ch1 Probability and Distributions

1.2 Set Theory

1.2.1. Find the union $C_1 \cup C_2$ and the intersection $C_1 \cap C_2$ of the two sets C_1 and C_2, where

(a) $C_1 = \{0, 1, 2, \}$, $C_2 = \{2, 3, 4\}$.

(b) $C_1 = \{x : 0 < x < 2\}$, $C_2 = \{x : 1 \leq x < 3\}$.

(c) $C_1 = \{(x, y) : 0 < x < 2, 1 < y < 2\}$, $C_2 = \{(x, y) : 1 < x < 3, 1 < y < 3\}$.

1.2.3. List all possible arrangements of the four letters $m, a, r,$ and y. Let C_1 be the collection of the arrangements in which y is in the last position. Let C_2 be the collection of the arrangements in which m is in the first position. Find the union and the intersection of C_1 and C_2.

1.2.6. If a sequence of sets $C_1, C_2, C_3, \ldots$ is such that $C_k \subset C_{k+1}$, $k = 1, 2, 3, \ldots$, the sequence is said to be a *nondecreasing sequence*. Give an example of this kind of sequence of sets.

1.2.7. If a sequence of sets $C_1, C_2, C_3, \ldots$ is such that $C_k \supset C_{k+1}$, $k = 1, 2, 3, \ldots$, the sequence is said to be a *nonincreasing sequence*. Give an example of this kind of sequence of sets.

1.3 The Probability Set Function

1.3.6. If the sample space is $\mathcal{C} = \{c : -\infty < c < \infty\}$ and if $C \subset \mathcal{C}$ is a set for which the integral $\int_C e^{-|x|}\, dx$ exists, show that this set function is not a probability set function. What constant do we multiply the integrand by to make it a probability set function?

1.3.8. Let C_1, C_2, and C_3 be three mutually disjoint subsets of the sample space $\mathcal{C}$. Find $P[(C_1 \cup C_2) \cap C_3]$ and $P(C_1^c \cup C_2^c)$.

1.3.11. A person has purchased 10 of 1000 tickets sold in a certain raffle. To determine the five prize winners, five tickets are to be drawn at random and without replacement. Compute the probability that this person wins at least one prize. *Hint:* First compute the probability that the person does not win a prize.

1.3.13. Three distinct integers are chosen at random from the first 20 positive integers. Compute the probability that: (a) their sum is even; (b) their product is even.

1.4 Conditional Probability and Independence

1.4.5. A hand of 13 cards is to be dealt at random and without replacement from an ordinary deck of playing cards. Find the conditional probability that there are at least three kings in the hand given that the hand contains at least two kings.

1.4.10. In an office there are two boxes of computer disks: Box C_1 contains seven Verbatim disks and three Control Data disks, and box C_2 contains two Verbatim disks and eight Control Data disks. A person is handed a box at random with prior probabilities $P(C_1) = \frac{2}{3}$ and $P(C_2) = \frac{1}{3}$, possibly due to the boxes' respective locations. A disk is then selected at random and the event C occurs if it is from Control Data. Using an equally likely assumption for each disk in the selected box, compute $P(C_1|C)$ and $P(C_2|C)$.

1.4.19. Cards are drawn at random and with replacement from an ordinary deck of 52 cards until a spade appears.

(a) What is the probability that at least four draws are necessary?

(b) Same as part (a), except the cards are drawn without replacement.

1.4.22. Players A and B play a sequence of independent games. Player A throws a die first and wins on a "six." If he fails, B throws and wins on a "five" or "six." If he fails, A throws and wins on a "four," "five," or "six." And so on. Find the probability of each player winning the sequence.

1.5 Random Variables

1.5.2. For each of the following, find the constant c so that $p(x)$ satisfies the condition of being a pmf of one random variable X.

(a) $p(x) = c(\frac{2}{3})^x$, $x = 1, 2, 3, \ldots$, zero elsewhere.

1.5.5. Let us select five cards at random and without replacement from an ordinary deck of playing cards.

(a) Find the pmf of X, the number of hearts in the five cards.

1.6 Discrete Random Variables

1.6.2. Let a bowl contain 10 chips of the same size and shape. One and only one of these chips is red. Continue to draw chips from the bowl, one at a time and at random and without replacement, until the red chip is drawn.

(a) Find the pmf of X, the number of trials needed to draw the red chip.

1.6.3. Cast a die a number of independent times until a six appears on the up side of the die.

(a) Find the pmf $p(x)$ of X, the number of casts needed to obtain that first six.

(b) Show that $\sum_{x=1}^{\infty} p(x) = 1$.

1.6.8. Let X have the pmf $p(x) = (\frac{1}{2})^x$, $x = 1, 2, 3, \ldots$, zero elsewhere. Find the pmf of $Y = X^3$.

1.7 Continuous Random Variables

1.7.1. Let a point be selected from the sample space $\mathcal{C} = \{c : 0 < c < 10\}$. Let $C \subset \mathcal{C}$ and let the probability set function be $P(C) = \int_C \frac{1}{10}\, dz$. Define the random variable X to be $X(c) = c^2$. Find the cdf and the pdf of X.

1.7.2. Let the space of the random variable X be $\mathcal{C} = \{x : 0 < x < 10\}$ and let $P_X(C_1) = \frac{3}{8}$, where $C_1 = \{x : 1 < x < 5\}$. Show that $P_X(C_2) \leq \frac{5}{8}$, where $C_2 = \{x : 5 \leq x < 10\}$.

1.7.10. Let $0 < p < 1$. A $(100p)$th **percentile** (**quantile** of order p) of the distribution of a random variable X is a value ξ_p such that $P(X < \xi_p) \leq p$ and $P(X \leq \xi_p) \geq p$. Find the 20th percentile of the distribution that has pdf $f(x) = 4x^3$, $0 < x < 1$, zero elsewhere.
Hint: With a continuous-type random variable X, $P(X < \xi_p) = P(X \leq \xi_p)$ and hence that common value must equal p.

1.7.13. Consider the cdf $F(x) = 1 - e^{-x} - xe^{-x}$, $0 \leq x < \infty$, zero elsewhere. Find the pdf, the mode, and the median (by numerical methods) of this distribution.

1.7.22. Let X have the uniform pdf $f_X(x) = \frac{1}{\pi}$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the pdf of $Y = \tan X$. This is the pdf of a **Cauchy distribution**.

1.7.23. Let X have the pdf $f(x) = 4x^3$, $0 < x < 1$, zero elsewhere. Find the cdf and the pdf of $Y = -\ln X^4$.

1.7.24. Let $f(x) = \frac{1}{3}$, $-1 < x < 2$, zero elsewhere, be the pdf of X. Find the cdf and the pdf of $Y = X^2$.
Hint: Consider $P(X^2 \leq y)$ for two cases: $0 \leq y < 1$ and $1 \leq y < 4$.

1.8 Expectation of a Random Variable

1.8.4. Let X be a number selected at random from a set of numbers $\{51, 52, \ldots, 100\}$. Approximate $E(1/X)$.
Hint: Find reasonable upper and lower bounds by finding integrals bounding $E(1/X)$.

1.8.6. Let X have the pdf $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere. Consider a random rectangle whose sides are X and $(1 - X)$. Determine the expected value of the area of the rectangle.

1.8.8. Let $f(x) = 2x$, $0 < x < 1$, zero elsewhere, be the pdf of X.

(a) Compute $E(1/X)$.

(b) Find the cdf and the pdf of $Y = 1/X$.

(c) Compute $E(Y)$ and compare this result with the answer obtained in part (a).

1.8.10. Let X have a Cauchy distribution which has the pdf

$$f(x) = \frac{1}{\pi} \frac{1}{x^2+1}, \quad -\infty < x < \infty.$$

Then X is symmetrically distributed about 0 (why?). Why isn't $E(X) = 0$?

1.9 Some Special Expectations

1.9.2. Let $p(x) = (\frac{1}{2})^x$, $x = 1, 2, 3, \ldots$, zero elsewhere, be the pmf of the random variable X. Find the mgf, the mean, and the variance of X.

1.9.4. If the variance of the random variable X exists, show that

$$E(X^2) \geq [E(X)]^2.$$

1.9.6. Let the random variable X have mean μ, standard deviation σ, and mgf $M(t)$, $-h < t < h$. Show that

$$E\left(\frac{X-\mu}{\sigma}\right) = 0, \quad E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = 1,$$

1.9.16. Let the random variable X have pmf

$$p(x) = \begin{cases} p & x = -1, 1 \\ 1-2p & x = 0 \\ 0 & \text{elsewhere,} \end{cases}$$

where $0 < p < \frac{1}{2}$. Find the measure of kurtosis as a function of p. Determine its value when $p = \frac{1}{3}$, $p = \frac{1}{5}$, $p = \frac{1}{10}$, and $p = \frac{1}{100}$. Note that the kurtosis increases as p decreases.

1.9.17. Let $\psi(t) = \log M(t)$, where $M(t)$ is the mgf of a distribution. Prove that $\psi'(0) = \mu$ and $\psi''(0) = \sigma^2$. The function $\psi(t)$ is called the **cumulant generating function**.

1.9.20. Let X be a random variable of the continuous type with pdf $f(x)$, which is positive provided $0 < x < b < \infty$, and is equal to zero elsewhere. Show that

$$E(X) = \int_0^b [1 - F(x)]\, dx,$$

where $F(x)$ is the cdf of X.

1.9.23. Let X have the cdf $F(x)$ that is a mixture of the continuous and discrete types, namely

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x+1}{4} & 0 \leq x < 1 \\ 1 & 1 \leq x. \end{cases}$$

Determine reasonable definitions of $\mu = E(X)$ and $\sigma^2 = \text{var}(X)$ and compute each. *Hint:* Determine the parts of the pmf and the pdf associated with each of the discrete and continuous parts, and then sum for the discrete part and integrate for the continuous part.

1.10 Important Inequalities

1.10.2. Let X be a random variable such that $P(X \leq 0) = 0$ and let $\mu = E(X)$ exist. Show that $P(X \geq 2\mu) \leq \frac{1}{2}$.

1.10.4. Let X be a random variable with mgf $M(t)$, $-h < t < h$. Prove that

$$P(X \geq a) \leq e^{-at}M(t), \quad 0 < t < h,$$

and that

$$P(X \leq a) \leq e^{-at}M(t), \quad -h < t < 0.$$

Hint: Let $u(x) = e^{tx}$ and $c = e^{ta}$ in Theorem 1.10.2. *Note:* These results imply that $P(X \geq a)$ and $P(X \leq a)$ are less than or equal to their respective least upper bounds for $e^{-at}M(t)$ when $0 < t < h$ and when $-h < t < 0$.

1.10.5. The mgf of X exists for all real values of t and is given by

$$M(t) = \frac{e^t - e^{-t}}{2t}, \quad t \neq 0, \quad M(0) = 1.$$

Use the results of the preceding exercise to show that $P(X \geq 1) = 0$ and $P(X \leq -1) = 0$. Note that here h is infinite.