Chapter 11

Analysis of Variance (ANOVA)

Outline

- **One-way ANOVA**
  - Model assumption
  - Inferences regarding linear combinations of means
  - ANOVA F test
  - Partioning of Sums of Squares
- **Two-way ANOVA**
  - Without interactions
  - With interactions

- Fish toxin data

ONEWAY ANALYSIS OF VARIANCE

Toxin 1	Toxin 2	Toxin 3	Control
28	33	18	11
23	36	21	14
14	34	20	11
27	29	22	16
	31	24	
	34		

One-way ANOVA

- Model assumption

$$Y_{i1}, Y_{i2}, \cdots, \ Y_{i,n_i} \sim N(\theta_i, \sigma^2), \quad i = 1, 2, \ldots, k.$$

$$\{Y_{ij}, i = 1, 2, \cdots, k, \ j = 1, 2, \cdots, n_i\} \text{ are independent.}$$

$$\Leftrightarrow \quad Y_{ij} = \theta_i + \varepsilon_{ij},$$

$$\varepsilon_{ij} \overset{i.i.d}{\sim} N(0, \sigma^2), \quad i = 1, 2, \cdots, k, \ j = 1, 2, \cdots, n_i$$

$$\Leftrightarrow \quad Y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \ \ with \sum_{i=1}^{k} \tau_i = 0$$

$$\varepsilon_{ij} \overset{i.i.d}{\sim} N(0, \sigma^2), \quad i = 1, 2, \cdots, k, \ j = 1, 2, \cdots, n_i$$

- Relaxed assumption

  $\mathrm{E}\epsilon_{ij} = 0, \mathrm{Var}\,\epsilon_{ij} = \sigma_i^2 < \infty$, for all i, j. $\mathrm{Cov}(\epsilon_{ij}, \epsilon_{i'j'}) = 0$

  for all i, i', j, and j' unless $i = i'$ and $j = j'$.

- Could do only point estimation, e.g. LSE

Treatments				
1	2	3	$\ldots$	k
y_{11}	y_{21}	y_{31}	$\cdots$	y_{k1}
y_{12}	y_{22}	y_{32}	$\cdots$	y_{k2}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	y_{k3}
				$\vdots$
		y_{3n_3}		
y_{1n_1}				
	y_{2n_2}			y_{kn_k}

LSE estimators

$$Y_{ij} = \theta_i + \varepsilon_{ij}$$

- Minimize

$$SSE = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \hat{Y}_{ij})^2 = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \theta_i)^2$$

- LSE estimators

$$\hat{\theta}_i = \bar{Y}_{i\bullet} = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij}, \qquad i = 1, 2, \cdots, k$$

MLE estimators for normal error model

- For normal error model, the likelihood

$$L = (\sqrt{2\pi\sigma^2})^{-N} \exp\left(-\sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{(y_{ij}-\theta_i)^2}{2\sigma^2} \right)$$

$$l = \ln L = -\frac{N}{2}\ln 2\pi\sigma^2 - \sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{(y_{ij}-\theta_i)^2}{2\sigma^2}$$

where $N = n_1 + n_2 + \ldots + n_k$

- MLE estimators for the parameters

$$\hat{\theta}_i = \bar{Y}_{i\bullet} = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij}, \quad \hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij}-\bar{Y}_{i\bullet})^2 = \frac{SS_W}{N}$$

- Unbiased or biased?

$$\bar{Y}_{i\bullet} \sim N(\theta_i, \sigma^2 / n_i), \qquad E(\hat{\theta}_i) = E(\bar{Y}_{i\bullet}) = \theta_i \quad \boxed{\text{unbiased}}$$

$$\frac{(n_i - 1)S_i^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i\bullet})^2 \sim \chi^2(n_i - 1), \quad \{S_i^2, \mathrm{i} = 1, \ldots, \mathrm{k}\} \textit{ are } \text{independent}$$

$$\Rightarrow \frac{SS_W}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i\bullet})^2 \sim \chi^2(N - k)$$

$$E(\hat{\sigma}^2) = \frac{E(SS_W)}{N} = \frac{N-k}{N}\sigma^2 \quad \boxed{\text{biased}}$$

MS$_\mathrm{w}$

Unbiased: $\hat{\sigma}^2_{\mathrm{ub}} = \dfrac{SS_W}{N-k} = \dfrac{1}{N-k}\displaystyle\sum_{i=1}^{k}(n_i - 1)S_i^2 \triangleq S_p^2$

Confidence Intervals

$\bar{Y}_{i\bullet}$ is independent with S_i^2, *and* SS_W as well.

$$\frac{\bar{Y}_{i\bullet}-\theta_i}{S_p/\sqrt{n_i}}=\frac{(\bar{Y}_{i\bullet}-\theta_i)/\sqrt{\sigma^2/n_i}}{\sqrt{(SS_w/\sigma^2)/(N-k)}}\sim t(N-k)$$

$$P\left(\left|\frac{\bar{Y}_{i\bullet}-\theta_i}{S_p/\sqrt{n_i}}\right|<t_{N-k,1-\alpha/2}\right)=1-\alpha \quad \Rightarrow$$

CI for θ_i: $\quad \bar{Y}_{i\bullet}\pm t_{N-k,1-\alpha/2}S_p/\sqrt{n_i}$

$$\frac{SS_W}{\sigma^2}\sim\chi^2(N-k),$$

$$P\{\chi^2_{N-k,\alpha/2}<\frac{SS_W}{\sigma^2}<\chi^2_{N-k,1-\alpha/2}\}=1-\alpha \quad \Rightarrow$$

CI for σ^2: $\quad \left(\frac{SS_W}{\chi^2_{N-k,1-\alpha/2}},\frac{SS_W}{\chi^2_{N-k,\alpha/2}}\right)$

$$Y_{ij} = \theta_i + \varepsilon_{ij} \Leftrightarrow Y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \ \ with \sum_{i=1}^{k} \tau_i = 0$$

- The relationship

$$\mu = \frac{1}{k}\sum_{i=1}^{k} \theta_i, \quad \tau_i = \theta_i - \mu$$

- MLE or LSE estimators

$$\hat{\mu} = \frac{1}{k}\sum_{i=1}^{k} \overline{Y}_{i\bullet}, \qquad \hat{\tau}_i = \overline{Y}_{i\bullet} - \hat{\mu}$$

ANOVA hypothesis

$$H_0: \quad \theta_1 = \theta_2 = \cdots = \theta_k,$$

Theorem 11.2.5 *Let* $\theta = (\theta_1, \ldots, \theta_k)$ *be arbitrary parameters. Then*

$$\theta_1 = \theta_2 = \cdots = \theta_k \Leftrightarrow \sum_{i=1}^{k} a_i \theta_i = 0 \quad \textit{for all } \mathbf{a} \in \mathcal{A},$$

where $\mathcal{A}$ *is the set of constants satisfying* $\mathcal{A} = \{\mathbf{a} = (a_1, \ldots, a_k) : \sum a_i = 0\}$; *that is, all contrasts must satisfy* $\sum a_i \theta_i = 0$.

$$H_0: \quad \theta_1 = \theta_2 = \cdots = \theta_k,$$

$$H_1: \quad \theta_i \neq \theta_j, \text{ for some } i, j.$$

$$\Updownarrow$$

$$H_0: \quad \sum_{i=1}^{k} a_i\theta_i = 0 \quad \text{for all } (a_1, \ldots, a_k) \text{ such that } \sum_{i=1}^{k} a_i = 0$$

$$H_1: \quad \sum_{i=1}^{k} a_i\theta_i \neq 0 \quad \text{for some } (a_1, \ldots, a_k) \text{ such that } \sum_{i=1}^{k} a_i = 0.$$

Inferences regarding linear combinations of means

$$\bar{Y}_{i\bullet} \sim N(\theta_i, \sigma^2 / n_i),$$

- For constants **a**=(a_1, a_2,…, a_k),

$$E\left(\sum_{i=1}^{k} a_i \bar{Y}_{i\bullet}\right) = \sum_{i=1}^{k} a_i \theta_i, \quad \mathrm{var}\left(\sum_{i=1}^{k} a_i \bar{Y}_{i\bullet}\right) = \sigma^2 \sum_{i=1}^{k} (a_i^2 / n_i$$

$$\frac{\sum_{i=1}^{k} a_i \bar{Y}_{i\bullet} - \sum_{i=1}^{k} a_i \theta_i}{\sqrt{\sigma^2 \sum_{i=1}^{k} (a_i^2 / n_i)}} \sim N(0,1)$$

$$\frac{N-k}{\sigma^2} S_p^2 = \frac{SS_W}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i\bullet})^2 \sim \chi^2(N-k)$$

$$T_{\mathbf{a}} = \frac{\sum_{i=1}^{k} a_i \bar{Y}_{i\bullet} - \sum_{i=1}^{k} a_i \theta_i}{\sqrt{S_p^2 \sum_{i=1}^{k} (a_i^2 / n_i)}} = \frac{\left(\sum_{i=1}^{k} a_i \bar{Y}_{i\bullet} - \sum_{i=1}^{k} a_i \theta\right) / \sqrt{\sigma^2 \sum_{i=1}^{k} (a_i^2 / n_i)}}{\sqrt{(SS_w / \sigma^2) / (N-k)}} \sim t(N-k)$$

- CI for $\sum_{i=1}^{k} a_i \theta_i$

$$\sum_{i=1}^{k} a_i \bar{Y}_{i\bullet} \pm t_{N-k,1-\alpha/2} S_p \sqrt{\sum_{i=1}^{k} (a_i^2 / n_i)}$$

- Testing for $H_0 : \sum_{i=1}^{k} a_i \theta_i = 0 \quad H_1 : \sum_{i=1}^{k} a_i \theta_i \neq 0$

$$T_{\mathbf{a}}^{*} = \frac{\sum_{i=1}^{k} a_i \bar{Y}_{i\bullet}}{\sqrt{S_p^2 \sum_{i=1}^{k} (a_i^2 / n_i)}} \overset{H_0}{\sim} t(N-k)$$

Rejection region: $|T_{\mathbf{a}}^{*}| > t_{N-k,1-\alpha/2}$

- **Example 1:** $H_0:\theta_1=\theta_2 \quad H_1:\theta_1\neq\theta_2$

  $\mathbf{a}=(1, -1, 0,\ldots, 0)$

$$T_1 \triangleq T_{\mathbf{a}}^* = \frac{\sum_{i=1}^{k} a_i \bar{Y}_{i\cdot}}{\sqrt{S_p^2 \sum_{i=1}^{k} (a_i^2/n_i)}} = \frac{\bar{Y}_{1\cdot}-\bar{Y}_{2\cdot}}{\sqrt{S_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \overset{H_0}{\sim} t(N-k)$$

Rejection region: $|T_1|>t_{N-k,1-\alpha/2}$

The difference between this test and the two-sample t test?

- **Example 2**:

$$H_0: \theta_1 = \frac{1}{2}(\theta_2 + \theta_3) \quad H_1: \theta_1 \neq \frac{1}{2}(\theta_2 + \theta_3)$$

$\mathbf{a}$=(1, –0.5, –0.5, 0,…, 0)

$$T_2 \triangleq T_{\mathbf{a}}^* = \frac{\sum_{i=1}^{k} a_i \bar{Y}_{i\cdot}}{\sqrt{S_p^2 \sum_{i=1}^{k} (a_i^2 / n_i)}} = \frac{\bar{Y}_{1\cdot} - 0.5\bar{Y}_{2\cdot} - 0.5\bar{Y}_{3\cdot}}{\sqrt{S_p^2 \left(\frac{1}{n_1} + \frac{1}{4n_2} + \frac{1}{4n_3} \right)}} \overset{H_0}{\sim} t(N-k)$$

Rejection region:

$$|T_2| > t_{N-k, 1-\alpha/2}$$

- **Example 3:**

$$H_0: \theta_1 = \theta_2 \text{ and } \theta_1 = \frac{1}{2}(\theta_2 + \theta_3) \quad H_1: \theta_1 \neq \theta_2 \text{ or } \theta_1 \neq \frac{1}{2}(\theta_2 + \theta_3)$$

$$\Leftrightarrow \quad H_0: \theta_1 = \theta_2 = \theta_3, \qquad H_1: \theta_1 \neq \theta_2 \text{ or } \theta_1 \neq \theta_3 \text{ or } \theta_2 \neq \theta_3$$

Rejection region:

$$|T_1| > c, \ \ or \ \ |T_2| > c \ \Leftrightarrow \ \max\left(|T_1|, |T_2|\right) > c$$

c is determined by the distribution of $\max(|T_1|,|T_2|)$, and α (the size of the test, or the type I error).

Page 380: Union-intersection and Intersection-Union tests

$$H_{0\gamma}: \theta \in \Theta_\gamma \text{ versus } H_{1\gamma}: \theta \in \Theta_\gamma^c.$$

rejection region for the test of $H_{0\gamma}$ is $\{\mathbf{x}: T_\gamma(\mathbf{x}) \in R_\gamma\}$.

(1) Union-intersection test

$$H_0: \theta \in \bigcap_{\gamma \in \Gamma} \Theta_\gamma.$$

Here Γ is an arbitrary index set that may be finite or infinite

Then the rejection region for the union–intersection test is

$$\bigcup_{\gamma \in \Gamma} \{\mathbf{x}: T_\gamma(\mathbf{x}) \in R_\gamma\}.$$

In some situations a simple expression for the rejection region

$$\bigcup_{\gamma\in\Gamma}\{\mathbf{x}: T_\gamma(\mathbf{x}) > c\} = \{\mathbf{x}: \sup_{\gamma\in\Gamma} T_\gamma(\mathbf{x}) > c\}.$$

(2) Intersection-union test

$$H_0: \theta \in \bigcup_{\gamma\in\Gamma} \Theta_\gamma.$$

Then the rejection region $\bigcap_{\gamma\in\Gamma}\{\mathbf{x}: T_\gamma(\mathbf{x}) \in R_\gamma\}.$

In some situations a simple expression for the rejection region

$$\bigcap_{\gamma\in\Gamma}\{\mathbf{x}: T_\gamma(\mathbf{x}) \geq c\} = \{\mathbf{x}: \inf_{\gamma\in\Gamma} T_\gamma(\mathbf{x}) \geq c\}.$$

ANOVA F test

$$H_0: \quad \theta_1 = \theta_2 = \cdots = \theta_k,$$

$$\Longleftrightarrow \quad H_0: \quad \sum_{i=1}^{k} a_i\theta_i = 0 \text{ for all } \mathbf{a} \in \mathcal{A}$$

where $\mathcal{A} = \{\mathbf{a} = (a_1, \ldots, a_k) : \sum_{i=1}^{k} a_i = 0\}$.

$$\Longleftrightarrow \quad H_0: \theta \in \bigcap_{\mathbf{a} \in \mathcal{A}} \Theta_{\mathbf{a}},$$

$$\Theta_{\mathbf{a}} = \{\theta = (\theta_1, \ldots, \theta_k): \sum_{i=1}^{k} a_i\theta_i = 0\}.$$

Union-intersection test

- For any given **a**,

$$H_{0_{\mathbf{a}}}: \quad \theta \in \Theta_{\mathbf{a}} \qquad \text{versus} \qquad H_{1_{\mathbf{a}}}: \quad \theta \notin \Theta_{\mathbf{a}}$$

- Test statistics

$$T_{\mathbf{a}}^{*} = \frac{\sum_{i=1}^{k} a_i \bar{Y}_{i\cdot}}{\sqrt{S_p^2 \sum_{i=1}^{k} (a_i^2 / n_i)}} \overset{H_0}{\sim} t(N-k)$$

Rejection region: $|T_{\mathbf{a}}^{*}| > t_{N-k,1-\alpha/2}$

Thus, rejection region for $H_0: \theta \in \bigcap_{\mathbf{a} \in \mathcal{A}} \Theta_{\mathbf{a}}$,

$$\sup_{\mathbf{a} \in A} |T_{\mathbf{a}}^{*}| > c \quad \text{or} \quad \sup_{\mathbf{a} \in A} T_{\mathbf{a}}^{*2} > c$$

Lemma 11.2.7 *Let $(v_1, \ldots, v_k)$ be constants and let $(c_1, \ldots, c_k)$ be positive constants. Then, for $\mathcal{A} = \{\mathbf{a} = (a_1, \ldots, a_k) : \sum a_i = 0\}$,*

$$\max_{\mathbf{a} \in \mathcal{A}} \left\{ \frac{\left(\sum_{i=1}^k a_i v_i\right)^2}{\sum_{i=1}^k a_i^2 / c_i} \right\} = \sum_{i=1}^k c_i (v_i - \bar{v}_c)^2,$$

where $\bar{v}_c = \sum c_i v_i / \sum c_i$. The maximum is attained at any $\mathbf{a}$ of the form $a_i = K c_i (v_i - \bar{v}_c)$, where K is a nonzero constant.

Note that

$$T_{\mathbf{a}} = \frac{\sum_{i=1}^k a_i \bar{Y}_{i\bullet} - \sum_{i=1}^k a_i \theta_i}{\sqrt{S_p^2 \sum_{i=1}^k (a_i^2 / n_i)}} = \frac{\sum_{i=1}^k a_i \bar{U}_i}{\sqrt{S_p^2 \sum_{i=1}^k (a_i^2 / n_i)}}$$

$$T_{\mathbf{a}}^2 = \frac{\left(\sum_{i=1}^k a_i \bar{U}_i\right)^2}{S_p^2 \sum_{i=1}^k (a_i^2 / n_i)}, \quad where\ \bar{U}_i = \bar{Y}_{i\bullet} - \theta_i$$

- **Theorem 11.2.8.**

$$\max_{\mathbf{a}\in A} T_{\mathbf{a}}^2 = \frac{\sum_{i=1}^{k} n_i\left((\bar{Y}_{i\bullet} - \bar{\bar{Y}}) - (\theta_i - \bar{\theta})\right)^2}{S_p^2} \sim (k-1)F(k-1, N-k)$$

$$\text{where } \bar{\bar{Y}} = \frac{1}{N}\sum_{i=1}^{k} n_i \bar{Y}_{i\bullet}, \quad \bar{\theta} = \frac{1}{N}\sum_{i=1}^{k} n_i\theta_i$$

Proof: $$\frac{N-k}{\sigma^2}S_p^2 = \frac{SS_W}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i\bullet})^2 \sim \chi^2(N-k)$$

$$\text{Let } \bar{U}_i = \bar{Y}_{i\bullet} - \theta_i, \bar{\bar{U}} = \frac{1}{N}\sum_{i=1}^{k} n_i \bar{U}_i$$

$$V_i = \sqrt{n_i}\bar{U}_i, \quad \bar{V} = \frac{1}{k}\sum_{i=1}^{k} V_i$$

$$\bar{U}_i \sim N(0, \sigma^2 / n_i) \qquad V_i \mathrel{\hat{=}} \sqrt{n_i}\,\bar{U}_i \overset{i.i.d}{\sim} N(0, \sigma^2)$$

$$\sum_{i=1}^{k} n_i \left((\bar{Y}_{i\bullet} - \bar{\bar{Y}}) - (\theta_i - \bar{\theta}) \right)^2 = \sum_{i=1}^{k} n_i \left(\bar{U}_i - \bar{\bar{U}} \right)^2$$

$$= \sum_{i=1}^{k} \left(V_i - \bar{V} \right)^2 \sim \sigma^2 \chi^2 (k-1)$$

- The denominator and the numerator are independent.

$$\frac{1}{k-1} \max_{\mathbf{a} \in A} T_{\mathbf{a}}^2 = \frac{\dfrac{1}{\sigma^2} \sum_{i=1}^{k} n_i \left((\bar{Y}_{i\bullet} - \bar{\bar{Y}}) - (\theta_i - \bar{\theta}) \right)^2 / (k-1)}{\dfrac{SS_W}{\sigma^2} / (N-k)}$$

$$\sim F(k-1, N-k)$$

ANOVA F test

$$H_0: \quad \theta_1 = \theta_2 = \cdots = \theta_k,$$

$$H_1: \quad \theta_i \neq \theta_j, \text{ for some } i, j.$$

- Test statistics

$$\max_{\mathbf{a}\in A} T_{\mathbf{a}}^{*2} = \frac{\sum_{i=1}^{k} n_i \left(\bar{Y}_{i\bullet} - \bar{\bar{Y}}\right)^2}{S_p^2} \overset{H_0}{\sim} (k-1)F(k-1, N-k)$$

or

$$F = \frac{1}{k-1} \max_{\mathbf{a}\in A} T_{\mathbf{a}}^{*2} = \frac{SS_B/(k-1)}{SS_W/(N-k)} \overset{H_0}{\sim} F(k-1, N-k)$$

$$F = \frac{SS_B / (k-1)}{SS_W / (N-k)} = \frac{MS_B}{MS_W} \overset{H_0}{\sim} F(k-1, N-k)$$

$$where\ SS_B = \sum_{i=1}^{k} n_i \left(\bar{Y}_{i\bullet} - \bar{\bar{Y}} \right)^2 \overset{H_0}{\sim} \sigma^2 \chi^2 (k-1)$$

$$SS_W = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i\bullet})^2 \sim \sigma^2 \chi^2 (N-k)$$

- Reject H_0 if $F > F_{k-1, N-k, 1-\alpha}$

Simultaneous Estimation of Contrasts

Pairwise contrasts

(1) CI for estimating single parameter $\theta_i - \theta_j$

$\mathbf{a}=(a_1, a_2, \ldots, a_k)$ with $a_i=1$, $a_j=-1$, $a_l=0$ $(l \neq i, j)$

$$T_{ij} \triangleq T_{\mathbf{a}} = \frac{\sum_{i=1}^{k} a_i \bar{Y}_{i\bullet} - \sum_{i=1}^{k} a_i \theta_i}{\sqrt{S_p^2 \sum_{i=1}^{k} (a_i^2 / n_i)}} = \frac{(\bar{Y}_{i\bullet} - \bar{Y}_{j\bullet}) - (\theta_i - \theta_j)}{\sqrt{S_p^2 \left(\frac{1}{n_i} + \frac{1}{n_j} \right)}} \sim t(N-k)$$

$$P\left(\left| T_{ij} \right| < t_{N-k, 1-\alpha/2} \right) = 1 - \alpha \quad \Rightarrow$$

$1-\alpha$ CI for $\theta_i - \theta_j$: $\quad (\bar{Y}_{i\bullet} - \bar{Y}_{j\bullet}) \pm t_{N-k, 1-\alpha/2} S_p \sqrt{\left(\frac{1}{n_i} + \frac{1}{n_j} \right)}$

Let $C_{ij} = \left\{ \theta_i - \theta_j \in (\bar{Y}_{i\bullet} - \bar{Y}_{j\bullet}) \pm t_{N-k,1-\alpha/2} S_p \sqrt{1/n_i + 1/n_j} \right\}$

Then $P(C_{ij}) = 1 - \alpha$

(2) Bonferroni's CIs for simultaneously estimating several parameters $\{\theta_i - \theta_j\}$

- Bonferroni Inequality

$$P\left(\bigcap_{i=1}^{m} A_i\right) = 1 - P\left(\bigcup_{i=1}^{m} \bar{A}_i\right) \geq 1 - \sum_{i=1}^{m} P(\bar{A}_i) = \sum_{i=1}^{m} P(A_i) - (m-1)$$

- E.g. The family confidence level for the CIs C_{12} , C_{23} for estimating $\theta_1 - \theta_2$, $\theta_2 - \theta_3$ is

$$P(C_{12} \,\&\, C_{23}) \geq 2(1-\alpha) - 1 = 1 - 2\alpha$$

- For m events C_{ij}s

$$P\left(\bigcap_{ij} C_{ij}\right) \geq \sum_{i,j} P(C_{ij}) - (m-1) = m(1-\alpha) - (m-1) = 1 - m\alpha$$

Bonferroni method

- If we want to find m joint CIs for m parameters $\theta_i - \theta_j$, which satisfies

$$P\left(\bigcap_{ij} C_{ij}\right) \geq 1 - \alpha^*$$

we can construct each single 1–α CI, where $\boxed{\alpha = \alpha^* / m.}$

- Using Bonferroni method to construct CIs for $\theta_1 - \theta_2$ and $\theta_2 - \theta_3$ with family confidence level 95%, we need set-up 97.5% confidence interval for each $\theta_1 - \theta_2$ and $\theta_2 - \theta_3$.

(3) Scheffe's CIs for simultaneously estimating infinite parameters $\left\{\sum_{i=1}^{k} a_i\theta_i\right\}$

Theorem 11.2.10 Under the ANOVA assumption (normal + homoscedasticity),

$$P\left(\bigcap_{\mathbf{a}\in A} C_{\mathbf{a}}\right) = 1-\alpha$$

where

$$C_{\mathbf{a}} = \left\{\sum_{i=1}^{k} a_i\theta_i \in \sum_{i=1}^{k} a_i\bar{Y}_{i\bullet} \pm M\sqrt{S_p^2\sum_{i=1}^{k}(a_i^2/n_i)}\right\}$$

$$M = \sqrt{(k-1)F_{k-1,N-k,1-\alpha}}$$

- Proof:

$$T_{\mathbf{a}} = \frac{\sum_{i=1}^{k} a_i \bar{Y}_{i\bullet} - \sum_{i=1}^{k} a_i \theta_i}{\sqrt{S_p^2 \sum_{i=1}^{k} (a_i^2 / n_i)}} \sim t(N-k)$$

$$\max_{\mathbf{a} \in A} T_{\mathbf{a}}^2 \sim (k-1) F(k-1, N-k)$$

$$M = \sqrt{(k-1) F_{k-1, N-k, 1-\alpha}}$$

$$P\left(\bigcap_{\mathbf{a} \in A} C_{\mathbf{a}}\right) = P\left(\bigcap_{\mathbf{a} \in A} \{ |T_{\mathbf{a}}| < M \}\right) = P\left(T_{\mathbf{a}}^2 < M^2, \ for\ all\ \mathbf{a} \in A\right)$$

$$= P\left(\max_{\mathbf{a} \in A} T_{\mathbf{a}}^2 < M^2\right) = P\left(\frac{1}{k-1} \max_{\mathbf{a} \in A} T_{\mathbf{a}}^2 < F_{k-1, N-k, 1-\alpha}\right) = 1 - \alpha$$

- Scheffe's CIs for for simultaneously estimating all pairwise contrasts $\{\theta_i - \theta_j\}$

$$(\bar{Y}_{i\bullet} - \bar{Y}_{j\bullet}) \pm M S_p \sqrt{1/n_i + 1/n_j}$$

$$\text{where } M = \sqrt{(k-1)F_{k-1, N-k, 1-\alpha}}$$

- Compare: Bonferroni's CIs for simultaneously estimating all pairwise contrasts $\{\theta_i - \theta_j\}$

$$(\bar{Y}_{i\bullet} - \bar{Y}_{j\bullet}) \pm B S_p \sqrt{1/n_i + 1/n_j}$$

$$\text{where } B = t_{N-k, 1-\alpha/(2m)},$$

m is number of all pairwise comparisons.

Partitioning of Sum of Squares

$$SS_T = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \bar{\bar{Y}})^2 = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i\bullet} + \bar{Y}_{i\bullet} - \bar{\bar{Y}})^2$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i\bullet})^2 + \sum_{i=1}^{k} n_i \left(\bar{Y}_{i\bullet} - \bar{\bar{Y}}\right)^2$$

$$= SS_W + SS_B$$

ONEWAY ANALYSIS OF VARIANCE

Toxin 1	Toxin 2	Toxin 3	Control
28	33	18	11
23	36	21	14
14	34	20	11
27	29	22	16
	31	24	
	34		

SST=SSW + SSB

For normal model,

$$SS_T = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \bar{\bar{Y}})^2 \sim \sigma^2\chi^2(N-1,\delta)$$

$$N = n_1 + n_2 + \ldots + n_k, \quad \delta = \frac{1}{\sigma^2}\sum_{i=1}^{n} n_i(\theta_i - \bar{\theta})^2$$

$$SS_W = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_{i\bullet})^2 \sim \sigma^2\chi^2(N-k,0)$$

$$SS_B = \sum_{i=1}^{k} n_i\left(\bar{Y}_{i\bullet} - \bar{\bar{Y}}\right)^2 \sim \sigma^2\chi^2(k-1,\delta)$$

SS_B is independent with SS_W, and

$$F = \frac{SS_B/(k-1)}{SS_W/(N-k)} = \frac{MS_B}{MS_W} \sim F(k-1, N-k, \delta)$$

$$H_0: \quad \theta_1 = \theta_2 = \cdots = \theta_k, \qquad H_1: \quad \theta_i \neq \theta_j, \text{ for some } i, j.$$

$$SS_T = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{\bar{Y}})^2 \overset{H_0}{\sim} \sigma^2 \chi^2 (N-1)$$

$$SS_W = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i\bullet})^2 \sim \sigma^2 \chi^2 (N-k)$$

$$SS_B = \sum_{i=1}^{k} n_i \left(\bar{Y}_{i\bullet} - \bar{\bar{Y}} \right)^2 \overset{H_0}{\sim} \sigma^2 \chi^2 (k-1)$$

$$F = \frac{SS_B / (k-1)}{SS_W / (N-k)} = \frac{MS_B}{MS_W} \overset{H_0}{\sim} F(k-1, N-k)$$

- Reject H_0 if $F > F_{k-1, N-k, 1-\alpha}$

Table 11.2.1. *ANOVA table for oneway classification*

Source of variation	Degrees of freedom	Sum of squares	Mean square	F statistic
Between treatment groups	$k-1$	SSB = $\sum n_i(\bar{y}_i - \bar{\bar{y}})^2$	MSB = SSB/$(k-1)$	$F = \frac{\text{MSB}}{\text{MSW}}$
Within treatment groups	$N-k$	SSW = $\sum\sum(y_{ij} - \bar{y}_{i\cdot})^2$	MSW = SSW/$(N-k)$	
Total	$N-1$	SST = $\sum\sum(y_{ij} - \bar{\bar{y}})^2$		

- For fish toxin data

Source of variation	Degrees of freedom	Sum of squares	Mean square	F statistic
Treatments	3	995.90	331.97	26.09
Within	15	190.83	12.72	
Total	18	1,186.73		

Reject H_0

Likelihood ration test (LRT)

$H_0: \quad \theta_1 = \theta_2 = \cdots = \theta_k, \qquad H_1: \quad \theta_i \neq \theta_j, \text{ for some } i, j.$

$$L = (\sqrt{2\pi\sigma^2})^{-N} \exp\left(-\sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{(y_{ij} - \theta_i)^2}{2\sigma^2} \right)$$

m.l.e in Θ : $\hat{\theta}_i = \bar{Y}_{i\bullet}$,

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i\bullet})^2 = \frac{SS_W}{N}$$

m.l.e in Θ_0 : $\hat{\theta}_i = \bar{\bar{Y}}$,

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{\bar{Y}})^2 = \frac{SS_T}{N}$$

$$L(\hat{\Theta}_0) = (\sqrt{2\pi SS_T / N})^{-n} \exp\left(-N/2\right),$$

$$L(\hat{\Theta}) = (\sqrt{2\pi SS_W / N})^{-n} \exp\left(-N/2\right)$$

$$\lambda = \frac{L(\hat{\Theta}_0)}{L(\hat{\Theta})} = \left(\frac{SS_W}{SS_T}\right)^{N/2} = \left(\frac{SS_W}{SS_W + SS_B}\right)^{N/2} = \left(\frac{1}{1 + SS_B / SS_W}\right)^{N/2}$$

$$\lambda < \lambda_0 \Leftrightarrow \frac{SS_B}{SS_W} > c^* \Leftrightarrow F = \frac{MS_B}{MS_W} = \frac{SS_B / (k-1)}{SS_W / (N-k)} > c$$

Rejection region

$\{\lambda > \lambda_0\} = \{F > c\}$

Given α， $c = F_{k-1, N-k, 1-\alpha}$

Two-way ANOVA

		Factor B			
		1	2	… j …	b
Factor A	1				
	2				
	… i …				
	a				

Two-way ANOVA without interaction

Restaurant ratings

	A	B	C	D
Rater 1	70	61	82	74
Rater 2	77	75	88	76
Rater 3	76	67	90	80
Rater 4	80	63	96	76
Rater 5	84	66	92	84
Rater 6	78	68	98	86

- 2 classification factors are considered
  - Restaurant (4 levels); Rater (6 levels)
- There is no replicate at each cell

Additive model

- Let $X_{ij} \sim N(\mu_{ij}, \sigma^2)$, $1 \leq i \leq a$, $1 \leq j \leq b$ be indep. with

$$\mu_{ij} = \mu + \alpha_i + \beta_j$$

satisfying

$$\sum \alpha_i = 0, \quad \sum \beta_j = 0.$$

- Here

$$\mu = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\mu_{ij},$$

$$\mu_{i\bullet} = \frac{1}{b}\sum_{j=1}^{b}\mu_{ij}, \quad \mu_{\bullet j} = \frac{1}{a}\sum_{i=1}^{a}\mu_{ij},$$

$\alpha_i = \mu_{i\bullet} - \mu,$ effect of the i^{th} level of factor A.

$\beta_j = \mu_{\bullet j} - \mu$ effect of the j^{th} level of factor B.

MLE estimators

- Likelihood $n=ab$

$$L = (\sqrt{2\pi\sigma^2})^{ab} \exp\left(\sum_{j=1}^{b} \sum_{i=1}^{a} \frac{(x_{ij} - \mu - \alpha_i - \beta_j)^2}{2\sigma^2} \right)$$

- MLE estimators for the parameters under $\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = 0$

$$\hat{\mu} = \bar{x}_{\bullet\bullet}, \quad \hat{\alpha}_i = \bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet}, \quad \hat{\beta}_j = \bar{x}_{\bullet j} - \bar{x}_{\bullet\bullet},$$

$$\hat{\sigma}^2 = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} (x_{ij} - \bar{x}_{i\bullet} - \bar{x}_{\bullet j} + \bar{x}_{\bullet\bullet})^2$$

$$\boxed{\hat{\mu}_{ij} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j = \bar{x}_{i\bullet} + \bar{x}_{\bullet j} - \bar{x}_{\bullet\bullet}}$$

$$
H_0 : \left.\begin{array}{c}
\mu_{11} = \cdots = \mu_{1b} \\
\mu_{21} = \cdots = \mu_{2b} \\
\cdots\cdots\cdots\cdots \\
\mu_{a1} = \cdots = \mu_{ab}
\end{array}\right\} \Longleftrightarrow
$$

$$
\beta_1 = \cdots = \beta_b = 0.
$$

$$H_0: \left.\begin{array}{c} \mu_{11} = \cdots = \mu_{a1} \\ \mu_{12} = \cdots = \mu_{a2} \\ \cdots\cdots\cdots\cdots \\ \mu_{1b} = \cdots = \mu_{ab} \end{array}\right\} \Longleftrightarrow$$

$$\alpha_1 = \cdots = \alpha_a = 0.$$

- 2-way ANOVA model :

$$X_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij},$$

$$with \ \sum_{i=1}^{a} \alpha_i = 0, \ \sum_{j=1}^{b} \beta_j = 0,$$

$$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2).$$

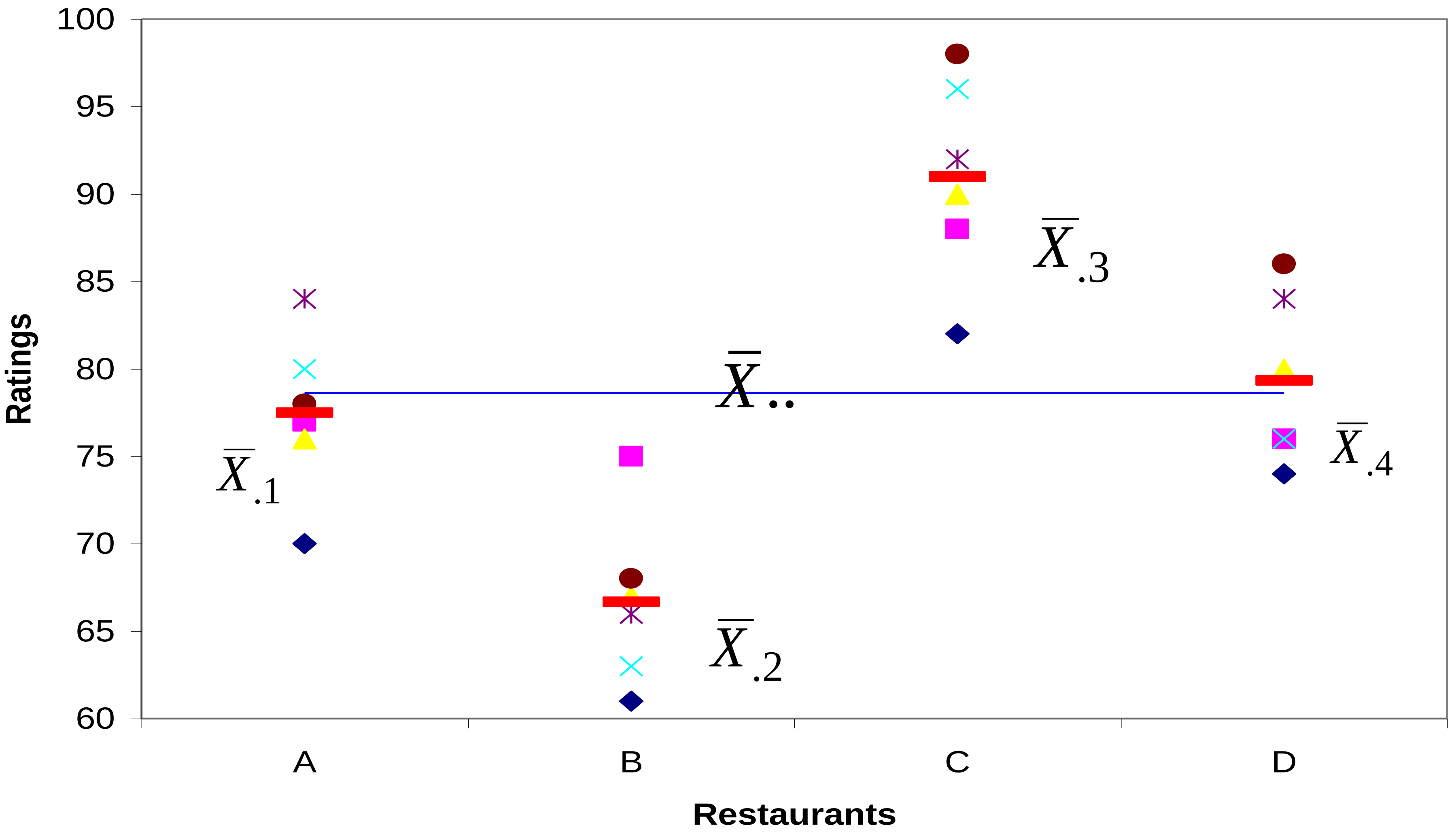
$$H_{A0}: \quad \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0$$

$$H_{B0}: \quad \beta_1 = \beta_2 = \cdots = \beta_b = 0$$

Restaurant ratings

	A	B	C	D	**Total**	**Mean**
Rater 1	70	61	82	74	287	71.75
Rater 2	77	75	88	76	316	79.00
Rater 3	76	67	90	80	313	78.25
Rater 4	80	63	96	76	315	78.75
Rater 5	84	66	92	84	326	81.50
Rater 6	78	68	98	86	330	82.50
Total	465	400	546	476	1887	
Mean	77.50	66.67	91.00	79.33		
Grand mean	78.63					

Scatter plot of restaurant ratings



Partitioning of Sum of Squares

$$SST = \sum_{j=1}^{b} \sum_{i=1}^{a} (X_{ij} - \bar{X}..)^2 \qquad df(T) = ab - 1$$

$$SSA = b \sum_{i=1}^{a} (\bar{X}_{i\bullet} - \bar{X}_{\bullet\bullet})^2 \qquad df(A) = a - 1$$

$$SSB = a \sum_{j=1}^{b} (\bar{X}_{\bullet j} - \bar{X}_{\bullet\bullet})^2 \qquad df(B) = b - 1$$

$$SSE = \sum_{j=1}^{b} \sum_{i=1}^{a} (X_{ij} - \bar{X}_{i\bullet} - \bar{X}_{\bullet j} + \bar{X}..)^2 \qquad df(E) = (a-1)(b-1)$$

SST=SSA+SSB+SSE

dfT=dfA+dfB+dfE

SST=SSA+SSB+SSE

$$df(A) = a-1 \qquad df(T) = b-1 \qquad df(E) = (a-1)(b-1)$$

$$F_A = \frac{SSA \big/ (a-1)}{SSE \big/ [(a-1)(b-1)]} = \frac{MSA}{MSE} \overset{H_{A0}}{\sim} F(a-1, (a-1)(b-1))$$

$$F_B = \frac{SSB \big/ (b-1)}{SSE \big/ [(a-1)(b-1)]} = \frac{MSB}{MSE} \overset{H_{B0}}{\sim} F(b-1, (a-1)(b-1))$$

- Test H_0 use LRT.

$$H_0 : \beta_1 = \cdots = \beta_b = 0$$

$$\Theta = \{\mu_{ij} = \mu + \alpha_i + \beta_j, \\ \sum \alpha_i = 0, \sum \beta_j = 0\}$$

$$\Theta_0 = \{\mu_{ij} = \mu + \alpha_i, \sum \alpha_i = 0\}.$$

m.l.e in Θ_0 : $\hat{\mu} = \bar{x}_{\bullet\bullet}, \quad \hat{\alpha}_i = \bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet},$

$$\hat{\mu}_{ij} = \hat{\mu} + \hat{\alpha}_i = \bar{x}_{i\bullet}, \quad \hat{\sigma}^2 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}(x_{ij} - \bar{x}_{i\bullet})^2$$

m.l.e in Θ : $\hat{\mu} = \bar{x}_{\bullet\bullet},$

$$\hat{\alpha}_i = \bar{x}_{i\bullet} - \bar{x}_{\bullet\bullet}, \qquad \hat{\beta}_j = \bar{x}_{\bullet j} - \bar{x}_{\bullet\bullet},$$

$$\hat{\mu}_{ij} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j = \bar{x}_{i\bullet} + \bar{x}_{\bullet j} - \bar{x}_{\bullet\bullet},$$

$$\hat{\sigma}^2 = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}(x_{ij} - \bar{x}_{i\bullet} - \bar{x}_{\bullet j} + \bar{x}_{\bullet\bullet})^2 = \frac{SSE}{ab}$$

$$L(\hat{\Theta}) =$$

$$\left(\frac{ab/(2\pi)}{\sum_{i,j}(X_{ij} + \overline{X_{..}} - \overline{X_{i.}} - \overline{X_{.j}})^2} \right)^{-\frac{ab}{2}}$$

$$\cdot e^{-\frac{ab}{2}}.$$

$$L(\hat{\Theta}_0) = (\frac{ab/(2\pi)}{\sum\sum(X_{ij} - \overline{X_{i.}})^2})^{-\frac{ab}{2}} e^{-\frac{ab}{2}}.$$

$$\lambda = \frac{L(\hat{\Theta}_0)}{L(\hat{\Theta})} = \left(\frac{SS_E}{\sum_{i=1}^{a}\sum_{j=1}^{b}(x_{ij} - \bar{x}_{i\bullet})^2} \right)^{ab/2}$$

$$= \left(\frac{SS_E}{SS_E + SS_B} \right)^{ab/2} = \left(\frac{1}{1 + SS_B / SS_E} \right)^{ab/2}$$

$$\lambda < \lambda_0 \Leftrightarrow \frac{SS_B}{SS_E} > c^* \Leftrightarrow F = \frac{MS_B}{MS_E} = \frac{SS_B / (b-1)}{SS_E / [(a-1)(b-1)]} > c$$

$$\boldsymbol{c = F_{1-\alpha}(b-1,(a-1)(b-1))}$$

Test:

$$H_0 : \alpha_1 = \cdots = \alpha_a = 0$$

use

$$F_A = \frac{SSA/(a-1)}{SSE/[(a-1)(b-1)]} = \frac{MSA}{MSE} \overset{H_{A0}}{\sim} F(a-1, (a-1)(b-1))$$

Reject H0 if $F_A > F_{1-\alpha}(a-1, (a-1)(b-1))$

Two-way ANOVA with interaction

Example: A chemical engineer is studying the effects of various reagents(试剂) and catalyst (催化剂) on the yield of a certain process.

Catalyst	Reagent		
	1	2	3
A	86.8 82.4 86.7 83.5	93.4 85.2 94.8 83.1	77.9 89.6 89.9 83.7
B	71.9 72.1 80.0 77.4	74.5 87.1 71.9 84.1	87.5 82.7 78.3 90.1
C	65.5 72.4 76.6 66.7	66.7 77.1 76.7 86.1	72.7 77.8 83.5 78.8
D	63.9 70.4 77.2 81.2	73.7 81.6 84.2 84.9	79.8 75.7 80.5 72.9

- 2 classification factors are considered
  - catalyst (4 levels);
  - reagents (3 levels)
- There are replicates at each cell.

$$X_{ijk} \sim N(\mu_{ij}, \sigma^2), \quad i = 1, \ldots, a; \quad j = 1, \ldots, b; \quad k = 1, \ldots, c.$$

$$\Leftrightarrow X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}, \; e_{ijk} \sim N(0, \sigma^2)$$

$$with \; \sum_{i=1}^{a} \alpha_i = 0, \; \sum_{j=1}^{b} \beta_j = 0, \sum_{i=1}^{a} \gamma_{ij} = \sum_{j=1}^{b} \gamma_{ij} = 0$$

$$\text{where } \mu = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \mu_{ij}, \; \mu_{i\bullet} = \frac{1}{b} \sum_{j=1}^{b} \mu_{ij}, \; \mu_{\bullet j} = \frac{1}{a} \sum_{i=1}^{a} \mu_{ij},$$

$\alpha_i = \mu_{i\bullet} - \mu,$ effect of the i^{th} level of factor A.

$\beta_j = \mu_{\bullet j} - \mu$ effect of the j^{th} level of factor B.

$\gamma_{ij} = \mu_{ij} - \mu_{i\bullet} - \mu_{\bullet j} + \mu$ effect of interaction of A_i and B_j

Effects for 2-way

main effect associated with the treatment A_i (first factor):

$$\alpha_i = \mu_{i\bullet} - \mu, \ \ i = 1, 2, ..., a$$

main effect associated with treatment B_j (second factor):

$$\beta_j = \mu_{\bullet j} - \mu, \quad j = 1, 2, ..., b$$

The interaction is defined as,

$$\gamma_{ij} = \mu_{ij} - \mu_{i.} - \mu_{.j} + \mu$$

MLE estimators

$$N=abc$$

$$L=(\sqrt{2\pi\sigma^2})^N \exp\left(\sum_{k=1}^{c}\sum_{j=1}^{b}\sum_{i=1}^{a}\frac{(x_{ijk}-\mu_{ij})^2}{2\sigma^2}\right) \Rightarrow \hat{\mu}_{ij}=\bar{x}_{ij\bullet}$$

or $$L=(\sqrt{2\pi\sigma^2})^N \exp\left(\sum_{k=1}^{c}\sum_{j=1}^{b}\sum_{i=1}^{a}\frac{(x_{ijk}-\mu-\alpha_i-\beta_j-\gamma_{ij})^2}{2\sigma^2}\right)$$

under $\sum_{i=1}^{a}\alpha_i=0,\ \sum_{j=1}^{b}\beta_j=0,\ \sum_{i=1}^{a}\sum_{j=1}^{b}\gamma_{ij}=0$

$$\Rightarrow \quad \hat{\mu}=\bar{x}_{\bullet\bullet\bullet},\quad \hat{\alpha}_i=\bar{x}_{i\bullet\bullet}-\bar{x}_{\bullet\bullet\bullet},\quad \hat{\beta}_j=\bar{x}_{\bullet j\bullet}-\bar{x}_{\bullet\bullet\bullet},$$

$$\hat{\gamma}_{ij}=\bar{x}_{ij\bullet}-\bar{x}_{i\bullet\bullet}-\bar{x}_{\bullet j\bullet}+\bar{x}_{\bullet\bullet\bullet},\quad \hat{\sigma}^2=\frac{1}{N}\sum_{k=1}^{c}\sum_{i=1}^{a}\sum_{j=1}^{b}(x_{ijk}-\bar{x}_{ij\bullet})^2$$

The model is:

$$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}, \quad \varepsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$$

$$i = 1, 2, \ldots, a; \quad j = 1, 2, \ldots, b; \quad k = 1, 2, \ldots, c.$$

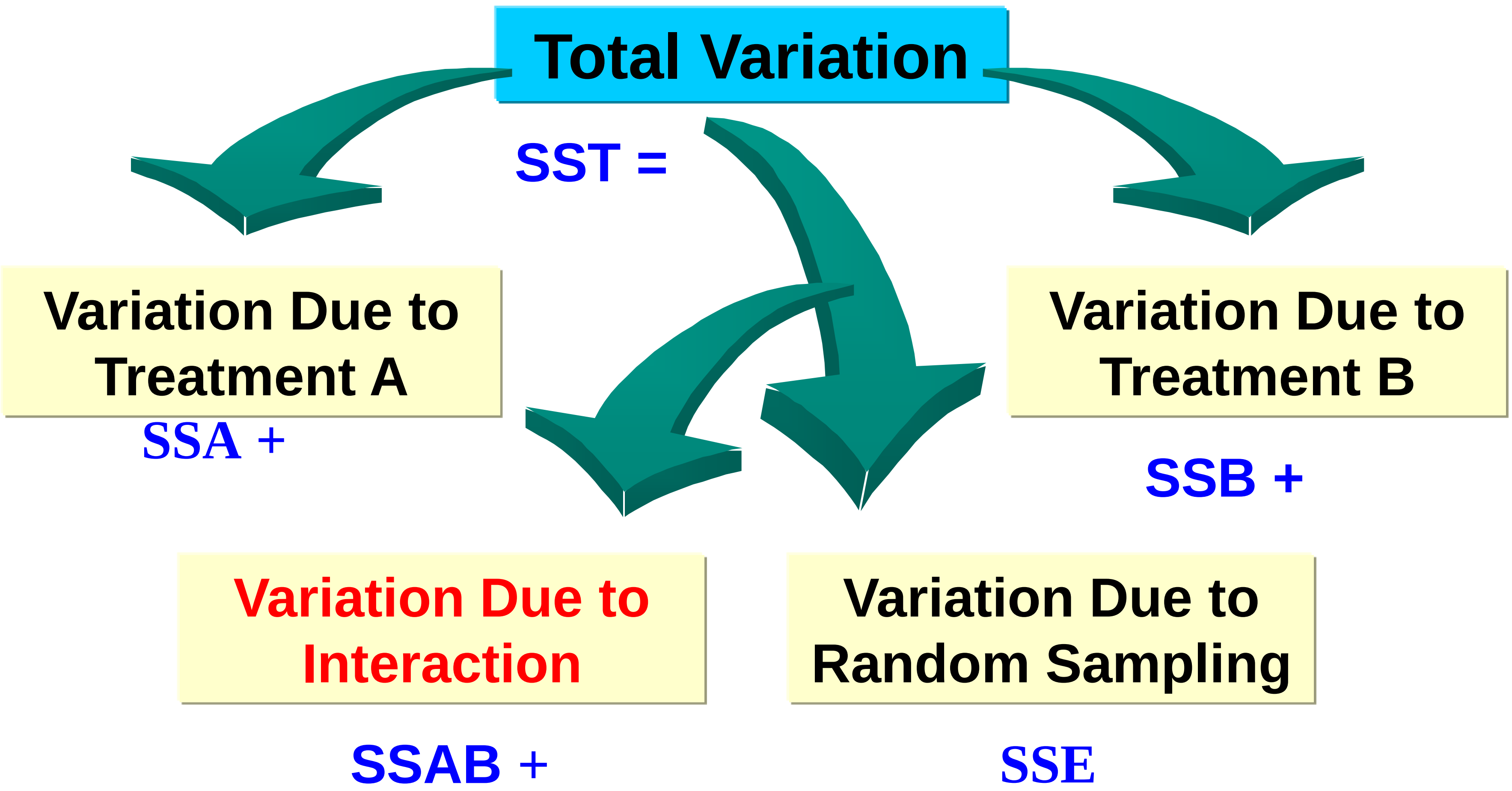
$$with \ \sum_{i=1}^{a} \alpha_i = 0, \ \sum_{j=1}^{b} \beta_j = 0, \sum_{i=1}^{a} \gamma_{ij} = \sum_{j=1}^{b} \gamma_{ij} = 0$$

$$H_{A0}: \quad \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0$$

$$H_{B0}: \quad \beta_1 = \beta_2 = \cdots = \beta_b = 0$$

$$H_{AB0}: \quad \gamma_{11} = \gamma_{12} = \cdots = \gamma_{ab} = 0$$

Partitioning of Variation



Partitioning of Variation

$$SST = \sum_{k=1}^{c}\sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ijk} - \bar{X}_{\dots})^2 \qquad df(T) = abc - 1$$

$$SSA = bc\sum_{i=1}^{a}(\bar{X}_{i\bullet\bullet} - \bar{X}_{\bullet\bullet\bullet})^2 \qquad df(A) = a - 1$$

$$SSB = ac\sum_{j=1}^{b}(\bar{X}_{\bullet j\bullet} - \bar{X}_{\bullet\bullet\bullet})^2 \qquad df(B) = b - 1$$

$$SSAB = c\sum_{j=1}^{b}\sum_{i=1}^{a}(\bar{X}_{ij\bullet} - \bar{X}_{i\bullet\bullet} - \bar{X}_{\bullet j\bullet} + \bar{X}_{\bullet\bullet\bullet})^2 \qquad df(AB) = (a-1)(b-1)$$

$$SSE = \sum_{k=1}^{c}\sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ijk} - \bar{X}_{ij\bullet})^2 \qquad df(E) = ab(c-1)$$

Generalization

$$N=\sum_{i=1}^{a}\sum_{j=1}^{b}n_{ij},\; n_{i\bullet}=\sum_{j=1}^{b}n_{ij},\; n_{\bullet j}=\sum_{i=1}^{a}n_{ij}$$

$$SST=\sum_{j=1}^{b}\sum_{i=1}^{a}\sum_{k=1}^{n_{ij}}(X_{ijk}-\bar{X}_{\dots})^2 \qquad df(T)=N-1$$

$$SSA=\sum_{i=1}^{a}n_{i\bullet}(\bar{X}_{i\bullet\bullet}-\bar{X}_{\bullet\bullet\bullet})^2 \qquad df(A)=a-1$$

$$SSB=\sum_{j=1}^{b}n_{\bullet j}(\bar{X}_{\bullet j\bullet}-\bar{X}_{\bullet\bullet\bullet})^2 \qquad df(T)=b-1$$

$$SSAB=\sum_{j=1}^{b}\sum_{i=1}^{a}n_{ij}(\bar{X}_{ij\bullet}-\bar{X}_{i\bullet\bullet}-\bar{X}_{\bullet j\bullet}+\bar{X}_{\bullet\bullet\bullet})^2 \qquad df(AB)=(a-1)(b-1)$$

$$SSE=\sum_{j=1}^{b}\sum_{i=1}^{a}\sum_{k=1}^{n_{ij}}(X_{ijk}-\bar{X}_{ij\bullet})^2 \qquad df(E)=N-ab$$

$$F_A = \frac{SSA/(a-1)}{SSE/(n-ab)} = \frac{MSA}{MSE} \overset{H_{A0}}{\sim} F(a-1, N-ab)$$

$$F_B = \frac{SSB/(b-1)}{SSE/(n-ab)} = \frac{MSB}{MSE} \overset{H_{B0}}{\sim} F(b-1, N-ab)$$

$$F_{AB} = \frac{SSAB/[(a-1)(b-1)]}{SSE/[N-ab]} = \frac{MSAB}{MSE}$$

$$\overset{H_{AB0}}{\sim} F((a-1)(b-1), N-ab)$$

Two-Way ANOVA Summary Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
A (Row)	$a - 1$	SSA	MSA	$\frac{MSA}{MSE}$
B (Column)	$b - 1$	SSB	MSB	$\frac{MSB}{MSE}$
AB (Interaction)	$(a-1)(b-1)$	SSAB	MSAB	$\frac{MSAB}{MSE}$
Error	$N-ab$	SSE	MSE	
Total	$N - 1$	SST		