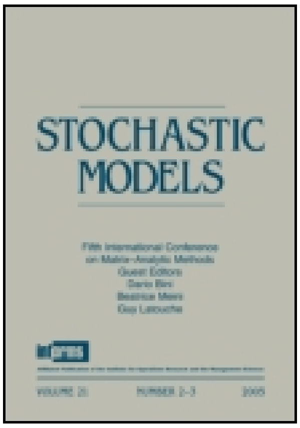

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Numerical calculation of stable densities and distribution functions

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NUMERICAL CALCULATION OF STABLE DENSITIES AND DISTRIBUTION FUNCTIONS

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Abstract

Computational formulas are given for stable densities and distribution functions in Zolotarev's (M) parameterization. These formulas are used in a software package called STABLE to calculate general stable densities, distribution functions and quantiles.

Key words: stable distributions, densities, distribution functions, modes, numerical quadrature.

1 Introduction

A practical difficulty in working with stable distributions is the lack of closed formulas for most stable densities and distribution functions. A general stable distribution is determined by four parameters: an index of stability α, a skewness parameter β, a scale σ and a location parameter μ and is generally specified in terms of it's characteristic function. The parameterization most commonly used for stable distributions, see Samorodnitsky and Taqqu (1994), is the following one: $X \sim S_\alpha(\sigma, \beta, \mu)$ if the characteristic function of Y is given by

$$E\exp(itY) = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha \left[1 - i\beta(\tan\frac{\pi\alpha}{2})(\operatorname{sign} t)\right] + i\mu t\} & \alpha \neq 1 \\ \exp\{-\sigma|t| \left[1 + i\beta\frac{2}{\pi}(\operatorname{sign} t)\ln|t|\right] + i\mu t\} & \alpha = 1. \end{cases} \tag{1}$$

The range of parameters are $0 < \alpha \leq 2$, $\sigma > 0$, $-1 \leq \beta \leq 1$, and $\mu \in \mathbf{R}$. Equation (1) is a slight variation of parameterization (A) in Zolotarev (1986).

There have been several efforts to compute stable densities and quantiles: Holt and Crow (1975) tabulate densities for selected values of α and β; Worsdale (1975) and Panton (1992) tabulate symmetric stable distribution functions; Brothers, DuMouchel and Paulson (1983) and Paulson and Delehanty (1993) tabulate fractiles of general stable distributions; and McCulloch and Panton (1996) tabulate densities and fractiles for the maximally skewed distributions. Efficient algorithms for approximating the density and distribution function in the symmetric case when $\alpha > 0.85$ have been recently developed by McCulloch (1996).

In Nolan (1997), we gave an algorithm for computing the general nonsymmetric density when $\alpha \geq 0.9$. Limitations of that program (described below) motivated us to develop better algorithms. One method we tried was to use the tabulated densities of a maximally skewed ($\beta = 1$) stable distribution from McCulloch and Panton (1996) and express the density $f(x; \alpha, \beta)$ for intermediate cases ($-1 < \beta < 1$) as a convolution of two maximally skewed densities ($\beta = 1$ and $\beta = -1$). This program worked reasonably well for central values of x, but understandably did not work well on the tails of the distribution. Since tail densities are important for maximum likelihood estimation and tail fractiles are generally more important than central ones, that program had limited usefulness.

Next, we tried directly implementing the integral formulas for densities and distribution functions in §2.2 of Zolotarev (1986). This approach had challenging numerical problems and worked poorly when α was near 1. Finally, we decided that the best numerical approach was to use the (M) parameterization of Zolotarev (1986) and derive similar integral formulas for the density and distribution function in that form. This paper describes those formulas and a program that implements them. The result is a general program that accurately computes general stable densities, distribution function and quantiles for essentially all values of the parameters.

2 Expressions for stable densities and distribution functions

This section states the theoretical formulas used for numerical calculations; for completeness full proofs are given. For numerical purposes, it is preferable to calculate densities and distribution functions for a standardized stable random variable X in Zolotarev's (M) parameterization. Such a r.v. has characteristic function (pg. 11, Zolotarev (1986))

$$E \exp(itX) = \begin{cases} \exp\{-|t|^\alpha \left[1 + i\beta(\operatorname{sign} t)(\tan \frac{\pi\alpha}{2})(|t|^{1-\alpha} - 1)\right]\} & \alpha \neq 1 \\ \exp\{-|t| \left[1 + i\beta(\operatorname{sign} t)\frac{2}{\pi} \ln |t|\right]\} & \alpha = 1. \end{cases} \tag{2}$$

The formulas we use for computing the density $f(x;\alpha,\beta)$ and distribution function $F(x;\alpha,\beta) = P(X \le x)$ are variations of Zolotarev's integral formulas. The formulas in §2.2 of Zolotarev (1986) are stated for a standardized stable random variable in the (B) representation, where the scale, shift and skewness are of a different form. To state formulas in the (M) parameterization, define

$$
\begin{aligned}
\zeta &= \zeta(\alpha,\beta) = \begin{cases} -\beta \tan \frac{\pi\alpha}{2} & \alpha \neq 1 \\ 0 & \alpha = 1, \end{cases} \\
\theta_0 &= \theta_0(\alpha,\beta) = \begin{cases} \frac{1}{\alpha}\arctan(\beta \tan \frac{\pi\alpha}{2}) & \alpha \neq 1 \\ \frac{\pi}{2} & \alpha = 1 \end{cases} \\
c_1(\alpha,\beta) &= \begin{cases} \frac{1}{\pi}\left(\frac{\pi}{2} - \theta_0\right) & \alpha < 1 \\ 0 & \alpha = 1 \\ 1 & \alpha > 1, \end{cases} \\
V(\theta;\alpha,\beta) &= \begin{cases} (\cos \alpha\theta_0)^{\frac{1}{\alpha-1}} \left(\frac{\cos\theta}{\sin \alpha(\theta_0+\theta)}\right)^{\frac{\alpha}{\alpha-1}} \frac{\cos(\alpha\theta_0+(\alpha-1)\theta)}{\cos\theta} & \alpha \neq 1 \\ \frac{2}{\pi}\left(\frac{\frac{\pi}{2}+\beta\theta}{\cos\theta}\right) \exp\left(\frac{1}{\beta}\left(\frac{\pi}{2}+\beta\theta\right)\tan\theta\right) & \alpha = 1, \beta \neq 0. \end{cases}
\end{aligned}
$$

While it may be more direct to write V as a function of the alternative skewness parameter θ_0, we prefer to stress the dependence on β. The range of θ_0 depends on α, making it less intuitive than β.

Theorem 1 *Let X have characteristic function (2). The density and distribution function of X are given by:*
(a) When $\alpha \neq 1$ and $x > \zeta$,

$$f(x;\alpha,\beta) = \frac{\alpha(x-\zeta)^{\frac{1}{\alpha-1}}}{\pi|\alpha-1|} \int_{-\theta_0}^{\frac{\pi}{2}} V(\theta;\alpha,\beta) \exp\left(-(x-\zeta)^{\frac{\alpha}{\alpha-1}} V(\theta;\alpha,\beta)\right) d\theta$$

and

$$F(x;\alpha,\beta) = c_1(\alpha,\beta) + \frac{\operatorname{sign}(1-\alpha)}{\pi} \int_{-\theta_0}^{\frac{\pi}{2}} \exp\left(-(x-\zeta)^{\frac{\alpha}{\alpha-1}} V(\theta;\alpha,\beta)\right) d\theta.$$

(b) When $\alpha \neq 1$ and $x = \zeta$,

$$f(\zeta;\alpha,\beta) = \frac{\Gamma(1+\frac{1}{\alpha})\cos(\theta_0)}{\pi(1+\zeta^2)^{1/(2\alpha)}}$$

and

$$F(\zeta;\alpha,\beta) = \frac{1}{\pi}\left(\frac{\pi}{2} - \theta_0\right).$$

(c) When $\alpha \neq 1$ and $x < \zeta$,

$$f(x;\alpha,\beta) = f(-x;\alpha,-\beta)$$

and

$$F(x;\alpha,\beta) = 1 - F(-x;\alpha,-\beta).$$

(d) When $\alpha = 1$,

$$f(x;1,\beta) = \begin{cases} \frac{1}{2|\beta|} e^{-\frac{\pi x}{2\beta}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(\theta;1,\beta) \exp\left(-e^{-\frac{\pi x}{2\beta}} V(\theta;1,\beta)\right) d\theta & \beta \neq 0 \\ \frac{1}{\pi(1+x^2)} & \beta = 0 \end{cases}$$

and

$$F(x;1,\beta) = \begin{cases} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left(-e^{-\frac{\pi x}{2\beta}} V(\theta;1,\beta)\right) d\theta & \beta > 0 \\ \frac{1}{2} + \frac{1}{\pi} \arctan x & \beta = 0 \\ 1 - F(x;\alpha,-\beta) & \beta < 0. \end{cases}$$

For clarity, we break the proof up into steps. First, we give an intermediate formula for the density. For this, define

$$\begin{aligned} h(t;u,\alpha,\beta) &= \begin{cases} ut + \zeta t^\alpha & \alpha \neq 1 \\ ut + \beta \frac{2}{\pi} t \ln t & \alpha = 1, \end{cases} \\ p(u;\alpha,\beta) &= \frac{1}{\pi} \int_0^\infty \cos(h(t;u,\alpha,\beta)) e^{-t^\alpha} dt, \\ q(u;\alpha,\beta) &= \frac{1}{\pi} \int_0^\infty \sin(h(t;u,\alpha,\beta)) e^{-t^\alpha} dt. \end{aligned}$$

The following theorem shows that the function p is central to the evaluation of one dimensional stable densities. In Abdul-Hamid and Nolan (1996), it is shown that both p and q can be used in the evaluation of multivariate stable densities.

Theorem 2 *For any $0 < \alpha < 2$ and any $-1 \leq \beta \leq 1$, the density of a standardized α-stable r.v. with characteristic function (2) is*

$$f(x;\alpha,\beta) = p(x - \zeta; \alpha, \beta).$$

Proof: When $\alpha \neq 1$, the inversion formula for characteristic functions shows that

$$\begin{aligned} f(x;\alpha,\beta) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} e^{-|t|^\alpha \left[1 + i\beta (\operatorname{sign} t)(\tan \frac{\pi\alpha}{2})(|t|^{1-\alpha} - 1)\right]} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|t|^\alpha - i\left[xt - \zeta(t - (\operatorname{sign} t)|t|^\alpha)\right]} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|t|^\alpha} \cos\left[(x-\zeta)t + \zeta(\operatorname{sign} t)|t|^\alpha\right] dt \\ &= \frac{1}{\pi} \int_0^\infty e^{-t^\alpha} \cos\left[(x-\zeta)t + \zeta t^\alpha\right] dt \\ &= p(x - \zeta; \alpha, \beta). \end{aligned}$$

When $\alpha = 1$, we likewise have

$$\begin{aligned} f(x;\alpha,\beta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} e^{-|t|\left[1+i\beta(\text{sign } t)\frac{2}{\pi}\ln|t|\right]}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|t|-i\left[xt+\beta\frac{2}{\pi}t\ln|t|\right]}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|t|}\cos\left[xt+\beta\frac{2}{\pi}t\ln|t|\right]dt \\ &= \frac{1}{\pi}\int_{0}^{\infty} e^{-t}\cos\left[xt+\beta\frac{2}{\pi}t\ln|t|\right]dt \\ &= p(x;1,\beta). \end{aligned}$$

□

The numerical evaluation of p (and q) is difficult using the above formulas because the interval of integration is infinite and the integrands oscillate an infinite number of times. An earlier program, described in Nolan (1997), evaluated stable densities directly from the expression above for p. It works by truncating the integral to a finite region (based on the desired accuracy and the decay of $e^{-t^{\alpha}}$), splitting the region of integration up into intervals where the cosine term is positive and negative (found numerically), and summing numerical approximations to the integral over each of those pieces. That program works reasonably well when $|x|$ is not large and when $\alpha > 0.9$. When $|x|$ is large the number of oscillations is large and when α is small the region of integration grows - both cases make the integral hard to evaluate accurately. Furthermore, it is difficult to get a small relative error in such calculations, and that formula gives no direct formula for the distribution function. The following theorem gives an alternative integral expression for p that eliminates these problems.

Theorem 3 *When $\alpha \neq 1$ and $u > 0$,*

$$p(u;\alpha,\beta) = \frac{\alpha}{\pi|\alpha-1|}u^{\frac{1}{\alpha-1}}\int_{-\theta_0}^{\frac{\pi}{2}} V(\theta;\alpha,\beta)\exp\left(-u^{\frac{\alpha}{\alpha-1}}V(\theta;\alpha,\beta)\right)d\theta.$$

When $\alpha = 1$ and $\beta > 0$,

$$p(u;1,\beta) = \frac{1}{2\beta}e^{-\frac{\pi u}{2\beta}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(\theta;1,\beta)\exp\left(-e^{-\frac{\pi u}{2\beta}}V(\theta;1,\beta)\right)d\theta.$$

Proof:

Step 1. First consider $\alpha \neq 1$. Extend h to the complex plane (cut along, say, the negative real axis) by $h(z) = h(z;u,\alpha,\zeta) = z^{\alpha} + i(uz + \zeta z^{\alpha})$. Note that $p(u;\alpha,\beta) = \Re\int_0^{\infty}\exp(-h(z))dz$.

Step 2. Find a path γ in the complex plane that avoids the cut and connects 0 to ∞ with $\Im\exp(-h(z)) = 0$ along γ. To find such a path, set $z = re^{i\theta}$, then

$$\begin{aligned} h(z) &= h(re^{i\theta}) = r^\alpha e^{i\alpha\theta} + i(ure^{i\theta} + \zeta r^\alpha e^{i\alpha\theta}) \\ &= r^\alpha(1+i\zeta)e^{i\alpha\theta} + iure^{i\theta} \\ &= r^\alpha(1+i\zeta)(\cos\alpha\theta + i\sin\alpha\theta) + iur(\cos\theta + i\sin\theta) \\ &= [r^\alpha(\cos\alpha\theta - \zeta\sin\alpha\theta) - ur\sin\theta] + i\,[r^\alpha(\zeta\cos\alpha\theta + \sin\alpha\theta) + ur\cos\theta] \end{aligned}$$

Substituting $\zeta = -\tan\alpha\theta_0 = -\sin\alpha\theta_0/\cos\alpha\theta_0$ yields $\cos\alpha\theta - \zeta\sin\alpha\theta = \cos\alpha(\theta-\theta_0)/\cos\alpha\theta_0$ and $\zeta\cos\alpha\theta + \sin\alpha\theta = \sin\alpha(\theta-\theta_0)/\cos\alpha\theta_0$. So

$$h(re^{i\theta}) = \left[r^\alpha\left(\frac{\cos\alpha(\theta-\theta_0)}{\cos\alpha\theta_0}\right) - ur\sin\theta\right] + i\left[r^\alpha\left(\frac{\sin\alpha(\theta-\theta_0)}{\cos\alpha\theta_0}\right) + ur\cos\theta\right] \tag{3}$$

To have $\Im h(re^{i\theta}) = 0$ requires that $r^\alpha\sin\alpha(\theta-\theta_0)/\cos\alpha\theta_0 = -ur\cos\theta$, or $r = [ur(\theta)]^{\frac{1}{\alpha-1}}$, where $r(\theta) = -\cos\theta\cos\alpha\theta_0/\sin\alpha(\theta_0-\theta)$. The desired contour is

$$\gamma = \{z(\theta) = [ur(\theta)]^{\frac{1}{\alpha-1}}e^{i\theta} : -\frac{\pi}{2} < \theta < \theta_0\}.$$

(When $\alpha \le 1$ and $\beta = 1$, $\alpha\theta_0 = -\frac{\pi}{2}$ and this path starts at a point on the negative imaginary axis. In this case, we can extend γ by adjoining the line segment along the imaginary axis from the origin to this point. This extra piece does not affect the argument below.)

Step 3. Along the path γ, (3) shows

$$\begin{aligned} h(z(\theta)) &= \Re h(z(\theta)) = [ur(\theta)]^{\frac{\alpha}{\alpha-1}}\left(\frac{\cos\alpha(\theta-\theta_0)}{\cos\alpha\theta_0} - u[ur(\theta)]^{\frac{1}{\alpha-1}}\sin\theta\right) \\ &= u^{\frac{\alpha}{\alpha-1}}r(\theta)^{\frac{1}{\alpha-1}}\left(r(\theta)\frac{\cos\alpha(\theta-\theta_0)}{\cos\alpha\theta_0} - \sin\theta\right) \\ &= u^{\frac{\alpha}{\alpha-1}}r(\theta)^{\frac{1}{\alpha-1}}\left(\cos\theta\cot\alpha(\theta_0-\theta) - \sin\theta\right) \\ &= u^{\frac{\alpha}{\alpha-1}}r(\theta)^{\frac{1}{\alpha-1}}\left(\frac{\cos\theta\sin\alpha(\theta_0-\theta) - \sin\theta\cos\alpha(\theta_0-\theta)}{\cos\alpha(\theta_0-\theta)}\right) \\ &= u^{\frac{\alpha}{\alpha-1}}r(\theta)^{\frac{1}{\alpha-1}}\sin[\alpha(\theta_0-\theta)-\theta]/\cos\alpha(\theta_0-\theta) \\ &= u^{\frac{\alpha}{\alpha-1}}[\cos\theta\cos\alpha\theta_0/\sin\alpha(\theta_0-\theta)]^{\frac{1}{\alpha-1}}\frac{\sin[\alpha(\theta_0-\theta)-\theta]}{\cos\alpha(\theta_0-\theta)} \\ &= u^{\frac{\alpha}{\alpha-1}}V(-\theta;\alpha,\beta) \end{aligned}$$

Also,

$$\begin{aligned} \Re dz(\theta) &= d(u^{\frac{1}{\alpha-1}}r(\theta)^{\frac{1}{\alpha-1}}\cos\theta) = u^{\frac{1}{\alpha-1}}d(\cos^2\theta\cos\alpha\theta_0/\sin\alpha(\theta_0-\theta)) \\ &= u^{\frac{1}{\alpha-1}}\frac{-\alpha}{|\alpha-1|}V(-\theta;\alpha,\beta)d\theta \end{aligned}$$

Step 4. Derive the formula for p by integrating along the path γ. A standard monodromy theorem, e.g. Ahlfors (1966), shows that the resulting integral equals p:

$$\begin{aligned} p(u;\alpha,\beta) &= \Re\frac{1}{\pi}\int_\gamma e^{-h(z)}dz \\ &= \frac{-\alpha}{\pi|\alpha-1|}u^{\frac{1}{\alpha-1}}\int_{-\frac{\pi}{2}}^{\theta_0} V(-\theta;\alpha,\beta)\exp(-u^{\frac{\alpha}{\alpha-1}}V(-\theta;\alpha,\beta))d\theta. \end{aligned}$$

Substituting $\phi=-\theta$ finishes this part of the proof.

Step 5. Now consider the $\alpha=1$ and $\beta>0$ case. Here $h(z)=uz+\beta\frac{2}{\pi}z\ln z$, and

$$\begin{aligned} h(re^{i\theta}) &= re^{i\theta}[u+\beta\frac{2}{\pi}\ln(re^{i\theta})] \\ &= r[u\cos\theta+\beta\frac{2}{\pi}(\ln r\cos\theta-\theta\sin\theta)] \\ &\qquad +ir[\beta\frac{2}{\pi}(\theta\cos\theta+\ln r\sin\theta)+u\sin\theta] \end{aligned}$$

To have $\Im h(re^{i\theta})=0$ requires that $\beta\frac{2}{\pi}(\ln r)\sin\theta=-[\beta\frac{2}{\pi}\theta\cos\theta+u\sin\theta]$, or that $r(\theta)=\exp\left[-\left(\frac{\pi u}{2\beta}+\theta\tan\theta\right)\right]$. The contour of integration in this case is

$$\gamma=\{z(\theta)=r(\theta)e^{i\theta}:-\frac{\pi}{2}<\theta<\frac{\pi}{2}\}.$$

Along this path,

$$h(z(\theta))=\Re h(z(\theta))=r(\theta)[u\cos\theta+\beta\frac{2}{\pi}(\ln r(\theta)\cos\theta-\theta\sin\theta)]=e^{-\frac{\pi u}{2\beta}}V(\theta;1,\beta).$$

$$\Re dz(\theta)=d(r(\theta)\cos\theta)=\frac{1}{2\beta}e^{-\frac{\pi u}{2\beta}}V(\theta;1,\beta).$$

Hence

$$p(u;1,\beta)=\Re\frac{1}{\pi}\int_\gamma e^{-h(z)}dz=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-\frac{\pi u}{2\beta}}V(\theta;1,\beta)\exp(-e^{-\frac{\pi u}{2\beta}}V(\theta;1,\beta))d\theta.$$

$\square$

Proof of Theorem 1: (a) The density expression is just a restatement of Theorems 2 and 3. The distribution function is the integral of the density formula.

(b) $h(t;0,\alpha,\beta)=\zeta t^\alpha$ so

$$f(\zeta;\alpha,\beta) = (1/\pi)\int_0^\infty\cos(\zeta t^\alpha)\exp(-t^\alpha)dt$$

$$
\begin{aligned}
&= (1/\pi)\int_0^\infty \cos(\zeta v)\exp(-v)v^{(1/\alpha)-1}dv \\
&= \Gamma(1+(1/\alpha))\cos((\arctan\zeta^2)/\alpha)/(\pi(1+\zeta^2)^{1/(2\alpha)}) \\
&= \Gamma(1+(1/\alpha))\cos\theta_0/(\pi(1+\zeta^2)^{1/(2\alpha)})
\end{aligned}
$$

The distribution function expression follows by taking $x \downarrow \zeta$ in (a): when $\alpha < 1$, the integrand converges to 0 pointwise, so $F(\zeta;\alpha,\beta) = c_1(\alpha,\beta)$. When $\alpha > 1$, the integrand converges to 1 pointwise and $F(\zeta;\alpha,\beta) = c_1(\alpha,\beta) - (\frac{\pi}{2} - (-\theta_0))/\pi = (\frac{\pi}{2} - \theta_0)/\pi$.
(c) It is straightforward that $\zeta(\alpha,-\beta) = -\zeta(\alpha,\beta)$ and $h(t;-u,\alpha,-\beta) = h(t;u,\alpha,\beta)$, so $p(-u;\alpha,-\beta) = p(u;\alpha,\beta)$.
(d) When $\alpha = 1$ and $\beta = 0$, this is the Cauchy distribution. When $\beta > 0$, the expression for the density is Theorem 3; the $\beta < 0$ case is similar. Integrating that formula leads to the distribution function formula. □

We note that computational formulas for density derivatives $f^{(n)}(x;\alpha,\beta)$ follow by repeated differentiation of the density formula in Theorem 1.

3 Numerical considerations

Computing stable densities and distributions involves numerically evaluating the integrals in Theorem 1. Without loss of generality, we assume $x > \zeta$ when $\alpha \neq 1$ and $\beta > 0$ when $\alpha = 1$. It is useful to rewrite the formulas as:

$$f(x;\alpha,\beta) = c_2(x,\alpha,\beta)\int_{-\theta_0}^{\frac{\pi}{2}} g(\theta;x,\alpha,\beta)\exp(-g(\theta;x,\alpha,\beta))d\theta \tag{4}$$

and

$$F(x;\alpha,\beta) = c_1(\alpha,\beta) + c_3(\alpha)\int_{-\theta_0}^{\frac{\pi}{2}} \exp(-g(\theta;x,\alpha,\beta))d\theta,$$

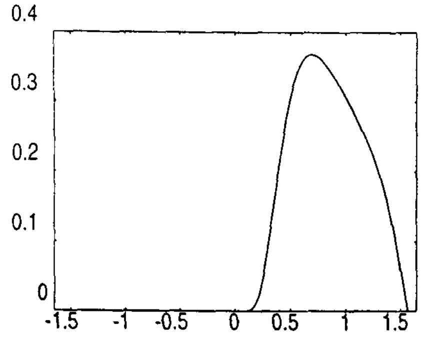
where

$$c_2(x,\alpha,\beta) = \begin{cases} \alpha/(\pi|\alpha-1|(x-\zeta)) & \alpha \neq 1 \\ 1/(2|\beta|) & \alpha = 1, \end{cases}$$

$$c_3(\alpha) = \begin{cases} \text{sign}\,(1-\alpha)/\pi & \alpha \neq 1 \\ 1/\pi & \alpha = 1, \end{cases}$$

$$g(\theta;x,\alpha,\beta) = \begin{cases} (x-\zeta)^{\frac{\alpha}{\alpha-1}}V(\theta;\alpha,\beta) & \alpha \neq 1 \\ e^{-\frac{\pi x}{2\beta}}V(\theta;1,\beta) & \alpha = 1. \end{cases}$$

To evaluate (4), first consider the case where $\alpha \neq 1$. It is known from the derivation of Theorem 3 (or the Appendix of Buckle (1995)) that $V(\cdot;\alpha,\beta)$ is continuous, positive, strictly monotonic (increasing when $\alpha < 1$, decreasing when $\alpha > 1$), is 0 at one endpoint and $+\infty$ at the other. Thus $g(\cdot;x,\alpha,\beta)$ has



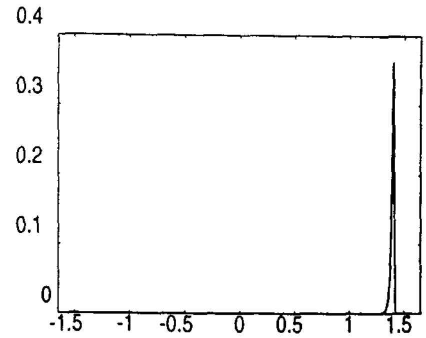


Figure 1: Graphs of the integrand functions $g(\cdot; 1, 1.5, -0.5) \exp(-g(\cdot; 1, 1.5, -0.5))$ (left), and $g(\cdot; 10, 0.99, 1) \exp(-g(\cdot; 10, 0.99, 1))$ (right).

the same properties and the integrand $g(\cdot)\exp(-g(\cdot))$ looks like Figure 1. In all cases the integrand starts at 0 when $\theta = -\theta_0$, increases monotonically to a maximum of $(1/e)$ at the (unique) point θ_2 where $g(\theta_2; x, \alpha, \beta) = 1$, and then decreases monotonically to 0 at $\theta = \frac{\pi}{2}$. Note that the integrand is a continuous, bounded, non-oscillating function and that the region of integration is a bounded interval.

Numerical integration techniques are well developed for such functions. The program STABLE uses the adaptive quadrature routine DQDAG from the IMSL subroutine library routine, see IMSL (1985), to evaluate these integrals. In some cases, e.g. the right side of Figure 1, the integrand can be very peaked, and the quadrature program can miss the spike and underestimate the integral. To avoid this problem, this program locates the peak (by numerically finding θ_2 where $g(\theta_2; x, \alpha, \beta) = 1$). The integral is then evaluated in two pieces: from $-\theta_0$ to θ_2 and from θ_2 to $\frac{\pi}{2}$.

When $\alpha = 1$, the following technical lemma (given without proof) shows that a similar approach works.

Lemma 1 *Fix $\beta \neq 0$. Then as $\alpha \to 1$, $-\theta_0 \to -\frac{\pi}{2}$ and $g(\theta; x, \alpha, \beta) \to g(\theta; x, 1, \beta)$.*

The distribution function is easier to approximate because the integrands are better behaved. When $\alpha < 1$, the integrand in the formula for $F(x; \alpha, \beta)$ starts at 1 when $\theta = -\theta_0$ and decreases monotonically to 0 when $\theta = \frac{\pi}{2}$. When $\alpha > 1$, the integrand increases monotonically from 0 to 1 over the region of integration.

STABLE works for most values of the (α, β) parameter space. The output has been compared to existing tables of densities and distribution functions

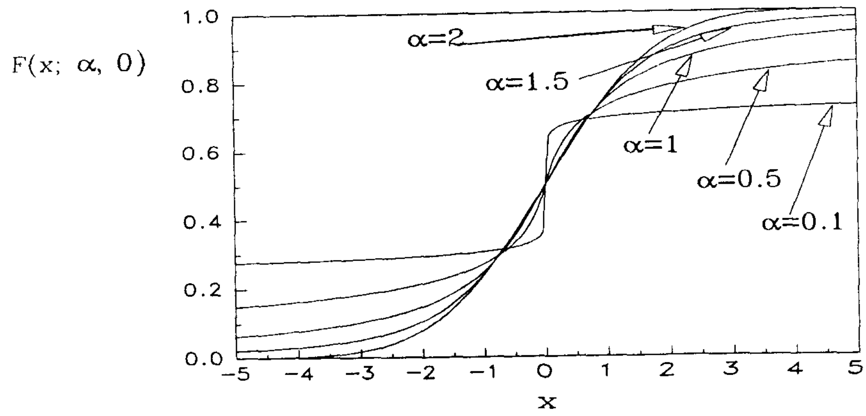


Figure 2: Plot of the symmetric stable distribution function $F(\cdot; \alpha, 0)$, α is indicated on the graph.

and agrees to the printed accuracy. For α very close to 1, i.e. $0 < |\alpha - 1| < 0.02$, or α close to zero, there are some numerical difficulties. When α is close to 1 but not equal 1, the integrands in the formulas for the density and distribution function change very rapidly and it is difficult to accurately approximate the integrals. When α is close to zero, the densities have a steep spike at the mode $m(\alpha, \beta)$. We are not aware of tables for these parameter values and it is hard to assess the accuracy of STABLE. For example, when $\alpha = 0.1$ and $\beta = 0$, $f(0; 0.1, 0) = 1.155 \times 10^6$, while $f(0.01; 0.1, 0) = 1.666$ - a change of six orders of magnitude when x changes by 0.01! Further calculations show $P(|X| \leq 0.01) = .2244$ and $P(|X| > 5) = .5527$, which is very different from a standardized Gaussian (or Cauchy) distribution: most of the mass is on the tails and much of what's left is concentrated very near the mode. Figure 2 shows the contrast between the distributions with small α and those with moderate/large α.

With the module STABLE, maximum likelihood estimation of stable parameters is possible from data. If one has a large sample, then using these routines may be slow. It is possible to precompute stable densities on a grid of x, α and β values and interpolate among these values for very high speed approximations of stable densities. However, to guarantee accuracy in such a scheme requires more work. In applications, one can probably restrict the range of α, and by symmetry β can be restricted to between 0 and 1. However, it is not clear how far such a table would have to extend in the x variable because we do not have accurate tail approximations for stable densities. While it is known that the densities have a power decay asymptotically, it is not yet known how large $|x|$ must be before that approximation is accurate. We will discuss maximum likelihood estimation and tail behavior in future work.

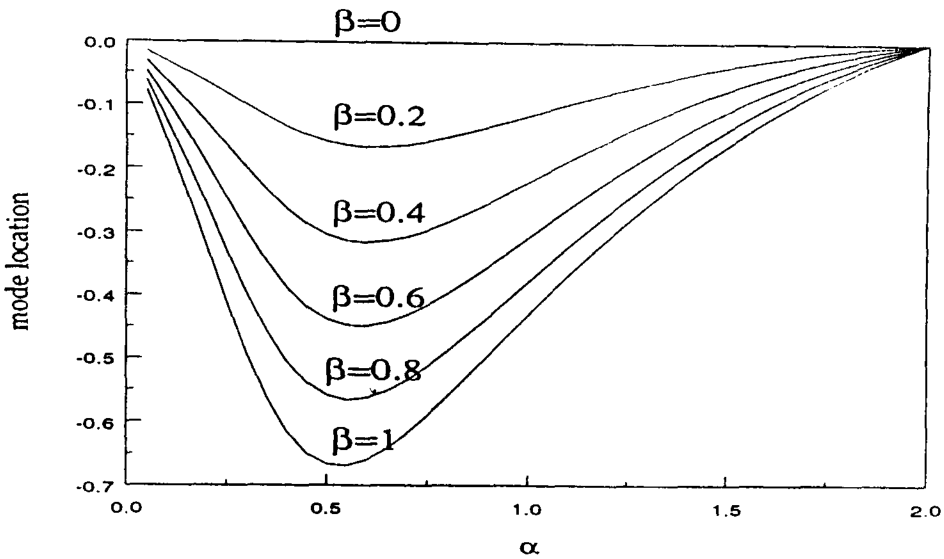


Figure 3: Plot of the mode location $m(\alpha, \beta)$ as a function of α and β.

4 Modes and parameterizations of stable distributions

All stable densities are unimodal, see §2.7 of Zolotarev (1986) for references. In fact, they are bell-shaped with the density having a positive derivative to the left of the mode and a negative derivative to the right of the mode. Previous work on locating the mode has used parameterization (C), because the characteristic function is simpler. However, there are drawbacks to such an approach: in the (A) or (C) parameterization with $\beta \neq 0$, the mode of a stable density tends to $(\operatorname{sign} \beta)\infty$ as $\alpha \uparrow 1$ and to $-(\operatorname{sign} \beta)\infty$ as $\alpha \downarrow 1$. The (M) parameterization does not have this problem because densities are jointly continuous in α and β.

Using the high quality density calculations possible with the STABLE module, we have numerically located the mode of a standardized stable random variable with parameters α and β, denoted by $m(\alpha, \beta)$, in the (M) representation. Symmetry arguments show that $m(\alpha, 0) = 0$ and $m(\alpha, -\beta) = -m(\alpha, \beta)$, so it suffices to locate the mode for $0 < \beta \leq 1$. Since each density is smooth and the derivative is zero exactly once, this is not a difficult numerical problem. While it is possible to numerically evaluate the derivative, we chose to use density values only, and to use a quadratic interpolation method to find the maximum. Specifically, we used the IMSL routine DUVMIF to numerically locate the minimum of $-f(x; \alpha, \beta)$. The results are plotted in Figure 3. The estimated accuracy in the computed values of $m(\alpha, \beta)$ is 0.0001.

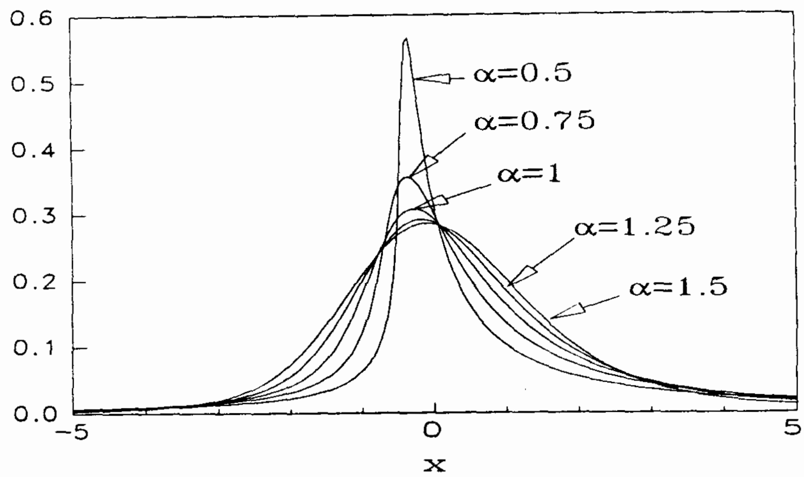


Figure 4: Plot of stable densities $f(\cdot;\alpha,\beta)$ for an $S^0_\alpha(1,0.5,0)$ distribution, α is indicated on the plot.

The values in Figure 3 have been stored in a table for the grid $\alpha = 0.05$ to 2 (step 0.05) and $\beta = 0$ to 1 (step 0.1). For speed, the function SMODE in the STABLE module returns a value for the mode by linear interpolation from this table. Numerical information on the height of the density at the mode and some partial analytical results on the mode are given in Nolan (1996).

The program has built in 3 parameterizations for stable distributions. The routines to calculate a stable density or distribution function take an argument, called `iparam`, that determines which parameterization is used. The notation below is from Nolan (1996), where more detail can be found on these parameterizations.

- $S^0_\alpha(\sigma,\beta,\mu^0)$ (`iparam=0`) A scale and location family based on the (M) parameterization: if Z has characteristic function (2), then $X^0 \stackrel{d}{=} \sigma Z + \mu^0 \sim S^0_\alpha(\sigma,\beta,\mu^0)$. Figure 4 shows some sample densities in this parameterization.

- $S_\alpha(\sigma,\beta,\mu)$ (`iparam=1`) The standard parameterization specified by the characteristic function (1). Figure 5 shows some sample densities in this parameterizations.

- $S^*_\alpha(\sigma^*,\beta,\mu^*)$ (`iparam=2`) A variation of the $S^0_\alpha(\sigma,\beta,\mu^0)$ parameterization that is shifted to have mode at μ^* and have scale that agrees with the standard deviation in the Gaussian case and the standard scale in the Cauchy case. Specifically, $X^* \stackrel{d}{=} \alpha^{-1/\alpha}\sigma^*(Z - m(\alpha,\beta)) + \mu^* \sim$

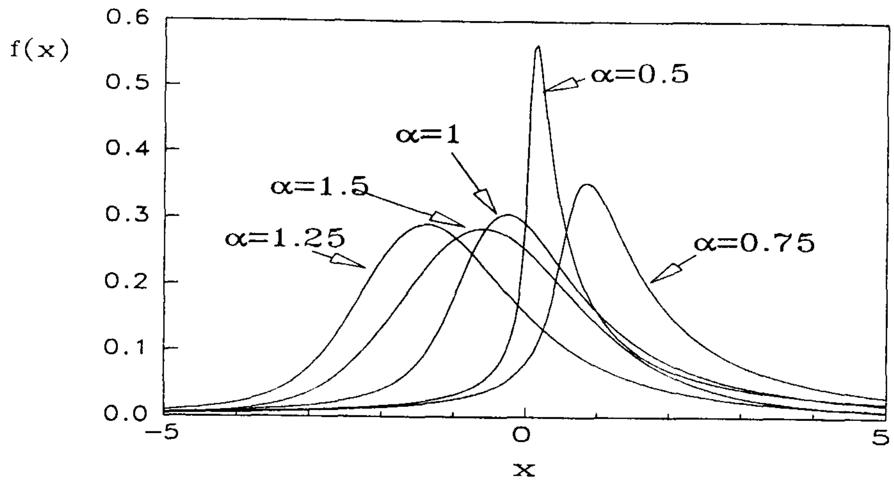


Figure 5: Plot of stable densities $f(\cdot;\alpha,\beta)$ for an $S_\alpha(1,0.5,0)$ distribution, α is indicated on the plot.

$S^*_\alpha(\sigma^*,\beta,\mu^*)$, where Z has characteristic function (2). This parameterization uses the mode as the most significant feature of a stable density and the scale change emphasizes the fact that stable distributions with higher α are more concentrated near the origin, while those with lower α have heavier tails. See Figure 6.

All computations are done in the S^0 parameterization, which is the simplest case for which the densities and distribution functions are continuous in all parameters, and for which an exact formula is known for the terms in the density.

5 Description of the program STABLE

Version 1 of STABLE consists of about 900 lines of FORTRAN source code. The entry points for the module are:

- SPDF - calculates stable probability density function at a vector of x values using the formulas in Theorem 1.
- SCDF - calculates stable cumulative distribution function at a vector of x values using the formulas in Theorem 1.
- SQUANT - calculates stable quantiles for a vector of probabilities. The algorithm makes repeated calls to SCDF to bracket the desired quantile, then does a binary search to locate the quantile.

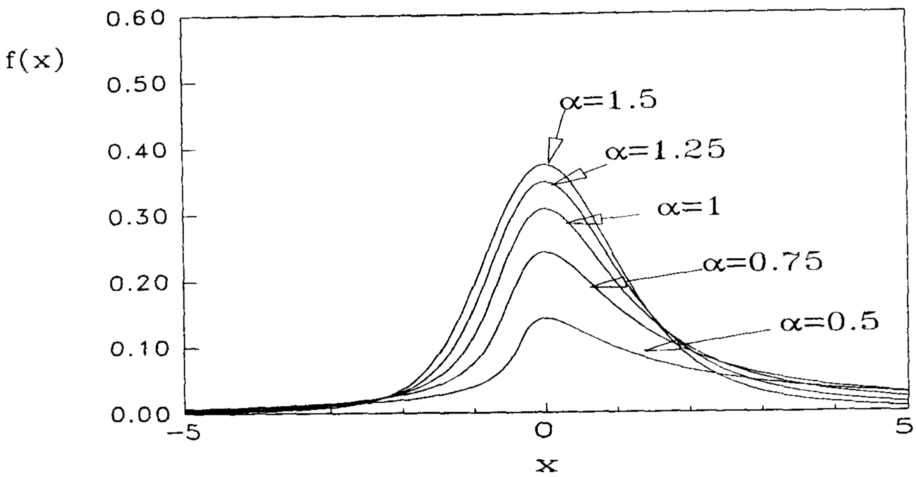


Figure 6: Plot of stable densities $f(\cdot;\alpha,\beta)$ for an $S^*_\alpha(1,0.5,0)$ distribution, α is indicated on the plot.

- SMODE - calculates the mode of a standardized stable density in the S^0 parameterization. See Section 4 for information on this function.
- SVER - returns a string with version information about the STABLE software.

For estimates of tail densities and tails of the distribution function, it is important to get relative, not absolute, accuracy in the calculations. The integral formulas in Theorem 1, in contrast to the formulas in Theorem 2, facilitate this. Relative accuracy is set to 10^{-6}. This accuracy can be changed by recompiling the source code; as usual, increasing the accuracy will generally slow down execution times. For speed, the program uses the known formulas for the density and distribution function for the Cauchy, Gaussian and Lévy distributions (the distribution functions for the latter two involve calls to the standard error function erf($\cdot$)). The main routines SPDF and SCDF are vector routines, allowing for parallel processing on suitable machines.

The STABLE module calls several external routines for numerical calculations: DERF (double precision error function), DGAMMA (double precision gamma function), and DQDAG (double precision adaptive quadrature subroutine). The current implementation uses the proprietary IMSL subroutines for these purposes, see IMSL (1985). Future implementations will use public domain routines for these functions, and the source will be made available enabling interfaces to S-Plus, Matlab, Mathematica, Gauss, etc. In the mean time, an executable version of STABLE for DOS is available from the WWW:

open http://www.cas.american.edu/~jpnolan and click on the link to stable software. That program has a driver routine that uses most of the features of the module. Under the DOS operating system, execution times are about 200 density evaluations per second (on a 60 MHz Pentium PC).

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