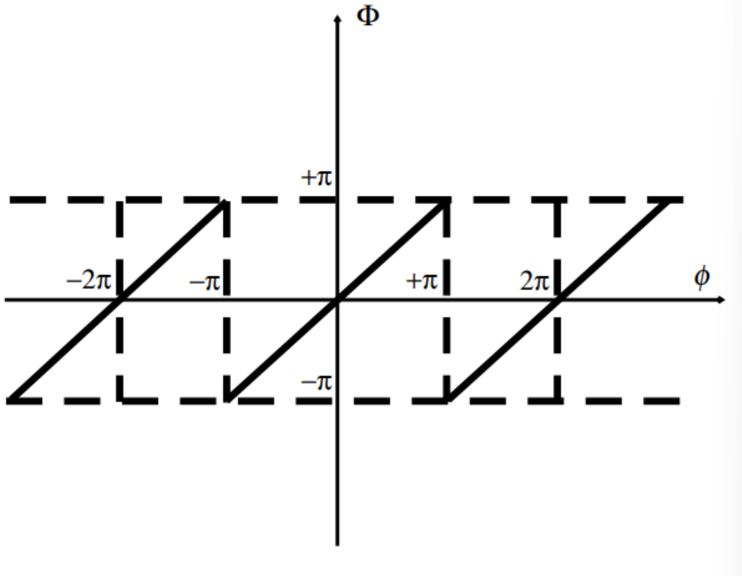
Tugas 6

1. Buktikan hubungan persamaan berikut ini benar

$$[\Phi, L_z] = i\left(1 - 2\pi \sum_{n=-\infty}^{n=\infty} \delta[\phi - (2n+1)\pi]\right)$$

Dengan $\hbar = 1$.

Jawaban :



Gambar 1. Grafik fungsi periodik Φ

Dengan menggunakan $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\frac{\partial}{\partial\phi}$ dan Φ diekstrak dari grafik periodik tersebut. Untuk menentukan grafik tersebut dengan fungsi kompleks *fourier series* berikut

$$\Phi(\phi) = 2\sum_{n=1}^{n=\infty} (-1)^{n+1}\frac{\sin n\phi}{n} = \sum_{n=1}^{n=\infty} (-1)^{n+\frac{1}{2}}\frac{e^{in\phi} - e^{-in\phi}}{n}$$

Sehingga

$$[\Phi, \hat{L}_z]|\psi_n\rangle = \left[\Phi, i\frac{\partial}{\partial\phi}\right]|\psi_n\rangle$$

$$= \Phi\left(-i\frac{\partial}{\partial\phi}\right)|\psi_n\rangle - \left(-i\frac{\partial}{\partial\phi}\right)\Phi|\psi_n\rangle$$

$$= -i\Phi\frac{\partial}{\partial\phi}|\psi_n\rangle + i\left(\frac{\partial\Phi}{\partial\phi}|\psi_n\rangle + \Phi\frac{\partial}{\partial\phi}|\psi_n\rangle\right)$$

$$= i\frac{\partial\Phi}{\partial\phi}|\psi_n\rangle$$

$$[\Phi, \hat{L}_z] = i\frac{\partial\Phi}{\partial\phi}$$

Lalu, memasukkan nilai Φ ke dalam persamaan di atas menjadi

$$[\Phi, \hat{L}_z] = i\frac{\partial}{\partial \phi}\left(2\sum_{n=1}^{n=\infty}(-1)^{n+1}\frac{\sin n\phi}{n}\right)$$

$$= -2i\sum_{n=1}^{\infty}(-1)^n \cos n\phi$$

Mengingat persamaan eksponensial untuk nilai minus 1 yaitu $-1 = e^{\pm i(2n+1)\pi}$

$$[\Phi, \hat{L}_z] = -2i\sum_{n=1}^{\infty}\left(e^{\pm i(2n+1)\pi}\right)^n\left(\frac{e^{in\phi} + e^{-in\phi}}{2}\right)$$

$$= -i\sum_{n=1}^{\infty}\left(e^{in(\phi-(2n+1)\pi)} + e^{-in(\phi-(2n+1)\pi)}\right)$$

$$= -i\sum_{n=-\infty}^{\infty}\left(e^{in(\phi-(2n+1)\pi)} - 1\right)$$

$$= i - i\sum_{n=-\infty}^{\infty} e^{in(\phi-(2n+1)\pi)}$$

Sumasi di atas bisa diganti sebagai fungsi delta dyrac dari persamaan

$$f(x) = 2\pi\delta(x) \to \delta(x) = \frac{1}{2\pi}\left(\sum_{n=-\infty}^{\infty} e^{inx}\right)$$

$$[\Phi, \hat{L}_z] = i\big(1 - 2\pi\delta(\phi - (2n+1)\pi)\big)$$

2. Buktikan bahwa $\hat{C}^2 + \hat{S}^2 = 1 - \frac{1}{2}|0\rangle\langle 0|$

Jawaban :

Dengan menggunakan $\hat{C} = \frac{1}{2}\left(\hat{E} + \hat{E}^{+}\right)$, $\hat{S} = \frac{1}{2i}\left(\hat{E} - \hat{E}^{+}\right)$, $\hat{E} = \sum_{n=0}^{\infty}|n\rangle\langle n+1|$, dan $\hat{E}^{+} = \sum_{n=0}^{\infty}|n+1\rangle\langle n|$ maka

$$\hat{C}^2 + \hat{S}^2 = \frac{1}{4}\left(\hat{E}^2 + \hat{E}\hat{E}^{+} + \hat{E}^{+}\hat{E} + (E^{+})^2\right) - \frac{1}{4}\left(\hat{E}^2 - \hat{E}\hat{E}^{+} - \hat{E}^{+}\hat{E} + (E^{+})^2\right)$$

$$= \frac{1}{2}\left(\hat{E}\hat{E}^{+} + \hat{E}^{+}\hat{E}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0,m=0}^{\infty}|n\rangle\langle n+1||m+1\rangle\langle m| + \sum_{n=0,m=0}^{\infty}|n+1\rangle\langle n||m\rangle\langle m+1|\right)$$

$$= \frac{1}{2}\left(\sum_{n=0,m=0}^{\infty}|n\rangle\delta_{n+1,m+1}\langle m| + \sum_{n=0,m=0}^{\infty}|n+1\rangle\delta_{n,m}\langle m+1|\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} |n\rangle\langle n| + \sum_{n=0}^{\infty} |n+1\rangle\langle n+1|\right)$$

Karena bagian kanan seharusnya dimulai dari $|1\rangle\langle 1|$ maka jika kita mengambil dari persamaan relasi kekomplitan yaitu $\sum_a \phi^*\phi|\phi\rangle\langle\phi| = 1$ akan ada satu bagian koordinat matriks yang tidak ikut pada persamaan di atas ke dalam relasi kekomplitan yaitu $|0\rangle\langle 0|$. Sehingga hasilnya menjadi

$$\hat{C}^2 + \hat{S}^2 = \frac{1}{2}(1 + 1 - |0\rangle\langle 0|)$$

$$= 1 - \frac{1}{2}|0\rangle\langle 0|$$

--Latihan--

1) Tunjukkan bahwa

$$E|n\rangle = \begin{cases} |n-1\rangle, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

$$E|n\rangle = \frac{1}{\sqrt{\hat{a}\hat{a}^+}}\hat{a}|n\rangle$$

Jawaban :

$$E|n\rangle = \frac{1}{\sqrt{1 + \hat{a}^+\hat{a}}}\sqrt{n}|n-1\rangle$$

Dengan ekspansi binomial dimana $(1+x)^n = 1 + nx + \cdots$ maka

$$E|n\rangle = \sqrt{n}(1 + \hat{a}^+\hat{a})^{-\frac{1}{2}}|n-1\rangle$$

$$= \sqrt{n}\left(1 - \frac{1}{2}\hat{a}^+\hat{a} + \cdots\right)|n-1\rangle$$

$$= \sqrt{n}\left(|n-1\rangle - \frac{1}{2}a^+\sqrt{n-1}|n-2\rangle\right)$$

$$= \sqrt{n}\left(|n-1\rangle - \frac{1}{2}\sqrt{n-1}\sqrt{n-1}|n-1\rangle\right)$$

$$= \sqrt{n}\left(1 - \frac{1}{2}(n-1)\right)|n-1\rangle$$

$$= \sqrt{n}(1 + n - 1)^{-\frac{1}{2}}|n-1\rangle$$

$$E|n\rangle = |n-1\rangle$$

Atau

$$E|n\rangle = 0 \text{ ketika } n = 0$$

2) Buktikan bahwa

$$[C, n] = iS$$

Dengan $\hat{E} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|$, $\hat{n} = \sum_{n=0}^{\infty} n|n\rangle\langle n|$, $\hat{C} = \frac{1}{2}\left(\hat{E} + \hat{E}^{+}\right)$, dan $\hat{S} = \frac{1}{2i}\left(\hat{E} - \hat{E}^{+}\right)$

$$[C, n] = \left[\frac{1}{2}\left(\hat{E} + \hat{E}^{+}\right), n\right]$$

$$= \frac{1}{2}\left(En - nE + E^{+}n - nE^{+}\right)$$

Untuk masing-masing persamaan bisa ditulis terlebih dahulu

$$En = \sum_{n=0}^{\infty} |n\rangle\langle n+1| \sum_{m=0}^{\infty} m|m\rangle\langle m|$$

$$= \sum_{n=0,m=0}^{\infty} |n\rangle\langle n+1|\, m|m\rangle\langle m|$$

$$= \sum_{n=0,m=0}^{\infty} m|n\rangle\delta_{n+1,m}\,\langle m|$$

$$En = \sum_{n=0}^{\infty} (n+1)|n\rangle\,\langle n+1|$$

Dan

$$nE = \sum_{m=0}^{\infty} m|m\rangle\langle m| \sum_{n=0}^{\infty} |n\rangle\langle n+1|$$

$$nE = \sum_{n=0}^{\infty} (n)|n\rangle\,\langle n+1|$$

Maka

$$[C, n] = \frac{1}{2} 2\left(E - E^{+}\right) = iS$$

3) Carilah nilai *corresponding phase distributions* $P(\phi)$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right)$$

$$|\phi\rangle = |0\rangle + e^{i\phi}|1\rangle + \cdots$$

Maka

$$P(\phi) = \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2$$

$$= \frac{1}{2\pi}\left|\frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|e^{-i\phi}\right)\left(|0\rangle + e^{i\theta}|1\rangle + \cdots\right)\right|^2$$

$$= \frac{1}{4\pi}\left|\langle 0|0\rangle + 0 + 0 + \left\langle 1\middle|e^{-i\phi+i\theta}\middle|1\right\rangle + 0\right|^2$$

$$= \frac{1}{4\pi}\left|1 + e^{i(\theta-\phi)}\right|^2$$

$$= \frac{1}{4\pi}\left(1 + e^{i(\theta-\phi)}\right)\left(1 + e^{-i(\theta-\phi)}\right)$$

$$= \frac{1}{4\pi}\left(1 + e^{-i(\theta-\phi)} + e^{i(\theta-\phi)} + 1\right)$$

$$= \frac{1}{4\pi}(2 + 2\cos(\theta-\phi))$$

$$P(\phi) = \frac{1}{2\pi}(1 + \cos(\theta-\phi)) = \frac{1}{\pi}\cos^2\frac{(\theta-\phi)}{2}$$