Maximum Likelihood Detection:BPSK

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Contents

0.1 Maximum Likelihood

0.1.1 Generate equiprobable $X \in \{1, -1\}$.

0.1.2 Generate

$$Y = AX + N, \tag{0.1.2.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

0.1.3 Plot Y using a scatter plot.

Solution: X, Y and the scatter plot of Y (*Fig.* 0.1.3.1) can be generated using the below code,

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codes/ch3_scatter.py
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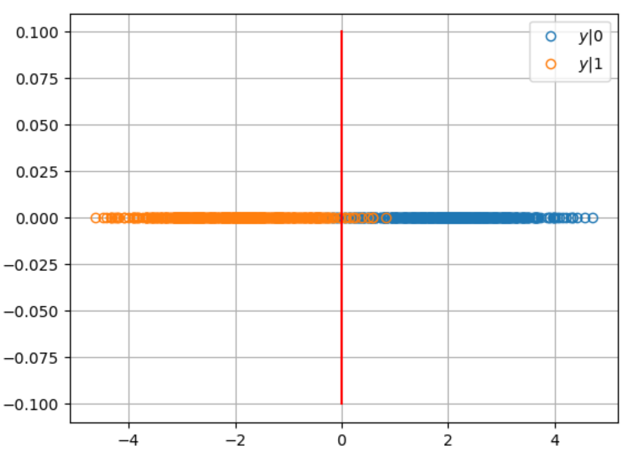

Figure 0.1.3.1: Scatter plot of Y

0.1.4 Guess how to estimate X from Y.
Solution:

$$y \underset{-1}{\overset{1}{\gtrless}} 0 \tag{0.1.4.1}$$

0.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{0.1.5.1}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{0.1.5.2}$$

Solution: Based on the decision rule in (0.1.4.1),

$$\begin{aligned}
\Pr\left(\hat{X} = -1 | X = 1\right) &= \Pr\left(Y < 0 | X = 1\right) \\
&= \Pr\left(AX + N < 0 | X = 1\right) \\
&= \Pr\left(A + N < 0\right) \\
&= \Pr\left(N < -A\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Pr\left(\hat{X} = 1 | X = -1\right) &= \Pr\left(Y > 0 | X = -1\right) \\
&= \Pr\left(N > A\right)
\end{aligned}$$

Since $N \sim \mathcal{N}(0, 1)$,

$$\Pr\left(N < -A\right) = \Pr\left(N > A\right) \tag{0.1.5.3}$$

$$\implies P_{e|0} = P_{e|1} = \Pr\left(N > A\right) \tag{0.1.5.4}$$

0.1.6 Find P_e assuming that X has equiprobable symbols.
Solution:

$$P_e = \Pr\left(X = 1\right) P_{e|1} + \Pr\left(X = -1\right) P_{e|0} \tag{0.1.6.1}$$

Since X is equiprobable

(0.1.6.2)

$$P_e = \frac{1}{2} P_{e|1} + \frac{1}{2} P_{e|0} \tag{0.1.6.3}$$

Substituting from (0.1.5.4)

$$P_e = \Pr\left(N > A\right) \tag{0.1.6.4}$$

Given a random varible $X \sim \mathcal{N}(0, 1)$ the Q-function is defined as

$$Q(x) = \Pr\left(X > x\right) \tag{0.1.6.5}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right) du. \tag{0.1.6.6}$$

(0.1.6.7)

Using the Q-function, P_e is rewritten as

$$P_e = Q(A) \tag{0.1.6.8}$$

0.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution: The theoretical P_e is plotted in *Fig.* 0.1.7.1, along with numerical estimations from generated samples of Y. The below code is used for the plot,

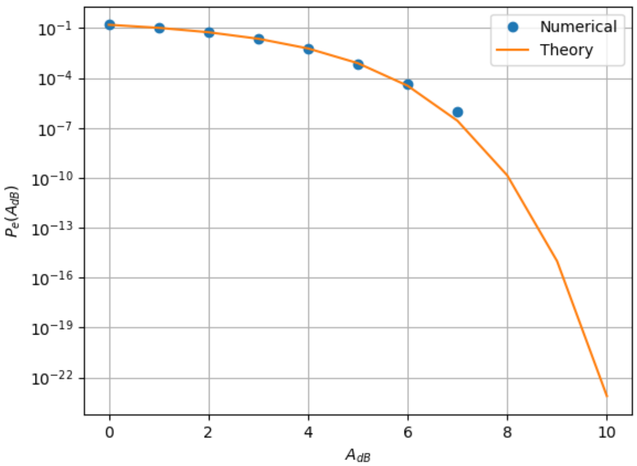

Figure 0.1.7.1: P_e versus A plot

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codes/chapter3/ch3_varyA.py
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0.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution: Given the decision rule,

$$y \underset{-1}{\overset{1}{\gtrless}} \delta \tag{0.1.8.1}$$

$$\begin{aligned} P_{e|0} &= \Pr\left(\hat{X} = -1|X = 1\right) \\ &= \Pr\left(Y < \delta|X = 1\right) \\ &= \Pr\left(AX + N < \delta|X = 1\right) \\ &= \Pr\left(A + N < \delta\right) \\ &= \Pr\left(N < -A + \delta\right) \\ &= \Pr\left(N > A - \delta\right) \\ &= Q(A - \delta) \end{aligned}$$

$$\begin{aligned} P_{e|1} &= \Pr\left(\hat{X} = 1|X = -1\right) \\ &= \Pr\left(Y > \delta|X = -1\right) \\ &= \Pr\left(N > A + \delta\right) \\ &= Q(A + \delta) \end{aligned}$$

Using (0.1.6.3), P_e is given by

$$P_e = \frac{1}{2}Q(A + \delta) + \frac{1}{2}Q(A - \delta) \tag{0.1.8.2}$$

Using the integral for Q-function from (0.1.6.6),

$$P_e = k(\int_{A+\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du + \int_{A-\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du) \tag{0.1.8.3}$$

where k is a constant

Differentiating (0.1.8.3) wrt δ (using Leibniz's rule) and equating to 0, we get

$$\begin{aligned} \exp\left(-\frac{(A+\delta)^2}{2}\right) - \exp\left(-\frac{(A-\delta)^2}{2}\right) &= 0 \\ \frac{\exp\left(-\frac{(A+\delta)^2}{2}\right)}{\exp\left(-\frac{(A-\delta)^2}{2}\right)} &= 1 \\ \exp\left(-\frac{(A+\delta)^2 - (A-\delta)^2}{2}\right) &= 1 \\ \exp\left(-2A\delta\right) &= 1 \end{aligned}$$

Taking ln on both sides

$$\begin{aligned} -2A\delta &= 0 \\ \implies \delta &= 0 \end{aligned}$$

P_e is maximum for $\delta = 0$

0.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{0.1.9.1}$$

Solution: Since X is not equiprobable, P_e is given by,

$$\begin{aligned} P_e &= (1-p)P_{e|1} + pP_{e|0} \\ &= (1-p)Q(A+\delta) + pQ(A-\delta) \end{aligned}$$

Using the integral for Q-function from (0.1.6.6),

$$\begin{aligned} P_e = k((1-p)\int_{A+\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du+ \\ p\int_{A-\delta}^{\infty} \exp\left(-\frac{u^2}{2}\right) du) \end{aligned} \tag{0.1.9.2}$$

where k is a constant.
Following the same steps as in problem 0.1.8, δ for maximum P_e evaluates to,

$$\delta = \frac{1}{2A}\ln\left(\frac{1}{p} - 1\right) \tag{0.1.9.3}$$

0.1.10 Repeat the above exercise using the MAP criterion.
Solution: The MAP rule can be stated as

$$\text{Set } \hat{x} = x_i \text{ if} \tag{0.1.10.1}$$
$$p_X(x_k)p_Y(y|x_k) \text{ is maximum for } k = i$$

For the case of BPSK, the point of equality between $p_X(x=1)p_Y(y|x=1)$ and $p_X(x=-1)p_Y(y|x=-1)$ is the optimum threshold. If this threshold is δ, then

$$\begin{aligned} &pp_Y(y|x=1) > (1-p)p_Y(y|x=-1) \text{ when } y > \delta \\ &pp_Y(y|x=1) < (1-p)p_Y(y|x=-1) \text{ when } y < \delta \end{aligned}$$

The above inequalities can be visualized in below figure for p = 0.3 and A = 3. Given $Y = AX + N$ where $N \sim$

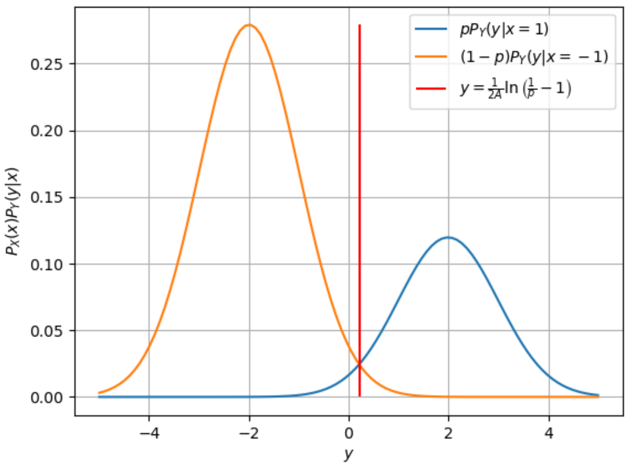

Figure 0.1.10.1: $p_X(X = x_i)p_Y(y|x = x_i)$ versus y plot for $X \in \{-1, 1\}$

$\mathcal{N}(0, 1)$, the optimum threshold is found as solution to the below equation

$$p \exp\left(-\frac{(y_{eq} - A)^2}{2}\right) = (1-p)\exp\left(-\frac{(y_{eq} + A)^2}{2}\right) \tag{0.1.10.2}$$

Solving for y_{eq}, we get

$$y_{eq} = \delta = \frac{1}{2A}\ln\left(\frac{1}{p} - 1\right) \tag{0.1.10.3}$$

which is same as δ obtained in problem 0.1.9