Problem 1: Checking the Heisenbeg Equation of Motion

$$
\begin{aligned}
\frac{d\hat{O}(t)}{dt} &= \frac{d}{dt}\left[e^{i\hat{H}t/\hbar}\hat{O}(0)e^{-i\hat{H}t/\hbar}\right] \\
&= \frac{i}{\hbar}\hat{H}e^{i\hat{H}t/\hbar}\hat{O}(0)e^{-i\hat{H}t/\hbar} - \frac{i}{\hbar}e^{i\hat{H}t/\hbar}\hat{O}(0)\hat{H}e^{-i\hat{H}t/\hbar}
\end{aligned}
$$

Now, an operator commutes with any function of it, so

$$
\begin{aligned}
\frac{d\hat{O}(t)}{dt} &= \frac{i}{\hbar}\hat{H}e^{i\hat{H}t/\hbar}\hat{O}(0)e^{-i\hat{H}t/\hbar} - \frac{i}{\hbar}e^{i\hat{H}t/\hbar}\hat{O}(0)e^{-i\hat{H}t/\hbar}\hat{H} \\
&= \frac{i}{\hbar}\hat{H}\hat{O}(t) - \frac{i}{\hbar}\hat{O}(t)\hat{H} \\
&= \frac{i}{\hbar}\left[\hat{H}, \hat{O}(t)\right]
\end{aligned} \tag{1}
$$

Problem 2: The BCH Formula

$$
\begin{aligned}
e^{A}Be^{-A} &= \left[I + A + \frac{A^2}{2} + \frac{A^3}{6} + \dots\right] B \left[I - A + \frac{A^2}{2} - \frac{A^3}{6} + \dots\right] \\
&= B - BA + \frac{BA^2}{2} - \frac{BA^3}{6} + \cdots + AB - ABA + \frac{ABA^2}{2} - \frac{ABA^3}{6} + \cdots + \frac{A^2B}{2} - \frac{A^2BA}{2} + \frac{A^2BA^2}{4} - \frac{A^2BA^3}{12} + \dots \\
&\quad + \frac{A^3B}{6} - \frac{A^3BA}{6} + \frac{A^3BA^2}{12} - \frac{A^3BA^3}{36} + \dots \\
&= B + \{AB - BA\} + \left\{\frac{BA^2}{2} - ABA + \frac{A^2B}{2}\right\} + \left\{\frac{BA^3}{6} + \frac{ABA^2}{2} - \frac{A^2BA}{2} + \frac{A^3B}{6}\right\} + \dots \\
&= B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots \\
&\equiv \sum_{n=0}^{\inf} \frac{[A^{(n)},B]}{n!}
\end{aligned} \tag{2}
$$

Here, we have defined

$$
[A^{(N)},B] \equiv \underbrace{[A,[A,[\dots,[A},B]]]]_{n\text{ times}}; \qquad [A^{(0)},B] \equiv B \tag{3}
$$