Problem 1: Gerry & Knight, Page 40, Problem 6

6. Consider the superposition state $|\psi_{01}\rangle = \alpha|0\rangle + \beta|1\rangle$ where α and β are complex and satisfy $|\alpha|^2 + |\beta|^2 = 1$. Calculate the variances of the quadrature operators $\hat{X}_1$ and $\hat{X}_2$. Are there any values of the parameters α and β for which either of the quadrature

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References 41

variances become *less* than for a vacuum state? If so, check to see if the uncertainty principle is violated. Repeat with the state $|\psi_{02}\rangle = \alpha|0\rangle + \beta|2\rangle$.

Let us consider

$$
\begin{aligned}
X_1 &= \frac{a+a^\dagger}{2} \\
X_2 &= \frac{a-a^\dagger}{2i}
\end{aligned}
\tag{1}
$$

and the superposition state

$$
\psi = \alpha\,|0\rangle + \beta\,|1\rangle \tag{2}
$$

Now,

$$
\begin{aligned}
a\left|\psi\right\rangle &= a\left(\alpha\,|0\rangle + \beta\,|1\rangle\right) = \beta\,|0\rangle \\
a^\dagger\left|\psi\right\rangle &= a^\dagger\left(\alpha\,|0\rangle + \beta\,|1\rangle\right) = a\,|1\rangle + \beta\sqrt{2}\,|2\rangle
\end{aligned}
\tag{3}
$$

so

$$
\begin{aligned}
\left\langle\psi\,|a|\,\psi\right\rangle &= \left(\alpha\,|0\rangle + \beta\,|1\rangle\right)\cdot\left(\beta\,|0\rangle\right) = \alpha\beta^* \\
\left\langle\psi\left|a^\dagger\right|\psi\right\rangle &= \left(\alpha\,|0\rangle + \beta\,|1\rangle\right)\cdot\left(\alpha\,|1\rangle + \beta\sqrt{2}\,|2\rangle\right) = \alpha^*\beta \\
\left\langle\psi\left|a^\dagger a^\dagger\right|\psi\right\rangle &= \left(\beta\,|0\rangle\right)\cdot\left(\alpha\,|1\rangle + \beta\sqrt{2}\,|2\rangle\right) = 0 \\
\left\langle\psi\left|a^\dagger a\right|\psi\right\rangle &= \left(\beta\,|0\rangle\right)\cdot\left(\beta\,|0\rangle\right) = \left|\beta\right|^2 \\
\left\langle\psi\left|aa^\dagger\right|\psi\right\rangle &= \left(\alpha\,|1\rangle + \beta\sqrt{2}\,|2\rangle\right)\cdot\left(\alpha\,|1\rangle + \beta\sqrt{2}\,|2\rangle\right) = |\alpha|^2 + 2\left|\beta\right|^2 \\
\left\langle\psi\,|aa|\,\psi\right\rangle &= \left(\alpha\,|1\rangle + \beta\sqrt{2}\,|2\rangle\right)\cdot\left(\beta\,|0\rangle\right) = 0
\end{aligned}
\tag{4}
$$

The normalization condition $|\alpha|^2 + \left|\beta\right|^2 = 1$ implies that we can write

$$
\beta \equiv \sqrt{1-|\alpha|^2}\,e^{i\phi} \tag{5}
$$

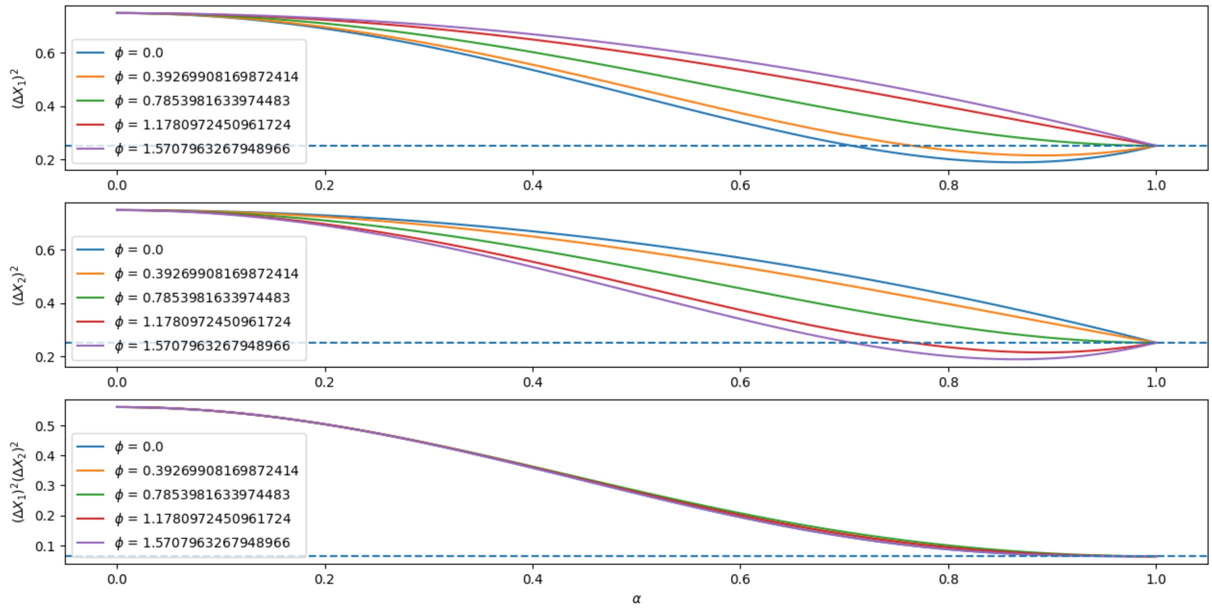
Furthermore, we can make α real by multiplying ψ with some global phase (which also redefines ϕ). We obtain

$$
\begin{aligned}
\langle X_1\rangle &= \frac{\langle\psi\,|a+a^\dagger|\,\psi\rangle}{2} = \frac{\alpha\beta^* + \alpha^*\beta}{2} = \alpha\sqrt{1-\alpha^2}\cos\phi \\
\langle X_2\rangle &= \frac{\langle\psi\,|a-a^\dagger|\,\psi\rangle}{2i} = \frac{\alpha\beta^* - \alpha^*\beta}{2i} = \alpha\sqrt{1-\alpha^2}\sin(-\phi) \\
\left\langle X_1^2\right\rangle &= \frac{\left\langle\psi\left|\left(a+a^\dagger\right)^2\right|\psi\right\rangle}{4} = \frac{|\alpha|^2 + 3\left|\beta\right|^2}{4} = \frac{\alpha^2 + 3(1-\alpha^2)}{4} = \frac{3-2\alpha^2}{4} \\
\left\langle X_2^2\right\rangle &= -\frac{\left\langle\psi\left|\left(a-a^\dagger\right)^2\right|\psi\right\rangle}{4} = \frac{|\alpha|^2 + 3\left|\beta\right|^2}{4} = \frac{3-2\alpha^2}{4}
\end{aligned}
\tag{6}
$$

and thus

$$
\begin{aligned}
(\Delta X_1)^2 &= \left\langle X_1^2\right\rangle - \langle X_1\rangle^2 = \frac{3-2\alpha^2}{4} - \alpha^2(1-\alpha^2)\cos^2\phi \\
(\Delta X_2)^2 &= \left\langle X_2^2\right\rangle - \langle X_2\rangle^2 = \frac{3-2\alpha^2}{4} - \alpha^2(1-\alpha^2)\sin^2\phi
\end{aligned}
\tag{7}
$$

Here is a plot of the variation of the variance of both quadratures with respect to α and ϕ (note that $\alpha \leq 1$):



For the vacuum state, we know that

$$\left(\Delta X_{1,2}\right)^2 = \frac{1}{4} \tag{8}$$

so there are actually values of α and ϕ for which our variance is lower than the vacuum value. To see if the uncertainty principle is violated we need to check if

$$(\Delta X_1)^2 (\Delta X_2)^2 \geq \frac{1}{16} \tag{9}$$

is satisfied. We can see from the bottom plot that **the uncertainty principle is not violated for all values of α and ϕ.**