1. Consider the superposition state $|\psi_{01}\rangle = \alpha|0\rangle + \beta|1\rangle$ where α and β are complex and satisfy $|\alpha|^2 + |\beta|^2 = 1$. Calculate the variance of the quadrature operators $\hat{X}_1$ and $\hat{X}_2$. Are there any values of the parameters α and β for which either of the quadrature variances become less than for a vacuum state?

Jawaban :

$\hat{X}_1 = \frac{1}{2}(a^+ + a)$

$\hat{X}_2 = \frac{i}{2}(a^+ - a)$

$$a|n\rangle = \sqrt{n}|n-1\rangle \text{ dan } a^+|n\rangle = \sqrt{n+1}|n+1\rangle$$

Pertama mencari dahulu nilai ekspektasi dari

$$\left\langle \left(\hat{X}_1\right)^2 \right\rangle = \left\langle \psi_{01} \left| \left(\frac{1}{2}(a^+ + a)\right)^2 \right| \psi_{01} \right\rangle$$

$$= \frac{1}{4}\langle\psi_{01}|(a^+)^2 + a^+a + aa^+ + (a)^2|\psi_{01}\rangle$$

$$= \frac{1}{4}\langle\psi_{01}|((a^+)^2 + a^+a + aa^+ + (a)^2|(\alpha|0\rangle + \beta|1\rangle)$$

$$= \frac{1}{4}\langle\psi_{01}|\left(\sqrt{2}\alpha|2\rangle + 0 + \alpha|0\rangle + 0 + \sqrt{6}\beta|3\rangle + \beta|1\rangle + 2\beta|1\rangle + 0\right)$$

$$= \frac{1}{4}\left(\langle 0|\alpha^* + \langle 1|\beta^*\left(\sqrt{2}\alpha|2\rangle + \alpha|0\rangle + \sqrt{6}\beta|3\rangle + \beta|1\rangle + 2\beta|1\rangle\right)\right)$$

$$= \frac{1}{4}(|\alpha|^2 + 3|\beta|^2)$$

$$\left\langle \left(\hat{X}_1\right)^2 \right\rangle = \frac{1}{4}(1 + 2|\beta|^2) = \frac{1}{4}(3 - 2|\alpha|^2)$$

$$\left\langle \hat{X}_1 \right\rangle^2 = \left(\left\langle \psi_{01} \left| \frac{1}{2}(a^+ + a) \right| \psi_{01} \right\rangle\right)^2$$

$$= \frac{1}{4}(\langle\psi_{01}|(a^+ + a)(\alpha|0\rangle + \beta|1\rangle))^2$$

$$= \frac{1}{4}\left(\langle\psi_{01}|\left(\alpha|1\rangle + 0 + \beta\sqrt{2}|2\rangle + \beta|0\rangle\right)\right)^2$$

$$= \frac{1}{4}\left((\langle 0|\alpha^* + \langle 1|\beta^*)\left(\alpha|1\rangle + \beta\sqrt{2}|2\rangle + \beta|0\rangle\right)\right)^2$$

$$\left\langle \hat{X}_1 \right\rangle^2 = \frac{1}{4}(\alpha^*\beta + \beta^*\alpha)^2$$

Maka nilai dari <u>*variances*</u> untuk *quadrature* operator adalah

$$\left(\Delta\hat{X}_1\right)^2 = \left\langle\left(\hat{X}_1\right)^2\right\rangle - \langle\hat{X}_1\rangle^2$$

$$= \frac{1}{4}(1+2|\beta|^2) - \frac{1}{4}(\alpha^*\beta + \beta^*\alpha)^2$$

Dengan mengubah $|\beta|^2 + |\alpha|^2 = 1 \rightarrow \alpha = \sqrt{1-|\beta|^2}e^{i\phi}$ dan $\alpha^* = \sqrt{1-|\beta|^2}e^{-i\phi}$

$$\left(\Delta\hat{X}_1\right)^2 = \frac{1}{4}(1+2|\beta|^2) - \frac{1}{4}\left(\sqrt{1-|\beta|^2}e^{-i\phi}\beta + \beta^*\sqrt{1-|\beta|^2}e^{i\phi}\right)^2$$

$$= \frac{1}{4}(1+2|\beta|^2) - \frac{1}{4}\left((1-|\beta|^2)e^{-2i\phi}\beta^2 + (\beta^*)^2(1-|\beta|^2)e^{2i\phi} + 2(1-|\beta|^2)|\beta|^2\right)$$

$$= \frac{1}{4}\left((1+2|\beta|^2) - (1-|\beta|^2)\left(\beta^2e^{-2i\phi} + (\beta^*)^2e^{2i\phi} + 2|\beta|^2\right)\right)$$

$$\left(\Delta\hat{X}_1\right)^2 = \frac{1}{4}(1+2|\beta|^4 - (1-|\beta|^2)|\beta|^2 2\cos 2\phi)$$

Atau jika ingin mencari α dimana $\beta = \sqrt{1-|\alpha|^2}e^{i\phi}$ dan $\beta^* = \sqrt{1-|\alpha|^2}e^{-i\phi}$

$$\left(\Delta\hat{X}_1\right)^2 = \frac{1}{4}(3-2|\alpha|^2) - \frac{1}{4}\left(\sqrt{1-|\alpha|^2}e^{-i\phi}\alpha + \alpha^*\sqrt{1-|\alpha|^2}e^{i\phi}\right)^2$$

$$= \frac{1}{4}(3-2|\alpha|^2) - \frac{1}{4}\left((1-|\alpha|^2)e^{-2i\phi}\alpha^2 + (\alpha^*)^2(1-|\alpha|^2)e^{2i\phi} + 2(1-|\alpha|^2)|\alpha|^2\right)$$

$$= \frac{1}{4}\left((3-2|\alpha|^2) - (1-|\alpha|^2)\left(\alpha^2e^{-2i\phi} + (\alpha^*)^2e^{2i\phi} + 2|\alpha|^2\right)\right)$$

$$\left(\Delta\hat{X}_1\right)^2 = \frac{1}{4}(3-4|\alpha|^2 + 2|\alpha|^4 - (1-|\alpha|^2)|\alpha|^2 2\cos 2\phi)$$

Untuk $\hat{X}_2$ maka mencari dulu nilai ekspektasi masing-masing yaitu

$$\left\langle\left(\hat{X}_2\right)^2\right\rangle = \left\langle\psi_{01}\left|\left(\frac{i}{2}(a^+ - a)\right)^2\right|\psi_{01}\right\rangle$$

$$= -\frac{1}{4}\langle\psi_{01}|(a^+)^2 - a^+a - aa^+ + (a)^2|\psi_{01}\rangle$$

$$= -\frac{1}{4}\langle\psi_{01}|((a^+)^2 - a^+a - aa^+ + (a)^2|(\alpha|0\rangle + \beta|1\rangle)$$

$$= -\frac{1}{4}\langle\psi_{01}|\left(\sqrt{2}\alpha|2\rangle - 0 - \alpha|0\rangle + 0 + \sqrt{6}\beta|3\rangle - \beta|1\rangle - 2\beta|1\rangle + 0\right)$$

$$= -\frac{1}{4}\left(\langle 0|\alpha^* + \langle 1|\beta^*\left(\sqrt{2}\alpha|2\rangle - \alpha|0\rangle + \sqrt{6}\beta|3\rangle - \beta|1\rangle - 2\beta|1\rangle\right)\right)$$

$$= -\frac{1}{4}(-|\alpha|^2 - 3|\beta|^2)$$

$$\left\langle\left(\hat{X}_2\right)^2\right\rangle = \frac{1}{4}(1+2|\beta|^2)$$

$$\langle \hat{X}_2 \rangle^2 = \left(\left\langle \psi_{01} \left| \frac{i}{2} (a^+ - a) \right| \psi_{01} \right\rangle \right)^2$$

$$= -\frac{1}{4} (\langle \psi_{01} | (a^+ - a)(\alpha|0\rangle + \beta|1\rangle))^2$$

$$= -\frac{1}{4} \left(\langle \psi_{01} | \big(\alpha|1\rangle - 0 + \beta\sqrt{2}|2\rangle - \beta|0\rangle \big) \right)^2$$

$$= -\frac{1}{4} \left((\langle 0|\alpha^* + \langle 1|\beta^*) \big(\alpha|1\rangle + \beta\sqrt{2}|2\rangle - \beta|0\rangle \big) \right)^2$$

$$\langle \hat{X}_2 \rangle^2 = -\frac{1}{4} (\beta^* \alpha - a^* \beta)^2$$

Maka nilai dari *variances* untuk *quadrature* operator adalah

$$\left(\Delta \hat{X}_2\right)^2 = \left\langle \left(\hat{X}_2\right)^2 \right\rangle - \langle \hat{X}_2 \rangle^2$$

$$= \frac{1}{4} (1 + 2|\beta|^2) + \frac{1}{4} (\beta^* \alpha - \alpha^* \beta)^2$$

Jika ingin mencari α dimana $\beta = \sqrt{1 - |\alpha|^2} e^{i\phi}$ dan $\beta^* = \sqrt{1 - |\alpha|^2} e^{-i\phi}$

$$\left(\Delta \hat{X}_2\right)^2 = \frac{1}{4} (3 - 2|\alpha|^2) + \frac{1}{4} \left(\sqrt{1 - |\alpha|^2} e^{-i\phi} \alpha - \alpha^* \sqrt{1 - |\alpha|^2} e^{i\phi} \right)^2$$

$$= \frac{1}{4} (3 - 2|\alpha|^2) + \frac{1}{4} \left((1 - |\alpha|^2) e^{-2i\phi} \alpha^2 + (\alpha^*)^2 (1 - |\alpha|^2) e^{2i\phi} - 2(1 - |\alpha|^2)|\alpha|^2 \right)$$

$$= \frac{1}{4} \left((3 - 2|\alpha|^2) + (1 - |\alpha|^2) \big(\alpha^2 e^{-2i\phi} + (\alpha^*)^2 e^{2i\phi} - 2|\alpha|^2 \big) \right)$$

$$\left(\Delta \hat{X}_2\right)^2 = \frac{1}{4} (3 - 4|\alpha|^2 + 2|\alpha|^4 + (1 - |\alpha|^2)|\alpha|^2 2 \cos 2\phi)$$

Lalu *plotting* bagian hasil analitik melalui *python* dan di-*upload* di github.