Bivariate Random Variables: FSK

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Contents

0.1 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{0.1.0.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.1.0.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{0.1.0.3}$$

0.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{0.1.1.1}$$

on the same graph using a scatter plot.
Solution: The scatter plot in Fig. 0.1.1.1 is generated using the below code,

```
codes/ch5_scatter.py
```

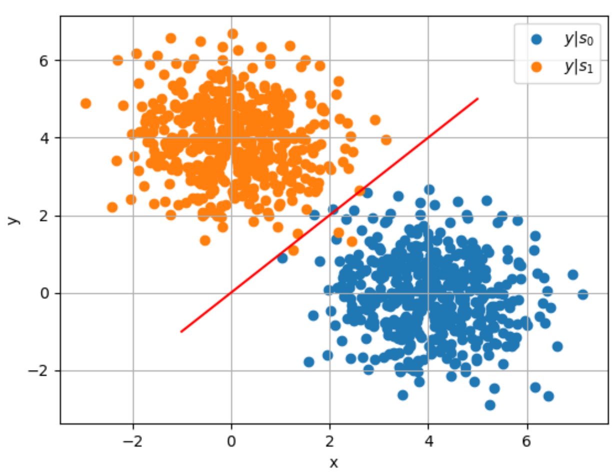

Figure 0.1.1.1: Scatter plot of $\mathbf{y}|\mathbf{s}_0$ and $\mathbf{y}|\mathbf{s}_1$

0.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.
Solution: Let $\mathbf{y} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}^T$. Then the decision rule is

$$y_1 \underset{1}{\overset{0}{\gtrless}} y_2 \tag{0.1.2.1}$$

$\mathbf{y}|\mathbf{s}_i$ is a random vector with each of its components normally distributed. The PDF of $\mathbf{y}|\mathbf{s}_i$ is given by,

$$p_{\mathbf{y}|\mathbf{s}_i}(\mathbf{y}) = \frac{1}{2\pi\sqrt{|\mathbf{\Sigma}|}} \exp\left(-\frac{1}{2}(\mathbf{y} - \mathbf{s}_i)^\top \mathbf{\Sigma}^{-1} (\mathbf{y} - \mathbf{s}_i)\right) \tag{0.1.2.2}$$

Where $\mathbf{\Sigma}$ is the covariance matrix. Substituting $\mathbf{\Sigma} = \sigma\mathbf{I}$,

$$p_{\mathbf{y}|\mathbf{s}_i}(\mathbf{y}) = \frac{1}{2\pi\sigma} \exp\left(-\frac{1}{2\sigma}(\mathbf{y} - \mathbf{s}_i)^\top \mathbf{I} (\mathbf{y} - \mathbf{s}_i)\right) \tag{0.1.2.3}$$

$$= \frac{1}{2\pi\sigma} \exp\left(-\frac{1}{2\sigma}(\mathbf{y} - \mathbf{s}_i)^\top (\mathbf{y} - \mathbf{s}_i)\right) \tag{0.1.2.4}$$

Assuming equiprobable symbols, use MAP rule in (**??**) to find optimum decision. Since there are only two possible symbols $\mathbf{s}_0$ and $\mathbf{s}_1$, the optimal decision criterion is found by equating $p_{\mathbf{y}|\mathbf{s}_0}$ and $p_{\mathbf{y}|\mathbf{s}_1}$.

$$p_{\mathbf{y}|\mathbf{s}_0} = p_{\mathbf{y}|\mathbf{s}_1}$$

$$\implies \exp\left(-\frac{1}{2\sigma}(\mathbf{y} - \mathbf{s}_0)^\top (\mathbf{y} - \mathbf{s}_0)\right) = \exp\left(-\frac{1}{2\sigma}(\mathbf{y} - \mathbf{s}_1)^\top (\mathbf{y} - \mathbf{s}_1)\right)$$

$$\implies (\mathbf{y} - \mathbf{s}_0)^\top (\mathbf{y} - \mathbf{s}_0) = (\mathbf{y} - \mathbf{s}_1)^\top (\mathbf{y} - \mathbf{s}_1)$$
$$\implies \mathbf{y}^\top\mathbf{y} - 2\mathbf{s}_0^\top\mathbf{y} + \mathbf{s}_0^T\mathbf{s}_0 = \mathbf{y}^\top\mathbf{y} - 2\mathbf{s}_1^\top\mathbf{y} + \mathbf{s}_1^T\mathbf{s}_1$$
$$\implies 2(\mathbf{s}_1 - \mathbf{s}_0)^\top \mathbf{y} = \|\mathbf{s}_1\|^2 - \|\mathbf{s}_0\|^2$$
$$\implies (\mathbf{s}_1 - \mathbf{s}_0)^\top \mathbf{y} = 0$$
$$\implies \begin{pmatrix} -1 \\ 1 \end{pmatrix}^\top \mathbf{y} = 0$$

0.1.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{0.1.3.1}$$

with respect to the SNR from 0 to 10 dB.
Solution: The blue dots in Fig. 0.1.4.1 are the P_e versus SNR plot. It is generated using the below code,

```
codes/ch5_snr.py
```

0.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

Solution: Using the decision rule from (0.1.2.1),

$$\begin{aligned} P_e &= \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \\ &= \Pr\left(y_1 < y_2 | \mathbf{x} = \mathbf{s}_0\right) \\ &= \Pr\left(A + n_1 < n_2\right) \\ &= \Pr\left(n_1 - n_2 < -A\right) \end{aligned} \tag{0.1.4.1}$$

Let $Z = n_1 - n_2$ where $n_1, n_2 \sim \mathcal{N}\left(0, \sigma^2\right)$. The PDF of X is given by,

$$\begin{aligned} p_Z(z) &= p_{n_1}(n_1) * p_{-n_2}(n_2) \\ &= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} e^{-\frac{(t-z)^2}{2\sigma^2}}\, dt \\ &= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2+t^2}{2\sigma^2}}\, dt \\ &= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{(2t-z)^2+z^2}{2(\sqrt{2}\sigma)^2}}\, dt \\ &= \frac{1}{2\pi\sigma^2} e^{-\frac{z^2}{2(\sqrt{2}\sigma)^2}} \int_{-\infty}^{\infty} e^{-\frac{(2t-z)^2}{2(\sqrt{2}\sigma)^2}}\, dt \\ &= \frac{e^{-\frac{z^2}{2(\sqrt{2}\sigma)^2}}}{\sqrt{2\pi}\sqrt{2}\sigma} \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2(\sqrt{2}\sigma)^2}}\, dk \\ &= \frac{e^{-\frac{z^2}{2(\sqrt{2}\sigma)^2}}}{\sqrt{2\pi}\sqrt{2}\sigma} \end{aligned} \tag{0.1.4.2}$$

From (0.1.4.2), $Z \sim \mathcal{N}\left(0, 2\sigma^2\right)$. Substituting $\sigma = 1$, $Z \sim \mathcal{N}(0, 2)$. (0.1.4.1) can be further simplified as,

$$\begin{aligned} P_e &= \Pr\left(Z < -A\right) \\ &= \Pr\left(Z > A\right) \\ &= Q\left(\frac{A}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{A}{\sqrt{2}}}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \end{aligned}$$

Fig. 0.1.4.1 compares the theoretical and simulation plots.

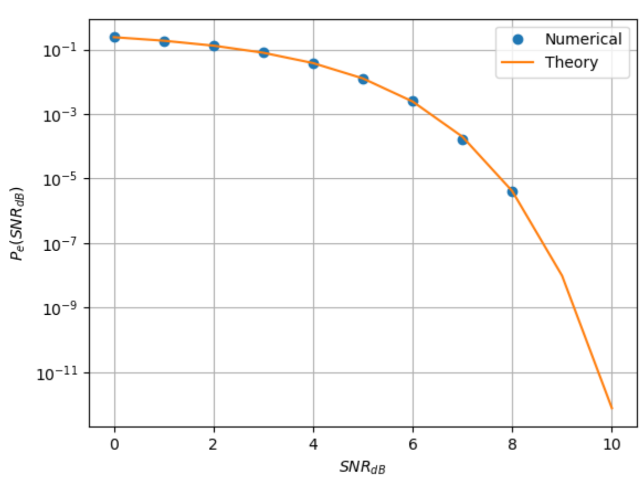

Figure 0.1.4.1: P_e versus SNR plot for FSK