Solutions to *Elements of Set Theory*[*]

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Contents

1 Introduction

Exercise 1.1. *Which of the following become true when $\in$ is inserted in place of the blank? Which become true when $\subseteq$ is inserted?*

(a) $\{\emptyset\}$ ____ $\{\emptyset, \{\emptyset\}\}$.

[*]by Herbert Enderton. I make no claim to correctness, as I'm just learning as I write these.

(b) $\{\emptyset\}$ ____ $\{\emptyset, \{\{\emptyset\}\}\}$.

(c) $\{\{\emptyset\}\}$ ____ $\{\emptyset, \{\emptyset\}\}$.

(d) $\{\{\emptyset\}\}$ ____ $\{\emptyset, \{\{\emptyset\}\}\}$.

(e) $\{\{\emptyset\}\}$ ____ $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$.

Solution. Choices (a) and (d) become true when $\in$ is inserted in place of the blank. Choices (b) and (c) become true when $\subseteq$ is inserted in place of the blank. Choice (e) is not true in either case. □

Exercise 1.2. *Show that not two of the three sets $\emptyset$, $\{\emptyset\}$, and $\{\{\emptyset\}\}$ are equal to each other.*

Solution. Note that $\{\emptyset\} \nsubseteq \emptyset$, and $\{\{\emptyset\}\} \nsubseteq \emptyset$. Also, $\{\{\emptyset\}\} \nsubseteq \{\emptyset\}$. Hence no two are equal. □

Exercise 1.3. *Show that if $B \subseteq C$, then $\mathscr{P}B \subseteq \mathscr{P}C$.*

Solution. Suppose $A \in \mathscr{P}B$. Then $A \subseteq B$, and thus $A \subseteq C$, as containment is transitive. Hence $A \in \mathscr{P}C$. □

Exercise 1.4. *Assume that x and y are members of a set B. Show that $\{\{x\}, \{x, y\}\} \in \mathscr{P}\mathscr{P}B$.*

Proof. Since x and y are members of B, it follows that $\{x\} \subseteq B$ and $\{x, y\} \subseteq B$. So $\{x\}$ and $\{x, y\} \in \mathscr{P}B$, and thus $\{\{x\}, \{x, y\}\} \subseteq \mathscr{P}B$, so $\{\{x\}, \{x, y\}\} \in \mathscr{P}\mathscr{P}B$. □

Exercise 1.5. *Define the rank of a set c to be the least α such that $c \subseteq V_\alpha$. Compute the rank of $\{\{\emptyset\}\}$. Compute the rank of $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.*

Solution. Observe that $V_{\alpha+1} = \mathscr{P}V_\alpha$. Taking $V_0 = A = \emptyset$, it follows that

$$V_1 = \mathscr{P}V_0 = \{\emptyset, \{\emptyset\}\}.$$

Hence the rank of $\{\{\emptyset\}\}$ is 1. Futhermore,

$$V_2 = \mathscr{P}V_1 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}.$$

Thus $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ has rank 2. □

Exercise 1.6. *We have stated that $V_{\alpha+1} = A \cup \mathscr{P}V_\alpha$. Prove this at least for $\alpha < 3$.*

Solution. By definition, for $\alpha = 0$,

$$V_1 = V_0 \cup \mathscr{P}V_0 = A \cup \mathscr{P}V_0.$$

For $\alpha = 1$,

$$V_2 = V_1 \cup \mathscr{P}V_1 = A \cup \mathscr{P}V_0 \cup \mathscr{P}V_1 = A \cup \mathscr{P}V_1.$$

The last equality follows from Exercise 1.3, as $V_0 \subseteq V_1$, and thus $\mathscr{P}V_0 \cup \mathscr{P}V_1 = \mathscr{P}V_1$. The case for $\alpha = 2$ follows similarly. □

Exercise 1.7. *List all the members of V_3. List all the members of V_4.*

Solution. Without listing them, V_3 has 16 members, and V_4 has 32 members. $\square$

2 Axioms and Operations

Exercise 2.1. *Assume that A is the set of integers divisible by 4. Similarly assume that B and C are the sets of integers divisible by 9 and 10, respectively. What is in $A \cap B \cap C$?*

Solution. The set $A \cap B \cap C$ consists precisely of the numbers divisible by the least common multiple of 4, 9 and 10. That is, it is the set of all multiples of 180. $\square$

Exercise 2.2. *Give an example of sets A and B for which $\bigcup A = \bigcup B$ but $A \neq B$.*

Solution. Take $A = \{\{a\}, \{b, c\}\}$ and $B = \{\{a\}, \{b\}, \{c\}\}$. $\square$

Exercise 2.3. *Show that every member of a set A is a subset of $\bigcup A$.*

Solution. If $A = \emptyset$, then the statement is vacuously true. So suppose A is nonempty, and take some $b \in A$. Again, if b is empty, then the statement holds. So suppose b is nonempty. Then for any $t \in b$, $t \in \bigcup A$ by the Union Axiom. In other words,

$$\forall t (t \in b \Rightarrow t \in \bigcup A).$$

This is precisely the definition that $b \subseteq \bigcup A$. $\square$

Exercise 2.4. *Show that if $A \subseteq B$, then $\bigcup A \subseteq \bigcup B$.*

Solution. Recall that $x \in \bigcup A \Leftrightarrow (\exists b \in A)\ x \in b$. Take $x \in \bigcup A$. So by definition, there exists $b \in A$ such that $x \in b$. Since $A \subseteq B$, $b \in B$ as well. Then by definition, $x \in \bigcup B$. $\square$

Exercise 2.5. *Assume that every member of $\mathscr{A}$ is a subset of B. Show that $\bigcup \mathscr{A} \subseteq B$.*

Solution. Take $a \in \bigcup \mathscr{A}$. So by the Union Axiom, there exists $b \in A$ such that $a \in b$. By assumption, $b \subseteq B$, and thus by definition of subset, $a \in B$. $\square$

Exercise 2.6. *(a) Show that for any set A, $\bigcup \mathscr{P} A = A$.*

(b) Show that $A \subseteq \mathscr{P} \bigcup A$. Under what conditions does equality hold?

Solution. For (a), take $x \in \bigcup \mathscr{P} A$. So there exists some $b \in \mathscr{P} A$ such that $x \in b$. Since $b \in \mathscr{P} A$, $b \subseteq A$, and thus $x \in A$. Also, since $A \subseteq A$, by definition of the power set, $A \in \mathscr{P} A$. Then by Exercise 2.3, it follows that $A \subseteq \bigcup \mathscr{P} A$.

For (b), take $x \in A$. Again by Exercise 2.3, $x \subseteq \bigcup A$, so by definition of the power set, $x \in \mathscr{P} \bigcup A$. Equality holds when A has form $\{\emptyset\}$. If $A = \{\emptyset\}$, then $\bigcup A = \emptyset$, and so $\mathscr{P} \bigcup A = \{\emptyset\}$. If $A = \{a, \emptyset\}$ for some $a \in A$, then $\bigcup A = a$, and so $\mathscr{P} \bigcup A = \mathscr{P} a$, which gives all subsets of a, which may not equal A. Note that if $A = \{a, b\}$ for $a, b \in A$ and a, b nonempty, then $\bigcup A = a \cup b$, so $\mathscr{P} \bigcup A$ contains $a \cup b$, and thus equality does not hold in this case. $\square$

Exercise 2.7. *(a) Show that for any sets A and B,*

$$\mathscr{P}A \cap \mathscr{P}B = \mathscr{P}(A \cap B).$$

(b) Show that $\mathscr{P}A \cup \mathscr{P}B \subseteq \mathscr{P}(A \cup B)$. Under what conditions does equality hold?

Solution. Take $x \in \mathscr{P}A \cap \mathscr{P}B$. Then $x \subseteq A$ and $x \subseteq B$, so in either case, $x \subseteq A \cap B$. Hence $x \in \mathscr{P}(A \cap B)$. Take $y \in \mathscr{P}(A \cap B)$. Thus $y \subseteq A \cap B$, and so $y \subseteq A$ and $y \subseteq B$, so $y \in \mathscr{P}A \cap \mathscr{P}B$.

For (b), take $x \in \mathscr{P}A \cup \mathscr{P}B$. Then $x \subseteq A$ or $x \subseteq B$, so in either case, $x \subseteq A \cup B$. Hence $x \in \mathscr{P}(A \cup B)$. Furthermore, equality holds if either A or B is a subset of the other. Without loss of generality, suppose $B \subseteq A$. If $x \subseteq A \cup B$, then $x \subseteq A$ since $A \cup B = A$. The same argument holds if $B \subseteq A$. However, suppose $B \nsubseteq A$. Consider then the case where $B = \{a, b\}$ and $A = \{a, c\}$. Then let $x = \{b, c\}$. Then $x \in \mathscr{P}(A \cup B)$, but $x \notin \mathscr{P}A \cup \mathscr{P}B$. ☐

Exercise 2.8. *Show that there is no set to which every singleton (that is, every set of the form $\{x\}$) belongs.*

Solution. Suppose $\mathscr{X}$ is the set of all singletons. Then $\bigcup \mathscr{X}$ is the set of all sets, which leads to Russell's paradox. ☐

Exercise 2.9. *Give an example of sets a and B for which $a \in B$ but $\mathscr{P}a \notin \mathscr{P}B$.*

Solution. Let $a = \{\emptyset\}$ and $B = \{\{\emptyset\}\}$. Clearly $a \in B$. Then $\mathscr{P}a = \{\emptyset, \{\emptyset\}\}$ and $\mathscr{P}B = \{\emptyset, \{\{\emptyset\}\}\}$, so $\mathscr{P}a \notin \mathscr{P}B$. ☐

Exercise 2.10. *Show that if $a \in B$, then $\mathscr{P}a \in \mathscr{P}\mathscr{P}\bigcup B$.*

Solution. First, recall from Exercise 2.6 that for any set A, $A \subseteq \mathscr{P}\bigcup A$. So in this case, $B \subseteq \mathscr{P}\bigcup B$. Hence if $a \in B$, then $a \in \mathscr{P}\bigcup B$, that is, $a \subseteq \bigcup B$. Then observe that $\mathscr{P}a \subseteq \mathscr{P}\bigcup B$, by Exercise 1.3. From $\mathscr{P}a \subseteq \mathscr{P}\bigcup B$ it follows that $\mathscr{P}a \in \mathscr{P}\mathscr{P}\bigcup B$. ☐

Exercise 2.11. *Show that for any sets A and B,*

$$A = (A \cap B) \cup (A - B) \quad \textit{and} \quad A \cup (B - A) = A \cup B$$

Solution. $A \subseteq (A \cup B) \cup (A - B)$ is obvious. Now take $x \in (A \cap B) \cup (A - B)$. If $x \in A \cap B$, then $x \in A$. If $x \in A - B$, again $x \in A$.

Since $B - A \subseteq B$, we have $A \cup (B - A) \subseteq A \cup B$ by the monotonicity properties. Take $x \in A \cup B$. If $x \in A$, then clearly $x \in A \cup (B - A)$. If $x \in B$, then by the first part of the exercise, then either $x \in A \cap B$, whence $x \in A$, or $x \in B - A$. ☐

Exercise 2.12. *Verify the following identity:*

$$C - (A \cap B) = (C - A) \cup (C - B).$$

Solution. Take $x \in C - (A \cap B)$. Then $x \in C$, but $x \in A \cap B$. Note that it is not the case that $x \in A$ and $x \in B$. If $x \in A$ but $x \notin B$, then $x \in C - B$. If $x \notin A$ but $x \in B$, then $x \in C - A$. Finally, if $x \notin A$ and $x \notin B$, then $x \in C - (A \cap B) = (C - A) \cup (C - B)$.

Now take $x \in (C - A) \cup (C - B)$. If $x \in C - A$, then $x \notin A$, and thus $x \notin A \cap B$. Then $x \in C - (A \cap B)$. A similar argument handles the case where $x \in C - B$. $\square$

Exercise 2.13. *Show that if $A \subseteq B$, then $C - B \subseteq C - A$.*

Solution. Recall that if $A \subseteq B$, then $-B \subseteq -A$. Take $x \in C - B$. Since $x \notin B$, then $x \in -B$, and so $x \in -A$. It follows that $x \in C - A$. $\square$

Exercise 2.14. *Show by exmaple that for some sets A, B, and C, the set $A - (B - C)$ is different from $(A - B) - C$.*

Solution. Let $A = \{a, d, e, f\}$, $B = \{b, d, e, g\}$, and $C = \{c, e, f, g\}$. Then $B - C = \{b, d\}$, so $A - (B - C) = \{a, e, f\}$. But $A - B = \{a, f\}$, so $(A - B) - C = \{a\}$. $\square$

Exercise 2.15. *Define the symmetric difference $A + B$ of sets A and B to be the set $(A - B) \cup (B - A)$.*

(a) Show that $A \cap (B + C) = (A \cap B) + (A \cap C)$.

(b) Show that $A + (B + C) = (A + B) + C$.

Solution. First observe that in general, $A \cap (B - C) = (A \cap B) - (A \cap C)$. Hence

$$\begin{aligned} A \cap (B + C) &= A \cap [(B - C) \cup (C - B)] \\ &= [A \cap (B - C)] \cup [A \cap (C - B)] \\ &= [(A \cap B) - (A \cap C)] \cup [(A \cap C) - (A \cap B)] \\ &= (A \cap B) + (A \cap C) \end{aligned}$$

Part (b) can best be shown with a membership table. First observe that

$$\begin{aligned} A + (B + C) &= A + [(B - C) \cup (C - B)] \\ &= [A - [(B - C) \cup (C - B)]] \cup [[(B - C) \cup (C - B)] - A] \end{aligned}$$

and

$$\begin{aligned} (A + B) + C &= [(A - B) \cup (B - A)] + C \\ &= [[(A - B) \cup (B - A)] - C] \cup [C - [(A - B) \cup (B - A)]] \end{aligned}$$

From this it is easy to verify the following membership table.

A	B	C	A+(B+C)	(A+B)+C
✓	✓	✓	✓	✓
✓	✓			
✓		✓		
	✓	✓		
✓			✓	✓
	✓		✓	✓
		✓	✓	✓

□

Exercise 2.16. *Simplify:*

$$[(A \cup B \cup C) \cap (A \cup B)] - [(A \cup (B - C)) \cap A].$$

Solution. Since $(A \cup B) \subseteq (A \cup B \cup C)$, it follows that $(A \cup B \cup C) \cap (A \cup B) = (A \cup B)$. For the same reason, $(A \cup (B - C)) \cap A = A$. So the expression simplifies to $(A \cup B) - A$, which further simplifies to $B - A$. □

Exercise 2.17. *Show that the following four conditions are equivalent.*

(a) $A \subseteq B$,

(b) $A - B = \emptyset$,

(c) $A \cup B = B$,

(d) $A \cap B = A$.

Solution. If $A \subseteq B$, then there is no element in A that is not also in B, hence $A - B = \emptyset$. Also, if $A - B = \emptyset$, then if there is any $x \in A$, then $x \in B$ as well. So (a) and (b) are equivalent. Also (a) is clearly equivalent to both (c) and (d). □

Exercise 2.18. *Assume that A and B are subsets of S. List all of the different sets that can be made from these three by use of the binary operations $\cup$, $\cap$, and $-$.*

Solution. A few that spring to mind are $A, B, S, A \cup B, A \cap B, A - B, B - A, S - A, S - B, S - (A \cup B), S - (A \cap B), S - (A - B), S - (B - A), S - S = \emptyset$. □

Exercise 2.19. *Is $\mathscr{P}(A - B)$ always equal to $\mathscr{P}A - \mathscr{P}B$? Is it ever equal to $\mathscr{P}A - \mathscr{P}B$?*

Solution. Recall that $\emptyset$ is in the power set of any set. So $\emptyset \in \mathscr{P}(A - B)$, as well as $\mathscr{P}B$. It follows that $\emptyset \notin \mathscr{P}A - \mathscr{P}B$, and thus the two will never be equal. □

Exercise 2.20. *Let A, B, and C be sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.*

Solution. Take $x \in B$. There are two subcases. If $x \in A$, then $x \in A \cap B$, and thus $x \in A \cap C$, so $x \in C$. If $x \notin A$, then $x \in A \cup B$ regardless, so $x \in A \cup C$, from which it follows that $x \in C$. In either case, $x \in C$, so $B \subseteq C$. A symmetric argument shows that $C \subseteq B$, and so $B = C$. □

Exercise 2.21. *Show that* $\bigcup(A \cup B) = \bigcup A \cup \bigcup B$.

Solution. Take $x \in \bigcup(A \cup B)$. Hence $\exists y \in (A \cup B)$ such that $x \in y$. Now $y \in A$ or $y \in B$, and so $x \in \bigcup A$ or $x \in \bigcup B$. In either case, $x \in \bigcup A \cup \bigcup B$.

Take $x \in \bigcup A \cup \bigcup B$. If $x \in \bigcup A$, then $\exists y \in A$ such that $x \in y$, Since $y \in A \cup B$, $x \in \bigcup(A \cup B)$ as well. A similar argument handles the case where $x \in \bigcup B$. □

Exercise 2.22. *Show that if A and B are nonempty sets, then* $\bigcap(A \cup B) = \bigcap A \cap \bigcap B$.

Solution. Take $x \in \bigcap(A \cup B)$. So for all $y \in A \cup B$, $x \in y$. In particular, for all $a \in A$, $x \in a$, so $x \in \bigcap A$, and for all $b \in B$, $x \in b$, so $x \in \bigcap B$.

Now take $x \in \bigcap A \cap \bigcap B$. So $x \in \bigcap A$ and $x \in \bigcap B$. Then for any $c \in A \cup B$, $c \in A$ or $c \in B$. It follows that $x \in c$. Hence $x \in \bigcap(A \cup B)$. □

Exercise 2.23. *Show that if $\mathscr{B}$ is nonempty, then* $A \cup \bigcap \mathscr{B} = \bigcap\{A \cup X \mid X \in \mathscr{B}\}$.

Solution. First observe that for any $X \in \mathscr{B}$, $\bigcap \mathscr{B} \subseteq X$. Hence $A \cup \bigcap \mathscr{B} \subseteq A \cap X$ for all $X \in \mathscr{B}$, and thus $A \cup \bigcap \mathscr{B} \subseteq \bigcap\{A \cup X \mid X \in \mathscr{B}\}$. For the reverse containment, if $x \in \mathscr{P}X$ for all $X \in \mathscr{A}$, then $x \subseteq X$ for all X, and thus $x \subseteq \bigcap \mathscr{A}$. Hence $x \in \mathscr{P} \bigcap \mathscr{A}$.

Now take any $x \in \bigcap\{A \cup X \mid X \in \mathscr{B}\}$. So $x \in A \cup X$ for all $X \in \mathscr{B}$. If $x \in A$, the containment holds. Otherwise, if $x \notin A$, we must have $x \in X$ for all $X \in \mathscr{B}$, that is, $x \in \bigcap \mathscr{B}$, and the equality follows. □

Exercise 2.24. *(a) Show that if $\mathscr{A}$ is nonempty, then* $\mathscr{P} \bigcap \mathscr{A} = \bigcap\{\mathscr{P}X \mid X \in \mathscr{A}\}$.

(b) Show that

$$\bigcup\{\mathscr{P}X \mid X \in \mathscr{A}\} \subseteq \mathscr{P} \bigcup \mathscr{A}.$$

Under what conditions does equality hold?

Solution. Take $x \in \mathscr{P} \bigcup \mathscr{A}$. So $x \subseteq \bigcup \mathscr{A}$. But observe that for any $X \in \mathscr{A}$, $\bigcap \mathscr{A} \subseteq X$, and so $x \subseteq X$. That is, $x \in \mathscr{P}X$, so $x \in \bigcap\{\mathscr{P}X \mid X \in \mathscr{A}\}$.

For (b), take $x \in \bigcup\{\mathscr{P}X \mid X \in \mathscr{A}\}$. So $x \in \mathscr{P}X$ for some $X \in \mathscr{A}$. It follows that $x \in X$, but $X \subseteq \bigcup \mathscr{A}$, so $x \subseteq \bigcup \mathscr{A}$, so $x \in \mathscr{P} \bigcup \mathscr{A}$. Now take $x \in \mathscr{P} \bigcup \mathscr{A}$. So $x \subseteq \bigcup \mathscr{A}$. In order for $x \in \bigcup\{\mathscr{P}X \mid X \in \mathscr{A}\}$, one needs $x \subseteq X$. If $\bigcup \mathscr{A} \subseteq X$ for some $X \in \mathscr{A}$, then equality will hold. □

Exercise 2.25. *Is* $A \cup \bigcup \mathscr{B}$ *always the same as* $\bigcup\{A \cup X \mid X \in \mathscr{B}\}$*? If not, then under what conditions does equality hold?*

Solution. Yes, they are always the same. Take $x \in A \cup \bigcup \mathscr{B}$. If $x \in A$, then $x \in A \cup X$ for all $X \in \mathscr{B}$, so $x \in \bigcup\{A \cup X \mid X \in \mathscr{B}\}$. If $x \in \bigcup \mathscr{B}$, then $x \in X$ for some $X \in \mathscr{B}$, and thus $x \in A \cup X$ for that particular X.

If $x \in \bigcup\{A \cup X \mid X \in \mathscr{B}\}$, then $x \in A \cup X$ for some $X \in \mathscr{B}$. If $x \in A$ we are done. If $x \notin A$, then we must have $x \in X$ for some $X \in \mathscr{B}$, and thus $x \in \bigcup \mathscr{B}$. □

Exercise 2.26. *Consider the following sets: $A = \{3, 4\}$, $B = \{4, 3\} \cup \emptyset$, $C = \{4, 3\} \cup \{\emptyset\}$, $D = \{x \mid x^2 - 7x + 12 = 0\}$, $E = \{\emptyset, 3, 4\}$, $F = \{4, 4, 3\}$, $G = \{4, \emptyset, \emptyset, 3\}$. For each pair of sets specify whether or not the sets are equal.*

Solution. First, simplify the sets. We have $A = \{3, 4\}$, $B = \{3, 4\}$, $C = \{3, 4, \emptyset\}$, $D = \{3, 4\}$, $E = \{\emptyset, 3, 4\}$, $F = \{3, 4\}$, $G = \{4, \emptyset, 3\}$. From this, it is clear which sets are equal and which are not. □

Exercise 2.27. *Give an example of sets A and B for which $A \cap B$ is nonempty and*

$$\bigcap A \cap \bigcap B \neq \bigcap (A \cap B).$$

Solution. Take $A = \{\emptyset, \{\emptyset\}\}$ and $B = \{\{\emptyset\}\}$. Then $A \cap B = \{\{\emptyset\}\}$. But $\bigcap A = \emptyset$, so $\bigcap A \cap \bigcap B = \emptyset$. However, $\bigcap(A \cap B) = \bigcap\{\{\emptyset\}\} = \{\emptyset\}$. □

Exercise 2.28. *Simplify:*

$$\bigcup\{\{3, 4\}, \{\{3\}, \{4\}\}, \{3, \{4\}\}, \{\{3\}, 4\}\}.$$

Solution.

$$\begin{aligned}\bigcup\{\{3, 4\}, \{\{3\}, \{4\}\}, \{3, \{4\}\}, \{\{3\}, 4\}\} &= \{3, 4, \{3\}, \{4\}, 3, \{4\}, \{3\}, 4\} \\ &= \{3, 4, \{3\}, \{4\}\}\end{aligned}$$

□

Exercise 2.29. *Simplify:*

(a) $\bigcap\{\mathscr{P}\mathscr{P}\mathscr{P}\emptyset, \mathscr{P}\mathscr{P}\emptyset, \mathscr{P}\emptyset, \emptyset\}$.

(b) $\bigcap\{\mathscr{P}\mathscr{P}\mathscr{P}\{\emptyset\}, \mathscr{P}\mathscr{P}\{\emptyset\}, \mathscr{P}\{\emptyset\}\}$.

Solution. For (a), since $\emptyset$ is included in the family, any member of the intersection must a member of $\emptyset$, which gives $\bigcap\{\mathscr{P}\mathscr{P}\mathscr{P}\emptyset, \mathscr{P}\mathscr{P}\emptyset, \mathscr{P}\emptyset, \emptyset\} = \emptyset$.

For (b), observe that $\mathscr{P}\{\emptyset\}$ is a subset of both $\mathscr{P}\mathscr{P}\mathscr{P}\{\emptyset\}$ and $\mathscr{P}\mathscr{P}\{\emptyset\}$, and hence $\bigcap\{\mathscr{P}\mathscr{P}\mathscr{P}\{\emptyset\}, \mathscr{P}\mathscr{P}\{\emptyset\}, \mathscr{P}\{\emptyset\}\} = \mathscr{P}\{\emptyset\}$. □

Exercise 2.30. *Let A be the set $\{\{\emptyset\}, \{\{\emptyset\}\}\}$. Evaluate the following:*

(a) $\mathscr{P}A$,

(b) $\bigcup A$,

(c) $\mathscr{P}\bigcup A$,

(d) $\bigcup \mathscr{P} A$.

Solution. Indeed,

(a) $\mathscr{P} A = \{\emptyset, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$.

(b) $\bigcup A = \{\emptyset, \{\emptyset\}\}$.

(c) $\mathscr{P} \bigcup A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

(d) Recall that taking a union of the power set of some set essentially returns said set. Hence $\bigcup \mathscr{P} A = A$.

$\square$

Exercise 2.31. *Let B be the set $\{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{\emptyset\}\}$. Evaluate the following sets.*

(a) $\bigcup B$,

(b) $\bigcap B$,

(c) $\bigcap \bigcup B$,

(d) $\bigcup \bigcap B$.

Solution. Indeed,

(a) $\bigcup B = \{1, 2, 3, \emptyset\}$,

(b) $\bigcap B = \emptyset$,

(c) $\bigcap \bigcup B = \bigcap\{1, 2, 3, \emptyset\} = \{1, 2, 3, \emptyset\}$,

(d) $\bigcup \bigcap B = \bigcup \emptyset = \emptyset$.

$\square$

Exercise 2.32. *Let S be the set $\{\{a\}, \{a, b\}\}$. Evaluate and simplify:*

(a) $\bigcup \bigcup S$,

(b) $\bigcap \bigcap S$,

(c) $\bigcap \bigcup S \cup (\bigcup \bigcup S - \bigcup \bigcap S)$.

Solution. First, $\bigcup \bigcup S = \bigcup\{a, b\} = a \cup b$, and $\bigcap \bigcap S = \bigcap\{a\} = a$. For (c), first note $\bigcap \bigcup S = a \cap b$ and $\bigcup \bigcap S = a$. Thus $\bigcap \bigcup S \cup (\bigcup \bigcup S - \bigcup \bigcap S) = (a \cap b) \cup (a \cup b - a) = (a \cap b) \cup (b - a) = b$. $\square$

Exercise 2.33. *With S as in the preceding exercise, evaluate $\bigcup(\bigcup S - \bigcap S)$ when $a \neq b$ and when $a = b$.*

Solution. Suppose $a \neq b$. Then $\bigcup S = \{a, b\}$ and $\bigcap S = \{a\}$, so

$$\bigcup(\bigcup S - \bigcap S) = \bigcup(\{a, b\} - \{a\}) = \bigcup(\{b\}) = b.$$

If $a = b$, then $S = \{\{a\}\} = \{\{b\}\}$. In either case, $\bigcup S - \bigcap S = \emptyset$, so $\bigcup(\bigcup S - \bigcap S) = \bigcup \emptyset = \emptyset$. $\square$

Exercise 2.34. *Show that $\{\emptyset, \{\emptyset\}\} \in \mathscr{P}\mathscr{P}\mathscr{P}S$ for every set S.*

Solution. Let S be any set. Since $\emptyset \subseteq S$, we have $\emptyset \in \mathscr{P}S$. It follows that $\{\emptyset\} \subseteq \mathscr{P}S$, and since clearly $\emptyset \subseteq \mathscr{P}S$, we have $\emptyset, \{\emptyset\} \in \mathscr{P}\mathscr{P}S$. Hence $\{\emptyset, \{\emptyset\}\} \subseteq \mathscr{P}\mathscr{P}S$ so $\{\emptyset, \{\emptyset\}\} \in \mathscr{P}\mathscr{P}\mathscr{P}S$. $\square$

Exercise 2.35. *Assume that $\mathscr{P}A = \mathscr{P}B$. Prove that $A = B$.*

Solution. Take $x \in A$. Then $\{x\} \subseteq A$, so $\{x\} \in \mathscr{P}A = \mathscr{P}B$. Thus $\{x\} \subseteq B$ or $x \in B$. So $A \subseteq B$ and a symmetric argument shows $B \subseteq A$, so $A = B$. Note this show that any set has a unique power set. $\square$

Exercise 2.36. *Verify that for all sets the following are correct.*

(a) $A - (A \cap B) = A - B$.

(b) $A - (A - B) = A \cap B$.

Solution. Take $x \in A - (A \cap B)$. Hence $x \in A$ but $x \notin B$, for if $x \in B$, then $x \in A \cap B$, a contradiction. For the other containment, note $A \cap B \subseteq B$, and so $A - (A \cap B) \supseteq A - B$.

For (b), take $x \in A - (A - B)$, so $x \in A$ but $x \notin A - B$, which implies $x \notin A$ or $x \in B$. Since $x \in A$, we must have $x \in B$, so $x \in A \cap B$. For the reverse containment, if $x \in A \cap B$, then $x \in A$, and since $x \in B$, clearly $x \notin A - B$. So $x \in A - (A - B$. $\square$

Exercise 2.37. *Show that for all sets the following equations hold.*

(a) $(A \cup B) - C = (A - C) \cup (B - C)$.

(b) $A - (B - C) = (A - B) \cup (A \cap C)$.

(c) $(A - B) - C = A - (B \cup C)$.

Solution. (a) Take $x \in (A \cup B) - C$. If $x \in A$, then $x \in A - C$. If $x \in B$, then $x \in B - C$. Conversely, if $x \in A - C$, then $x \in (A \cup B) - C$, and similarly for the case if $x \in B - C$.

(b) Take $x \in A - (B - C)$, if $x \notin B - C$, then either $x \notin B$ or $x \in C$. In the first case, $x \in A - B$, in the second, $x \in A \cap C$. Conversely, if $x \in A - B$, then $x \notin B$, so $x \notin B - C$, so $x \in A - (B - C)$. If $x \in A \cap C$, then $x \notin B - C$.

(c) If $x \in (A - B) - C$, then $x \in A$, but $x \notin B$ and $x \notin C$, so $x \notin B \cup C$. Conversely, if $x \in A - (B \cup C)$, then $x \notin B$ and $x \notin C$, so $x \in A - B$ and $x \notin C$.

$\square$

Exercise 2.38. *Prove that for all sets the following are valid.*

(a) $A \subseteq C \vee B \subseteq C \Leftrightarrow A \cup B \subseteq C$.

(b) $C \subseteq A \vee C \subseteq B \Leftrightarrow C \subseteq A \cap B$.

Solution. Assuming $A \subseteq C \vee B \subseteq C$, then for any $x \in A \cup B$, $x \in C$ if $x \in A$ or if $x \in B$. Conversely, since both A and B are subsets of $A \cup B$, we have $A \subseteq C$ and $B \subseteq C$.

Now assuming $C \subseteq A \vee C \subseteq B$, for any $x \in C$ one has $x \in A$ and $x \in B$, so $x \in A \cap B$. Conversely, $A \cap B \subseteq A$ and $A \cap B \subseteq B$, and so $C \subseteq A \vee C \subseteq B$. □

3 Relations and Functions

3.1 Ordered Pairs

Exercise 3.1. *Suppose that we attempted to generalize the Kuratowski definitions of orrdered pairs to ordered triples by defining*

$$\langle x, y, z\rangle^* = \{\{x\}, \{x, y\}, \{x, y, z\}\}.$$

Show that this definition is unsuccessful by giving examples of objects u, v, w, x, y, z with $\langle x, y, z\rangle^* = \langle u, v, w\rangle^*$ but with either $y \neq v$ or $z \neq w$ (or both).

Solution. Take $x = 1, y = 2, z = 1$ and $u = 1, v = 2, w = 2$. Then

$$\langle x, y, z\rangle^* = \{\{1\}, \{1, 2\}, \{1, 2, 1\}\} = \{\{1\}, \{1, 2\}\}$$

and

$$\langle u, v, w\rangle^* = \{\{1\}, \{1, 2\}, \{1, 2, 2\}\} = \{\{1\}, \{1, 2\}\}$$

but $z \neq w$. □

Exercise 3.2. *(a) Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.*

(b) Show that if $A \times B = A \times C$ and $A \neq \emptyset$, then $B = C$.

Solution. Take $\langle x, y\rangle \in A \times (B \cup C)$. So $y \in B \cup C$. If $y \in B$, then $\langle x, y\rangle \in A \times B$ and if $y \in C$, then $\langle x, y\rangle \in A \times C$. Conversely, if $\langle x, y\rangle \in A \times B$, then $y \in B \cup C$, so $\langle x, y\rangle \in A \times (B \cup C)$, and similary for if $y \in C$.

For (b), take $x \in B$, so for some $a \in A$, $\langle a, x\rangle \in A \times B$. Hence $\langle a, x\rangle = \langle a', c\rangle$ for some $a' \in A$ and $c \in C$. By Theorem 3A, $x = c$, so $x \in C$, and thus $B \subseteq C$. A symmetric argument shows $C \subseteq B$. □

Exercise 3.3. *Show that $A \times \bigcup \mathscr{B} = \bigcup\{A \times X \mid X \in \mathscr{B}\}$.*

Solution. Take $\langle x, y\rangle \in A \times \bigcup \mathscr{B}$. So $y \in \bigcup \mathscr{B}$, and thus there exists some $X \in \mathscr{B}$ such that $y \in X$. Hence $\langle x, y\rangle \in A \times X$, and thus $\langle x, y\rangle \in \bigcup\{A \times X \mid X \in \mathscr{B}\}$. Conversely, if $\langle x, y\rangle \in A \times X$ for some $X \in \mathscr{B}$, then $y \in \bigcup \mathscr{B}$, and so $\langle x, y\rangle \in A \times \bigcup \mathscr{B}$. □

Exercise 3.4. *Show that there is no set to which every ordered pair belongs.*

Solution. Suppose that such a set exists. So in particular, for every set X, the ordered pair $\langle X, X\rangle = \{\{X\}, \{X, X\}\} = \{\{X\}\}$ is in this set. It follows by a subset axiom that this set contains the set of all singleton sets, which, by Exercise 2.8, has been shown to not exist. Hence we reach a contradiction, so there is no set to which every ordered pair belongs. $\square$

Exercise 3.5. *(a) Assume that A and B are given sets, and show that there exists a set C such that for any y,*

$$y \in C \Leftrightarrow y = \{x\} \times B \quad \textit{for some } x \textit{ in } A.$$

In other words, show that $\{\{x\} \times B \mid x \in A\}$ is a set.

(b) With A, B, and C as above, show that $A \times B = \bigcup C$.

Solution. First observer that for any $x \in A$, $\{x\} \times B \subseteq A \times B$. It follows that $\{x\} \times B \in \mathscr{P}(A \times B)$. Hence $\{\{x\} \times B \mid x \in A\} \subseteq \mathscr{P}(A \times B)$. Now since $A \times B$ is indeed a set, $\mathscr{P}(A \times B)$ is also a set, and thus by a subset axiom, we can construct

$$\{w \in \mathscr{P}(A \times B) \mid w = \{x\} \times B \text{ for some } x \in A\}.$$

More formally, the following is an axiom,

$$\forall A \forall B \forall D \exists C \forall w (w \in C \Leftrightarrow w \in D \wedge \exists a (a \in A \wedge w = \{x\} \times B)),$$

and we can instantiate with $A = A$, $B = B$, and $D = \mathscr{P}(A \times B)$.

Furthermore, $A \times B = \bigcup C$. For take $y \in A \times B$. So $y = \langle a, b\rangle$ for some $a \in A$ and $b \in B$. In particular, $y \in \{a\} \times B$, so $y \in \bigcup\{\{x\} \times B \mid x \in A\}$. Conversely, take $y \in \bigcup\{\{x\} \times B \mid x \times A\}$. So $y \in \{x\} \times B$ for some $x \in A$. But $\{x\} \times B \subseteq A \times B$, and thus $y \in A \times B$. $\square$

3.2 Relations

Exercise 3.6. *Show that a set A is a relation iff $A \subseteq domA \times ranA$.*

Solution. Suppose that A is a relation. So by definition, A is a set of ordered pairs. If A is the empty relation, the containment holds trivially. Otherwise, take $\langle x, y\rangle \in A$. Clearly, there exists y such that $\langle x, y\rangle \in A$ so $x \in \text{dom } A$, and similarly, there exists x such that $\langle x, y\rangle \in A$, so $y \in \text{ran } A$. So $A \subseteq \text{dom } A \times \text{ran } A$.

Conversely, suppose $A \subseteq \text{dom } A \times \text{ran } A$. So A is a set of ordered pairs, and by definition, a relation. $\square$

Exercise 3.7. *Show that if R is a relation, then fld $R = \bigcup\bigcup R$.*

Solution. Take $x$fld R. Recall that fld $R = \text{dom } R \cup \text{ran } R$, and by Lemma 3D, dom $R \subseteq \bigcup\bigcup R$ and ran $R \subseteq \bigcup\bigcup R$, so by Exercise 2.38, fld $R \subseteq \bigcup\bigcup R$.

Conversely, take $x \in \bigcup\bigcup R$. Hence there exists some $y \in \bigcup R$ such that $x \in y$ and there exists some $z \in R$ such that $y \in z$. Now $z = \langle a, b\rangle$ for $a \in \text{dom } R$ and $b \in \text{ran } R$. If

$y = \{a\}$, then $x = a$, and so $x \in \text{dom } R$, and hence $x \in \text{fld } R$. If $y \in \{a, b\}$, then either $x = a$, whence $x \in \text{dom } R$, and hence $x \in \text{fld } R$, or $x = b$, whence $x \in \text{ran } R$, and hence $x \in \text{fld } R$. □

Exercise 3.8. *Show that for any set $\mathscr{A}$:*

$$dom \bigcup \mathscr{A} = \bigcup \{dom\ R \mid R \in \mathscr{A}\},$$

$$ran \bigcup \mathscr{A} = \bigcup \{ran\ R \mid R \in \mathscr{A}\}.$$

Solution. Take $x \in \text{dom} \bigcup A$. So $\exists y \langle x, y \rangle \in \bigcup \mathscr{A}$. Since $\langle x, y \rangle \in \bigcup \mathscr{A}$, for some $R \in \mathscr{A}$, $\langle x, y \rangle \in R$, so $x \in \text{dom } R$, and thus $x \in \bigcup \{\text{dom } R \mid R \in \mathscr{A}\}$. For some $R \in \mathscr{A}$, take $x \in \text{dom } R$, so $\exists y \langle x, y \rangle \in R$. Hence $\{\langle x, y \rangle\} \subseteq R$, but $R \subseteq \bigcup \mathscr{A}$, and so $\{\langle x, y \rangle\} \subseteq \bigcup \mathscr{A}$, so $\langle x, y \rangle \in \bigcup \mathscr{A}$, so $x \in \text{dom} \bigcup \mathscr{A}$.

The argument for the range is similar. Take $x \in \text{ran} \bigcup \mathscr{A}$. So $\exists t \langle t, x \rangle \in \bigcup \mathscr{A}$. So for some $R \in \mathscr{A}$, $\langle t, x \rangle \in R$, so $x \in \text{ran } R$. Now for any $R \in \mathscr{A}$, take $x \in R$. Hence $\exists t \langle t, x \rangle \in R$, so by the same reasoning as above, $\{\langle t, x \rangle\} \subseteq R \subseteq \bigcup \mathscr{A}$, and hence $\langle t, x \rangle \in \bigcup \mathscr{A}$, so $x \in \text{ran} \bigcup \mathscr{A}$. □

Exercise 3.9. *Discuss the result of replacing the union operation by the intersection operation in the preceding problem.*

Solution. Suppose $x \in \bigcap \mathscr{A}$. So $\exists y \langle x, y \rangle \in \bigcap \mathscr{A}$, that is, $\langle x, y \rangle \in R$ for all $R \in \mathscr{A}$. Hence $x \in \bigcap \{\text{dom } R \mid R \in \mathscr{A}\}$. However, suppose $x \in \bigcap \{\text{dom } R \mid R \in \mathscr{A}\}$. All one may deduce from this is that for each $R \in \mathscr{A}$, there exists some y_R such that $\langle x, y_R \rangle \in R$, but this does not imply that there is some y such that $\langle x, y \rangle \in R$ for all $R \in \mathscr{A}$. Indeed, consider $\mathscr{A} = \{\{\langle 2, 3 \rangle\}, \{\langle 2, 5 \rangle\}\}$. So $2 \in \text{dom } R$ for all $R \in \mathscr{A}$, but there is no single element such that $\langle 2, y \rangle \in R$ for all R. So all we may conclude is that

$$\text{dom} \bigcap \mathscr{A} \subseteq \bigcap \{\text{dom } R \mid R \in \mathscr{A}\}.$$

By similar reasoning, we see that

$$\text{ran} \bigcap \mathscr{A} \subseteq \bigcap \{\text{ran } R \mid R \in \mathscr{A}\}.$$

The reverse containment does not holds, for consider the set $\mathscr{A} = \{\{\langle 2, 3 \rangle\}, \{\langle 5, 3 \rangle\}\}$. Then 3ran R for all $R \in \mathscr{A}$, but there is no such t such that $\langle t, 3 \rangle \in \text{ran} \bigcap \mathscr{A}$. □

3.3 n-ary Relations

Exercise 3.10. *Show that an ordered 4-tuple is also an ordered m-tuple for every positive integer m less than 4.*

Solution. First observe that by definition, $\langle x, y, z \rangle = \langle \langle x, y \rangle, z \rangle$ and so

$$\langle x, y, z \rangle = \langle \langle x, y \rangle, z \rangle = \{\{\langle x, y \rangle\}, \{\langle x, y \rangle, z\}\}.$$

So just as $\langle x, y\rangle$ is just a set, so is $\langle x, y, z\rangle$. Hence by definition

$$\langle x_1, x_2, x_3, x_4\rangle = \langle\langle x_1, x_2, x_3\rangle, x_4\rangle,$$

so $\langle x_1, x_2, x_3, x_4\rangle$ is an ordered pair, with first coordinate $\langle x_1, x_2, x_3\rangle$ and second coordinate x_4. Note that it is indeed ordered, since

$$\begin{aligned}\langle\langle x_1, x_2, x_3\rangle, x_4\rangle = \langle\langle y_1, y_2, y_3\rangle, y_4\rangle &\Leftrightarrow \langle x_1, x_2, x_3, x_4\rangle = \langle y_1, y_2, y_3, y_4\rangle \\ &\Leftrightarrow x_1 = y_1,\ x_2 = y_2,\ x_3 = y_3,\ x_4 = y_4 \\ &\Leftrightarrow \langle x_1, x_2, x_3\rangle = \langle y_1, y_2, y_3\rangle \text{ and } x_4 = y_4\end{aligned}$$

Similarly, by definition,

$$\begin{aligned}\langle x_1, x_2, x_3, x_4\rangle &= \langle\langle\langle x_1, x_2\rangle, x_3\rangle, x_4\rangle \\ &= \langle\langle x_1, x_2\rangle, x_3, x_4\rangle,\end{aligned}$$

where we have applied the fact that $\langle x, y, z\rangle = \langle\langle x, y\rangle, z\rangle$, with $x = \langle x_1, x_2\rangle$, $y = x_3$, and $z = x_4$. It follows that $\langle x_1, x_2, x_3, x_4\rangle$ is also an ordered triple. Finally, by definition, $\langle\langle x_1, x_2, x_3, x_4\rangle\rangle = \langle x_1, x_2, x_3, x_4\rangle$, so it is also a 1-tuple. □

3.4 Functions

Exercise 3.11. *Prove the following version (for functions) of the extensionality principle: Assume that F and G are functions, dom F = dom G, and $F(x) = G(x)$ for all x in the common domain. Then $F = G$.*

Solution. Take $\langle x, F(x)\rangle \in F$. So $x \in \text{dom } F = \text{dom } G$, and thus $\langle x, G(x)\rangle \in G$. But $F(x) = G(x)$, and so $\langle x, F(x)\rangle = \langle x, G(x)\rangle$, so $\langle x, F(x)\rangle \in G$ and thus $F \subseteq G$. A symmetric argument shows $G \subseteq F$, and so $F = G$. □

Exercise 3.12. *Assume that f and g are functions and show that*

$$f \subseteq g \Leftrightarrow \textit{dom } f \subseteq \textit{dom } g \wedge (\forall x \in \textit{dom } f) f(x) = g(x).$$

Solution. Suppose $f \subseteq g$. Now take $x \in \text{dom } f$. Then $\langle x, f(x)\rangle \in f$, so $\langle x, g(x)\rangle \in g$. Thus $x \in \text{dom } g$, so dom $f \subseteq$ dom g. Moreover, $\langle x, g(x)\rangle \in g$ so since g is a function, one must have that $\langle x, f(x)\rangle = \langle x, g(x)\rangle$, and thus $f(x) = g(x)$.

Conversely, take $\langle x, f(x)\rangle \in f$. Then $x \in \text{dom } f$, so by assumption, $f(x) = g(x)$. It follows that $\langle x, f(x)\rangle = \langle x, g(x)\rangle$, so since $x \in \text{dom } g$, $\langle x, f(x)\rangle \in g$, and thus $f \subseteq g$. □

Exercise 3.13. *Assume that f and g are functions with $f \subseteq g$ and dom $g \subseteq$ dom f. Show that $f = g$.*

Solution. By Exercise 3.12, it suffices to show that

$$(\forall x \in \text{dom } g) g(x) = f(x)$$

from which the containment $g \subseteq f$ follows. Take $x \in \text{dom } g$. Then $x \in \text{dom } f$ so $\langle x, f(x)\rangle \in f$. Since $f \subseteq g$, $\langle x, f(x)\rangle \in g$. Furthermore, since $x \in \text{dom } g$, $\langle x, g(x)\rangle \in g$. Since g is a function, that fact that $\langle x, f(x)\rangle, \langle x, g(x)\rangle \in g$ implies that $g(x) = f(x)$, as desired. □

Exercise 3.14. *Assume that f and g are functions.*

(a) Show that $f \cap g$ is a function.

(b) Show that $f \cup g$ is a function iff $f(x) = g(x)$ for every x in $(\text{dom } f) \cap (\text{dom } g)$.

Solution. Assume that $\langle x, y\rangle \in f \cap g$ and $\langle x, y'\rangle \in f \cap g$. In particular, $\langle x, y\rangle \in f \cap g$ and $\langle x, y'\rangle \in f$, so since f is a function, $y = y'$, and hence $f \cap g$ is a function.

For (b), first suppose $f \cup g$ is a function. Take any $x \in (\text{dom } f) \cap (\text{dom } g)$. Thus $\langle x, f(x)\rangle \in f$ and $\langle x, g(x)\rangle \in g$. So $\langle x, f(x)\rangle, \langle x, g(x)\rangle \in f \cup g$, and since $f \cup g$ is a function, $f(x) = g(x)$.

Conversely, suppose that $\langle x, y\rangle$ and $\langle x, y'\rangle$ are in $f \cup g$. There are several cases to consider. If $\langle x, y\rangle \in f$ and $\langle x, y'\rangle \in f$, then $y = y'$ since f is a function. Similarly, if both pairs are in g, then $y = y'$ since g is a function. Finally, if $\langle x, y\rangle$ is in f and the $\langle x, y'\rangle$ is in g, then $x \in (\text{dom } f) \cap (\text{dom } g)$, and so by hypothesis, $y = f(x) = g(x) = y'$. The same holds if the pairs happen to be in the other set. In any case, $y = y'$, and so $f \cup g$ is a function. $\square$

Exercise 3.15. *Let $\mathscr{A}$ be a set of functions such that for any f and g in $\mathscr{A}$, either $f \subseteq g$ or $g \subseteq f$. Show that $\bigcup \mathscr{A}$ is a function.*

Solution. Take $\langle x, y\rangle, \langle x, y'\rangle \in \bigcup \mathscr{A}$. Now for some $f \in \mathscr{A}$, $\langle x, y\rangle \in f$ and for some $g \in \mathscr{A}$, $\langle x, y'\rangle \in g$. Without loss of generality, suppose $f \subseteq g$, and so $\langle x, y\rangle \in g$. It follows that $y = g(x) = y'$, and hence $\mathscr{A}$ is a function. $\square$

Exercise 3.16. *Show that there is no set to which every function belongs.*

Solution. Suppose such a set exists, so let $\mathscr{F}$ denote the set of all functions. Then by a subset axiom,
$\exists C \forall f (f \in C \Leftrightarrow f \in \mathscr{F} \wedge (\forall x \forall y)((x \in \text{dom } f \wedge y \in \text{dom } f \Rightarrow x = y) \wedge (\forall z)(z \in \text{dom } f \Rightarrow \langle z, z\rangle \in f)))$.

Hence such a subset C consists of functions $f \colon \{x\} \to \{x\} \colon x \mapsto x$. That is, functions of the form

$$f = \{\langle x, x\rangle\} = \{\{x\}, \{x, x\}\} = \{\{\{x\}\}\}.$$

Note then that $\bigcup\bigcup f = \{x\}$, and in particular, $\bigcup\bigcup C$ is the set of all singletons, which has been shown to not exist. Hence no such set $\mathscr{F}$ exists. $\square$

Exercise 3.17. *Show that the composition of two single-rooted sets is again single-rooted. Conclude that the composition of two one-to-one functions is again one-to-one.*

Solution. Let F and G be two single-rooted sets. Suppose that $x(F \circ G)y$ and $x'(F \circ G)y$. So there exists t and t' such that $xGt \wedge tFy$ and $x'Gt' \wedge t'Fy$. Since F is single-rooted, $t = t'$, and then since G is single-rooted, from xGt and $x'Gt'$ we see $x = x'$. Hence $F \circ G$ is single-rooted.

Since injective functions are just special cases of single-rooted relations, we conclude that the composition of injections is again injective. $\square$

Exercise 3.18. *Let R be the set*

$$\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}.$$

Evaluate the following: $R \circ R$, $R \restriction \{1\}$, $R^{-1} \restriction \{1\}$, $R[\![\{1\}]\!]$, *and* $R^{-1}[\![\{1\}]\!]$.

Solution. By inspection, we have

$$R \circ R = \{\langle 0,2\rangle, \langle 0,3\rangle, \langle 1,3\rangle\}.$$

Also, $R \restriction \{1\} = \{\langle 1,2\rangle, \langle 1,3\rangle\}$, $R^{-1} \restriction= \{\langle 1,0\rangle\}$, $R[\![\{1\}]\!] = \{2,3\}$, and $R^{-1}[\![\{1\}]\!] = \{0\}$. □

Exercise 3.19. *Let*

$$A = \{\langle \emptyset, \{\emptyset, \{\emptyset\}\}\rangle, \langle \{\emptyset\}, \emptyset\rangle\}.$$

Evaluate each of the following: $A(\emptyset), A[\![\emptyset]\!], A[\![\{\emptyset\}]\!], A[\![\{\emptyset, \{\emptyset\}\}]\!], A^{-1}, A \circ A, A \restriction \emptyset, A \restriction \{\emptyset\}, A \restriction \{\emptyset, \{\emptyset\}\}, \bigcup\bigcup A$.

Solution. First, note that A is indeed a function. Hence,

$$A(\emptyset) = \{\emptyset, \{\emptyset\}\}.$$

However, there are no ordered pairs $\langle u, v\rangle$ such that $u \in \emptyset$, thus,

$$A[\![\emptyset]\!] = \emptyset.$$

Since A is a function,

$$A[\![\{\emptyset\}]\!] = A(\emptyset).$$

Also,

$$A[\![\{\emptyset, \{\emptyset\}\}]\!] = \{A(\emptyset), A(\{\emptyset\})\} = \{\{\emptyset, \{\emptyset\}\}, \emptyset\}.$$

Flipping the pairs,

$$A^{-1} = \{\langle \{\emptyset, \{\emptyset\}\}, \emptyset\rangle, \langle \emptyset, \{\emptyset\}\rangle\}.$$

Taking $\emptyset$ as the intermediary,

$$A \circ A = \{\langle \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\rangle\}.$$

By similar reasons to the computation of $A[\![\emptyset]\!]$,

$$A \restriction \emptyset = \emptyset.$$

However,

$$A \restriction \{\emptyset\} = \{\langle \emptyset, \{\emptyset, \{\emptyset\}\}\rangle\}.$$

and

$$A \restriction \{\emptyset, \{\emptyset\}\} = A.$$

Finally, first note that

$$\begin{aligned}\bigcup A &= \langle \emptyset, \{\emptyset, \{\emptyset\}\}\rangle \cup \langle \{\emptyset\}, \emptyset\rangle \\ &= \{\{\emptyset\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\} \cup \{\{\{\emptyset\}\}, \{\{\emptyset\}, \emptyset\}\} \\ &= \{\{\emptyset\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \{\{\emptyset\}\}, \{\{\emptyset\}, \emptyset\}\}\end{aligned}$$

and thus

$$\begin{aligned}\bigcup\bigcup A &= \{\emptyset\} \cup \{\emptyset, \{\emptyset, \{\emptyset\}\}\} \cup \{\{\emptyset\}\}] \cup \{\{\emptyset\}, \emptyset\} \\ &= \{\emptyset, \emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}, \{\emptyset\}, \emptyset\} \\ &= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\end{aligned}$$

□

Exercise 3.20. *Show that $F \restriction A = F \cap (A \times \text{ran } F)$.*

Solution. By definition, $F \restriction A = \{\langle u, v\rangle \mid uFv \wedge u \in A\}$. Hence such $\langle u, v\rangle \in F$, so $F \restriction A \subseteq F$. Also, $u \in A$ and thus $v \in \text{ran } F$, so $\langle u, v\rangle \in A \times \text{ran } F$, so $F \restriction A \subseteq A \times \text{ran } F$, and altogether, $F \restriction\subseteq F \cap (A \times \text{ran } F)$.

Now take $\langle u, v\rangle \in F \cap (A \times \text{ran } F)$. So $\langle u, v\rangle \in A \times \text{ran } F$, so $u \in A$. Also, $\langle u, v\rangle \in F$, so uFv. Together, these imply $\langle u, v\rangle \in F \restriction A$, and the equality follows. □

Exercise 3.21. *Show that $(R \circ S) \circ T = R \circ (S \circ T)$ for any sets R, S, and T.*

Solution. Take $\langle x, y\rangle \in (R \circ S) \circ T$. Hence there exists t such that

$$x(R \circ S)t \wedge tTy.$$

This implies there exists s such that

$$xRs \wedge sSt \wedge tTy.$$

So $xRs \wedge s(S \circ T)y$, and thus $\langle x, y\rangle \in R \circ (S \circ T)$. The reverse containment follows similarly. □

Exercise 3.22. *Show that the following are correct for any sets.*

(a) $A \subseteq B \Leftrightarrow F[\![A]\!] \subseteq F[\![B]\!]$.

(b) $(F \circ G)[\![A]\!] = F[\![G[\![A]\!]]\!]$.

(c) $Q \restriction (A \cup B) = (Q \restriction A) \cup (Q \restriction B)$.

Solution. Indeed,

(a) Take $v \in F[\![A]\!]$. So $\langle u, v\rangle \in F$ for some $u \in A$. By assumption, $A \subseteq B$, so $u \in B$ as well, and thus $v \in F[\![B]\!]$.

(b) Take $v \in (F \circ G)[\![A]\!]$. So $u(F \circ G)v$ and $u \in A$ for some u. Hence there exists some t such that $uGt \wedge tFv$. So $t \in G[\![A]\!]$, and thus $v \in F[\![G[\![A]\!]]\!]$.

Conversely, take $y \in F[\![G[\![A]\!]]\!]$. So for some $x \in G[\![A]\!]$, xFy and for some $z \in A$, zGx. Then $z(F \circ G)y$, and so $y \in (F \circ G)[\![A]\!]$.

(c) Take $v \in Q \upharpoonright (A \cup B)$. So for some $u \in A \cup B$, uQv. If $u \in A$, $v \in Q \upharpoonright A$, and if $u \in B$, then $v \in Q \upharpoonright B$. In either case, $Q \upharpoonright (A \cup B) \subseteq (Q \upharpoonright A) \cup (Q \upharpoonright B)$.

Now take $y \in (Q \upharpoonright A) \cup (Q \upharpoonright B)$. If $y \in Q \upharpoonright A$, then there is some $x \in A$ such that xQy, and hence $y \in Q \upharpoonright (A \cup B)$. The case where $y \in B$ follows similarly.

□

Exercise 3.23. *Let I_A be the identity function on the set A. Show that for any sets B and C,*

$$B \circ I_A = B \upharpoonright A \quad \textit{and} \quad I_A[\![C]\!] = A \cap C.$$

Solution. First take $\langle x, y \rangle \in B \circ I_A$. So there exists t such that $xI_At \wedge tBy$. But since I_A is the identity function, $t = x$. So xBy, and so $\langle x, y \rangle \in B \upharpoonright A$.

Conversely, take $\langle x, y \rangle \in B \upharpoonright A$. So xBy and $x \in A$. In particular, $xI_Ax \wedge xBy$, so $x(B \circ I_A)y$, that is, $\langle x, y \rangle \in B \circ I_A$.

For the other equality, first take $x \in I_A[\![C]\!]$. So for some $c \in C$, $\langle c, x \rangle \in I_A$. Note that $c \in \text{dom } I_A = A$, so $c \in A$. And since I_A is the identity on A, we must have $c = x$, so $x \in C$, and $x \in \text{ran } I_A = A$, so $x \in A \cap C$.

Conversely, take $x \in A \cap C$. Since $x \in A$, $\langle x, x \rangle \in I_A$, but since $x \in C$, it follows that $x \in I_A[\![C]\!]$. □

Exercise 3.24. *Show that for a function F, $F^{-1}[\![A]\!] = \{x \in \text{dom } F \mid F(x) \in A\}$.*

Solution. Take $x \in F^{-1}[\![A]\!]$. So for some $y \in A$, $\langle y, x \rangle \in F^{-1}$ and so $\langle x, y \rangle \in F$. Thus $y = F(x) \in A$, and $x \in \text{dom } F$, as desired.

Conversely, take $v \in \{x \in \text{dom } F \mid F(x) \in A\}$. So $\langle x, F(u) \rangle \in F$, and thus $\langle F(u), u \rangle \in F^{-1}$. Since $F(u) \in A$, it follows that $u \in F^{-1}[\![A]\!]$. □

Exercise 3.25. *(a) Assume that G is a one-to-one function. Show that $G \circ G^{-1}$ is $I_{ran\ G}$, the identity function on ran G.*

(b) Show that the result of part (a) holds for any function G, not necessarily one-to-one.

Solution. Suppose that $x(G \circ G^{-1})y$. Hence there exists t such that $xG^{-1}t \wedge tGy$. So $tGx \wedge tGy$. Now $x \in \text{ran } G$, and since G is a function, $x = y$. Hence $G \circ G^{-1}$ is $I_{\text{ran } G}$. Note that nowhere is injectivity used, and thus (a) holds for any function G. □

Exercise 3.26. *Prove the second halves of parts (a) and (b) of Theorem 3K. That is, prove that*

$$F[\![\bigcup \mathscr{A}]\!] = \bigcup \{F[\![\mathscr{A}]\!] \mid A \in \mathscr{A}\} \quad \textit{and} \quad F[\![\bigcap \mathscr{A}]\!] = \bigcap \{F[\![\mathscr{A}]\!] \mid A \in \mathscr{A}\}.$$

Solution. Take $v \in F[\![\bigcup \mathscr{A}]\!]$. So there exists $u \in \bigcup \mathscr{A}$ such that uFv. Since $u \in \bigcup \mathscr{A}$, there exists an $A \in \mathscr{A}$ such that $u \in A$. So $v \in F[\![A]\!]$ for this particular A, so $u \in \bigcup\{F[\![\mathscr{A}]\!] \mid A \in \mathscr{A}\}$. Conversely, take $v \in \bigcup\{F[\![\mathscr{A}]\!] \mid A \in \mathscr{A}\}$. So for some $A \in \mathscr{A}$, $v \in F[\![A]\!]$, and thus there is some $u \in A$ such that uFv. Since $u \in A$ and $A \subseteq \bigcup \mathscr{A}$, $u \in \bigcup \mathscr{A}$, and thus $v \in F[\![\bigcup \mathscr{A}]\!]$.

For (b), take $v \in F[\![\mathscr{A}]\!]$. So uFv for some $u \in \bigcap \mathscr{A}$. That is, for every $A \in \mathscr{A}$, $u \in A$. It follows that $v \in F[\![A]\!]$ for each $A \in \mathscr{A}$, and so the containment holds. To see that equality holds when F is single-rooted, take $y \in \bigcap\{F[\![A]\!] \mid A \in \mathscr{A}\}$. So for each $A \in \mathscr{A}$, $y \in F[\![A]\!]$, and thus there exists $u_A \in A$ such that $\langle u_A, y\rangle \in F$. Since F is single-rooted, these u_A are all equal, so denoting $u_A = u$, one has $u \in \bigcap \mathscr{A}$, and hence $y \in F[\![\bigcap \mathscr{A}]\!]$. $\square$

Exercise 3.27. *Show that dom $(F \circ G) = G^{-1}[\![\text{dom } F]\!]$ for any sets F and G.*

Solution. Take $x \in \text{dom } (F \circ G)$. So there exists y such that $x(F \circ G)y$. Moreover, there exists t, such that $xGt \wedge tFy$, and thus $tG^{-1}x$. Since $t \in \text{dom } F$, we have $x \in G^{-1}[\![\text{dom } F]\!]$.

Now take $x \in G^{-1}[\![\text{dom } F]\!]$. So for some $t \in \text{dom } F$, $tG^{-1}x$, and so xGt. Since $t \in \text{dom } F$, there exists some y such that tFy, and thus $x(F \circ G)y$, so $x \in \text{dom } (F \circ G)$. $\square$

Exercise 3.28. *Assume that f is a one-to-one function from A into B, and that G is the function with dom $G = \mathscr{P}A$ defined by the equation $G(X) = f[\![X]\!]$. Show that G maps $\mathscr{P}A$ one-to-one into $\mathscr{P}B$.*

Solution. First note that G indeed maps $\mathscr{P}A$ into $\mathscr{P}B$. For $X \in \mathscr{P}A$, $G(X) = f[\![X]\!] = \{v \mid ufv, u \in X\}$, so $v \in B$, and thus $f[\![X]\!] \subseteq B$, so $f[\![X]\!] \in \mathscr{P}B$.

Now suppose $G(X) = G(Y)$ for $X, Y \in \mathscr{P}A$, and thus $f[\![X]\!] = f[\![Y]\!]$. Take $x \in X$. Since f is defined on all of A, and $X \subseteq A$, we have $\langle x, f(x)\rangle \in f$, so $f(x) \in f[\![X]\!]$. Then $f(x) \in f[\![Y]\!]$, and thus for some $y \in Y$, $\langle y, f(x)\rangle \in f$, but since f is injective, $y = x$. So $x \in Y$, and thus $X \subseteq Y$. A parallel argument shows $Y \subseteq X$, and so $X = Y$, and thus $G\colon \mathscr{P}A \to \mathscr{P}B$ is injective. $\square$

Exercise 3.29. *Assume that $f\colon A \to B$ and define a function $G\colon B \to \mathscr{P}A$ by*

$$G(b) = \{x \in A \mid f(x) = b\}.$$

Show that if f maps A onto B, then G is one-to-one. Does the converse hold?

Solution. Suppose that $G(b) = G(b')$. Note that $G(b)$ is nonempty, since f is surjective. Now take $x \in G(b)$, so $f(x) = b$. Since $x \in G(b')$, $f(x) = b'$ also, so $b = b'$. Essentially, and $b \in B$ has a unique set of preimages.

However, the converse does not hold. Take $A = \{1\}$, and $B = \{1, 2\}$, and suppose $G(1) = \{1\}$ and $G(2) = \emptyset$. So G is injective, however, from the definition of G, we have that $f(1) = 1$, but 2 has no preimage under f, so f does not map A onto B. $\square$

Exercise 3.30. *Assume that $F\colon \mathscr{P}A \to \mathscr{P}A$ and that F has the monotonicity property:*

$$X \subseteq Y \subseteq A \Leftrightarrow F(X) \subseteq F(Y).$$

Define

$$B = \bigcap\{X \subseteq A \mid F(X) \subseteq X\} \quad \textit{and} \quad C = \bigcup\{X \subseteq A \mid X \subseteq F(X)\}.$$

(a) Show that $F(B) = B$ and $F(C) = C$.

(b) Show that if $F(X) = X$, then $B \subseteq X \subseteq C$.

Solution. Observe, by Theorem 3K, for $X \subseteq A$,

$$F(B) = F\left(\bigcap_{F(X)\subseteq X} X\right) \subseteq \bigcap_{F(X)\subseteq X} F(X) \subseteq \bigcap_{F(X)\subseteq X} X = B.$$

Hence $F(B) \subseteq B$, and thus by the monotonicity property, $F(F(B)) \subseteq F(B)$, which implies $F(B) \supseteq B$, and thus $F(B) = B$.

Again by Theorem 3K,

$$F(C) = F\left(\bigcup_{X\subseteq F(X)} X\right) = \bigcup_{X\subseteq F(X)} F(X) \supseteq \bigcup_{X\subseteq F(X)} X = C.$$

So $C \subseteq F(C)$, and thus by montonicity, $F(C) \subseteq F(F(C))$, from which it follows that $F(C) \subseteq C$, so $F(C) = C$.

For (b), if $F(X) = X$, then $F(X) \subseteq X$, and so $B \subseteq X$, as X is one of the sets in the intersection. Also, $X \subseteq F(X)$, so $X \subseteq C$, as X is one of the sets in the union. Thus $B \subseteq X \subseteq C$. □

3.5 Infinite Cartesian Products

Exercise 3.31. *Show that from the first form of the axiom of choice we can prove the second form, and conversely.*

Solution. Assume the first form. Let I be any set and let H be any function such that dom $H = I$, and $H(i) \neq \emptyset$ for all $i \in I$. Define a relation $R \subseteq I \times \bigcup_{i\in I} H(i)$ by

$$\langle i, x\rangle \in R \Leftrightarrow x \in H(i).$$

By assumption, there exists a function $G \subseteq R$ with dom $G =$ dom $R = I$, as for each $i \in I$, $i \in$ dom R since $H(i)$ is nonempty. So for all $\langle i, G(i)\rangle \in G$, $\langle i, G(i)\rangle \in R$, and thus by the definition of R, $G(i) \in H(i)$. It follows that $G \in \prod_{i\in I} H(i)$, so $\prod_{i\in I} H(i) \neq \emptyset$. Thus the second form follows from the first.

Conversely, let R be any relation, and denote dom $R = I$. Define a function

$$H\colon I \to \mathscr{P}(\text{ran } R)\colon i \mapsto H(i) := \{x \in \text{ran } R \mid iRx\}.$$

In particular, H is a function with domain I, and $H(i) \neq \emptyset$ for all $i \in I$. So by the second form, $\prod_{i\in I} H(i) \neq \emptyset$, so take $G \in \prod_{i\in I} H(i)$. Hence dom $G = I$, and for all $i \in I$, $G(i) \in H(i)$. Also, for any $\langle i, G(i)\rangle \in G$, $G(i) \in H(i) \subseteq$ ran R, and so $\langle i, G(i)\rangle \in R$, so $G \subseteq R$. Hence the two statements of the Axiom of Choice are equivalent.[1] □

[1]Thanks to Arturo Magidin for his hints on this exercise.

Exercise 3.32.

(a) Show that R is symmetric iff $R^{-1} \subseteq R$.

(b) Show that R is transitive iff $R \circ R \subseteq R$.

Solution. Suppose R is symmetric. Take $\langle x, y \rangle \in R^{-1}$. So $xR^{-1}y$, and thus yRx, but since R is symmetric, xRy, so $\langle x, y \rangle \in R$. Conversely, suppose $R^{-1} \subseteq R$. Suppose xRy, then $yR^{-1}x$, and so $\langle y, x \rangle \in R^{-1} \subseteq R$, so yRx, and R is symmetric.

Now suppose R is transitive. Take $\langle x, z \rangle \in R \circ R$. Thus there exists y such that $xRy \wedge yRz$, and since R is transitive, xRz, so $\langle x, z \rangle \in R$. Conversely, suppose $R \circ R \subseteq R$, and suppose $xRy \wedge yRz$. Thus $\langle x, z \rangle \in R \circ R$, so $\langle x, z \rangle \in R$, and thus xRz, so R is transitive. □

Exercise 3.33. *Show that R is a symmetric and transitive relation iff $R = R^{-1} \circ R$.*

Solution. First suppose that R is symmetric and transitive. Take $\langle x, y \rangle \in R$. So xRy, and thus yRx. By transitivity, we have yRy. Thus $yR^{-1}y$ and from $xRy \wedge y^R{-}1y$ we have $x(R^{-1} \circ R)y$, so $R \subseteq R^{-1} \circ R$. Now take $\langle x, z \rangle \in R^{-1} \circ R$. Hence there exists y such that $xRy \wedge yR^{-1}z$. But from $yR^{-1}z$ implies zRy, which implies yRz, as R is symmetric, and thus from $xRy \wedge yRz$ we have xRz, and thus $R = R^{-1} \circ R$.

Conversely, assume $R = R^{-1} \circ R$. Suppose xRy. So $x(R^{-1} \circ R)y$, and thus there exists z such that $xRz \wedge zR^{-1}y$. Hence

$$xRz \wedge zR^{-1}y \Rightarrow xRz \wedge yRz \Rightarrow zR^{-1}x \wedge yRz \Rightarrow y(R^{-1} \circ R)x \Rightarrow yRx$$

and so R is symmetric.

Now suppose $xRy \wedge yRz$. Hence $x(R^{-1} \circ R)y \wedge y(R^{-1} \circ R)z$. Hence there exist s and t such that $xRs \wedge sR^{-1}y$ and $yRt \wedge tR^{-1}z$. In particular, since R is symmetric, we have $sRy \wedge yR^{-1}t$, and so $s(R^{-1} \circ R)t$, so sRt. Then observe,

$$\begin{aligned} sRt \Rightarrow tR^{-1}s &\Rightarrow zRt \wedge tR^{-1}s \\ &\Rightarrow zRs \Rightarrow sR^{-1}z \\ &\Rightarrow xRs \wedge sR^{-1}z \quad \text{since } xRs \\ &\Rightarrow x(R^{-1} \circ R)z \\ &\Rightarrow xRz \end{aligned}$$

so R is transitive. □

Exercise 3.34. *Assume that $\mathscr{A}$ is a nonemptyset, every member of which is a transitive relation.*

(a) Is the set $\bigcap \mathscr{A}$ a transitive relation?

(b) Is $\bigcup \mathscr{A}$ a transitive relation?

Solution. Yes, $\bigcap \mathscr{A}$ is a transitive relation. Suppose $\langle x, y\rangle, \langle y, z\rangle \in \bigcap \mathscr{A}$. It follows that for every $A \in \mathscr{A}$, $\langle x, y\rangle, \langle y, z\rangle \in A$ as well, and since A is transitive, $\langle x, z\rangle \in A$ as well. Hence $\langle x, z\rangle \in \bigcap \mathscr{A}$, so $\bigcap \mathscr{A}$ is transitive.

This is not so for $\bigcup \mathscr{A}$. Suppose $\mathscr{A} = \{A_1, A_2\}$, where

$$A_1 = \{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}$$

and

$$A_2 = \{\langle b, d\rangle, \langle d, e\rangle, \langle b, e\rangle\}.$$

So both A_1 and A_2 are transitive, but

$$\bigcup \mathscr{A} = \{\langle a, b\rangle, \langle b, c\rangle, \langle a, c\rangle, \langle b, d\rangle, \langle d, e\rangle, \langle b, e\rangle\},$$

which is not transitive, since $\langle a, d\rangle \notin \bigcup \mathscr{A}$. □

Exercise 3.35. *Show that for any R and x, we have $[x]_R = R[\![\{x\}]\!]$.*

Solution. Take $y \in [x]_R$. So $\langle x, y\rangle \in R$ and by definition, $y \in R[\![\{x\}]\!]$. Conversely, take $y \in R[\![\{x\}]\!]$. Then for some $t \in \{x\}$, $\langle t, y\rangle \in R$. Clearly $t = x$, so $\langle x, y\rangle \in R$, and thus $y \in [x]_R$. □

Exercise 3.36. *Assume that $f\colon A \to B$ and that R is an equivalence relation on B. Define Q to be the set*

$$\{\langle x, y\rangle \in A \times A \mid \langle f(x), f(y)\rangle \in R\}.$$

Show that Q is an equivalence relation on A.

Solution. For any $x \in A$, $f(x) \in B$, so $\langle f(x), f(x)\rangle \in R$. Hence $\langle x, x\rangle \in Q$, so Q is reflexive on A. Now suppose $\langle x, y\rangle \in Q$. Then $\langle f(x), f(y)\rangle \in R$, so $\langle f(y), f(x)\rangle \in R$ since R is symmetric, so $\langle y, x\rangle \in Q$, and hence Q is symmetric. Finally, suppose $\langle x, y\rangle, \langle y, z\rangle \in Q$. So $\langle f(x), f(y)\rangle, \langle f(y), f(z)\rangle \in R$, and thus $\langle f(x), f(z)\rangle \in R$ since R is transitive, so $\langle x, z\rangle \in Q$ and Q is transitive. Thus Q is an equivalence relatoin on A. □

Exercise 3.37. *Assume that Π is a partition of a set A. Define the relation R_Π as follows:*

$$xR_\Pi y \Leftrightarrow (\exists B \in \Pi)(x \in B \wedge y \in B).$$

Show that R_Π is an equivalence relation on A.

Solution. For any $x \in A$, there exists some $B \in \Pi$ such that $x \in B$, but the definition of a partition. Hence $xR_\Pi x$, so R_Π is reflexive. Now suppose that $xR_\Pi y$. So there exists some $B \in \Pi$ such that $x \in B \wedge y \in B$, which obviously implies $y \in B \wedge x \in B$, so $yR_\Pi x$, so R_Π is symmetric.

Now suppose $xR_\Pi y$ and $yR_\Pi z$. Hence there exist $B, C \in \Pi$ such that $x \in B \wedge y \in B$ and $y \in C \wedge z \in C$. But note that $y \in B \cap C$, and hence we must have $B = C$, since no two distinct sets in a partition intersect. Thus $x \in C \wedge z \in C$, so $xR_\Pi z$, and R_Π is transitive. □

Exercise 3.38. *Theorem 3P shows that A/R is a partition of A whenever R is an equivalence relation on A. Show that if we start with the equivalence relation R_Π of the preceding exercise, then the partition A/R_Π is just Π.*

Solution. Take any $[x] \in A/R_\Pi$. Note for any $y \in [x]$, there exists a $B \in \Pi$ such that $x, y \in B$, by the definition of R_Π. Fix this B. For any $z \in [x]$, we have that $yR_\Pi z$, since R_Π is an equivalence relation. By the reasoning the previous exercise, $z \in B$ as well. So any two elements of $[x]$ are in this fixed B, and thus $[x] \subseteq B$. Moreover, for any $b \in B$, clearly $bR_\Pi x$, and thus $b \in [x]$. Hence $[x] = B$. So it follows that any equivalence class in A/R_Π equals some set in the partition Π.

On the other hand, take any $C \in \Pi$. Now C is nonempty, so take $m \in C$. By definition, this C is a subset of $[m]$, since any element in C are R_Π-related to m. However, by the same reasoning in the above paragraph, this $m \in C$, and thus $[m] \subseteq C$. This shows that any equivalence class in A/R_Π is equal to some $C \in \Pi$, and any $C \in \Pi$ is equal to some equivalence class in A/R_Π. So Π and A/R_Π are families with the same sets as members, and thus the same. $\square$

Exercise 3.39. *Assume that we start with an equivalence relation R on A and define Π to be the partition A/R. Show that R_Π, as defined in Exercise 3.37 is just R.*

Solution. First take $\langle x, y\rangle \in R$. Then since Π is a partition consisting of equivalences classes of R, $x, y \in B$ for some unique $B \in \Pi$. Then by the definition of R_Π, so $\langle x, y\rangle \in R_\Pi$, and thus $R \subseteq R_\Pi$.

Now take $\langle x, y\rangle \in R_\Pi$, and so there exists some $B \in \Pi$ such that $x \in B$ and $y \in B$. Now B is an equivalence class of R, so $B = [x]_R = [y]_R$, and thus by Lemma 3N, xRy, so $\langle x, y\rangle \in R$. Hence $R = R_\Pi$. $\square$

Exercise 3.40. *Define an equivalence relation R on the set P of positive integers by*

$$mRn \Leftrightarrow m \text{ and } n \text{ have the same number of prime factors.}$$

Is there a function $f\colon P/R \to P/R$ such that $f([n]_R) = [3n]_R$ for each n?

Solution. Define a function $F\colon P \to P\colon n \mapsto 3n$. Then recall from Theorem 3Q that such a function f exists iff F is compatible with R. However, consider the positive integers 2 and 3. Note $2R3$ since both 2 and 3 have exactly one prime factor. However, $F(2) = 6 = 2 \cdot 3$ and $F(3) = 9 = 3^2$, so $F(2)$ has 2 prime factors, but $F(3)$ still only has 1. Thus $\langle F(2), F(3)\rangle \notin R$, so F is not compatible with R, so no such function f exists. $\square$

Exercise 3.41. *Let $\mathbb{R}$ be the set of real numbers and define the relation Q on $\mathbb{R} \times \mathbb{R}$ by $\langle u, v\rangle Q\langle x, y\rangle$ iff $u + y = x + v$.*

(a) Show that Q is an equivalence relation on $\mathbb{R} \times \mathbb{R}$.

(b) Is there a function $G\colon (\mathbb{R} \times \mathbb{R})/Q \to (\mathbb{R} \times \mathbb{R})/Q$ satisfying the equation

$$G([\langle x, y\rangle]_Q) = [\langle x + 2y, y + 2x\rangle]_Q?$$

Solution. Q is reflexive on $\mathbb{R} \times \mathbb{R}$ due to the commutativity of $+$ on $\mathbb{R}$. Suppose $\langle u, v\rangle Q\langle x, y\rangle$, so $u + y = x + v$. Then clearly $x + v = u + y$, so $\langle x, y\rangle Q\langle u, v\rangle$, so Q is symmetric. Now assume $\langle u, v\rangle Q\langle x, y\rangle \wedge \langle x, y\rangle Q\langle w, z\rangle$. Thus

$$u + y = x + v \wedge x + z = w + y$$

from which we see that

$$u + z = x + v - y + z = (x + z - y) + v = w + v$$

which implies $\langle u, v\rangle Q\langle w, z\rangle$. Hence Q is transitive.

Define a function

$$F\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}\colon \langle x, y\rangle \mapsto \langle x + 2y, y + 2x\rangle.$$

Then if $\langle u, v\rangle Q\langle x, y\rangle$, we have $u + y = x + v$. This implies

$$u + y + 2(x + v) = x + v + 2(u + y),$$

which in turn implies

$$(u + 2v) + (y + 2x) = (x + 2y) + (v + 2u).$$

That is, $F(\langle u, v\rangle)QF(\langle x, y\rangle)$. So F is compatible with Q, and thus there exists such a function G by Theorem 3Q. □

Exercise 3.42. *State precisely the "analogous results" mentioned in Theorem 3Q.*

Solution. The analogous results are:

Assume that R is an equivalence relation on A and that $F\colon A \times A \to A$. If F is compatible with R, then there exists a unique $\hat{F}\colon A/R \times A/R \to A/R$ such that

$$\hat{F}([x], [y]) = [F(x, y)] \quad \text{for all } x, y \in A.$$

If F is not compatible with R, then no such $\hat{F}$ exists.

Of course, we must extend the definition of compatibility. Note that we would like to have the following commutative diagram:

$$\begin{array}{ccc} A \times A & \xrightarrow{F} & A \\ \downarrow & & \downarrow \\ A/R \times A/R & \xrightarrow[\hat{F}]{} & A/R \end{array}$$

From this, we see that if $\langle x, y\rangle$ and $\langle u, v\rangle$ have the same image under $A \times A \to (A/R) \times (A/R)$, that is, $\langle [x], [y]\rangle = \langle [u], [v]\rangle$, then we we also like $[F\langle x, y\rangle] = [F\langle u, v\rangle]$. This suggests that we define that F is compatible with R if for any $x, y, u, v \in R$,

$$xRy \wedge uRv \implies F\langle x, y\rangle RF\langle u, v\rangle.$$

Assume that F is compatible with R, and we will prove that such a $\hat{F}$ exists. We require that $\langle\langle[x],[y]\rangle,[F(x,y)]\rangle \in \hat{F}$, so define $\hat{F}$ as

$$\hat{F} = \{\langle\langle[x],[y]\rangle,[F(x,y)]\rangle \mid x,y \in A\}.$$

To see that $\hat{F}$ to indeed be a function, consider the pairs $\langle\langle[x],[y]\rangle,[F\langle x,y\rangle]\rangle$ and $\langle\langle[u],[v]\rangle,[F\langle u,v\rangle]\rangle$ in $\hat{F}$. Then we have the implications

$$\begin{aligned}\langle[x],[y]\rangle = \langle[u],[v]\rangle &\implies [x]=[u] \wedge [y]=[v]\\ &\implies xRu \wedge yRv\\ &\implies F\langle x,y\rangle RF\langle u,v\rangle\\ &\implies [F\langle x,y\rangle] = [F\langle u,v\rangle]\end{aligned}$$

So $\hat{F}$ is a function. It is clear that dom $\hat{F} = A/R \times A/R$ and ran $\hat{F} \subseteq A/R$. Also $\hat{F}\langle[x],[y]\rangle = [F\langle x,y\rangle]$, since $\hat{F}\langle[x],[y]\rangle = [F\langle x,y\rangle] \in \hat{F}$. Also, $\hat{F}$ is unique, for suppose for some $G\colon A/R \times A/R \to A/R$ the same condition holds. Then for any $x,y \in A$,

$$G\langle[x],[y]\rangle = [F\langle[x],[y]\rangle] = \hat{F}\langle[x],[y]\rangle$$

so $F = G$.

Now suppose that F is not compatible. By incompatibility, there are some pairs $\langle x,y\rangle,\langle u,v\rangle \in A\times A$ such that $xRu \wedge yRv$ but it is not case that $F\langle x,y\rangle RF\langle u,v\rangle$. Hence $[x]=[u]$, $[y]=[v]$, but $[F\langle x,y\rangle] \neq [F\langle u,v\rangle]$. But we must have

$$\hat{F}\langle[x],[y]\rangle = [F\langle x,y\rangle] \quad \text{and} \quad \hat{F}\langle[u],[v]\rangle = [F\langle u,v\rangle]$$

which is impossible since the left sides are equal, since $\hat{F}$ is assumed to be a function, but the right sides are not equal. □

3.6 Ordering Relations

Exercise 3.43. *Assume that R is a linear ordering on a set A. Show that R^{-1} is also a linear ordering on A.*

Solution. Suppose that $xR^{-1}y \wedge yR^{1-}z$. Then $yRx \wedge zRy$, and since R is transitive, zRx and so $xR^{-1}z$. Furthermore, for any x,y, exactly one of xRy, $x=y$, or yRx holds, and thus exactly one of $yR^{-1}x$, $x=y$, or $xR^{-1}y$ holds. □

Exercise 3.44. *Assume that $<$ is a linear ordering on a set A. Assume that $f\colon A\to A$ and that f has the property that whenever $x<y$, then $f(x)<f(y)$. Show that f is one-to-one and that whenever $f(x)<f(y)$, then $x<y$.*

Solution. Suppose that $f(x)=f(y)$. So $f(x) \not< f(y)$, and thus $x \not< y$. Similarly, $f(y) \not< f(x)$, and thus $y \not< x$. Since $<$ satisifies the trichotomy property, we must have that $x=y$, so f is injective.

Now suppose that $f(x)<f(y)$, but $x \not< y$. If $x=y$, then $f(x)=f(y)$, a contradiction. If $y<x$, then $f(y)<f(x)$, again a contradiction to the trichotomy property. Hence $x<y$ is the only possibility. □

Exercise 3.45. *Assume that $<_A$ and $<_B$ are linear orderings on A and B, respectively. Define the binary relation $<_L$ on the Cartesian product $A \times B$ by:*

$$\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle \quad \textit{iff either} \quad a_1 <_A a_2 \vee (a_1 = a_2 \wedge b_1 <_B b_2).$$

Show that $<_L$ is a linear ordering on $A \times B$.

Solution. Suppose $\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle \wedge \langle a_2, b_2\rangle <_L \langle a_3, b_3\rangle$. So $a_1 <_A a_2 \vee (a_1 = a_2 \wedge b_1 <_B b_2)$ and $a_2 <_A a_3 \vee (a_2 = a_3 \wedge b_2 <_B b_3)$. First suppose that $a_1 <_A a_2$ and $a_2 <_A a_3$, so by transitivity, $a_1 <_A a_3$, and thus $\langle a_1, b_1\rangle <_L \langle a_3, b_3\rangle$. Now suppose that $a_1 <_A a_2$ and $a_2 = a_3 \wedge b_2 <_B b_3$, and hence $a_1 <_A a_3$, so again $\langle a_1, b_1\rangle <_L \langle a_3, b_3\rangle$.

Now suppose $a_1 = a_2 \wedge b_1 <_B b_2$ and $a_2 <_A a_3$. Thus $a_1 <_A a_3$, so $\langle a_1, b_1\rangle <_L \langle a_3, b_3\rangle$. Finally, if $a_1 = a_2 \wedge b_1 <_B b_2$ and $a_2 = a_3 \wedge b_2 <_B b_3$, then $a_1 = a_3$, and $b_1 <_B b_3$, so $\langle a_1, b_1\rangle <_L \langle a_3, b_3\rangle$, and thus $<_L$ is transitive.

To show the trichotomy property, assume that $\langle a_1, b_1\rangle \not<_L \langle a_2, b_2\rangle$ and $\langle a_1, b_1\rangle \not<_L \langle a_2, b_2\rangle$. In particular, $a_1 \not<_A a_2$ and $a_1 \neq a_2$, so since $<_A$ is a linear ordering, one must have $a_2 <_A a_1$, and so $\langle a_2, b_2\rangle <_L \langle a_1, b_1\rangle$. Thus we see at least one of the three possibility holds. To see that at most one possibility holds, suppose $\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle$. Now clearly $\langle a_1, b_1\rangle \neq \langle a_2, b_2\rangle$, since we must have that either $a_1 <_A a_2$ or $b_1 <_B b_2$. Also, $\langle a_2, b_2\rangle \not<_L \langle a_1, b_1\rangle$ since if $a_1 <_A a_2$, then $a_2 \not<_A a_1$ and $a_2 \neq a_1$. And if $a_1 = a_2 \wedge b_1 <_B b_2$, then $a_2 \not<_A a_1$ and $b_2 \not<_B b_1$. A symmetric argument show that if $\langle a_2, b_2\rangle <_L \langle a_1, b_1\rangle$ holds, then neither of the other two possibilities do.

Also, if $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$, then $a_1 \not<_A a_2$ and $b_1 \not<_B b_2$, and $a_2 \not<_A a_1$ and $b_2 \not<_B b_1$, so neither of the other two possibilities hold. Hence $<_L$ is a linear ordering. □

Exercise 3.46. *Evaluate the following sets:*

(a) $\bigcap\bigcap\langle x, y\rangle$.

(b) $\bigcap\bigcap\bigcap\{\langle x, y\rangle\}^{-1}$.

Solution.

$$\begin{aligned}\bigcap\bigcap\langle x, y\rangle &= \bigcap\bigcap\{\{x\}, \{x, y\}\}\\ &= \bigcap\{x\} = x\end{aligned}$$

Denote $R = \{\langle x, y\rangle\}$, so $R^{-1} = \{\langle x, y\rangle\}^{-1} = \{\langle y, x\rangle\}$. Thus,

$$\begin{aligned}\bigcap\bigcap\bigcap\{\langle x, y\rangle\}^{-1} &= \bigcap\bigcap\bigcap\{\langle y, x\rangle\}\\ &= \bigcap\bigcap\langle y, x\rangle\\ &= y \quad \text{by part (a)}\end{aligned}$$

□

Exercise 3.47.

(a) *Find all of the functions from* $\{0,1,2\}$ *into* $\{3,4\}$.

(b) *Find all of the functions from* $\{0,1,2\}$ *into* $\{3,4,5\}$.

Solution. Without listing them, there are 2^3 functions from $\{0,1,2\}$ into $\{3,4\}$ and $3 \cdot 2 \cdot 1$ functions from $\{0,1,2\}$ into $\{3,4,5\}$. □

Exercise 3.48. *Let* T *be the set* $\{\emptyset, \{\emptyset\}\}$.

(a) *Find all of the ordered pairs, in any, in* $\mathscr{P}T$.

(b) *Evaluate and simplify:* $(\mathscr{P}T)^{-1} \circ (\mathscr{P}T \restriction \{\emptyset\})$

Solution. Firstly,

$$\mathscr{P}T = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}.$$

Hence there are no ordered pairs in $\mathscr{P}T$. As such, there are no v such that $\emptyset \mathscr{P}T v$, and thus $\mathscr{P}T \restriction \{\emptyset\} = \emptyset$. It follows immediately that $(\mathscr{P}T)^{-1} \circ (\mathscr{P}T \restriction \{\emptyset\}) = \emptyset$ as well. □

Exercise 3.49. *Find as many equivalence relations as you can on the set* $\{0,1,2\}$.

Solution. Recall that the number of equivalence relations on a set is the same as the number of distinct partitions of that set. We can partition $\{0,1,2\}$ in 5 ways, namely:

- $\{0\}, \{1\}, \{2\}$
- $\{0\}, \{1,2\}$
- $\{1,\}, \{0,2\}$
- $\{2\}, \{0,1\}$
- $\{0,1,2\}$

These correspond to the following equivalence relations, with each set in the partition corresponding to an equivalence class. We have

- $R_1 = \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle\}$
- $R_2 = \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \langle 1,2\rangle, \langle 2,1\rangle\}$
- $R_3 = \{\langle 1,1\rangle, \langle 0,0\rangle, \langle 2,2\rangle, \langle 0,2\rangle, \langle 2,0\rangle\}$
- $R_4 = \{\langle 2,2\rangle, \langle 0,0\rangle, \langle 1,1\rangle, \langle 0,1\rangle, \langle 1,0\rangle\}$
- $R_5 = \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 0,2\rangle, \langle 2,0\rangle, \langle 1,2\rangle, \langle 2,1\rangle\}$

□

Exercise 3.50.

(a) *Find a linear ordering on* $\{0,1,2,3\}$ *that contains the ordered pairs* $\langle 0,3\rangle$ *and* $\langle 2,1\rangle$.

(b) *Now find a different one meeting the same conditions.*

Solution. In an abuse of notation, consider the linear ordering $0 < 2 < 1 < 3$. Indeed, $0 < 3$ and $2 < 1$, as desired. Another possibility is the linear ordering $2 < 0 < 1 < 3$. $\square$

Exercise 3.51. *Find as many linear orderings as possible on the set* $\{0, 1, 2\}$ *that contain the pair* $\langle 2, 1\rangle$.

Solution. By the trichotomy property, we see that any two distinct elements must be related in some way. Hence we have three possible linear orderings with $2 < 0$, in particular they are $1 < 2 < 0$, $2 < 1 < 0$, and $2 < 0 < 1$. $\square$

Exercise 3.52. *Suppose that* $A \times B = C \times D$. *Under what conditions can we conclude that* $A = C$ *and* $B = D$*?*

Solution. We must have that A, B, C, D are all nonempty. If this is the case, for any $\langle a, b\rangle \in A \times B$, $\langle a, b\rangle \in C \times D$, so $a \in C$ and $b \in D$ and thus $A \subseteq C$ and $B \subseteq D$. A parallel argument shows that $C \subseteq A$ and $D \subseteq B$. However, if some of the sets are empty, say $B = D = \emptyset$, then, for example, $\mathbb{Q} \times B = \mathbb{R} \times D = \emptyset$, but clearly $\mathbb{Q} \neq \mathbb{R}$. $\square$

Exercise 3.53. *Show that for any set* R *and* S *we have* $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$, $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$, *and* $(R - S)^{-1} = R^{-1} - S^{-1}$.

Solution. Suppose $\langle r, s\rangle \in (R \cup S)^{-1}$, so $\langle s, r\rangle \in R \cup S$. If $\langle s, r\rangle \in R$, then $\langle r, s\rangle \in R^{-1}$, and if by the same reasoning, if $\langle s, r\rangle \in S$, then $\langle r, s\rangle \in S^{-1}$. Hence $(R \cup S)^{-1} \subseteq R^{-1} \cup S^{-1}$. Conversely, if $\langle r, s\rangle \in R^{-1} \cup S^{-1}$, then if $\langle r, s\rangle \in R^{-1}$, then $\langle s, r\rangle \in R$, so $\langle s, r\rangle \in R \cup S$, so $\langle r, s\rangle \in (R \cup S)^{-1}$, thus implying equality. A completely parallel argument shows the equality $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.

Take $\langle r, s\rangle \in (R - S)^{-1}$, so $\langle s, r\rangle \in R - S$, and thus $\langle s, r\rangle \in R$ and $\langle s, r\rangle \notin S$. Hence $\langle r, s\rangle \in R^{-1}$ but $\langle r, s\rangle \notin S^{-1}$, so $\langle r, s\rangle \in R^{-1} - S^{-1}$. The reverse containment follows, since each of these prior steps is reversable. $\square$

Exercise 3.54. *Prove that the following equations hold for any sets.*

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(c) $A \times (B - C) = (A \times B) - (A \times C)$.

Solution. The proofs of (a) and (b) are straightforward. For (c) take $\langle x, y\rangle \in A \times (B - C)$. Then $x \in A$ and $y \in B - C$, and thus $y \in B$ but $y \notin C$, so $\langle x, y\rangle \in A \times B$ but $\langle x, y\rangle \notin A \times C$. The converse follows similarly. $\square$

Exercise 3.55. *Answer "yes" or "no." Where the answer is negative, supply a counterexample.*

(a) Is it always true that $(A \times A) \cup (B \times C) = (A \cup B) \times (A \cup C)$*?*

(b) Is it always true that $(A \times A) \cap (B \times C) = (A \cap B) \times (A \cap C)$*?*

Solution. The answer to (a) is no. Take $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, and consider the pair $\langle 1, 3\rangle \in (A \cup B) \times (A \cup C)$. However, clearly $\langle 1, 3\rangle \notin A \times A$ and $\langle 1, 3\rangle \notin B \times C$ since $3 \notin A$ and $1 \notin B$, respectively. It is not hard to see that the answer to (b) is yes. □

Exercise 3.56. *Answer "yes" or "no." Where the answer is negative, supply a counterexample.*

(a) Is dom $(R \cup S)$ always the same as dom $R \cup$ dom S?

(b) Is dom $(R \cap S)$ always the same as dom $R \cap$ dom S?

Solution. The answer to (a) is yes. Take $x \in \text{dom } R \cup S$, then there is some y such that $\langle x, y\rangle \in R \cup S$. If $\langle x, y\rangle \in R$, then $x \in \text{dom } R$, and similarly for S. Conversely, if $x \in \text{dom } R \cup \text{dom } S$, then if $x \in \text{dom } R$, there exists a y such that $\langle x, y\rangle \in R \subseteq R \cup S$, and similarly $\langle x, y\rangle \in S \subseteq R \cup S$ if $x \in \text{dom } S$.

For (b), the answer is no. Consider $R = \{\langle 1, 2\rangle\}$, $S = \{\langle 1, 3\rangle\}$, and thus $R \cap S = \emptyset$. Now $1 \in \text{dom } R \ \cap \ \text{dom } S$, but $1 \notin \text{dom } R \cap S$, since $R \cap S = \emptyset$, and hence dom $R \cap S = \emptyset$. □

Exercise 3.57. *Answer "yes" or "no." Where the answer is negative, supply a counterexample.*

(a) Is $R \circ (S \cup T)$ always the same as $(R \circ S) \cup (R \circ T)$?

(b) Is $R \circ (S \cap T)$ always the same as $(R \circ S) \cap (R \circ T)$?

Solution. The answer to (a) is yes. For (b), let $R = \{\langle 3, 2\rangle, \langle 4, 2\rangle\}$, $S = \{\langle 1, 3\rangle\}$ and $T = \{\langle 1, 4\rangle\}$. So $S \cap T = \emptyset$, and thus $R \circ (S \cap T) = \emptyset$ as well. However by inspection, $\langle 1, 2\rangle \in (R \circ S) \cap (R \circ T)$, and thus the two sets are not equal. □

Exercise 3.58. *Give an example to show that $F[\![F^{-1}[\![S]\!]]\!]$ is not always the same as S.*

Solution. Take $S = \{1\}$ and $F = \{\langle 2, 3\rangle\}$, so $F^{-1} = \{\langle 3, 2\rangle\}$. Now $F^{-1}[\![S]\!] = \emptyset$, as 1 is not in the domain of F. It follows then that $F[\![F^{-1}[\![S]\!]]\!] = \emptyset$ as well. □

Exercise 3.59. *Show that for any sets $Q \restriction (A \cap B) = (Q \restriction A) \cap (Q \restriction B)$ and $Q \restriction (A - B) = (Q \restriction A) - (Q \restriction B)$.*

Solution. Take $\langle u, v\rangle \in Q \restriction (A \cap B)$, so there exists $u \in A \cap B$ such that uQv. Since $u \in A$ and $u \in B$, $\langle u, v\rangle \in Q \restriction A$ and $\langle u, v\rangle \in Q \restriction B$. Conversely, if $\langle u, v\rangle \in (Q \restriction A) \cap (Q \restriction B)$, then uQv and $u \in A$ and $u \in B$, so $u \in A \cap B$, so $\langle u, v\rangle \in Q \restriction (A \cap B)$.

If $\langle u, v\rangle \in Q \restriction (A - B)$. So uQv and $u \in A$ but $u \notin B$. So $\langle u, v\rangle \in Q \restriction A$, but $\langle u, v\rangle \notin Q \restriction B$ so $\langle u, v\rangle \in (Q \restriction A) - (Q \restriction B)$. Conversely, take $\langle u, v\rangle \in Q \restriction (A - B) = (Q \restriction A) - (Q \restriction B)$, so uQv but $u \in A$ and $u \notin B$. So $u \in A - B$, and thus $\langle u, v\rangle \in Q \restriction (A - B)$. □

Exercise 3.60. *Prove that for any sets $(R \circ S) \restriction A = R \circ (S \restriction A)$.*

Solution. Take $\langle u, v\rangle \in (R \circ S) \restriction A$. Then $u \in A$ and $u(R \circ S)v$, so there exists some t such that $uSt \wedge tRv$. Now $\langle u, t\rangle \in S \restriction A$ and $\langle t, v\rangle \in R$. Hence $\langle u, v\rangle \in R \circ (S \restriction A)$.

Conversely, take $\langle u, v\rangle \in R \circ (S \restriction A)$. Hence there exists t such that $u(S \restriction A)t$ and tRv. Now $\langle u, t\rangle \in (S \restriction A)$ implies $u \in A$ and uSt, and so $u(R \circ S)v$, which implies $\langle u, v\rangle \in (R \circ S) \restriction A$. $\square$

4 Natural Numbers

4.1 Inductive Sets

Exercise 4.1. *Show that $1 \neq 3$, i.e., that $\emptyset^+ \neq \emptyset^{+++}$.*

Solution. By defintion of the successor, we have

$$\emptyset^+ = \{\emptyset\} \quad \text{and} \quad \emptyset^{+++} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}.$$

Now take, for instance, $\{\emptyset\} \in \emptyset^{+++}$. However, $\{\emptyset\} \notin \emptyset^+$, and thus by Extensionality, $\emptyset^+ \neq \emptyset^{+++}$. $\square$

4.2 Peano's Postulates

Exercise 4.2. *Show that if a is a transitive set, then a^+ is also a transitive set.*

Solution. It suffices to show that for all $x \in a^+$, $x \subseteq a$. Recall $a^+ = a \cup \{a\}$. Take $x \in a^+$. If $x \in a$, then $x \subseteq a \subseteq a^+$, so $x \subseteq a^+$. If $x \in \{a\}$, then $x = a$, and again $x \subseteq a^+$. Hence a^+ is transitive. $\square$

Exercise 4.3.

(a) Show that if a is a transitive set, then $\mathscr{P}a$ is also a transitive set.

(b) Show that if $\mathscr{P}a$ is a transitive set, then a is also a transitive set.

Solution. Recall that if a is transitive, then $a \subseteq \mathscr{P}a$. So for any $x \in \mathscr{P}a$, we have $x \subseteq a$, and hence $x \subseteq \mathscr{P}a$. Thus $\mathscr{P}a$ is a transitive set.

For (b), if $\mathscr{P}a$ is transitive, then $\bigcup \mathscr{P}a \subseteq \mathscr{P}a$. Since $\bigcup \mathscr{P}a = a$, we have $a \subseteq \mathscr{P}a$. So for any $x \in a$, $x \in \mathscr{P}a$, and thus $x \subseteq a$, so a is a transitive set. $\square$

Exercise 4.4. *Show that if a is a transitive set, then $\bigcup a$ is also a transitive set.*

Solution. Take $x \in \bigcup a$. Since a is transitive, $\bigcup a \subseteq a$, and thus $x \in a$. Now take $y \in x$. Recall that $\bigcup a = \{b \mid (\exists c \in a)(b \in c)\}$. It follows that $y \in \bigcup a$, as $y \in x \in a$. Hence $x \subseteq \bigcup a$, and $\bigcup a$ is a transitive set. $\square$

Exercise 4.5. *Assume that every member of $\mathscr{A}$ is a transitive set.*

(a) Show that $\bigcup \mathscr{A}$ is a transitive set.

(b) Show that $\bigcap \mathscr{A}$ is a transitive set (assuming that $\mathscr{A}$ is nonempty.)

Solution. Take $x \in \bigcup \mathscr{A}$, so $x \in A$ for some $A \in \mathscr{A}$. Now take $y \in x$, then $y \in x \in A$ and so $y \in A$ since A is a transitive set. It follows that $y \in \bigcup \mathscr{A}$, and thus $x \subseteq \bigcup \mathscr{A}$, so $\bigcup \mathscr{A}$ is a transitive set.

Now take $x \in \bigcap \mathscr{A}$. So for all $A \in \mathscr{A}$, $x \in A$. Take any $y \in x$, and thus since each A is a transitive set, $y \in A$ for all $A \in \mathscr{A}$, so $y \in \bigcap \mathscr{A}$. Hence $\bigcap \mathscr{A}$ is a transitive set. $\square$

Exercise 4.6. *Prove the converse to Theorem 4E: If $\bigcup(a^+) = a$, then a is a transitive set.*

Solution. Suppose that $\bigcup(a^+) = a$, we then have the following equalities:

$$a = \bigcup(a^+) = \bigcup(a \cup \{a\}) = \bigcup a \cup \bigcup\{a\} = \bigcup a \cup a.$$

It follows that $\bigcup a \subseteq a$, which is equivalent to saying a is a transitive set. $\square$

4.3 Recursion on ω

Exercise 4.7. *Complete part 4 of the proof of the recursion theorem on ω.*

Solution. If h_1 and h_2 satisfy the conclusion of the theorem, then

$$h_1(0) = h_2(0) = a$$

so $0 \in S$. Now suppose that $k \in S$, so that $h_1(k) = h_2(k)$. We then have

$$h_1(k^+) = F(h_1(k)) = F(h_2(k)) = h_2(k^+).$$

So $k^+ \in S$, and hence S is inductive. So $S = \omega$, so $h_1 = h_2$. $\square$

Exercise 4.8. *Let f be a one-to-one function from A into A, and assume that $c \in A - \operatorname{ran} f$. Define $h\colon \omega \to A$ by recursion:*

$$h(0) = c, \qquad h(n^+) = f(h(n)).$$

Show that h is one-to-one.

Solution. Let

$$T = \{n \in \omega \mid (\forall m, n) m \neq n \implies h(m) \neq h(n)\}.$$

Suppose $m \neq 0$. Then by Theorem 4C, $m = p^+$ for some p. Note then that $h(m) = h(p^+) = f(h(p)) \neq c$, since $c \notin \operatorname{ran} f$. So $h(0) = c \neq h(m)$, and thus $0 \in T$. Now suppose $k \in T$, and that $h(k^+) = h(m)$. By the same reasoning above, $m \neq 0$, so $m = p^+$ for some p. Then,

$$f(h(k)) = h(k^+) = h(m) = h(p^+) = f(h(p)).$$

Since f is injective, $h(k) = h(p)$, and thus $k = p$, since $k \in T$. So $k^+ = p^+ = m$, and thus $k^+ \in T$. So T is inductive, and thus $T = \omega$, so h is one-to-one. $\square$

Exercise 4.9. *Let f be a function from B into B, and assume that $A \subseteq B$. We have two possible methods for constructing the "closure" C of A under f. First define C^* to be the intersection of the closed supersets of A:*

$$C^* = \bigcap \{X \mid A \subseteq X \subseteq B \wedge f[\![X]\!] \subseteq X\}.$$

Alternatively, we could apply the recursion theorem to obtain the function h for which

$$h(0) = A \qquad h(n^+) = h(n) \cup f[\![h(n)]\!].$$

Define C_* to be $\bigcup$ ran h; in other words

$$C_* = \bigcup_{i \in \omega} h(i).$$

Show that $C^* = C_*$.

Solution. Note that for all $i \in \omega$, $f[\![h(i)]\!] \subseteq h(i^+)$. Thus,

$$f[\![C_*]\!] = \bigcup_{i \in \omega} f[\![h(i)]\!] \subseteq \bigcup_{i \in \omega} h(i^+) \subseteq \bigcup_{i \in \omega} h(i) = C_*.$$

It follows that $C^* \subseteq C_*$, since clearly $A \subseteq C_* \subseteq B$, and so C_* is one of the sets being intersected to form C^*.

Now let

$$T = \{n \in \omega \mid h(n) \subseteq C^*\}.$$

Note $0 \in T$, since $h(0) = A \subseteq C^*$, as each set X being intersected contains A. Now assume $k \in T$. Since $h(k^+) = h(k) \cup f[\![h(k)]\!]$, and $h(k) \subseteq C^*$ by hypothesis, it suffices to show that $f[\![h(k)]\!] \subseteq C^*$. Since $h(k) \subseteq C^*$, $h(k) \subseteq X$ for all X being intersected, and thus $f[\![h(k)]\!] \subseteq f[\![X]\!] \subseteq X$. Hence $f[\![h(k)]\!] \subseteq C^*$ for similar reasoning as above. Hence $h(k^+) \subseteq C^*$ and $k^+ \in T$. So T is inductive, and thus $T = \omega$. Since $h(n) \subseteq C^*$ for all n, $C_* \subseteq C^*$, and equality follows. □

Exercise 4.10. *In Exercise 4.9, assume that B is the set of real numbers, $f(x) = x^2$, and A is the closed interval $[\frac{1}{2}, 1]$. What is the set called C^* and C_*?*

Solution. Note that any set X in the intersection forming C^* must contain $1/2$. Since X is also closed under f, $1/4 \in X$, and thust $1/16 \in X$, and so on. Essentially, if $1/2^n \in X$, then $1/2^{n+1} \in X$. Since this sequence converges to 0, any set in the intersection must contain the interval $[0, 1]$. Since $[0, 1]$ itself is closed under f, and is contained in every set in the intersection, we have $C^* = C_* = [0, 1]$. □

Exercise 4.11. *In Exercise 4.9, assume that B is the set of real numbers, $f(x) = x - 1$, and $A = \{0\}$. What is the set called C^* and C_*?*

Solution. Since $0 \in X$ for each X in the intersection, $-1 \in X$ as well, and thus $-2 \in X$ an so on. So each set must contain the set $\{n \in \mathbb{Z} \mid n \leq 0\}$. Since this set is closed under f, we have that it is equal to C^*. □

Exercise 4.12. *Formulate an analogue to Exercise 4.9 for a function $f: B \times B \to B$.*

Solution. An analogue for the closure of A under f would be the set

$$C^* = \{X \mid A \subseteq X \subseteq B \wedge f[\![X \times X]\!] \subseteq X\}.$$

$\square$

4.4 Arithmetic

Exercise 4.13. *Let m and n be natural numbers such that $m \cdot n = 0$. Show that either $m = 0$ or $n = 0$.*

Solution. By the contrapositive, assume $m \neq 0$ and $n \neq 0$. Since $n \neq 0$, $n = p^+$ for some p. Then

$$m \cdot n = M_m(n) = M_m(p^+) = M_m(p) + m \neq 0.$$

The last inequality holds, for if it were equality, then $0 = M_m(p^+) = M_m(p)^+ = \sigma(M_m(p))$, and thus $0 \in \operatorname{ran} \sigma$, contrary to the first of the Peano axioms. $\square$

Exercise 4.14. *Call a natural number even if it has the form $2 \cdot m$ for some m. Call it odd if it has the form $(2 \cdot p) + 1$ for some p. Show that each natural number is either even or odd, but never both.*

Solution. Recall from Theorem 4K that $+$ and $\cdot$ are commutative. First, let us make two observations.

- For all $n \in \omega$, $n^+ = n + 1$. To see this, let

  $$T = \{n \in \omega \mid n^+ = n + 1\}.$$

  Note that $0 \in T$, since $0^+ = 1 = 0+1$, with this last equality following from Theorem 4K(2). Suppose $k \in T$, so $k^+ = k + 1$. Then $k^{++} = (k + 1)^+ = k^+ + 1$, again by Theorem 4K(2). Hence $T = \omega$.

- For all $n \in \omega$, $n \cdot 1 = n$. To see this, let

  $$S = \{n \in \omega \mid n \cdot 1 = n\}.$$

  Note $0 \in S$ by (M1). Suppose $k \in S$, so $k \cdot 1 = k$. Then, by (M2) and the first observation,

  $$k^+ \cdot 1 = 1 \cdot k^+ = 1 \cdot k + 1 = k \cdot 1 + 1 = k + 1 = k^+.$$

  Hence $S = \omega$.

Now let
$$A = \{n \in \omega \mid \exists m(n = 2 \cdot m) \vee \exists p(n = 2 \cdot p + 1)\}.$$
This is the set of natural numbers that are either even or odd. Note that $0 \in A$, since $0 = 2 \cdot 0$. So assume $k \in A$. If k is even, then $k = 2 \cdot m$, so $k^+ = (2 \cdot p)^+ = 2 \cdot p + 1$ by the first observation. So k^+ is odd. If k is odd, then $k = 2 \cdot p + 1$, so by the second observation and the distributive law,
$$k^+ = (2 \cdot p + 1)^+ = 2 \cdot p + 1^+ = 2 \cdot p + 2 = 2 \cdot p + 2 \cdot 1 = 2 \cdot (p + 1).$$
Hence k^+ is even, so $k^+ \in A$. Hence $A = \omega$, and thus all natural numbers are either even or odd.

Now let
$$B = \{n \in \omega \mid \neg(\exists m(n = 2 \cdot m) \wedge \exists p(n = 2 \cdot p + 1))\},$$
the set of all natural numbers that are not both even and odd. We have seen that 0 is even. 0 is not odd, for if $0 = 2 \cdot p + 1$, then $0 = (2 \cdot p)^+ = \sigma(2 \cdot p)$, but then $0 \in \operatorname{ran} \sigma$, contrary to the first Peano axiom. Hence $0 \in B$. Suppose $k \in B$. Suppose k is odd but not even, so $k = 2 \cdot p + 1$ for some p. As seen before, k^+ is thus even. However, k^+ is not odd, for if $k^+ = 2 \cdot m + 1$ for some m, then since the successor function is injective, we have
$$k^+ = 2 \cdot m + 1 = (2 \cdot m)^+ \implies k = 2 \cdot m$$
contrary to the fact that k is not even. Suppose k is even, but not odd. So suppose $k = 2m \neq 2n + 1$. Then since σ is injective,
$$k^+ = (2 \cdot m)^+ \neq (2 \cdot n + 1)^+ \Rightarrow k^+ = 2 \cdot m + 1 \neq 2 \cdot n + 1^+ = 2 \cdot n + 2 = 2 \cdot n + 2 \cdot 1 = 2 \cdot (n + 1).$$
So k^+ is odd, but not even. So by induction, $B = \omega$. □

Exercise 4.15. *Prove the associative law of addition*
$$m + (n + p) = (m + n) + p.$$

Solution. Let
$$A = \{p \in \omega \mid m + (n + p) = (m + n) + p\}.$$
Note that $m + (n + 0) = m + n = (m + n) + 0$, so $0 \in A$. So assume $k \in A$. Then,
$$m + (n + k^+) = m + (n + k)^+ = (m + (n + k))^+ = ((m + n) + k)^+ = (m + n) + k^+.$$
So $k^+ \in A$, and thus $A = \omega$. □

Exercise 4.16. *Prove the commutative law for multiplication*
$$m \cdot n = n \cdot m.$$

Solution. First, we prove two preliminary facts.

- Let
$$A = \{n \in \omega \mid 0 \cdot n = 0\}.$$
Note that $0 \cdot 0 = 0$ by (M1), so $0 \in A$. Assume that $k \in A$, so $0 \cdot k = 0$. Thus
$$0 \cdot k^+ = 0 \cdot k + 0 = 0 + 0 = 0,$$
so $k^{\in} A$, and thus $A = \omega$.

- Now let
$$B = \{n \in \omega \mid m^+ \cdot n = m \cdot n + n\}.$$
Note by (M1), $m^+ \cdot 0 = 0 = m\dot{0} = m \cdot 0 + 0$, so $0 \in B$. Assume $k \in B$, so $m^+ \cdot k = m \cdot k + k$. Then
$$\begin{aligned} m^+ \cdot k^+ &= m^+ \cdot k + m^+ && \text{by M2} \\ &= m \cdot k + k + m^+ && \text{since } k \in B \\ &= m \cdot k + (k + m^+) \\ &= m \cdot k + (k^+ + m) && \text{by the second result} \\ &= m \cdot k + m + k^+ && \text{commutativity of addition} \\ &= m \cdot k^+ + k^+ \end{aligned}$$

Now let
$$C = \{m \in \omega \mid m \cdot n = n \cdot m\}.$$
By the first result, we have $0 \cdot n = 0 = n \cdot 0$, so $0 \in C$. Suppose $k \in C$, so $k \cdot n = n \cdot k$. Then by (M2),
$$k^+ \cdot n = k \cdot n + n = n \cdot k + n = n + k^+.$$
Hence $k^+ \in C$, so $C = \omega$. □

Exercise 4.17. *Prove that $m^{n+p} = m^n \cdot m^p$.*

Solution. Let
$$A = \{p \in \omega \mid m^{n+p} = n^n \cdot m^p\}.$$
Using the results from the previous exercises, observe that
$$m^{n+0} = m^n = m^n \cdot 1 = m^n \cdot m^0$$
so $0 \in A$. Now assume that $k \in A$, so $m^{n+k} = m^n \cdot m^k$. Then by (A2) and (E2),
$$m^{n+k^+} = m^{(n+k)^+} = m^{n+k} \cdot m = m^n \cdot m^k \cdot m = m^n \cdot m^{k^+}.$$
Hence $k^+ \in A$, so $A = \omega$. □

4.5 Ordering on ω

Exercise 4.18. *Simplify:* $\in_\omega^{-1} [\![\{7,8\}]\!]$.

Solution. Observe by definition that

$$\in_\omega^{-1} [\![\{7,8\}]\!] = \{v \mid \exists u \in \{7,8\}, u \in_\omega^{-1} v\} = \{v \mid \exists u \in \{7,8\}, v \in_\omega u\}.$$

Hence $\in_\omega^{-1} [\![\{7,8\}]\!] = \{0,1,2,3,4,5,6,7\}$. ☐

Exercise 4.19. *Prove that if m is a natural number and d is a nonzero number, then there exist numbers q and r such that $m = (d \cdot q) + r$ and r is less than d.*

Solution. Let

$$B = \{m \in \omega \mid \exists q, r((m = (d \cdot q) + r) \wedge r \in d)\}.$$

Since $0 = d \cdot 0 + 0$, and since $0 \in d$, and so $0 \in B$. Now assume that $k \in B$, so there exist q and r such that $k = (d \cdot q) + r$ for some $r \in d$. It follows that $k^+ = ((d \cdot q) + r)^+ = d \cdot q + r^+$. By Lemma 4L, since $r \in d$, $r^+ \in d^+$, and thus $r^+ \in\!\!\!\!= d$. Now if $r^+ \in d$, the conditions are satisfied, so $k^+ \in B$. If $r^+ = d$, then

$$k^+ = (d \cdot q) + d = (d \cdot q) + d \cdot 1 = d \cdot (q+1) = d \cdot (q+1) + 0,$$

and thus $k^+ \in B$, so B is inductive, and thus $B = \omega$. ☐

Exercise 4.20. *Let A be a nonempty subset of ω such that $\bigcup A = A$. Show that $A = \omega$.*

Solution. First, observe that $0 \in A$, since A is nonempty, for any such $p \in A$, $0 \in p$, and thus $0 \in \bigcup A = A$. Now suppose that $k \in A$. Then $k \in \bigcup A$, and so $k \in a$ for some $a \in A$. If $a = k^+$, we have $k^+ \in A$. Note that we can not have $a \in k^+$, otherwise, $a \in\!\!\!\!= k$, which contradicts the trichotomy law. Hence $k^+ \in a$, and so $k^+ \in \bigcup A = A$. In either case, $k^+ \in A$, so A is inductive. Hence $A = \omega$. ☐

Exercise 4.21. *Show that no natural number is a subset of any of its elements.*

Solution. Let n be any natural number. If $n = 0$, the conclusion holds vacuously, so suppose $n \neq 0$. Let $m \in n$ be any member of n. If $n \subseteq m$, then by Corollary 4M, $n \in\!\!\!\!= m$, which contradicts trichotomy. Hence no natural number is a subset of any of its elements. ☐

Exercise 4.22. *Show that for any natural numbers m and p we have $m \in m + p^+$.*

Solution. Since $p^+ \neq 0$, we have $0 \in p^+$. Then by Theorem 4N, $0 + m \in p^+ + m$. Then by commutativity of addition and (A1), we have $m \in m + p^+$. ☐

Exercise 4.23. *Assume that m and n are natural numbers with m less than n. Show that there is some p in ω for which $m + p^+ = n$.*

Solution. First note that $n \neq 0$, since $m \in n$. If $m = 0$, then we can take $0+p^+ = p^+ = n$, and we know such a p exists since n is nonzero. Now suppose $k \in n$, and that for some p, $k + p^+ = n$. Since $k \in n$, $k^+ \underline{\in} n$. If $k^+ = n$, then the conclusion holds trivially, since $k^+ \notin n$. If $k^+ \in n$, then observe that

$$k + p^+ = (k + p)^+ = k^+ + p = n.$$

Since $k^+ \neq n$, we have $p \neq 0$, and thus $p = q^+$ for some q. Hence $k^+ + q^+ = n$, and thus conclusion holds for all $m \in n$. Essentially, since we know $0 + n = n$, we can then find a p such that $1 + p^+ = n$, and from this we can find a q such that $2 + q^+ = n$, and so on for all $m \in n$. This must eventually terminate, as n is finite. □

Exercise 4.24. *Assume that $m + n = p + q$. Show that*

$$m \in p \Leftrightarrow q \in n.$$

Solution. Suppose $m \in p$. By the previous exercise, there exists some r such that $m + r^+ = p$. Hence $m + n = m + r^+ + q$. Due to the cancellation and commutativity of addition, we then have $n = q + r^+$. It follows immediately from the previous exercises that $q \in n$. A completely parallel argument shows that if $q \in n$, then $m \in p$. □

Exercise 4.25. *Assume that $n \in m$ and $q \in p$. Show that*

$$(m \cdot q) + (n \cdot p) \in (m \cdot p) + (n \cdot q).$$

Solution. Since $q \in p$, from Exercise 23 we have that $p = q + s^+$ for some s. Then note

$$\begin{aligned}
n \in m &\implies n \cdot s^+ \in m \cdot s^+ \\
&\implies m \cdot q + n \cdot q + n \cdot s^+ \in m \cdot q + n \cdot q + m \cdot s^+ \\
&\implies m \cdot q + n \cdot (q + s^+) \in m \cdot (q + s^+) + n \cdot q \\
&\implies (m \cdot q) + (n \cdot p) \in (m \cdot p) + (n \cdot q).
\end{aligned}$$

□

Exercise 4.26. *Assume that $n \in \omega$ and $f\colon n^+ \to \omega$. Show that ran f has a largest element.*

Solution. Let

$$A = \{n \in \omega \mid f\colon n^+ \to \omega \text{ has a largest element.}\}.$$

If $n = 0$, then $n^+ = 1$, so ran $f = \{f(0)\}$, and thus trivially possesses a largest element. So suppose $k \in A$, and thus for $f\colon k^+ \to \omega$, max(ran f) exists. Recall $k^{++} = k^+ \cup \{k^+\}$, and consider $f\colon k^+ \cup \{k^+\} \to \omega$. Not $f[\![k^+]\!]$ possesses a largest element, call it M. The only other element in ran f is $f(k^+)$. If $f(k^+) = M$, then $\max \operatorname{ran} f = M = f(k^)$. If $f(k^+) \in M$, then max(ran f) $= M$. If $M \in f(k^+)$, then by transitivity, max(ran f) $= f(k^+)$. In any case, ran f possesses the largest element, so $k^+ \in A$. So for any $n \in \omega$ and $f\colon n^+ \to \omega$, ran f has a largest element.

Alternatively, recall that any natural number is just the set of smaller natural numbers. As such, n^+ is a set with $n^+ = n+1$ elements. Since f is a function, ran f can have at most $n+1$ elements. Thus ran f is a finite set, and let us say that it has j^+ elements. If ran f has 1 element, then trivially it has a largest element. Otherwise, take two distinct elements m and n in ran f. By the trichotomy law, $\{m, n\}$ has a largest element. So suppose that any subset A of j or fewer elements of ran f has a largest element, say $\max A$. Denoting k as the element in the set ran $f \setminus A$, we see that the largest element of ran f is the larger of $\max A$ and k, which is sure to exist by trichotomy. $\square$

Exercise 4.27. *Assume that A is a set, G is a function, and f_1 and f_2 map ω into A. Further assume that for each n in ω both $f_1 \restriction n$ and $f_2 \restriction n$ belong to dom G and*

$$f_1(n) = G(f_1 \restriction n) \wedge f_2(n) = G(f_2 \restriction n).$$

Show that $f_1 = f_2$.

Solution. It suffices to show that f_1 and f_2 agree on all elements in ω. Let

$$B = \{m \in \omega \mid f_1(m) = f_2(m)\}.$$

Note that $f_1 \restriction 0 = \{\langle u, v\rangle \mid f(u) = v \wedge u \in 0\} = \emptyset$, since there are no elements in 0. Similarly, $f_2 \restriction 0 = \emptyset$, and so

$$f_1(0) = G(f_1 \restriction 0) = G(\emptyset) = G(f_2 \restriction 0) = f_2(0)$$

and thus $0 \in B$. So assume that $k \in B$, and hence $f_1(k) = f_2(k)$. Then observe that

$$f_1(k^+) = G(f_1 \restriction k^+) = G(f_1 \restriction k) \cup G(f_1 \restriction \{k\}) = G(f_1 \restriction k) \cup G(\langle k, f_1(k)\rangle)$$

and

$$f_2(k^+) = G(f_2 \restriction k^+) = G(f_2 \restriction k) \cup G(f_2 \restriction \{k\}) = G(f_2 \restriction k) \cup G(\langle k, f_2(k)\rangle).$$

By the induction hypothesis, $G(f_1 \restriction k) = G(f_2 \restriction k)$, and since $f_1(k) = f_2(k)$, then $G(\langle k, f_1(k)\rangle) = G(\langle k, f_2(k)\rangle)$. It follows that $f_1(k^+) = f_2(k^+)$, and so $k^+ \in B$. So $B = \omega$, and $f_1 = f_2$. $\square$

Exercise 4.28. *Rewrite the proof of Theorem 4G using, in place of induction, the well ordering of ω.*

Solution. Let

$$T = \{n \in \omega \mid n \not\subseteq \omega\}.$$

Suppose $m := \min T$ exists, so $m \not\subseteq \omega$. Now $m \neq 0$, since $0 \subseteq \omega$. Thus $m = p^+$ for some p. Then $m = p \cup \{p\}$. Since $p \in m$, we must have $p \subseteq \omega$. Clearly $\{p\} \subseteq \omega$. It follows that $m \subseteq \omega$, a contradiction. Hence T has no least element, and thus by the well-ordering principle, $T = \emptyset$. It follows that for all $n \in \omega$, $n \subseteq \omega$, and thus ω is transitive. $\square$

4.6 Review Exercise

Exercise 4.29. *Write an expression for the set 4 using only symbols $\emptyset$, $\{$, $\}$, and commas.*

Solution. Recall $4 = \{0, 1, 2, 3\}$. Hence

$$4 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}.$$

$\square$

Exercise 4.30. *What is $\bigcup 4$? What is $\bigcap 4$?*

Solution. By Theorem 4E, $\cup(a^+) = a$, and thus $\cup 4 = 3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Also, $\cap 4 = 0 \cap 1 \cap 2 \cap 3 = 0$, since any set intersected with $\emptyset$ is again $\emptyset$. $\square$

Exercise 4.31. *What is $\bigcup\bigcup 7$?*

Solution. Again by Theorem 4E, we have

$$\bigcup\bigcup 7 = \bigcup 6 = 5.$$

$\square$

Exercise 4.32.

(a) Let $A = \{1\}$. Calculate A^+ and $\bigcup(A^+)$.

(b) What is $\bigcup(\{2\}^+)$?

Solution. First,

$$A^+ = \{1\} \cup \{\{1\}\} = \{1, \{1\}\} = \{\{\emptyset\}, \{\{\emptyset\}\}\}.$$

Hence

$$\bigcup(A^+) = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} = 2.$$

Also, $\{2\}^+ = \{2\} \cup \{\{2\}\} = \{2, \{2\}\}$. So

$$\bigcup(\{2\}^+) = 2 \cup \{2\} = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = 3.$$

$\square$

Exercise 4.33. *Which of the following sets are transitive? (For each set S that is not transitive, specify a member of $\bigcup S$ not belonging to S.)*

(a) $\{0, 1, \{1\}\}$.

(b) $\{1\}$.

(c) $\langle 0, 1\rangle$.

Solution.

(a) Note

$$\cup\{0,1,\{1\}\} = 0 \cup 1 \cup \{1\} = 1 \cup \{1\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} = 2 = \{0,1\}.$$

Since $\{0,1\} \subseteq \{0,1,\{1\}\}$, this set is transitive.

(b) $\bigcup\{1\} = 1 = \{\emptyset\}$. Note that $\{\emptyset\} \not\subseteq \{1\} = \{\{\emptyset\}\}$, since $\emptyset \notin \{\{\emptyset\}\}$.

(c) Note $\bigcup\langle 0,1\rangle = \bigcup\{\{0\},\{0,1\}\} = \bigcup\{2\} = 2$. Since $2 = \{0,1\} \not\subseteq \{\{0\},\{2\}\}$, since $\emptyset \in 2$, but $\emptyset \notin \{\{0\},\{2\}\}$.

□

Exercise 4.34. *Find suitable a, b, etc. making each of the following sets transitive.*

(a) $\{\{\{\emptyset\}\}, a, b\}$.

(b) $\{\{\{\{\emptyset\}\}\}, c, d, e\}$.

Solution. For (a), let $a = \emptyset$ and $b = \{\emptyset\}$. Then

$$\bigcup\{\{\{\emptyset\}\}, \emptyset, \{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}.$$

Since $\{\emptyset, \{\emptyset\}\} \subseteq \{\{\{\emptyset\}\}, \emptyset, \{\emptyset\}\}$, the set is transitive.

For (b), let $c = \emptyset$, $d = \{\emptyset\}$, and $e = \{\{\emptyset\}\}$. Then

$$\bigcup\{\{\{\{\emptyset\}\}\}, \emptyset, \{\emptyset\}, \{\{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}.$$

Since $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\} \subseteq \{\{\{\{\emptyset\}\}\}, \emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$, the set is transitive. □

Exercise 4.35. *Let S be the set $\langle 1,0\rangle$.*

(a) Find a transitive set T_1 for which $S \subseteq T_1$.

(b) Find a transitive set T_2 for which $S \in T_2$.

Solution. Let

$$T_1 = \{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, \emptyset, \{\emptyset\}\}$$

and

$$T_2 = \{\{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}\}.$$

It is straightforward to verify that both these sets satisfy the conditions of the problem. □

Exercise 4.36. *There is a function $h\colon \omega \to \omega$ for which $h(0) = 3$ and $h(n^+) = 2 \cdot h(n)$. What is $h(4)$?*

Solution. One has the following set of implications:

$$\begin{aligned} h(0) = 3 &\implies h(1) = 2 \cdot h(0) = 6 \\ &\implies h(2) = 2 \cdot h(1) = 12 \\ &\implies h(3) = 2 \cdot h(2) = 24 \\ &\implies h(4) = 2 \cdot h(3) = 48 \end{aligned}$$

□

Exercise 4.37. *We will say that a set S has n elements (where $n \in \omega$) iff there is a one-to-one function from n onto S. Assume that A has m elements and B has n elements.*

(a) Show that A and B are disjoint, then $A \cup B$ has $m + n$ elements.

(b) Show that $A \times B$ has $m \cdot n$ elements.

Solution. Let $f_1 \colon m \to A$ and $f_2 \colon n \to B$ be two possible bijections. Define a new bijection $g \colon m + n \to A \cup B$ as such:

$$g(a) = \begin{cases} f_1(a) & \text{if } a \in m \\ f_2(b) & \text{otherwise, where } a = m + b. \end{cases}$$

This equation is a bijection, and since A and B are disjoint, there will be a unique image for each $a \in m + n$.

Define also a bijection $h \colon m \cdot n \to A \times B$ as such:

$$g(a) = \langle f_1(i), f_2(j) \rangle \quad \text{if } m \cdot i \subseteq a \in m \cdot i^+ \text{ and } a = (m \cdot i) + j.$$

Essentially for all intervals of m, we holds and element of A fixed, and lets the second coordinate range through all n possibilities for B, before moving to the next interval. □

Exercise 4.38. *Assume that h is the function from ω into ω for which $h(0) = 1$ and $h(n^+) = h(n) + 3$. Give an explicit expression for $h(n)$.*

Solution. I claim that $h(n) = 3 \cdot n + 1$ for all $n \in \omega$. Note that $h(0) = 3 \cdot 0 + 1$. Assume that $h(k) = 3 \cdot k + 1$. Then $h(k^+) = h(k) + 3 = 3 \cdot k + 1 + 3 = 3 \cdot (k + 1) + 1 = 3 \cdot k^+ + 1$. So by induction, $h(n) = 3 \cdot n + 1$ for all $n \in \omega$. □

Exercise 4.39. *Assume that h is the function from ω into ω for which $h(0) = 1$ and $h(n^+) = h(n) + (2 \cdot n) + 1$. Give an explicit (not recursive expression for $h(n)$.*

Solution. After calculating the first few values, I claim $h(n) = n \cdot n + 1$. Note $h(0) = 0 \cdot 0 + 1 = 1$. So assume $h(k) = k \cdot k + 1$. Then

$$\begin{aligned} h(k^+) &= h(k) + 2 \cdot k + 1 \\ &= k \cdot k + 1 + 2 \cdot k + 1 \\ &= k \cdot k + 2 \cdot k + 2 \\ &= k \cdot k + k + (k + 1) + 1 \\ &= k \cdot (k + 1) + (k + 1) + 1 \\ &= (k + 1) \cdot (k + 1) + 1 \\ &= k^+ \cdot k^+ + 1 \end{aligned}$$

Hence $h(n) = n \cdot n + 1$ for all $n \in \omega$. □

Exercise 4.40. *Assume that h is the function from ω into ω defined by $h(n) = 5 \cdot n + 2$. Express $h(n^+)$ in terms of $h(n)$ as simply as possible.*

Solution. Note that

$$h(n^+) = 5 \cdot n^+ + 2 = 5 \cdot n + 5 + 2 = 5 \cdot n + 2 + 5 = h(n) + 5.$$

□

5 Construction of the Real Numbers

5.1 Integers

Exercise 5.1. *Is there a function $F\colon \mathbb{Z} \to \mathbb{Z}$ satisfying the equation*

$$F([\langle m, n\rangle]) = [\langle m + n, n\rangle]?$$

Solution. Let

$$\hat{F}\colon \omega \times \omega \to \omega \times \omega\colon \langle m, n\rangle \mapsto \langle m + n, n\rangle.$$

By Theorem 3Q, it is enough to show that G is not compatible with $\sim$ to show that such a function F does not exist. Take for example $\langle 5, 2\rangle$ and $\langle 3, 0\rangle$. Then clearly $\langle 5, 3\rangle \sim \langle 3, 0\rangle$. Note that $\hat{G}(\langle 5, 2\rangle) = \langle 5{+}2, 2\rangle = \langle 7, 2\rangle$ and $\hat{F}(\langle 3, 0\rangle) = \langle 3, 0\rangle$. Hence $\hat{F}(\langle 5, 2\rangle) \not\sim \hat{F}(\langle 3, 0\rangle)$, so $\hat{F}$ is not compatible with $\sim$, and thus no such F exists. □

Exercise 5.2. *Is there a function $G\colon \mathbb{Z} \to \mathbb{Z}$ satisfying the equation*

$$G([\langle m, n\rangle]) = [\langle m, m\rangle]?$$

Solution. Let

$$\hat{G}\colon \omega \times \omega \to \omega \times \omega\colon \langle m, n\rangle \mapsto \langle m, m\rangle.$$

Suppose $\langle m, n\rangle \sim \langle m', n'\rangle$. Note that $G(\langle m, n\rangle) \sim G(\langle m', n'\rangle)$, since $\langle m, n\rangle \sim \langle m', n'\rangle$ as clearly $m + m' = m' + m$. Thus $\hat{G}$ is compatible with $\sim$, and so such a function G exists. In fact, it is simply the function that maps all integers to 0. □

Exercise 5.3. *Is there a function $H\colon \mathbb{Z} \to \mathbb{Z}$ satisfying the equation*

$$H([\langle m, n\rangle]) = [\langle n, m\rangle]?$$

Solution. Let

$$\hat{H}\colon \omega \times \omega \to \omega \times \omega\colon \langle m, n\rangle \mapsto \langle n, m\rangle.$$

Note that if $\langle m, n\rangle \sim \langle m', n'\rangle$ then $m + n' = m' + n$, and so $n + m' = n' + m$, and hence $\langle n, m\rangle \sim \langle n', m'\rangle$, or $\hat{G}(\langle m, n\rangle) \sim \hat{G}(\langle m', n'\rangle)$. Since $\hat{H}$ is compatible with $\sim$, and thus by Theorem 3Q, such a function H exists. □

Exercise 5.4. *Prove that $+_{\mathbb{Z}}$ is associative.*

Solution. Let $a = [\langle m, n\rangle]$, $b = [\langle p, q\rangle]$, and $c = [\langle r, s\rangle]$. Then

$$\begin{aligned}(a +_{\mathbb{Z}} b) +_{\mathbb{Z}} c &= ([\langle m, n\rangle] +_{\mathbb{Z}} [\langle p, q\rangle]) +_{\mathbb{Z}} [\langle r, s\rangle] \\ &= ([\langle m + p, n + q\rangle]) +_{\mathbb{Z}} [\langle r, s\rangle] \\ &= [\langle (m + p) + r, (n + q) + s\rangle] \\ &= [\langle m + (p + r), n + (q + s)\rangle] \qquad \text{by the associativity of natural numbers} \\ &= [\langle m, n\rangle] +_{\mathbb{Z}} [\langle p + r, q + s\rangle] \\ &= [\langle m, n\rangle] +_{\mathbb{Z}} ([\langle p, q\rangle] +_{\mathbb{Z}} [\langle r, s\rangle]) \\ &= a +_{\mathbb{Z}} (b +_{\mathbb{Z}} c)\end{aligned}$$

□

Exercise 5.5. *Give a formula for subtraction of integers:*

$$[\langle m, n\rangle] - [\langle p, q\rangle] =?$$

Solution. Informally, we would like to have that

$$(m - n) - (p - q) = (m + q) - (n + p)$$

which suggests that we should define subtraction of integers by the formula

$$[\langle m, n\rangle] - [\langle p, q\rangle] = [\langle m + q, n + p\rangle].$$

Note this is well defined. Suppose $\langle m, n\rangle \sim \langle m', n'\rangle$ and $\langle p, q\rangle \sim \langle p', q'\rangle$. So

$$m + n' = m' + n \quad \text{and} \quad p + q' = p' + q.$$

Adding the first equation to the reverse of the second yields

$$m + q + n' + p' = m' + q' + n + p$$

which implies $\langle m + q, n + p\rangle \sim \langle m' + q', n' + p'\rangle$. This justifies the definition of $-_{\mathbb{Z}}$. □

Exercise 5.6. *Show that $a \cdot_{\mathbb{Z}} 0_{\mathbb{Z}} = 0_{\mathbb{Z}}$ for every integer a.*

Solution. Let $a = [\langle m, n\rangle]$. Then

$$a \cdot_{\mathbb{Z}} 0_{\mathbb{Z}} = [\langle m, n\rangle] \cdot_{\mathbb{Z}} [\langle 0, 0\rangle] = [\langle m \cdot 0 + n \cdot 0, m \cdot 0 + n \cdot 0\rangle] = [\langle 0, 0\rangle] = 0_{\mathbb{Z}},$$

where we use the fact that $0 \cdot k = 0$ for $k \in \omega$, and $0 + 0 = 0$. □

Exercise 5.7. *Show that*

$$a \cdot_{\mathbb{Z}} (-b) = (-a) \cdot_{\mathbb{Z}} b = -(a \cdot_{\mathbb{Z}} b)$$

for all integers a and b.

Solution. Let $a = [\langle m, n\rangle]$ and $b = [\langle p, q\rangle]$. Then

$$a \cdot_{\mathbb{Z}} (-b) = [\langle m, n\rangle] \cdot_{\mathbb{Z}} [\langle q, p\rangle] = [\langle mq + np, mp + nq\rangle]$$

and

$$(-a) \cdot_{\mathbb{Z}} b = [\langle n, m\rangle] \cdot_{\mathbb{Z}} [\langle p, q\rangle] = [\langle np + mq, nq + mp\rangle].$$

Hence by the commutativity of $+$ and $\cdot$ on ω, we have $a \cdot_{\mathbb{Z}} (-b) = (-a) \cdot_{\mathbb{Z}} b$. Also, observe

$$a \cdot_{\mathbb{Z}} b + (-a) \cdot_{\mathbb{Z}} b = [\langle mp + nq + np + mq, mq + np + nq + mp\rangle] = [\langle 0, 0\rangle] = 0_{\mathbb{Z}}.$$

Since inverses are unique, we must have that $(-a) \cdot_{\mathbb{Z}} b = -(a \cdot_{\mathbb{Z}} b)$ and so

$$a \cdot_{\mathbb{Z}} (-b) = (-a) \cdot_{\mathbb{Z}} b = -(a \cdot_{\mathbb{Z}} b).$$

□

Exercise 5.8. *Prove parts (a), (b), and (c) of Theorem 5ZL.*

Solution. For $m, n \in \omega$,

(a) $E(m + n) = [\langle m + n, 0\rangle] = [\langle m, 0\rangle] +_{\mathbb{Z}} [\langle n, 0\rangle] = E(m) + E(n)$.

(b) Since $0 \cdot k = 0$ for $k \in \omega$,

$$E(mn) = [\langle mn, 0\rangle] = [\langle m, 0\rangle] \cdot_{\mathbb{Z}} [\langle n, 0\rangle] = E(m) \cdot_{\mathbb{Z}} E(n).$$

(c) Note

$$m \in n \iff m + 0 \in n + 0 \iff [\langle m, 0\rangle] <_{\mathbb{Z}} [\langle n, 0\rangle] \iff E(m) <_{\mathbb{Z}} E(n).$$

□

Exercise 5.9. *Show that*

$$[\langle m, n\rangle] = E(m) - E(n)$$

for all natural numbers m and n.

Solution. Indeed,

$$[\langle m, n\rangle] = [\langle m, 0\rangle] +_{\mathbb{Z}} [\langle 0, n\rangle] = [\langle m, 0\rangle] - [\langle n, 0\rangle] = E(m) - E(n).$$

□

5.2 Rational Numbers

Exercise 5.10. *Show that $r \cdot_{\mathbb{Q}} 0_{\mathbb{Q}} = 0_{\mathbb{Q}}$ for every rational number r.*

Solution. Let $r = [\langle a, b\rangle]$, with $b \neq 0_{\mathbb{Z}}$. Then

$$r \cdot_{\mathbb{Q}} 0_{\mathbb{Q}} = [\langle a, b\rangle] \cdot_{\mathbb{Q}} [\langle 0, 1\rangle] = [\langle a \cdot 0, b \cdot 1\rangle] = [\langle 0, b\rangle] = [\langle 0, 1\rangle] = 0_{\mathbb{Q}}.$$

□

Exercise 5.11. *Give a direct proof that if $r \cdot_{\mathbb{Q}} s = 0_{\mathbb{Q}}$, then either $r = 0_{\mathbb{Q}}$ or $s = 0_{\mathbb{Q}}$.*

Solution. Let $r = [\langle a, b\rangle]$ and $s = [\langle c, d\rangle]$, and suppose $r \cdot_{\mathbb{Q}} s = 0_{\mathbb{Q}}$. Then

$$\begin{aligned} r \cdot_{\mathbb{Q}} s = 0_{\mathbb{Q}} &\iff [\langle ac, bd\rangle] = [\langle 0, 1\rangle] \\ &\iff ac \cdot_{\mathbb{Z}} 1 = bd \cdot_{\mathbb{Z}} 0 \\ &\iff ac = 0_{\mathbb{Z}} \\ &\iff a = 0_{\mathbb{Z}} \vee b = 0_{\mathbb{Z}} \qquad \text{since } \mathbb{Z} \text{ is an integral domain.} \end{aligned}$$

If $a = 0_{\mathbb{Z}}$, then $r = 0_{\mathbb{Q}}$, and likewise for s if $b = 0_{\mathbb{Z}}$. □

Exercise 5.12. *Show that*

$$r <_{\mathbb{Q}} 0_{\mathbb{Q}} \quad \textit{iff} \quad 0_{\mathbb{Q}} <_{\mathbb{Q}} -r.$$

Solution. Let $r = [\langle a, b\rangle]$ with $b \neq 0_{\mathbb{Z}}$. Suppose $r <_{\mathbb{Q}} 0_{\mathbb{Q}}$, so

$$\begin{aligned} [\langle a, b\rangle] <_{\mathbb{Q}} [\langle 0, 1\rangle] &\iff a \cdot 1 < 0 \cdot b = 0 \\ &\iff a < 0 \\ &\iff 0 < -a \\ &\iff 0 \cdot b < (-a) \cdot 1 \\ &\iff 0_{\mathbb{Q}} <_{\mathbb{Q}} -r \end{aligned}$$

since $-r = [\langle -a, b\rangle]$. □

Exercise 5.13. *Give a new proof of the cancellation law for $+_{\mathbb{Z}}$ using Theorem 5ZD instead of Theorem 5ZJ.*

Solution. Theorem 5ZD tells us that additive inverses for integers exist. So suppose $a +_{\mathbb{Z}} c = b +_{\mathbb{Z}} c$. The additive inverse of c, call it $-c$, exists, and adding it to both sides yields the implications

$$a +_{\mathbb{Z}} c +_{\mathbb{Z}} (-c) = b +_{\mathbb{Z}} c +_{\mathbb{Z}} (-c) \implies a +_{\mathbb{Z}} 0_{\mathbb{Z}} = b +_{\mathbb{Z}} 0_{\mathbb{Z}} \implies a = b.$$

□

Exercise 5.14. *Show that the ordering of rations is dense, i.e., between any two rationals there is a third one:*

$$p <_{\mathbb{Q}} s \Rightarrow (\exists r)(p <_{\mathbb{Q}} r <_{\mathbb{Q}} s).$$

Solution. Let $p = [\langle a, b\rangle]$ and $s = [\langle c, d\rangle]$, with $b, d > 0_{\mathbb{Z}}$. Note that this implies $ad <_{\mathbb{Z}} cb$, as well as $abd <_{\mathbb{Z}} bbc$ and $add <_{\mathbb{Z}} bcd$ since b and d are positive. Now let

$$r = (p +_{\mathbb{Q}} s) \div [\langle 2, 1\rangle] = [\langle ad + cb, bd\rangle] \cdot_{\mathbb{Q}} [\langle 1, 2\rangle] = [\langle ad + cb, 2bd\rangle].$$

Using the inequalities mentioned above, observe

$$2abd = abd + abd <_{\mathbb{Z}} abd + bbc = b \cdot (ad + cb)$$

from which it follows that $p <_{\mathbb{Q}} r$. Also,

$$(ad + cb) \cdot d = add + bcd < bcd + bcd = 2bcd$$

from which it follows that $r <_{\mathbb{Q}} s$, and so $p <_{\mathbb{Q}} r <_{\mathbb{Q}} s$, showing that the ordering of the rational numbers is dense. □

Exercise 5.15. *In Theorem 5RB, show that $\bigcup A$ is closed downward and has no largest element.*

Solution. Suppose $q \in \bigcup A$, so for some $x \in A$, $q \in A$. Take any $r < q$, then since x is downward closed, $r \in x$. It follows that $r \in \bigcup A$, so $\bigcup A$ is closed downward.

Now take any $p \in \bigcup A$. So $p \in x$ for some $x \in A$. Since x has no largest element, there is some $q \in x$ such that $p < q$. Of course, $q \in \bigcup A$ as well. Hence for any $p \in \bigcup A$, we can always find some element $q \in \bigcup A$ larger than p, so $\bigcup A$ has no largest element. □

Exercise 5.16. *In Lemma 5RC, show that $x +_R y$ has no largest element.*

Solution. Take any $q + r \in x +_R y$. So $q \in x$ and $r \in y$. Since neither x nor y has a largest element, there exist $q' \in x$ and $r' \in y$ such that $q < q'$ and $r < r'$. Since addition preserves order in the rationals, $q + r < q' + r' \in x +_R y$. Hence $x +_R y$ has no largest element. □

Exercise 5.17. *Assume that a is a positive integer. Show that for any integer b there is some k in ω with*

$$b < a \cdot E(k).$$

Solution. In the argument that follows, $<$ shall be used to denote $<_Z$. Firstly, if $b \leq 0$, then clearly $b < a$, so take $k = 1$. Then $b < a = a \cdot 1_Z = a \cdot E(1)$.

Now suppose that $b > 0$. Let $b = [\langle m, n\rangle]$. Since $b = [\langle m, n\rangle] > [\langle 0, 0\rangle]$, we have $0 + n = n \in m = 0 + m$. Then $b = [\langle m, n\rangle] < [\langle m + 1, 0\rangle] = E(m + 1)$, since $m + 0 = m \in (m + 1) + n$, as $m \in m + 1$ and $0 \in n$. Since $0 < a$, we have by Theorem 5ZJ that $b \cdot a < E(m + 1) \cdot a$. Now since $1 \leq a$, we also have $b = b \cdot 1 \leq b \cdot a$. It follows then that

$$b \leq b \cdot a < E(m + 1) \cdot a.$$

By transitivity and commutativity of multiplication, we then have that $b < a \cdot E(m + 1)$, so take $k = m + 1$. □

Exercise 5.18. *Assume that p is a positive rational number. Show that for any rational number r there is some k in ω with*

$$r < p \cdot E(E(k)).$$

(Here k is in ω, $E(k)$ is the corresponding integer, and $E(E(k))$ is the corresponding rational.)

Solution. First, if $r \leq 0$, then $r < p$, and so $r < p = p \cdot 1_Q = p \cdot E(E(1))$ since

$$E(E(1)) = E([\langle 1, 0 \rangle]) = E(1_Z) = [\langle 1, 1 \rangle] = 1_Q.$$

Now suppose $0 < r$. Denote $r = [\langle m, n \rangle]$. Note that $[\langle 0, 1 \rangle] < [\langle m, n \rangle]$ implies $0 \cdot n < m \cdot 1$, so m is positive. Furthermore, denote $p = [\langle s, t \rangle]$, with $s > 0$ since p is positive. Consider $E(E((m+1) \cdot t)) = [\langle (m+1) \cdot t, 1 \rangle]$. I claim $r < p \cdot E(E((m+1) \cdot t))$. Observe

$$\begin{aligned} p \cdot E(E((m+1) \cdot t)) &= [\langle s, t \rangle] \cdot [\langle (m+1) \cdot t, 1 \rangle] \\ &= [\langle s \cdot (m+1) \cdot t, t \rangle] \\ &= [\langle s \cdot (m+1), 1 \rangle] && \text{by canceling the } t. \end{aligned}$$

Now $m < m+1$, and since n and s are both positive integers, $1 \leq n \cdot s$. Then by Theorem 5QJ we have

$$m = m \cdot 1 \leq m \cdot (n \cdot s) < (m+1) \cdot n \cdot s.$$

from which it follows that $[\langle m, n \rangle] < [\langle s \cdot (m+1), 1 \rangle]$, so $r < p \cdot E(E((m+1) \cdot t))$. Hence take $k = (m+1) \cdot t$. □

Exercise 5.19. *Assume that p is a positive rational number. Show that for any real number x there is some rational q in x such that*

$$p + q \notin x.$$

Solution. We will assume that the following properties hold for the rationals: if $p > 0$ and $q < 0$, then $pq < 0$. Also, $(-p)q = p(-q) = -(pq)$, both of which is easily verifiable by appling the definitions of orderings on $\mathbb{Q}$, $\mathbb{Z}$, and ω.

Now, a modification of the result of the previous exercise. I claim that if p is a positive rational, and r is any rational, then there exists an integer $n \in \mathbb{Z}$ such that $p \cdot E(n) < r$. If $r \geq 0_Q$, then we make take any $n < 0_Z$, and hence $E(n) < 0_Q$, so $p \cdot E(n) < r$. If $r < 0_Q$, then observe that

$$p \cdot E(n) < r \iff -r < -(p \cdot E(n)) = p \cdot (-E(n))$$

as we may add the additive inverse of each to both sides, and the equality follows from one of our originally stated properties. Now $-r > 0_Q$, so it follows that $-E(n) > 0_Q$ as well. So $-E(n) = E(E(k))$ for some $k \in \omega$, so we know $-r < p \cdot (-E(n))$ holds by the previous exercise. Hence $p \cdot E(n) < r$ for some $n \in \mathbb{Z}$.

Now since $x \neq \mathbb{Q}$, we know there is some $r \notin x$. By the previous exercise, there is some $k \in \omega$ such that $r < p \cdot E(E(k))$. Hence $p \cdot E(E(k)) \notin x$. Similarly, since $x \neq \emptyset$, there is some rational $s \in x$, and we know there is some $n \in \mathbb{Z}$ such that $p \cdot E(n) < s$, and thus $p \cdot E(n) \in x$. Now consider the set

$$A = \{m \in \mathbb{Z} \mid p \cdot E(m) \notin x\}.$$

Note that this set is nonempty, but bounded below by n, since for any m such that $p \cdot E(m) \notin x$, we must have $m > n$, since x is downward closed. It follows that there is a least such m such that $p \cdot E(m) \notin x$, and thus $p \cdot E(m-1) \in x$. We may take $q = p \cdot E(m-1)$, and thus

$$p + q = p + p \cdot E(m-1) = p \cdot (1_Q + E(m-1)) = p \cdot (E(1) + E(m-1)) = p \cdot E(m),$$

and thus $p + q \notin x$. [2] □

Exercise 5.20. *Show that for any real number x, we have $0_R \leq_R |x|$.*

Solution. To prove this statement is equivalent to proving

$$\{r \in \mathbb{Q} \mid r < 0\} \subseteq \{p \in \mathbb{Q} \mid (\exists s > p) - s \notin x\}.$$

Take $r \in 0_R$, so $r < 0$. If $r \in x$, we are done, so suppose $r \notin x$. Since $r < 0$, it follows that for all $y \in x$, $y < 0$, for if $y \geq 0$, then we would have that $r \in x$. Hence x contains only negative rational numbers. Let $s = 0$. Then $s > r$, $-s = 0$ (since $0 + 0 = 0$), and hence $-s \notin x$. So $r \in -x$, thus proving the containment. □

Exercise 5.21. *Show that if $x <_R y$, then there is a rational number r with*

$$x <_R E(r) <_R y.$$

Solution. First, we will prove the archimedean property of $\mathbb{R}$. That is, if $x, y \in \mathbb{R}$ and $x > 0_R$, then there exists a positive $n \in \mathbb{Z}$ such that $y <_R x \cdot E(E(n))$.

Let

$$A = \{x \cdot E(E(n)) \mid n \in \mathbb{Z} \wedge n > 0_Z\}.$$

Suppose that the above claim is not true, so it is not the case that there exists an n such that $x \cdot E(E(n)) >_R y$. That is, $x \cdot E(E(n)) \leq_R y$ for all n, so y is an upper bound of A. By Theorem 5RB, A has a supremum, so let $\alpha = \sup A$. Since $x > 0_R$, it follows that $\alpha - x < \alpha$, and thus $\alpha - x$ is not an upper bound of A. So

$$\alpha - x < x \cdot E(E(m)) \quad \text{for some } m \in \mathbb{Z}.$$

So

$$\alpha < x \cdot E(E(m)) + x \cdot E(E(1)) = x \cdot (E(E(m)) + E(E(1))) = x \cdot E(E(m+1)),$$

as the composition of homomorphisms is again a homomorphism. Hence we reach a contradiction, since α is an upper bound of A, so the archimedean property holds.

[2] Thanks to Andres Caicedo for his suggestions and sketch proof.

Now suppose $x <_R y$, so $0 <_R y - x$. By the archemedean property, there exists an $n \in \mathbb{Z}$ such that

$$E(E(n)) \cdot (y - x) >_R 1.$$

Furthermore, there exist $m_1, m_2 \in \mathbb{Z}$ such that $m_1 >_R x \cdot E(E(n))$ and $E(E(m_2)) >_R x \cdot (-E(E(n)))$. Hence

$$-E(E(m_2)) <_R E(E(n)) \cdot x <_R E(E(m_1)).$$

Now $x \cdot E(E(n))$ is a real number, and so taking $p = 1$ in Exercise 5.18, we see there is an integer k such that $E(E(k)) \in x$ but $E(E(k+1)) \notin x$. That is,

$$E(E(m-1)) <_R x \cdot E(E(n)) <_R E(E(m))$$

where we write $k = m + 1$. So

$$x \cdot E(E(n)) <_R E(E(m)) \leq_R x \cdot E(E(n)) + 1 <_R y \cdot E(E(n))$$

and thus

$$x \cdot E(E(n)) <_R E(E(m)) <_R y \cdot E(E(n)).$$

Now since n is nonzero, both $E(n)$ and $E(E(n))$ are also nonzero, since it is the composition of injections. Then by Theorem 5RI(d), there is a nonzero real y, such that $x \cdot y = 1_R$. Denote such a y as $E(E(n))^{-1}$. Note that $E(E(n)^{-1}) \cdot E(E(n)) = E(E(n)^{-1} \cdot E(n)) = E(1_Q) = 1_R$, since E is a homomorphism, and then since E is injective, we have that $E(E(n))^{-1} = E(E(n)^{-1})$. So altogether,

$$E(E(n)^{-1}) \cdot E(E(n)) = E(E(n)^{-1} \cdot E(n)),$$

with $E(n)^{-1} \cdot E(n) \in \mathbb{Q}$. So multiplying through yields

$$x <_R (E(n)^{-1} \cdot E(n)) \cdot E(m) <_R y$$

Taking this to be the desired r, we have

$$x <_R E(E(n)^{-1} \cdot E(n) \cdot E(m)) <_R y.$$

$\square$

Exercise 5.22. *Assume that $x \in \mathbb{R}$. How do we know that $|x| \in \mathbb{R}$?*

Solution. Recall $|x| = x \cup -x$. Since $x \neq \emptyset$ and $-x \neq \emptyset$, clearly $|x| \neq \emptyset$. Suppose now that $x <_R 0_R$. Then for any $q \in x$, $0 > q$, and $-0 = 0 \notin x$, since $0 \notin 0_R$, and consequently not in x. Hence $q \in -x$, and thus $x \subseteq -x$. Hence $|x| = x \cup -x = -x \neq \mathbb{Q}$.

Suppose instead that $x \geq_R 0$. Suppose $r \geq 0$, then if $s > r$, then $s > 0$, and so $-s < 0$. Thus $s \in x$, so $r \notin -x$. So if $r \in -x$, then necessarily $r < 0$, and so $r \in x$. Hence $-x \subseteq x$, and so $|x| = x \cup -x = x \neq \mathbb{Q}$. Hence $\emptyset \neq |x| \neq \mathbb{Q}$.

Now suppose $q \in |x|$. Suppose $r < q$. If $q \in x$, then $r \in x$ since x is downward closed, so $r \in |x|$. Similarly, if $q \in -x$, then $r \in -x$,since $-x$ is downward closed, so $r \in |x|$.

Now take any $p \in |x|$. If $p \in x$, there exists some $p' \in x$ such that $p' > p$, since x has no greatest element. If $p \in -x$, there exists some $p'' \in -x$ such that $p'' > p$, since $-x$ has no greatest element. In either case, p' or p'' is in $|x|$, and so $|x|$ has no greatest element. Altogether, $|x| \in \mathbb{R}$. $\square$