The P-representation

For an operator $\hat{B}$,

$$\hat{B} \equiv \int B_P\left(\alpha, \alpha^*\right)|\alpha\rangle\langle\alpha| \, d^2\alpha \tag{1}$$

The expectation value is given by

$$\begin{aligned}
\left\langle \hat{B} \right\rangle &= \operatorname{Tr}\left(\hat{B}\hat{\rho}\right) \\
&= \sum_n \langle n| \int B_P\left(\alpha, \alpha^*\right)|\alpha\rangle \ \langle\alpha|\hat{\rho}|n\rangle \, d^2\alpha \\
&= \int B_P\left(\alpha, \alpha^*\right) \ \langle\alpha|\hat{\rho}|\alpha\rangle \, d^2\alpha
\end{aligned} \tag{2}$$

where we have used the completeness relation. Let us now define the Husimi Q function:

$$\mathcal{Q}\left(\alpha\right) = \frac{\langle\alpha|\hat{\rho}|\alpha\rangle}{\pi} \tag{3}$$

For $\hat{B} = \hat{I}$, we obtain

$$\int \mathcal{Q}\left(\alpha\right) d^2\alpha = 1 \tag{4}$$

We can also define the corresponding Q-representation for $\hat{B}$ as

$$B_{\mathcal{Q}}(\alpha, \alpha^*) \equiv \ \langle\alpha|\hat{B}|\alpha\rangle = e^{-|\alpha|^2} \sum_{nm} \frac{B_n m}{\sqrt{n!m!}} \left(\alpha^*\right)^n \left(\alpha\right)^m \tag{5}$$

where we have used the definition of coherent states. The matrix element with respect to the number state basis is obtained from this definition. We can combine this with the P-representation of the density matrix by writing

$$\begin{aligned}
\left\langle \hat{B} \right\rangle &= \operatorname{Tr}\left(\hat{B}\hat{\rho}\right) \\
&= \operatorname{Tr}\left\{\hat{B}P\left(\alpha\right)|\alpha\rangle\langle\alpha| \, d^2\alpha\right\} \\
&= \int P\left(\alpha\right) \ \langle\alpha|\hat{B}|\alpha\rangle \, d^2\alpha \\
&= \int P\left(\alpha\right) B_{\mathcal{Q}}(\alpha, \alpha^*) d^2\alpha
\end{aligned} \tag{6}$$

Note that $P\left(\alpha\right)$ can go outside the brakets since it is just a function.

The Wigner Function

For an arbitrary density operator $\hat{\rho}$, the corresponding Wigner function is defined as

$$W(q,p) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \left\langle q - \frac{1}{2}x \middle| \hat{\rho} \middle| q + \frac{1}{2}x \right\rangle e^{ipx/\hbar} dx \tag{7}$$

If the state is pure, then $\hat{\rho} = |\psi\rangle\langle\psi|$. Now,

$$\langle x|\psi\rangle = \psi(x) \tag{8}$$

In words, the inner product projects the state into the position space. With this we can write

$$W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \psi^*\left(q + \frac{1}{2}x\right) \psi\left(q - \frac{1}{2}x\right) e^{ipx/\hbar} dx \tag{9}$$

Let us integrate the Wigner function over p. We obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} W(q,p) dp &= \int_{-\infty}^{\infty} \psi^*\left(q + \frac{1}{2}x\right) \psi\left(q - \frac{1}{2}x\right) \left[\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ipx/\hbar} dp\right] dx \\
&= \int_{-\infty}^{\infty} \psi^*\left(q + \frac{1}{2}x\right) \psi\left(q - \frac{1}{2}x\right) \delta(x) dx
\end{aligned} \tag{10}$$

The delta function makes the contributions from $x \neq 0$ vanish, leaving us with

$$\int_{-\infty}^{\infty} W(q,p) dp = \psi^*\left(q\right) \psi\left(q\right) = \left|\psi(q)\right|^2 \tag{11}$$

An equivalent definition of the same Wigner function is given by

$$W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \varphi^* \left(p + \frac{1}{2}k \right) \varphi \left(p - \frac{1}{2}k \right) e^{ikq/\hbar} dk \tag{12}$$

Using this definition, we can find in the similar way that

$$\int_{-\infty}^{\infty} W(q,p) dq = |\varphi(p)|^2 \tag{13}$$