Problem 1: The optical Bloch equation

One definition of the coherent state is

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{1}$$

We know that

$$|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}} |0\rangle \tag{2}$$

This allows us to write

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\left(\alpha \hat{a}^\dagger\right)^n}{n!} |0\rangle \tag{3}$$

But this summation is just the Taylor expansion of $\exp\left(\alpha \hat{a}^\dagger\right)$, so

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} |0\rangle \tag{4}$$

This looks pretty nice, but we can go further. Note that we can write

$$|0\rangle = \sum_{k=0}^{\infty} \frac{(-\alpha^* \hat{a})^k}{k!} |0\rangle \tag{5}$$

This can be easily seen by the fact that the only nonzero term is the one with $k = 0$. Applying any number of a on $|0\rangle$ will result in 0. But this summation is just the Taylor expansion of $\exp(-\alpha^* a)$. Putting this into (4), we obtain

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} |0\rangle \tag{6}$$

Finally, we can use the identity

$$e^{\hat{A}+\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{\frac{1}{2}[\hat{A},\hat{B}]} \tag{7}$$

which holds under the condiiton that

$$\begin{aligned} \left[\hat{A}, \hat{B}\right] &\neq 0 \\ \left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] &= 0 \\ \left[\hat{B}, \left[\hat{A}, \hat{B}\right]\right] &= 0 \end{aligned} \tag{8}$$

This identity is exactly satisfied in (6) with $\hat{A} = -\alpha \hat{a}^\dagger$ and $\hat{B} = \alpha^* \hat{a}$. Let us check this out:

$$\begin{aligned} \left[-\alpha \hat{a}^\dagger, \alpha^* \hat{a}\right] |\alpha\rangle &= -\alpha \hat{a}^\dagger \alpha^* \hat{a} |\alpha\rangle + \alpha^* \hat{a} \alpha \hat{a}^\dagger |\alpha\rangle \\ &= |\alpha|^2 \left[\hat{a}, \hat{a}^\dagger\right] |\alpha\rangle \\ &= |\alpha|^2 |\alpha\rangle \end{aligned} \tag{9}$$

The result is a constant, so the other two equations in (8) are also satisfied. Applying (7) on (6) (remember that $\exp\left(|\alpha|^2\right)$ is just a number; we can move it around) gives us

$$|\alpha\rangle = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle \equiv D(\alpha) |0\rangle \tag{10}$$

Here $D(\alpha)$ is known as the **displacement operator**, and this equivalent definition can be called the **vacuum displacement definition**.

Problem 2: The thermal state density matrix at absolute zero

We have

$$\begin{aligned} \rho &= \frac{e^{-\beta \hat{H}}}{Z} \\ &= \frac{\sum_{n=0}^{\infty} e^{-\beta E_n} |n\rangle \langle n|}{\sum_{m=0}^{\infty} e^{-\beta E_m}} \end{aligned} \tag{11}$$

Since $\cdots > E_2 > E_1 > E_0$, the term that tends toward zero at the **slowest** rate as β gets large is $e^{-\beta E_0}$. It is thus the only term that survives and we can write

$$\rho \approx \frac{e^{-\beta E_0} |0\rangle \langle 0|}{e^{-\beta E_0}} = |0\rangle \langle 0|, \qquad \text{for large } \beta \tag{12}$$