Ch 8. Hypothesis Testing

Intro

Definition

1. Any statement about the unknown parameter $\boldsymbol{\theta}$ is called a *hypothesis*
2. One of the complementary hypothesis is called *Null Hypothesis* (denoted by H_0) and other is called *Alternative Hypothesis* (denoted by H_1 or H_A).

$\triangleright$ Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\theta_1, \sigma^2)$ - regular diet program, $Y_1, \cdots, Y_n \overset{iid}{\sim} N(\theta_2, \sigma^2)$ - Caloric restricted diet program

$$H_0 : \theta_1 = \theta_2 \quad vs \quad H_1 : \theta_1 > \theta_2$$

$\triangleleft$ Note: Θ_0 and Θ_1 are often called *Null* and *Alternative* space of parameter and the hypotheses are expressed as

$$H_0 : \boldsymbol{\theta} \in \Theta_0 \quad vs \quad \boldsymbol{\theta} \in \Theta_1$$

Definition

A hypothesis that completely specifies the distribution of $X_1, \cdots, X_n$ is called a *simple hypothesis* otherwise it is called *composite hypothesis*.

$\triangleright$ Example: $\theta_1 = \theta_2$, $\theta_1 = \theta_2 = 2$, $\theta_1 > \theta_2$.

- After observing $X_1 = x_1, \cdots, X_n = x_n$, we need to decide which hypothesis, H_0 or H_1, we will accept. Let $\mathfrak{X}$ denote the set of all possible realization of $X_1, \cdots, X_n$. Testing function (rule) plays the same role as estimator in point estimation.

Definition

1. A function $\phi : \mathfrak{X} \to [0,1]$ is called a *testing function.*
2. If a testing function takes a values in $\{0,1\}$, i.e. $\phi : \mathfrak{X} \to \{0,1\}$, it is called a *simple testing function.*

$\triangleleft$ Note: The interpretation of definition 1 is that after observing $X_1 = x_1, \cdots, X_n = x_n$, reject H_0 with probability $\phi(x_1, \cdots, x_n)$ and accept H_0 with probability $1 - \phi(x_1, \cdots, x_n)$. This is called a randomized procedure.

Definition

- $R_\phi = \{\mathbf{x} : \phi(\mathbf{x}) = 1\}$ is called the *rejection region* or *critical region*
- $A_\phi = \{\mathbf{x} : \phi(\mathbf{x}) = 0\}$ is called the *acceptance region*

Q: How to compare several testing function ? or How to construct a good testing functions ?

• Errors in Testing

		True status of Nature	
		H_0 is true	H_1 is true
Action	Accept H_0	O.K.	Type II error
	Reject H_0	Type I error	O.K.

– Type I error: Reject H_0 when H_0 is true

– Type II error: Accept H_0 when H_0 is false

Q: How to compare several testing function ? or How to construct a good testing functions ?

- Errors in Testing

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– Type I error: Reject H_0 when H_0 is true
– Type II error: Accept H_0 when H_0 is false

Definition

The *power function* $\beta_\phi(\theta)$ of a test $\phi(\mathbf{x})$ is the function defined as

$$\beta_\phi(\theta) = P_\theta[\phi(\mathbf{X}) = 1] = E_\theta[\phi(\mathbf{X})] = P_\theta(\mathbf{X} \in R_\phi)$$

◁ Note:

- $\theta \in \Theta_0$, $\beta_\phi(\theta) = \; Pr[Type \;\; I \;\; error \;\;]$.
  $\theta \in \Theta_1$, $\beta_\phi(\theta) = 1 - Pr[Type \;\; II \;\; error]$.
- $\sup_{\theta \in \Theta_0} \beta_\phi(\theta)$ is called the *size of the test* ϕ. Thus, any test such that $\sup_{\theta \in \Theta_0} \beta_\phi(\theta) = \alpha$ is called as a *size* α *test*.
- Test ϕ such that $\sup_{\theta \in \Theta_0} \beta_\phi(\theta) \leq \alpha$ is called a *level* α *test*.

Example Suppose $X_1, \ldots, X_n$ is a random sample from $N(\theta, 1)$. We're interested in testing the hypotheses

$$H_0: \ \theta \leq 10 \qquad H_1: \ \theta > 10.$$

$$\Theta = (-\infty, \infty) \qquad \Theta_0 = (-\infty, 10]$$

$$\Theta_0^c = (10, \infty)$$

For example:

- X_i might be a measure of product quality when a new process is used.
- The average quality measure using the old process is 10.
- H_0 says that the new process is no better than the old.
- H_1 says the new process *is* better than the old.

A sufficient statistic in this model is

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Also, we know that $\bar{X}$ is both the MLE and the UMVUE of θ. A sensible test would have the following form:

Take action a_1 if $\bar{x} \geq c_n$, and

take action a_0 if $\bar{x} < c_n$,

where $\bar{x}$ is the observed value of $\bar{X}$ and c_n is some constant larger than 10.

Type I error: *Conclude new process is better when it isn't.*

Type II error: *Conclude new process is no better than the old when in fact it is better.*

Example (continued) Suppose we use a test function ϕ as follows:

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \bar{x} \geq 10 + \frac{1.645}{\sqrt{n}} \\ 0, & \text{otherwise.} \end{cases}$$

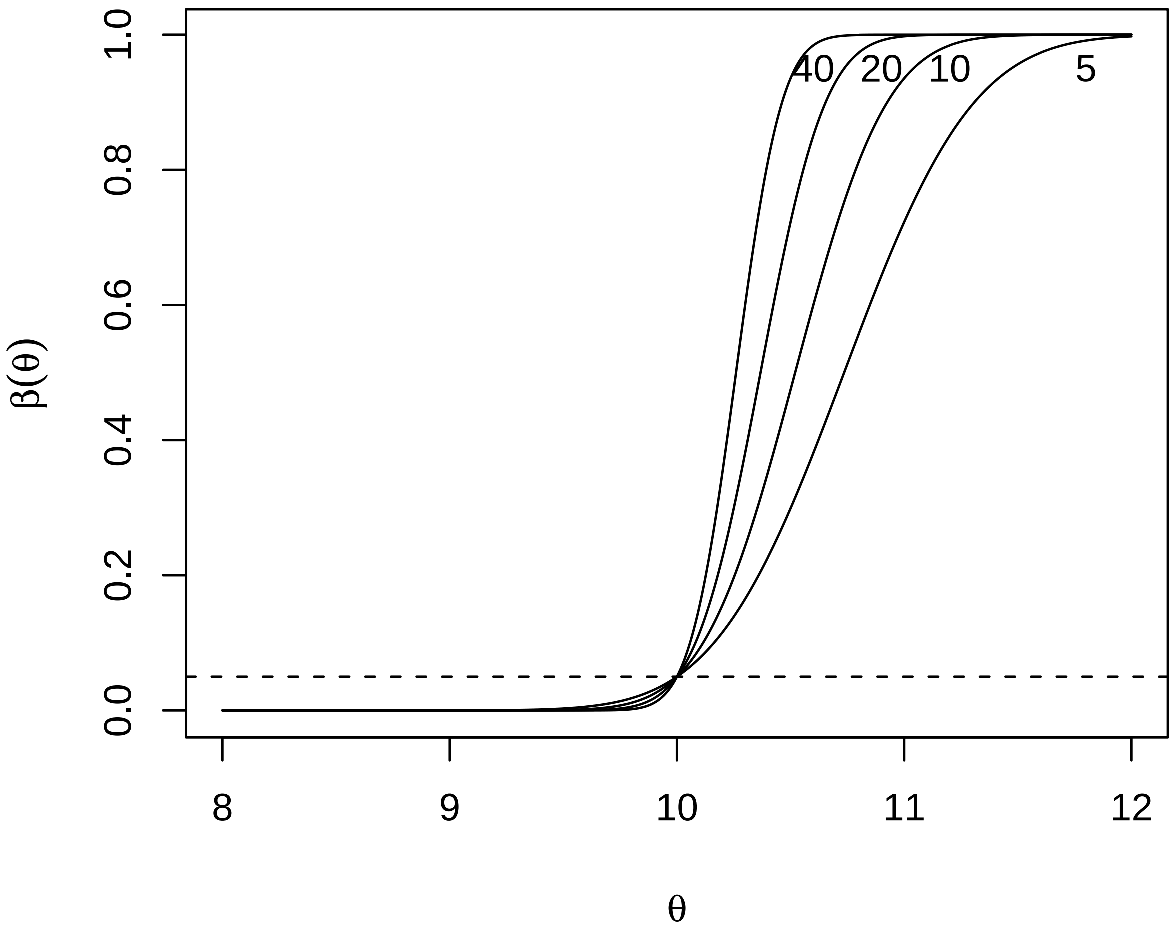
$$\begin{aligned} \beta(\theta) &= P_\theta\left(\bar{X} \geq 10 + \frac{1.645}{\sqrt{n}}\right) \\ &= P_\theta\left(\frac{\bar{X} - \theta}{1/\sqrt{n}} \geq \sqrt{n}(10 - \theta) + 1.645\right) \\ &= P(Z \geq \sqrt{n}(10 - \theta) + 1.645), \end{aligned}$$

where $Z \sim N(0, 1)$.

Remarks

- $\beta(10) = 0.05$ for each n.
- $\beta(\theta)$ increases monotonically, from 0 at $\theta = -\infty$ to 1 at $\theta = \infty$.
- So, the size of the test is 0.05, no matter the value of n.

Power curves for Example



The numbers beside the curves indicate sample size, n.

Ch 8. Hypothesis Testing

Evaluating the test

▷ Example: $X_1, \cdots, X_n \overset{iid}{\sim}$ exponential(θ).

$$H_0 : \theta \geq 1 \quad vs \quad H_1 : \theta < 1$$

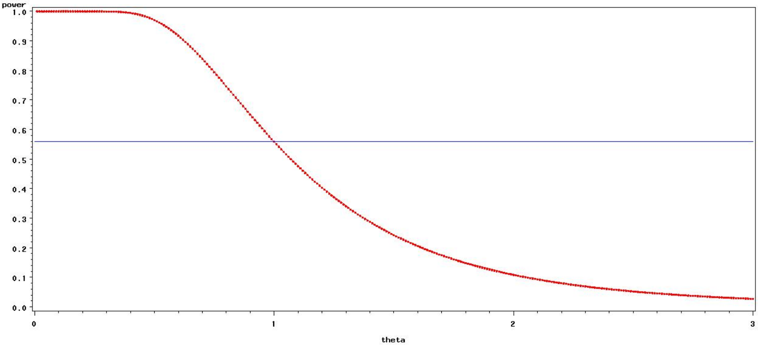
Consider a test function

$$\phi(\mathbf{x}) = \begin{cases} 1 & \bar{x} < 1 \\ 0 & elsewhere. \end{cases}$$

$$n\bar{X} = \sum_{i=1}^{n} X_i \sim \Gamma(n, \theta)$$

$$\frac{2n\bar{X}}{\theta} \sim \Gamma(n, 2), \text{ i.e. } \chi^2(2n)$$

Ch 8. Hypothesis Testing
Evaluating the test



Ch 8. Hypothesis Testing

Evaluating the test

▷ Example: $X \sim$ Binomial$(2,\theta)$

$$H_0 : \theta = \frac{1}{2} \quad vs \quad H_1 : \theta = \frac{3}{4}$$

Find a test of size 0.

▷ Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$.

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta \neq \theta_0$$

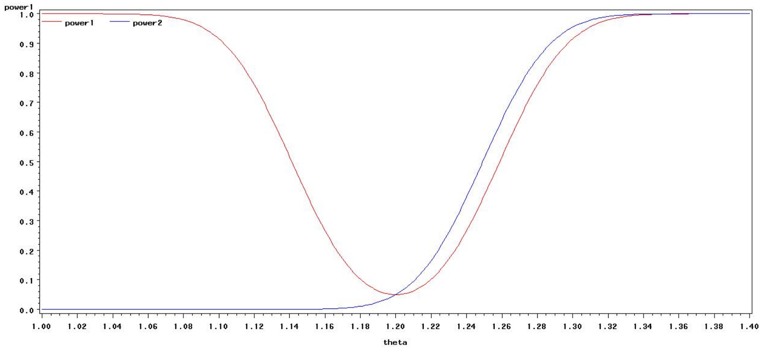
Consider the following two test functions

$$\phi_1(\mathbf{x}) = \begin{cases} 1 & |\bar{x} - \theta_0| > c_1 \\ 0 & elsewhere. \end{cases}$$

$$\phi_2(\mathbf{x}) = \begin{cases} 1 & \bar{x} > \theta_0 + c_2 \\ 0 & elsewhere. \end{cases}$$

Ch 8. Hypothesis Testing

Evaluating the test



Definition

A test function $\phi[\mathbf{X} = (X_1, \cdots, X_n)]$ is said to be the *most powerful* test of size α for testing

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1$$

if

1. $E_{\theta_0}[\phi(\mathbf{X})] = \alpha$, $[\beta_\phi(\theta_0) = \alpha.]$
2. for any other test function $\tilde{\phi}(\mathbf{X})$ with $E_{\theta_0}[\tilde{\phi}(\mathbf{X})] \leq \alpha$,

$$E_{\theta_1}[\phi(\mathbf{X})] \geq E_{\theta_1}[\tilde{\phi}(\mathbf{X})], \quad [\beta_\phi(\theta_1) \geq \beta_{\tilde{\phi}}(\theta_1)]$$

MP test has the smallest probability of type II error among all test rules with probability of type I error no bigger than α.

▷ Example: $H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1$

$X = x$	0	1	2
$p(x\|\theta_0)$	0.05	0.05	0.90
$p(x\|\theta_1)$	0.90	0.08	0.02
$p(x\|\theta_1)/p(x\|\theta_0)$	18	1.6	0.022

Size $\alpha = 0.05$ tests?

Find the MP test of size 0.05? Choose the test that has the largest/smallest ratio?

Theorem (Neyman-Pearson Lemma)

$X_1, \cdots, X_n$ has a joint pdf/pmf $f(\mathbf{x}|\theta)$, $\theta \in \Theta$. Consider the testing the hypotheses,

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1$$

Then, for any $0 \leq \alpha \leq 1$, there exist a MP test of size α given below;

$$\phi(\mathbf{x}) = \begin{cases} 1 & \textit{if} \quad f(\mathbf{x}|\theta_1) > kf(\mathbf{x}|\theta_0), \\ \gamma & \textit{if} \quad f(\mathbf{x}|\theta_1) = kf(\mathbf{x}|\theta_0), \\ 0 & \textit{if} \quad f(\mathbf{x}|\theta_1) < kf(\mathbf{x}|\theta_0), \end{cases}$$

where the constants k and γ are chose to satisfy

$$E_{\theta_0}[\phi(\mathbf{X})] = \beta_\phi(\theta_0) = \alpha.$$

We will prove the theorem for the continuous case. In this case, we take r=0. There exists k>0 such that P($\phi(x)=1$)= α

Let $\phi'(x)$ be the test function of any other α level test

Since $0 \le \phi'(\mathbf{x}) \le 1$, (1) implies that

$$\left[\phi(\mathbf{x}) - \phi'(\mathbf{x})\right]\left[f\left(\mathbf{x}|\theta_1\right) - kf\left(\mathbf{x}|\theta_0\right)\right] \ge 0$$

$$0 \le \int \left[\phi(\mathbf{x}) - \phi'(\mathbf{x})\right]\left[f\left(\mathbf{x}|\theta_1\right) - kf\left(\mathbf{x}|\theta_0\right)\right] d\mathbf{x}$$

$$= \beta(\theta_1) - \beta'(\theta_1) - k\left[\beta(\theta_0) - \beta'(\theta_0)\right]. \quad (3)$$

$$\le \beta(\theta_1) - \beta'(\theta_1)$$

◁ Note:

1. The MP test ϕ reject H_0 if the likelihood ratio

$$L = \frac{f(\mathbf{x}|\theta_1)}{f(\mathbf{x}|\theta_0)}$$

   is large.

2. In general, there may be more than one choice of k and γ that $\beta_\phi(\theta_0) = \alpha$. Then each is MP test of size α.
3. When $f(\mathbf{x}|\theta_1)/f(\mathbf{x}|\theta_0)$ has a continuous distribution under the null, H_0, $\gamma = 0$ is usually taken and considered as the MP test of size α.

Example Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, 1)$, and suppose $\theta_0 < \theta_1$. Find the most powerful size α test of

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta = \theta_1,$$

and the power of this test.

$$f(\boldsymbol{x}|\theta) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(x_i - \theta)^2\right).$$

$$f(\boldsymbol{x}|\theta_1) > k f(\boldsymbol{x}|\theta_0) \quad \text{iff}$$

$$\log f(\boldsymbol{x}|\theta_1) - \log f(\boldsymbol{x}|\theta_0) > \log k = k'.$$

$$\log f(\boldsymbol{x}|\theta_1) - \log f(\boldsymbol{x}|\theta_0) =$$
$$-\textstyle\sum_{i=1}^{n}(x_i - \theta_1)^2/2 + \sum_{i=1}^{n}(x_i - \theta_0)^2/2$$

The last quantity exceeds k' iff

$$\bar{x} > \frac{(\theta_1^2 - \theta_0^2)/2 + k'/n}{(\theta_1 - \theta_0)}.$$

The test function of the most powerful test thus has the form

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bar{x} > c \\ 0, & \bar{x} < c, \end{cases}$$

where c is determined from $E_{\theta_0}[\phi(\boldsymbol{X})] = \alpha$.

$$\begin{aligned} E_{\theta_0}[\phi(\boldsymbol{X})] &= P_{\theta_0}(\bar{X} > c) \\ &= P_{\theta_0}\left(\frac{\bar{X} - \theta_0}{1/\sqrt{n}} > \sqrt{n}(c - \theta_0)\right) = \alpha. \end{aligned}$$

This implies that $\sqrt{n}(c - \theta_0) = z$, or $c = \theta_0 + z_\alpha/\sqrt{n}$. So, we reject H_0 iff $\bar{x} > \theta_0 + z_\alpha/\sqrt{n}$.

Power of the test is

$$P_{\theta_1}\left(\bar{X} > \theta_0 + z_\alpha/\sqrt{n}\right) =$$

$$P_{\theta_1}\left(\frac{\bar{X}-\theta_1}{1/\sqrt{n}} > z_\alpha - \sqrt{n}(\theta_1 - \theta_0)\right) =$$

$$1 - \Phi\left(z_\alpha - \sqrt{n}(\theta_1 - \theta_0)\right).$$

What happens to the power as $n \to \infty$?

▷ Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

$$H_0 : \mu = \mu_0 \quad vs \quad H_1 : \mu = \mu_1 (< \mu_0)$$

Find the MP-test of size α.

$$L(\theta) = \left(\frac{1}{2\pi}\right)^{n/2} \exp\{-\sum_{i=1}^{n} \frac{(x_i - \theta)^2}{2\sigma^2}\}$$

$$\frac{L(\theta_1)}{L(\theta_0)} > k \Longrightarrow \quad \bar{x} < \frac{(2\sigma^2 \log k)/n - \theta_0^2 + \theta_1^2}{2(\theta_1 - \theta_0)}.$$

$$\alpha = P_{\theta_0}(\bar{X} < c). \quad \bar{X} < c = -\sigma z_\alpha/\sqrt{n} + \theta_0.$$

Example Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli rv's with success probability θ. Test

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta = \theta_1, \quad \theta_1 > \theta_0.$$

Define $c_1 = \log(\theta_1/\theta_0)$ and

$$c_2 = -\log\left((1-\theta_1)/(1-\theta_0)\right).$$

Then

$$\log L(\boldsymbol{x}|\theta_0, \theta_1) = (c_1 + c_2)\sum_{i=1}^{n} x_i - nc_2.$$

Since c_1 and c_2 are positive,

$$L(\boldsymbol{x}|\theta_0, \theta_1) > k \quad \text{iff} \quad \sum_{i=1}^{n} x_i > k'.$$

When H_0 is true, $\sum_{i=1}^n X_i$ has a bin(n, θ_0) distribution. The possible sizes for tests of the form

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & L(\boldsymbol{x}|\theta_0, \theta_1) > k, \\ 0, & L(\boldsymbol{x}|\theta_0, \theta_1) < k \end{cases}$$

are 0 and $\sum_{i=j}^{n} \binom{n}{i} \theta_0^i (1-\theta_0)^{n-i}$, $j = 0, 1, \ldots, n$.

Let k' be such that $j-1 < k' < j$, where j is an integer and $0 \le j \le n$. The test

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \sum_{i=1}^{n} x_i > k', \\ 0, & \sum_{i=1}^{n} x_i < k' \end{cases}$$

has size $\alpha = E_{\theta_0}[\phi(\boldsymbol{X})]$, which is

$$\sum_{i=j}^{n} \binom{n}{i} \theta_0^i (1-\theta_0)^{n-i}.$$

The power of the test is

$$E_{\theta_1}[\phi(\boldsymbol{X})] = \sum_{i=j}^{n} \binom{n}{i} \theta_1^i (1-\theta_1)^{n-i}.$$

For a specified α, we may need to be satisfied with a *level* α test.

Example continued: *Randomized tests*

Let $n = 10$ and $\theta_0 = 1/2$. Suppose we would like our test to have size .05. Let $T = \sum_{i=1}^{10} X_i$, which has distribution $\text{bin}(10, 1/2)$ under H_0.

$$P(T = 10) = (1/2)^{10} = 0.0009766$$

$$\begin{aligned} P(T \geq 9) &= 10(1/2)^{10} + 0.0009766 \\ &= 0.0107422 \end{aligned}$$

$$\begin{aligned} P(T \geq 8) &= 45(1/2)^{10} + 0.0107422 \\ &= 0.0546875 \end{aligned}$$

Let $\phi_R(x) = P(\text{rejecting } H_0 | T = x)$. The form of the randomized test is

$$\phi_R(x) = \begin{cases} 1, & x = 9, 10, \\ p, & x = 8, \\ 0, & x \leq 7, \end{cases}$$

where p is chosen so that the size of the test is .05.

We have

$$\begin{aligned}
\alpha &= E_{\theta_0}[\phi_R(T)] \\
&= \sum_{x=0}^{10} \phi_R(x) \binom{10}{x} (1/2)^{10} \\
&= p45(1/2)^{10} + P_{\theta_0}(T \geq 9).
\end{aligned}$$

In order for α to be .05, we need

$$p = \frac{[.05 - P_{\theta_0}(T \geq 9)]}{45(1/2)^{10}} = .8933.$$

Given a data set, if $T = 8$, we could generate a value from the $U(0,1)$ distribution (using a computer), and reject H_0 iff the generated value is less than .8933.

Definition

Let $f(\mathbf{x}|\theta)$, $\theta \in \Theta$ be the joint pdf/pmf of $X_1, \cdots, X_n$. Let Θ_0 and Θ_1 be the nonempty disjoint subsets of Θ. A test rule $\phi(\mathbf{x})$ is said to be an *uniformly most powerful (UMP)* test of size α for testing

$$H_0 : \theta \in \Theta_0 \quad vs \quad H_1 : \theta \in \Theta_1$$

if

1. $\max_{\theta \in \Theta_0} E_\theta[\phi(\mathbf{X})] = \alpha$
2. for any other test $\tilde{\phi}(\mathbf{x})$ with $\max_{\theta \in \Theta_0} E_\theta[\tilde{\phi}(\mathbf{X})] \leq \alpha$, we have

   $$E_\theta[\phi(\mathbf{X})] \geq E_\theta[\tilde{\phi}(\mathbf{X})]$$

   for each $\theta \in \Theta_1$.

◁ Note:

1. A UMP test has the smallest probability of type II error for every $\theta \in \Theta_1$among all the test with size $\leq \alpha$.
2. Condition 2 is a really strong requirement. Unlike the simple versus simple case, UMP test may not exist for composite H_0 and for composite H_1.
3. NP lemma can be used to show that UMP test does not exist or identify the UNP test if it exists. HOW ?

a. Fix $\theta_0 \in \Theta_0$ appropriately (usually boundary of Θ_0).

b. Choose any $\theta_1 \in \Theta_1$

c. Then find a MP test of size α, $\phi(\mathbf{x})$, for

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1.$$

If

i $\phi(\mathbf{x})$ does not depend on θ_1

ii $\max_{\theta \in \Theta_0} E_\theta[\phi(\mathbf{X})] = \alpha$

then $\phi(\mathbf{x})$ is the UMP-test of size α.

▷ Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

$$H_0 : \mu = \mu_0 \quad vs \quad H_1 : \mu > \mu_0$$

$$H_0 : \mu \leq \mu_0 \quad vs \quad H_1 : \mu > \mu_0$$

For the first case,

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \bar{x} \geq \theta_0 + z_\alpha/\sqrt{n} \\ 0, & \bar{x} < \theta_0 + z_\alpha/\sqrt{n}. \end{cases}$$

This test does not depend on θ_1. Does it have level α when used to test H_0 vs. H_1 in the 2nd case?

Let $\tilde{\theta} < \theta_0$.

$$P_{\tilde{\theta}}\left(\bar{X} \geq \theta_0 + \frac{z_\alpha}{\sqrt{n}}\right) =$$

$$P_{\tilde{\theta}}\left(\frac{\bar{X} - \tilde{\theta}}{1/\sqrt{n}} \geq \sqrt{n}(\theta_0 - \tilde{\theta}) + z_\alpha\right) =$$

$$P(Z \geq \sqrt{n}(\theta_0 - \tilde{\theta}) + z_\alpha) < \alpha$$

since $\theta_0 > \tilde{\theta}$. This explains why θ_0 was used in H_0'.

In many cases the most powerful level α test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ $(\theta_1 > \theta_0)$ *will* depend on θ_1. In such cases a UMP test for $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$ does not exist. However, there is a large class of distributions for which UMP tests do exist.

▷ Example: $X_1, \cdots, X_n \sim f(x|\lambda)$.

$$f(x|\lambda) = \lambda e^{-\lambda x}, \quad x > 0$$

$$H_0 : \lambda \leq \lambda_0 \quad vs \quad H_1 : \lambda > \lambda_0$$

$$L(\lambda) = \lambda^n \exp\left[-\lambda \sum_{i=1}^{n} x_i\right] = \lambda^n \exp[-n\lambda \bar{x}]$$

$$\frac{L(\lambda_1)}{L(\lambda_0)} = \left(\frac{\lambda_1}{\lambda_0}\right)^n \exp\left(-n(\lambda_1 - \lambda_0)\bar{x}\right) > k \Leftrightarrow \bar{x} < c^* \Leftrightarrow 2n\lambda_0 \bar{x} < c$$

$$2n\lambda_0 \bar{X} \overset{\lambda=\lambda_0}{\sim} \chi^2(2n) \qquad \phi(x_1, \cdots, x_n) = \begin{cases} 1, & 2n\lambda_0 \bar{x} < \chi^2_{0.05}(2n) \\ 0, & \text{otherwise} \end{cases}$$

Definition

Let $f(\mathbf{x}|\theta)$, $\theta \in \Theta$ be the joint pdf/pmf of $X_1, \cdots, X_n$. The family is said to have *Monotone Likelihood Ratio (MLR)* in a statistic $T(\mathbf{X})$ if, for all $\theta'' > \theta'$, $\theta'', \theta' \in \Theta$, there exist a nondecreasing function of T, g, such that

$$L = \frac{f(\mathbf{x}|\theta'')}{f(\mathbf{x}|\theta')} = g_{\theta',\ \theta''}[T(\mathbf{x})]$$

in a support of $\mathbf{x}$.

◁ Note:

- if $g_{\theta',\ \theta''}(x)$ is decreasing then $g_{\theta',\ \theta''}(-x)$ is increasing.
- if $f(\mathbf{x}|\theta'') > 0$ and $f(\mathbf{x}|\theta') = 0$ then $L = \infty$.

Theorem

Let $X_1, \cdots, X_n$ have joint pdf/pmf $f(\mathbf{x}|\theta)$, $\theta \in \Theta$. Assume the family has MLR in $T(\mathbf{X})$. Then

1. *A UMP test of size α for*

$$H_0 : \theta \leq \theta_0 \quad vs \quad H_1 : \theta > \theta_0$$

   is given by

$$\phi(\mathbf{x}) = \begin{cases} 1, & T(\mathbf{x}) > k, \\ \gamma, & T(\mathbf{x}) = k, \\ 0, & T(\mathbf{x}) < k, \end{cases}$$

   where k and γ are determined by

$$P_{\theta_0}[T(\mathbf{X}) > k] + \gamma P_{\theta_0}[T(\mathbf{X}) = k] = \alpha.$$

Theorem (-Continued)

2. *A UMP test of size α for*

$$H_0 : \theta \geq \theta_0 \quad vs \quad H_1 : \theta < \theta_0$$

is given by

$$\phi(\mathbf{x}) = \begin{cases} 1, & T(\mathbf{x}) < k, \\ \gamma, & T(\mathbf{x}) = k, \\ 0, & T(\mathbf{x}) > k, \end{cases}$$

where k and γ are determined by

$$P_{\theta_0}[T(\mathbf{X}) < k] + \gamma P_{\theta_0}[T(\mathbf{X}) = k] = \alpha.$$

Example Let $X_1, \ldots, X_n$ be i.i.d. $N(0, \theta)$. $\Theta = \{\theta : \theta > 0\}$. Find UMP test of

$$H_0: \ \theta \geq \theta_0 \quad \text{vs.} \quad H_1: \theta < \theta_0.$$

Check for the MLR property.

$$f(\boldsymbol{x}|\theta) = \exp\left(-\frac{1}{2\theta}\sum_{i=1}^{n} x_i^2 - \frac{n}{2}\log(2\pi\theta)\right)$$

$$\frac{f(\boldsymbol{x}|\theta'')}{f(\boldsymbol{x}|\theta')} \uparrow \quad \text{in} \ \sum_{i=1}^{n} x_i^2 \quad \text{if} \ \theta'' > \theta'$$

Since $1/\theta$ is a decreasing function of θ, Theorems 9 and 10 tell us that the UMP level α test of H_0 vs. H_1 has the form

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} x_i^2 \leq c^* \\ 0, & \text{if } \sum_{i=1}^{n} x_i^2 > c^*. \end{cases}$$

The constant c^* is such that

$$P_{\eta_0}\left(\sum_{i=1}^{n} X_i^2 \leq c^*\right) = \alpha.$$

When $\theta = \theta_0$, we know that $X_i/\sqrt{\theta_0} \sim N(0,1)$, and so $X_i^2/\theta_0 \sim \chi_1^2$ and $\sum_{i=1}^n X_i^2/\theta_0 \sim \chi_n^2$.

It follows that $c^* = \chi_{n,\alpha}^2\,\theta_0$, where $\chi_{n,p}^2$ is the $100p$th percentile of the χ_n^2 distribution.

So, we have found the most powerful level α test of H_0' vs. H_1', and hence of H_0 vs. H_1.

$$\beta(\theta) = P_\theta\left[\chi_n^2 \le \left(\frac{\theta_0}{\theta}\right)\chi_{n,\alpha}^2\right]$$

Limiting cases: $\theta = 0$ and $\theta = \infty$.

▷ Example: $X_1, \cdots X_n \overset{iid}{\sim} f(x|\theta)$, $f(x|\theta) = c(\theta)h(x)\exp[w(\theta)t(x)]$

$$T^*(\mathbf{X}) = \sum_{i=1}^{n} t(X_i)$$

if $w(\theta)$ is an increasing function of θ $\quad let T(X)=T^*(X)$

if $w(\theta)$ is an increasing function of θ $\quad let T(X)=-T^*(X)$

$$H_0 : \theta \leq \theta_0 \quad vs \quad H_1 : \theta > \theta_0$$

$$\phi(\mathbf{x}) = \begin{cases} 1, & T(\mathbf{x}) > k, \\ \gamma, & T(\mathbf{x}) = k, \\ 0, & T(\mathbf{x}) < k, \end{cases}$$

$$H_0 : \theta \geq \theta_0 \quad vs \quad H_1 : \theta < \theta_0$$

$$\phi(\mathbf{x}) = \begin{cases} 1, & T(\mathbf{x}) < k, \\ \gamma, & T(\mathbf{x}) = k, \\ 0, & T(\mathbf{x}) > k, \end{cases}$$

Example: *UMP tests do not always exist*

Let the probability model be as in previous Example
Want to test

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_1 : \theta \neq \theta_0.$$

$$\Theta_0 = \{\theta_0\} \qquad \Theta_0^c = (0, \infty) \cap \{\theta_0\}^c$$

MP test of $H_0' : \theta = \theta_0$ vs. $H_1' : \theta = \theta_1$ for $\theta_1 > \theta_0$ is

$$\phi_1(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n x_i^2 \geq c \\ 0, & \text{if } \sum_{i=1}^n x_i^2 < c. \end{cases}$$

MP test of $H_0'' : \theta = \theta_0$ vs. $H_1'' : \theta = \theta_2$ for $\theta_2 < \theta_0$ is

$$\phi_2(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n x_i^2 \leq c_1 \\ 0, & \text{if } \sum_{i=1}^n x_i^2 > c_1. \end{cases}$$

Assume ϕ^* is UMP for testing H_0 vs. H_1. Then it is most powerful for $H_0' : \theta = \theta_0$ vs. $H_1' : \theta = \theta_1$ and hence agrees with ϕ_1 by N-P lemma.

Also, it must agree with ϕ_2 by the same logic. But $\phi_1 \neq \phi_2$, which yields a contradiction. Hence, there is no UMP test.

When a UMP test doesn't exist, one can look at a smaller class of tests and try to find the most powerful test within the smaller class.

Examples of such tests:

Class of unbiased tests A test is said to be unbiased if $\beta(\theta) \geq$ size of test for all $\theta \in \Theta_0^c$.

i.e. power function of a test satisfies

$$\beta(\theta) \leq \alpha \quad \text{if } \theta \in \Theta_0 ,$$
$$\beta(\theta) \geq \alpha \quad \text{if } \theta \in \Theta_1 .$$

Definition

Let $X_1, \cdots, X_n$ have joint pdf/pmf $f(\mathbf{x}|\theta)$, $\theta \in \Theta$. Let Θ_0 be a proper subset of Θ. Define the likelihood ratio

$$\lambda(\mathbf{x}) = \frac{\sup_{\theta \in \Theta_0} f(\mathbf{x}|\theta)}{\sup_{\theta \in \Theta} f(\mathbf{x}|\theta)}.$$

Then the Likelihood Ratio Test (LRT) of size α for testing $H_0 : \theta \in \Theta_0 \quad vs \quad H_1 : \theta \in \Theta_0^c$ is

$$\phi(\mathbf{x}) = \begin{cases} 1, & \lambda(\mathbf{x}) < k, \\ \gamma, & \lambda(\mathbf{x}) = k, \\ 0, & \lambda(\mathbf{x}) > k, \end{cases}$$

where k and γ satisfy $\sup_{\theta \in \Theta_0} E_\theta[\phi(\mathbf{x})] = \alpha$.

◁ Note:

1. Let $\hat{\theta}_0$ be the MLE of θ under H_0 and $\hat{\theta}$ be the MLE of θ without any restriction. Then,

$$\lambda(\mathbf{x}) = \frac{f(\mathbf{x}|\hat{\theta}_0)}{f(\mathbf{x}|\hat{\theta})}.$$

2. $0 \leq \lambda(\mathbf{x}) \leq 1$.

▷ Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 is known.

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta \neq \theta_0$$

Likelihood ratio tests are especially useful intwo situations:

(i) Two-sided tests

(ii) Tests in the presence of nuisance parameters

Example (Likelihood ratio test for the mean in normal pdf)

$X_1,\cdots,X_n$ (iid) from a $N(\mu, \sigma^2)$ distribution, where $-\infty<\mu<\infty$, $\sigma>0$. Consider the hypotheses

$$H_0 : \mu = \mu_0 \qquad H_1 : \mu \neq \mu_0,$$

where μ_0 is specified. The likelihood function

$$L = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\sum_{i=1}^{n}\frac{(x_i-\overline{x})^2}{2\sigma^2}\right]\exp\left[-\sum_{i=1}^{n}\frac{(\overline{x}-\mu)^2}{2\sigma^2}\right].$$

(1) σ (> 0) is known (p408, 8.37)

$$\Theta = \{\mu : -\infty < \mu < \infty\}; \quad \Theta_0 = \{\mu_0\}.$$

m.l.e in Θ : $\hat{\mu} = \bar{X}$

Restricted m.l.e in Θ_0 : $\hat{\mu} = \mu_0$

$$\lambda = \frac{L(\mu_0)}{L(\bar{X})} = \exp\{-(2\sigma^2)^{-1} n(\bar{X} - \mu_0)^2\}.$$

$$-2\ln\lambda = \frac{n(\bar{X}-\mu_0)^2}{\sigma^2} \triangleq Z^2.$$

$$\lambda \le \lambda_0 \Leftrightarrow |Z| \ge c = \sqrt{-2\ln\lambda_0}$$

$$Z = \frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}} \overset{H_0}{\sim} N(0,1)$$

Therefore the reject region is

$$|Z| = \sqrt{-2\ln\lambda} \ge \Phi^{-1}(1-\frac{\alpha}{2}).$$

(2) σ (> 0) is unknown (p408, 8.37,8.38)

$$\Theta = \{(\mu, \sigma^2): -\infty < \mu < \infty, \sigma^2 > 0\}$$

$$\Theta_0 = \{(\mu_0, \sigma^2): \sigma^2 > 0\}.$$

$$H_0 : \mu_1 = \mu_0, \sigma^2 > 0 \quad vs \quad H_1 : \mu \neq \mu_0, \sigma^2 > 0$$

m.l.e in Θ :

$$\hat{\mu} = \bar{X} \text{ and } \hat{\sigma}^2 = (1/n)\sum_{i=1}^{n}(X_i - \bar{X})^2,$$

$$L(\hat{\mu}, \hat{\sigma}^2) = \frac{1}{(2\pi e \hat{\sigma}^2)^{n/2}}$$

m.l.e in Θ_0 :

$$\hat{\mu}^* = \mu_0 \ and \ \hat{\sigma}^{*2} = (1/n)\sum_{i=1}^{n} (X_i - \mu_0)^2,$$

$$L(\hat{\mu}^*, \hat{\sigma}^{*2}) = \frac{1}{(2\pi e \hat{\sigma}^{*2})^{n/2}}$$

Therefore, the likelihood ratio test statistic

$$\lambda = \left(\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\sum_{i=1}^{n} (X_i - \mu_0)^2} \right)^{n/2} = \left(\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\sum_{i=1}^{n} (X_i - \bar{X})^2 + n(\bar{X} - \mu_0)^2} \right)^{n/2}.$$

$$\lambda \le \lambda_0 \Leftrightarrow \lambda^{-2/n} = 1 + \frac{n(\bar{X} - \mu_0)^2}{\sum_{i=1}^{n} (X_i - \bar{X})^2} \ge c,$$

$$\because T = \frac{\bar{X} - \mu_0}{S / \sqrt{n}} \overset{H_0}{\sim} t(n-1)$$

$\alpha = P\{|T| \ge t_{1\text{-}\alpha/2}(n-1)\} \Leftrightarrow$ Reject H_0 if $|T| \ge t_{1\text{-}\alpha/2}(n-1)$.

Example: (p409,8.41)

$X_1, \cdots, X_n$ iid. $\sim N(\theta_1, \theta_3)$,

$Y_1, \cdots, Y_m$ iid. $\sim N(\theta_2, \theta_3)$,

where $\Theta = \{(\theta_1, \theta_2, \theta_3) : \theta_1, \theta_2 \in R^1, \theta_3 > 0\}$

$$H_0 : \theta_1 = \theta_2, \qquad H_1 : \theta_1 \neq \theta_2.$$

$$L(\theta) = \left(\frac{1}{2\pi\theta_3}\right)^{\frac{n+m}{2}} \exp\{-\frac{1}{2\theta_3}$$

$$[\sum^{n}(x_i - \theta_1)^2 + \sum^{m}(y_i - \theta_2)^2]\}$$

$$\sup_{\theta\in\Theta_0} L(\theta) = \left(\frac{1}{2\pi e\omega}\right)^{\frac{mn}{2}},$$

where

$$\omega = \frac{\sum_{i=1}^{n}(x_i - u)^2 + \sum_{i=1}^{m}(y_i - u)^2}{m+n},$$

$$u = \frac{n\bar{x} + m\bar{y}}{m+n}.$$

$$\sup_{\theta\in\Theta} L(\theta) = \left(\frac{1}{2\pi e\omega'}\right)^{\frac{mn}{2}}$$

with

$$\omega' = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{i=1}^{m}(y_i - \bar{y})^2}{m+n}.$$

$$\lambda = \frac{\sup_{\theta \in \Theta_0} L(\theta)}{\sup_{\theta \in \Theta} L(\theta)} = \left(\frac{\omega'}{\omega} \right)^{\frac{n+m}{2}},$$

$$\frac{\omega'}{\omega} = \frac{(n-1)S_x^2 + (m-1)S_y^2}{(n-1)S_x^2 + (m-1)S_y^2 + \frac{nm}{m+n}(\bar{X} - \bar{Y})^2}$$

$$= \frac{1}{1 + \frac{nm}{m+n} \frac{(\bar{X} - \bar{Y})^2}{(n-1)S_x^2 + (m-1)S_y^2}}$$

$$= \frac{m+n-2}{(m+n-2) + T^2}.$$

where

$$T = \frac{\sqrt{\frac{nm}{m+n}}(\bar{X} - \bar{Y})}{\frac{(n-1)S_x^2 + (m-1)S_y^2}{m+n-2}}.$$

- If H_0 holds,

$$T \sim^{H_0} t(n+m-2).$$

Therefore

$$\lambda \le \lambda_0 \Leftrightarrow |T| \ge c$$

with $\alpha = P(|T| \ge c; H_0)$.

- $n = 10, m = 6, \alpha = 0.05, \Rightarrow c = 2.145$.

Example (Likelihood ratio test for the **variance** in normal pdf)

$X_1, \cdots, X_n$ (iid) from a $N(\mu, \sigma^2)$ distribution, where $-\infty < \mu < \infty$, $\sigma > 0$. Consider the hypotheses

$$H_0 : \sigma^2 = \sigma_0^2 \qquad H_1 : \sigma^2 \neq \sigma_0^2$$

where σ_0 is specified. The likelihood function

$$L = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp\left[-\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2\sigma^2} \right]$$

(1) μ is known

$$\Theta = \{ \sigma^2 : \sigma^2 > 0\}, \quad \Theta_0 = \{\sigma_0^2\}$$

$$\text{m.l.e in } \Theta : \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$$

$$\text{m.l.e in } \Theta_0 : \; \hat{\sigma}^2_{(0)} = \sigma_0^2$$

$$\lambda = \frac{L(\sigma_0^2)}{L(\hat{\sigma}^2)} = \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \exp\left[-\frac{n}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^2} - 1\right)\right] = \left(\frac{Q}{n}\right)^{n/2} \exp\left(-\frac{Q}{2} + \frac{n}{2}\right)$$

$$Q = \frac{n\hat{\sigma}^2}{\sigma_0^2} = \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma_0}\right)^2 \overset{H_0}{\sim} \chi^2(n)$$

$$\frac{d(\ln\lambda(Q))}{dQ}=\frac{n}{2Q}-\frac{1}{2}$$

$$Q<1/n,\ \frac{d(\ln\lambda)}{dQ}>0,\ \lambda(Q)\uparrow;\quad Q>1/n,\ \frac{d(\ln\lambda)}{dQ}<0,\ \lambda(Q)\downarrow$$

Therefore the reject region

$$\lambda(Q)=\left(\frac{Q}{n}\right)^{n/2}\exp\left(-\frac{Q}{2}+\frac{n}{2}\right)\le k\Leftrightarrow Q\le c_1\ or\ Q\ge c_2$$

Let $f(x)$ be pdf of $\chi^2(n)$. Then c_1,c_2 satisfy

$$\begin{cases}\int_{c_1}^{c_2} f(x)dx=1-\alpha\\ \lambda(c_1)=\lambda(c_2)\end{cases}\Rightarrow\begin{cases}\int_{c_1}^{c_2} f(x)dx=1-\alpha\\ c_1^{\,n/2}e^{-c_1/2}=c_2^{\,n/2}e^{-c_2/2}\end{cases}$$

For convenience, we take c_1,c_2 as $\chi^2_{\alpha/2}(n)$, $\chi^2_{1-\alpha/2}(n)$

(2) μ is unknown

$$\Theta = \{(\mu, \sigma^2): -\infty < \mu < \infty, \sigma^2 > 0\}$$

$$\Theta_0 = \{(\mu, \sigma_0^2): -\infty < \mu < \infty\}.$$

m.l.e in Θ : $\hat{\mu} = \bar{X}$ *and* $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2,$

m.l.e in Θ_0 : $\hat{\mu} = \bar{X}$ *and* $\hat{\sigma}^2_{(0)} = \sigma_0^2,$

$$\lambda = \frac{L(\bar{X}, \sigma_0^2)}{L(\bar{X}, \hat{\sigma}^2)} = \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \exp\left[-\frac{n}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^2} - 1\right)\right] = \left(\frac{Q}{n}\right)^{n/2} \exp\left(-\frac{Q}{2} + \frac{n}{2}\right)$$

$$Q = \frac{n\hat{\sigma}^2}{\sigma_0^2} = \frac{1}{\sigma_0^2}\sum_{i=1}^{n}(X_i - \bar{X})^2 \overset{H_0}{\sim} \chi^2(n-1)$$

Similarly, the reject region

$$\lambda(Q) = \left(\frac{Q}{n}\right)^{n/2} \exp\left(-\frac{Q}{2} + \frac{n}{2}\right) \le k \Leftrightarrow Q \le c_1 \ or \ Q \ge c_2$$

Let $f(x)$ be pdf of $\chi^2(n-1)$. Then c_1, c_2 satisfy

$$\begin{cases} \int_{c_1}^{c_2} f(x)dx = 1-\alpha \\ \lambda(c_1) = \lambda(c_2) \end{cases} \Rightarrow \begin{cases} \int_{c_1}^{c_2} f(x)dx = 1-\alpha \\ {c_1}^{n/2} e^{-c_1/2} = {c_2}^{n/2} e^{-c_2/2} \end{cases}$$

For convenience, we take c_1, c_2 as $\chi^2{}_{\alpha/2}(n-1)$, $\chi^2{}_{1-\alpha/2}(n-1)$

p-values

Instead of reporting that we reject or do not reject a null hypothesis, we can report what is called the p-value of the test.

The p-value is the probability (under the null hypothesis) of seeing a value of the test statistic as extreme as the one that we have observed.

The p-value is also the smallest possible level at which the null hypothesis would be rejected. Thus it gives more information than just that we reject or don't reject at a certain level. It tells us how significant the evidence was in the sample.

If we reject the null hypothesis for large values of the test statistic, then the p-value is a statistic defined by

$$p(\mathbf{x}) = \sup_{\theta \in \Theta_0} P_\theta\left(W(\mathbf{X}) \geq W(\mathbf{x})\right),$$

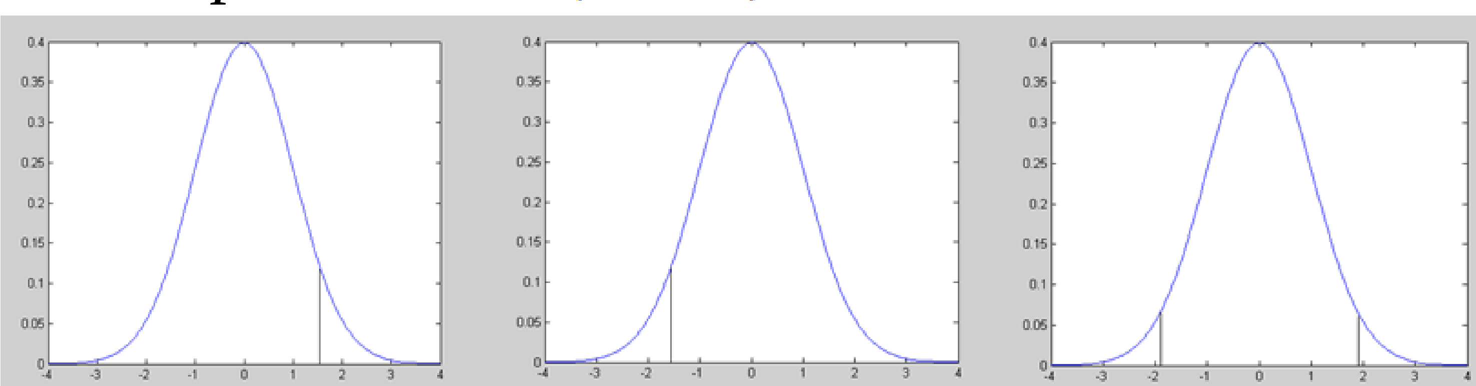
where $W(\mathbf{x})$ is the observed value of the test statistic.

Whereas the choice of the level α is subjective, one can report the p-value associated with the test and allow the user to determine whether the results are significant enough to reject H_0.

Example

Let $X_1, \ldots, X_n$ be a random sample from $N\left(\mu, \sigma^2\right)$, where σ^2 is known. Suppose that we are testing $H_0 : \mu \le \mu_0$ against $H_1 : \mu > \mu_0$ and we observe $\dfrac{\left(\bar{x} - \mu_0\right)}{\sigma / \sqrt{n}} = 3$, then the p-value is $P\left(Z > 3\right) = .0013.$

If $\dfrac{\left(\bar{x} - \mu_0\right)}{\sigma / \sqrt{n}} = 2$, then the p-value is $P\left(Z > 2\right) = .0228.$ If $\dfrac{\left(\bar{x} - \mu_0\right)}{\sigma / \sqrt{n}} = 1.5$, then the p-value is $P(Z > 1.5) = 0.0668$



$H_1 : \mu > \mu_0$	$H_1 : \mu < \mu_0$	$H_1 : \mu \ne \mu_0$
$P_{\mu_0}\{Z > z\}$	$P_{\mu_0}\{Z < z\}$	$P_{\mu_0}\{\lvert Z\rvert < \lvert z\rvert\}$

Bayesian tests

From a Bayesian point of view, θ is considered as random. We can then consider

$$P\left(\theta \in \Theta_0 \middle| \mathbf{x}\right) \text{ and } P\left(\theta \in \Theta_0^c \middle| \mathbf{x}\right)$$

using the posterior distribution.

One approach to Bayesian hypothesis testing is to reject the null hypothesis if

$$P\left(\theta \in \Theta_0^c \middle| \mathbf{x}\right) > P\left(\theta \in \Theta_0 \middle| \mathbf{x}\right).$$

This is the same as $P\left(\theta \in \Theta_0^c \middle| \mathbf{x}\right) > 1/2$.

Note that $P\left(\theta \in \Theta_0^c \middle| \mathbf{x}\right) = \int_{\Theta_0^c} \pi\left(\theta \middle| \mathbf{x}\right) d\theta$.

From a frequentist point of view, probabilities

$$P\left(\theta \in \Theta_0 \middle| \mathbf{x}\right) \text{ and } P\left(\theta \in \Theta_0^c \middle| \mathbf{x}\right)$$

aren't very meaningful because θ is considered as fixed, so that either the null hypothesis is true or it isn't (and we don't know the value of the parameter to determine which is correct).

Example

Let $X_1,\ldots,X_n$ be i.i.d. $N(\mu,\sigma^2)$ (σ^2 known) and $\pi(\mu) \sim N\left(\theta,\tau^2\right)$.

We have shown that $\pi\left(\mu|\mathbf{x}\right) \sim N\left(\dfrac{n\tau^2\overline{x}+\sigma^2\theta}{n\tau^2+\sigma^2},\dfrac{\sigma^2\tau^2}{n\tau^2+\sigma^2}\right)$.

We test $H_0:\mu \le \mu_0$ vs. $H_1:\mu > \mu_0$.

We reject if $P\left(\mu > \mu_0|\mathbf{x}\right) > 1/2$.

If $\mu_0 > \dfrac{n\tau^2\overline{x} + \sigma^2\theta}{n\tau^2 + \sigma^2}$ then the area under the curve to the right of μ_0 is less than 1/2.

If $\mu_0 < \dfrac{n\tau^2\overline{x} + \sigma^2\theta}{n\tau^2 + \sigma^2}$ then the area under the curve to the right of μ_0 is greater than 1/2.

So we reject H_0 if

$$\mu_0 < \frac{n\tau^2\overline{x} + \sigma^2\theta}{n\tau^2 + \sigma^2} \Leftrightarrow n\tau^2\mu_0 + \sigma^2\mu_0 < n\tau^2\overline{x} + \sigma^2\theta$$

$$\Leftrightarrow \mu_0 + \frac{\sigma^2(\mu_0 - \theta)}{n\tau^2} < \overline{x}.$$

(Note, if our prior mean was $\theta = \mu_0$, then a priori we would be putting equal weight on H_0 and H_1. We would reject H_0 if $\overline{x} > \mu_0$.)

Homework: p402~412

8.6 8.10 8.12 8.19 8.29 8.33 8.39