The optical Bloch equation

Previously we have found the equations describing a two level system inside an electric field subject to relaxation processes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{11}-\rho_{22}\right)=\frac{2i\mu_d\mathscr{E}}{\hbar}\left(\rho_{21}-\rho_{21}^*\right)-\frac{\left(\rho_{11}-\rho_{22}\right)-\left(\rho_{11}-\rho_{22}\right)_0}{T_1} \tag{1}$$

$$\frac{\mathrm{d}\rho_{21}}{\mathrm{d}t}=-i\omega_{21}\rho_{21}+\frac{i\mu_d\mathscr{E}}{\hbar}\left(\rho_{11}-\rho_{22}\right)-\frac{\rho_{21}}{T_2} \tag{2}$$

where $\hbar\omega\equiv E_2-E_1$ and $\mu_d\equiv\mu_{12}=\mu_{21}$. Here $(\ )_0$ denotes the equilibrium value. We are interested in the behavior of the system under an oscillating electric field, for example

$$\mathscr{E}=\frac{\mathscr{E}_0}{2}\left(e^{i\omega t}+e^{-i\omega t}\right) \tag{3}$$

Let us first define the quantity

$$\beta_{21}\equiv\rho_{21}e^{i\omega t} \tag{4}$$

Substituting (3) into (1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{11}-\rho_{22}\right)=\frac{2i\mu_d}{\hbar}\frac{\mathscr{E}_0}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)\left(\rho_{21}-\rho_{21}^*\right)-\frac{\left(\rho_{11}-\rho_{22}\right)-\left(\rho_{11}-\rho_{22}\right)_0}{T_1} \tag{5}$$

Let us take a look at

$$\left(e^{i\omega t}+e^{-i\omega t}\right)\left(\rho_{21}-\rho_{21}^*\right)=\beta_{21}-\beta_{21}^*e^{2i\omega t}+\beta_{21}e^{-2i\omega t}-\beta_{21}^* \tag{6}$$

Solving a differential equation involves integrating it over the time period under consideration. Over a reasonably large timescale we can ignore the faster oscillating term as it evaluates towards zero. We can thus write, under this so-called **rotating wave approximation**,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{11}-\rho_{22}\right)=\frac{i\mu_d\mathscr{E}_0}{\hbar}\left(\beta_{21}-\beta_{21}^*\right)-\frac{\left(\rho_{11}-\rho_{22}\right)-\left(\rho_{11}-\rho_{22}\right)_0}{T_1} \tag{7}$$

By multiplying (2) by $e^{i\omega t}$, substituting (3) and (4) in, then using RWA, we obtain

$$\begin{aligned}
\left(\frac{\mathrm{d}\rho_{21}}{\mathrm{d}t}\right)e^{i\omega t}&=-i\omega_{21}\rho_{21}e^{i\omega t}+\frac{i\mu_d}{\hbar}\frac{\mathscr{E}_0}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)\left(\rho_{11}-\rho_{22}\right)e^{i\omega t}-\frac{\rho_{21}e^{i\omega t}}{T_2}\\
\frac{\mathrm{d}\beta_{21}}{\mathrm{d}t}-i\omega\beta_{21}&=-i\omega_{21}\beta_{21}+\frac{i\mu_d}{\hbar}\frac{\mathscr{E}_0}{2}\left(e^{2i\omega t}+1\right)\left(\rho_{11}-\rho_{22}\right)-\frac{\beta_{21}}{T_2}\\
\frac{\mathrm{d}\beta_{21}}{\mathrm{d}t}&=i\left(\omega-\omega_{21}\right)\beta_{21}+\frac{i\mu_d\mathscr{E}_0}{2\hbar}\left(\rho_{11}-\rho_{22}\right)-\frac{\beta_{21}}{T_2}
\end{aligned} \tag{8}$$

Equation (7) and (8) are known as the **optical Bloch equations**.

We are interestes in the steady state solution, so we want to solve

$$0=2i\Omega\left(\beta_{21}-\beta_{21}^*\right)-\frac{\left(\rho_{11}-\rho_{22}\right)-\left(\rho_{11}-\rho_{22}\right)_0}{T_1} \tag{9}$$

and

$$0=i\left(\omega-\omega_{21}\right)\beta_{21}+i\Omega\left(\rho_{11}-\rho_{22}\right)-\frac{\beta_{21}}{T_2} \tag{10}$$

where we have defined $\mu_d\mathscr{E}_0/2\hbar\equiv\Omega$.

Rearranging (10) gives us

$$\beta_{21}=\frac{i\Omega\left(\rho_{11}-\rho_{22}\right)}{\frac{1}{T_2}-i\left(\omega-\omega_{21}\right)}=\Omega T_2\left(\rho_{11}-\rho_{22}\right)\frac{i-T_2\left(\omega-\omega_{21}\right)}{1+T_2^2\left(\omega-\omega_{21}\right)^2} \tag{11}$$

from which we can write

$$\beta_{21}-\beta_{21}^*=\Omega T_2\left(\rho_{11}-\rho_{22}\right)\frac{2i}{1+T_2^2\left(\omega-\omega_{21}\right)^2} \tag{12}$$

Substituting this into (9), we obtain

$$\begin{aligned} 0 &= \left(\rho_{11}-\rho_{22}\right)\frac{-4\Omega^2 T_2}{1+T_2^2\left(\omega-\omega_{21}\right)^2}-\frac{\left(\rho_{11}-\rho_{22}\right)-\left(\rho_{11}-\rho_{22}\right)_0}{T_1} \\ \left(\rho_{11}-\rho_{22}\right) &= \left(\rho_{11}-\rho_{22}\right)_0\frac{1+T_2^2\left(\omega-\omega_{21}\right)^2}{1+T_2^2\left(\omega-\omega_{21}\right)^2+4\Omega^2 T_1 T_2} \end{aligned} \tag{13}$$

Putting this into (11) we obtain

$$\beta_{21}=\Omega T_2\left(\rho_{11}-\rho_{22}\right)_0\frac{i-T_2\left(\omega-\omega_{21}\right)}{1+T_2^2\left(\omega-\omega_{21}\right)^2+4\Omega^2 T_1 T_2} \tag{14}$$

(13) and (14) are the solutions to the optical Bloch equations. We can use these results to write out the response of the system to the field. The polarization is given by

$$P=\varepsilon_0\chi_e\mathscr{E} \tag{15}$$

We can use $\chi_e\equiv\chi_e`+i\chi_e``$ and put in $\mathscr{E}$ to write

$$P=\varepsilon_0\mathscr{E}_0\left(\chi_e`\cos\left(\omega t\right)+\chi_e``\sin\left(\omega t\right)\right) \tag{16}$$

We can also equate this to

$$P=\Delta n\left\langle\mu_e\right\rangle=2\mu_d\left[\mathrm{Re}\left(\beta_{21}\right)\cos\left(\omega t\right)+\mathrm{Im}\left(\beta_{21}\right)\sin\left(\omega t\right)\right] \tag{17}$$

where $\Delta n=n\left(\rho_{11}-\rho_{22}\right)$ (and hence $\Delta n_0=n\left(\rho_{11}-\rho_{22}\right)_0$) is the number density difference between the two levels, with n the total number density of particles. Using (16), (17), and (14), we obtain

$$\begin{aligned} \chi_e` &= \frac{\mu_d^2 T_2\Delta n_0}{\varepsilon_0\hbar}\frac{T_2\left(\omega_{21}-\omega\right)}{1+T_2^2\left(\omega-\omega_{21}\right)^2+4\Omega^2 T_1 T_2} \\ \chi_e`` &= \frac{\mu_d^2 T_2\Delta n_0}{\varepsilon_0\hbar}\frac{1}{1+T_2^2\left(\omega-\omega_{21}\right)^2+4\Omega^2 T_1 T_2} \end{aligned} \tag{18}$$

We can relate these quantities to the complex permittivity of the material, via the relation

$$D=\varepsilon_0\mathscr{E}+P=\varepsilon_0\mathscr{E}+\varepsilon_0\chi_e\mathscr{E}=\varepsilon\mathscr{E} \tag{19}$$

from which we can write

$$\varepsilon=\varepsilon`+i\varepsilon``=\varepsilon_0\left(1+\chi_e\right) \tag{20}$$

By comparison with (18), we can easily write

$$\begin{aligned} \varepsilon` &= \varepsilon_0+\frac{\mu_d^2 T_2\Delta n_0}{\hbar}\frac{T_2\left(\omega_{21}-\omega\right)}{1+T_2^2\left(\omega-\omega_{21}\right)^2+4\Omega^2 T_1 T_2} \\ \varepsilon`` &= \frac{\mu_d^2 T_2\Delta n_0}{\hbar}\frac{1}{1+T_2^2\left(\omega-\omega_{21}\right)^2+4\Omega^2 T_1 T_2} \end{aligned} \tag{21}$$