Coherent phonons in a two-band semiconductor

The Hamiltonian is given by (see here)

$$H = \sum_{\boldsymbol{k},\alpha} \varepsilon_{\alpha\boldsymbol{k}} c^{\dagger}_{\alpha\boldsymbol{k}} c_{\alpha\boldsymbol{k}} + \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} b^{\dagger}_{\boldsymbol{q}} b_{\boldsymbol{q}} + \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} \left(b_{\boldsymbol{q}} + b^{\dagger}_{-\boldsymbol{q}} \right) c^{\dagger}_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} \tag{1}$$

Let us define

$$D_{\boldsymbol{q}} \equiv b_{\boldsymbol{q}} + b^{\dagger}_{-\boldsymbol{q}} \tag{2}$$

as some quantity representing the displacement of the phonons. We would like to calculate

$$\frac{\mathrm{d}D_{\boldsymbol{q}}}{\mathrm{d}t} = i[H, D_{\boldsymbol{q}}] \tag{3}$$

Let us first consider the RHS and calculate the commutator term-by-term. Some useful formulas include the commutator and anticommutator relations for the creation and annihilation operators, and the commutator identities. They are used in the following calculations.

The first term:

$$\begin{aligned}
\left[\sum_{\boldsymbol{k},\alpha} \varepsilon_{\alpha\boldsymbol{k}} c^{\dagger}_{\alpha\boldsymbol{k}} c_{\alpha\boldsymbol{k}}, b_{\boldsymbol{q}'} + b^{\dagger}_{-\boldsymbol{q}'} \right] &= \sum_{\boldsymbol{k},\alpha} \varepsilon_{\alpha\boldsymbol{k}} c^{\dagger}_{\alpha\boldsymbol{k}} c_{\alpha\boldsymbol{k}} \left(b_{\boldsymbol{q}'} + b^{\dagger}_{-\boldsymbol{q}'} \right) - \left(b_{\boldsymbol{q}'} + b^{\dagger}_{-\boldsymbol{q}'} \right) \sum_{\boldsymbol{k},\alpha} \varepsilon_{\alpha\boldsymbol{k}} c^{\dagger}_{\alpha\boldsymbol{k}} c_{\alpha\boldsymbol{k}} \\
&= \sum_{\boldsymbol{k},\alpha} \varepsilon_{\alpha\boldsymbol{k}} c^{\dagger}_{\alpha\boldsymbol{k}} c_{\alpha\boldsymbol{k}} b_{\boldsymbol{q}'} + \sum_{\boldsymbol{k},\alpha} \varepsilon_{\alpha\boldsymbol{k}} c^{\dagger}_{\alpha\boldsymbol{k}} c_{\alpha\boldsymbol{k}} b^{\dagger}_{-\boldsymbol{q}'} - \sum_{\boldsymbol{k},\alpha} b_{\boldsymbol{q}'} \varepsilon_{\alpha\boldsymbol{k}} c^{\dagger}_{\alpha\boldsymbol{k}} c_{\alpha\boldsymbol{k}} - \sum_{\boldsymbol{k},\alpha} b^{\dagger}_{-\boldsymbol{q}'} \varepsilon_{\alpha\boldsymbol{k}} c^{\dagger}_{\alpha\boldsymbol{k}} c_{\alpha\boldsymbol{k}} \\
&= \sum_{\boldsymbol{k},\alpha} \varepsilon_{\alpha\boldsymbol{k}} \left[c^{\dagger}_{\alpha\boldsymbol{k}} c_{\alpha\boldsymbol{k}}, b_{\boldsymbol{q}'} \right] + \sum_{\boldsymbol{k},\alpha} \varepsilon_{\alpha\boldsymbol{k}} \left[c^{\dagger}_{\alpha\boldsymbol{k}} c_{\alpha\boldsymbol{k}}, b^{\dagger}_{-\boldsymbol{q}'} \right] \\
&= 0
\end{aligned} \tag{4}$$

The second term:

$$\begin{aligned}
\left[\sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} b^{\dagger}_{\boldsymbol{q}} b_{\boldsymbol{q}}, b_{\boldsymbol{q}'} + b^{\dagger}_{-\boldsymbol{q}'} \right] &= \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} b^{\dagger}_{\boldsymbol{q}} b_{\boldsymbol{q}} \left(b_{\boldsymbol{q}'} + b^{\dagger}_{-\boldsymbol{q}'} \right) - \left(b_{\boldsymbol{q}'} + b^{\dagger}_{-\boldsymbol{q}'} \right) \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} b^{\dagger}_{\boldsymbol{q}} b_{\boldsymbol{q}} \\
&= \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} b^{\dagger}_{\boldsymbol{q}} b_{\boldsymbol{q}} b_{\boldsymbol{q}'} + \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} b^{\dagger}_{\boldsymbol{q}} b_{\boldsymbol{q}} b^{\dagger}_{-\boldsymbol{q}'} - \sum_{\boldsymbol{q}} b_{\boldsymbol{q}'} \omega_{\boldsymbol{q}} b^{\dagger}_{\boldsymbol{q}} b_{\boldsymbol{q}} - \sum_{\boldsymbol{q}} b^{\dagger}_{-\boldsymbol{q}'} \omega_{\boldsymbol{q}} b^{\dagger}_{\boldsymbol{q}} b_{\boldsymbol{q}} \\
&= \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} \left[b^{\dagger}_{\boldsymbol{q}} b_{\boldsymbol{q}}, b_{\boldsymbol{q}'} \right] + \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} \left[b^{\dagger}_{\boldsymbol{q}} b_{\boldsymbol{q}}, b^{\dagger}_{-\boldsymbol{q}'} \right] \\
&= \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} \left(b^{\dagger}_{\boldsymbol{q}} [b_{\boldsymbol{q}}, b_{\boldsymbol{q}'}] + \left[b^{\dagger}_{\boldsymbol{q}}, b_{\boldsymbol{q}'} \right] b_{\boldsymbol{q}} \right) + \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} \left(b^{\dagger}_{\boldsymbol{q}} \left[b_{\boldsymbol{q}}, b^{\dagger}_{-\boldsymbol{q}'} \right] + \left[b^{\dagger}_{\boldsymbol{q}}, b^{\dagger}_{-\boldsymbol{q}'} \right] b_{\boldsymbol{q}} \right) \\
&= \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} \left(b^{\dagger}_{\boldsymbol{q}} [b_{\boldsymbol{q}}, b_{\boldsymbol{q}'}] + \left[b^{\dagger}_{\boldsymbol{q}}, b_{\boldsymbol{q}'} \right] b_{\boldsymbol{q}} \right) + \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} \left(b^{\dagger}_{\boldsymbol{q}} \left[b_{\boldsymbol{q}}, b^{\dagger}_{-\boldsymbol{q}'} \right] + \left[b^{\dagger}_{\boldsymbol{q}}, b^{\dagger}_{-\boldsymbol{q}'} \right] b_{\boldsymbol{q}} \right) \\
&= -\omega_{\boldsymbol{q}'} b_{\boldsymbol{q}'} + \omega_{-\boldsymbol{q}'} b^{\dagger}_{-\boldsymbol{q}'}
\end{aligned} \tag{5}$$

The first of the last term:

$$
\begin{aligned}
\left[\sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} b_{\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}, b_{\boldsymbol{q}'} + b^\dagger_{-\boldsymbol{q}'}\right] &= \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} b_{\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} \left(b_{\boldsymbol{q}'} + b^\dagger_{-\boldsymbol{q}'}\right) - \left(b_{\boldsymbol{q}'} + b^\dagger_{-\boldsymbol{q}'}\right) \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} b_{\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} \\
&= \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} b_{\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} b_{\boldsymbol{q}'} + \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} b_{\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}'} \\
&\quad - \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} b_{\boldsymbol{q}'} M_{\boldsymbol{k}\boldsymbol{q}} b_{\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} - \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}'} M_{\boldsymbol{k}\boldsymbol{q}} b_{\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} \\
&= \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} \left[b_{\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}, b_{\boldsymbol{q}'}\right] + \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} \left[b_{\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}, b^\dagger_{-\boldsymbol{q}'}\right] \\
&= \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} \left(b_{\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} [c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}, b_{\boldsymbol{q}'}] + b_{\boldsymbol{q}} \left[c^\dagger_{\alpha\boldsymbol{k}}, b_{\boldsymbol{q}'}\right] c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} + [b_{\boldsymbol{q}}, b_{\boldsymbol{q}'}] c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}\right) \\
&\quad + \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} \left(b_{\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} \left[c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}, b^\dagger_{-\boldsymbol{q}'}\right] + b_{\boldsymbol{q}} \left[c^\dagger_{\alpha\boldsymbol{k}}, b^\dagger_{-\boldsymbol{q}'}\right] c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} + \left[b_{\boldsymbol{q}}, b^\dagger_{-\boldsymbol{q}'}\right] c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}\right) \\
&= M_{\boldsymbol{k},-\boldsymbol{q}'} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}-\boldsymbol{q}'}
\end{aligned} \tag{6}
$$

The second of the last term:

$$
\begin{aligned}
\left[\sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}, b_{\boldsymbol{q}'} + b^\dagger_{-\boldsymbol{q}'}\right] &= \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} \left(b_{\boldsymbol{q}'} + b^\dagger_{-\boldsymbol{q}'}\right) - \left(b_{\boldsymbol{q}'} + b^\dagger_{-\boldsymbol{q}'}\right) \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} \\
&= \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} b_{\boldsymbol{q}'} + \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}'} \\
&\quad - \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} b_{\boldsymbol{q}'} M_{\boldsymbol{k}\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} - \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}'} M_{\boldsymbol{k}\boldsymbol{q}} b^\dagger_{-\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} \\
&= \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} \left[b^\dagger_{-\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}, b_{\boldsymbol{q}'}\right] + \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} \left[b^\dagger_{-\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}, b^\dagger_{-\boldsymbol{q}'}\right] \\
&= \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} \left(c^\dagger_{\alpha\boldsymbol{k}} [c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}, b_{\boldsymbol{q}'}] + b^\dagger_{-\boldsymbol{q}} \left[c^\dagger_{\alpha\boldsymbol{k}}, b_{\boldsymbol{q}'}\right] c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} + \left[b^\dagger_{-\boldsymbol{q}}, b_{\boldsymbol{q}'}\right] c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}\right) \\
&\quad + \sum_{\alpha,\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{q}} \left(b^\dagger_{-\boldsymbol{q}} c^\dagger_{\alpha\boldsymbol{k}} \left[c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}, b^\dagger_{-\boldsymbol{q}'}\right] + b^\dagger_{-\boldsymbol{q}} \left[c^\dagger_{\alpha\boldsymbol{k}}, b^\dagger_{-\boldsymbol{q}'}\right] c_{\alpha,\boldsymbol{k}+\boldsymbol{q}} + \left[b^\dagger_{-\boldsymbol{q}}, b^\dagger_{-\boldsymbol{q}'}\right] c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}+\boldsymbol{q}}\right) \\
&= -M_{\boldsymbol{k},-\boldsymbol{q}'} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}-\boldsymbol{q}'}
\end{aligned} \tag{7}
$$

The equation of motion is then

$$
\frac{\mathrm{d}D_{\boldsymbol{q}'}}{\mathrm{d}t} = -i\omega_{\boldsymbol{q}'} b_{\boldsymbol{q}'} + i\omega_{-\boldsymbol{q}'} b^\dagger_{-\boldsymbol{q}'} + iM_{\boldsymbol{k},-\boldsymbol{q}'} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}-\boldsymbol{q}'} - iM_{\boldsymbol{k},-\boldsymbol{q}'} c^\dagger_{\alpha\boldsymbol{k}} c_{\alpha,\boldsymbol{k}-\boldsymbol{q}'} \tag{8}
$$