Chapter 5
Properties of a Random Sample

Contents

5.1 Basic Concepts of Random Samples

Definition 5.1.1 The random variables $X_1, \ldots, X_n$ are called a *random sample of size n from the population* $f(x)$ if $X_1, \ldots, X_n$ are mutually independent random variables and the marginal pdf or pmf of each X_i is the same function $f(x)$. Alternatively, $X_1, \ldots, X_n$ are called *independent and identically distributed random variables with pdf or pmf* $f(x)$. This is commonly abbreviated to iid random variables.

- iid=**independent** and **identically distributed**

then the joint pdf or pmf is

$$f(x_1, \ldots, x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta), \quad (5.1.2)$$

5.2 Sums of Random Variables from a Random Sample

- Some basic concepts
- Some basic definitions
- Some basic tools

Basic concept

Definition 5.2.1

Let $X_1 \ldots\ldots X_n$ be a random sample of size n from a population and let $T(x_1 \ldots. x_n)$ be a real-valued or vector-valued or function whose domain includes the sample space of$(X_1,\ldots,X_n)$. Then Y=T$(X_1,\ldots,X_n)$ is called a **statistic**. The probability distribution of a statistic Y is called the **sampling distribution of Y**.

The only restriction:
Statistic cannot be a function of a parameter.

Basic Definitions

Definition 5.2.2

The sample mean is the arithmetic average of the value in a random sample. It is usually denoted by

$$\overline{X} = \frac{X_1 + \cdots + X_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Definition 5.2.3

The sample variance is the statistic defined by

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Sample standard deviation

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

Basic tools

- Theorem 5.2.4 Let $x_1, \cdots x_n$ be any numbers and $\bar{x} = (x_1 + \cdots x_n)/n$. Then

a. $\min_a \sum_{i=1}^n (x_i - a)^2 = \sum_{i=1}^n (x_i - \bar{x})^2$

b. $(n-1)s^2 = \sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n x_i^2 - n\bar{x}^2$

- Proof:

a. To prove part(a),just add and subtract $\overline{x}$ to get

$$\sum_{i=1}^{n}(x_i - a)^2 = \sum_{i=1}^{n}(x_i - \overline{x} + \overline{x} - a)^2$$

$$= \sum_{i=1}^{n}(x_i - \overline{x})^2 + 2\sum_{i=1}^{n}(x_i - \overline{x})(\overline{x} - a) + \sum_{i=1}^{n}(\overline{x} - a)^2$$

$$= \sum_{i=1}^{n}(x_i - \overline{x})^2 + \sum_{i=1}^{n}(\overline{x} - a)^2$$

$$= \sum_{i=1}^{n}(x_i - \overline{x})^2 + n(\overline{x} - a)^2$$

- b. To prove part(b),just take a=0 in the above equation:

$$\sum_{i=1}^{n}(x_i - a)^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - a)^2$$

✓ One of this equation’s significance is that we can get a more simple method to calculate the sample variance S^2

Three useful tools for studying the distributional properties of statistics.

- Theorem 5.2.6 Let $X_1,\ldots,X_n$ be a random sample from a population with mean μ and variance $\sigma^2 < \infty$. Then

  a. $E\overline{X} = \mu,$ b. $\operatorname{var}(\overline{X}) = \dfrac{\sigma^2}{n},$ c. $ES^2 = \sigma^2.$

- Proof: a.

$$E\overline{X} = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n}nEX_1 = \mu.$$

✱b. To prove part(b),similar to above, we have

$$Var(\overline{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} n Var(X_1) = \frac{\sigma^2}{n}$$

- c. we can use Theorem 5.2.4 ,we have

$$ES^2 = E\left(\frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right]\right) = \frac{1}{n-1}(nEX_1^2 - nE\overline{X}^2)$$

$$= \frac{1}{n-1}\left(n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right) = \sigma^2.$$

5.3 Sampling from the Normal Distribution

- 5.3.1

  Properties of the Sample Mean and Variance
- 5.3.2

  The Derived Distributions: Student’s t and Snedecor’s F

5.3.1 Properties of the Sample Mean and Variance

- Theorem 5.3.1. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Then

  (a) $\overline{X}$ and S^2 are independent rv's,

  (b) $\overline{X}$ has a $N(\mu, \sigma^2/n)$ distribution,

  (c). $(n-1)S^2/\sigma^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2 \sim \chi^2(n-1)$

- Proof:

  (b) is obvious so we focus on (a) and (c).

We first prove (a). Write :

$$\sum_{i=2}^{n}(X_i - \overline{X}) = -(X_1 - \overline{X})$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

$$= \frac{1}{n-1}\left((X_1 - \overline{X})^2 + \sum_{i=2}^{n}(X_i - \overline{X})^2\right)$$

$$= \frac{1}{n-1}\left(\left(\sum_{i=2}^{n}(X_i - \overline{X})\right)^2 + \sum_{i=2}^{n}(X_i - \overline{X})^2\right)$$

Thus S^2 can be written as a function only of $(X_2 - \overline{X}, \cdots, X_n - \overline{X})$.

If we can show that these rv's are joint Independent of $\overline{X}$ then we are done.

$$\operatorname{cov}(\bar{X}, X_i - \overline{X}) = \operatorname{cov}(\bar{X}, X_i) - \operatorname{var}(\bar{X}) = \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0$$

$$(\bar{X}, X_2 - \overline{X}, \cdots, X_n - \overline{X}) \sim \text{MultiNormal},$$

如果$(X_1, X_2, \cdots, X_n)'$服从n维正态分布, 那么$X_1, X_2, \cdots, X_n$相互独立等价于它们两两不相关.

$$\bar{X} \perp (X_2 - \overline{X}, \cdots, X_n - \overline{X})$$

✱In order to prove part (c) ,we introduce the Lemma 5.3.2

- (1) If Z is a n(0,1) random variable, then $Z^2 \sim \chi^2(1)$ (p52: 2.1.7)
- (2) If $X_1,\ldots X_n$ are independent and $X_i \sim \chi^2(v_i)$, then $X_1+\ldots+X_n \sim \chi^2(v_1+\cdots+v_n)$ (p183: 4.6.8)

● If $X_1,\ldots X_n$ are independent and $X_i \sim n(0,1)$, then

$$X_1^2+\ldots+X_n^2 \sim \chi^2(n)$$

$$(k+1)\bar{X}_{k+1} = k\bar{X}_k + X_{k+1};$$

$$\sum_{i=1}^{k+1}(X_i - \overline{X}_k)^2 = \sum_{i=1}^{k+1}(X_i - \overline{X}_{k+1})^2 + (k+1)(\overline{X}_{k+1} - \overline{X}_k)^2$$

$$\Rightarrow \quad kS_{k+1}^2 = (k-1)S_k^2 + \frac{k}{k+1}(X_{k+1} - \bar{X}_k)^2$$

Now consider n=2, $X_2 - X_1 \sim N(0, 2\sigma^2)$,

$$\frac{S_2^2}{\sigma^2} = \frac{(X_1 - \bar{X}_2)^2 + (X_2 - \bar{X}_2)^2}{\sigma^2} = \left(\frac{X_2 - X_1}{\sqrt{2}\sigma}\right)^2 \sim \chi^2(1)$$

According to the induction hypothesis,

$$\frac{(k-1)S_k^2}{\sigma^2} \sim \chi^2(k-1)$$

Now consider $n=k+1$, $S_k^2 \perp (X_{k+1} - \bar{X}_k)$

$$X_{k+1} - \bar{X}_k \sim N\left(0, \frac{k+1}{k}\sigma^2\right)$$

$$\frac{k}{k+1} \cdot \frac{(X_{k+1} - \bar{X}_k)^2}{\sigma^2} = \left(\frac{X_{k+1} - \bar{X}_k}{\sigma\sqrt{(k+1)/k}}\right)^2 \sim \chi^2(1)$$

$$\therefore \frac{kS_{k+1}^2}{\sigma^2} = \frac{(k-1)S_k^2}{\sigma^2} + \frac{k}{k+1}\frac{(X_{k+1} - \bar{X}_k)^2}{\sigma^2} \sim \chi^2(k)$$

5.3.2 The Distributions: Student's t and Snedecor's F

- In particular, in most practical cases the variance, σ^2, is unknown. Thus , to get any idea of the variability of $\overline{X}$ (as an estimate of μ), it is necessary to estimate this variance.

Considering the quantity:

$$\frac{\overline{X}-\mu}{S/\sqrt{n}}$$

Which we can use to inference about μ when σ is unknown.

Structure of t distribution

Suppose W**~N(0, 1),** V**~**χ^2**(*r*),** W, V are independent, then

$$T = \frac{W}{\sqrt{V/r}} \sim t(r).$$

t(r) is called t-distribution with degree freedom *r*.

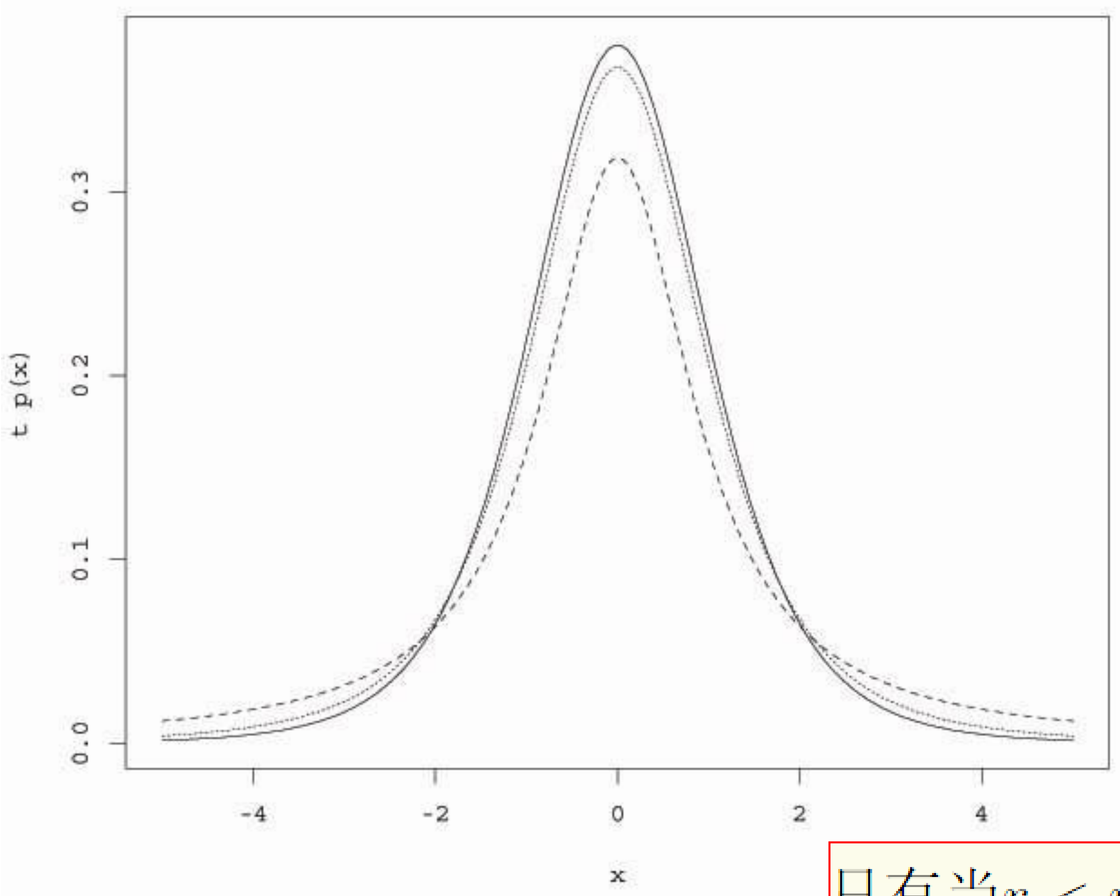
The p.d.f.

$$f(t) = \frac{\Gamma(\frac{r+1}{2})}{\sqrt{r\pi}\,\Gamma(\frac{r}{2})}\left(1+\frac{t^2}{r}\right)^{-\frac{r+1}{2}}, \quad -\infty < t < \infty$$

r=1: Cauchy distribution

The figure of *f* (t) looks like the figure of pdf of **N(0, 1).**

But its tail is thicker.



$$E[t(n)] = 0, \quad n > 1$$

$$\mathrm{Var}[t(n)] = \frac{n}{n-2}, \quad n > 2$$

只有当$r < n \ (n > 1)$时，r阶矩才存在.

We have

$$\frac{\overline{X}-\mu}{S/\sqrt{n}}=\frac{\left(\overline{X}-\mu\right)/\left(\sigma/\sqrt{n}\right)}{\sqrt{S^2/\sigma^2}}$$

- So, we can easily derive the distribution of the quantity

$$\frac{\overline{X}-\mu}{S/\sqrt{n}}\sim t(n-1)$$

(p223: Definition 5.3.4)

Structure of F-distribution

Suppose **U ~χ^2(r_1), V~χ^2(r_2), U, V** are independent. Then

$$F = \frac{U/r_1}{V/r_2} \sim F(r_1, r_2).$$

F(r_1, r_2) is called an F-distribution with degrees of freedom r_1 and r_2. Its pdf is

$$h(y) = \begin{cases} \dfrac{\Gamma(\frac{r_1+r_2}{2})(r_1/r_2)^{r_1/2} y^{\frac{r_1}{2}-1}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})(1+\frac{r_1}{r_2}y)^{(r_1+r_2)/2}}, & y>0 \\ 0, & y \le 0 \end{cases}$$

Let $X_1, \ldots. X_n$ be a random sample from a $n\left(\mu_x, \sigma_x^2\right)$ Population, and let $Y_1, \ldots Y_m$ be a random sample From an independent $n\left(\mu_y, \sigma_y^2\right)$ population. The random variance

$$F = \frac{S_X{}^2 / \sigma_X{}^2}{S_Y{}^2 / \sigma_Y{}^2} \sim F(n-1, m-1)$$

(P224: Definition 5.3.6)

5.4 Order Statistics

✱Definition 5.4.1

The order statistics of a random sample $X_1,\ldots X_n$ are the sample values placed in ascending order, They are denoted by $X_{(1)},\ldots,X_{(n)}$.

$$X_{(1)}=\text{smallest of } X_1,X_2,\cdots,X_n$$
$$X_{(2)}=\text{second smallest of } X_1,X_2\,\cdots,X_n$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$X_{(n)}=\text{largest of } X_1,X_2,\cdots,X_n$$

Certain functions of order statistics, $X_{(1)},\ldots,X_{(n)}$ are important statistics themselves. A few of these are:

(a) $X_{(n)} - X_{(1)}$, which is called the ***sample range***;

(b) $(X_{(n)} + X_{(1)}) / 2$, which is called the ***sample midrange***;

(c) $X_{((n+1)/2)}$ if n is odd, or $X_{(n/2)} + X_{(n/2+1)}$, which is called the ***median*** of the r. s.

The difference between sample median and sample mean.

- Theorem 5.4.3 Let $X_1,\ldots X_n$ be a random sample from a discrete distribution with pmf $f_X(x_i)=p_i$, where $x_1<x_2<\ldots$are the possible values of X in ascending order, Define
- $P_0=0$, $P_1=p_1$, $P_2=p_1+p_2$, …… $P_{\mathrm{i}}=p_1+p_2+\ldots+p_{\mathrm{i}}$, ……
- Then

$$P(X_{(j)} \le x_i) = \sum_{k=j}^{n} \binom{n}{k} P_i^k (1-P_i)^{n-k}$$

$$P\left(X_{(j)} = x_i\right) = \sum_{k=j}^{n} \binom{n}{k} \left[P_i^k (1-P_i)^{n-k} - P_{i-1}{}^k (1-P_{i-1})^{n-k} \right]$$

- Theorem 5.4.4 Let $X_{(1)},\ldots,X_{(n)}$ denote the order statistics of a random sample, $X_1,\ldots,X_n$, from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$. Then the pdf of $X_{(j)}$ is

$$f_{X(j)}(x)=\frac{n!}{(j-1)!(n-j)!}f_X(x)\left[F_X(x)\right]^{j-1}\left[1-F_X(x)\right]^{n-j}$$

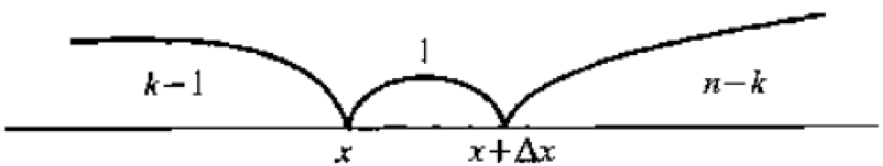


图 5.3.5 $x_{(k)}$ 取值的示意图

$$\because f_k(x)\Delta x=P(x<X_{(k)}\le x+\Delta x)$$

$$=\binom{1}{n}f(x)\Delta x\binom{n-1}{k-1}[F(x)]^{k-1}[1-F(x)]^{n-k}$$

$$=\frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k}f(x)\Delta y$$

- The distribution of $X_{(1)}$ and $X_{(n)}$ may be obtained directly.

$$F_{X_{(n)}}(x) = \mathrm{P}\left(X_{(n)} \le x\right) = \mathrm{P}\left(X_i \le x \text{ for all i}\right) = \left[F_X(x)\right]^n$$

Thus $f_{X_{(n)}}(x) = n\left[F_X(x)\right]^{n-1} f_X(x)$

$$1 - F_{X_{(1)}}(x) = \mathrm{P}\left(X_{(1)} > x\right) = \mathrm{P}\left(X_i > x \text{ for all i}\right) = \left[1 - F_X(x)\right]^n$$

Thus $f_{X_{(1)}}(x) = n\left[1 - F_X(x)\right]^{n-1} f_X(x)$.

- Theorem 5.4.6 (Still for continuous case)

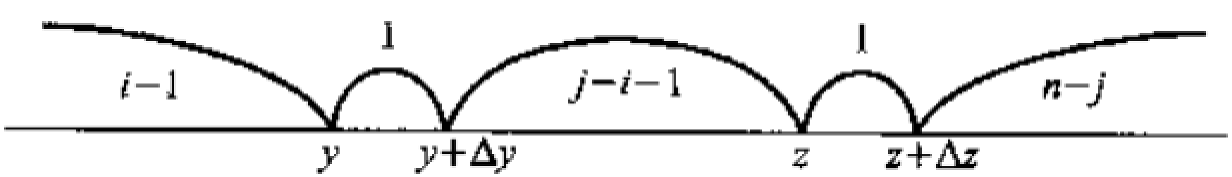
  The joint pdf of $X_{(i)}$ and $X_{(j)}$, 1<=i<j<=n, is

$$f_{X(i),X(j)}(y,z)=\frac{n!}{(i-1)!(j-1-i)!(n-j)!}f_X(y)f_X(z)\left[F_X\left(y\right)\right]^{i-1}$$

$$\times\left[F_X\left(z\right)-F_X\left(y\right)\right]^{j-1-i}\left[1-F_X\left(z\right)\right]^{n-j}\ for-\infty<\mu<\nu<\infty$$

$$\because f_{ij}(y,z)\Delta y\Delta z=P(y<X_{(i)}\le y+\Delta y, z<X_{(j)}\le z+\Delta z)$$

$$=\binom{n}{i-1,1,j-i-1,1,n-j}[F(y)]^{i-1}f(y)\Delta y[F(z)-F(y)]^{j-i-1}f(z)\Delta z[1-F(z)]^{n-j}$$



5.5 Convergence Concepts

- Convergence of random variables
  - **Convergence in Distribution**
  - **Convergence in Probability**
  - **Convergence in r-order mean**
  - **Almost sure convergence**

- Law of Large Numbers

- Central Limit Theorem

Convergences

Def. Let $\{X_n\}$ be a sequence of r. vs. We say

(1) X_n **convergence in distribution, (**or **converge weakly**) to X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

$$X_n \xrightarrow{D} X$$

(2) X_n **converges in probability** to X if for any ε>0

$$\lim_{n\to\infty} P[|X_n - X| \geq \varepsilon] = 0,$$

$$X_n \xrightarrow{P} X$$

(3) X_n **convergence in r-order (r>0)** to X if

$$\lim_{n\to\infty} E\left(|X_n - X|^r\right) = 0,$$

$$X_n \xrightarrow{r} X$$

(4) X_n **converges almost surely, or almost everywhere, or with probability 1, or strongly** towards X if

$$P[\lim_{n\to\infty} X_n = X] = 1.$$

$$X_n \xrightarrow{a.s.} X$$

Properties

(1) **If** b is constant, $X_n \xrightarrow{D} b \Leftrightarrow X_n \xrightarrow{P} b.$

(2) $X_n \xrightarrow{P} X, \text{ and } Y_n \xrightarrow{P} Y. \quad \text{Then} \quad \boxed{X_n \pm (\times) Y_n \xrightarrow{P} X \pm (\times) Y.}$

(3) $X_n \xrightarrow{P} X,$ and the real function g is continuous. Then

$\boxed{g(X_n) \xrightarrow{P} g(X).}$ **(p233 Theorem 5.5.4)**

(4) (p239 Theorem 5.5.17: Slutsky's Theorem)

$$If \; X_n \xrightarrow{D} X, Y_n \xrightarrow{P} a, \; then \; X_n Y_n \xrightarrow{D} aX, X_n + Y_n \xrightarrow{D} X + a.$$

Law of large numbers (L.L.N)

- Let $\{X_n\}$ be a sequence of i.i.d. rvs with finite mean $\mu < \infty$. Then the sample mean converges to μ

$$\overline{X}_n = \frac{X_1 + X_2 + \dots X_n}{n} \longrightarrow \mu, \; n \to \infty$$

  – Weak L.L.N: $\bar{X}_n \xrightarrow{P} \mu.$

  – Strong L.L.N $\bar{X}_n \xrightarrow{a.s.} \mu.$

- **Remark:** (1) for i.i.d. case, assumption of finite variance $\sigma^2 < \infty$ is **not necessary**. Large or infinite variance will make the convergence slower, but the weak or strong LLN holds anyway.
- (2) for not i.i.d. case, under Markovian assumption,

$$\operatorname{var} \bar{X}_n \to 0 \Rightarrow \quad \bar{X}_n \xrightarrow{P} \mu.$$

Examples

$$If\ X_1,\dots,X_n \overset{iid}{\sim} X, E(X)=\mu<\infty, then,$$ from LLN,

(1) $\bar{X} \xrightarrow{P} \mu;$

(2) if furthermore $var(X)=\sigma^2<\infty, then$

$$S^2 \xrightarrow{P} \sigma^2;\ S \xrightarrow{P} \sigma.$$

In fact, $E(X^2)=\mu^2+\sigma^2<\infty,$ from LLN,

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2=\frac{1}{n}\left(\sum_{i=1}^{n}X_i^2-n\bar{X}_n^2\right)\xrightarrow{p}E[X_1^2]-\mu^2=\sigma^2.$$

$$S^2=\frac{n}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\xrightarrow{p}\sigma^2$$

$$S\xrightarrow{P}\sigma$$

Central limit theorem

Central Limit Theorem (C.C.L.)

Let $\{X_n\}$ be a sequence of i.i.d. rvs with a finite mean μ and a finite positive variance σ^2. Then

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1), (n \to \infty)$$

According to **Slutsky's Theorem**,

$$\frac{\overline{X}_n - \mu}{S/\sqrt{n}} \xrightarrow{D} N(0,1), (n \to \infty)$$

consistency

- **Definition**: Any statistic that converges in probability to a parameter is called a **consistent estimator** of that parameter .

- **S^2 is consistent estimator of σ^2;**
- **S is consistent estimator of σ**

Delta Method

- The rth-order Taylor series expansion of $g(x)$ about a

$$T_r(x) = \sum_{i=0}^{r} \frac{g^{(i)}(a)}{i!}(x-a)^i + \text{Remainder.}$$

- T is a r.v with $E(T)=\theta$.

$$g(T) = g(\theta) + g'(\theta)(T-\theta) + \text{Remainder}$$

$$g(T) \approx g(\theta) + g'(\theta)(T-\theta) + \frac{g''(\theta)}{2}(T-\theta)^2 + \text{Remainder}$$

- Example 5.5.23 An iid sample $\{X_i\}$ with $EX_i=\mu\neq 0$, and $\mathrm{var}(X_i)=\sigma^2$.

$$\frac{1}{\bar{\bar{X}}}=\frac{1}{\mu}-\frac{1}{\mu^2}(\bar{X}-\mu)+\frac{1}{\mu^3}(\bar{X}-\mu)^2+\text{Remainder}$$

When n is large,

$$E\left(\frac{1}{\bar{\bar{X}}}\right)\approx\frac{1}{\mu} \qquad or: E\left(\frac{1}{\bar{\bar{X}}}\right)\approx\frac{1}{\mu}+\frac{\sigma^2}{n\mu^3}$$

$$\mathrm{var}\left(\frac{1}{\bar{\bar{X}}}\right)\approx\frac{1}{\mu^4}\mathrm{var}(\bar{X})=\frac{\sigma^2}{n\mu^4}$$

Delta Method

Theorem 5.5.24 (Delta method) Let Y_n be a sequence of rvs that satisfies

$$\sqrt{n}(Y_n - \theta) \xrightarrow{D} N(0, \sigma^2), (n \to \infty)$$

For a given function g, a specific value θ, suppose that g′ (θ) exists and is not 0, then

$$\sqrt{n}\left\{g(Y_n) - g(\theta)\right\} \xrightarrow{D} N(0, g'(\theta)^2 \sigma^2), (n \to \infty)$$

Corollary The often-used special case of the Theorem is $Y_n = \overline{X}_n$ in which $\{X_n\}$ be i.i.d. rvs, by C.C.L.

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2), (n \to \infty) \quad \textbf{(C.C.L.)}$$

thus

$$\sqrt{n}\left\{g(\overline{X}_n) - g(\mu)\right\} \xrightarrow{D} N(0, g'(\mu)^2 \sigma^2), (n \to \infty)$$

- Example 5.5.25 X with EX=$\mu \neq 0$,

$$\sqrt{n}\left(\frac{1}{\bar{X}} - \frac{1}{\mu}\right) \to \mathrm{n}\left(0, \left(\frac{1}{\mu}\right)^4 \mathrm{Var}_\mu X_1\right)$$

$$\boxed{\frac{\sqrt{n}\left(\frac{1}{\bar{X}} - \frac{1}{\mu}\right)}{\left(\frac{1}{\bar{X}}\right)^2 S} \to \mathrm{n}(0,1)}$$

Theorem 5.5.26 (Second-order Delta Method) *Let Y_n be a sequence of random variables that satisfies $\sqrt{n}(Y_n - \theta) \to \mathrm{n}(0, \sigma^2)$ in distribution. For a given function g and a specific value of θ, suppose that $g'(\theta) = 0$ and $g''(\theta)$ exists and is not 0. Then*

$$n[g(Y_n) - g(\theta)] \to \sigma^2 \frac{g''(\theta)}{2} \chi_1^2 \text{ in distribution.} \tag{5.5.13}$$

Let $T_1, \dots, T_k$ be random variables with means $\theta_1, \dots, \theta_k$, and define $\mathbf{T} = (T_1, \dots, T_k)$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$.

$$g(\mathbf{t}) = g(\boldsymbol{\theta}) + \sum^{k} g_i'(\boldsymbol{\theta})(t_i - \theta_i) + \text{Remainder}.$$

$$\mathrm{E}_{\boldsymbol{\theta}} g(\mathbf{T}) \approx g(\boldsymbol{\theta})$$

$$\begin{aligned}
\mathrm{Var}_{\boldsymbol{\theta}}\, g(\mathbf{T}) &\approx \mathrm{E}_{\boldsymbol{\theta}}\left([g(\mathbf{T}) - g(\boldsymbol{\theta})]^2\right) \\
&\approx \mathrm{E}_{\boldsymbol{\theta}}\left(\left(\sum_{i=1}^{k} g_i'(\boldsymbol{\theta})(T_i - \theta_i)\right)^2\right)
\end{aligned}$$

$$= \sum_{i=1} [g_i'(\boldsymbol{\theta})]^2 \mathrm{Var}_{\theta}\, T_i + 2\sum_{i>j} g_i'(\boldsymbol{\theta}) g_j'(\boldsymbol{\theta}) \mathrm{Cov}_{\theta}(T_i, T_j),$$

$$= \sum_{i,j=1}^{k} g_i'(\theta) g_j'(\theta) \mathrm{Cov}(T_i, T_j)$$

Theorem 5.5.28 (Multivariate Delta Method) *Let* $\mathbf{X}_1, \ldots, \mathbf{X}_n$ *be a random sample with* $\mathrm{E}(X_{ij}) = \mu_i$ *and* $\mathrm{Cov}(X_{ik}, X_{jk}) = \sigma_{ij}$. *For a given function* g *with continuous first partial derivatives and a specific value of* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)$ *for which* $\tau^2 = \Sigma\Sigma\sigma_{ij}\frac{\partial g(\boldsymbol{\mu})}{\partial \mu_i} \cdot \frac{\partial g(\boldsymbol{\mu})}{\partial \mu_j} > 0$,

$$\sqrt{n}[g(\bar{X}_1, \ldots, \bar{X}_s) - g(\mu_1, \ldots, \mu_p)] \to \mathrm{n}(0, \tau^2) \text{ in distribution .}$$

Homework: P257~263

- 5.11, 5.13, 5.16, 5.17, 5.21, 5.22, 5.44