Tugas 10

Mencari nilai komutasi antara Hamiltonian dengan density matriks berikut

$$[H,\rho] = \sum_{i,j} \rho_{ij}\left(\left[-\frac{\gamma B_z}{2}(|0\rangle\langle 0| - |1\rangle\langle 1|), |i\rangle\langle j|\right]\right)$$

$$= -\frac{\gamma B_z}{2}\left(\sum_{i,j} \rho_{ij}\big((|0\rangle\langle 0| - |1\rangle\langle 1|)|i\rangle\langle j|\big) - \sum_{i,j} \rho_{ij}\big(|i\rangle\langle j|(|0\rangle\langle 0| - |1\rangle\langle 1|)\big)\right)$$

$$= -\frac{\gamma B_z}{2}\left(\sum_{i,j} \rho_{ij}(|0\rangle\delta_{0i}\langle j| - |1\rangle\delta_{1i}\langle j|) - \sum_{i,j} \rho_{ij}\big(|i\rangle\delta_{j0}\langle 0| - |i\rangle\delta_{j1}\langle 1|\big)\right)$$

$$= -\frac{\gamma B_z}{2}\left(\left(\sum_{j}(\rho_{0j}|0\rangle\langle j| - \rho_{1j}|1\rangle\langle j|)\right) - \left(\sum_{i}(\rho_{i0}|i\rangle\langle 0| - \rho_{i1}|i\rangle\langle 1|)\right)\right)$$

$$[H,\rho] = -\frac{\gamma B_z}{2}\sum_{i}(\rho_{0i}|0\rangle\langle i| - \rho_{1i}|1\rangle\langle i| - \rho_{i0}|i\rangle\langle 0| + \rho_{i1}|i\rangle\langle 1|)$$

SOAL LATIHAN

1) Dengan diketahui

$$H = \frac{\Delta}{2}\sigma_z + \Omega\sigma_x = \frac{\Delta}{2}(|0\rangle\langle 0| - |1\rangle\langle 1|) + \Omega(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

Maka

$$[H,\rho] = \sum_{i,j} \rho_{ij}\left(\left[\frac{\Delta}{2}(|0\rangle\langle 0| - |1\rangle\langle 1|) + \Omega(|0\rangle\langle 1| + |1\rangle\langle 0|), |i\rangle\langle j|\right]\right)$$

$$= \frac{\Delta}{2}\sum_{i}(\rho_{0i}|0\rangle\langle i| - \rho_{1i}|1\rangle\langle i| - \rho_{i0}|i\rangle\langle 0| + \rho_{i1}|i\rangle\langle 1|) + \sum_{i,j}\rho_{ij}(\Omega[(|0\rangle\langle 1| + |1\rangle\langle 0|), |i\rangle\langle j|])$$

Untuk bagian merah maka

$$\sum_{i,j}\rho_{ij}(\Omega[(|0\rangle\langle 1| + |1\rangle\langle 0|), |i\rangle\langle j|]) = \Omega\left(\sum_{i,j}\rho_{ij}\big((|0\rangle\langle 1| + |1\rangle\langle 0|)|i\rangle\langle j|\big) - \sum_{i,j}\rho_{ij}\big(|i\rangle\langle j|(|0\rangle\langle 1| + |1\rangle\langle 0|)\big)\right)$$

$$= \Omega\left(\sum_{i,j}\rho_{ij}(|0\rangle\delta_{i1}\langle j| + |1\rangle\delta_{i0}\langle j|) - \sum_{i,j}\rho_{ij}\big(|i\rangle\delta_{0j}\langle 1| + |i\rangle\delta_{1j}\langle 0|\big)\right)$$

$$= \Omega\sum_{i}(\rho_{1i}|0\rangle\langle i| + \rho_{0i}|1\rangle\langle i| - \rho_{i0}|i\rangle\langle 1| - \rho_{i1}|i\rangle\langle 0|)$$

Memasukkan kembali ke persamaan komutasi awal untuk bagian merah menjadi

$$= \frac{\Delta}{2}\sum_{i}(\rho_{0i}|0\rangle\langle i| - \rho_{1i}|1\rangle\langle i| - \rho_{i0}|i\rangle\langle 0| + \rho_{i1}|i\rangle\langle 1|) + \Omega\sum_{i}\big(\rho_{1i}|0\rangle\langle i| + \rho_{0i}|1\rangle\langle i| - \rho_{i0}|i\rangle\langle 1| - \rho_{i1}|i\rangle\langle 0|\big)$$

$$= \sum_i \left(\left(\frac{\Delta}{2} \rho_{0i} + \Omega \rho_{1i} \right) |0\rangle\langle i| + \left(\Omega \rho_{0i} - \frac{\Delta}{2} \rho_{1i} \right) |1\rangle\langle i| - \left(\Omega \rho_{i1} + \frac{\Delta}{2} \rho_{i0} \right) |i\rangle\langle 0| + \left(\frac{\Delta}{2} \rho_{i1} - \Omega \rho_{i0} \right) |i\rangle\langle 1| \right)$$

Mencari nilai $\dot{\rho}$ setiap fungsi

$$\dot{\rho}_{11} = -i\langle 1|[H,\rho]|1\rangle = -i\left(\left(\Omega \rho_{01} - \frac{\Delta}{2} \rho_{11} \right) + \left(\frac{\Delta}{2} \rho_{11} - \Omega \rho_{10} \right) \right) = -i\Omega(\rho_{01} - \rho_{10})$$

$$\dot{\rho}_{00} = -i\langle 0|[H,\rho]|0\rangle = -i\left(\left(\frac{\Delta}{2} \rho_{00} + \Omega \rho_{10} \right) - \left(\Omega \rho_{01} + \frac{\Delta}{2} \rho_{00} \right) \right) = -i\Omega(\rho_{10} - \rho_{01})$$

$$\dot{\rho}_{01} = -i\langle 0|[H,\rho]|1\rangle = -i\left(\left(\frac{\Delta}{2} \rho_{01} + \Omega \rho_{11} \right) + \left(\frac{\Delta}{2} \rho_{01} - \Omega \rho_{00} \right) \right) = -i\big(\Delta \rho_{01} + \Omega(\rho_{11} - \rho_{00})\big)$$

$$\dot{\rho}_{10} = -i\langle 1|[H,\rho]|0\rangle = -i\left(\left(\Omega \rho_{00} - \frac{\Delta}{2} \rho_{10} \right) - \left(\Omega \rho_{11} + \frac{\Delta}{2} \rho_{10} \right) \right) = -i\big(-\Delta \rho_{01} + \Omega(\rho_{00} - \rho_{11})\big)$$

Maka didapat

$$\dot{\rho} = -i\begin{pmatrix} \Omega(\rho_{10} - \rho_{01}) & \big(\Delta \rho_{01} + \Omega(\rho_{11} - \rho_{00})\big) \\ \big(-\Delta \rho_{01} + \Omega(\rho_{00} - \rho_{11})\big) & \Omega(\rho_{01} - \rho_{10}) \end{pmatrix}$$

Sehingga dari persamaan di atas dapat dibuat menjadi

$$\begin{aligned} \dot{\rho}_{11} - \dot{\rho}_{00} &= -i\Omega(\rho_{01} - \rho_{10}) + i\Omega(\rho_{10} - \rho_{01}) \\ &= i2\Omega(\rho_{10} - \rho_{01}) \\ &= 4\Omega\big(\mathrm{im}(\rho_{01})\big) \end{aligned}$$

dan

$$\begin{aligned} \ddot{\rho}_{01} &= -i\left(\Delta \dot{\rho}_{01} + \Omega\left(4\Omega\big(\mathrm{im}(\rho_{01})\big)\right)\right) \\ &= -i\Delta \dot{\rho}_{01} - i4\Omega^2\big(\mathrm{im}(\rho_{01})\big) \end{aligned}$$

$$re(\ddot{\rho}_{01}) + i\big(\mathrm{im}(\ddot{\rho}_{01})\big) = -i\Delta\left(re(\dot{\rho}_{01}) + i\big(im(\dot{\rho}_{01})\big)\right) - i4\Omega^2\big(\mathrm{im}(\rho_{01})\big)$$

Memisahkan bagian real dan imajiner menjadi

$$re(\ddot{\rho}_{01}) = \Delta im(\dot{\rho}_{01})$$

dan

$$\mathrm{im}(\ddot{\rho}_{01}) = -\Delta\big(re(\dot{\rho}_{01})\big) - 4\Omega^2\big(\mathrm{im}(\rho_{01})\big)$$

Untuk bagian real dapat diintegralkan menjadi

$$re(\dot{\rho}_{01}) = \Delta im(\rho_{01}) + C$$

Dengan C adalah konstanta integrasi. Maka dapat disubstitusikan pers. di atas menjadi

$$\mathrm{im}(\ddot{\rho}_{01}) = -\Delta(\Delta \mathrm{im}(\rho_{01}) + C) - 4\Omega^2\big(\mathrm{im}(\rho_{01})\big)$$

$$\mathrm{im}(\ddot{\rho}_{01}) + (4\Omega^2 + \Delta^2)\big(\mathrm{im}(\rho_{01})\big) + \Delta C = 0$$

Solusi dari persamaan diferensial di atas adalah

$$\mathrm{im}\big(\rho_{01}(t)\big) = A\cos(\omega t) + B\sin(\omega t) - C\frac{\Delta}{\omega^2} \text{ dengan } \omega^2 = 4\Omega^2 + \Delta^2$$

Substitusi solusi fungsi imajiner ke dalam pers. real yang telah diintegrasi menjadi

$$re(\dot{\rho}_{01}) = \Delta(A\cos(\omega t) + B\sin(\omega t)) + \left(1 - \frac{\Delta^2}{\omega^2}\right)C$$

Hasil integrasi adalah sebagai berikut

$$re\big(\rho_{01}(t)\big) = \frac{\Delta A}{\omega}\sin(\omega t) - \frac{\Delta B}{\omega}\cos(\omega t) + \left(1 - \frac{\Delta^2}{\omega^2}\right)Ct + D$$

Dengan *D* adalah konstanta integrasi. Maka persamaan kompleks dari ρ_{01} sebagai berikut

$$\begin{aligned}\rho_{01} &= re(\rho_{01}) + i\big(\mathrm{im}(\rho_{01})\big)\\ &= \frac{\Delta A}{\omega}\sin(\omega t) - \frac{\Delta B}{\omega}\cos(\omega t) + \left(1 - \frac{\Delta^2}{\omega^2}\right)Ct + D + i\left(A\cos(\omega t) + B\sin(\omega t) - C\frac{\Delta}{\omega^2}\right)\end{aligned}$$

$$\rho_{01} = \left(\frac{\Delta}{\omega}A + iB\right)\sin\omega t + \left(-\frac{\Delta}{\omega}B + iA\right)\cos\omega t + \left(1 - \frac{\Delta^2}{\omega^2}\right)Ct - iC\frac{\Delta}{\omega^2} + D$$

Mengingat kembali bahwa $(\rho_{01})^* = \rho_{10}$ maka

$$\rho_{10} = \left(\frac{\Delta}{\omega}A - iB\right)\sin\omega t + \left(-\frac{\Delta}{\omega}B - iA\right)\cos\omega t + \left(1 - \frac{\Delta^2}{\omega^2}\right)Ct + iC\frac{\Delta}{\omega^2} + D$$

Lalu untuk sisanya maka

$$\begin{aligned}\dot{\rho}_{11} &= -i\Omega(\rho_{01} - \rho_{10})\\ &= -i\Omega\Bigg(\left(\frac{\Delta}{\omega}A + iB\right)\sin\omega t + \left(-\frac{\Delta}{\omega}B + iA\right)\cos\omega t + \left(1 - \frac{\Delta^2}{\omega^2}\right)Ct - iC\frac{\Delta}{\omega^2} + D\\ &\qquad - \left(\left(\frac{\Delta}{\omega}A - iB\right)\sin\omega t + \left(-\frac{\Delta}{\omega}B - iA\right)\cos\omega t + \left(1 - \frac{\Delta^2}{\omega^2}\right)Ct + iC\frac{\Delta}{\omega^2} + D\right)\Bigg)\end{aligned}$$

$$\dot{\rho}_{11} = 2\Omega\left(B\sin\omega t + A\cos\omega t - C\frac{\Delta}{\omega^2}\right)$$

Maka hasil integrasinya adalah

$$\rho_{11} = 2\Omega\left(-\frac{A}{\omega}\sin\omega t + \frac{B}{\omega}\cos\omega t - C\frac{\Delta}{\omega^2}t\right) + E$$

Dengan E adalah konstanta integrasi. Selanjutnya untuk bagian satunya lagi maka

$$\begin{aligned}\dot{\rho}_{00} &= -i\Omega\big(\rho_{10} - \rho_{01}\big)\\ &= -i\Omega\Bigg(\left(\left(\frac{\Delta}{\omega}A - iB\right)\sin\omega t + \left(-\frac{\Delta}{\omega}B - iA\right)\cos\omega t + \left(1 - \frac{\Delta^2}{\omega^2}\right)Ct + iC\frac{\Delta}{\omega^2} + D\right)\\ &\qquad - \left(\frac{\Delta}{\omega}A + iB\right)\sin\omega t + \left(-\frac{\Delta}{\omega}B + iA\right)\cos\omega t + \left(1 - \frac{\Delta^2}{\omega^2}\right)Ct - iC\frac{\Delta}{\omega^2} + D\Bigg)\end{aligned}$$

$$= -i\Omega\left(-2iB\sin\omega t - 2iA\cos\omega t + 2iC\frac{\Delta}{\omega^2}\right)$$

$$\dot{\rho}_{00} = 2\Omega\left(-B\sin\omega t - A\cos\omega t + C\frac{\Delta}{\omega^2}\right)$$

Maka hasil integrasinya adalah

$$\rho_{00} = 2\Omega\left(\frac{A}{\omega}\sin\omega t - \frac{B}{\omega}\cos\omega t + C\frac{\Delta}{\omega^2}t\right) + F$$

Dengan F adalah konstanta integrasi. Maka dari sini sudah bisa terbentuk nilai *density matrix* dalam bentuk matriks berikut :

$$\rho = \begin{pmatrix} 2\Omega\left(\frac{A}{\omega}\sin\omega t - \frac{B}{\omega}\cos\omega t + C\frac{\Delta}{\omega^2}t\right) + F & \left(\frac{\Delta}{\omega}A + iB\right)\sin\omega t + \left(-\frac{\Delta}{\omega}B + iA\right)\cos\omega t + \left(1 - \frac{\Delta^2}{\omega^2}\right)Ct - iC\frac{\Delta}{\omega^2} + D \\ \left(\frac{\Delta}{\omega}A - iB\right)\sin\omega t + \left(-\frac{\Delta}{\omega}B - iA\right)\cos\omega t + \left(1 - \frac{\Delta^2}{\omega^2}\right)Ct + iC\frac{\Delta}{\omega^2} + D & 2\Omega\left(-\frac{A}{\omega}\sin\omega t + \frac{B}{\omega}\cos\omega t - C\frac{\Delta}{\omega^2}t\right) + E \end{pmatrix}$$

Namun, masih ada informasi yang kurang yaitu konstanta integrasi (C, D, E, dan F) maupun konstanta solusi ODE (A dan B) belum ditemukan. Maka dengan menganggap nilai density matrix dalam fungsi waktu keadaan $t = 0$ sebagai berikut

$$\rho(0) = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Didapat 4 persamaan berikut

$$\rho_{00}(0) = -\frac{2\Omega \mathrm{B}}{\omega} + \mathrm{F} = 1$$

$$\rho_{01}(0) = -\frac{\Delta}{\omega}B + iA - iC\frac{\Delta}{\omega^2} + D = 0$$

$$\rho_{10}(0) = -\frac{\Delta}{\omega}B - iA + iC\frac{\Delta}{\omega^2} + D = 0$$

$$\rho_{11}(0) = \frac{2\Omega B}{\omega} + E = 0$$

Sehingga hasil sederhananya menjadi

$$-\frac{4\Omega B}{\omega} - E + F = 1$$

$$F = 1 + E + \frac{4\Omega B}{\omega} \rightarrow F = 1 + \frac{2\Omega B}{\omega}$$

Dan

$$E + F = 1$$

$$E + 1 + E + \frac{4\Omega B}{\omega} = 1$$

$$2E = -4\Omega B \rightarrow E = -\frac{2\Omega B}{\omega}$$

Kemudian

$$2iA - 2iC\frac{\Delta}{\omega^2} = 0 \rightarrow A = C\frac{\Delta}{\omega^2}$$

$$-2\frac{\Delta}{\omega}B + 2D = 0 \rightarrow D = \frac{\Delta}{\omega}B$$

Lalu mengingat pers. dengan bagian turunan terhadap waktu berikut

$$\dot{\rho}_{01} = -i\left(\Delta\rho_{01} + \Omega(\rho_{11} - \rho_{00})\right)$$

$$\dot{\rho}_{01}(0) = -i\left(\Delta\rho_{01}(0) + \Omega\left(\rho_{11}(0) - \rho_{00}(0)\right)\right)$$

$$\dot{\rho}_{01}(0) = i\Omega(1-0) - i\Delta\rho_{01}(0)$$

$$\left(\frac{\Delta}{\omega}A + iB\right) + \left(1 - \frac{\Delta^2}{\omega^2}\right)C = i\Omega - i\Delta\left(\left(-\frac{\Delta}{\omega}B + iA\right) - iC\frac{\Delta}{\omega^2} + D\right)$$

$$\left(\frac{\Delta}{\omega}C\frac{\Delta}{\omega^2} + iB\right) + \left(1 - \frac{\Delta^2}{\omega^2}\right)C = i\Omega - i\Delta\left(\left(-\frac{\Delta}{\omega}B + iC\frac{\Delta}{\omega^2}\right) - iC\frac{\Delta}{\omega^2} + \frac{\Delta}{\omega}B\right)$$

$$\frac{\Delta^2}{\omega^3}C + iB + \left(1 - \frac{\Delta^2}{\omega^2}\right)C = i\Omega$$

$$B = \Omega + i\frac{\omega(\omega^2 - \Delta^2) + \Delta^2}{\omega^2}C$$

Mengingat bahwa bagian real hanya

$$re(\dot{\rho}_{01}) = \Delta im(\rho_{01})$$

Maka

$$C = 0$$

Sehingga

$$A = 0$$

$$B = \Omega$$

$$D = \frac{\Delta}{\omega}\Omega$$

$$E = -\frac{2\Omega^2}{\omega}$$

$$F = 1 + \frac{2\Omega^2}{\omega}$$

Dengan begitu, hasil *density matrix* dapat disederhanakan menjadi

$$\rho = \begin{pmatrix} 1 - \frac{2\Omega^2}{\omega}(\cos\omega t - 1) & \frac{\Delta}{\omega}\Omega(1 - \cos\omega t) + i\Omega\sin\omega t \\ \frac{\Delta}{\omega}\Omega(1 - \cos\omega t) - i\Omega\sin\omega t & \frac{2\Omega^2}{\omega}(\cos\omega t - 1\ ) \end{pmatrix}$$