Random Numbers

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Contents

0.1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

0.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
codes/include/coeffs.h
codes/src/uni_gen_stat.c
```

0.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{0.1.2.1}$$

Solution: The following code plots Fig. 0.1.2.1

```
codes/src/cdf_plot_uni.py
```

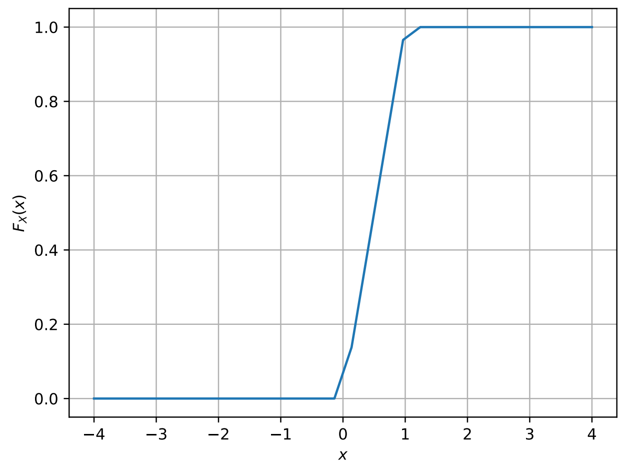

Figure 0.1.2.1: The CDF of U

0.1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$F_U(x) = \int_{-\infty}^{x} f_U(x)\,dx \tag{0.1.3.1}$$

For the uniform random variable U, $f_U(x)$ is given by

$$f_U(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & elsewhere \end{cases} \tag{0.1.3.2}$$

Substituting (0.1.3.2) in (0.1.3.1), $F_U(x)$ is found to be

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 0 \end{cases} \tag{0.1.3.3}$$

0.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{0.1.4.1}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{0.1.4.2}$$

Write a C program to find the mean and variance of U.
Solution: The following code prints the mean and variance of U

```
codes/src/uni_gen_stat.c
```

0.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{0.1.5.1}$$

Solution: For a random variable X, the mean μ_X and variance σ_X^2 are given by

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x dF_U(x) \tag{0.1.5.2}$$

$$\sigma_X^2 = E\left[X^2\right] - \mu_X^2 = \int_{-\infty}^{\infty} x^2 dF_U(x) - \mu_X^2 \tag{0.1.5.3}$$

Substituting the CDF of U from (0.1.3.3) in (0.1.5.2) and

(0.1.5.3), we get

$$\mu_U = \frac{1}{2} \tag{0.1.5.4}$$

$$\sigma_U^2 = \frac{1}{12} \tag{0.1.5.5}$$

which match with the values printed in problem 0.1.4

0.2 Central Limit Theorem

0.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{0.2.1.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program.

```
codes/include/coeffs.h
codes/src/gau_gen_stat.c
```

0.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 0.2.2.1

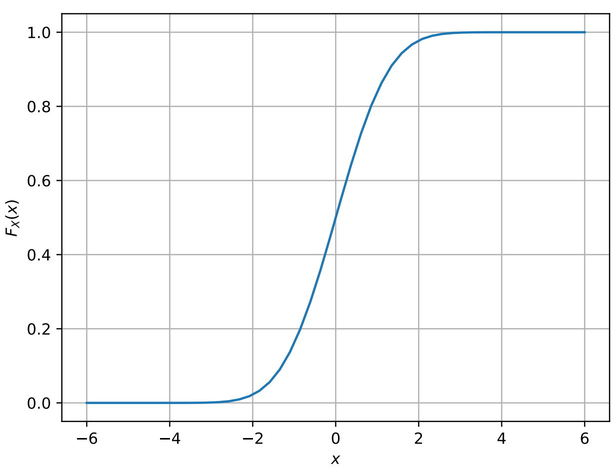

Figure 0.2.2.1: The CDF of X

The properties of a CDF are

$$F_X(-\infty) = 0 \tag{0.2.2.1}$$

$$F_X(\infty) = 1 \tag{0.2.2.2}$$

$$\frac{dF_X(x)}{dx} \geq 0 \tag{0.2.2.3}$$

0.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{0.2.3.1}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 0.2.3.1 using the code below

```
codes/src/cdf_pdf_plot_gau.py
```

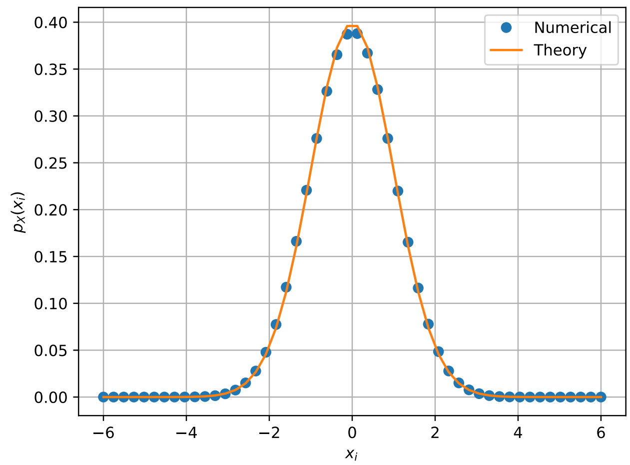

Figure 0.2.3.1: The PDF of X

The properties of PDF are

$$f_X(x) \geq 0 \tag{0.2.3.2}$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \tag{0.2.3.3}$$

0.2.4 Find the mean and variance of X by writing a C program. **Solution:** The following code prints the mean and variance of X

```
codes/src/gau_gen_stat.c
```

0.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{0.2.5.1}$$

repeat the above exercise theoretically.
Solution: Substituting the PDF from (0.2.5.1) in (0.1.5.2),

$$\mu_X = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{0.2.5.2}$$

Using

(0.2.5.3)

$$\int x \cdot \exp\left(-ax^2\right) \mathrm{d}x = -\frac{1}{2a} \cdot \exp\left(-ax^2\right) \tag{0.2.5.4}$$

$$\mu_X = \frac{1}{\sqrt{2\pi}} \left[-\exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{\infty} \tag{0.2.5.5}$$

$$\mu_X = 0 \tag{0.2.5.6}$$

Substituting μ_X and the PDF in (0.1.5.3) to compute variance,

$$\sigma_X^2 = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{0.2.5.7}$$

Substituting

$$t = \frac{x^2}{2}, \tag{0.2.5.8}$$

$$\sigma_X^2 = \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}} \exp(-t)\, dt \tag{0.2.5.9}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{3}{2}-1} \exp(-t)\, dt$$

Using the gamma function

$$\Gamma(x) = \int_0^{\infty} z^{x-1} \cdot e^{-z}\, \mathrm{d}z \tag{0.2.5.10}$$

$$\sigma_X^2 = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \tag{0.2.5.11}$$

$$= \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2}$$

$$= 1$$

0.3 From Uniform to Other

0.3.1 Generate samples of

$$V = -2\ln(1-U) \tag{0.3.1.1}$$

and plot its CDF.
Solution: The samples for U are loaded from uni.dat file generated in problem 0.1.4. The CDF of V is plotted in Fig. 0.3.1.1 using the code below,

```
codes/src/function_1.py
```

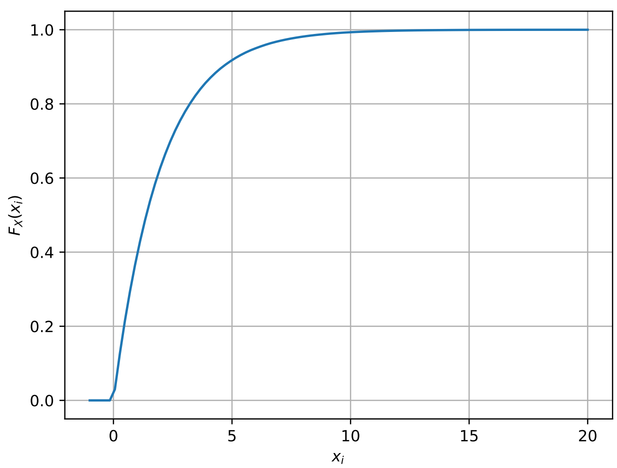

Figure 0.3.1.1: The CDF of V

0.3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V < x) \tag{0.3.2.1}$$

$$= P(-2\ln(1-U) < x) \tag{0.3.2.2}$$

$$= P(U < 1 - e^{\frac{-x}{2}}) \tag{0.3.2.3}$$

$$= F_U(1 - e^{\frac{-x}{2}}) \tag{0.3.2.4}$$

Using $F_U(x)$ defined in (0.1.3.3),

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{\frac{-x}{2}} & x \geq 0 \end{cases} \tag{0.3.2.5}$$

0.4 Triangular Distribution

0.4.1 Generate

$$T = U_1 + U_2 \tag{0.4.1.1}$$

Solution: Download the following files and execute the C program.

```
codes/include/coeffs.h
codes/src/two_uni_gen.c
```

0.4.2 Find the CDF of T.
Solution: Loading the samples from uni1.dat and uni2.dat in python, the CDF is plotted in Fig. 0.4.2.1

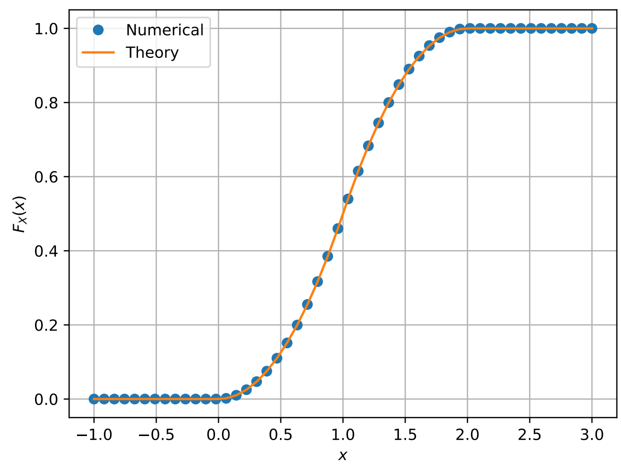

Figure 0.4.2.1: The CDF of T

0.4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 0.4.3.1 using the code below

```
codes/src/function_2.py
```

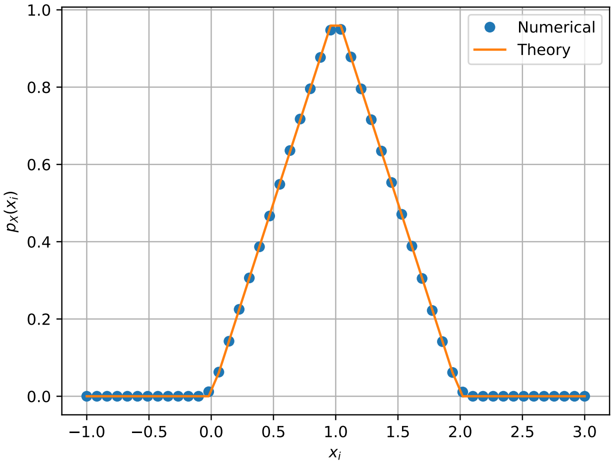

Figure 0.4.3.1: The PDF of T

0.4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Since T is the sum of two independant random variables $U1$ and $U2$, the PDF of T is given by

$$p_T(x) = p_{U1}(x) * p_{U2}(x) \tag{0.4.4.1}$$

Using the PDF of U from (0.1.3.2), the convolution results in

$$p_T(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & x > 2 \end{cases} \tag{0.4.4.2}$$

The CDF of T is found using (0.1.3.1) by replacing U with T. Evaluating the integral for the piecewise function $p_T(x)$,

$$F_T(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \le x \le 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \le x \le 2 \\ 1 & x > 2 \end{cases} \tag{0.4.4.3}$$

0.4.5 Verify your results through a plot.
Solution: The theoretical and numerical plots for the CDF and PDF of T closely match in Fig. 0.4.2.1 and Fig. 0.4.3.1