Forced oscillator

Let us consider

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + F(t)\left(a^\dagger + a\right) \tag{1}$$

We write

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{i}{\hbar}[H, a] \\ &= \frac{i}{\hbar}\left\{\hbar\omega\left[a^\dagger a, a\right] + F(t)\left[a^\dagger + a, a\right]\right\} \\ &= \frac{i}{\hbar}\{-\hbar\omega a - F(t)\} \\ &= -i\omega a - \frac{iF(t)}{\hbar}\end{aligned} \tag{2}$$

We can identify

$$\begin{aligned} i\omega &\equiv P(t) \\ -\frac{iF(t)}{\hbar} &\equiv Q(t)\end{aligned} \tag{3}$$

to arrive at the form

$$\dot{a} + P(t)a = Q(t) \tag{4}$$

whose solution is

$$a = e^{-I}\int Q(t)e^{I}\mathrm{d}t + ce^{-I} \tag{5}$$

where

$$I = \int P(t)\mathrm{d}t \tag{6}$$

with the integration constant c explicitly written. We have

$$I = \int i\omega\mathrm{d}t = i\omega t \tag{7}$$

so

$$\int Q(t)e^{I}\mathrm{d}t = -\frac{i}{\hbar}\int F(t)e^{i\omega t}\mathrm{d}t \tag{8}$$

and thus

$$a = \left[a(0) - \frac{i}{\hbar}\int F(t)e^{i\omega t}\mathrm{d}t\right]e^{-i\omega t} \tag{9}$$

where we have written $a(0)$ instead of c. Similarly, we have

$$a^\dagger = \left[a^\dagger(0) + \frac{i}{\hbar}\int F(t)e^{-i\omega t}\mathrm{d}t\right]e^{i\omega t} \tag{10}$$