A more general analysis of atom-light interaction using the Jaynes-Cummings model

We consider a two-level atomic state initially in the state

$$|\psi(0)\rangle_{\text{atom}} = C_g\,|g\rangle + C_e\,|e\rangle \tag{1}$$

and a photon initially in the state

$$|\psi(0)\rangle_{\text{light}} = \sum_{n=0}^{\infty} C_n\,|n\rangle \tag{2}$$

The initial atom-light composite state is

$$|\psi(0)\rangle = |\psi(0)\rangle_{\text{atom}}\,|\psi(0)\rangle_{\text{light}} = \sum_{n=0}^{\infty} C_g C_n\,|g\rangle\,|n\rangle + C_e C_n\,|e\rangle\,|n\rangle \tag{3}$$

There are some possibilities for the state the system can be in at a later time. If the system is initially in the $|g\rangle\,|n\rangle$ state, then it will be in either $|g\rangle\,|n\rangle$ or $|e\rangle\,|n-1\rangle$ state. If it is initially in the $|e\rangle\,|n\rangle$ state, then the possibilities are $|e\rangle\,|n\rangle$ or $|g\rangle\,|n+1\rangle$. And this is for some value of n. Generally, we have

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} A_n(t)\,|g\rangle\,|n\rangle + B_n(t)\,|e\rangle\,|n-1\rangle + C_n(t)\,|e\rangle\,|n\rangle + D_n(t)\,|g\rangle\,|n+1\rangle \tag{4}$$

The interaction Hamiltonian is given by

$$\hat{H}_{\text{II}} = \lambda\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right) \tag{5}$$

Putting (4) and (5) into

$$\frac{\partial\,|\psi\rangle}{\partial t} = -i\hat{H}_{\text{II}}\,|\psi\rangle \tag{6}$$

we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \dot{A}_n\,|g\rangle\,|n\rangle + \dot{B}_n\,|e\rangle\,|n-1\rangle + \dot{C}_n\,|e\rangle\,|n\rangle + \dot{D}_n\,|g\rangle\,|n+1\rangle \\ = -i\lambda\sum_{n=0}^{\infty} A_n\,|e\rangle\,\sqrt{n}\,|n-1\rangle + B_n\,|g\rangle\,\sqrt{n}\,|n\rangle + C_n\,|g\rangle\,\sqrt{n+1}\,|n+1\rangle + D_n\,|e\rangle\,\sqrt{n+1}\,|n\rangle\end{aligned} \tag{7}$$

Matching the terms, we have

$$\dot{A}_n = -i\lambda\sqrt{n}B_n \tag{8}$$

$$\dot{B}_n = -i\lambda\sqrt{n}A_n \tag{9}$$

$$\dot{C}_n = -i\lambda\sqrt{n+1}D_n \tag{10}$$

$$\dot{D}_n = -i\lambda\sqrt{n+1}C_n \tag{11}$$

Solving (8) and (9) simultaneously, with the initial conditions $A(0) = C_g C_n$ and $B(0) = 0$, we obtain

$$\begin{aligned}A_n(t) &= C_g C_n \cos\left(\lambda t\sqrt{n}\right) \\ B_n(t) &= -iC_g C_n \sin\left(\lambda t\sqrt{n}\right)\end{aligned} \tag{12}$$

Solving (10) and (11) simulatneously with the initial conditions $C(0) = C_e C_n$ and $D(0) = 0$, we obtain

$$\begin{aligned}C_n(t) &= C_e C_n \cos\left(\lambda t\sqrt{n+1}\right) \\ D_n(t) &= -iC_e C_n \sin\left(\lambda t\sqrt{n+1}\right)\end{aligned} \tag{13}$$

We finally obtain

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} C_g C_n \cos\left(\lambda t\sqrt{n}\right)|g\rangle\,|n\rangle - iC_g C_n \sin\left(\lambda t\sqrt{n}\right)|e\rangle\,|n-1\rangle + C_e C_n \cos\left(\lambda t\sqrt{n+1}\right) - iC_e C_n \sin\left(\lambda t\sqrt{n+1}\right)|g\rangle\,|n+1\rangle \tag{14}$$