Problem 1

We have

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 = \frac{1}{2m}\left[\hat{p}^2 + \left(m\omega\hat{q}\right)^2\right]$$
$$a = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{q} + i\hat{p}\right)$$
$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{q} - i\hat{p}\right)$$

We write

$$\begin{aligned} a^\dagger a &= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{q} - i\hat{p}\right)\frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{q} + i\hat{p}\right) \\ &= \frac{1}{2\hbar m\omega}\left[\left(m\omega q\right)^2 + \hat{p}^2 + im\omega\left[\hat{q},\hat{p}\right]\right] \end{aligned}$$

Since $[\hat{q},\hat{p}] = i\hbar$, we have

$$a^\dagger a = \frac{1}{2\hbar m\omega}\left[\left(m\omega q\right)^2 + \hat{p}^2 - \hbar m\omega\right] \tag{1}$$

Similarly,

$$aa^\dagger = \frac{1}{2\hbar m\omega}\left[\left(m\omega q\right)^2 + \hat{p}^2 + \hbar m\omega\right] \tag{2}$$

Evidently,

$$\begin{aligned} [a,a^\dagger] &= aa^\dagger - a^\dagger a \\ &= \frac{1}{2\hbar m\omega}\left[\left(m\omega q\right)^2 + \hat{p}^2 + \hbar m\omega\right] - \frac{1}{2\hbar m\omega}\left[\left(m\omega q\right)^2 + \hat{p}^2 - \hbar m\omega\right] \quad = 1 \end{aligned} \tag{3}$$

Problem 2

Take a look at (1). This is just the Hamiltonian, divided by $\hbar\omega$ and subtracted by $\frac{1}{2}$. We thus have

$$\begin{aligned} a^\dagger a &= \frac{\hat{H}}{\hbar\omega} - \frac{1}{2} \\ \hat{H} &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \end{aligned} \tag{4}$$

Problem 3

We have

$$\begin{aligned} [\hat{H},a] &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)a - a\hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \\ &= \hbar\omega\left(a^\dagger aa + \frac{1}{2}a - aa^\dagger a - \frac{1}{2}a\right) \\ &= -\hbar\omega\left[a,a^\dagger\right]a \\ &= -\hbar\omega a \end{aligned} \tag{5}$$

and

$$\begin{aligned} \left[\hat{H},a^\dagger\right] &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)a^\dagger - a^\dagger\hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \\ &= \hbar\omega\left(a^\dagger aa^\dagger + \frac{1}{2}a^\dagger - a^\dagger a^\dagger a - \frac{1}{2}a^\dagger\right) \\ &= \hbar\omega a^\dagger\left[a,a^\dagger\right] \\ &= \hbar\omega a^\dagger \end{aligned} \tag{6}$$

Problem 4

By definition

$$\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$$

so

$$\hat{L}_i = \sum_{jk}\varepsilon_{ijk}\hat{r}_j\hat{p}_k$$

The identities needed are

$$\begin{aligned} \left[\hat{r}_i,\hat{p}_j\right] &= i\hbar\delta_{ij} \\ [AB,CD] &= A[B,C]D + [A,C]BD + CA[B,D] + C[A,D]B \end{aligned} \tag{7}$$

We have

$$
\begin{aligned}
\left[\hat{L}_x, \hat{L}_y\right] &= \left[\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z\right] \\
&= \left[\hat{y}\hat{p}_z, \hat{z}\hat{p}_x\right] - \left[\hat{y}\hat{p}_z, \hat{x}\hat{p}_z\right] - \left[\hat{z}\hat{p}_y, \hat{z}\hat{p}_x\right] + \left[\hat{z}\hat{p}_y, \hat{x}\hat{p}_z\right]
\end{aligned} \tag{8}
$$

Using the identities in (7), we are left with

$$
\begin{aligned}
\left[\hat{L}_x, \hat{L}_y\right] &= \hat{y}\hat{p}_x[\hat{p}_z, \hat{z}] + \hat{x}\hat{p}_y[\hat{z}, \hat{p}_z] \\
&= i\hbar\left(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x\right) \\
&= i\hbar\hat{L}_z
\end{aligned} \tag{9}
$$

Cycling through the indices, we end up with

$$
\begin{aligned}
[\hat{L}_y, \hat{L}_z] &= i\hbar\hat{L}_x \\
[\hat{L}_z, \hat{L}_x] &= i\hbar\hat{L}_y
\end{aligned} \tag{10}
$$

Problem 5

Useful identities are (9), (10), and

$$
\begin{aligned}
[AB, C] &= A[B, C] + [A, C]B \\
[A, A^n] &= 0
\end{aligned} \tag{11}
$$

We have

$$
\begin{aligned}
[\hat{L}^2, \hat{L}_z] &= [\hat{L}_x^2, \hat{L}_z] + [\hat{L}_y^2, \hat{L}_z] + [\hat{L}_z^2, \hat{L}_z] \\
&= \hat{L}_x[\hat{L}_x, \hat{L}_z] + [\hat{L}_x, \hat{L}_z]\hat{L}_x + \hat{L}_y[\hat{L}_y, \hat{L}_z] + [\hat{L}_y, \hat{L}_z]\hat{L}_y \\
&= \hat{L}_x(-\hat{L}_y) + (-\hat{L}_y)\hat{L}_x + \hat{L}_y\hat{L}_x + \hat{L}_x\hat{L}_y \qquad = 0
\end{aligned} \tag{12}
$$

The result is similar for $\hat{L}_x$ and $\hat{L}_y$.