations are close to the analysis.

5. The python code is available in

```
/codes/ch1.py
```

Chapter 2

Random Number

2.1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1. Generate 10^6 samples of U using a C program and save into a file called uni.dat

   .

   Solution: Download the following files and execute the C program.

   ```
   codes/include/coeffs.h
   codes/src/uni_gen_stat.c
   ```

2. Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{2.1}$$

   Solution: The following code plots Fig. 2.1

codes/src/2.1.py

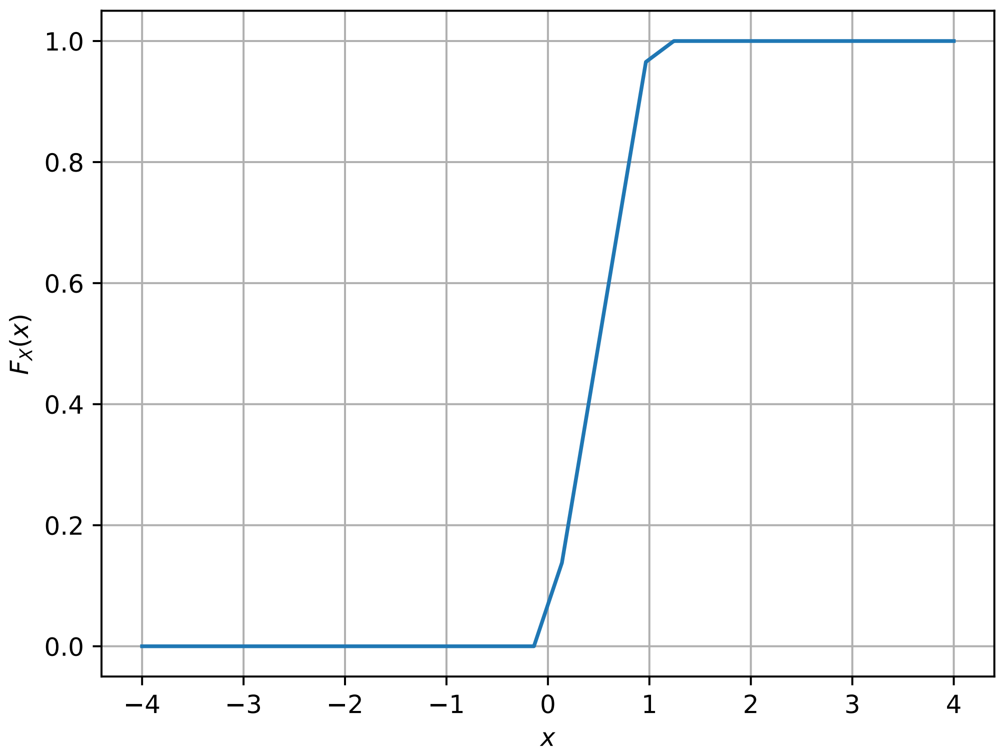

Figure 2.1: The CDF of U

3. Find a theoretical expression for $F_U(x)$.

   Solution:

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\, dx \tag{2.2}$$

For the uniform random variable U, $f_U(x)$ is given by

$$f_U(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & elsewhere \end{cases} \tag{2.3}$$

Substituting (2.3) in (2.2), $F_U(x)$ is found to be

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 0 \end{cases} \tag{2.4}$$

4. The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{2.5}$$

and its variance as

$$\operatorname{var}[U] = E[U - E[U]]^2 \tag{2.6}$$

Write a C program to find the mean and variance of U.

Solution: The following code prints the mean and variance of U

```
codes/src/uni_gen_stat.c
```

5. Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{2.7}$$

Solution: For a random variable X, the mean μ_X and variance σ_X^2 are given by

$$\mu_X = E\left[X\right] = \int_{-\infty}^{\infty} x dF_U(x) \tag{2.8}$$

$$\sigma_X^2 = E\left[X^2\right] - \mu_X^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \mu_X^2 \tag{2.9}$$

Substituting the CDF of U from (2.4) in (2.8) and (2.9), we get

$$\mu_U = \frac{1}{2} \tag{2.10}$$

$$\sigma_U^2 = \frac{1}{12} \tag{2.11}$$

which match with the values printed in problem 4

2.2. Central Limit Theorem

1. Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.12}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
codes/include/coeffs.h
codes/src/gau_gen_stat.c
```

2. Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

   Solution: The CDF of X is plotted in Fig. 2.2

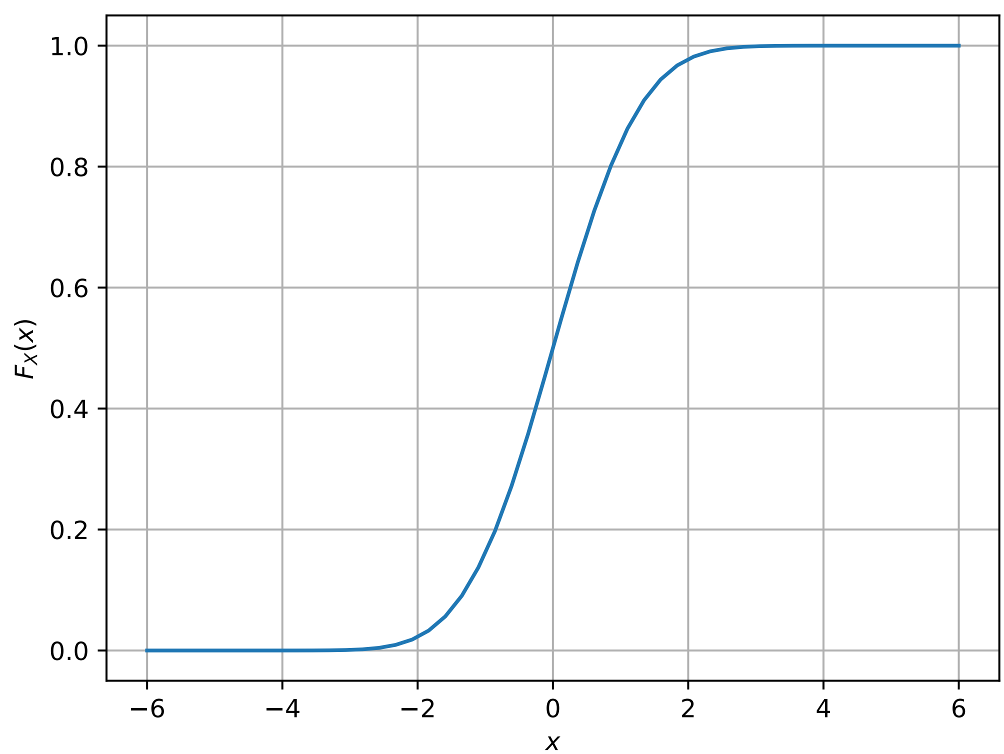

Figure 2.2: The CDF of X

The properties of a CDF are

$$F_X(-\infty) = 0 \tag{2.13}$$

$$F_X(\infty) = 1 \tag{2.14}$$

$$\frac{dF_X(x)}{dx} \geq 0 \tag{2.15}$$

3. Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.16}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
codes/src/2.2.py
```

The properties of PDF are

$$f_X(x) \geq 0 \tag{2.17}$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \tag{2.18}$$

4. Find the mean and variance of X by writing a C program. **Solution:** The following code prints the mean and variance of X

```
codes/src/gau_gen_stat.c
```

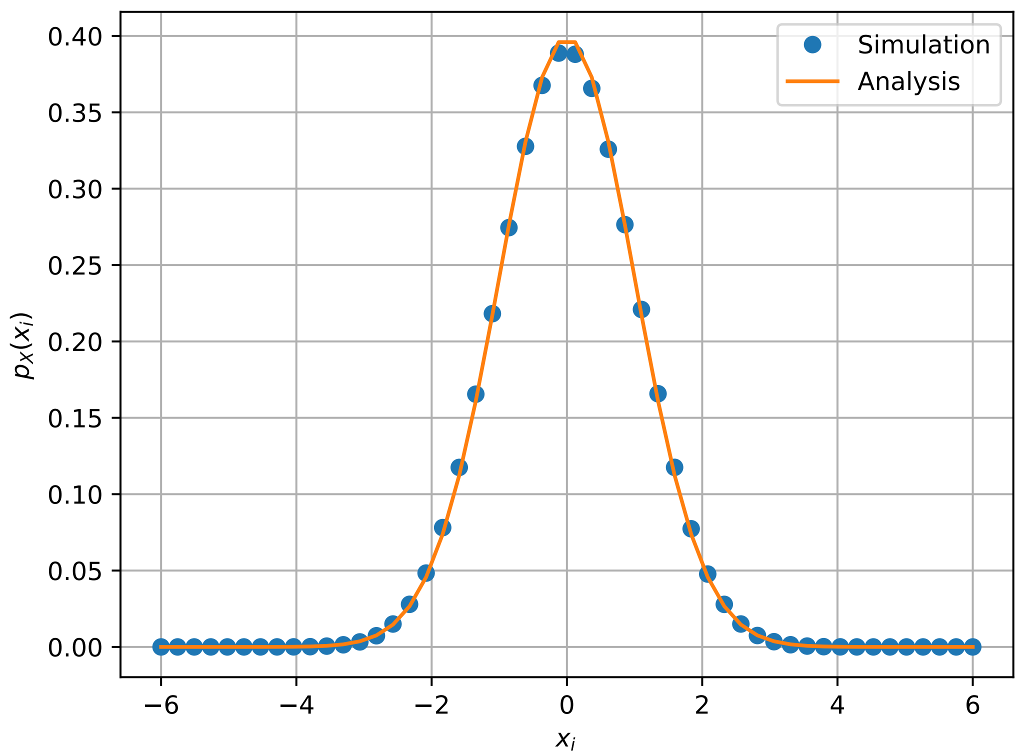

Figure 2.3: The PDF of X

5. Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.19}$$

repeat the above exercise theoretically.

Solution: Substituting the PDF from (2.19) in (2.8),

$$\mu_X = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\, dx \tag{2.20}$$

Using

$$\tag{2.21}$$

$$\int x \cdot \exp\left(-ax^2\right) \mathrm{d}x = -\frac{1}{2a} \cdot \exp\left(-ax^2\right) \tag{2.22}$$

$$\mu_X = \frac{1}{\sqrt{2\pi}} \left[-\exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{\infty} \tag{2.23}$$

$$\mu_X = 0 \tag{2.24}$$

Substituting μ_X and the PDF in (2.9) to compute variance,

$$\sigma_X^2 = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.25}$$

Substituting

$$t = \frac{x^2}{2}, \tag{2.26}$$

$$\sigma_X^2 = \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}} \exp\left(-t\right) \, dt \tag{2.27}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{3}{2}-1} \exp\left(-t\right) \, dt$$

Using the gamma function

$$\Gamma(x) = \int_0^\infty z^{x-1} \cdot e^{-z} \, \mathrm{d}z \tag{2.28}$$

$$\begin{aligned}\sigma_X^2 &= \frac{2}{\sqrt{\pi}} \Gamma(\frac{3}{2}) \qquad (2.29)\\ &= \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \\ &= 1\end{aligned}$$

2.3. From Uniform to Other

1. Generate samples of

$$V = -2 \ln (1 - U) \tag{2.30}$$

and plot its CDF.

Solution: The samples for U are loaded from uni.dat file generated in problem 4. The CDF of V is plotted in Fig. 2.4 using the code below,

```
codes/src/2.3.py
```

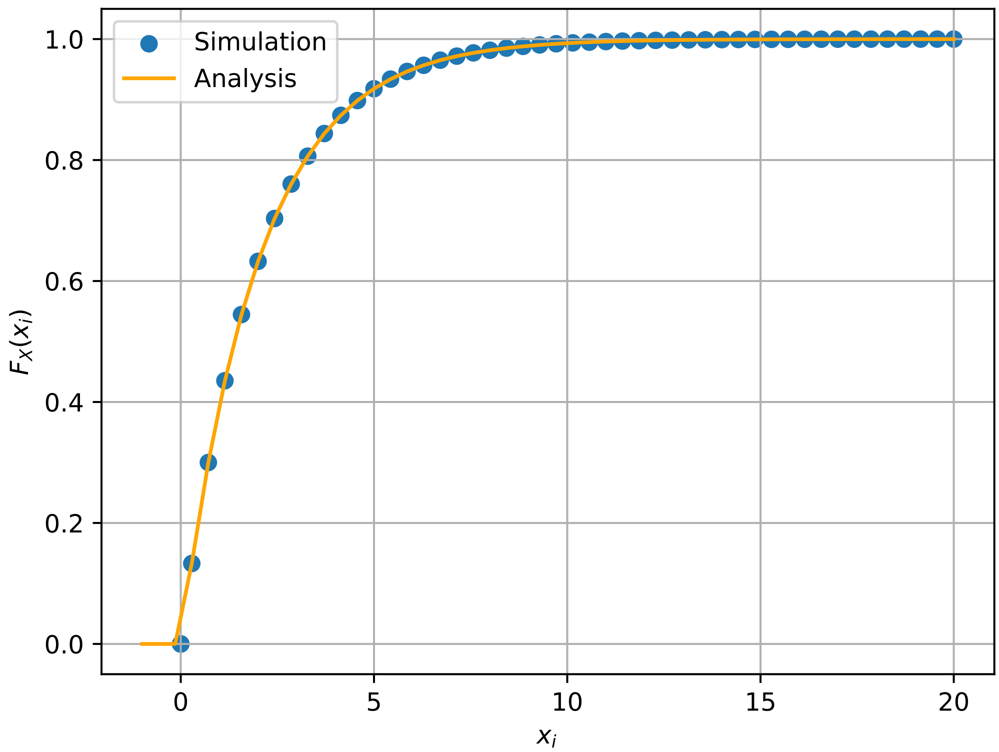

Figure 2.4: The CDF of V

2. Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V < x) \tag{2.31}$$

$$= P(-2\ln(1-U) < x) \tag{2.32}$$

$$= P(U < 1 - e^{\frac{-x}{2}}) \tag{2.33}$$

$$= F_U(1 - e^{\frac{-x}{2}}) \tag{2.34}$$

Using $F_U(x)$ defined in (2.4),

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{\frac{-x}{2}} & x \geq 0 \end{cases} \tag{2.35}$$

2.4. Triangular Distribution

1. Generate

$$T = U_1 + U_2 \tag{2.36}$$

   Solution: Download the following files and execute the C program.

```
codes/include/coeffs.h
codes/src/two_uni_gen.c
```

2. Find the CDF of T.

   Solution: Loading the samples from uni1.dat and uni2.dat in python, the CDF is plotted in Fig. 2.5

3. Find the PDF of T.

   Solution: The PDF of T is plotted in Fig. 2.6 using the code below

```
codes/src/2.4.py
```

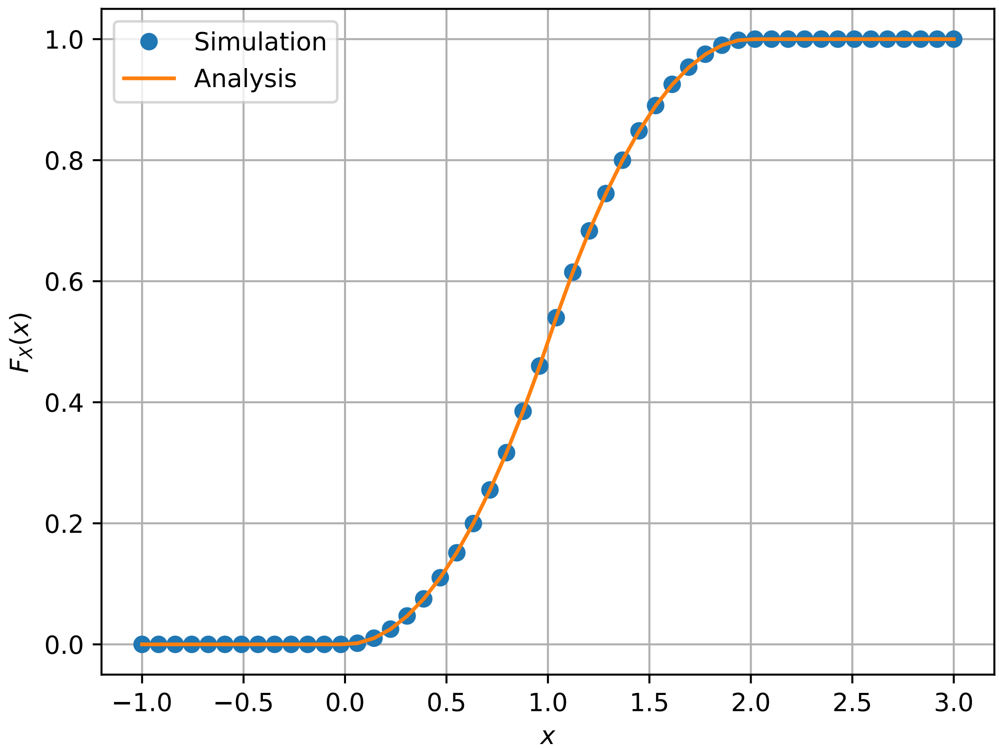

Figure 2.5: The CDF of T

4. Find the theoretical expressions for the PDF and CDF of T.

   Solution: Since T is the sum of two independant random variables $U1$ and $U2$, the PDF of T is given by

$$p_T(x) = p_{U1}(x) * p_{U2}(x) \tag{2.37}$$

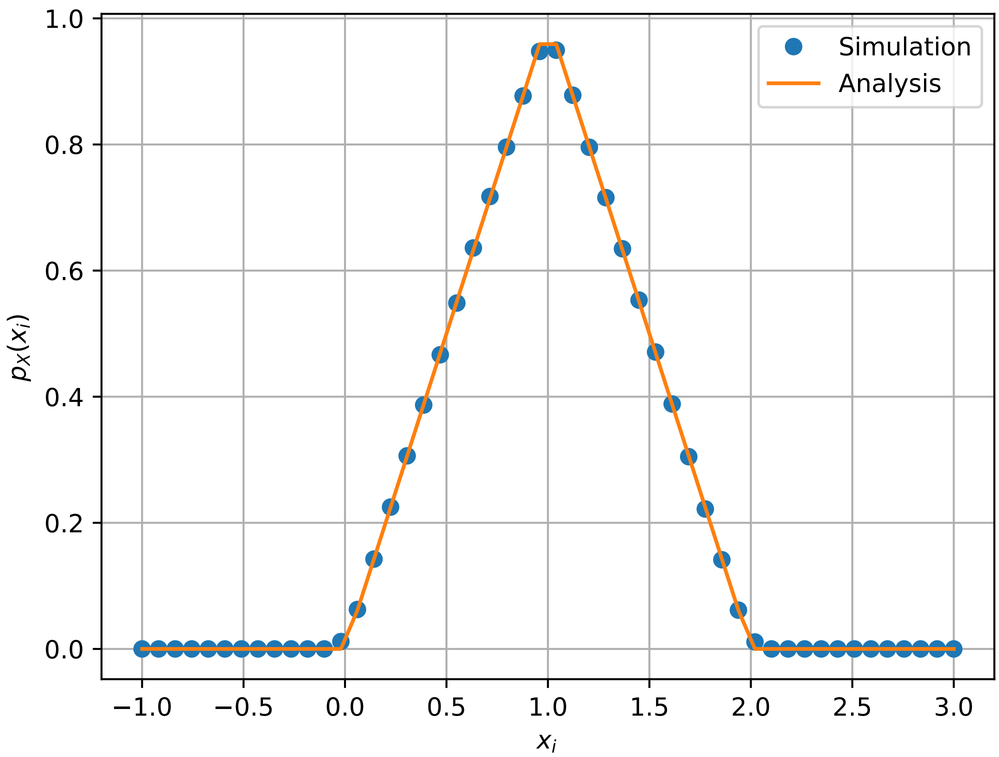

Figure 2.6: The PDF of T

Using the PDF of U from (2.3), the convolution results in

$$p_T(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & x > 2 \end{cases} \tag{2.38}$$

The CDF of T is found using (2.2) by replacing U with T. Evaluating the integral for the piecewise function $p_T(x)$,

$$F_T(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \le x \le 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \le x \le 2 \\ 1 & x > 2 \end{cases} \tag{2.39}$$

5. Verify your results through a plot.

   Solution: The theoretical and numerical plots for the CDF and PDF of T closely match in Fig. 2.5 and Fig. 2.6

Chapter 3

Maximum Likelihood Detection: BPSK

3.1. Maximum Likelihood

1. Generate equiprobable $X \in \{1, -1\}$.

2. Generate
$$Y = AX + N, \tag{3.1}$$
where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

3. Plot Y using a scatter plot.

Solution: X, Y and the scatter plot of Y ($Fig.$ 3.1) can be generated using the below code, codes/ch3_scatter.py

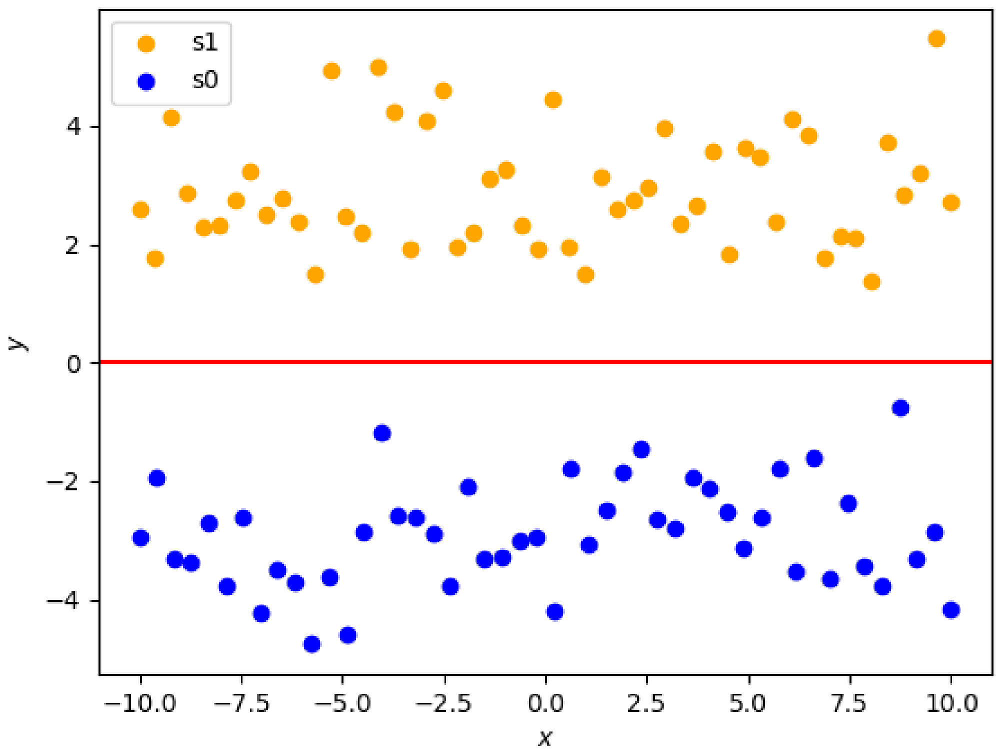

Figure 3.1: Scatter plot of Y

4. Guess how to estimate X from Y.

   Solution:

$$y \underset{-1}{\overset{1}{\gtrless}} 0 \tag{3.2}$$

5. Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{3.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{3.4}$$

Solution: Based on the decision rule in (3.2),

$$\begin{aligned}
\Pr\left(\hat{X} = -1 | X = 1\right) &= \Pr\left(Y < 0 | X = 1\right) \\
&= \Pr\left(AX + N < 0 | X = 1\right) \\
&= \Pr\left(A + N < 0\right) \\
&= \Pr\left(N < -A\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Pr\left(\hat{X} = 1 | X = -1\right) &= \Pr\left(Y > 0 | X = -1\right) \\
&= \Pr\left(N > A\right)
\end{aligned}$$

Since $N \sim \mathcal{N}\left(0, 1\right)$,

$$\Pr\left(N < -A\right) = \Pr\left(N > A\right) \tag{3.5}$$

$$\implies P_{e|0} = P_{e|1} = \Pr\left(N > A\right) \tag{3.6}$$

6. Find P_e assuming that X has equiprobable symbols.

Solution:

$$P_e = \Pr\left(X = 1\right) P_{e|1} + \Pr\left(X = -1\right) P_{e|0} \tag{3.7}$$

Since X is equiprobable

$$\tag{3.8}$$

$$P_e = \frac{1}{2} P_{e|1} + \frac{1}{2} P_{e|0} \tag{3.9}$$

Substituting from (3.6)

$$P_e = \Pr\left(N > A\right) \tag{3.10}$$

Given a random varible $X \sim \mathcal{N}\left(0, 1\right)$ the Q-function is defined as

$$Q(x) = \Pr\left(X > x\right) \tag{3.11}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du. \tag{3.12}$$

$$\tag{3.13}$$

Using the Q-function, P_e is rewritten as

$$P_e = Q(A) \tag{3.14}$$

7. Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

   Solution: The theoretical P_e is plotted in $Fig.$ 3.2, along with numerical estimations from generated samples of Y. The below code is used for the plot,

```
codes/chapter3/ch3_varyA.py
```

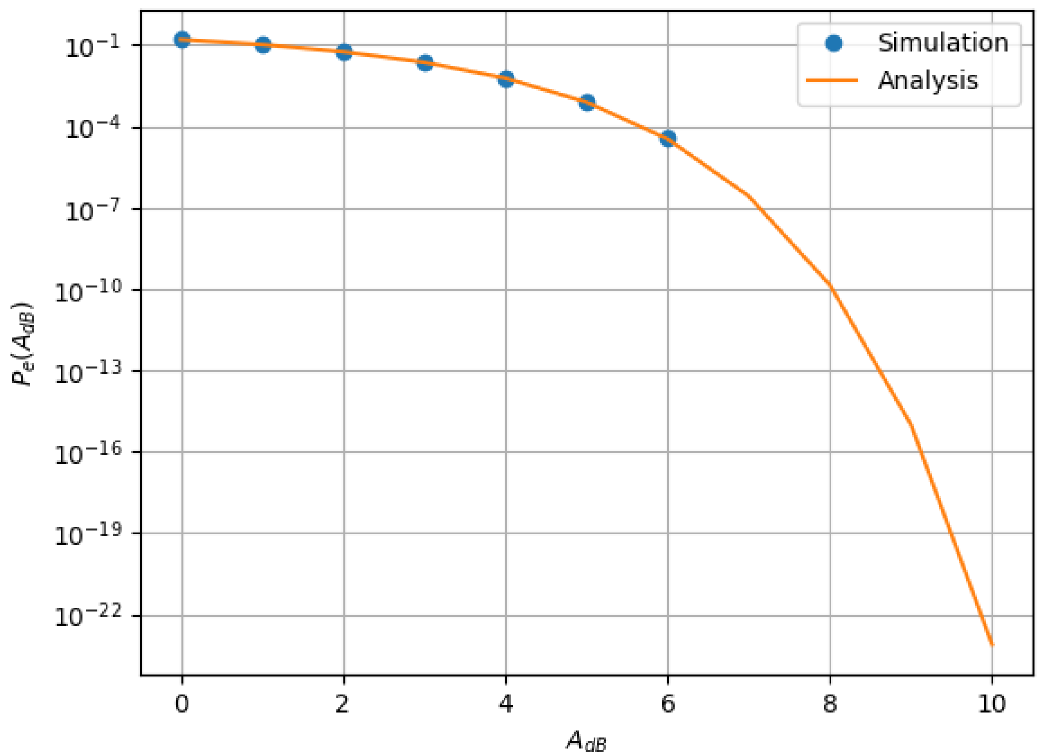

Figure 3.2: P_e versus A plot

8. Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

Solution: Given the decision rule,

$$y \underset{-1}{\overset{1}{\gtrless}} \delta \tag{3.15}$$

$$
\begin{aligned}
P_{e|0} &= \Pr\left(\hat{X} = -1 | X = 1\right) \\
&= \Pr\left(Y < \delta | X = 1\right) \\
&= \Pr\left(AX + N < \delta | X = 1\right) \\
&= \Pr\left(A + N < \delta\right) \\
&= \Pr\left(N < -A + \delta\right) \\
&= \Pr\left(N > A - \delta\right) \\
&= Q(A - \delta)
\end{aligned}
$$

$$
\begin{aligned}
P_{e|1} &= \Pr\left(\hat{X} = 1 | X = -1\right) \\
&= \Pr\left(Y > \delta | X = -1\right) \\
&= \Pr\left(N > A + \delta\right) \\
&= Q(A + \delta)
\end{aligned}
$$

Using (3.9), P_e is given by

$$
P_e = \frac{1}{2}Q(A + \delta) + \frac{1}{2}Q(A - \delta) \tag{3.16}
$$

Using the integral for Q-function from (3.12),

$$
P_e = k(\int_{A+\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du + \int_{A-\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du) \tag{3.17}
$$

where k is a constant

Differentiating (3.17) wrt δ (using Leibniz's rule) and equating to 0, we get

$$\exp\left(-\frac{(A+\delta)^2}{2}\right) - \exp\left(-\frac{(A-\delta)^2}{2}\right) = 0$$

$$\frac{\exp\left(-\frac{(A+\delta)^2}{2}\right)}{\exp\left(-\frac{(A-\delta)^2}{2}\right)} = 1$$

$$\exp\left(-\frac{(A+\delta)^2-(A-\delta)^2}{2}\right) = 1$$

$$\exp\left(-2A\delta\right) = 1$$

Taking ln on both sides

$$-2A\delta = 0$$

$$\implies \delta = 0$$

P_e is maximum for $\delta = 0$

9. Repeat the above exercise when

$$p_X(0) = p \qquad (3.18)$$

Solution: Since X is not equiprobable, P_e is given by,

$$\begin{aligned}P_e &= (1-p)P_{e|1} + pP_{e|0} \\ &= (1-p)Q(A+\delta) + pQ(A-\delta)\end{aligned}$$

Using the integral for Q-function from (3.12),

$$P_e = k((1-p)\int_{A+\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du + p\int_{A-\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du) \quad (3.19)$$

where k is a constant.

Following the same steps as in problem 8, δ for maximum P_e evaluates to,

$$\delta = \frac{1}{2A} \ln\left(\frac{1}{p} - 1\right) \quad (3.20)$$

10. Repeat the above exercise using the MAP criterion.

Solution: The MAP rule can be stated as

$$\text{Set } \hat{x} = x_i \text{ if} \quad (3.21)$$

$p_X(x_k)p_Y(y|x_k)$ is maximum for $k = i$

For the case of BPSK, the point of equality between $p_X(x=1)p_Y(y|x=1)$ and

$p_X(x=-1)p_Y(y|x=-1)$ is the optimum threshold. If this threshold is δ, then

$pp_Y(y|x=1) > (1-p)p_Y(y|x=-1)$ when $y > \delta$

$pp_Y(y|x=1) < (1-p)p_Y(y|x=-1)$ when $y < \delta$

The above inequalities can be visualized in below figure for p = 0.3 and A = 3. Given $Y = AX + N$ where $N \sim \mathcal{N}(0,1)$, the optimum threshold is found as

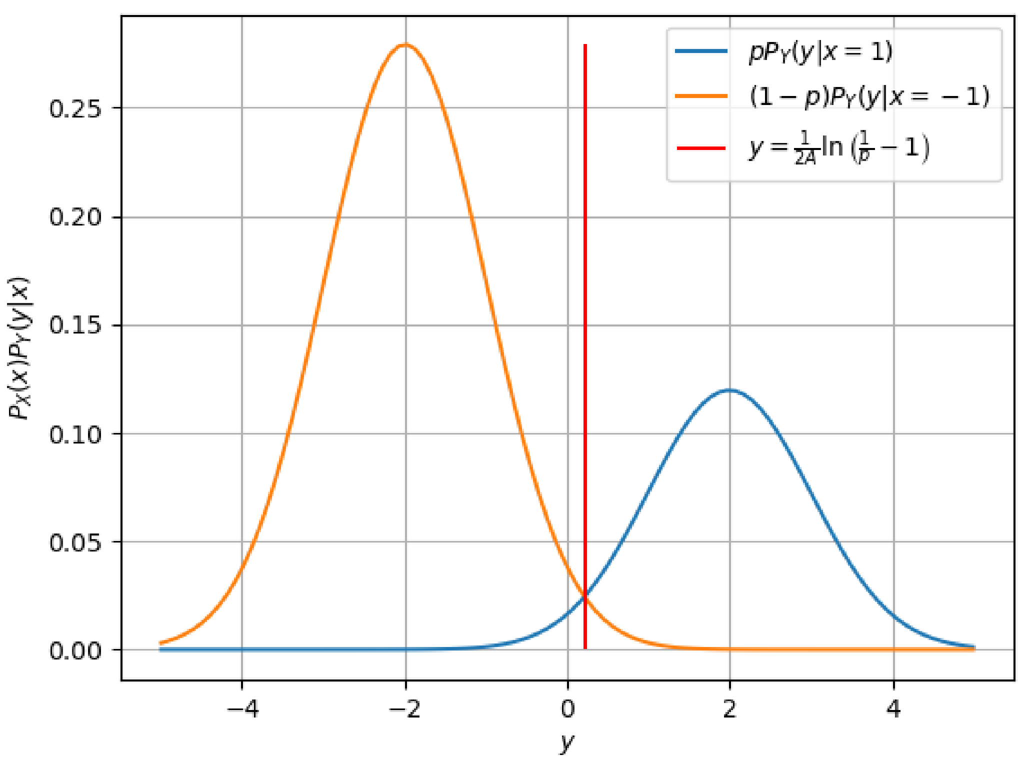

Figure 3.3: $p_X(X=x_i)p_Y(y|x=x_i)$ versus y plot for $X \in \{-1,1\}$

solution to the below equation

$$p \exp\left(-\frac{(y_{eq} - A)^2}{2}\right) = (1-p) \exp\left(-\frac{(y_{eq} + A)^2}{2}\right) \tag{3.22}$$

Solving for y_{eq}, we get

$$y_{eq} = \delta = \frac{1}{2A} \ln\left(\frac{1}{p} - 1\right) \tag{3.23}$$

which is same as δ obtained in problem 9

Chapter 4

Transformation of Random Variables

4.1. Gaussian to Other

1. Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{4.1}$$

   Solution: The CDF and PDF of V are plotted in Fig. 4.1 and Fig. 4.2 respectively using the below code

```
codes/ch4_squares.py
```

2. If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{4.2}$$

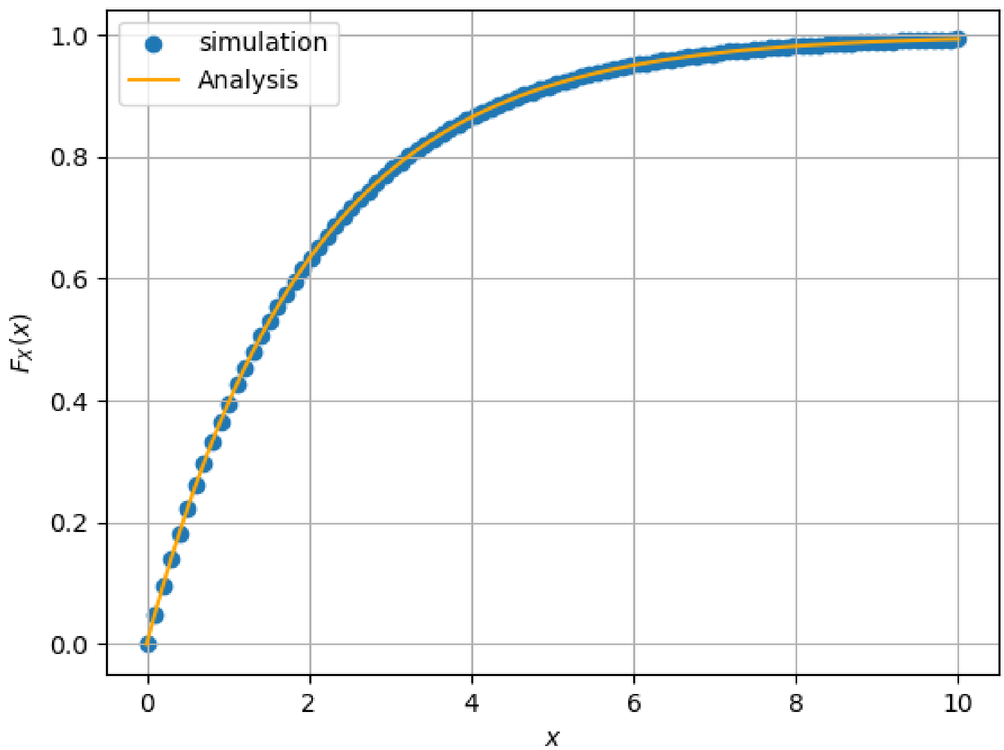

Figure 4.1: CDF of V

find α.

Solution: Let $Z = X^2$ where $X \sim \mathcal{N}(0,1)$. Defining the CDF for Z,

$$\begin{aligned}
P_Z(z) &= \Pr(Z < z) \\
&= \Pr\left(X^2 < z\right) \\
&= \Pr\left(-\sqrt{z} < X < \sqrt{z}\right) \\
&= \int_{-\sqrt{z}}^{\sqrt{z}} p_X(x)\, dx
\end{aligned}$$

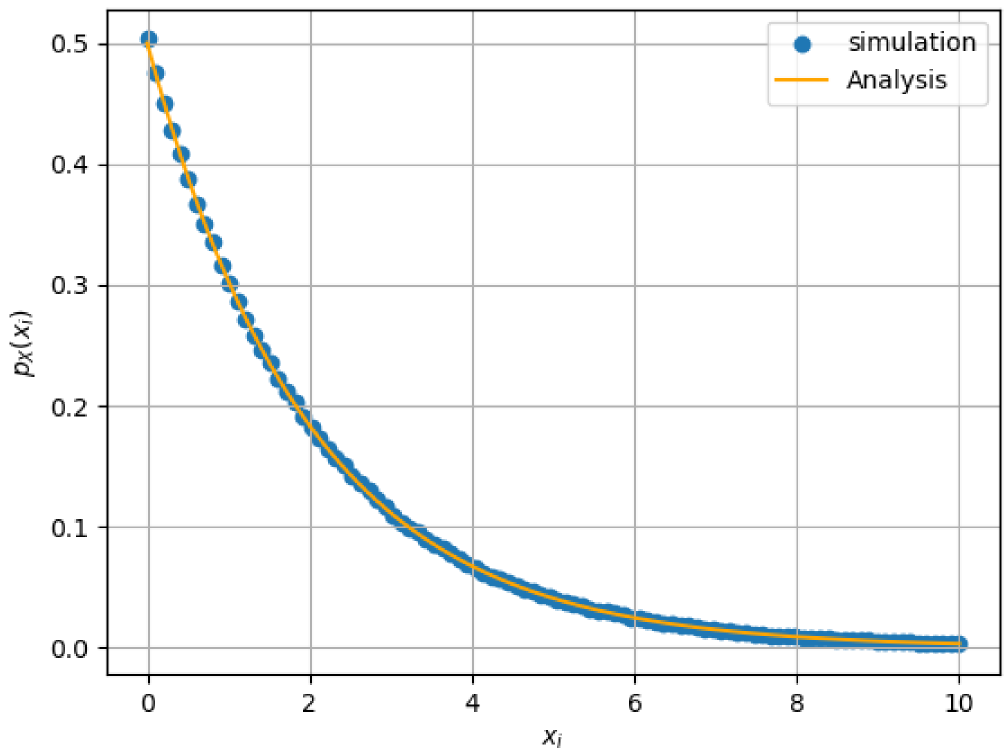

Figure 4.2: PDF of V

Using (2.16), the PDF of Z is given by

$$\frac{d}{dz}P_Z(z) = p_Z(z)$$
$$= \frac{p_X(\sqrt{z}) + p_X(-\sqrt{z})}{2\sqrt{z}} \text{ (Using Lebniz's rule)} \tag{4.3}$$

Substituting the standard gaussian density function $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ in (4.3),

$$p_Z(z) = \begin{cases} \frac{1}{\sqrt{2\pi z}} e^{-\frac{z}{2}} & z \geq 0 \\ 0 & z < 0 \end{cases} \tag{4.4}$$

The PDF of X_1^2 and X_2^2 are given by (4.4). Since V is the sum of two independant random variables,

$$\begin{aligned} p_V(v) &= p_{X_1^2}(x_1) * p_{X_2^2}(x_2) \\ &= \frac{1}{2\pi} \int_0^v \frac{e^{-\frac{x}{2}}}{\sqrt{x}} \frac{e^{-\frac{v-x}{2}}}{\sqrt{v-x}} \, dx \\ &= \frac{e^{-\frac{v}{2}}}{2\pi} \int_0^v \frac{1}{\sqrt{x(v-x)}} \, dx \\ &= \frac{e^{-\frac{v}{2}}}{2\pi} \left[-\arcsin \left(\frac{v-2x}{v} \right) \right]_0^v \\ &= \frac{e^{-\frac{v}{2}}}{2\pi} \pi \\ &= \frac{e^{-\frac{v}{2}}}{2} \text{ for } v \geq 0 \end{aligned}$$

$F_V(v)$ can be obtained from $p_V(v)$ using (2.2)

$$\begin{aligned} F_V(v) &= \frac{1}{2} \int_0^v \exp\left(-\frac{v}{2}\right) \\ &= 1 - \exp\left(-\frac{v}{2}\right) \text{ for } v \geq 0 \end{aligned} \tag{4.5}$$

Comparing (4.5) with (4.2), $\alpha = \frac{1}{2}$

3. Plot the CDF and PDF of

$$A = \sqrt{V} \tag{4.6}$$

Solution: The CDF and PDF of A are plotted in

$$Fig.\ 4.3$$

and

$$Fig.\ 4.4$$

respectively using the below code

```
codes/ch4_sqrt.py
```

The CDF of A is given by,

$$F_A(a) = \Pr(A < a) \tag{4.7}$$

$$= \Pr\left(\sqrt{V} < a\right) \tag{4.8}$$

$$= \Pr\left(V < a^2\right) \tag{4.9}$$

$$= F_V\left(a^2\right) \tag{4.10}$$

$$= 1 - \exp\left(-\frac{a^2}{2}\right) \tag{4.11}$$

Using (2.16), the PDF is found to be

$$p_A(a) = a\exp\left(-\frac{a^2}{2}\right) \tag{4.12}$$

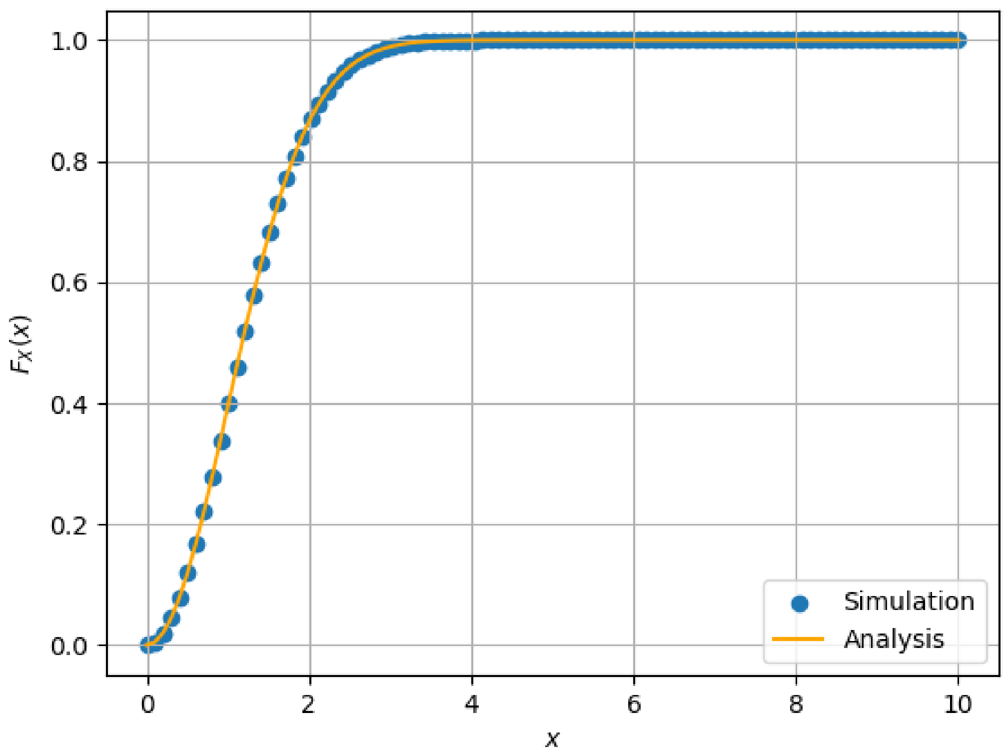

Figure 4.3: CDF of A

4.2. Conditional Probability

1. Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{4.13}$$

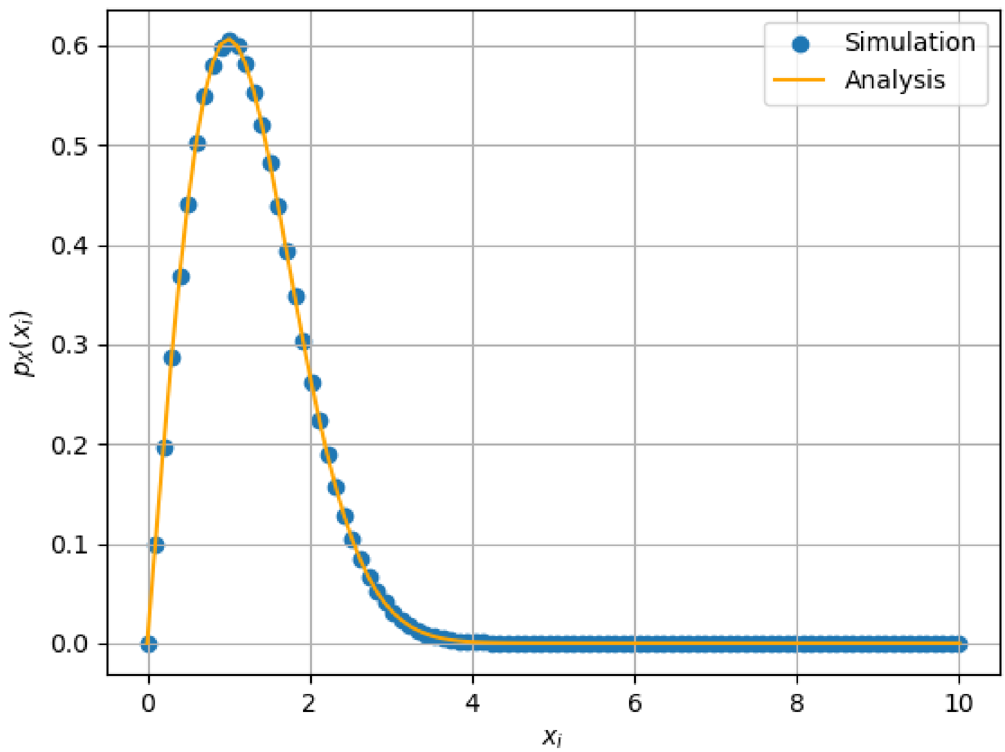

Figure 4.4: PDF of A

for

$$Y = AX + N, \tag{4.14}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}\left(0, 1\right), X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

Solution: The blue dots in Fig. 4.5 is the required plot. The below code is used to generate the plot,

```
codes/ch4/ch4_err.py
```

2. Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

   Solution: Assuming the decision rule in (3.2), when N is constant, P_e is given by

$$\begin{aligned}
P_e &= \Pr\left(\hat{X} = -1 | X = 1\right) \\
&= \Pr\left(Y < 0 | X = 1\right) \\
&= \Pr\left(AX + N < 0 | X = 1\right) \\
&= \Pr\left(A + N < 0\right) \qquad (4.15) \\
&= \Pr\left(A < -N\right) \\
&= \begin{cases} F_A(-N) & N \geq 0 \\ 0 & N < 0 \end{cases} \qquad (4.16)
\end{aligned}$$

   For a Rayleigh random variable X with $E\left[X^2\right] = \gamma$, the PDF and CDF are given by

$$p_X(x) = \frac{2x}{\gamma} \exp\left(-\frac{x^2}{\gamma}\right) \text{ for } x \geq 0 \qquad (4.17)$$

$$F_X(X) = 1 - \exp\left(-\frac{x^2}{\gamma}\right) \text{ for } x \geq 0 \qquad (4.18)$$

   Substituting (4.18) in (4.16),

$$P_e(N) = \begin{cases} 1 - \exp\left(-\frac{N^2}{\gamma}\right) & N \geq 0 \\ 0 & N < 0 \end{cases} \qquad (4.19)$$

3. For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)\, dx \tag{4.20}$$

Find $P_e = E[P_e(N)]$.

Solution: Using $P_e(N)$ from (4.19),

$$\begin{aligned} P_e &= \int_{-\infty}^{\infty} P_e(x)p_N(x)\, dx \\ &= \int_0^{\infty} \left(1 - e^{-\frac{x^2}{\gamma}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \end{aligned}$$

$$P_e = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{x^2}{2}}\, dx \\ - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \exp\left(-x^2\left(\frac{1}{\gamma} + \frac{1}{2}\right)\right) dx$$

$$P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}}$$

4. Plot P_e in problems 1 and 3 on the same graph w.r.t γ. Comment.

Solution: P_e plotted in same graph in Fig. 4.5. The value of P_e is much higher when the channel gain A is Rayleigh distributed than the case where A is a constant (compare with Fig. 3.2).

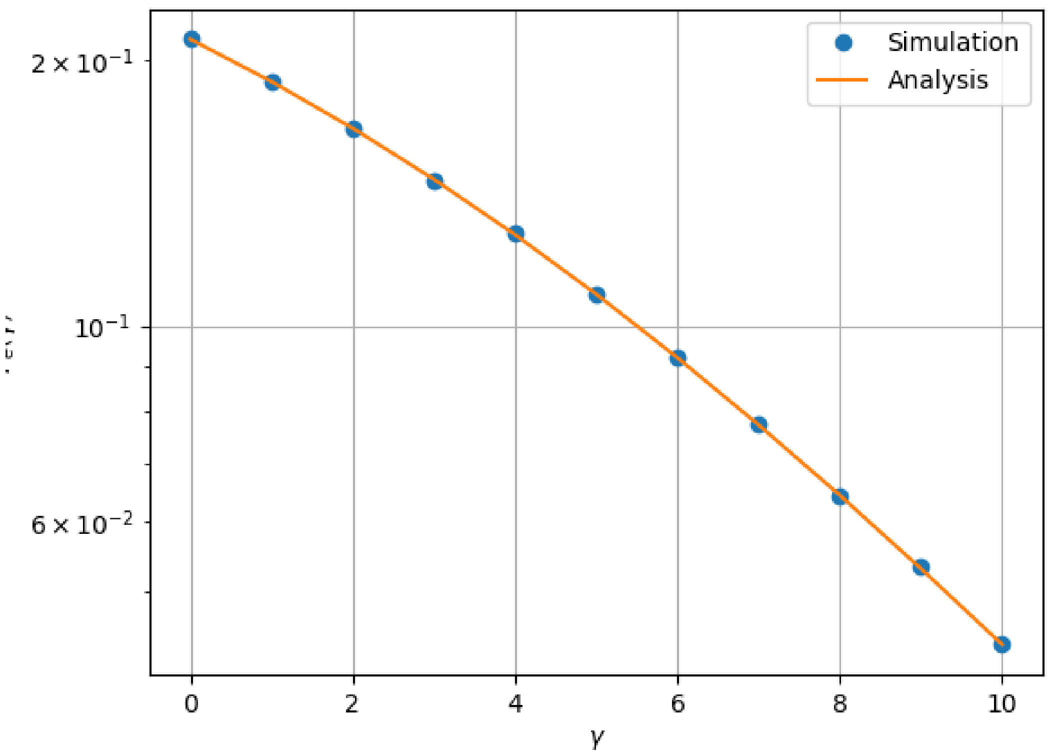

Figure 4.5: P_e versus γ

From (4.15), P_e is given by

$$P_e = \Pr\left(A + N < 0\right) \tag{4.21}$$

One method of computing (4.15) is by finding the PDF of $Z = A + N$ (as the convolution of the individual PDFs of A and N) and then integrating $p_Z(z)$ from $-\infty$ to 0. The other method is by first computing P_e for constant N and then finding the expectation of $P_e(N)$. Both provide the same result but the computation of integrals is simpler when using the latter method.

Chapter 5

Bivariate Random Variables: FSK

5.1. Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{5.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{5.3}$$

1. Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{5.4}$$

on the same graph using a scatter plot.

Solution: The scatter plot in Fig. 5.1 is generated using the below code, codes/ch5_scatter.py

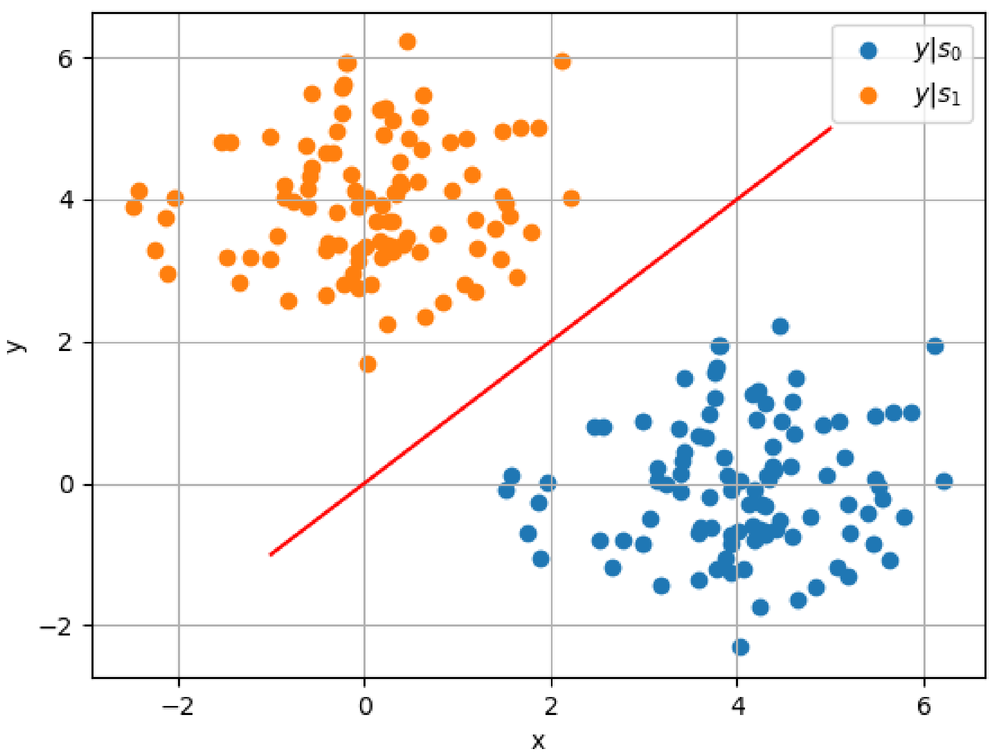

Figure 5.1: Scatter plot of $\mathbf{y}|\mathbf{s}_0$ and $\mathbf{y}|\mathbf{s}_1$

2. For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

   Solution: Let $\mathbf{y} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}^T$. Then the decision rule is

$$y_1 \underset{1}{\overset{0}{\gtrless}} y_2 \tag{5.5}$$

$\mathbf{y}|\mathbf{s}_i$ is a random vector with each of its components normally distributed. The

PDF of $\mathbf{y}|\mathbf{s}_i$ is given by,

$$p_{\mathbf{y}|\mathbf{s}_i}(\mathbf{y}) = \frac{1}{2\pi\sqrt{\left|\boldsymbol{\Sigma}\right|}} \exp\left(-\frac{1}{2}(\mathbf{y}-\mathbf{s}_i)^\top \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\mathbf{s}_i)\right) \tag{5.6}$$

Where $\boldsymbol{\Sigma}$ is the covariance matrix. Substituting $\boldsymbol{\Sigma} = \sigma\mathbf{I}$,

$$p_{\mathbf{y}|\mathbf{s}_i}(\mathbf{y}) = \frac{1}{2\pi\sigma} \exp\left(-\frac{1}{2\sigma}(\mathbf{y}-\mathbf{s}_i)^\top \mathbf{I}(\mathbf{y}-\mathbf{s}_i)\right) \tag{5.7}$$

$$= \frac{1}{2\pi\sigma} \exp\left(-\frac{1}{2\sigma}(\mathbf{y}-\mathbf{s}_i)^\top (\mathbf{y}-\mathbf{s}_i)\right) \tag{5.8}$$

Assuming equiprobable symbols, use MAP rule in (3.21) to find optimum decision. Since there are only two possible symbols $\mathbf{s}_0$ and $\mathbf{s}_1$, the optimal decision criterion is found by equating $p_{\mathbf{y}|\mathbf{s}_0}$ and $p_{\mathbf{y}|\mathbf{s}_1}$.

$$p_{\mathbf{y}|\mathbf{s}_0} = p_{\mathbf{y}|\mathbf{s}_1}$$

$$\implies \exp\left(-\frac{1}{2\sigma}(\mathbf{y}-\mathbf{s}_0)^\top (\mathbf{y}-\mathbf{s}_0)\right) = \exp\left(-\frac{1}{2\sigma}(\mathbf{y}-\mathbf{s}_1)^\top (\mathbf{y}-\mathbf{s}_1)\right)$$

$$\implies (\mathbf{y}-\mathbf{s}_0)^\top (\mathbf{y}-\mathbf{s}_0) = (\mathbf{y}-\mathbf{s}_1)^\top (\mathbf{y}-\mathbf{s}_1)$$

$$\implies \mathbf{y}^\top\mathbf{y} - 2\mathbf{s}_0^\top\mathbf{y} + \mathbf{s}_0^T\mathbf{s}_0 = \mathbf{y}^\top\mathbf{y} - 2\mathbf{s}_1^\top\mathbf{y} + \mathbf{s}_1^T\mathbf{s}_1$$

$$\implies 2(\mathbf{s}_1-\mathbf{s}_0)^\top \mathbf{y} = \|\mathbf{s}_1\|^2 - \|\mathbf{s}_0\|^2$$

$$\implies (\mathbf{s}_1-\mathbf{s}_0)^\top \mathbf{y} = 0$$

$$\implies \begin{pmatrix} -1 \\ 1 \end{pmatrix}^\top \mathbf{y} = 0$$

3. Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{5.9}$$

with respect to the SNR from 0 to 10 dB.

Solution: The blue dots in Fig. 5.2 are the P_e versus SNR plot. It is generated using the below code,

```
codes/ch5_snr.py
```

4. Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

Solution: Using the decision rule from (5.5),

$$
\begin{aligned}
P_e &= \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \\
&= \Pr\left(y_1 < y_2 | \mathbf{x} = \mathbf{s}_0\right) \\
&= \Pr\left(A + n_1 < n_2\right) \\
&= \Pr\left(n_1 - n_2 < -A\right) \qquad (5.10)
\end{aligned}
$$

Let $Z = n_1 - n_2$ where $n_1, n_2 \sim \mathcal{N}\left(0, \sigma^2\right)$. The PDF of X is given by,

$$
\begin{aligned}
p_Z(z) &= p_{n_1}(n_1) * p_{-n_2}(n_2) \\
&= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-\frac{(t-z)^2}{2\sigma^2}}\, dt \\
&= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2+t^2}{2\sigma^2}}\, dt \\
&= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{(2t-z)^2+z^2}{2(\sqrt{2}\sigma)^2}}\, dt \\
&= \frac{1}{2\pi\sigma^2} e^{-\frac{z^2}{2(\sqrt{2}\sigma)^2}} \int_{-\infty}^{\infty} e^{-\frac{(2t-z)^2}{2(\sqrt{2}\sigma)^2}}\, dt \\
&= \frac{e^{-\frac{z^2}{2(\sqrt{2}\sigma)^2}}}{\sqrt{2\pi}\sqrt{2}\sigma} \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2(\sqrt{2}\sigma)^2}}\, dk \\
&= \frac{e^{-\frac{z^2}{2(\sqrt{2}\sigma)^2}}}{\sqrt{2\pi}\sqrt{2}\sigma} \qquad (5.11)
\end{aligned}
$$

From (5.11), $Z \sim \mathcal{N}\left(0, 2\sigma^2\right)$. Substituting $\sigma = 1$, $Z \sim \mathcal{N}\left(0, 2\right)$. (5.10) can be further simplified as,

$$P_e = \Pr\left(Z < -A\right)$$
$$= \Pr\left(Z > A\right)$$
$$= Q\left(\frac{A}{\sqrt{2}}\right)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{A}{\sqrt{2}}}^{\infty} \exp\left(-\frac{x^2}{2}\right)\, dx$$

Fig. 5.2 compares the theoretical and simulation plots.

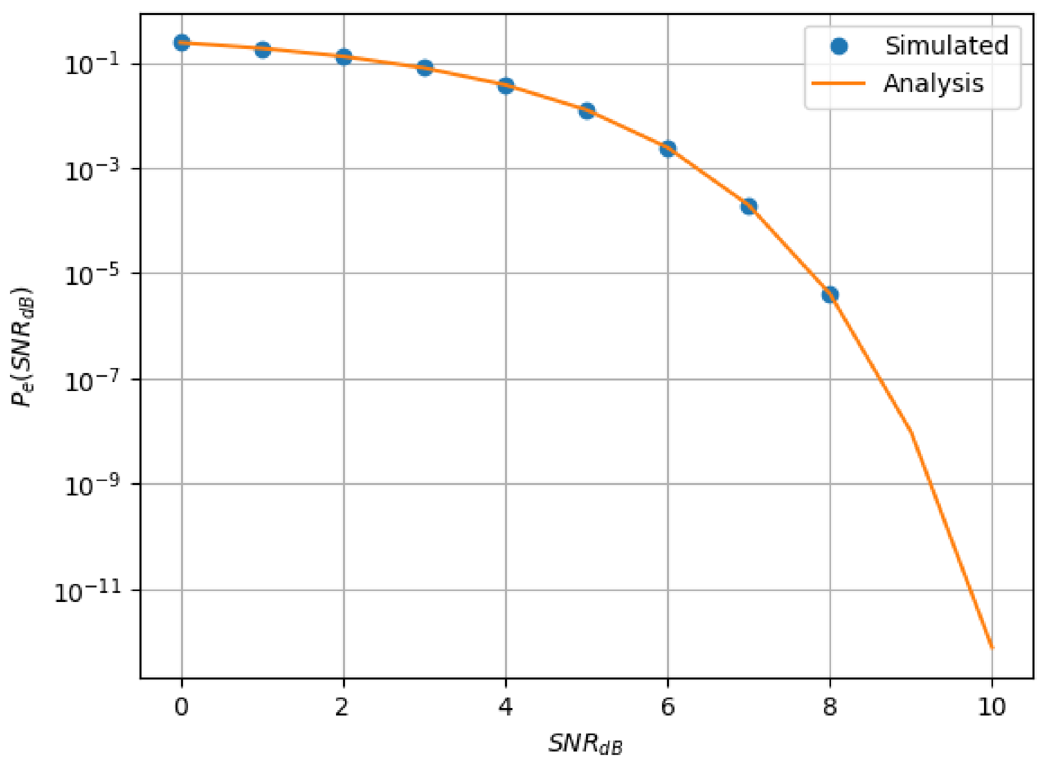

Figure 5.2: P_e versus SNR plot for FSK