ch5

5.1 Convergence in Probability

5.1.3. Let W_n denote a random variable with mean μ and variance b/n^p, where $p > 0$, μ, and b are constants (not functions of n). Prove that W_n converges in probability to μ.
Hint: Use Chebyshev's inequality.

5.1.5. Let $X_1, \ldots, X_n$ be iid random variables with common pdf

$$f(x) = \begin{cases} e^{-(x-\theta)} & x > \theta \quad -\infty < \theta < \infty \\ 0 & \text{elsewhere.} \end{cases} \tag{5.1.3}$$

This pdf is called the **shifted exponential**. Let $Y_n = \min\{X_1, \ldots, X_n\}$. Prove that $Y_n \to \theta$ in probability, by first obtaining the cdf of Y_n.

5.1.7. For Exercise 5.1.5, obtain the mean of Y_n. Is Y_n an unbiased estimator of θ? Obtain an unbiased estimator of θ based on Y_n.

5.1.3 For all $\epsilon > 0$,

$$P(|W_n - \mu| \geq \epsilon) \leq \frac{b}{n^p \epsilon^2} \to 0,$$

as $n \to \infty$.

5.1.5 Note that $Y_n \geq t \Leftrightarrow X_i \geq t$, for all $i = 1, 2, \ldots, n$. Hence, for $t > \theta$, the fact that $X_1, X_2, \ldots, X_n$ are iid implies

$$\begin{aligned} P(|Y_n - \theta| \leq \epsilon) &= P(Y \leq \epsilon + \theta) = 1 - e^{-n(\epsilon+\theta-\theta)} \\ 1 - e^{-n\epsilon} \to 1, \end{aligned}$$

as $n \to \infty$.

5.1.7 The density of Y_n is $f(y) = n\exp\{-n(y-\theta)\}$ for $y > \theta$. Hence,

$$\begin{aligned} E[Y_n] &= n\int_\theta^\infty y e^{-n(y-\theta)}\, dy \\ &= \int_0^\infty \left(\frac{z}{n} + \theta\right) e^{-z}\, dz \\ &= \frac{1}{n}\int_0^\infty z^{2-1} e^{-z}\, dz + \theta \int_0^\infty e^{-z}\, dz = \frac{1}{n} + \theta, \end{aligned}$$

where the integral on the second line results from the substitution $z = n(y-\theta)$. Based on this result $Y_n - \frac{1}{n}$ is an unbiased estimate of θ.

5.2 Convergence in Distribution

5.2.2. Let Y_1 denote the minimum of a random sample of size n from a distribution that has pdf $f(x) = e^{-(x-\theta)}$, $\theta < x < \infty$, zero elsewhere. Let $Z_n = n(Y_1 - \theta)$. Investigate the limiting distribution of Z_n.

5.2.4. Let Y_2 denote the second smallest item of a random sample of size n from a distribution of the continuous type that has cdf $F(x)$ and pdf $f(x) = F'(x)$. Find the limiting distribution of $W_n = nF(Y_2)$.

5.2.5. Let the pmf of Y_n be $p_n(y) = 1$, $y = n$, zero elsewhere. Show that Y_n does not have a limiting distribution. (In this case, the probability has "escaped" to infinity.)

5.2.2

$$\begin{aligned}
g_1(y_1) &= ne^{-n(y_1-\theta)}, \quad 0 < y_1 < \infty \\
z &= n(y_1 - \theta) \text{ and } \frac{dy_1}{dz} = \frac{1}{n}, \\
h_n(z) &= e^{-z} \text{ and } H_n(z) = 1 - e^{-z}, 0 < z < \infty \\
\lim_{n\to\infty} H_n(z) &= \begin{cases} 1 - e^{-z} & 0 < z < \infty \\ 0 & \text{elsewhere.} \end{cases}
\end{aligned}$$

5.2.4

$$\begin{aligned}
g_2(y_2) &= n(n-1)F(y_2)[1 - F(y_2)]^{n-2} f(y_2), -\infty < y_2 < \infty \\
w &= nF(y_2) \Rightarrow \frac{dy_2}{dw} = \frac{1}{nf(y_2)}. \\
h(w) &= \frac{n-1}{n} w(1 - w/n)^{n-2}, 0 < w < n \\
\lim_{n\to\infty} H_n(w) &= \lim_{n\to\infty} \int_0^w \frac{n-1}{n} z(1 - z/n)^{n-2}\, dz \\
&= \int_0^w ze^{-z}\, dz,
\end{aligned}$$

which is a $\Gamma(2, 1)$ cdf.

5.2.5

$$\begin{aligned}
F_n(y) &= \begin{cases} 0 & y < n \\ 1 & n \le y. \end{cases} \\
\lim_{n\to\infty} F_n(y) &= 0, \quad -\infty < y < \infty.
\end{aligned}$$

There is no cdf which equals this limit at every point of continuity.

5.2.11. Let the random variable Z_n have a Poisson distribution with parameter $\mu = n$. Show that the limiting distribution of the random variable $Y_n = (Z_n - n)/\sqrt{n}$ is normal with mean zero and variance 1.

5.2.18. Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of a random sample (see Section 5.2) from a distribution with pdf $f(x) = e^{-x}$, $0 < x < \infty$, zero elsewhere. Determine the limiting distribution of $Z_n = (Y_n - \log n)$.

5.2.11

$$\begin{aligned}\lim_{n\to\infty} E[e^{t(Z_n-n)/\sqrt{n}}] &= \lim_{n\to\infty}\{e^{-tsqrtn}\exp[n(e^{t/\sqrt{n}}-1)]\} \\ &= \lim_{n\to\infty}\left\{\exp\left[-t/\sqrt{n}+n\left(t/\sqrt{n}+\frac{t^2}{2n}+\frac{t^3}{6n^{3/2}}-\cdots\right)\right]\right\}. \\ &= \lim_{n\to\infty}\left[\exp\left(\frac{t^2}{2}+\frac{t^3}{6n^{1/2}}\cdots\right)\right] = \exp(t^2/2),\end{aligned}$$

which is the mgf of $N(0,1)$.

5.2.18 Note that $Y_n \le t \Leftrightarrow X_i \le t$, for all $i = 1, 2, \ldots, n$. Hence, for $0 < t$, the fact that $X_1, X_2, \ldots, X_n$ are iid implies

$$\begin{aligned}P(Y_n \le t+\log n) &= (P(X_1 \le t+\log n))^n \\ &= \left[1-e^{-(t+\log n)}\right]^n \\ &= \left[1-e^{-t}\frac{1}{n}\right]^n \to \exp\{-e^{-t}\},\end{aligned}$$

as $n \to \infty$.

5.3 Central Limit Theorem

5.3.2. Let $\overline{X}$ denote the mean of a random sample of size 128 from a gamma distribution with $\alpha = 2$ and $\beta = 4$. Approximate $P(7 < \overline{X} < 9)$.

5.3.3. Let Y be $b(72, \frac{1}{3})$. Approximate $P(22 \le Y \le 28)$.

5.3.2

$$\text{var}(\overline{X}) = (2)(4^2)/128 = 1/4 \text{ and } E(\overline{X}) = (2)(4) = 8;$$

$$P\left(\frac{7-8}{1/2} < \frac{\overline{X}-8}{1/2} < \frac{9-8}{1/2}\right) \approx \Phi(2) - \Phi(-2).$$

5.3.3

$$P(21.5 < Y < 28.5) \approx \Phi\left(\frac{28.5-24}{4}\right) - \Phi\left(\frac{21.5-24}{4}\right),$$

because $E(Y) = 24$ and $\text{var}(Y) = 16$.

5.3.5. Let Y denote the sum of the observations of a random sample of size 12 from a distribution having pmf $p(x) = \frac{1}{6}$, $x = 1, 2, 3, 4, 5, 6$, zero elsewhere. Compute an approximate value of $P(36 \le Y \le 48)$.
Hint: Since the event of interest is $Y = 36, 37, \ldots, 48$, rewrite the probability as $P(35.5 < Y < 48.5)$.

5.3.5

$$E(X) = 3.5 \text{ and } \text{var}(X) = 35/12 \ \Rightarrow\ E(Y) = 42 \text{ and } \text{var}(Y) = 35.$$

Hence,

$$P(35.5 < Y < 48.5) \approx \Phi\left(\frac{48.5-42}{\sqrt{35}}\right) - \Phi\left(\frac{35.5-42}{\sqrt{35}}\right).$$

5.3.12. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean μ. Thus, $Y = \sum_{i=1}^{n} X_i$ has a Poisson distribution with mean $n\mu$. Moreover, $\overline{X} = Y/n$ is approximately $N(\mu, \mu/n)$ for large n. Show that $u(Y/n) = \sqrt{Y/n}$ is a function of Y/n whose variance is essentially free of μ.

5.3.12

$$
\begin{aligned}
u(\overline{X}) &\approx v(\overline{X}) = u(\mu) + u'(\mu)(\overline{X}), \\
\mathrm{var}[v(\overline{X})] &= [u'(\mu)]^2(\mu/n) = c, \\
u'(\mu) &= c_1/\sqrt{\mu}, \text{ a solution is } u(\mu) = c_2\sqrt{\mu}.
\end{aligned}
$$

Taking $c_2 = 1$, we have $u(\overline{X}) = \sqrt{\overline{X}}$.