Problem 1: Matrix representation of operators in linear algebra

The completeness relation

$$\sum_n |n\rangle \langle n| = 1 \tag{1}$$

tells us that the set of vectors $\{|n\rangle\}$ spans its vector space, and that any vector in that space can be defined as some linear combination of them. There is, however, something else useful that comes with this relation. Let us consider an operator $\hat{A}$ and apply the relation twice like so:

$$\hat{A} = \sum_{m,n} |m\rangle \langle m| \hat{A} |n\rangle \langle n| \tag{2}$$

If N is the dimensionality of the vector space, then we have N^2 terms in the summation. We can arrange the $\langle m| \hat{A} |n\rangle$ into a $N \times N$ square matrix, forming the matrix representation of the operator. Formally,

$$\hat{A}_{mn} = \langle m| \hat{A} |n\rangle \tag{3}$$

Problem 2: Expectation value in terms of the density matrix

We again make use of the completeness relation:

$$\begin{aligned} \langle A\rangle &= \langle \psi | \hat{A} | \psi \rangle \\ &= \langle \psi | \left(\sum_j |j\rangle \langle j| \right) \hat{A} \left(\sum_i |i\rangle \langle i| \right) |\psi\rangle \\ &= \sum_{i,j} \langle \psi | j \rangle \langle j | \hat{A} | i \rangle \langle i | \psi \rangle \end{aligned} \tag{4}$$

But note that

$$\begin{aligned} \langle j | \hat{A} | i \rangle &= A_{ji} \\ \langle i | \psi \rangle \langle \psi | j \rangle &= \rho_{ij} \end{aligned} \tag{5}$$

Thus,

$$\langle A\rangle = \sum_{i,j} \rho_{ij} \hat{A}_{ji} = \sum_i \left(\rho \hat{A}\right)_{ii} = \mathrm{tr}\left(\rho \hat{A}\right) \tag{6}$$

Problem 3: Miller's QM Problem 14.3.1 We have

$$\begin{aligned} |\psi_1\rangle &= |H\rangle \quad &\rightarrow \quad &20\% \text{ chance} \\ |\psi_2\rangle &= 0.6|H\rangle + 0.8i|V\rangle \quad &\rightarrow \quad &80\% \text{ chance} \end{aligned} \tag{7}$$

where $|H\rangle$ and $|V\rangle$ are the orthonormal polarization basis state vectors.

As we are asked to put $\langle H| \rho |H\rangle$ as the top left element of the density matrix, we shall use

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8}$$

Our density matrix is

$$\begin{aligned} \rho &= 0.2 |\psi_1\rangle \langle \psi_1| + 0.8 |\psi_2\rangle \langle \psi_2| \\ &= 0.2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + 0.8 \begin{pmatrix} 0.6 \\ 0.8i \end{pmatrix} \begin{pmatrix} 0.6 & 0.8i \end{pmatrix} \\ &= \begin{pmatrix} 0.488 & 0.384i \\ 0.384i & -0.512 \end{pmatrix} \end{aligned} \tag{9}$$

Problem 4: Miller's QM Problem 14.3.2

(a) & (b)

With the z-axis as the axis of quantization, and the usual convention that

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ i \end{pmatrix} \tag{10}$$

we have

$$\begin{aligned}
\rho &= 0.5\,|s_x\rangle\langle s_x| + 0.5\,|s_y\rangle\langle s_y| \\
&= 0.5\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)\left(\langle\uparrow| + \langle\downarrow|\right) + 0.5\left(|\uparrow\rangle + i\,|\downarrow\rangle\right)\left(\langle\uparrow| + i\,\langle\downarrow|\right) \\
&= 0.5\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} + 0.5\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 2 & 1-i \\ 1+i & 2 \end{pmatrix}
\end{aligned} \tag{11}$$

(c)

This one is easy, since we are already familiar with the Pauli matrices in this basis. If $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are the Cartesian basis vectors, then

$$\begin{aligned}
\hat{\boldsymbol{\mu}} &= g\mu_B\left(\sigma_x\boldsymbol{x} + \sigma_y\boldsymbol{y} + \sigma_z\boldsymbol{z}\right) \\
&= g\mu_B\begin{pmatrix} \boldsymbol{z} & \boldsymbol{x} - i\boldsymbol{y} \\ \boldsymbol{x} + i\boldsymbol{y} & -\boldsymbol{z} \end{pmatrix}
\end{aligned} \tag{12}$$

(d)

$$\begin{aligned}
\langle\boldsymbol{\mu}\rangle &= \mathrm{tr}\left(\rho\hat{\boldsymbol{\mu}}\right) \\
&= \frac{g\mu_B}{4}\,\mathrm{tr}\left(\begin{pmatrix} 2 & 1-i \\ 1+i & 2 \end{pmatrix}\begin{pmatrix} \boldsymbol{z} & \boldsymbol{x} - i\boldsymbol{y} \\ \boldsymbol{x} + i\boldsymbol{y} & -\boldsymbol{z} \end{pmatrix}\right) \\
&= \frac{g\mu_B}{4}\left(2\boldsymbol{z} + (1-i)(\boldsymbol{x} + i\boldsymbol{y}) + (1+i)(\boldsymbol{x} - i\boldsymbol{y}) + 2(-\boldsymbol{z})\right) \\
&= \frac{g\mu_B}{4}(2, 2, 0)
\end{aligned} \tag{13}$$

It is exactly what we found in problem 14.1.1.