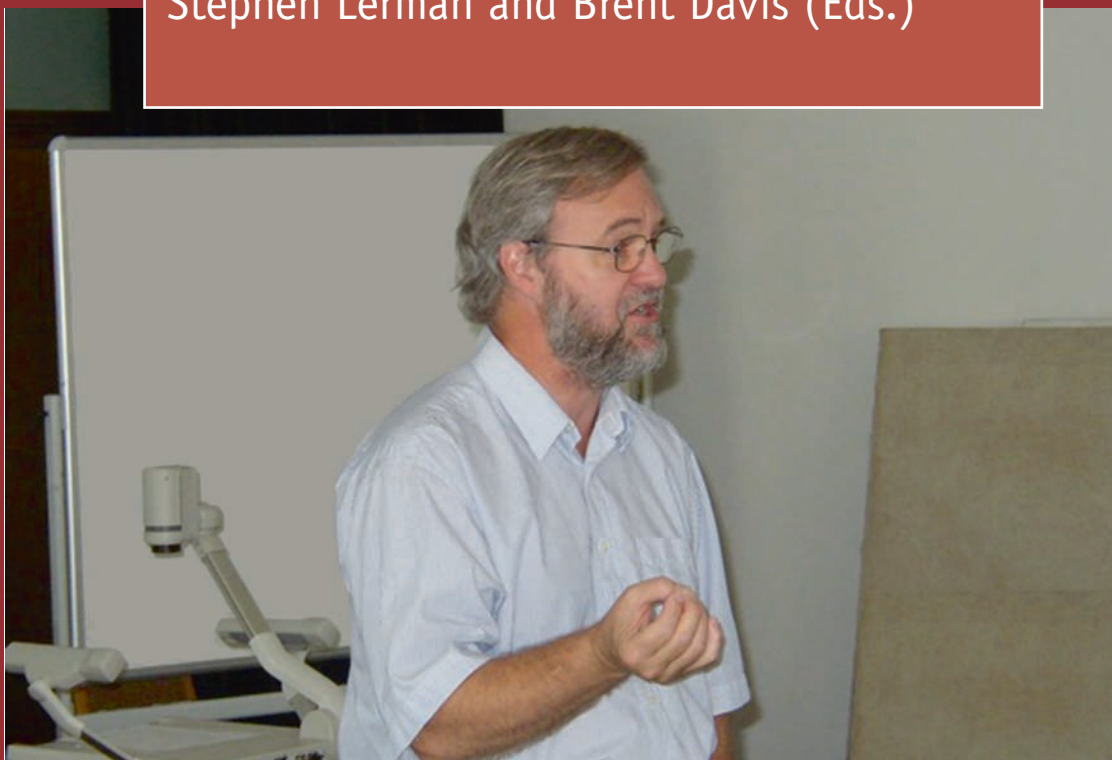


Mathematical Action & Structures of Noticing

Studies on John Mason's Contribution
to Mathematics Education

Stephen Lerman and Brent Davis (Eds.)



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*Studies on John Mason's Contribution to Mathematics
Education*

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PREFACE

This book is not, and at the same time is, a tribute book to the enormous contributions made by our colleague and friend John Mason to the research field of mathematics education, and to mathematical pedagogy, across the world.

It is not a tribute book because every chapter is a report of research and thinking by the authors, not simply a statement of appreciation. Some are accounts of mathematical activity; some are reports of research studies; others are attempts to develop new ideas and perspectives. All engage with how John's ideas have influenced thinking and research and the authors demonstrate how they have taken those ideas forward, or changed them to extend their own research and thinking.

At the same time it is a tribute book because it is a collection of chapters by authors with a specific focus, that of how their work, as presented here, has been inspired by John through his substantial opus and his vibrant presence in a network of mathematics educators.

In the opening chapter, John provides a sense of that body of work, contextualized in a personal account of the evolution of the field of mathematics education over recent decades. Before we go any further, we should mention that this autobiographical chapter on his years of his work in mathematics and mathematics education was actually written by him for 'another purpose'; at least that is what he was told when invited to write that reflective piece. It was always intended to be the opening chapter to this book. Our confidence that no one could craft a better introduction more than outweighed the anxiety associated with the small deception needed to secure the chapter. (That was not our only anxiety as editors. Another was that, to the point of going to press, we seem to have managed to keep this project a secret from John. That was no small feat. As the list of contributing authors reveals, John is not just well known, but well connected.)

John's introductory chapter renders the task of writing a Preface a very simple one, that of explaining how we have structured the book. We chose three themes that have permeated the research that has grown out of his ideas:

Thinking Mathematically

The Discipline of Noticing: Mathematical Pedagogy and
Pedagogic Mathematics

Variation and Mathematical Structure.

It's not easy to separate John's work into three, apparently separate ideas, nor has it been easy looking at each of the chapters and allocating them to one of the parts of the book. Like John's own work across these topics, there are overlaps,

synergies and resonances across the labels. Nevertheless, we hope we have done justice to the chapters written and to John's work, in our classification.

The book ends with a chapter by Anne Watson. At the beginning we consulted substantially with Anne, but once the outline of the book project had become clear we took over the whole responsibility for it. Anne's chapter looks at her own and John's work as it is currently developing. After all, Anne knows John rather well!

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1. MATHEMATICS EDUCATION: THEORY, PRACTICE & MEMORIES OVER 50 YEARS

INTRODUCTION

Having been asked to look back over my life in mathematics education, I take the liberty of recalling some of the most stimulating moments as they come back to me, in an attempt to analyse what mathematics education has been about for me. In particular I want to suggest that while the field has maintained and even widened the gap between theory and practice, it is incumbent upon us to keep in front of us that the purpose of our work is to understand and contribute to student learning of mathematics. One way I have consistently attempted to do this is to try to *preach what I practice* rather than the other way round.

One of my great amusements is that in the UK I am, I believe, seen mainly as a theorist, someone who works with ideas and tries to make them available to others. By contrast, in other countries I believe I am seen as immensely and intensely practical. In my defence, in the UK I can point to numerous publications that have offered teachers practical actions to initiate when working with learners. But as I have often said to staff on arrival at the Open University, you have 12 to 18 months to establish a reputation in the university; after that it is very hard to alter. So too in the academic world more generally. For example, it seems that when people attend one lecture-presentation, they assume that that is all the person can talk about, when actually most of us are willing to engage with a wide range of issues at different phases of mathematics education.

SOME HISTORICAL ACCOUNTS

I say ‘memories over 50 years’ in the title, because I started tutoring at the age of 15 at the request of my mathematics teacher Geoff Steele. It was only later that I realised just how profoundly he had influenced me through his stimulation and challenge. Many years later still I discovered that he had had no training in teaching nor much in mathematics: he worked to keep just ahead of me, writing out in longhand theorems in projective geometry, leading me through Hall & Knight on continued fractions, and engaging me in Susanne K. Langer’s *Symbolic Logic*. I came to value very highly the contact I had with topics that did not appear in the formal curriculum until late university, and on this I base my recommendations that students be challenged sideways, in breadth and depth, rather than accelerated through the curriculum. It was true that I twice ‘skipped’ ahead, but this provided space to consolidate and explore broadly, and I am ever so grateful for it.

My first attempts at tutoring were of course completely naïve. I explained things when students I was tutoring got stuck. I tutored high school students in my first year at university, and then started supporting students in my college in the years below me. I discovered later that they used to go first to my friend across the hall who knew how to do the problems; then they would come to me and watch me struggle, asking them questions about what theorems they knew and so on! In my third and fourth years I tutored for the university. It was here that I discovered the effectiveness of *being mathematical with and in front of learners*, although I didn't formulate this slogan until much later. I would face a class of students stuck on a problem about which I knew almost nothing. So I would ask them to read the problem out loud, then to tell me from their notes what the technical terms meant, and what theorems they had in their notes concerning them. In every case I eventually 'saw' how to resolve the problem (and to my retrospective regret) I would then show them how to do the problem. At least they saw a slightly more experienced learner struggling publicly, so they could pick up some practices for themselves.

In graduate school I was shown George Pólya's (1965) film *Let Us Teach Guessing* on the Friday afternoon before teaching a semester class that began at 7:45 each morning, starting on the Monday. As I later realised, the film released in me many of the practices used with me by Geoff Steele, and resonated vibrantly with practices I had developed spontaneously so as not to look a complete fool in tutorials. I got agreement with the class that they would work my way on Mondays; on Tuesdays to Thursdays we would then 'finish the chapter of the week', and on Fridays I would do revision problems with and for them. Within a week or two we were working 'my way' until at least part way through Wednesdays, finishing the chapter Thursdays, and revising on Fridays.

So what did it mean to 'work my way'? My recollection is that I would ask them questions to get them thinking. I would construct examples and generally cajole them. I would summarise in technical language what I thought they had begun to 'see' or appreciate. I felt that I was engaging them in the thinking. I am sure that an observer would have seen me getting them to 'guess what was in my mind', but the class was interactive and I hope challenging.

On my arrival at the Open University I realised that distance education was the antithesis of what I thought learning was about, but I settled down to write material. It was then that I realised how problematic teaching mathematics is, and that I had been enculturated as a structuralist through the influence of Bourbaki on many of my lecturers. One advantage was that, unlike the practices in a face-to-face institution where each lecture is an event to be survived, I frequently found myself at my desk wondering what example to use, what definition to use, what theorems to put in what order, and how best to interlace examples with abstractions and generalities.

As I was one of the few people in the faculty who had worked with Pólya's ideas, I was asked to organise the summer schools. I instituted sessions such as *investigations* (inspired by what I discovered going on in primary schools in the UK and based on Pólya's film), *mental callisthenics* (reproducing sessions I had

experienced myself in primary school), *surgeries* (where students could come and ask for help on any aspect of the course), *tutor revelations* (in which a tutor would work through some typical questions while exposing the inner thoughts, choices and incantations accompanying the solution) as well as lectures and tutorials. I even tried sessions called *tutor bashing* in which a tutor poses a question to another tutor who then works at it from cold in front of the students before posing a question to the next tutor in sequence. This was an attempt to show students that tutors were mortal and fallible and that mathematics does not flow perfectly out of the pen in its completed form.

It took me several years to realise that what was obvious to me about how the mathematical practices described by Pólya, such as specialising & generalising, imagining & expressing, conjecturing & convincing, were not obvious to many of our tutors. Thus some tutors adopted a “don’t let them leave for coffee break without giving them the formula, because they’ll just get it from others anyway” approach, and recommended this to tutors new to the summer schools. Furthermore on investigation it transpired that students in tutorials where the tutor did not like investigations usually came out not liking them, and students with tutors who did like them usually came out enjoying them. The stance, beliefs and attitudes of the tutor could be highly influential. This later resonated with the pioneering work of Alba Thompson (1984, 1992) working with teachers.

In 1973–74 I spent nearly a year with some 125 others in a house in Gloucestershire under the direction of J.G. Bennett, mathematician, scientist, linguist and seeker. It was a pivotal year for me, crystallizing many awarenesses and awakening me to many others. Ever since then I have been reconstructing the ideas and practices I encountered. In particular, I experienced deeply an experiential approach to enquiry, which for me extended through mathematics to every aspect of life and thought. Many years later I decided to devote time to reconstructing those practices expressed in the domain of mathematics education in particular, but applicable in any caring profession. I called it the Discipline of Noticing (Bennett, 1976; Mason, 1996, 2002). My idea was to provide teachers and other carers with a philosophically well-founded method and theoretical framework for researching their own practice, that is, for working on themselves.

I began my research life in combinatorial geometry, and achieved a certain notoriety within the rather small community of like-minded scholars as someone with a practical, example-rich approach to tackling really difficult mathematical problems from a structural perspective. It was after an event run by a colleague, Johnny Baker, in which teachers from secondary mathematics and science departments reported on their experience of teaching and using mathematics in schools, that I realised that although the mathematical problems I worked on were difficult, very few people cared, whereas the problems in mathematics education are essentially unsolvable, but a very large number of people care. So I turned my attention explicitly to mathematics education, and set myself three years to establish a reputation in the field. However, I have never lost my interest in, no, my addiction to, working on mathematics. I always have some mathematical problem in the back of my mind that I work on in otherwise idle moments. This serves to keep me in touch with my

own experience and sensitises me to struggles that others may have with different topics, concepts or problems.

It was while designing a course for mathematics teachers that I persuaded my colleagues that it would be a good idea for teachers to engage in mathematical thinking for themselves at their own level each week. But then a decision had to be made as to how to choose what problems to offer them, and how to structure that experience. We found that we had plenty of advice to offer, in my case gleaned from Pólya, Bennett and awareness of my own experience. Together Leone Burton and I decided to write a book about problem solving, in order to assist us in developing a structural framework for making suitable selections for the course. Leone introduced me to Kaye Stacey, and *Thinking Mathematically* (Mason *et al.*, 1982) was born. Unsurprisingly, perhaps, I never had any opportunity to teach a course from it, though others have and continue to do so more than 25 years on.

My non-academic activities in the 70s brought me into contact with a wide range of group activities exploring sensitivity training and body-mind connections. I encountered a wide range of practices and authors such as Abraham Maslow (1971). I immediately recognised from him that I was really interested in what is possible, what *could* be, rather than *what is the case currently*. Since at the Open University there were no students on campus to use as subjects, nor easy contact with schools, it matched my situation to be more interested in what is possible through examining my own experience while also working with teachers in in-service professional development sessions. One effect was that, with only a few exceptions, I only ever worked with teachers and students on a one-off basis rather than over an extended period of time.

Soon after I arrived at the Open University I discovered the Association of Teachers of Mathematics, and began to go to their meetings. I encountered there a group working on mental imagery and this fired my imagination, literally. I adopted and adapted practices that Dick Tahta and others used with posters and animations, to use with videotape of classroom interactions, and this was the basis for what our group constructed as an approach to using video. At ICME 5 in Adelaide I discovered this was called or at least akin to constructivism. Combined with my experiential orientation, this set me up to resonate strongly with Ernst von Glasersfeld when I met him in Montréal in 1984.

In Adelaide I also met Guy Brousseau and extended contact with Nichol Balacheff. Impressed by several of the constructs they used, in particular *transposition didactique* (Chevallard, 1985), *situation didactique* (Brousseau, 1984, 1997) and the *didactic tension*, and *epistemological obstacles* (Bachelard, 1938), it took me a long time to appreciate even the most surface features of the deeply analytic frame that informed their impressive research. However, my own experience in working with groups of teachers experientially had shown me that offering results of research enquiries to others is in itself highly problematic.

When I was a graduate student there were frequent calls for research on the most effective way to teach mathematics to undergraduates: 3 lectures + 2 tutorials per week, or 5 lectures per week, or 3 tutorials and 2 lectures or what? It was evident to me that the issue depended on too many factors connected with the setting, the

individuals, the expectations, and the practices within lecturing and tutorials to be able to declare one better than another universally. Whereas most mathematicians that I knew were seeking a mathematical-type of theorem with definitive conclusions, I was convinced that any value system would be situation dependent. I found statistical findings deeply unsatisfying, because either they would agree with my own prejudices, in which case they told me nothing, or they would contradict those biases, in which case I would reject them as being unsuitable or irrelevant to my situation. I felt perfectly at home with the impossibility of mathematical-like theorems in mathematics education, because of the presence of human will, intention and ideals.

When I started working with teachers in the UK I soon realised that making use of ‘findings’ was problematic for others as well. Proposing a research finding (qualitative or quantitative) that is close to current practice is likely to be responded to by assimilation without noticing any subtle differences. Proposing a research finding that challenges thinking or which is not immediately compatible with practices, is unlikely to stimulate people to try the idea much less adopt it simply because it is a research finding. There has to be something that catches attention, either because it seems implausible and so motivates checking, or because it appears to match perceived current needs.

In my own case, even if I do try something, I am most likely to modify it to fit with my perspective and approach. Indeed, there is no ‘it’ as such, only my reconstruction. This corresponds with my view of classroom incidents, indeed incidents and events generally: there is no ‘event’ as such, merely the stories told about it, whether at first, second, or later hand. On the other hand, if the finding fits with my experience, it is likely to seem ‘obvious’, so I am likely to pay no attention to any slight differences that might in fact be significant. Instead, I feel reassured and carry on. Thus for me a successful professional development session is one in which participants can actually imagine themselves acting differently in some situation in the future which they recognise. This statement is much more significant than it may appear at first sight. I emphasise *imagine themselves*, for what I learned from Bennett was the immense power of mental imagery for preparing actions to take place in the future.

What does seem to be helpful is prompting people to experience something which sheds light on their past experience and offers to inform their future choices. I see professional development as personal enquiry, stimulated and supported by work with colleagues, but essentially a psychological issue with a socio-cultural ecology. I have however always resisted pushing this as far, for example, as my one-time colleague Barbara Jaworski (2003, 2006, 2007) has done. I am content to indicate possibilities to others rather than trying to maximise efficiency and efficacy. For me, change is such a delicate matter that it must be left to individuals within their various communities, to the extent that “I cannot change others; I can work on changing myself”, and even that is far from easy!

My interests have always been in supporting others in fostering and sustaining mathematical thinking in their students. I have at various times concentrated on mental imagery, modelling, problem solving, and language, but these have been but byways in getting to grips with the nature and role of attention. I have, for

example, found it convenient to shift my discourse from *processes of problem solving* to *exploiting natural powers*, finding that the same ideas (imagining & expressing, specialising & generalising, conjecturing & convincing, among others) continue to be potent as long as they are expressed in a context which resonates with people's experience and a discourse that they recognise. Early in my reading of mathematics education books and papers I recognised that each generation has to re-express insights in their own vernacular, even though these insights have been expressed before. Indeed, each person has to re-experience and re-construct for her- or himself. This contrasts with mathematics in which it is possible to be directed along a 'highway' towards problems at the boundary without traversing all of the country in between. Mathematics education is not like that, and perhaps never will be, at least until we establish common ways of working. I take up this theme in the next section.

In the early 1980s I had the chance to attend a number of seminars led by Caleb Gattegno when he tried to re-vivify his *science of education* (Gattegno, 1987, 1990) in the mathematics education community in England. I found his approach attractive, with a good deal in common with what I had learned from Bennett, but leading to a rather different cosmology. I began to get a taste of what it is like when an experienced 'grey-beard' assembles their to-them-coherent-and-comprehensive framework or theory. Whereas when the fragments were being worked on and described there is often considerable interest amongst colleagues; once the whole is assembled, people don't really want to know. I ran into this phenomenon again when reading Richard Skemp's later book (1979), where again my experience was one of interest in some of his distinctions, without appreciating all of them, or the way they all fit together. Reading Jean Piaget, Zoltan Dienes, Hans Freudenthal, David Ausubel, Frédérique Papy and Humberto Maturana all had similar effects on me, partly perhaps, because I came across their work after or near the end of their careers. I found many specific distinctions of great value, but resisted taking on board their over-arching theories.

I assume that the issue is one of subordination. Philosophers are trained to suppress their own thinking in order to 'think like' the philosophers they are studying. In mathematics education, the intention is to improve the experience of learners, and this pragmatic dimension may contribute to a reluctance to let go of one's own stance in order to enter, absorb and fully appreciate the stance of someone else. Several French colleagues have given me the impression that in France they are more used to subordinating to an established theory, whereas Northern European, anglo-saxon cultures appear to be more pragmatic and less theory-oriented in assembling their own personal framework or theory that 'works for them'.

THE RISE OF MATHEMATICS EDUCATION

Others are more scholarly at researching the ebb and flow, the waxing and waning of salient constructs in mathematics education. My memory is that in the 1970s and early 1980s research interest focused on students. I was, naturally, caught up in the Pólya-inspired *problem solving* discourse of *processes*, as manifested in *Thinking*

Mathematically. As data accumulated, attention turned to student errors and misconceptions. I recall the breath of fresh air when Douglas McLeod and Verna Adams (1989) edited a book on affect and problem solving, and Alba Thompson championed devoting attention to teachers' beliefs as influencing both how people teach, and what students learn. As I look back now, it seems to me that one of the reasons for each generation revisiting and re-constructing classic insights and awareness is that, as well as participating in a process of personal re-construction, each generation finds itself dissatisfied with the explanatory and-or remedy power of the current discourse and foci. The discourse is drained of its power to inform choices? Each generation seeks fresh fields for explanation as to why students, on the whole, do not learn mathematics effectively or efficiently, and latterly, why teachers do not teach what they know and why they know so little of what it is necessary to know in order to teach effectively.

On the one hand we have Henri Poincaré's position of being mystified as to why perfectly rational people can fail to succeed at the perfectly rational discipline of mathematics, and on the other hand we have generations of students convinced that either fractions or algebra was a watershed of their involvement in mathematics. Clearly rationality is not the central feature of most people's psyche. One of the many things that has impressed me about Open University students over the years is that when I used to ask students on our mathematics courses why they were going to all that effort, I almost always got the reply "always liked mathematics at school; never could do it, mind you, but always wanted to know more". Something touches people, even if it remains dormant.

Attention in mathematics education research has shifted variously between the structure of and inherent obstacles in specific topics, psychological aspects of learning mathematics, psychological aspects of teaching mathematics, sociological aspects of teaching and learning mathematics, acts of teaching, teachers' beliefs and how they influence learners, the historical-socio-cultural forces at work in and through institutions, and the content and format of teacher education courses, not to say the obstacles encountered by novice teachers due to weak mathematical background, and the destructive forces of school practices and government policies on the ideals and aspirations of novice teachers emerging from teacher education courses, to name but a few. Most of what is accepted as *research* involves making observations of others (what I call *extra-spective*) whether associated with deliberate interventions or not. Observations and transcripts are turned into data by being selected for analysis. Analysis then applies a framework for making distinctions, or generates or modifies such a framework.

It is tempting to say that we (the community of mathematics educators, scholars and researchers) have accumulated a great deal of data. We have individually, though perhaps not collectively, drawn a multitude of distinctions and formulated a plethora of constructs to analyse and account for what has been observed. But what have we really got to show for all this effort? Publications proliferate faster than I for one can read them, much less take them in and integrate them. Rarely do we get evidence that the framework has enabled teachers to modify their practice and so influence student learning. So what is the mathematics education enterprise?

THE ENTERPRISE OF MATHEMATICS EDUCATION

On the surface, it is reasonable to expect that those engaged in mathematics education research and scholarship have as their aim the improvement of conditions for learning (and hence for teaching) mathematics more effectively, at every age and stage. Consequently evidence of effectiveness must lie ultimately in improvement in learner experience and performance, in both the short term and long term.

Of course there is an immediate obstacle, for there is little or no agreement as to what constitutes evidence of learner experience, much less evidence of improvement. Since it is easiest to gauge by scores on tests, national and international studies administer tests and pronounce on the results. Questionnaires and even interviews with selected subjects can be carried out. But there is a fundamental difficulty. Test results only indicate what subjects did on one occasion under one set of circumstances with or without specific training in preparation. Interviews at best reveal only what the interviewer probes and selects due to their sensitivities, and questionnaire responses are highly dubious indicators of what lies beneath surface reactions to specific questions. As soon as you identify an indicator of mathematical thinking or other mathematical competence or success, it should take a competent teacher at most two years to work out how to train students to answer those ‘types’ of questions. It all comes back to Guy Brousseau’s notion of the *didactic contract* as manifested by the *didactic tension* and its parallel, the *assessment tension*:

The more clearly and specifically the teacher (assessor) indicates the behaviour sought, the easier it is for the learner to display that behaviour without generating it from themselves (understanding).

Put another way in a discourse derived from Caleb Gattegno, training behaviour is important and useful, but it is inflexible and even dangerous if it is not paralleled with educating awareness. It is awareness (what enables you to act, what you find ‘comes to mind’ in the way of actions) that guides and directs (en)action, using the energy arising from affect (by harnessing emotions). It is ever so tempting to train someone’s behaviour by giving them rules and mnemonics to memorise, and quantities of exercises on which to rehearse. But real learning only occurs when these form the basis for reflection and integration so that awareness is educated (as in the Confucian culture approach to teaching and learning). Alternatively, one can work on educating awareness and training behaviour together, through harnessing emotion, and this is the approach that I have endeavoured to practice, and through practicing, to articulate for myself and others so as to make the process more efficient over time. Because I am interested in what is possible, and because the only way of directing other people’s attention is through being aware of the focus of my own attention, I use *intra-spection* (between selves or between people) as distinct from *intro-spection* which got a negative connotation through the indulgences of people trying to develop phenomenological research methods in the early part of the 20th century.

One of the underlying tensions in mathematics education that I am aware of is that between a 'scientific stance' and a 'phenomenological stance'. Editors want their journals to contribute to the scientific development of knowledge. Journals have recently become so obsessed with theoretical frameworks that papers get longer and longer, without any growth in substance. I suspect that colleagues, especially editors, want to see mathematics education build a coherent and well-founded structure of knowledge. They like to see people building on each others' work, adding to and refining rather than starting afresh. I wonder however whether this is even possible, much less desirable, given the nature of focus of mathematics education, working as it does with human beings placed in institutional settings of various sorts, and exercising their wills and intentions through their dominant dispositions. I want to put a different case, the case of working with lived experience.

STRUCTURED AWARENESS

I have often thought and sometimes said, that when I am engaged in my enquiries, I enjoy it most when I am at the overlap between mathematics, psychology and sociology, philosophy and religion. There is something about working on a mathematical problem which is for me profoundly spiritual; something about working on teaching and learning that integrates all three traditional aspects of my psyche (awareness, behaviour and emotion, or more formally, cognition, enaction and affect) as well as will and intention, which themselves derive from ancient psycho-religious philosophies such as expressed in the Upanishads (Radhakrishnan, 1953) and the Bhagavad Gita (Mascaró, 1962; see also Raymond, 1972). I associate this sense of integration with an enhanced awareness, a sense of harmony and unity, a taste of freedom, which is in stark contrast to the habit and mechanicality of much of my existence. Even a little taste of freedom arising in a moment of participating in a choice, of responding freshly rather than reacting habitually is worth striving for.

One way to summarise such experiences is that, in the end, what I learn most about, is myself. This observation is not as solipsistic, isolating and idiosyncratic as it might seem, for in order to learn about myself I need to engage with others (who may, as is the case for hermits, be virtual), and I need to be supported and sustained in those enquiries. A suitable community can be invaluable, though an unsuitable community can be a millstone! I reached this conclusion through realising that when a researcher is reporting their data, and then analysing it, the distinctions they make, the relationships they notice, the properties they abstract all tell me as much about their own sensitivities to notice and dispositions to act as they do about the situation-data being analysed. Indeed I proposed an analogy to the Heisenberg principle in physics: the ratio of the precision of detail of analysis to the precision of detail about the researcher is roughly constant (Mason, 2002, p. 181).

The seeds of this observation were in working with teachers on classroom video, informed by techniques for working on mathematical animations (e.g. those of Jean Nicolet and Gattegno's reworking of them). The technique is to get participants to reconstruct as much of the film as they can after seeing it just once. When they

have made a good attempt, they have specific questions about portions they only partly recall, or where different people have different stories. So a second viewing makes sense, but only because there are specific questions. Applied to classroom video, we adopted a similar stance in order to counteract the common reaction of “I wouldn’t let that teacher in my classroom”, or “my low attainers are lower attaining than those low attainers” (Jaworski, 1989; Pimm, 1993). It seemed that teachers saw classroom video as a challenge to their identity and practices. By getting them to recount specific incidents briefly but vividly from the video with a minimum of judgement, evaluation, explaining, we found that they soon recognised incidents as being similar to incidents they had met in their own experience. So the videos became an entry into participants’ own past experience, and hence gave access to their lived experience. This makes so much more sense than critiquing the behaviour of some unknown teacher whose class may already have left school, so there is no way that their behaviour could be altered!

Incidents that strike a viewer usually resonate or trigger associations with incidents recalled from the past. Describing these to others briefly-but-vividly so as to resonate or trigger their own recollections provides a database of rich experiences which can be accessed through the use of pertinent labels. Often sameness and difference between re-constructed incidents has to be negotiated amongst colleagues, and this is what prompts probing beneath the surface. As Italo Calvino (1983, p. 55) said, “It is only after you come to know the surface of things that you venture to see what is underneath; but the surface is inexhaustible”.

I have come to recognise that Bennett (among many others over the centuries) was right when he highlighted the fundamental act of making distinctions. It is, after all, how organisms at all levels of complexity operate. Change is the experience of making distinctions over time; difference is the experience of distinction making in time. Evaluation is the experience of distinguishing relative intensities (as a ratio or scaling). Bennett went much further, amplifying Gurdjieff’s observation that ‘man is third force blind’ (see Orage, 1930). In other words, distinguishing things, this from that, is important, but locks you into tension or evaluation, and is just the beginning of what is possible. In order to appreciate how the world works (whether material, mental, symbolic or spiritual) it is necessary to become aware that actions require three impulses, something to initiate, something to respond, and something to mediate between these, to bring them into or hold them in relationship. The product of actions can then go on to serve to initiate, respond to, or mediate a further action. Bennett continued this neo-pythagorean analysis into the quality of numbers from 1 to 12 in his monumental four-volume work *The Dramatic Universe*, which he called *systematics*, long before ‘systems theory’ became a slogan. Perhaps because of my structural upbringing, I found myself resonating with his approach, to the extent that I could sometimes hear in the structure of his talks the ways in which he was systematically employing systemic qualities of a particular number.

PRECISION & REPLICATION

There is another issue concerning transforming observations into data and the degree of precision presented in research reports. Over the years there has been an evident growth in the length and complexity of papers in mathematics education. It used to be that some detailed transcripts along with some analysis stimulated colleagues to investigate the phenomenon in their own setting. A classic example would be the paper by Stanley Erlwanger (1973, reprinted in Cooney *et al.*, 2004) about Benny's encounter with fractions. Nowadays this paper would probably be rejected by journals as failing to present an adequate theoretical framework and discussion of method and ethics. Many papers are so heavily theory-laden in the opening sections that by the time I get to the substance I have forgotten exactly which parts of which theory are actually being employed, and indeed sometimes it is not even very easy to detect this. It seems to me that often only tiny fragments of theoretical frameworks are called upon. Indeed, I have no problem with this at all, because of my eclectically cherry-picking approach to understanding and practice: all I can ever do is be stimulated or sensitised to notice, that is to discern details not previously attended to, and through that discernment, raise questions deserving of enquiry. But if authors are selecting fragments, why not be straightforward about it? I go so far as to suggest that experience itself is fragmentary, despite consciousness and the collection of selves that make up consciousness and personality trying to develop stories to make it look continuous and coherent (Mason, 1986, 1988). This is the one detail on which I disagree with William James' notion of a 'stream of consciousness'. My own observations agree with Tor Nørretranders (1998) that these stories are a fabricated illusion.

I realise that editors have a commitment to building the scientific foundations of mathematics education, but I don't see that present practices are actually furthering the field, in the main. What we do have is a plethora of distinctions, sometimes several labels for at best subtly distinct distinctions, and sometimes the same label is used for different distinctions. What the field really needs is some agreement on ways of working, rather than on theoretical frames and stances. We need to build up a vocabulary for how we compare observations, turn them into data, and negotiate meaning amongst ourselves. This would then make it easier to offer similar distinctions to others including teachers, teacher educators and policy makers, and to negotiate similarities, differences and intensities. Caleb Gattegno offered his *science of education* but this is too radical for most to agree to; in the Discipline of Noticing I tried to offer a less radical and more practical foundation for ways of working; I am sure that others feel they have done the same. The problem in my view lies not in the fact that everyone discerns slightly differently, but that we don't have established ways of negotiating similarities and differences in what is noticed and in what triggers that noticing, and in what actions might then be called into play.

Despite the developments in style (I hesitate to use the word *improvement*) it is still rarely if ever possible to imagine much less actually carry out a replication of a study reported on in a mathematics education research paper in a journal. There is simply never enough detail. I happen to suspect that it would never be possible to

replicate a study exactly, precisely because of the complex range of factors comprising the traditional triad of student, teacher and mathematics all embedded in an institutional environment.

If it is either impossible or not necessary to be able to replicate the conditions of a study, what is it that we are gaining by reporting on our studies? My radical response to such a question is that what matters most is *educating awareness* by alerting me to something worth noticing because it then opens the way to choosing to respond rather than react with a more creative action than would otherwise be the case. I don't need all sorts of detailed data, because the more precise and fine-grained the detail, the less likely I am to pay attention to the over all phenomenon being instantiated, and so the less likely I am to recognise it again in the future and so choose to act differently.

REPRISE

I am genuinely perplexed about the role and nature of structure in a domain such as mathematics education. On the one hand, with my structural background, I find it really helpful to be able occasionally to invoke one or other structure in order to inform my thinking. But I have colleagues who resist such an approach, just as I have resisted accommodating the whole of other people's structured frameworks. It is too simplistic to say that each could be expounded and then tested by experiment to see which is 'best'. I am reminded of a sequence of lectures I was required to attend in my first year at university on *the leap of faith*. My recollection is that they were about the philosophical conundrum of how you cannot investigate or enquire into what it is like to believe something without actually believing it. Put in an overly extreme form perhaps, 'if you can critique it, you haven't experienced it fully'. Of course this is anathema to many in Western society, but increasingly popular to fundamentalists the world over.

My own experience is that I do not usually use my own frameworks systematically or mechanically, because they have been integrated into how I perceive the world and how my thinking progresses. Every so often it is useful to ask myself if I have taken all aspects of an action, an activity, a potentiality, a moment, a transformation into account, and this is when it can be fruitful to remind myself of the pertinent number and its structural qualities. More specifically, the Structure of a Topic framework (Griffin & Gates, 1989; Mason & Johnston-Wilder, 2004a, 2004b), based on Bennett's 'present moment' system associated with qualities of six is particularly useful when preparing to teach a topic. The six modes of interaction (expounding, explaining, exploring, examining, exercising and expressing) arising from the qualities of three alert me to possible forms of interaction and bring different interventions to mind (Mason, 1979).

I suspect that each of us does something similar. We act in the world; when some tension or disturbance arises, we resort to accustomed modes of thinking using whatever frameworks come to mind, and then carry on. We have habitual but slowly evolving forms of activity with which we feel comfortable; we are sensitised to certain aspects of potentiality in a situation; we stress certain aspects of the

present moment; and so on. Calvino (1983, p. 107) said something similar: “The universe is a mirror in which we can contemplate only what we have learned to know in ourselves”, which in turn resonates with a North American shaman Hyemeyohsts Storm (1985) who phrased it as “the Universe is the Mirror of the People”. The universe, whether material, imagined or symbolic, provides a mirror for seeing ourselves and so bringing possible actions to mind. Indeed, all we can see is in fact ourselves, in the sense that what we discern and relate is a reflection of ourselves. What professional development means is ongoing work to extend sensitivities, striving for a greater balance in the interplay of component features, so as to participate more fully in the evolution of awareness.

REFERENCES

- Bachelard, G. (1938/1980). *La formation de l'esprit scientifique*. Paris: J. Vrin.
- Bennett, J. (1956–1966). *The dramatic universe* (Vol. 1–4). London: Hodder & Stoughton.
- Bennett, J. (1976). *Noticing. Volume 2 of the sherborne theme talks series*. Sherborne, UK: Coombe Springs Press.
- Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H. Steiner (Ed.), *Theory of mathematics education* (pp. 110–119). Bielefeld, DE: Institut für Didaktik der Mathematik der Universität Bielefeld.
- Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactiques des mathématiques, 1970–1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Trans.). Dordrecht, NL: Kluwer.
- Calvino, I. (1983). *Mr Palomar*. London: Harcourt, Brace & Jovanovich.
- Carpenter, T., Dossey, J., & Koehler, J. (Eds.). (2004). *Classics in mathematics education research*. Reston, VA: NCTM.
- Chevallard, Y. (1985). *La transposition didactique*. Grenoble, FR: La Pensée Sauvage.
- Erlwanger, S. (1973). Benny's conception of rules and answers in IPI Mathematics. *Journal of Mathematical Behaviour*, 1(2), 7–26.
- Gattegno, C. (1987). *The science of education, part I: Theoretical considerations*. New York: Educational Solutions.
- Gattegno, C. (1990). *The science of education*. New York: Educational Solutions.
- Griffin, P., & Gates, P. (1989). *Project Mathematics UPDATE: PM753A,B,C,D, Preparing to teach angle, equations, ratio and probability*. Milton Keynes, UK: Open University.
- Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, 9(2), 187–211.
- Jaworski, B. (1989). *Using classroom videotape to develop your reaching*. Milton Keynes, UK: Centre for Mathematics Education, Open University.
- Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: Towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics*, 54(2–3), 249–282.
- Jaworski, B. (2007). Developmental research in mathematics teaching and learning. Developing learning communities based on inquiry and design. In P. Liljedahl (Ed.), *Canadian mathematics education study group proceedings, 2006 annual meeting* (pp. 3–16). Burnaby, BC: CMESG.
- Mascaró, J. (Trans.). (1962). *The bhagavad gita*. Harmondsworth, UK: Penguin.
- Maslow, A. (1971). *The farther reaches of human nature*. New York: Viking Press.
- Mason, J. (1979, Feb). Which medium, which message. *Visual Education*, 29–33.
- Mason, J. (1986). Probes and fragments. *For the Learning of Mathematics*, 6(2), 42–46.

- Mason, J. (1988). Fragments: The implications for teachers, learners and media users/researchers of personal construal and fragmentary recollection of aural and visual messages. *Instructional Science*, 17, 195–218.
- Mason, J. (1996). *Personal enquiry: Moving from concern towards research*. Milton Keynes, UK: Open University.
- Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: RoutledgeFalmer.
- Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. London: Addison Wesley.
- Mason, J., & Johnston-Wilder, S. (2004a). *Designing and using mathematical tasks*. Milton Keynes, UK: Open University. (2006 reprint: St. Albans, UK: QED).
- Mason, J., & Johnston-Wilder, S. (2004b). *Fundamental constructs in mathematics education*. London: RoutledgeFalmer.
- McLeod, D., & Adams, V. (Eds.). (1989). *Affect and mathematical problem solving: A new perspective*. New York: Springer-Verlag.
- Nørretranders, T. (1998). *The user illusion: Cutting consciousness down to size* (J. Sydenham, Trans.). London: Allen Lane.
- Orage, A. (1954). *Essays & aphorisms*. New York: Janus.
- Pimm, D. (1993). From should to could: Reflections on possibilities of mathematics teacher education. *For the Learning of Mathematics*, 13(2), 27–32.
- Pólya, G. (1962). *Mathematical discovery: On understanding, learning, and teaching problem solving* (combined ed.). New York: Wiley.
- Pólya, G. (1965). *Let us teach guessing* [film]. Washington, DC: Mathematical Association of America.
- Raymond, L. (1972). *To live within*. London: George Allen & Unwin.
- Radhakrishnan, S. (1953). *The principal upanishads*. London: George Allen & Unwin.
- Skemp, R. (1979). *Intelligence, learning and action*. Chichester, UK: Wiley.
- Storm, H. (1985). *Seven arrows*. New York: Ballantine.
- Thompson, A. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, 15, 105–127.
- Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 127–146). New York: MacMillan.

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Mathematical Action & Structures Of Noticing

Part 1:

THINKING MATHEMATICALLY

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THINKING MATHEMATICALLY

Easily the best known and most widely distributed of John's publications is his 1982 *Thinking Mathematically* (authored with Leone Burton and Kaye Stacey). We borrow not just the title of that book for this section, but its core theme.

Central to John's approach is that teaching and learning is about the development of mathematical thinking, not on acquiring mathematical knowledge *per se*. Indeed such knowledge is inert without it having been acquired through the challenge to think mathematically. The teacher's role is to support pupils to attend to the powers with which they are born, powers to discriminate between and to see similarity across objects, to conjecture, to inquire, and so on, and to develop those powers in the appropriate directions. For teachers to know what is meant by 'appropriate directions' requires them, indeed all of us involved in the field of mathematics education, to continue to experience those challenges to think mathematically and to sharpen our awareness of the range of features of mathematical thinking that John and his many colleagues have worked on over the years. John will often begin a talk with asking the audience to engage in a mathematical activity first, so that the points he is going to make in his talk are immediate for listeners.

This section begins with a chapter by David Tall in which he incorporates the theory of mathematical thinking into his own theory of the long-term development of the individual that includes the notion of proof. He relates this to Richard Skemp's work on mathematical knowledge and emotions.

The following two chapters, one by Derek Holton, Michael Thomas and Anthony Harradine, and the other by Tim Rowland, give us rich examples of mathematical activity through examples, emphasising and illustrating those features of mathematical thinking that have been described in John's writing.

The fourth chapter in this section, by Olive Chapman, describes a particular research study, drawn from a programme of research over many years, that of pre-service mathematics teachers learning about productive problem solving by their students through their own developing understanding of problems, the problem-solving process, problem-solving pedagogy and problem solving as inquiry-based teaching.

The final chapter in this part is by Elke Söbbeke and Heinz Steinbring, in which they argue that the main goal of mathematics education is to build an understanding of mathematical relations and structures and to practice mathematics on an elementary level, and to achieve this calls for a focus on the abstract and the general right from the start of children's mathematics education. They exemplify their argument through rich examples of young children's mathematical activity.

DAVID TALL

2. THE DEVELOPMENT OF MATHEMATICAL THINKING

Problem-Solving and Proof

INTRODUCTION

It was my privilege to use the book *Thinking Mathematically* (Mason, Burton & Stacey, 1982) for over a quarter of a century from its first publication in 1982 to my retirement in 2007. This was a life-changing experience. Before my encounter with this remarkable text I saw my objective as a mathematics educator to reflect on mathematical knowledge and present it to students in ways that would enable them to make sense of it. In my early career, I wrote books and course notes with this purpose in mind. On the publication of *Thinking Mathematically*, I chose to use the text as a course book for a course that I termed ‘Problem Solving’ for second and third year undergraduate mathematicians with a liberal sprinkling of computer scientists, mathematical physicists and others.

I remember my abject fear when I first met with these students. I was going to start with the first problem in the book, inviting the students to work out whether it was better to calculate a percentage discount before or after adding a percentage tax. My panic was noted by my secretary in those early days as I walked by her door looking nervous and she said, ‘You’re doing that problem-solving again, aren’t you?’

My fear arose because these were very able mathematics students and it was quite likely that they would say, ‘but you just multiply the two factors and multiplication is commutative.’ But none of them did.

Place someone in an unusual context and present him or her with a problem and it is likely that they will initially lose all sense of direction and need to build up their confidence. This happened to me and it happened to my students. Over time we developed confidence and an ability to anticipate what would happen. It turned the routine learning (or mis-learning) of mathematics into a dynamic act of self-construction and gave most of us concerned a deep sense of pleasure.

Each week we had a two-hour problem solving session with a class of forty to eighty students where I began by setting the scene with the objective of the class, using successive sections of the book then leaving the students to solve a particular problem illustrating the objective of the day. I also announced a ‘problem of the week’ for students who finished the problem of the day to keep them occupied. Initially some competitive students would move on to the problem of the week fairly quickly, but often they hadn’t solved the problem at all.

The book suggested three levels of explanation:

convince yourself,
convince a friend, and
convince an enemy.

Often the students had a story that clearly convinced themselves and even convinced their friends in the group, but by acting as an enemy I was able to begin to help them be more reflective about what they claimed, so that, over time, they began to question their ideas as a matter of course.

It was my belief that I should not try to solve the problems in advance. It was a distinct advantage to be caring but non-directive in my relationships with the students. *Not* knowing the ‘answer’ meant that I could change my approach from someone who shows how to do things and gives hints into someone who encourages the students to think for themselves. ‘Are you sure?’ ‘What does this tell you?’ ‘Is there another way of looking at it?’

At the same time I introduced the students to Richard Skemp’s theories of modes of building and testing and, more importantly, to his ideas of goals and anti-goals, to help the students reflect on their emotions to be able to reason why they felt as they did and use this knowledge to advantage.

SKEMP’S THREE MODES OF BUILDING AND TESTING

In his book *Intelligence, Learning and Action*, Richard Skemp (1979, p. 163) made a valuable distinction between different modes of building and testing conceptual structures shown in Table 1. He speaks of building and testing a personal ‘reality’ as opposed to the ‘actuality’ of the physical world. Mode (i) relates to the individual’s conception of the world we live in (‘actuality’), mode (ii) to the individual’s

Table 1. Skemp’s Modes of Building and Testing

REALITY CONSTRUCTION	
REALITY BUILDING	REALITY TESTING
Mode (i) from our own encounters with actuality: <i>experience</i>	Mode (i) against expectation of events in actuality: <i>experiment</i>
Mode (ii) from the realities of others: <i>communication</i>	Mode (ii) comparison with the realities of others: <i>discussion.</i>
Mode (iii) from within, by formation of higher order concepts: by extrapolation, imagination, intuition: <i>creativity</i>	Mode (iii) comparison with one’s own existing knowledge and beliefs: <i>internal consistency.</i>

relationships with others, and Mode (iii) to the individual's relationship with mathematics itself. There is a strong relationship with the levels of *Thinking Mathematically* (convince yourself, convince a friend, convince an enemy), in terms of order of levels, but not in a one-to-one fashion. Whereas Mode (i) refers to the personal perceptions of the world based on experience and reflections on actual experiments, the act to 'convince yourself' can involve any personal ideas that the individual may bring to bear on the problem in hand. However, in both cases, the onus is on the individual to use their own resources. Meanwhile Mode (ii) involves relationships with others, which would include both friends and 'enemies', where the latter are doubters who demand a higher level of rigour. Skemp's beautiful Mode (iii) involves the relationship of the human mind and spirit with mathematics, through creativity and internal consistency.

In *Thinking Mathematically*, the role of Mode (iii) is formulated in terms of an 'internal enemy', in which the individual learns to criticise their own creative thinking to seek self-improvement and internal consistency. The full list of levels of explanation in *Thinking Mathematically* is therefore:

- Convince yourself
- Convince a friend
- Convince an enemy
- Develop an internal enemy.

Long-term this leads to the desire to think mathematically by producing arguments that may begin with personal insights, are made clearer by discussions with a friend, then with an enemy whose purpose is to challenge the ideas put forward and make the deductions more rigorous. The ultimate goal is a personal level of consistency corresponding to a Mode (iii) relationship with the coherence of mathematical ideas themselves.

MATHEMATICS AND THE EMOTIONS

Thinking Mathematically focuses on the role of the emotions in mathematics, particularly in dealing with the high of an 'Aha!' experience which should be enjoyed before subjecting the insight to further scrutiny, and being 'Stuck', requiring a positive approach to analyse what has happened and how this can help to suggest alternative approaches.

In the middle of the twentieth century, psychologists separated the cognitive and affective domains (as, for instance, Bloom's famous *Taxonomy of Educational Objectives* distinguished three distinct domains: cognitive, affective and psychomotor). Richard Skemp stood out from the crowd by relating the cognitive and affective domains in terms of his (1979) theory of goals and anti-goals. A goal is an intention that is desired. It may be a short-term simple goal, for instance, to add two numbers together, or it may be a long-term major goal, for example, to succeed in mathematics. On the other hand, an anti-goal is something that is not desired and is to be avoided. For instance, a child may wish to avoid being asked a question in class because of a fear of being made to seem foolish. In general terms a goal is

something that increases the likelihood of survival, but an anti-goal is something to avoid along the way.

Children are born with a positive attitude to learning. They explore the world spontaneously, with great pleasure. But unpleasant experiences may cause them to avoid a repetition of that unpleasantness, which leads to the development of anti-goals.

In his theory of goal-oriented learning, Skemp formulated two distinct aspects of goals and anti-goals. One concerns the emotions sensed as one moves towards, or away from, a goal or anti-goal (represented by arrows in Figure 1). The other concerns an individual's overall sense of being able to achieve a goal, or avoid an anti-goal (representing by the smiling faces for a positive sense and frowning faces for a negative).

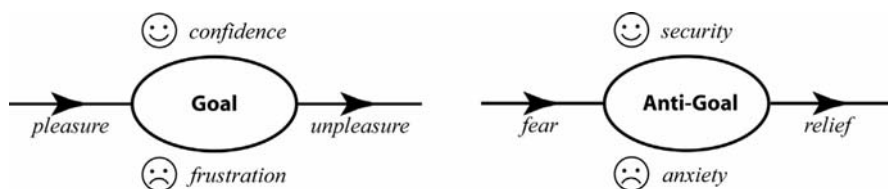


Figure 1. Emotions associated with goals and anti-goals

The emotions related to goals and anti-goals are very different. Believing one is able to achieve a goal is accompanied by a sense of confidence, whilst being unable to achieve a goal is accompanied by frustration. Moving towards a goal gives pleasure, whilst moving away gives unpleasure, in the sense employed earlier by Freud. It is subtly different from the more usual, but not equivalent, term 'displeasure'. Drifting away from a desired goal may not be 'unpleasant' in the sense that it is distasteful, it may simply generate a feeling that one is going on the wrong path and intimate the need to reconsider one's options.

By using Skemp's theoretical framework while working with the book *Thinking Mathematically*, I found it possible to have discussions about individuals' emotional reactions to mathematics, to recognize the different emotional signs and to use them to advantage. For instance the subtle difference between frustration and anxiety in being unable to solve a problem reveals the difference between a goal one desires positively and an anti-goal one wishes to avoid. Once the source of the problem is identified, it becomes possible to take action to move in a more appropriate direction.

PROOF ANXIETY

The one important word missing from *Thinking Mathematically* is 'proof'. In a private conversation, John told me that this was because of the reaction of students to the word in his summer schools working with Open University students. If the idea of 'proof' was mentioned, they froze. In Skemp's terminology this seems to be anxiety arising from a sense of not being able to avoid an anti-goal. Proof seems to

be something that these mature students had difficulty with, and they had long since seen it as a topic that they wished to avoid. If guided towards it, they felt a sense of fear, which could only be relieved by moving away from it again.

Thinking Mathematically is designed to give positive encouragement to students through strategies that are likely to lead to the pleasure of success and build confidence in the art of problem solving as a goal to be achieved, rather than an anti-goal to be avoided. So what is it that causes proof to become an anti-goal? To gain insight into this, it is helpful look at the long-term development of mathematical thinking.

Cognitive development of mathematical thinking

In a number of recent papers (e.g. Tall, 2008), I have followed the path of development of human thinking from mental facilities *set-before* birth and the subsequent experience *met-before* in our lives that affect our current thinking as it matures. Long-term we develop through refining our knowledge structures, coming to terms with complicated situations by focusing on important elements and naming them, so that we can talk about them and build ever more sophisticated meanings. Mason's insight of a *delicate shift of attention* plays its part in switching our thinking from the global complications to the essential aspects that turn out to be important. More generally it is the *discipline of noticing* that is important to seek to focus on essential ideas and gain insight into various problematic situations.

The framework that I have developed centres on the way in which we use words and symbols to *compress* knowledge into thinkable concepts, such as compressing counting processes into the concept of number or the likenesses of triangles into the principle of congruence in Euclidean geometry. Through experience and reflection, we build thinkable concepts into knowledge structures (schemas) that enable us to recognise situations when we attempt to solve new problems. Problem Solving arises when our knowledge structures are not sufficient to recognise the precise problem, or, if we have recognised it, to have the connections immediately available to solve it. To be more effective in mathematical thinking we therefore need to be aware of how our knowledge structures operate and how they develop over time.

As a pupil of Richard Skemp, I was taken by his simple analysis of the way the human mind works through *perception*, *action* and *reflection*, which gives us input through perception, output through action and makes mental links between the two through reflection. Skemp took his theory forward by suggesting that the mind operated at two levels, delta-one with physical perception and action, and delta-two with mental perception and action, linked together by reflection. I reflected on this structure and came to the conclusion that the distinctions between what we *perceive* through our senses and what we *conceive* in our mind are not as clear as we might wish them to be. So, rather than a two-stage theory, I saw a developing mental structure focusing on the complementary nature of perception and action and how it shifts from physical perceptions and actions to mental structures.

Quite recently (February 2008 to be more precise) I realised, to my astonishment, that our mathematical thinking could be seen to develop from just *three* mental facilities that are set-before our birth and which come to fruition

through our personal and social activities as we mature. I termed these three set-befores: *recognition*, *repetition* and *language*. Recognition is the human ability, which we share with many other species, of recognising similarities and differences that can be *categorised* as thinkable concepts. Repetition is the human ability, again shared with other species, of being able to learn to repeat sequences of actions in a single operation, such as see-grasp-suck, or the human operations of counting or solving linear equations. This is the basis of *procedural* knowledge. However, language enhances the set-befores of recognition and repetition. Recognition can be extended to give successive levels of thinking: forming thinkable concepts, then using those concepts as mental objects of attention to work at higher levels. Repetition can be compressed subtly through *encapsulation* of operations as thinkable concepts, denoted by symbols that can evoke either the underlying operation to perform, or the thinkable concept itself to be manipulated in its own right. These thinkable concepts that act dually, ambiguously and flexibly as process and concept are named *procepts*. As thinking processes become more sophisticated, language itself becomes increasingly powerful, leading to new formal ways of forming concepts through *definition* and mathematical *proof*.

This offers a framework for the development of mathematical knowledge structures, building on recognition, repetition and language, with compression into thinkable concepts through categorisation, encapsulation and definition, evolving through three distinct but interrelated mental worlds of mathematics that I term conceptual embodiment, proceptual symbolism and axiomatic formalism. Within the confines of this framework I usually compress the names to single words: embodiment, symbolism and formalism, while acknowledging that these terms have very different meanings in other theories.

This enables me to put the names together in new ways, such as formal embodiment, embodied symbolism, or formal symbolism. Indeed, the meanings of the two word phrases themselves depend on the direction travelled. Arithmetic arises from counting, adding, taking away, sharing as embodied operations that shift into symbolic embodiment. Representing number systems on the number line shifts back to give an embodied symbolism.

For instance, algebra builds from embodiment to symbolism through generalised arithmetic operations of combining, taking away, sharing, distributing, and so on. The reverse direction takes us from algebraic expressions and functions to graphs. These are quite different activities and, as we shall see later, there are a number of problematic aspects of these relationships.

The cognitive development of proof in the embodied world

We now turn our attention to see what the framework of embodiment, symbolism and formalism tells us about students' growing appreciation of proof.

In the embodied world of geometry, building on perception of figures and actions to make constructions gives us more specific insight into the nature of these figures. We already have the analysis of van Hiele to chart the development over the years. Give a child a plastic triangle, with equal sides and the child sees it as a whole and can touch and explore it to sense its corners, its sides and its angles. At

one and the same time, it has three equal sides *and* three equal angles. From this beginning, where a figure has simultaneous properties, the child moves through successive van Hiele levels where the meanings and relationships change in conception. I choose to describe these successive levels as:

Perception: recognising shapes

Description: verbalising some of the properties

Definition: prescribing figures in terms of selected properties

Euclidean Proof: using constructs such as congruent triangles to build up a coherent theoretical framework of Euclidean geometry

Rigour: Formulating other geometric structures in terms of set-theoretic axioms.

In school mathematics, we are mainly concerned with the first four levels up to the development of Euclidean proof. My major focus of attention is the shift from Description to Definition. It seems innocuous. One simply moves from specifying certain properties of a figure to giving a more focused definition. However, cognitively, there is a *huge* shift in meaning. The plastic triangle that the child describes as being equilateral with its three equal sides and three equal angles is now defined as having three equal sides. Full stop.

The child can *see* that an equilateral triangle also has three equal angles, but now it becomes necessary to *prove* that an equilateral triangle, as defined, really does have three equal angles, *as a consequence* of having three equal sides. The method of proof is quite technical. It goes like this. First establish the meaning of congruent triangles. (Two triangles are congruent if they have three corresponding properties: three sides, two sides and included angle, two angles and corresponding side, or right-angle, hypotenuse, one side.)

Effectively the notion of congruence depends on embodied actions. If two triangles ABC , XYZ have two sides equal $AB = XY$, $AC = XZ$ and included angle equal, $\angle A = \angle X$, then pick up triangle ABC and place it on triangle XYZ with vertex A placed on X , side AB placed on XY and angle A over angle X . Then, because the angles are equal, the side AC will lie directly over XY and, because the side-lengths are equal, point C will be coincident with X and point B will be coincident with Y . It follows that all the other corresponding aspects must be equal, including all corresponding angles, all corresponding sides and even the midpoints of the respective sides, the angle bisectors, and so on.

Now take a triangle ABC with equal sides AB , BC and, by constructing the midpoint M of the base AC , form two triangles ABM and CBM . These have corresponding sides equal, $AB = CB$ (given), $AM = CM$ (by construction), BM (common), so the triangles are congruent and, in particular, $\angle A = \angle C$. Q.E.D. Apply the same argument again, and if a triangle has three equal sides, *then* it has three equal angles.

There are some who appreciate the need for proof and get great pleasure out of the beauty of many aesthetic ideas in Euclidean geometry, such as the circle theorems where two angles subtended by the same chord in a circle are equal. But

the vast majority of learners have connections in their minds that *tell* them such things as the fact that an equilateral triangle has equal sides and equal angles, and so, *why do they need to 'prove' it*. The shift from description to definition and deduction is mystifying for many and forms an obstacle causing fear and anxiety. Indeed, the only way to cope with the problem is to use the met-before of repetition to learn the proofs as procedures by rote. It addresses the goal of passing examinations without attending to the goal of understanding.

The cognitive development of proof in the symbolic world

The symbolic world of arithmetic and algebra develops out of embodied actions of counting, adding, taking away, making a number of equal-sized groups, sharing, and so on. These are then symbolised and there is a shift of attention away from specific embodiments and towards the relationships between the symbols.

In the embodied world of counting, it is not initially obvious that addition is commutative. If a child is at a stage of 'count-on' then 8 + 2 by counting on two after 8 to get 9, 10 is much easier than count-on 8 after 2 to get 3, 4, 5, 6, 7, 8, 9, 10. The realisation that it is possible to perform the shorter count and get the same answer can be a pleasurable moment of insight.

Over time, experience shows that addition and multiplication are independent of order, and do not depend on the sequence in which the operations are performed, so that $3 + 4 + 2$ can be performed as $3 + 4$ is 7 then $7 + 2$ is 9, or as $4 + 2$ is 6 and $3 + 6$ is also 9. These are formulated as 'rules', though they are not rules that are to be imposed on numbers, but observations that have been noticed. Then there is the associative law that says that $3 \times (4 + 2)$ is the same as $3 \times 4 + 3 \times 2$, which gets more interesting in sums like $20 - 3 \times (4 - 2)$ being the same as $20 - 3 \times 4 + 3 \times 2$.

At this stage the learner has to deal with a range of principles in using the notation of arithmetic and how they operate in practice. These principles are then employed in algebra.

To 'prove' the formula for the difference between two squares, it is usual to start with $(a + b)(a - b)$ and to multiply it out using the 'distributive law' then use commutativity of multiplication to reorganise the expression and cancel ba and $-ab$ to get the final result:

$$\begin{aligned}(a + b)(a - b) &= a(a - b) + b(a - b) \\ &= a^2 - ab + ba - b^2 \\ &= a^2 - b^2\end{aligned}$$

The problem here is to know what is 'known' and what needs to be 'proved'. The 'laws' being quoted (if they are indeed spoken explicitly) depend on experience and build on all kinds of met-befores that are implicit within the mind. While it may be appropriate in the more sophisticated axiomatic formal world to build proofs on definitions and deductions, for the teenager struggling with algebra it may cause nothing but confusion.

My own view is that the shift from embodiment to symbolism that operates in whole number arithmetic is not as evident in the shift from embodiment to algebra. For the learner who has a flexible proceptual view of symbolism, algebra may be

an easy, even essentially trivial, application of generalised arithmetic. But for the learner who is already struggling with arithmetic and operates more in a time-dependent, procedural manner, it is likely to be highly complicated.

Letters may be used to represent unknown numbers in an equation such as $3x + 5 = 5x - 7$ or as units as in $120 \text{ cm} = 1.2 \text{ m}$. The famous ‘students and professors problem’ relating the number of students (S) to the number of professors (P) when there are 6 students for each professor should be written as $S = 6P$ using the algebraic meaning of letters. However, it is often interpreted as $1P = 6S$ in the units sense that 1 professor corresponds to 6 students.

The met-before that every arithmetic expression, such as $3 + 2$, 3.14×4.77 , or $\sqrt{2} + 1$, ‘has an answer’ is violated by algebraic expressions such as $3 + 2x$ that has no ‘answer’ unless x is known. So now the student who is bewildered by expressions that cannot be worked out is asked to manipulate them as if he or she knows what they are, when they have no meaning.

The interpretation of letters as objects that may help the student simplify $3a + 4b + 2a$ to $5a + 2b$ by thinking of a as ‘apple’ and b as ‘banana’, but it fails to give a meaning to the expression $3a - 5b$. (How can you take away 5 bananas when you only have 3 apples?)

The idea that an equation such as $5x + 1 = 3x + 5$ is a balance between 3 things and 5 on one side and 5 things and 1 on the other is seen as being widely meaningful to many students (Vlassis, 2002). Take $3x$ off both sides to get $2x + 1 = 5$, now take 1 off both sides to get $2x = 4$ and divide both sides by 2 to get the solution $x = 2$. But change the equation slightly to $3x + 5 = 5x - 7$ and suddenly it has no embodied meaning. How can you imagine a balance in which one side is $5x - 7$? How can you take 7 away from $5x$ when you don’t yet know what x is?

In so many ways, the shift from embodiment to symbolic algebra is a minefield of dysfunctional met-befores for so many learners. This does not lead to the goal of making sense of algebra to develop power in formulating and solving equations. Instead, algebra becomes a topic to be avoided at all costs, an anti-goal provoking fear and a sense of anxiety as one attempts to find any method possible to avoid failure. For so many it leads to dysfunctional ways of learning procedures to cope with the difficulties: the English use of BODMAS to remember the order of precedence of operation (Brackets, Of, Division, Multiplication, Addition, Subtraction), the American acronym FOIL to multiply out pairs of terms in brackets (First, Outside, Inside, Last), operations to solve equations such as ‘change sides, change signs; divide both sides by shifting the quantity to the other side and put it underneath.’ For so many, algebra is an anti-goal to be avoided at all costs.

Now we are beginning to build up a picture of what may be happening in school as children learn arithmetic, then algebra. For so many, the initial embodiments of putting together and sharing have a meaning in the actual world in which they live. But the many successive compressions in meaning from operation to flexible procept work for some but impose increasing pressures on others. Eddie Gray and I called this ‘the proceptual divide’ in which the flexible thinkers have a built-in engine to derive new facts from old based on their rich knowledge of relationships

between numbers, while others see increasing complication in all the detail and fall back on attempts to learn procedures by rote to cope with the pressures of testing.

Learning procedures by rote can be supporting in being able to perform routine calculations but procedural learning alone makes it more difficult to imagine flexible relationships between compressed concepts that are required in more sophisticated problem solving. As mathematics becomes more complicated for those who lack the rich flexible meanings, mathematics itself becomes an anti-goal to be avoided, creating a sense of anxiety and fear. More generally, mathematical proof, which requires a coherent grasp of ideas and how they are related, becomes problematic, both in geometry and in algebra.

GENERATING CONFIDENCE THROUGH *THINKING MATHEMATICALLY*

Given the relationship between cognitive success and emotional reactions, it becomes likely that one might attempt to improve students' abilities to think mathematically through organising situations in which they may experience success. Having experienced the good feelings generated in an open-ended problem-solving course myself, I was fortunate to be joined by Yudariah binte Mohammad Yusof, a university teacher from Malaysia who was concerned by the concentration on procedural learning in her students and the lack of a problem-solving ethic, other than that of becoming highly proficient at solving specific problems that would feature on the university examinations.

She took part in the Problem Solving course at Warwick University and trialled a questionnaire investigating student attitudes towards various aspects of mathematics and problem-solving. She then returned to Malaysia to teach the course and to research its effect on the students. (The details are given in Yusof & Tall 1996.) Half way through the course she telephoned me to express concern that her students continued to ask her what she wanted them to do, so that they could do well on the course. All I could say to her was that she should maintain the objective that the students needed to take control of their own working using the framework of *Thinking Mathematically*.

By the end of the course attitudes had changed dramatically. To identify what was meant by a 'desirable change', she asked the students' lecturers to fill in the questionnaires twice, once to indicate what they *expected* the students to say, once to say what they *preferred* the students to say. The direction of change from expected to preferred was taken to be a 'positive' change. In general *all* the changes in students' attitudes during the problem-solving course were positive, but when they returned to their normal mathematics lectures and were asked again six months later, the changes generally went back in the opposite direction. In other words, the problem-solving course took the students' attitudes in the direction desired by the staff, but when the staff themselves did the teaching, the attitudes of the students changed to the opposite direction.

My experience to date, through the work of research students carrying out studies in other countries and through my own links with communities around the world, is consistent with the global concern about the learning of mathematics. Some societies try to encourage meaningful learning through problem-solving,

some teach by rote to encourage proficiency with many variations in between. Everywhere the pressure to compete and succeed in tests is driving the policies of governments. Surely our job as mathematics educators is not just to increase percentages passing examinations but a wider and deeper concern to understand the nature of mathematical thinking, to identify precisely why it is so difficult for many and how it can be improved for each individual.

REFLECTIONS

The analysis given here shows the power of *Thinking Mathematically* to improve students' attitudes and improve students' self-confidence and pleasure in doing mathematics and thinking for themselves. However, this occurs in a context in which so many older students have anxieties in dealing with the most central of all mathematical concepts, the notion of *proof*. The analysis given here in terms of the development of mathematical thinking through increasingly sophisticated embodiment and symbolism reveals transitions that are required to make sense of increasingly sophisticated mathematical thinking. The apparently innocuous shift from description to definition in geometry violates earlier beliefs in the properties of figures that are 'known' as part of a global perception but now must be 'proved' from the selected definitional properties. The shift from arithmetic to algebra involves a range of met-befores where established beliefs need to be changed to make sense of the new ideas.

John Mason has led a personal crusade for everyone to think about mathematics in new ways and his methods have yielded success. Clearly the way forward is to increase students' confidence by giving them genuine experiences of successful thinking, for only then will they face new problems as a challenge rather than a source of anxiety and fear. There is still much to be done by future generations to extend the pathways already trodden.

REFERENCES

- Bloom, B. S. (1956). *Taxonomy of educational objectives, handbook I: The cognitive domain*. New York: David McKay.
- Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. Wokingham, UK: Addison Wesley.
- Skemp, R. R. (1979). *Intelligence, learning and action*. London: Wiley.
- Tall, D. O. (2008). The transition to formal thinking in mathematics. To appear in *Mathematics Education Research Journal*. Retrieved from <http://www.warwick.ac.uk/staff/David.Tall/pdfs/dot2008e-merj-3worlds.pdf>
- Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, 49(3), 341–359.
- Yusof, Y. B. M., & Tall, D. O. (1996). Conceptual and procedural approaches to problem solving. In L. Puig & A. Gutierrez (Eds.), *Proceedings of 20th meeting of international group for the psychology of mathematics education* (Vol. 4, pp. 3–10). Valencia, ES.

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