A more general analysis of atom-light interaction using the Jaynes-Cummings model

We consider a two-level atomic state initially in the state

$$|\psi(0)\rangle_{\text{atom}} = C_g |g\rangle + C_e |e\rangle \tag{1}$$

and a photon initially in the state

$$|\psi(0)\rangle_{\text{light}} = \sum_{n=0}^{\infty} C_n |n\rangle \tag{2}$$

The initial atom-light composite state is

$$|\psi(0)\rangle = |\psi(0)\rangle_{\text{atom}} |\psi(0)\rangle_{\text{light}} = \sum_{n=0}^{\infty} C_g C_n |g\rangle |n\rangle + C_e C_n |e\rangle |n\rangle \tag{3}$$

There are some possibilities for the state the system can be in at a later time. If the system is initially in the $|g\rangle |n\rangle$ state, then it will be in either $|g\rangle |n\rangle$ or $|e\rangle |n-1\rangle$ state. If it is initially in the $|e\rangle |n\rangle$ state, then the possibilities are $|e\rangle |n\rangle$ or $|g\rangle |n+1\rangle$. And this is for some value of n. Generally, we have

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} A_n(t) |g\rangle |n\rangle + B_n(t) |e\rangle |n-1\rangle + P_n(t) |e\rangle |n\rangle + Q_n(t) |g\rangle |n+1\rangle \tag{4}$$

The interaction Hamiltonian is given by

$$\hat{H}_{\text{II}} = \lambda \left(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger \right) \tag{5}$$

Putting (4) and (5) into

$$\frac{\partial |\psi\rangle}{\partial t} = -i \hat{H}_{\text{II}} |\psi\rangle \tag{6}$$

we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} \dot{A}_n |g\rangle |n\rangle + \dot{B}_n |e\rangle |n-1\rangle + \dot{P}_n |e\rangle |n\rangle + \dot{Q}_n |g\rangle |n+1\rangle \\ = -i\lambda &\sum_{n=0}^{\infty} A_n |e\rangle \sqrt{n} |n-1\rangle + B_n |g\rangle \sqrt{n} |n\rangle + P_n |g\rangle \sqrt{n+1} |n+1\rangle + Q_n |e\rangle \sqrt{n+1} |n\rangle \end{aligned} \tag{7}$$

Matching the terms, we have

$$\dot{A}_n = -i\lambda\sqrt{n} B_n \tag{8}$$

$$\dot{B}_n = -i\lambda\sqrt{n} A_n \tag{9}$$

$$\dot{P}_n = -i\lambda\sqrt{n+1} Q_n \tag{10}$$

$$\dot{Q}_n = -i\lambda\sqrt{n+1} P_n \tag{11}$$

Solving (8) and (9) simultaneously, with the initial conditions $A_n(0) = C_g C_n$ and $B_n(0) = 0$, we obtain

$$\begin{aligned} A_n(t) &= C_g C_n \cos\left(\lambda t \sqrt{n}\right) \\ B_n(t) &= -i C_g C_n \sin\left(\lambda t \sqrt{n}\right) \end{aligned} \tag{12}$$

Solving (10) and (11) simulatneously with the initial conditions $P_n(0) = C_e C_n$ and $Q_n(0) = 0$, we obtain

$$\begin{aligned} P_n(t) &= C_e C_n \cos\left(\lambda t \sqrt{n+1}\right) \\ Q_n(t) &= -i C_e C_n \sin\left(\lambda t \sqrt{n+1}\right) \end{aligned} \tag{13}$$

We finally obtain

$$\begin{aligned} |\psi(t)\rangle = \sum_{n=0}^{\infty} &C_g C_n \cos\left(\lambda t \sqrt{n}\right) |g\rangle |n\rangle - i C_g C_n \sin\left(\lambda t \sqrt{n}\right) |e\rangle |n-1\rangle \\ &+ C_e C_n \cos\left(\lambda t \sqrt{n+1}\right) |e\rangle |n\rangle - i C_e C_n \sin\left(\lambda t \sqrt{n+1}\right) |g\rangle |n+1\rangle \end{aligned} \tag{14}$$

This result is a bit different compared to equation (4.120) in Gerry & Knight. We can match the result by going back to (4) and shift the indices of the second and fourth term by writing

$$\sum_{m=0}^{\infty} B_m(t) \left|e\right\rangle \left|m-1\right\rangle \quad \rightarrow \quad \sum_{n=0}^{\infty} B_{n+1} \left|e\right\rangle \left|n\right\rangle \tag{15}$$

and

$$\sum_{m=0}^{\infty} Q_m(t) \left|g\right\rangle \left|m+1\right\rangle \quad \rightarrow \quad \sum_{n=0}^{\infty} Q_{n-1} \left|g\right\rangle \left|n\right\rangle \tag{16}$$

Note that $D_{n-1} \equiv 0$. If we look at (4), we can see that the fourth term starts with $D_0 \left|g\right\rangle \left|1\right\rangle$ so it is justified to "pad" the series with an extra term equal to zero. Meanwhile, it is obvious that $\left|e\right\rangle \left|n-1\right\rangle = 0$ for $n = 0$ (i.e. there is no spontaneous absorption) so it is safe to discard the first of the second term.

Using (15) and (16) on (14), we finally obtain

$$\begin{aligned} \left|\psi(t)\right\rangle &= \sum_{n=0}^{\infty} C_g C_n \cos\left(\lambda t\sqrt{n}\right) \left|g\right\rangle \left|n\right\rangle - iC_g C_{n+1} \sin\left(\lambda t\sqrt{n+1}\right) \left|e\right\rangle \left|n\right\rangle + C_e C_n \cos\left(\lambda t\sqrt{n+1}\right) \left|e\right\rangle \left|n\right\rangle - iC_e C_{n-1} \sin\left(\lambda t\sqrt{n}\right) \left|g\right\rangle \left|n\right\rangle \\ &= \sum_{n=0}^{\infty} \left\{ \left[C_g C_n \cos\left(\lambda t\sqrt{n}\right) - iC_e C_{n-1} \sin\left(\lambda t\sqrt{n}\right)\right] \left|g\right\rangle + \left[C_e C_n \cos\left(\lambda t\sqrt{n+1}\right) - iC_g C_{n+1} \sin\left(\lambda t\sqrt{n+1}\right)\right] \left|e\right\rangle \right\} \left|n\right\rangle \end{aligned} \tag{17}$$

which is equation (4.120) in Gerry & Knight.