Lecture #9 Homework #9 [2025-11-04]

Problem 9.1. Calculate the Landau parameters to leading order in $\lambda_{1,2}$ for a Fermi liquid with the contact interactions

(a) $V(\boldsymbol{x}-\boldsymbol{x}') = \lambda_1 \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$.

(b) $V(\boldsymbol{x}-\boldsymbol{x}') = -\lambda_2 \nabla^2 \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$ (so that $V(q) = \lambda_1 q^2$ in Fourier space).

(c) Taking the results of (a) and (b) literally, sketch the regions of the λ_1, λ_2 phase diagram where the Fermi surface becomes unstable.

Solution. For a spin-independent two-body interaction $V(\boldsymbol{q})$, the Landau interaction function in the forward direction

$$f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'} = \left[V(0) - V(|\boldsymbol{p}-\boldsymbol{p}'|)\right]\delta_{\sigma\sigma'}.$$

Decomposing into spin-symmetric f^s and spin-antisymmetric f^a parts

$$f^s(\theta) = V(0) - \frac{1}{2}V(q), \quad f^a(\theta) = -\frac{1}{2}V(q),$$

where $q = |\boldsymbol{p}-\boldsymbol{p}'| = 2k_F \sin(\theta/2)$, $\theta = \langle \boldsymbol{p}, \boldsymbol{p}' \rangle$. Expanding in Legendre polynomials

$$f^s(\theta) = \sum_l f_l^s P_l(\cos\theta), \quad f^a(\theta) = \sum_l f_l^a P_l(\cos\theta).$$

Hence, the Landau parameters are

$$F_l^s = N(0) f_l^s, \quad F_l^a = N(0) f_l^a.$$

where $N(0) = \frac{mk_F}{2\pi^2}$ is the density of states per spin at the Fermi level.

(a) In the case of $V(\boldsymbol{x}-\boldsymbol{x}') = \lambda_1 \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$.
Do the Fourier transform

$$V(\boldsymbol{q}) = V(0) = \lambda_1,$$

then

$$f_l^s(\theta) = f_0^s = \lambda_1 - \frac{\lambda_1}{2} = \frac{\lambda_1}{2}, \quad f_l^a(\theta) = f_0^a = -\frac{\lambda_1}{2}.$$

Hence, the Landau parameters are

$$F_0^s = \frac{N(0)\lambda_1}{2}, \quad F_0^a = -\frac{N(0)\lambda_1}{2}, \quad F_{l>0} = 0.$$

(b) In the case of $V(\boldsymbol{x}-\boldsymbol{x}') = -\lambda_2 \nabla^2 \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')$.
Do the Fourier transform

$$V(\boldsymbol{q}) = \lambda_2 q^2, \quad \text{and} \quad V(0) = 0,$$

then

$$f^s(\theta) = f^a(\theta) = -\frac{\lambda_2}{2} q^2 \xlongequal{q^2 = 2k_F^2(1-\cos\theta)} -\lambda_2 k_F^2 (1-\cos\theta).$$

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Expanding in Legendre polynomials

$$f^s(\theta) = -\lambda_2 k_F^2 + \lambda_2 k_F^2 \cos\theta = -\lambda_2 k_F^2 + \lambda_2 k_F^2 P_1(\cos\theta),$$

Then

$$f_0^s = f_0^a = -\lambda_2 k_F^2, \quad f_1^s = f_1^a = \lambda_2 k_F^2, \quad f_{l\geq 2} = 0.$$

Hence, the Landau parameters are

$$F_0^s = F_0^a = -N(0)\lambda_2 k_F^2, \quad F_1^s = F_1^a = N(0)\lambda_2 k_F^2, \quad F_{l>1} = 0.$$

(c) When both interactions are present, the Landau parameters add linearly

$$F_0^s = \frac{N(0)\lambda_1}{2} - N(0)\lambda_2 k_F^2, \quad F_0^a = -\frac{N(0)\lambda_1}{2} - N(0)\lambda_2 k_F^2, \quad F_1^s = F_1^a = N(0)\lambda_2 k_F^2.$$

The Pomeranchuk stability conditions are

$$1 + \frac{F_l^s}{2l+1} > 0, \quad 1 + \frac{F_l^a}{2l+1} > 0.$$

For $l = 0$: $1 + F_0^s > 0$, $1 + F_0^a > 0$; For $l = 1$: $1 + \frac{F_1^s}{3} > 0$, $1 + \frac{F_1^a}{3} > 0$. Define two coefficients $A = \frac{N(0)}{2}$, $B = N(0)k_F^2$, then

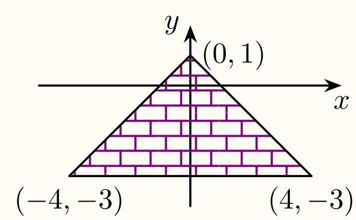
$$F_0^s = A\lambda_1 - B\lambda_2, \quad F_0^a = -A\lambda_1 - B\lambda_2, \quad F_1^s = F_1^a = B\lambda_2.$$

Let $x = A\lambda_1$, $y = B\lambda_2$, the stability conditions become

i. $1 + x - y > 0$ ii. $1 - x - y > 0$ iii. $1 + \frac{y}{3} > 0$.

Do the linear programming, the stable region in the (x, y)-plane is a triangle on the right. In terms of λ_1, λ_2, the stable region is inside the triangle with vertices

$$\left(-\frac{4}{A}, -\frac{3}{B}\right), \quad \left(\frac{4}{A}, -\frac{3}{B}\right), \quad \left(0, \frac{1}{B}\right).$$



Problem 9.2. Test your understanding of Landau's mass renormalization formula by generalizing it to include the effect of a magnetization. Suppose we introduce a second vector potential into (6.86)

$$A(\theta) \underset{|\boldsymbol{k}|\to 0}{\sim} \int_{k_F - k\cos\theta}^{k_F} \mathrm{d}q \frac{2\,\mathrm{i}\pi a^2}{\omega - v_F(|\boldsymbol{k}+\boldsymbol{q}| - q)} = \frac{2\,\mathrm{i}\pi a^2 k\cos\theta}{\omega - v_F k\cos\theta}. \tag{6.86}$$

that couples to the spin current, writing

$$\mathcal{H}[\mathbf{A}_N, \mathbf{W}] = \sum_\sigma \int \mathrm{d}^3x \frac{1}{2m} \psi_\sigma^\dagger(x)[(-\mathrm{i}\hbar\nabla - \mathbf{A}_N - \sigma\mathbf{W})^2]\psi_\sigma(x) + \hat{V}.$$

Whereas $\mathbf{A}_N$ couples to the current of particles, $\mathbf{W}$ couples to the (z component of the) spin current. Assume that V conserves spin current.

(a) By comparing the bare shift of the energies

$$\delta\epsilon^{(0)}_{\boldsymbol{p}\sigma} = -\frac{\boldsymbol{p}}{m}\cdot(\mathbf{A}_N + \sigma\mathbf{W}),$$

with the shift that results from interaction feedback,

$$\delta\epsilon_{\boldsymbol{p}\sigma} = -\frac{\boldsymbol{p}}{m^*}\cdot\mathbf{A}_N - \sigma\frac{\boldsymbol{p}}{m^*_s}\cdot\mathbf{W},$$

show that there are two different mass renormalizations,

$$\frac{m}{m^*} = \frac{1}{1+F^s_1}, \quad \frac{m}{m^*_s} = \frac{1}{1+F^a_1}.$$

(b) Show that, when the Fermi liquid is polarized, the masses of the "up" and "down" quasiparticles are now different, and given by

$$\frac{1}{m^*_\sigma} = \frac{1}{m}\left[\frac{1}{1+F^s_1} + \frac{M}{1+F^a_1}\right], \quad (\sigma = \uparrow,\downarrow),$$

where the magnetization $M = n_\uparrow - n_\downarrow$ is the difference of "up" and "down" densities.

Solution.

(a) *Proof.* In the steady state, the distribution function is a shifted Fermi sphere, then

$$\delta n_{\boldsymbol{p}\sigma} = -\frac{\partial n^0}{\partial\epsilon}(\boldsymbol{p}\cdot\boldsymbol{u}_\sigma),$$

where $\boldsymbol{u}_\sigma$ is the drift velocity for σ. Substitute it and $\delta\epsilon^{(0)}_{\boldsymbol{p}\sigma}$ to get the full quasiparticle energy shift

$$\delta\epsilon_{\boldsymbol{p}\sigma} = \delta\epsilon^{(0)}_{\boldsymbol{p}\sigma} + \sum_{\boldsymbol{p}'\sigma'} f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'}\delta n_{\boldsymbol{p}'\sigma'} = -\frac{\boldsymbol{p}}{m}\cdot(\mathbf{A}_N + \sigma\mathbf{W}) - \sum_{\boldsymbol{p}'\sigma'} f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'}\frac{\partial n^0}{\partial\epsilon'}(\boldsymbol{p}'\cdot\boldsymbol{u}_{\sigma'}).$$

Due to isotropy, the interaction term simplifies. The Landau interaction function is decomposed into spin-symmetric and spin-antisymmetric parts

$$f_{\boldsymbol{p}\sigma,\boldsymbol{p}'\sigma'} = f^s(\theta)\delta_{\sigma\sigma'} + f^a(\theta)(\sigma\sigma').$$

where θ is the angle between $\boldsymbol{p}$ and $\boldsymbol{p}'$. Expanding in Legendre polynomials

$$f^s(\theta) = \sum_l f^s_l P_l(\cos\theta), \quad f^a(\theta) = \sum_l f^a_l P_l(\cos\theta).$$

The Landau parameters are defined as:

$$F^s_l = N(0)f^s_l, \quad F^a_l = N(0)f^a_l,$$

where $N(0)$ is the density of states at the Fermi level per spin. After evaluating the angular integrals, the energy shift becomes

$$\delta\epsilon_{\boldsymbol{p}\sigma} = -\frac{\boldsymbol{p}}{m}\cdot(\mathbf{A}_N + \sigma\mathbf{W}) + \frac{1}{3}[F^s_1(\boldsymbol{p}\cdot\boldsymbol{u}_\sigma) + F^a_1\sigma(\boldsymbol{p}\cdot(\boldsymbol{u}_\uparrow - \boldsymbol{u}_\downarrow))].$$

In the steady state, the effective force on quasiparticles vanishes, i.e., $\delta\epsilon_{\boldsymbol{p}\sigma} = \boldsymbol{p} \cdot \boldsymbol{u}_\sigma$. Comparing with the formula above, we can obtain $\boldsymbol{u}_\sigma$

$$\boldsymbol{u}_\sigma = -\frac{1}{m^*}\mathbf{A}_N - \sigma\frac{1}{m_s^*}\mathbf{W}.$$

Then, comparing coefficients yields the desired relations

$$\frac{m}{m^*} = \frac{1}{1+F_1^s}, \quad \frac{m}{m_s^*} = \frac{1}{1+F_1^a}.$$

(b) *Proof.* When the system is magnetized ($M = n_\uparrow - n_\downarrow \neq 0$), the Fermi surfaces for spin-up and spin-down are different. The drift velocities become

$$\boldsymbol{u}_\uparrow = \boldsymbol{u} + \boldsymbol{u}_s, \quad \boldsymbol{u}_\downarrow = \boldsymbol{u} - \boldsymbol{u}_s,$$

where $\boldsymbol{u}$ is the charge drift velocity and $\boldsymbol{u}_s$ is the spin drift velocity. The energy shifts are

$$\delta\epsilon_{\boldsymbol{p}\uparrow} = -\frac{\boldsymbol{p}}{m^*}\cdot\mathbf{A}_N - \frac{\boldsymbol{p}}{m_s^*}\cdot\mathbf{W}, \quad \delta\epsilon_{\boldsymbol{p}\downarrow} = -\frac{\boldsymbol{p}}{m^*}\cdot\mathbf{A}_N + \frac{\boldsymbol{p}}{m_s^*}\cdot\mathbf{W}.$$

The effective masses for each spin species are then:

$$\frac{1}{m_\sigma^*} = \frac{1}{m}\left[\frac{1}{1+F_1^s} + \sigma\frac{M}{1+F_1^a}\right], \quad \text{for } \sigma = \uparrow, \downarrow,$$

where M is the magnetization. □

Lecture #10 Homework #10 [2025-11-11]

Problem 10.1. Interpreting the Fermi liquid via a Schrieffer-Wolff transformation. We start from a translationally invariant, spin-1/2 fermion system

$$\mathcal{H} = \mathcal{H} + \mathcal{H}_{\text{int}},$$

where

$$\mathcal{H}_0 = \sum_{k,\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma}, \quad \mathcal{H}_{\text{int}} = \frac{1}{2V} \sum_{k,k',q,\sigma,\sigma'} V_q c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'} c_{k\sigma}.$$

The Fermi sea $|\text{FS}\rangle$ is the ground state of $\mathcal{H}_0$, with all states $|k| < k_F$ filled. We assume V_q is weak and smooth near $q = 0$.

To do a canonical transformation for a Fermi liquid state, we want to integrate out (cancel out) **off-shell** excitations. This is different from previous cases, where the original interacting terms are canceled out entirely. Instead, for a Fermi liquid state, we write H_{int} schematically as

$$\mathcal{H}_{\text{int}} = \mathcal{H}_{\text{diag}} + H_{\text{off}},$$

where

- $\mathcal{H}_{\text{diag}}$ conserves the number of quasiparticles near the Fermi surface (i.e., forward scattering, density-density type).
- $\mathcal{H}_{\text{off}}$ mixes sectors with different numbers of particle-hole pairs (e.g., it creates or destroys particle-hole pairs relative to the Fermi sea).

Formally,

$$\mathcal{H}_{\text{diag}} = \frac{1}{2V} \sum_{\substack{k,k',q \\ \text{both } k,\, k^\dagger q \text{ near } k_F}} V_q c^\dagger_{k+q} c^\dagger_{k'-q} c_{k'} c_k,$$

and $\mathcal{H}_{\text{off}}$ is the rest.

Such separation can be formally done via the semi-classical variational method A simplified version is written

$$\mathcal{H}_{\text{diag}} = \frac{1}{2V} \sum_{\substack{k,k',q \\ \text{both } k = |k+q| = |k'-q| = |k'| = k_F,\, q \to 0}} V_q c^\dagger_{k+q} c^\dagger_{k'-q} c_{k'} c_k.$$

So you can see $\mathcal{H}_{\text{off}}$ is just $\mathcal{H}_{\text{int}}$ with at least one momentum that is away from the Fermi surface. We can just keep it in the form of $\mathcal{H}_{\text{int}}$.

We choose $\mathcal{S}$ such that

$$\mathcal{H}_{\text{off}} + [\mathcal{S}, \mathcal{H}_0] = 0.$$

This means $\mathcal{S}$ is chosen to cancel the leading-order off-diagonal part of $\mathcal{H}_{\text{int}}$ under the transformation.

Then the transformed Hamiltonian becomes

$$\mathcal{H}' = \mathcal{H}_0 + \mathcal{H}_{\text{diag}} + \frac{1}{2}[\mathcal{S}, \mathcal{H}_{\text{off}}] + \mathcal{O}(V^3),$$

where the new effective Hamiltonian is block-diagonal up to order V^2.
Constructing $\mathcal{S}$ explicitly.
The commutator $[\mathcal{S}, \mathcal{H}_0]$ acts as

$$[\mathcal{S}, \mathcal{H}_0] = \sum_{\alpha\beta} S_{\alpha\beta}(\epsilon_\alpha - \epsilon_\beta) c_\alpha^\dagger c_\beta,$$

so to satisfy $[\mathcal{S}, \mathcal{H}_0] = -\mathcal{H}_{\text{off}}$, we can write

$$\mathcal{S} = \sum_{\alpha\beta} \frac{(\mathcal{H}_{\text{off}})_{\alpha\beta}}{\epsilon_\alpha - \epsilon_\beta}.$$

This is the Schrieffer-Wolff generator, which mixes states differing in unperturbed energy. For the present problem, $\mathcal{S}$ connects a bare fermion state $c_{k\alpha}^\dagger|\text{FS}\rangle$ to configurations with one additional particle-hole pair.

Now, answer the following questions.

(a) Evaluate $[\mathcal{S}, c_{k\sigma}^\dagger]$;

(b) Transforming the creation operator - "ressing" the bare fermion. The transformed fermion creation operator is

$$\tilde{c}_{k\sigma}^\dagger = \mathrm{e}^{\mathcal{S}} c_{k\sigma}^\dagger \,\mathrm{e}^{-\mathcal{S}} = c_{k\sigma}^\dagger + [\mathcal{S}, c_{k\sigma}^\dagger] + \frac{1}{2}[\mathcal{S}, [\mathcal{S}, c_{k\sigma}^\dagger]] + \cdots$$

to the first order in V

$$\tilde{c}_{k\sigma}^\dagger \approx c_{k\sigma}^\dagger + [\mathcal{S}, c_{k\sigma}^\dagger].$$

(c) Compute $\tilde{c}_{k\sigma}^\dagger|\text{FS}\rangle$.

(d) Compute quasiparticle weight $Z_k \equiv |\langle \text{FS}|\tilde{c}_k \tilde{c}_k^\dagger|\text{FS}\rangle|^2$ from the canonical transformation.

Solution.

(a) Starting from the identity of $\mathcal{S}$,

$$[\mathcal{S}, \mathcal{H}_0] = -\mathcal{H}_{\text{off}}.$$

To get the matrix elements of $\mathcal{S}$, substitute the commutator above into the states $\langle m|$ and $|n\rangle$ ($m \neq n$)

$$-\langle m|\mathcal{H}_{\text{off}}|n\rangle = \langle m|[\mathcal{S}, \mathcal{H}_0]|n\rangle = \langle m|\mathcal{S}|n\rangle E_n - E_m\langle m|\mathcal{S}|n\rangle = (E_n - E_m)\langle m|\mathcal{S}|n\rangle,$$

where $\langle m|\mathcal{H}_0 = \langle m|E_m$ and $\mathcal{H}_0|n\rangle = E_n|n\rangle$. Then,

$$\langle m|\mathcal{S}|n\rangle = \frac{\langle m|\mathcal{H}_{\text{off}}|n\rangle}{E_m - E_n} \quad \text{for} \quad m \neq n,$$

where we set the diagonal elements $\langle n|\mathcal{S}|n\rangle = 0$. By inserting $|m\rangle\langle n|$ to form the identity $\mathbb{1}$, the operator $\mathcal{S}$ in Dirac notation can be expressed as

$$\mathcal{S} = \sum_m \sum_n |m\rangle \frac{\langle m|\mathcal{H}_{\text{off}}|n\rangle}{E_m - E_n} \langle n|$$

So, we can generate $\mathcal{S}$ from the four operators in $\mathcal{H}_{\text{off}}$, which is equivalent to the four operators in $\mathcal{H}_{\text{int}}$ but without the diagonal elements, that is

$$\mathcal{S} = \sum_{k,k',q,\sigma,\sigma'} S^{\sigma\sigma'}_{kk'q} c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'} c_{k\sigma},$$

Then, substitute it into the condition $[\mathcal{S}, \mathcal{H}_0] = -\mathcal{H}_{\text{off}}$ to determine $S^{\sigma\sigma'}_{kk'q}$. To distinguish the indices, we need to write $\mathcal{H}_0$ as

$$\mathcal{H}_0 = \sum_{p,\tau} \epsilon_p c^\dagger_{p\tau} c_{p\tau}$$

Then, the commutator is

$$[\mathcal{S}, \mathcal{H}_0] = \sum_{k,k',q,\sigma,\sigma'} S^{\sigma\sigma'}_{kk'q} \sum_{p,\tau} \epsilon_p [c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'} c_{k\sigma}, c^\dagger_{p\tau} c_{p\tau}].$$

Compute the kernel first

$$\begin{aligned}
[c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'} c_{k\sigma}, c^\dagger_{p\tau} c_{p\tau}] &= [c^\dagger_{k+q,\sigma}, c^\dagger_{p\tau} c_{p\tau}] c^\dagger_{k'-q,\sigma'} c_{k'\sigma'} c_{k\sigma} + c^\dagger_{k+q,\sigma} [c^\dagger_{k'-q,\sigma'}, c^\dagger_{p\tau} c_{p\tau}] c_{k'\sigma'} c_{k\sigma} \\
&\quad + c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} [c_{k'\sigma'}, c^\dagger_{p\tau} c_{p\tau}] c_{k\sigma} + c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'} [c_{k\sigma}, c^\dagger_{p\tau} c_{p\tau}].
\end{aligned}$$

The four terms

i. $[c^\dagger_{k+q,\sigma}, c^\dagger_{p\tau} c_{p\tau}] = -c^\dagger_{p\tau} \delta_{p,k+q} \delta_{\tau,\sigma}$

ii. $[c^\dagger_{k'-q,\sigma'}, c^\dagger_{p\tau} c_{p\tau}] = -c^\dagger_{p\tau} \delta_{p,k'-q} \delta_{\tau,\sigma'}$

iii. $[c_{k'\sigma'}, c^\dagger_{p\tau} c_{p\tau}] = c_{p\tau} \delta_{k'p} \delta_{\sigma'\tau}$

iv. $[c_{k\sigma}, c^\dagger_{p\tau} c_{p\tau}] = c_{p\tau} \delta_{kp} \delta_{\sigma\tau}$

where we used the anti-commutative properties of the Fermions

$$\{c_{i,j}, c^\dagger_{i',j'}\} = \delta_{i,i'} \delta_{j,j'}, \quad \{c_{i,j}, c_{i',j'}\} = \{c_{i,j}, c_{i',j'}\} = 0.$$

Substitute them into the second sum of the commutator

$$\begin{aligned}
\sum_{p,\tau} \epsilon_p [c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'} c_{k\sigma}, c^\dagger_{p\tau} c_{p\tau}] = &- \sum_{p,\tau} \epsilon_p c^\dagger_{p\tau} \delta_{p,k+q} \delta_{\tau,\sigma}\, c^\dagger_{k'-q,\sigma'} c_{k'\sigma'} c_{k\sigma} \\
&- \sum_{p,\tau} \epsilon_p\, c^\dagger_{k+q,\sigma}\, c^\dagger_{p\tau} \delta_{p,k'-q} \delta_{\tau,\sigma'}\, c_{k'\sigma'} c_{k\sigma} \\
&+ \sum_{p,\tau} \epsilon_p\, c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'}\, c_{p\tau} \delta_{k'p} \delta_{\sigma'\tau}\, c_{k\sigma} \\
&+ \sum_{p,\tau} \epsilon_p\, c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'}\, c_{p\tau} \delta_{kp} \delta_{\sigma\tau}\, .
\end{aligned}$$

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Due to the sifting property of the δ-function, the second sum of the commutator becomes (here take the minus sign out since $[\mathcal{S},\mathcal{H}_0]=-\mathcal{H}_{\text{off}}$)

$$\sum_{p,\tau}\epsilon_p[c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}c_{k\sigma},c^\dagger_{p\tau}c_{p\tau}]=-(\epsilon_{k+q}+\epsilon_{k'-q}-\epsilon_{k'}-\epsilon_k)c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}c_{k\sigma}.$$

Hence, the commutator $[\mathcal{S},\mathcal{H}_0]$ becomes

$$-\mathcal{H}_{\text{off}}=[\mathcal{S},\mathcal{H}_0]=-\sum_{k,k',q,\sigma,\sigma'}S^{\sigma\sigma'}_{kk'q}(\epsilon_{k+q}+\epsilon_{k'-q}-\epsilon_{k'}-\epsilon_k)c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}c_{k\sigma}$$

To compare with the expression of $\mathcal{H}_{\text{off}}$

$$\mathcal{H}_{\text{off}}=\frac{1}{2V}\sum_{k,k',q,\sigma,\sigma'}V_q c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}c_{k\sigma}\quad\text{where}\quad k\neq k',\ \sigma\neq\sigma'.$$

Then, we arrive at

$$\begin{aligned}\sum_{k,k',q,\sigma,\sigma'}S^{\sigma\sigma'}_{kk'q}c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}c_{k\sigma}&=\mathcal{S}\\&=\frac{1}{2V}\sum_{k,k',q,\sigma,\sigma'}\frac{V_q}{\epsilon_{k+q}+\epsilon_{k'-q}-\epsilon_k-\epsilon_{k'}}c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}c_{k\sigma}.\end{aligned}$$

This is the second-quantized form of $\mathcal{S}$. Now, calculate the commutator $[\mathcal{S},c^\dagger_{k\sigma}]$. Calculate the kernel

$$[c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}c_{k\sigma},c^\dagger_{k\sigma}]$$

first. To distinguish the indices, we take $c^\dagger_{k\sigma}\to c^\dagger_{p\tau}$

$$\begin{aligned}[c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}c_{k\sigma},c^\dagger_{p\tau}]&=c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}(c_{k'\sigma'}\{c_{k\sigma},c^\dagger_{p\tau}\}-\{c_{k'\sigma'},c^\dagger_{p\tau}\}c_{k\sigma})\\&=c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}\delta_{kp}\delta_{\sigma\tau}-c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}\delta_{k'p}\delta_{\sigma'\tau}c_{k\sigma},\end{aligned}$$

where $c^\dagger_{p\tau}$ only "knock" on $c_{k'\sigma'}$ and $c_{k\sigma}$ effectively due to the anti-commutative properties of the Fermions. We define the function

$$\Phi(p_1,p_2,q)=\frac{1}{\epsilon_{p_1+q}+\epsilon_{p_2-q}-\epsilon_{p_1}-\epsilon_{p_2}}.$$

Then, the commutator becomes

$$\begin{aligned}[\mathcal{S},c^\dagger_{p\tau}]&=\sum_{k,k',q,\sigma,\sigma'}\frac{V_q\Phi(k,k',q)}{2V}(c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}\delta_{kp}\delta_{\sigma\tau}-c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}\delta_{k'p}\delta_{\sigma'\tau}c_{k'\sigma'})\\&=\sum_{k',q,\sigma'}\frac{V_q\Phi(p,k',q)}{2V}c^\dagger_{p+q,\tau}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}-\sum_{k,q,\sigma}\frac{V_q\Phi(k,p,q)}{2V}c^\dagger_{k+q,\sigma}c^\dagger_{p-q,\tau}c_{k,\sigma}.\end{aligned}$$

Swap dummy variables in the second sum: $k\to k'$, $\sigma\to\sigma'$, $q\to -q$, and use $c^\dagger_{p+q}c^\dagger_{k'-q}=-c^\dagger_{k'-q}c^\dagger_{p+q}$

$$[\mathcal{S},c^\dagger_{p\tau}]=\sum_{k',q,\sigma'}\frac{V_q\Phi(p,k',q)}{2V}c^\dagger_{p+q,\tau}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}+\sum_{k,q,\sigma}\frac{V_{-q}\Phi(k',p,-q)}{2V}c^\dagger_{p+q,\tau}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'},$$

where $\Phi(p,k',q)=\Phi(k',p,-q)$. Hence,

$$[\mathcal{S},c^\dagger_{p\tau}]=\sum_{k',q,\sigma'}\frac{V_q+V_{-q}}{2V}\frac{c^\dagger_{p+q,\tau}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}}{\epsilon_{p+q}+\epsilon_{k'-q}-\epsilon_p-\epsilon_{k'}},\ [\mathcal{S},c^\dagger_{k\sigma}]=\sum_{k',q,\sigma'}\frac{V_q+V_{-q}}{2V}\frac{c^\dagger_{k+q,\sigma}c^\dagger_{k'-q,\sigma'}c_{k'\sigma'}}{\epsilon_{k+q}+\epsilon_{k'-q}-\epsilon_k-\epsilon_{k'}}.$$

(b) Substitute the result from (a) directly

$$\tilde{c}^\dagger_{k\sigma} \approx c^\dagger_{k\sigma} + \sum_{k',q,\sigma'} \frac{V_q + V_{-q}}{2V} \frac{c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'}}{\epsilon_{k+q} + \epsilon_{k'-q} - \epsilon_k - \epsilon_{k'}}.$$

(c) Substitute the result from (b) directly

$$\tilde{c}^\dagger_{k\sigma}|\mathsf{FS}\rangle = c^\dagger_{k\sigma}|\mathsf{FS}\rangle + \sum_{k',q,\sigma'} \frac{V_q + V_{-q}}{2V} \frac{c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k'\sigma'}}{\epsilon_{k+q} + \epsilon_{k'-q} - \epsilon_k - \epsilon_{k'}}.|\mathsf{FS}\rangle$$

(d) Evaluate the overlap $\langle \mathsf{FS}|c_k \tilde{c}^\dagger_k|\mathsf{FS}\rangle$ term by term

i. The zeroth order $\mathcal{O}(V^0)$

$$\langle \mathsf{FS}|c_k c^\dagger_k|\mathsf{FS}\rangle = 1.$$

Here assume $|k| > k_F$, so $c_k c^\dagger_k = 1$ in the Fermi sea.

ii. Denote the operator

$$\chi_k \equiv [\mathcal{S}, c^\dagger_k].$$

The first order $\mathcal{O}(V^1)$

$$\langle \mathsf{FS}|c_k \chi_k|\mathsf{FS}\rangle = \langle \mathsf{FS}|c_k[\mathcal{S}, c^\dagger_k]|\mathsf{FS}\rangle = 0,$$

where χ_k creates a *two-particle, one-hole state* relative to the Fermi sea. Applying c_k still leaves a net excitation, and the matrix element vanishes since the states are orthogonal ($\mathcal{S}$ is off-diagonal).

iii. The second order $\mathcal{O}(V^2)$

$$\frac{1}{2}\langle \mathsf{FS}|c_k[\mathcal{S}, \chi_k]|\mathsf{FS}\rangle = \frac{1}{2}\langle \mathsf{FS}|c_k[\mathcal{S}, [\mathcal{S}, c^\dagger_k]]|\mathsf{FS}\rangle = -\frac{1}{2}\langle \mathsf{FS}|\chi^\dagger_k \chi_k|\mathsf{FS}\rangle = -\frac{1}{2}|\chi_k|\mathsf{FS}\rangle|^2.$$

Since the generator $\mathcal{S}$ is anti-Hermitian ($\mathcal{S}^\dagger = -\mathcal{S}$), the second-order projection simplifies to $-\frac{1}{2}\langle \chi^\dagger_k \chi_k\rangle_{\mathsf{FS}}$. Substituting the explicit form of χ_k into the weight, we obtain

$$Z_k = 1 - \frac{1}{4V^2} \sum_{k',q,\sigma'} \frac{|V_q + V_{-q}|\theta(k_F - |k'|)\theta(|k+q| - k_F)\theta(|k'-q| - k_F)}{(\epsilon_{k+q} + \epsilon_{k'-q} - \epsilon_k - \epsilon_{k'})^2} < 1,$$

which brings a quasiparticle weight less than 1.