Tugas 8

1. Membuktikan bahwa elemen matriks ρ merupakan rata-rata dari perkalian koefisien ekspansi fungsi basis dengan konjugatnya.

   Jawaban :

   Dengan diketahui persamaan matriks untuk density operator sebagai berikut

$$\rho = \sum_j P_j \left|\psi_j\right\rangle\left\langle\psi_j\right|$$

   Maka untuk mencari **representasi matriks** atau **elemen matriks** bisa dilakukan

$$\rho_{uv} = \langle\phi_u|\rho|\phi_v\rangle = \left\langle \phi_u \middle| \sum_j P_j \left|\psi_j\right\rangle\left\langle\psi_j\right| \middle| \phi_v \right\rangle$$

   Mengingat dari *pure states* adalah $\left|\psi_j\right\rangle = \sum_u c_u^{(j)}|\phi_u\rangle$ maka

$$\begin{aligned}\langle\phi_u|\rho|\phi_v\rangle &= \sum_j P_j \langle\phi_u| \sum_u c_u^{(j)}|\phi_u\rangle \langle\phi_v| \sum_v \left(c_u^{(j)}\right)^* |v\rangle \\ &= \sum_j P_j c_u^{(j)} \left(c_v^{(j)}\right)^* \\ \rho_{uv} &= \overline{(c_u(c_v)^*)}\end{aligned}$$

   Dimana c_u dan c_v merupakan koefisien ekspansi fungsi basis dan fungsi basis konjugat.

2. Tunjukkan dengan detail bahwa nilai rata-rata ensemble untuk suatu operator $\hat{A}$ yang bekerja pada keadaan campuran adalah trace dari $\{\rho\hat{A}\}$

   Jawaban :

   Dari persamaan 14.4 pada buku didapat

$$\overline{\langle\hat{A}\rangle} = \sum_j P_j \langle\psi_j|\hat{A}|\psi_j\rangle$$

   Untuk mencari bahwa trace dari $\{\rho\hat{A}\}$ adalah nilai rata-rata ensemble maka

$$Tr(\rho\hat{A}) = \sum_q \langle\phi_q|\rho\hat{A}|\phi_q\rangle$$

   Dengan $\rho = \sum_j P_j\left|\psi_j\right\rangle\left\langle\psi_j\right|$ maka

$$Tr(\rho\hat{A}) = \sum_q \langle\phi_q| \sum_j P_j|\psi_j\rangle\langle\psi_j| \hat{A}|\phi_q\rangle$$

$$= \sum_q \sum_j P_j \langle \phi_q | \psi_j \rangle \langle \psi_j | \hat{A} | \phi_q \rangle$$

$$= \sum_q \sum_j P_j \langle \psi_j | \hat{A} | \phi_q \rangle \langle \phi_q | \psi_j \rangle$$

Mengingat relasi kekomplitan dalam bentuk sumasi $\sum_q |\phi_q\rangle\langle\phi_q| = 1$ maka

$$Tr(\rho\hat{A}) = \sum_j P_j \langle \psi_j | \hat{A} | \psi_j \rangle$$

Lalu berdasarkan persamaan 14.10 di buku dimana $Tr(\rho) = \sum_j P_j = 1$ maka

$$Tr(\rho\hat{A}) = 1 \times \langle \psi_j | \hat{A} | \psi_j \rangle = \overline{\langle \hat{A} \rangle}$$

3. 14.3.1 Suppose we have a set of photons in a mixed state, with probabilities $P_1 = 0.2$ and $P_2 = 0.8$ respectively of being in the two different pure states

$$|\psi_1\rangle = |\psi_H\rangle \text{ and } |\psi_2\rangle = \frac{3}{5}|\psi_H\rangle + \frac{4i}{5}|\psi_V\rangle$$

where $|\psi_H\rangle$ and $|\psi_V\rangle$ are the normalized and orthogonal basis states representing horizontal and vertical polarization respectively. ($|\psi_1\rangle$ therefore is a horizontally polarized state, and $|\psi_2\rangle$ is an elliptically polarized state.) Write the density matrix for this state, in the $|\psi_H\rangle$ and $|\psi_V\rangle$ basis, with $\langle \psi_H | \rho | \psi_H \rangle$ as the top left element.

Jawaban :

$$\rho = \sum_{i=1}^{2} P_i |\psi_i\rangle\langle\psi_i|$$

$$= P_1 |\psi_1\rangle\langle\psi_1| + P_2|\psi_2\rangle\langle\psi_2|$$

$$= \frac{1}{5}(|\psi_H\rangle\langle\psi_H|) + \frac{4}{5}\left(\frac{3}{5}|\psi_H\rangle + \frac{4i}{5}|\psi_V\rangle\right)\left(\frac{3}{5}\langle\psi_H| - \frac{4i}{5}\langle\psi_V|\right)$$

$$= \frac{1}{5}(|\psi_H\rangle\langle\psi_H|) + \frac{4}{5}\left(\frac{9}{25}|\psi_H\rangle\langle\psi_H| + \frac{12i}{25}(|\psi_V\rangle\langle\psi_H| - |\psi_H\rangle\langle\psi_V|) + \frac{16}{25}|\psi_V\rangle\langle\psi_V|\right)$$

$$= \frac{1}{125}(61|\psi_H\rangle\langle\psi_H| + 48i(|\psi_V\rangle\langle\psi_H| - |\psi_H\rangle\langle\psi_V|) + 64|\psi_V\rangle\langle\psi_V|)$$

Lalu kita ubah menjadi matriks jika $|\psi_H\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ dan $|\psi_V\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ seperti berikut

$$\rho = \frac{1}{125}\left(61\begin{pmatrix}1\\0\end{pmatrix}(1 \quad 0) + 48i\left(\begin{pmatrix}0\\1\end{pmatrix}(1 \quad 0) - \begin{pmatrix}1\\0\end{pmatrix}(0 \quad 1)\right) + 64\begin{pmatrix}0\\1\end{pmatrix}(0 \quad 1)\right)$$

$$= \frac{1}{125}\left(61\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix} + 48i\left(\begin{vmatrix}0 & 0\\1 & 0\end{vmatrix} - \begin{vmatrix}0 & 1\\0 & 0\end{vmatrix}\right) + 64\begin{vmatrix}0 & 0\\0 & 1\end{vmatrix}\right)$$

$$\rho = \frac{1}{125}\begin{vmatrix}61 & -48i\\48i & 64\end{vmatrix}$$

4\. 14.3.2 Consider the mixed spin state, with equal probabilities of the electrons being in the pure state $|s_x\rangle$ and the pure state $|s_y\rangle$. Here $|s_x\rangle$ and $|s_y\rangle$ are respectively spin states oriented along the $+x$ and $+y$ directions. (See Problem 14.1.1)

(i) Evaluate the density operator ρ on the z spin basis (i.e., $|\uparrow\rangle$ and $|\downarrow\rangle$)

(ii) Now write this density operator as a density matrix, with the term in $|\uparrow\rangle\langle\uparrow|$ in the top left element.

(iii) Taking the spin magnetic dipole moment operator to be $\hat{\boldsymbol{\mu}}_e = g\mu_B\hat{\boldsymbol{\sigma}}$, evaluate $\hat{\boldsymbol{\mu}}_e$ as a matrix on the same z spin basis (i.e., $|\uparrow\rangle$ and $|\downarrow\rangle$, with the element $\langle\uparrow|\hat{\boldsymbol{\mu}}_e|\uparrow\rangle$ in the top left corner.

(iv) Using the expression of the form $\overline{\langle A\rangle} = Tr(\rho\hat{A})$, evaluate the ensemble average expectation value for the spin magnetic dipole moment in this mixed state. [Hint: the answer should be the same as that for Problem 14.1.1 (ii)(a).]

Jawaban :

(i) Untuk menulis operator densitas menjadi

$$\rho = 0.5|s_x\rangle\langle s_x| + 0.5|s_y\rangle\langle s_y|$$

$$= \frac{1}{2}\left(\frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle)(\langle\uparrow| + \langle\downarrow|) + \frac{1}{2}(|\uparrow\rangle + i|\downarrow\rangle)(\langle\uparrow| - i\langle\downarrow|)\right)$$

$$= \frac{1}{4}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow| + i(|\downarrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow|) + |\downarrow\rangle\langle\downarrow|)$$

$$\rho = \frac{1}{4}(2(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) + (1+i)|\downarrow\rangle\langle\uparrow| + (1-i)|\uparrow\rangle\langle\downarrow|)$$

(ii) Mengubah menjadi bentuk matriks

$$\rho = \frac{1}{4}\left(2\left(\begin{pmatrix}1\\0\end{pmatrix}(1 \quad 0) + \begin{pmatrix}0\\1\end{pmatrix}(0 \quad 1)\right) + (1+i)\begin{pmatrix}0\\1\end{pmatrix}(1 \quad 0) + (1-i)\begin{pmatrix}1\\0\end{pmatrix}(0 \quad 1)\right)$$

$$= \frac{1}{4}\left(2\left(\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix} + \begin{vmatrix}0 & 0\\0 & 1\end{vmatrix}\right) + (1+i)\begin{vmatrix}0 & 0\\1 & 0\end{vmatrix} + (1-i)\begin{vmatrix}0 & 1\\0 & 0\end{vmatrix}\right)$$

$$\rho = \frac{1}{4}\begin{vmatrix}2 & 1-i\\1+i & 2\end{vmatrix}$$

(iii) Membentuk spin magnetic dipole moment operator menjadi matriks berikut

$$\hat{\boldsymbol{\mu}}_{\boldsymbol{e}} = g\mu_B(\sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z})$$

$$= g\mu_B\left(\begin{vmatrix}0 & 1\\1 & 0\end{vmatrix}\hat{x} + \begin{vmatrix}0 & -i\\i & 0\end{vmatrix}\hat{y} + \begin{vmatrix}1 & 0\\0 & -1\end{vmatrix}\hat{z}\right)$$

$$\hat{\boldsymbol{\mu}}_{\boldsymbol{e}} = g\mu_B\begin{vmatrix}\hat{z} & \hat{x}-i\hat{y}\\\hat{x}+i\hat{y} & -\hat{z}\end{vmatrix}$$

(iv) Mencari hasil trace berikut

$$\overline{\langle\hat{\mu}_e\rangle} = Tr(\rho.\hat{\mu}_e)$$

$$= Tr\left(\frac{g\mu_B}{4}\begin{vmatrix}2 & 1-i\\1+i & 2\end{vmatrix}\begin{vmatrix}\hat{z} & \hat{x}-i\hat{y}\\\hat{x}+i\hat{y} & -\hat{z}\end{vmatrix}\right)$$

$$= Tr\left(\frac{g\mu_B}{4}\begin{vmatrix}2\hat{z}-(1-i)(\hat{x}-i\hat{y}) & 2(\hat{x}-i\hat{y})+(1-i)\hat{z}\\(1+i)\hat{z}+2(\hat{x}+i\hat{y}) & (1+i)(\hat{x}+i\hat{y})-2\hat{z}\end{vmatrix}\right)$$

$$\overline{\langle\hat{\mu}_e\rangle} = \frac{g\mu_B}{4}(2,2,0) = \frac{g\mu_B}{2}(1,1,0) = \frac{g\mu_B}{\sqrt{2}}$$