# 4.1 Atom-field interactions

$$\hat{H}_0 = \frac{1}{2m}\hat{P} + V(r)$$

$$\hat{H}(r,t) = \frac{1}{2m}\left[\hat{P} + eA(r,t)\right]^2 - e\Phi(r,t) + V(r)$$

$$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$R = e^{\frac{-ie\chi}{\hbar}}$$

$$\psi' = R\psi$$

$$\begin{aligned} H'\psi' = H'R\psi &= i\hbar\frac{\partial}{\partial t}(R\,\psi) \\ &= i\hbar\left[R\frac{\partial\psi}{\partial t} + \left(\frac{\partial R}{\partial t}\right)\psi\right] \\ &= R\left[i\hbar\frac{\partial\psi}{\partial t}\right] + i\hbar\left(\frac{\partial R}{\partial t}\right)\psi \\ &= RH\psi + i\hbar\left(\frac{\partial R}{\partial t}\right)\psi \end{aligned}$$

$$H'R\psi R^+ = RHR^+ + i\hbar\left(\frac{\partial R}{\partial t}\right)R^+$$

$$H' = RHR^+ + i\hbar\left(\frac{\partial R}{\partial t}\right)R^+$$

$$H = \frac{1}{2m}\left[\hat{P} + eA\right]^2 - e\Phi + \mathrm{V}(r)$$

$$H' = RHR^+ + i\hbar\left(\frac{\partial R}{\partial t}\right)R^+$$

$$= e^{-\frac{ie\chi}{\hbar}}\,H\,e^{\frac{ie\chi}{\hbar}} + i\hbar\left(\frac{\partial}{\partial t}e^{-\frac{ie\chi}{\hbar}}\right)e^{\frac{ie\chi}{\hbar}}$$

$$= e^{-\frac{ie\chi}{\hbar}}\left[\frac{1}{2m}\left[\hat{P} + eA\right]^2 - e\Phi + \mathrm{V}(r)\right]e^{\frac{ie\chi}{\hbar}} + e\frac{\partial\chi}{\partial t}$$

$$= \frac{1}{2m}\left[\hat{P} + eA'\right]^2 - e\Phi + V(r) + e\frac{\partial\chi}{\partial t}$$

$$H' = \frac{1}{2m}\left[\hat{P} + eA'\right]^2 - e\left(\Phi - \frac{\partial\chi}{\partial t}\right) + \mathrm{V}(r)$$

hasil: $$H' = \frac{1}{2m}\left[\hat{P} + eA'\right]^2 - e\Phi' + \mathrm{V}(r)$$

$$\Phi' = \Phi - \frac{\partial\chi}{\partial t}$$

$$e^{-\frac{ie\chi}{\hbar}}[-e\Phi + \mathrm{V}(r)]e^{-\frac{ie\chi}{\hbar}} = -e\Phi + \mathrm{V}(r)$$

$$i\hbar\left(\frac{\partial}{\partial t}e^{-\frac{ie\chi}{\hbar}}\right)e^{\frac{ie\chi}{\hbar}} = i\hbar\frac{-ie}{\hbar}\,e^{-\frac{ie\chi}{\hbar}}\frac{\partial\chi}{\partial t}e^{\frac{ie\chi}{\hbar}}$$

$$= \frac{\partial\chi}{\partial t}\,e = e\frac{\partial\chi}{\partial t}$$

yang dimana $\hat{P} = i\hbar\nabla$

$$\begin{aligned} \left[\hat{P} + eA\right]^2 e^{\frac{ie\chi}{\hbar}}\psi &= \left[\hat{P}\hat{P} + eA\hat{P} + \hat{P}eA + e^2AA\right]e^{\frac{ie\chi}{\hbar}}\psi \\ &= \left[\hat{P}^2 + eA\hat{P} + \hat{P}eA + e^2A^2\right]e^{\frac{ie\chi}{\hbar}}\psi \end{aligned}$$

$$= -\hbar^2\nabla^2\left(e^{\frac{ie\chi}{\hbar}}\psi\right) - ie\hbar\nabla\cdot\left(Ae^{\frac{ie\chi}{\hbar}}\psi\right) - ie\hbar A\cdot\nabla\left(e^{\frac{ie\chi}{\hbar}}\psi\right) + e^2A^2e^{\frac{ie\chi}{\hbar}}\psi$$

Dengan,

$$\begin{aligned}
\nabla^2\left(e^{\frac{ie\chi}{\hbar}}\psi\right) &= -\frac{ie}{\hbar}\nabla\cdot\left(e^{\frac{ie\chi}{\hbar}}\psi\nabla\chi\right) + \nabla\cdot\left(e^{\frac{ie\chi}{\hbar}}\nabla\psi\right)\\
&= \frac{ie}{\hbar}(\nabla^2\chi)e^{\frac{ie\chi}{\hbar}}\psi + \frac{ie}{\hbar}(\nabla\chi)\cdot\nabla\left(e^{\frac{ie\chi}{\hbar}}\psi\right) + e^{\frac{ie\chi}{\hbar}}\psi\nabla^2\psi + \nabla\psi\cdot\nabla e^{\frac{ie\chi}{\hbar}}\\
&= \frac{ie}{\hbar}(\nabla^2\chi)e^{\frac{ie\chi}{\hbar}}\psi - \frac{e^2}{\hbar^2}(\nabla\chi)^2e^{\frac{ie\chi}{\hbar}}\psi + \frac{ie}{\hbar}e^{\frac{ie\chi}{\hbar}}(\nabla\chi)\cdot\nabla^2\psi + e^{\frac{ie\chi}{\hbar}}\nabla^2\psi + \frac{ie}{\hbar}e^{\frac{ie\chi}{\hbar}}(\nabla\chi)\cdot(\nabla\psi)\\
\nabla\left(e^{\frac{ie\chi}{\hbar}}\psi\right) &= \psi\nabla e^{\frac{ie\chi}{\hbar}} + e^{\frac{ie\chi}{\hbar}}\nabla\psi\\
&= -\frac{ie}{\hbar}e^{\frac{ie\chi}{\hbar}}(\nabla\chi)\psi + e^{\frac{ie\chi}{\hbar}}\nabla\psi\\
\nabla\cdot\left(Ae^{\frac{ie\chi}{\hbar}}\psi\right) &= (\nabla\cdot\mathrm{A})e^{\frac{ie\chi}{\hbar}}\psi + A\cdot\nabla\left(e^{\frac{ie\chi}{\hbar}}\psi\right)\\
&= (\nabla\cdot\mathrm{A})e^{\frac{ie\chi}{\hbar}}\psi + \frac{ie}{\hbar}e^{\frac{ie\chi}{\hbar}}(A\cdot\nabla\chi)\psi + e^{\frac{ie\chi}{\hbar}}A\cdot\nabla\psi\\
\mathrm{A}\cdot\nabla\left(e^{\frac{ie\chi}{\hbar}}\psi\right) &= -\frac{ie}{\hbar}e^{\frac{ie\chi}{\hbar}}(\mathrm{A}\cdot\nabla\chi)\psi + e^{\frac{ie\chi}{\hbar}}\mathrm{A}\cdot\nabla\psi
\end{aligned}$$

$A' = A + \nabla\chi$ dan perhatikan suku yang mengandung $\chi$, dapat ditulis

$\left(\hat{P}\cdot e\nabla\chi\right)\psi = -ie\hbar\nabla\cdot(\psi\nabla\chi) = -ie\hbar(\nabla^2\chi)\psi - ie\hbar\,\nabla\chi\cdot\nabla\psi$

$\left(e\nabla\chi\cdot\hat{P}\right)\psi = ie\hbar\nabla\chi\cdot\nabla\psi$

$$e^{-\frac{ie\chi}{\hbar}}\left[\hat{P}+eA\right]^2e^{\frac{ie\chi}{\hbar}}\psi = -\hbar^2\left[\frac{ie}{\hbar}(\nabla^2\chi)\psi - \frac{e^2}{\hbar^2}(\nabla\chi)^2\psi + \frac{2ie}{\hbar}(\nabla\chi)\cdot\nabla^2\psi + \nabla^2\psi + \frac{ie}{\hbar}(\nabla\chi)\cdot(\nabla\psi)\right]$$

$$\begin{aligned}
&\quad - ie\hbar\left[(\nabla\cdot\mathrm{A})\psi + \frac{ie}{\hbar}(A\cdot\nabla\chi)\psi + A\cdot\nabla\psi\right] - ie\hbar\left[\frac{ie}{\hbar}(\mathrm{A}\cdot\nabla\chi)\psi + \mathrm{A}\cdot\nabla\psi\right] + e^2A^2\psi\\
&= \{-\hbar^2\nabla^2\psi\} + \{[-ie\hbar(\nabla\cdot A)\psi - ie\hbar(A\cdot\nabla)\psi] + [ie\hbar(\nabla^2\chi)\psi - i\hbar\nabla\chi\cdot\nabla\psi]\}\\
&\qquad +\{[-ie\hbar A\cdot\nabla\psi] + [-ie\hbar\nabla\chi\cdot\nabla\psi]\}\\
&\qquad +\{[e^2A^2\psi] + [e^2(A\cdot\nabla\chi)\psi] + [e^2(A\cdot\nabla\chi)\psi] + [e^2(\nabla\chi)^2\psi]\}\\
&= \hat{P}^2\psi + \left\{\hat{P}\cdot(eA\psi) + \hat{P}\cdot(\psi e\nabla\chi)\right\} + \left\{e\left(A\cdot\hat{P}\right)\psi + \left(e\nabla\chi\cdot\hat{P}\right)\psi\right\} + \{e(A+\nabla\chi)\}^2\psi\\
&= \hat{P}^2\psi + \left\{\hat{P}\cdot e(A+\nabla\chi)\right\}\psi + \left\{e(A+\nabla\chi)\cdot\hat{P}\right\}\psi + \{e(A+\nabla\chi)\}^2\psi\\
&= \left[\hat{P}+eA'\right]^2\psi
\end{aligned}$$

$$R\left[\hat{P}+eA\right]^2R^+ = \left[\hat{P}+eA'\right]^2$$