For a statistical mixture of quantum states $\{|\psi_i\rangle\}$, ~~the density~~

the density operator is given by

$$\hat{\rho} = \sum_i \underbrace{p_i}_{\text{probability of finding the system in state } |\psi_i\rangle} |\psi_i\rangle\langle\psi_i|$$

Some properties:

$$\mathrm{Tr}(\hat{\rho}) = \sum_i p_i = 1$$

$$\overline{\langle \hat{O} \rangle} = \mathrm{Tr}(\hat{\rho}\hat{O}) = \sum_i p_i \langle\psi_i|\hat{O}|\psi_i\rangle$$

Using the completeness relation, we may define the matrix elements of $\hat{\rho}$ in a given basis:

$$\hat{\rho} = \left(\sum_m |m\rangle\langle m|\right) \hat{\rho} \left(\sum_n |n\rangle\langle n|\right) = \sum_{mn} |m\rangle \underbrace{\rho_{mn}}_{\rho_{mn} \equiv \langle m|\hat{\rho}|n\rangle} \langle n|$$

~~The diagonal~~ With $|n\rangle$ being the number states, ρ_{nn} is just the probability of finding n photons.

With the completeness relation for coherent states,

$$\underbrace{\int}_{\text{integral over the complex plane}} |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = 1$$

we instead have

$$\hat{\rho} = \int \langle\alpha'|\hat{\rho}|\alpha''\rangle\, |\alpha'\rangle\langle\alpha''| \frac{d^2\alpha'\, d^2\alpha''}{\pi^2}$$

Another representation is known as the Glauber-Sudarshan P representation. Let $P(\alpha)$ be a function such that the density matrix is diagonal in the basis $\{|\alpha\rangle\}$ of coherent states. Since $\hat{\rho}$ is hermitian, $P(\alpha)$ must be real. We have

$$\hat{\rho} \equiv \int P(\alpha)\, |\alpha\rangle\langle\alpha|\, d^2\alpha$$

$P(\alpha)$ is called the Glauber-Sudarshan P function. We see that

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}) &= \sum_n \langle n| \left(\int P(\alpha) |\alpha\rangle\langle\alpha|\, d^2\alpha \right) |n\rangle \\
&= \int P(\alpha) \sum_n \langle\alpha|n\rangle\langle n|\alpha\rangle\, d^2\alpha \\
&= \int P(\alpha)\, d^2\alpha \\
&= 1
\end{aligned}$$

While it seems like $P(\alpha)$ works as a probability distribution function, it has properties a "true" probability distribution does not have. There are states for which $P(\alpha)$ is negative or singular. Such states are called "nonclassical".

We may go the other way around, and define

"Nonclassical states are those for which the corresponding $P(\alpha)$ is negative in some region of phase space (the α-plane) or is more singular than a delta function".

$\hookrightarrow$ requires more work to remove singularity, e.g.

$\delta(x)$ is removed by $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$ (v.s.) $\frac{\partial \delta(x)}{\partial x}$ is removed by $\int_{-\infty}^{\infty} \underbrace{x}_{\text{more work}} \frac{\partial \delta(x)}{\partial x} = 1$

Coherent states, which are "quasi-classical" in that they describe states of the field having properties close to those of classical oscillating coherent fields, are "classical" by this definition since their P functions are delta functions.

Certain effects like quadrature and number squeezing can occur ONLY for nonclassical states. This is why squeezing ~~are~~ is called a nonclassical effect.

Let us start with

$$\langle -u|\hat{\rho}|u\rangle = \int P(\alpha) \langle -u|\alpha\rangle\langle\alpha|u\rangle\, d^2\alpha$$

With

$$\langle\beta|\alpha\rangle = e^{\frac{1}{2}(\beta^*\alpha - \beta\alpha^*)} e^{-\frac{1}{2}|\beta-\alpha|^2}$$

we have

$$\begin{aligned}
\langle -u|\alpha\rangle &= e^{\frac{1}{2}(-u^*\alpha + u\alpha^*)} e^{-\frac{1}{2}|-u-\alpha|^2} \\
&= e^{\frac{1}{2}(u\alpha^* - u^*\alpha - |u+\alpha|^2)} = e^{-u^*\alpha - \frac{1}{2}(|\alpha|^2+|u|^2)}
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|u\rangle &= e^{\frac{1}{2}(\alpha^* u - \alpha u^*)} e^{-\frac{1}{2}|\alpha-u|^2} \\
&= e^{\frac{1}{2}(\alpha^* u - \alpha u^* - |\alpha-u|^2)} = e^{\alpha^* u - \frac{1}{2}(|\alpha|^2+|u|^2)} \, ?
\end{aligned}$$

$$\begin{aligned}
\langle -u|\alpha\rangle\langle\alpha|u\rangle &\cancel{= e^{u\alpha^* - u^*\alpha - |u+\alpha|^2}} \\
&= e^{\alpha^* u - u^*\alpha - |\alpha|^2 - |u|^2}
\end{aligned}$$

Thus,

$$\langle -u|\hat{\rho}|u\rangle = e^{-|u|^2} \int P(\alpha)\, e^{-|\alpha|^2} e^{\alpha^* u - \alpha u^*}\, d^2\alpha$$

Let $\alpha \equiv a + ib$ and $u \equiv x + iy$ so that

$$\begin{aligned}
\alpha^* u - \alpha u^* &= (a - ib)(x + iy) - (a + ib)(x - iy) \\
&= ax + iay - ibx + by - ax + iay - ibx - by \\
&= 2i(ay - bx)
\end{aligned}$$

$$\langle -u|\hat{\rho}|u\rangle = e^{-|u|^2}\int P(\alpha)\, e^{-|\alpha|^2} e^{i(2ay)} e^{-i(2bx)}\, d^2\alpha$$

~~We can identify~~ So $e^{\alpha^* u - \alpha u^*}$ are just the exponential term appearing in the complex Fourier transform over the complex plane. We can thus identify

$$\left\{\langle -u|\hat{\rho}|u\rangle e^{|u|^2}\right\}_f = \int \left\{P(\alpha)e^{-|\alpha|^2}\right\} e^{\alpha^* u - \alpha u^*}\, d^2\alpha .$$

The reverse transform gives us

$$\boxed{P(\alpha)\, e^{-|\alpha|^2} = \frac{1}{\pi^2}\int e^{|u|^2}\langle -u|\hat{\rho}|u\rangle\, e^{\alpha u^* - \alpha^* u}\, d^2u}$$

Example

Let us consider a pure state:

$$\hat{\rho} = |\beta\rangle\langle\beta|$$

we have

$$\langle -u|\hat{\rho}|u\rangle = \langle u|\beta\rangle\langle\beta|u\rangle$$

$$= \exp\left[\tfrac{1}{2}\left((-u)^*\beta - (-u)\beta^*\right) - \tfrac{1}{2}|-u-\beta|^2\right] \exp\left[\tfrac{1}{2}\left(\beta^* u - \beta u^*\right) - \tfrac{1}{2}|\beta - u|^2\right]$$

$$= \exp\left[-|\beta|^2 - |u|^2 + \beta^* u - \beta u^*\right]$$

so

$$P(\alpha) = \frac{e^{|\alpha|^2 - |\beta|^2}}{\pi^2}\int e^{\beta^* u - \beta u^*} e^{\alpha u^* - \alpha^* u}\, d^2u$$

$$= \frac{e^{|\alpha|^2 - |\beta|^2}}{\pi^2}\int e^{u^*(\alpha-\beta) - u(\alpha^* - \beta^*)}\, d^2u$$

Notice that the exponential integrand has the same form as the $(e^{\alpha^* u - \alpha u^*})$ above which we know makes the form of a Fourier exponent. The whole

$$\frac{1}{\pi^2}\int \exp(\dots)\, d^2u$$

thing is thus simply the reverse Fourier transform of a constant. We may thus write

$$P(\alpha) = \delta^2(\alpha - \beta)$$

confirming our claim of the "classicality" of a coherent state.

[Hendry]