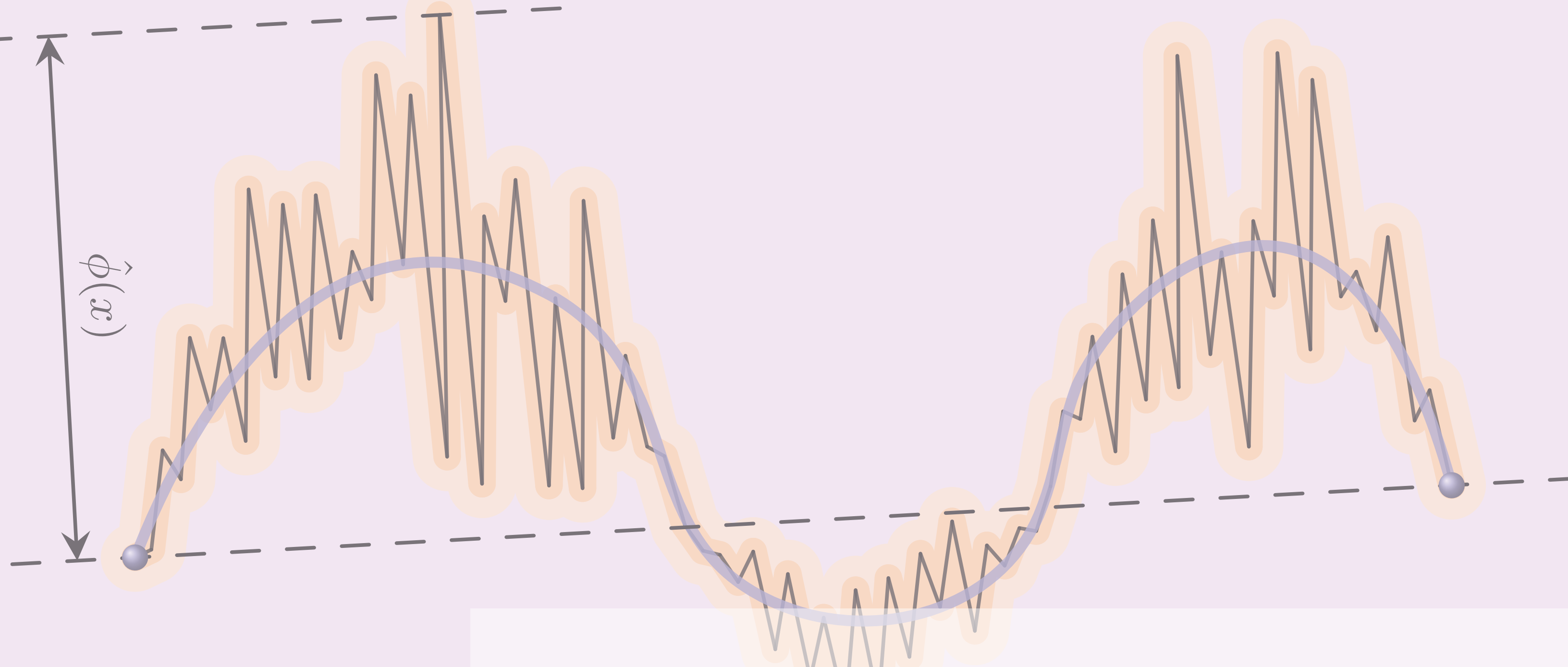
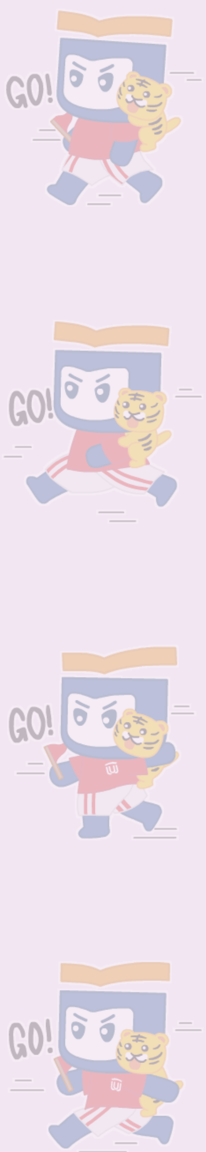


Quantum Many-Body Theory



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Contents

Chapter 1 Introduction

Prerequisite

Quantum Mechanics QM II and / or Advanced QM Double wave (hard): instanton gas

Statistical Mechanics

Solid State Physics

Mechanics (formal theory)

Electrodynamics mostly for transport theory

Reference Texts

See the References chapter: [1–10]. Textbooks by Ding [1], Coleman [2], Stefanucci and Leeuwen [5], and Bogolubov Jr [7] are recommanded.
In the Lecture Notes (Ding [1])

- Weak coupling theory will be standard
- Strong coupling ARE NOT standard, in developing

1.1 What QMB study

From Poisson Brackets $\{\quad\}_{PO} \Leftrightarrow$ Quantum commutators $[\quad]$.

Chapter 2 Gaussian Integral

2.1 Free Gaussian Integral

$$I = \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{e}^{-x^2} = \sqrt{\pi}, \; I^2 = \pi$$

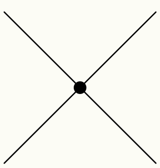
2.1.1 Perturbation expansion of $\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{e}^{-x^2 - gx^4}$

$$Z(g) = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int \mathrm{d}x x^{4n} \, \mathrm{e}^{-x^2} \tag{2.1}$$

$$\int x^{2m} \, \mathrm{e}^{-x^2} \, \mathrm{d}xi \tag{2.2}$$

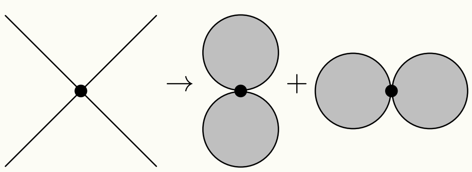
Perturbation expansion is actually divergent: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \Rightarrow \infty$
The series $Z(g)$ has zero convergence radius.

2.2 Feynman Diagram

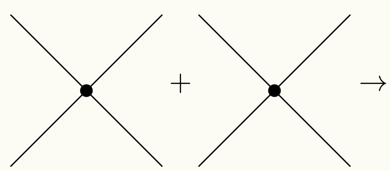
1. Vertice: gx^4 

2. Propagator: $< x, x >= \frac{1}{2}$

   (a) $n = 1$:

   (b) $n = 2$:

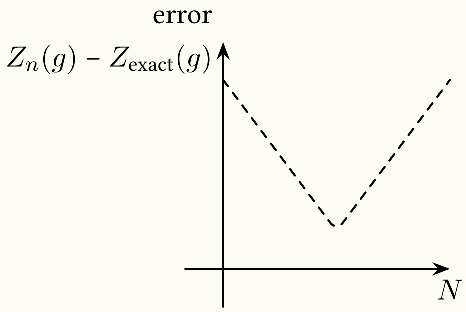
   

Symmetry factor.

2.3 Finite n still good for g-expansion

$$Z_{\text{exact}}(g) = \frac{1}{2} \sqrt{\frac{\pi}{g}} \, \mathrm{e}^{1/(8g)} K_{1/4}\left(\frac{1}{8g}\right) \tag{2.3}$$

$K_{1/4}$ is modified Bessel function.



Around $N = 13$: it starts going out.

Kondo Problem: to 1 or 2 order.

From (2.1), we have

$$Z(g) = \frac{1}{g^{1/4}} \int \mathrm{d}y e^{-y^4 - y^2/g^{1/2}} \tag{2.4}$$

$$= \frac{1}{g^{1/4}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{1}{g^{m/2}} \underbrace{\int \mathrm{d}y y^{2m} e^{-y^4}}_{\frac{1}{2}\frac{\Gamma\left(\frac{m}{2}+\frac{1}{4}\right)}{g^{m/2+1/4}}} \text{(Taylor Expansion)} \tag{2.5}$$

$$\left|\frac{a_{m+1}}{a_m}\right| \to \frac{1}{\sqrt{2g}} \cdot \frac{1}{\sqrt{m}} \to 0 \tag{2.6}$$

Then, $Z(1/g)$ cmverges $(0, +\infty)$

Large y, no matter how small g is, y^4 term dominates.

2.4 Multi-variable Gaussian Integral

Just means we have found

$$\int \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_N \, \mathrm{e}^{-[x][M][x]} = \frac{1}{2} \int \mathrm{d}y_1 \cdots \mathrm{d}y_N \, \mathrm{e}^{-[y]u^\dagger M u[y]} \tag{2.7}$$

(a) $[x] = (x_1, \ldots, x_N)$

(b) $[M] = (M_{ij})$.

Diagonalise M: $[x] \to u[y]$, $M \to \begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$, $u^\dagger M u = \begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$.

Add interaction:

$$\int \mathrm{d}x_1 \ldots \mathrm{d}x_N \, \mathrm{e}^{-\sum_{i=1}^{N} x_i^4 g} \tag{2.8}$$

Consider strong coupling expansion:

$$\int \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \mathrm{e}^{-\sum x_{i1}, x_{i2}, \ldots, x_{in} g_{12}^{34}} \tag{2.9}$$

$$\int \mathrm{d}y_1 \cdots \mathrm{d}y_N = \mathrm{e}^{-y^4} \tag{2.10}$$

Chapter 3 Second Quantization

3.1 QM of single particle

Dirac notation will be used in this book: ket $|\Phi\rangle$, bra $\langle\Phi|$, and operator $\hat{O}$. Wave function should be $\langle x|\Phi\rangle$.
For single particle QM, we have six axioms (postulates)
With hamiltonian $\hat{H}$

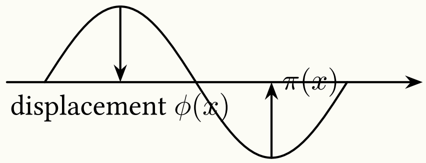
1. Description of a state $|\Phi\rangle \in \mathcal{H}$:
    - Using Vector Space
    - List
    - Orthogonality
    - Completeness
2. Observables: $\hat{A} \in \mathcal{H}$, $\hat{A}^\dagger = \hat{A}$.
3.
4. Measurements: Spectral Decomposion
    - a_n, eigenvalue of $\hat{A}$.
    - $P_n = |\langle a_n|\Phi\rangle|^2$
5. Post-measurement: collapse: $|\Phi\rangle_{\text{pre}} \xrightarrow{\text{measurement}} |a_n\rangle$
6. $\mathrm{i}\hbar\,\partial_t|\Psi\rangle = \mathcal{H}|\Psi\rangle$

Canonical(?) Quantization rules $x \to \hat{x}$, $p \to \hat{p}$, and $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$,
Then you can define Poisson bracket $\{\bar{x}, p\} = \mathbb{I} \Rightarrow [\hat{x}, \hat{p}] = \mathbb{I}$.

3.2 Canonical Quantisation

3.2.1

$|x\rangle$, $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$. $\hat{p} = -\mathrm{i}\hbar\,\partial_x$, $\langle x|\Phi\rangle$.
$\langle\hat{p}|\Phi\rangle$, $\hat{p} \to p$, $\hat{x} \to \mathrm{i}\hbar\,\partial p$.
For classical string like



$$\mathcal{H} = \int \mathrm{d}x \left(\frac{T}{2} (\nabla_x \phi(x))^2 + \frac{1}{2\rho} \pi^2(x) \right) \tag{3.1}$$ [1]

Exactly, $\mathrm{i}\{\tilde{\phi}, \tilde{\pi}\} = 1$.
Quantisation $[\hat{\phi}(x), \hat{\pi}(y)] = \mathrm{i}\hbar\delta(x - y)$.

3.2.2 Harmonic oscillator: zero-dim field theory

The hamiltonian, eigen energy and state are

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2,\ E_n = \left(n + \frac{1}{2}\right)\hbar\omega,\ |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle \tag{3.2}$$

Let

$$\xi = \frac{x}{\Delta x},\ p_\xi = \frac{p}{\Delta p},\ \Delta x = \sqrt{\frac{\hbar}{m\omega}},\ \Delta p = \sqrt{m\omega\hbar}$$

then we have

$$\begin{cases} a^\dagger = \frac{1}{\sqrt{2}}(\xi - \mathrm{i}p_\xi) \\ a = \frac{1}{\sqrt{2}}(\xi + \mathrm{i}p_\xi) \end{cases} \qquad \text{and} \qquad \begin{cases} [a, a^\dagger] = 1 \\ [a, a] = [a^\dagger, a^\dagger] = 0 \end{cases} \tag{3.3}$$

Do it differently, the time evolution

$$x(t) = \langle \phi_n(t) | \hat{x} | \phi_n(t) \rangle = \langle \phi_n | \mathrm{e}^{iHt/\hbar} \hat{x}\, \mathrm{e}^{-iHt/\hbar} | \phi_n \rangle = \langle \phi_n | \hat{x}(t) | \phi_n \rangle \tag{3.4}$$

and the EOM (Equation of Motion) is

$$-\mathrm{i}\hbar \frac{\mathrm{d}a(t)}{\mathrm{d}t} = [H, a(t)] = \omega a(t) \tag{3.5}$$

So,

$$\hat{x}(t) = \frac{\Delta x}{\sqrt{2}} [a(t) + a^\dagger(t)],\ \hat{p}(t) = \frac{\Delta p}{\mathrm{i}\sqrt{2}} [a(t) - a^\dagger(t)] \tag{3.6}$$

These type of operators are so called *field operators*.

$$\int_{-\Delta}^{+\Delta} \mathrm{d}x e^{-x^2 - gx^4} \text{Herry Ponc}\hat{o} = \lim_{N\to\infty} \lim_{\Delta\to\infty} \tag{3.7}$$

1. $j, (\pi_j)^2 + (D_j - D_{j+1})^2$: chiral phonon, theormal hall.

[1] Physically, it should have more terms

3.3 QM of many particles

Seventh Axiom: If you don't consider anything: we want to remove exchange degeneracy ⇒ symmetrization / antisymmetrization rule.
Two types wavefunctions. For Boson: permanent; for Fermison slater dert.
Each state is one particle in particle lable $\varphi_1^{\epsilon_1}, \varphi_2^{\epsilon_2}, \dots, \varphi_m^{\epsilon_m}$, which $\epsilon_i \approx \epsilon_j$, and state lable $1, 2, \dots, N$.

$$\begin{array}{c c} & \begin{array}{cccc} \varphi_1^{\epsilon_1} & \varphi_2^{\epsilon_2} & \cdots & \varphi_m^{\epsilon_m} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ N \end{array} & \begin{pmatrix} |1, \varphi_1^{\epsilon_1}\rangle & |1, \varphi_1^{\epsilon_2}\rangle & & \\ |2, \varphi_1^{\epsilon_1}\rangle & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

symmetrical polynomials.
We can get into a newer occupation number representation
label any state $|\varphi\rangle = |n_{\epsilon_1}(\text{occupation / energy No.}), n_{\epsilon_2}, \dots, n_{\epsilon_n}, \dots\rangle_{\text{boson / fermion}}$
occupation number representation: dependent on underlying first quantization basis.

$$N = n_{\epsilon_1} + \cdots + n_{\epsilon_n} + \cdots \tag{3.8}$$

3.4 Fock space

$$F = \{\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n, \cdots\} \tag{3.9}$$

Rock matrix:

$$\begin{array}{c} \begin{array}{cc} |\varphi, N=0\rangle & |\varphi, N=1\rangle \end{array} \\ \begin{pmatrix} H_0 & & & & \\ & H_1 & & & \\ & & \ddots & & \\ & & & H_N & \\ & & & & \ddots \end{pmatrix} \end{array} \tag{3.10}$$

3.5 Field operators

Suppose we have

$$|\Psi\rangle = |\qquad\rangle; \tag{3.11}$$

$\Psi(x)$: $\langle x|\Psi\rangle$.
Each state has a particular wave function,
$|x_1\rangle$ is given by $\Psi^\dagger(x_1)|0\rangle \Rightarrow \hat{x}|0\rangle = \hat{a} + \hat{a}^\dagger|0\rangle = |1\rangle$.
We are creating $a^\dagger|0\rangle$, $|x\rangle\langle x|1\rangle = |x_1\rangle\psi_1(x_1)$.

Project $a^\dagger$ onto $|x\rangle\langle x|$, that is

$$\psi^\dagger(x) = a^\dagger|x\rangle\langle x|$$

Similarly, if u want two particles at two different positions, u get

$$\psi^\dagger(x_2)\psi^\dagger(x_1)|0\rangle = |x_1, x_2\rangle = \psi^\dagger(x_2)|x_1\rangle \tag{3.12}$$

They are commutation

$$\psi^\dagger(x)\psi^\dagger(y) = \pm\psi^\dagger(y)\psi^\dagger(x) \Leftrightarrow [\psi^\dagger(x), \psi^\dagger(y)]_\mp{}^{2} = 0 \tag{3.13}$$

And finally, introduce

$$[\hat{\psi}(x), \hat{\psi}(y)] = 0, \hat{\psi}(x)|0\rangle = 0$$

For any state like $|\Psi\rangle \in F$, u also see like $\langle\Psi|\psi(x)|\psi\rangle = \langle\Psi|\psi^\dagger(x)|\psi\rangle = 0$.
$\langle\varphi, N|\varphi, N-1\rangle = 0$. And finally, we want to be able to $\langle y|x\rangle = \delta(x-y) = \langle y|\psi^\dagger(x)|0\rangle = \langle 0|\hat{\psi}(y)|x\rangle$. u can prove that: $\hat{\psi}(x)|y\rangle = \delta(x-y)$, and finally $[\hat{\psi}(x), \hat{\psi}(y)]_\mp = \delta(x-y)$.

3.6 Creation / annilation operators

For any state $\langle x|n\rangle = \varphi_n(x)$, and u can also $|n\rangle = \int \mathrm{d}x$.

IDENTITY IS VERY IMPORTANT! $\mathbb{1}$

$$\mathbb{1} = \int \mathrm{d}x|x\rangle\langle x|$$

Then we have

$$|n\rangle = \int \mathrm{d}x|x\rangle\langle x|n\rangle = \int \mathrm{d}x\varphi_n(x)|x\rangle = \int \mathrm{d}x\varphi_n(x)\psi^\dagger(x)|0\rangle$$

u can define $\Rightarrow d_n^\dagger = \int \mathrm{d}x\varphi_n(x)\psi^\dagger(x) \Rightarrow d_n^\dagger|0\rangle = |n\rangle$.
Similarly, $\Rightarrow \hat{d}_n \equiv \int \mathrm{d}x\varphi_n^*(x)\tilde{\psi}(x)$: $\Rightarrow \hat{d}_n|n\rangle = |0\rangle$. So $\hat{d}_n|n\rangle \to \hat{d}_n\hat{d}_n^\dagger|0\rangle$ (Homework)
Also prove that $[d_n, d_m^\dagger]_\pm = \langle n|m\rangle = \delta_{n,m}$
In other words, ...

Occupation number rep.

for $|x_1, ..., x_n\rangle$, if u change any particle like (reverse x_1 and x_n), u will get $(t)^p|x_1, x_2, ..., x_n\rangle$. $(t)^p$: boson & fermion.
interchange: $1\ 2\ 3 \to 2\ 1\ 3$ (Permutation) $\#P$ ($P_{1\to 2}$, $P_{2\to 1}$): $\#$ of perm speration needed.
When take $\mod{}_2(p)$,
$h_o:\ \epsilon_1, \epsilon_2, \ldots, \epsilon_i \to |n_1, n_2, \ldots, n_i, \ldots\rangle_{b(P)}$ $\Rightarrow$ Fock space: $H_0 \oplus H_1 \oplus \ldots H_N \oplus \ldots$
Creation and annilation operators: $d_{n_i}^\dagger, d_{n_i} \Leftrightarrow \psi^\dagger(x)|0\rangle = |x\rangle$
$d_n^\dagger|0\rangle = |n\rangle$, $d_n^\dagger = \frac{1}{\sqrt{n!}}(a^\dagger)^n$

[2] $[A, B]_\mp = AB \mp BA$

Identity

$$\mathbb{1} = \frac{1}{N!}\sum_{n_1,\ldots,n_i}|n_1,n_2,n_3,...\rangle\langle...| = \int \mathrm{d}x|x\rangle\langle x| = \int \mathrm{d}x\psi^\dagger(x)\cancel{|0\rangle\langle 0|}\psi(x) = \int \mathrm{d}x_1\ldots\mathrm{d}x_n|x_1,...,x_i\rangle\langle...|$$

3.7 Hamiltonians in second quantization

Operators of total number of particles

$$\hat{N} = \int \mathrm{d}x\hat{n}(\boldsymbol{x}),\ \hat{N}(\boldsymbol{x}) = \psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}),\ \hat{N} = \sum_n d_n^\dagger d_n, \hat{N}_n = d_n^\dagger d_n \tag{3.14}$$

$\hat{n}(\boldsymbol{x})|n_1\ldots...\rangle \to \sum|x_1,...,x_i,...\rangle$
Eignevalue is equal to $(\sum_i \delta(x-x_i))|x_1,\ldots,x_i,\ldots,x_N\rangle$
$\hat{h}_0$: $\hat{h}_0|\epsilon_1\rangle = \epsilon_i|\epsilon_i\rangle$
If we have N particles, for each of the particle, we have to specify a Hamiltonian

$$H_0 = \sum_{i=1}^{N}\hat{h}_0(\boldsymbol{x})i$$

In some occupation states, we have specify the occupation numbers

$$|n_1,\ldots,n_j,\ldots\rangle, \quad \text{while} \quad \sum n_j = N$$

and

$$H_0|n_1,\ldots,n_j,\ldots\rangle = \sum_j \epsilon_j n_j|...\rangle$$

$\oplus$ in book $\sum \epsilon_j|...\rangle$.
Then we can define $H_0|n...\rangle = (\epsilon_{n_1} + \epsilon_{nN})|...\rangle$

$$H_0 = \int \mathrm{d}x\,\mathrm{d}x'\psi^\dagger(x)\langle x|\hat{h}_0|x\rangle\hat{\psi}(x')$$

If $\hat{h}_0 = \hat{p}^2/2m$, then $\langle x|\hat{h}_0|x'\rangle = \delta(x-x_0)(-\frac{\nabla^2k^2}{2m})$ Then,

$$H_0 = \sum_n \int \mathrm{d}x\,\mathrm{d}x'\psi^\dagger(x)\underbrace{\langle x|\hat{h}|n\rangle}_{\varphi_n(x')}\underbrace{\langle n|x\rangle}_{\varphi_n^*(x')}\hat{\psi}(x') \tag{3.15}$$

$$= \sum_n \int \mathrm{d}x\,\mathrm{d}x'\epsilon_n\varphi_n(x)\psi^\dagger(x);\varphi_n^*(x')\hat{\psi}(x') \tag{3.16}$$

$$= \sum_n \epsilon_n d_n^\dagger d_n = \sum_n \epsilon_n\hat{N}_n \tag{3.17}$$

$$[\hat{N}_n, d_m^\dagger]_- = \delta_{nm}d_m^\dagger,\ [\hat{N}_n, d_m]_- = -\delta_{nm}d_n$$

Consider $[H_0, \hat{d}_m^\dagger] = \epsilon_m d_m^\dagger$, $[H_0, d_m] = -\epsilon_n d_m$, the we get spectral geneeratiy algtron $[H_0, \hat{O}] \propto \hat{O}$ while $H_o = \sum\epsilon_0|n\rangle\langle n|$ and $|m\rangle\langle m|$
Finally, we can write

$$H_0 = \int \mathrm{d}r\psi^\dagger(r)h_{r_0}(r,\nabla_r,\sim)\psi(r)$$

Here, $h = \langle r|\hat{h}_0|r\rangle$.

3.7.1 Interactivy Hamiltonian

Consider two body $\nu(\hat{x}_1, \hat{x}_2)$

$$H_{\text{int}}|x_1, ..., x_N..\rangle = \frac{1}{2}\left(\sum_{i\neq j}\nu(\hat{x}_i, \hat{x}_j)\right)|x_1, ..., x_N..\rangle = \left(\sum_{i>j}\nu(\hat{x}_i, \hat{x}_j)\right)|x_1, ..., x_N..\rangle$$

$$\hat{|}x_1, ..., x_n > \ \hat{n}(x)$$

We write $v(\hat{x}_i, \hat{x}_j) = \int \mathrm{d}x' \, \mathrm{d}x \delta(x - x_i)\delta(x' - x_j)v(x, x')$ and substitute it into H_{int}, then using $(\sum_i \delta(x - x_i))|x_1, ..., x_i..., x_N\rangle$ inversely, then we have

$$H_{\text{int}} = \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}x'(\nu(x, x')n(x)n(x') - \nu(x, x)n(x)) \tag{3.18}$$

$$= \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}x'\nu(x, x)\left(\psi^\dagger(x)\psi(x)\psi^\dagger(x')\psi(x') - \delta(x - x')\psi^\dagger(x)\psi(x)\right) \tag{3.19}$$

$$= \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}x'\nu(x, x')\psi^\dagger(x)\psi^\dagger(x')\psi(x)\psi(x') \tag{3.20}$$

3.7.2 Total Hamiltonian

Such as

$$\begin{aligned}\mathcal{H} &= \int \mathrm{d}x\,\mathrm{d}x'\psi^\dagger(x)\langle x|h_0|x'\rangle\psi'(x') + \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}x'\nu(x, x')\psi^\dagger(x)\psi^\dagger(x')\psi(x)\psi(x') \\ &= \sum_{ij} h_{ij} d_i^\dagger d_j + \frac{1}{2}\sum_{ijmn}\nu_{ijmn} d_i^\dagger d_j^\dagger d_m d_n\end{aligned}$$

Here

$$h_{ij} = \langle i|\hat{h}|j\rangle = \int \mathrm{d}r \varphi_i^* h_i(r, \nabla_r, \sim)\varphi_j(r) = h_{ji}^*, \ \langle x|i\rangle = \varphi_i(x)$$

and

$$\nu_{ijmn} = \int \mathrm{d}x\,\mathrm{d}x'\varphi_i^*(x)\varphi_j^*(x')\nu(x, x')\varphi_m(x')\varphi_n(x)$$

Chapter 4 Model Hamiltonians

4.1 Hierarchy of Hamiltonians

For solids like

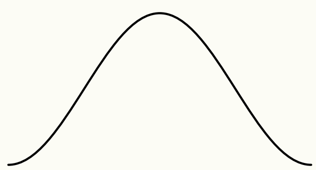
$$\cdots \circ \circ \circ \circ \circ \circ \cdots$$

$\sum_i N(R_i) \to$ periodic potential ($\leftarrow$ single electron $\leftrightarrow$ $quantumgeroetry$ $\to$ phonons).
$h_0 \xrightarrow{a\to 0} \partial_x \to k(-\infty, +\infty) \in (-\frac{\pi}{a}, \frac{\pi}{a}) \to$ Móire Nlm

4.2 Pariser-Parr-Pople Model (tight-binding + interaction)

4.2.1 Hydrogen molecule

For two hydrogen atoms



$$h_0 = (-\frac{\nabla_x^2}{2m}),\ \nu(x_1, x_2)$$

$$h_0 = \sum_{1,2,\ldots,n} \hat{n}_i(\epsilon_n^0)\epsilon_N^0 \to \epsilon(\hat{n}_1 + \hat{n}_2) + \sum_{\sigma,\sigma',nm} T_{1n}(R)(d_{1n\sigma} \dagger d_2 n\sigma + d_{2M'_\sigma} d_{M_\sigma})$$

$|\varphi_1^n\rangle$, $|\varphi_2^n\rangle$. $\langle\varphi_1^n|n_0|\varphi_2^m\rangle$.
$h_{\text{int}} = v(\hat{n}_{1\uparrow}n_{1\downarrow} + \hat{n}_{2\uparrow}n_{2\downarrow}) + \nu_{12}(\hat{n}_1 - 1)(\hat{n}_2 - 1)$.

4.3 Ammonia NH_3

4.4 Bloch electrons in 2nd Quantization

Single particle basis: Bloch state $|\boldsymbol{k}\sigma\rangle$. Wave function $\langle\boldsymbol{r}|\boldsymbol{k}\sigma\rangle = \psi_{\boldsymbol{k}\sigma}(\boldsymbol{r})$, $\psi_k = u_k(r)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot r}$
For single particle Hamiltonian

$$\mathcal{H}_0 = \sum_{\substack{k,\sigma \\ k',\sigma'}} \langle k\sigma|h_0|k'\sigma'\rangle d_k^\dagger d_{k'\sigma'} \Rightarrow \sum_{k\sigma} \epsilon(k) d_{k\sigma}^\dagger d_{k\sigma} \tag{4.1}$$

For tight-binding $\sum_{ij} t_{ij}|i\rangle\langle j|$

$$H_0 = \sum_{\substack{\sigma\sigma' \\ ij}} T_{ij\sigma\sigma'}\hat{d}_{i\sigma}^\dagger \hat{d}_{j\sigma'} \tag{4.2}$$

$$\langle r|\boldsymbol{k}\sigma\rangle \Leftrightarrow \langle r|i\sigma\rangle \tag{4.3}$$

which is called *(Maximally-localized) Wannier function*: exponentially atomic.

4.4.1 Generic lattice

The hamiltonian

$$\mathcal{H}_0 = \sum_{ij} t_{ij} c_{i\sigma} \dagger c_{j\sigma} + \sum_i \epsilon_0 c_i^\dagger c_i,\ H = \sum_k c_{k\alpha}^\dagger H_{\alpha\beta} \hat{C}_{k\beta},\ \hat{C}_k = \frac{1}{\sqrt{N}} \sum_{i,\alpha} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}(\boldsymbol{R}+\boldsymbol{b}_\alpha)} \hat{c}_{i,\boldsymbol{b}\alpha} \tag{4.4}$$

Here,

$$\boldsymbol{R}_{i,A} = \sum_{n_1,n_2} n_1\boldsymbol{a}_1 + n_2\boldsymbol{a}_2 + \boldsymbol{b}_A,\ \boldsymbol{R}_{i,B} = \sum_{n_1,n_2} n_1\boldsymbol{a}_1 + n_2\boldsymbol{a}_2 + \boldsymbol{b}_B,\ \boldsymbol{b}_A = (a,0),\ \boldsymbol{b}_B = (0,1)$$

α, β: sublattice, spins; orbitals index.

4.4.2 Graphene

$$\boldsymbol{a}_1 = \frac{a}{2}(3,\sqrt{3}),\ \boldsymbol{a}_2 = \frac{a}{2}(3,-\sqrt{3}),\ \boldsymbol{b}_B = a(1,0),\ \delta_1 = \boldsymbol{a}_1 + \boldsymbol{b}_B,\ \delta_2 = \frac{a}{2}(1-\sqrt{3}),\ \delta_3 = a(1,0). \tag{4.5}$$

$$\mathcal{H} = -t\sum(a_i^\dagger b_j + \text{h.c.}) \tag{4.6}$$

while $a_i = C_{R_i,A}$, $b = C_{R_i,B}$, and $h_k \approx \sum_\delta \mathrm{e}^{\mathrm{i}k\delta} + \mathrm{e}^{-\mathrm{i}k\delta}$ Then

$$H = \begin{pmatrix} a_k^\dagger & b_k^\dagger \end{pmatrix} \begin{pmatrix} 0 & h_k \\ h_k & 0 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

pumping:

$$h_k = t\Big[\mathrm{e}^{i(1,0)\cdot\boldsymbol{k}} + \mathrm{e}^{-\mathrm{i}(1,0)\cdot\boldsymbol{k}} + \mathrm{e}^{\mathrm{i}(0,1)\cdot\boldsymbol{k}} + \cdots\Big] = -2t(\cos k_x + \cos k_y) \tag{4.7}$$

and

$$H_k = -\sum 2t(\cos k_x + \cos k_y) c_k^\dagger c_k \tag{4.8}$$

h_k near some point (Dirac point): $h_k \sim (\boldsymbol{k} - \boldsymbol{K})\frac{\partial h_k}{\partial k}$, so it can be written as $h_k \sim \boldsymbol{v}_F \cdot \delta\boldsymbol{k} \leftrightarrow \partial_{\boldsymbol{x}}$

4.4.3 Flat-bannd

$h_k =$ const now.
Landau levels
Landau gauge:
Symmertic Gauge $|\phi_{LG}\rangle \Rightarrow \sum |\Phi_{SG}\rangle$
magnetic translation T_x, T_y: $a_x, a_y \propto \dot{l}_b$

4.4.4 Electron phonon coupling: BCS

Hamiltonian with solid:

$$\mathcal{H}_{\text{solid}} \begin{cases} \mathcal{H}_{\text{phono}} = \sum_{\boldsymbol{q}} \mathcal{E}(\boldsymbol{q}) b_q^\dagger b_q \\ \mathcal{H}_{\text{Bloch}} = \sum_k \mathcal{E}(k) c_k^\dagger c_k \end{cases} \tag{4.9}$$

$V(R_i, x_i)$

$$H_{e-p} = \sum_{k_1} \sum_{q\boldsymbol{G}} T(b_q^\dagger + b_{-q}^\dagger) \hat{a}_{k+q+\boldsymbol{G}}^\dagger \hat{a}_k \tag{4.10}$$

$$\bar{x}_j^* = \frac{1}{n} \sum_{i=1}^n x_{ij} - \bar{x}_j = \frac{1}{n} \left(\sum_{i=1}^n x_{ij} - \sum_{i=1}^n \bar{x}_j \right)$$

4.5 QM identical partical

1. $|1 : \phi_1\rangle$, $|2 : \chi_2\rangle$ -¿ occupation num. of condition

   - $|n_{\phi_1}, n_{\chi_2}\rangle$ in real space: $\phi \to \hat{x}$;

   After interchange particle label, there should be a sign: $(\pm 1)^p$: minus - fermion; boson: trivial. so we have

   $$(-1)^p \leftrightarrow [d_n^\dagger, d_m]_\mp : d_n^\dagger d_m \mp d_m d_n^\dagger$$

   $[d_n, d_m]_\mp = 0 = [d_n^\dagger, d_m^\dagger]$. $d_n^\dagger d_m^\dagger |0\rangle = -d_m^\dagger d_n^\dagger |0\rangle$

4.6 Electron-phonon coupling

$V_{\text{ion-ion}} = \sum V(R_i - R_j) \Rightarrow \sum_q \omega_q a_q^\dagger a_q$,
$V_{\text{el-el}} = \sum V(r_i - r_j) \Rightarrow \sum_i u n_{i\sigma} n_{iq} = + \cdots \sum_{ij} V n_i n_j$,
$V_{\text{el-ion}} = \sum V(R_i - r_j) \Rightarrow \sum_{\sigma q} (b_q^{-1} + b_{q-}) c_{k+q\sigma}^+ c_{k\sigma}$
corsplication: feed back effect (self-consistent eq.) == DFT

$$H_{ep} = \sum_{kq\sigma} T_{kq} (b_q + \hat{b}_{-q}) G_{k+q,\sigma}^\alpha a_{k\sigma}$$

MW_{ene}-field BCS

4.7 Effective $H_{a,g}$(femons)

4.8 canonical transformaission

Also called: Foldy-Wyen

4.9 effection H canonical transf

$$H = H_0 + \delta H$$

Broad prodive $\hat{T} = \mathrm{e}^{-\hat{\mathcal{S}}} \mathcal{H} \, \mathrm{e}^{\mathcal{S}}$. Expand:

$$\tilde{H} = H + [H, \hat{S}]_- + \frac{1}{2!}[[\hat{H}, \hat{S}]_- S]_- = H_0 + \delta H + [H_0, \hat{S}] + [\delta H, \hat{S}]$$

find $\hat{S} : [H_0, \hat{S}] + \Delta H = 0$. S is usually called generator.
$\Rightarrow H_0 + [\Delta H, S] + \frac{1}{2!}[([H_0, S] + [\delta H, S]), S] = H_0 + [\delta H, S] + \frac{1}{2}[-\delta H, S] + O(S^2) = H_0 + \frac{1}{2}[\delta H, S]$
So, from δH u get. $H = H_0 + \frac{1}{2}[\delta H, S]$.

4.10 For el-ph coupling $\hat{\mathsf{S}} = \sum_{\mathsf{kq}\sigma} \tau_{\mathsf{q}}(\mathsf{x}\mathsf{b}_{\mathsf{q}} + \mathsf{y}\mathsf{b}^{\dagger}_{-\mathsf{q}})\mathsf{a}^{*}_{\mathsf{k}+\mathsf{q}\sigma}\mathsf{a}_{\mathsf{k}}$

$$H_0 = \sum \epsilon_k a^{\dagger}_{k\sigma} a_{k\sigma}$$

The commutator to S is

$$[H_0, \hat{S}] = \sum \tau_q \{x(\epsilon(k+1) - \epsilon(k) - \hbar\omega_q)b_q + y(\epsilon(k+q) - \epsilon(k) + \hbar\omega_q)b^{\dagger}_{-q}\}a^{\dagger}_{k+q\sigma}a_{k\sigma}$$
$$= -H_{\text{el-ph}} \Rightarrow \begin{cases} x = (\epsilon(k) - \epsilon(k+1) + \hbar(\omega)q^{-1}) \\ y = (\epsilon(k) - \epsilon(k+1) - \hbar(\omega)q^{-1}) \end{cases}$$

then, the effective Hamiltonian

$$H_{\text{eff}} = \frac{1}{2}[\delta H, \hat{S}] = \frac{1}{2} \sum_{\substack{kk' \\ q\sigma\sigma'}} \tau_{-q}\tau_q(q - x)a^{\dagger}_{k+q}a_{k'-q}a_{k'\sigma'}a_{k\sigma}$$

$x = \frac{1}{\epsilon(k) - \epsilon(k+q) - \hbar\omega_q} \approx \delta(\epsilon(k) - \epsilon(k+1) - \hbar\omega_q)$.
Consider this δ-function

$$\int \mathrm{d}E = \frac{1}{E \pm \hbar\omega_q}\delta(E - \hbar\omega_q)$$
$$H_0 = \sum \epsilon_n a^{\dagger}_n a_n, \hat{G} = (E - H_0)^{-1} = G[E]$$

Green's function:

$$(E - H_{ph})^{-1} = G_{Phi}$$
$$\tilde{H} = H_0 + H_{eff} = \sum_k \epsilon_k \tilde{a}^{\dagger}_{k\sigma}\tilde{a}_{k\sigma} + \frac{1}{2}\sum_{kk',q\sigma\sigma'} T_q T_{-q}(y - x)\tilde{a}^{\dagger}_{k+q}\tilde{a}^{\dagger}_{k'-q}\tilde{a}_{k'\sigma'}\tilde{a}_{k\sigma}$$
$$\mathrm{e}^{-\hat{S}} H_0 \, \mathrm{e}^{\hat{S}} = \mathrm{e}^{-S} a^{\dagger}_k (e^S e^{-S}) a_k \, \mathrm{e}^S$$

$$\tilde{a}^{\dagger}_k = \mathrm{e}^{-S} a^{\dagger}_k \, \mathrm{e}^S = a^{\dagger} + [a^{\dagger}_k, S] + \ldots = a^{\dagger}_k + \sum_q (xb_q + yb_{-q})^{\dagger}, \ \tilde{a}_k = \mathrm{e}^S a_k \, \mathrm{e}^{-S}$$

4.11 From el-ph to BCS

at low-T, $k \in nearES$. $|T_q|^2(y-x) \sim< 0$, means for particular part of fermion, the attraction become attractive.

$$H = \sum_k \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} - v \sum_{kk'} c^+_{k\uparrow} c^+_{k\downarrow} c_{k'\downarrow} c_{k'\uparrow}.$$

This is called Richardson = RCS model Inoneomeanfield 1. not a sc. qs =¿ CFT 2. prove bcs, superconductivity spontuneous sym. treaking: add some like Bogoliubov.
Meanfield BCS model.
SBB: $H_{GB} = \sum \epsilon_k c^\dagger_{k\sigma} c_{k\sigma}$

4.12 Holstein model

1. $H_c = \sum_s \omega_0 (a^\dagger_s a_s + \frac{1}{2})$
2. $H_{\text{int}} = -q \sum_{s\sigma} d^\dagger_{s\sigma} d_{s\sigma} (a_s + a^*_s)$
3. $H_{el} = \sum_i (t_{ij} d^\dagger_{is\sigma} d_{js\sigma} + h.c.)$
4. Strong compling limi. Leads to heavy poloron.
5. Lang-Firsov transformation,(c.t.)

$$\hat{S} = -\mathrm{i} \frac{g}{\omega_0} \sum \hat{n}_{s\sigma} (a^\dagger_s - a_s)$$

$$H_{cp} = T \sum (B^\dagger_s B_S)$$

$\hat{B}_s = \mathrm{e}^{q/\omega_0}$

4.13 elec-el interaction

$$V_{el-el} = \sum V(r_i - r_j)$$

.

$$\Rightarrow H_A = \sum \epsilon_k c^{-1}_{k\sigma} c_{k\sigma} + u \hat{n}_{d\uparrow} n_{d\downarrow} + \sum \epsilon_d d^\dagger d + V \sum k (c^\dagger_{k\sigma} d\sigma + d^\dagger \sigma c_{k\sigma})$$

$$S = \sum_{k\sigma} (A_k c^\dagger_{k\sigma} d_\sigma - A_k d^\dagger_\sigma c_{k\sigma}) + \sum (B_A d_\sigma c^\dagger_k - B^\dagger_k c_{k\sigma d^\dagger_\sigma})$$

$$A_K = \frac{V_k}{\epsilon_k - \epsilon d}, \; B_k = \frac{V_k}{\epsilon_k - (\epsilon_d + b)}$$

then

$$H_{\text{eff}} = \frac{1}{2}[\delta H, S] = \frac{1}{2}[\delta H, S(H_0)]$$

while $H_0 = H_c + H_{n,d}$.

$$H_{\text{eff}} = J_k \boldsymbol{S}_d \cdot \boldsymbol{S}_{c,o} = \sum_{kk'} J_{kk'} (\boldsymbol{S}_d \cdot \sum_{\alpha\beta} c^\dagger_{k'\alpha} \frac{\boldsymbol{\sigma}_{\alpha\beta}}{2} c_{k\beta} + \frac{1}{4} \sum_\sigma c^\dagger_{k'\sigma} c_{k\sigma})$$

$$\boldsymbol{S}_d = \sum_{\alpha\beta} \hat{d}_\alpha \frac{\sigma_{\alpha\beta}}{2} \hat{d}_\beta$$

So,

$$S_z = \frac{1}{2}(d^\dagger_\uparrow d_\uparrow - d^\dagger_\downarrow d_\downarrow),\ S_z^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = d^\dagger_\uparrow d_\downarrow, S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = d^\dagger_\downarrow d_\uparrow$$

$$\boxed{\text{Hkondo-imparity} = J_k \boldsymbol{S}_d \boldsymbol{S}_{c,o}}$$

for $J_K > 0$, $J_K = \frac{4V^2}{U}$.
After the treatment, from anderson module to kondo modo:

$$H_{And} \to H_{Kondom} = \sum \epsilon_k c^\dagger_k c_k + J_k \boldsymbol{S}_\alpha \cdot \boldsymbol{S}_{c,N}$$

S_d is free moment, formation of a local moment.
J_k can become negative, so change the kondo limit, Anti FM – compling, to becoma a FM.

4.14 Hubbard model

$$H = \sum_{\sigma k} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + N \sum_i (n_\uparrow - \frac{1}{2})(n_\downarrow - \frac{1}{2}) + (\sum_{\sigma k} -\mu c^\dagger_{k\sigma} c_{k\sigma}) = -\sum_{ij\sigma} t_{ij} c^\dagger_{i\sigma} c_{j\sigma} \tag{4.11}$$

4.14.1 Schrieff-Wolff transformation (C.T.)

For $n = 1$, deal with Heisenberg model

$$H_{Heis} = \sum_{ij} J_{ij} \boldsymbol{S}_i \boldsymbol{S}_j \tag{4.12}$$

while $J_{ij} = \frac{4t_{ij}t_{ji}}{N} = \frac{4t^2}{N}$.
When doping, $t - J$ model

$$H = \sum_{ij} t_{ij} \tilde{c_{ik}} \tilde{c}_{j\sigma} + \sum J \boldsymbol{S}_i \cdot \boldsymbol{S}_j.$$

so-called Ggutzwillen projection, project out double-occupancy.
Gutzwiller approximation $\delta \sum t \hat{c}^\dagger_i c_{j\sigma} + \sum J(1 - 2\delta) \boldsymbol{S}_i \cdot \boldsymbol{S}_j$.

4.15 Many layers of QMB problems

1. reduce degrees of freedom (wrong way: CT, quasi particles; polaron)
2. methods to reduce DoF: link/crosscheck different methods path tn: Effective theory EFT.

Chapter 5 Methodology

5.1 Computation Gods

Q#1 We want to have an interacting Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$$

normally, we consider $\tilde{E}_0 > \langle V \rangle$.

$$\sum t_{ij} c_i^\dagger c_j + \epsilon_A - 2t(\cos kx + \cos ky)$$

$2Dt = W$. while V energy $un_\uparrow n_\downarrow$, $W > u$; or $u > W$ is for cookrate. $U \sim \delta t$.

(a) The first thing we will discuss is the Ground State (zero temperature); and we also need some thermo equilibrium states (finite T), ρ: density matrix.

(b) Observables ⇔ Experiments. $\langle \hat{O} \rangle$.

- Thermodynamic: specific head C_v; M (magnetisation), χ susceptibility.
- Second type of measurement is spectroscopy (Can be transform to many body Green Funcs): it's energy dependent. ARPES means Angel Result . Another called Raman (low frequency). X-ray. Using potitros (μ-SR); neutron.

For MB: $L \to \infty$, $N \to \infty$. infinity $- \langle x|n \rangle$.

Q#1.1 $\underset{\rho}{|\Psi\rangle}$: how to represent efficiently? →Quantum Statistical.
For a system: n, T, P, V, μ, S. Macros $F(n, \mu, T, ...)$ to get equation of states.
A "few" important (enough to determine physical properties) observables (parameter). For landau parameters.

5.2 How to compute

1. Represent eff.
   - Perturbatim, and something beyond that: rariation (functional), mens to minimize sth, (δF, δE) to some differential intense Equal group. In class method, EL - eq, δL.
     - unperturbed theory (H_0): pay attention to degeneracy
       quasi-degeneracy (FL), Gapless $\Delta E|_{L\to\infty} \to 0$.
     - generate perturbation sequences
       Feynman diagrem, bookkeeping = wilsm RG.

5.3 QM: formulation

- Hamiltonian Mechanics
- Lagrangian Mechanins: path integral

Pictures (under Hamiltonian)

- Schrödinger picture
- Heisenberg picture
- interaction picture (Dirac picture)

means where u keep the TIME EVOLUTION.
In Schrödinger picture:

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi_s(t)\rangle = H|\Psi_s(t)\rangle \tag{5.1}$$

and operators are time-independent.
In Heisenberg picture, states are time-independent.

$$-\mathrm{i}\frac{\partial}{\partial t}\hat{O}_H(t) = [H, O(t)]$$

and the relation

$$O_H(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar}\hat{O}_p\,\mathrm{e}^{-\mathrm{i}Ht/\hbar}$$

For Dirac picture, states change slowly

$$\mathrm{i}\,\partial_t|\Psi_I(t)\rangle = V_I(t)|\Psi_I(t)\rangle$$

and Evolution accroding to H

$$-\mathrm{i}\,\partial_t O_I(t) = [H_o, O_I(t)]$$

In some sense, they are equivalent

$$O_H(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar}\hat{O}_p\,\mathrm{e}^{-\mathrm{i}Ht/\hbar},\ O_I(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar}\hat{O}_I\,\mathrm{e}^{-\mathrm{i}Ht/\hbar}$$

5.3.1 Pictures

Schrödinger Picture:

$$\mathrm{i}\hbar\,\partial_t|\psi_s(t)\rangle = H|\psi_s(t)\rangle \tag{5.2}$$

Left: mixed state; Right: ρ_s. $-\mathrm{i}\hbar\,\partial_t\rho_s = [H, \rho_s]$
From the above, we derive

$$\mathrm{i}\hbar\,\partial_t U_s(t, t_0) = H_t U_s(t, t_0)$$

while $u \sim \mathrm{e}^{\mathrm{i}Ht/\hbar}$.

Time evolution operator.

$$|\Psi_s(t)\rangle = U_s(t, t_0)|\Psi_s(t_0)\rangle.$$

and $U^\dagger = U^{-1}, U(t_0, t_0) = \mathbb{1}$.
While $U(t, t_0) = U_s(t, t')U_s(t', t_0)$

$$U_s(t, t_0) = 1 - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}t_1 H_t \times U_s(t_1, t_0)$$

So, we can do the following

$$U_s(t, t_0) = 1 - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}t_1 H_t,\ U_s(t_1, t_0) = 1 - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1} \mathrm{d}t_2 U(t_1, t_0)$$

then,

$$U_s(t, t_0) = 1 - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}t_1 H_{t_1}\left(1 - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1} \mathrm{d}t_2 H_{t_2} U(t_0, t_1)\right) = \ldots$$

equivalent to the equation

$$x^2 - x + 1 = 0,\ x = \frac{-1}{1 - x} = \frac{-1}{1 - (\frac{-1}{1-\ldots})} \text{ Continued fraction}$$

recursion (von Neumann's series)

$$U_s(t, t_0) = 1 + \sum_{n=1}^{\infty} U_s^{(n)}(t, t_0), \tag{5.3}$$

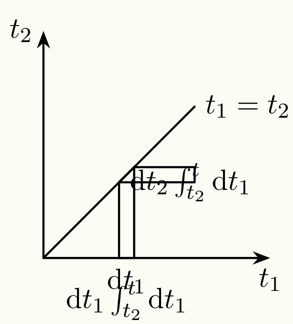
or

$$U_s^{(n)}(t, t_0) = \left(-\frac{\mathrm{i}}{\hbar}\right)^n \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_3 \cdots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n \times H_{t_1} H_{t_2} \cdots H_{t_n}.$$

while $t \geq t_1 \geq t_2 \cdots \geq t_n \geq t_0$.

5.3.2 Dyson's time-ordering operator

$$T_D(A(t_1)B(t_2)) = \begin{cases} A(t_1)B(t_2), & \text{for } t_1 > t_2 \\ B(t_2)A(t_1), & \text{for } t_2 > t_1 \end{cases}$$



$$\mathrm{d}t_2 \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 H_{t_1} H_{t_2} = \int_{t_3}^{t} \mathrm{d}t_2 \int_{t_2}^{t} \mathrm{d}t_1 H_{t_1} H_{t_2}$$

then, $t_1 \Leftrightarrow t_2$

$$\begin{aligned}
\int_{t_2}^{t} \mathrm{d}t_1 \int_{t_2}^{t_1} \mathrm{d}t_2 H_{t_1} H_{t_2} &= \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_1}^{t} \mathrm{d}t_2 H_{t_2} H_{t_1} \\
&= \frac{1}{2} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t (H_{t_1} H_{t_2} \theta(t_1 - t_2) + H_{t_2} H_{t_1} \theta(t_2 - t_1)) \\
&= \frac{1}{2!} \int_{t_0}^{t} \mathrm{d}t_1 \, \mathrm{d}t_2 T_D (H_{t_1} H_{t_2})
\end{aligned}$$

Generalize

$$U_s^{(n)}(t, t_0) = \frac{1}{n!} \left(-\frac{\mathrm{i}}{\hbar} \right)^n \int_{t_0}^{t} \int \mathrm{d}t_1 \cdots \mathrm{d}t_n T_D (H_{t_1} H_{t_2} H_{t_3} \cdots H_{t_n}) \tag{5.4}$$

$T_D(H_{t_1}, H_{t_2}, H_{t_3})$: $\theta(t_1 - t_2)$, $\theta(t_2 - t_1)$, $\theta(t_2 - t_3)$, $\theta(t_3 - t_2)$, $\theta(t_3 - t_1)$, $\theta(t_1 - t_3)$, like $2^3 = 8$. The formal expression

$$U_s(t, t_0) = T_D \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t_0}^{t} \mathrm{d}t' H_{t'} \right) \tag{5.5}$$

Now, let's try to

$$H = H_0 + V$$

5.3.3 Conventional pert

$$H_0 |\eta_n\rangle = \eta_n |\eta_n\rangle \tag{5.6}$$

$$H |E_0\rangle = E_0 |E_0\rangle \tag{5.7}$$

while

$$\Delta E_0 = E_0 - \eta_0 = \frac{\langle \eta_0 | V | E_0 \rangle}{\langle \eta_0 | E \rangle}$$

where $|E_0\rangle \approx |\eta_0\rangle$. define $P_0 = |\eta_0\rangle\langle\eta_0|$, $Q_0 = 1 - P_0 = \sum_{n=1}^{\infty} |\eta_n\rangle\langle\eta_n|$.

$$(D - H_0)|E_0\rangle = (D - \hat{H} + \hat{V})|E_0\rangle = (D - E_0 + \hat{V})|E_0\rangle$$

we get

$$|E_0\rangle = \frac{1}{D - H_0}(D - E_0 + V)|E_0\rangle. = P_0|E_0\rangle + D_0|E_0\rangle = |\eta_0\rangle\langle\eta_0|E_0\rangle + Q_0|E_0\rangle \tag{5.8}$$

and

$$|\tilde{E}_0\rangle = \frac{|E_0\rangle}{\langle \eta_0 | E_0 \rangle}$$

with this, we can write down the perturbation state

$$|\tilde{E}_0\rangle = |\eta_0\rangle + \frac{1}{D - H_0} Q_0 (D - E_0 + V)|\tilde{D}_0\rangle, \tag{5.9}$$

$$[H_0, P_0] = [Q_0, H_0] = 0, \tag{5.10}$$

while Q_0 stands for the projection for orthorgonal basis. Formally,

$$|\tilde{E}_0\rangle = \sum_{m=0}^{\infty} \left\{ \frac{1}{D - H_0} Q_0 (D - E_0 + V) \right\}^m |\eta_0\rangle \tag{5.11}$$

and $\Delta E_0 = \langle \eta_0 | V | \tilde{E}_0 \rangle$.

- Schrodinger pert: $D = \eta_0$, $\Delta E_0(m = 0) = \langle \eta_0 | \lambda v | \eta_0 \rangle \sim \lambda$
- $\Delta E_0(m = 1) = \frac{\sum |\langle \eta_0 | \lambda v | \eta_n \rangle|^2}{\eta_1 - \eta_n} \sim \lambda^2$, $\Delta E_0(m) \sim \lambda^{m+1}$.
- $\Delta E_0(m = 2) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\langle \eta_0 | V | \eta_m \rangle \langle \eta_m | V | \eta_n \rangle \langle \eta_n | V | \eta_0 \rangle}{(\eta_0 - \eta_m)(\eta_0 - \eta_n)} (\sim \lambda^3) - \Delta E_0 \sum \frac{|\langle \eta_0 | V | \eta_n \rangle|^2}{(\eta_0 - \eta_n)^2} (\sim \lambda^2)$ where $\Delta E_0 \sim (\lambda^0 + \lambda + \cdots + \lambda^m)$.

  $f(\lambda, \lambda^2, \lambda^3)$ mixing $(\lambda^1, \lambda^2, \lambda^3)$.

  interaction problem: diff-integral.

5.4 Brillouin-Wigner Pert.

Here, $D = E_0$.

5.4.1 Lippmann-Schwinger Equation

$$\hat{G} = \frac{1}{Z - H_0}, \hat{G}(z) = \frac{1}{Z - \hat{H}}.$$

we can write

$$Z - H_0 = Z - H + V$$

and

$$(1 + V(Z - H)^{-1})(Z - H) = (1 + \hat{V}\hat{G})(Z - H) = (1 + \hat{V}\hat{G})\hat{G}^{-1} = (Z - H_0)G$$

and $G_0^{-1}(Z - H_0)G = 1 + VG$, $\hat{G} = \hat{G}_0 + \hat{G}\hat{V}\hat{G}$. To expand it,

$$\hat{G} = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \cdots$$

5.5 Necessity of Gell-Mann-Lon (GML) theorem (No necessity ...)

Doing sth different

$$H_\alpha = H_0 + V\,\mathrm{e}^{-\alpha|t|},\ \alpha > 0 \tag{5.12}$$

scuicthing on the interactionL ADIABADIACALLY. there are two limits,

$$\lim_{t \to \pm\infty} H_\alpha = H_0; \tag{5.13}$$

$$\lim_{t \to 0} H_\alpha = H \tag{5.14}$$

in the first one, the eigenstate

$$|\psi_\alpha(t)\rangle|_{t\to\pm\infty} = \begin{cases} |\eta_0\rangle,\ t\to-\infty \\ e^{\mathrm{i}\varphi(\alpha)}|\eta_0 > t\to\infty \end{cases}$$

for the second one

$$|\Psi_\alpha(t)\rangle'|_{t\to 0} \to |\Psi_D(t=0)\rangle, .$$

and $H|\Psi_D\rangle = E|\Psi_D\rangle$. as $\alpha > 0$, $\alpha \to 0$, $\frac{\partial}{\partial\alpha}\big|_{\alpha\to 0}$.

$$V_D(t)\,\mathrm{e}^{-|t|\alpha} = \mathrm{e}^{\frac{1}{\hbar}H_0 t} V\,\mathrm{e}^{-\frac{1}{\hbar}H_0(t)}\,\mathrm{e}^{-|t|\alpha}$$

and we want to compute this: time evolution of:

$$U_\alpha^D(t,t_0) = \mathcal{T}_D\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t' V_D(t')\mathrm{e}^{-|t|\alpha}} = \sum_{n=0}^{\infty}\frac{1}{n!}\Big(-\frac{1}{\hbar}\Big)^n\int_{t_0}^{t}\int \mathrm{d}t_1\cdots\mathrm{d}t_n\,\mathrm{e}^{-\alpha(|t_1|+\cdots)} \tag{5.15}$$

$$T_d\{V_D(t_1),\cdots V_D(t_n)\}$$

$$|\Psi_\alpha^D(t)\rangle = U_\alpha^D(t,t_0)|\Psi_\alpha^D(t_0)\rangle\big|_{t_0\to\infty}$$

1. $|E_0^D\rangle = \lim_{\alpha\to 0}|\Psi_\alpha^D(0)|_V$ Gell-Mann-Low Theorem

$$\lim_{\alpha\to 0}\frac{U_\alpha^D(0,-\infty)|\eta_0\rangle}{\langle\eta_0|U_\alpha^D(0,-\infty)|\eta_0\rangle} = \lim_{\alpha\to 0}\frac{|\Psi_\alpha^D(0)\rangle}{\langle\eta_0|\Psi_\alpha^D(0)\rangle}$$

so

$$\langle\eta_0|\Psi_\alpha^D(0)\rangle \xrightarrow{\alpha\to 0} \exp\Big(-\mathrm{i}\frac{f(\lambda)}{\alpha}\Big).$$

Linked-Cluster Theorem

$$\begin{aligned} &\langle\Psi_\alpha^D(0)|\hat{A}^D(t)\hat{B}^D(t')|\Psi_\alpha^D(0)\rangle \\ \Rightarrow&\langle\eta_0|V^D(+\infty,0)\hat{A}(t)\hat{B}(t')U^D_{(0,-\infty)}|\eta\rangle \\ \Rightarrow&\langle\eta_0|T_D(V^D,\cdots),\hat{A}(t)\hat{B}(t')T_D(V^D\cdots)|\eta_0\rangle, \\ \Rightarrow&F(G_D^{cansal}) \end{aligned}$$

From this, we have

$$G^c = \frac{1}{G_v - \epsilon_k^0 - \Sigma}$$

$|\eta_0\rangle \longrightarrow G^c \longrightarrow |\Psi_\alpha^p(0)\rangle \longrightarrow |\eta_0\rangle$ Q:

5.6 No-Necssity of GML

From ρ_0: $|\Psi_0\rangle$, $H_0 + \delta H$, to $\rho = \rho_0 + \delta_\rho$. To describe MB system more efficientlym we need operator basis (orthogonal, complete) set. $\{|\eta_n\rangle\}$: inner product. $\mathrm{Tr}[u^\alpha, u^\beta] = \delta^{\alpha\beta}$. $\{u^\alpha\}$ is complete.

$$\hat{\rho} = \sum u^\alpha\langle u^\alpha\rangle$$

Fano regonance. and u find that

$$\rho^2 = \rho \Rightarrow \{\rho_0, \delta\rho\} = \delta\rho + \delta\rho^2$$

so called polynomial eq.

5.7 GML & proof

$$H = H_0 + V\,\mathrm{e}^{-\alpha|t|} = H_0 + \lambda\delta H$$

where D means in Dirac picture

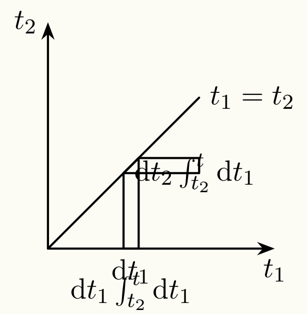
$$\lim_{\alpha\to 0}\frac{u_\alpha^D(0,-\infty)|\eta_0\rangle}{\langle\eta|u_\alpha^D(0,-\infty)|\eta_0\rangle} = \lim_{\alpha\to 0}\frac{|\psi_\alpha^D(0)\rangle}{\langle\eta_0|\psi_\alpha^D(0)\rangle}$$

Proof.

$$(H_0+V)\frac{|\psi_\alpha^D(0)\rangle}{\langle\eta_0|\psi_\alpha^D(0)\rangle} = \frac{\langle\eta_0|H|\psi_0^D(0)\rangle}{\langle\eta_0|\psi_0^D\rangle}\frac{|\psi_\alpha^D(0)\rangle}{\langle\eta_0|\psi_\alpha^D(0)\rangle}$$

$$\begin{aligned}(H_0-\eta_0)|\psi_\alpha^D(0)\rangle &= -\sum_n \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}\right)^{n-1}\int_{-\infty}^0\int_{-\infty}^0 \mathrm{d}t_1\cdots\mathrm{d}t_n\,\mathrm{e}^{-\alpha(|t_1|+\cdots+|t_n|)} \\ &= \cancelto{n\frac{\partial}{\partial t_n}}{\left(\sum_{j=1}^{n}\right)}\mathcal{T}_D(V_D(t_1)V_D(t_2)\cdots V_D(t_n))|\eta_0\rangle\end{aligned}$$

where $\left(\sum_{j=1}^n \frac{\partial}{\partial t_j}\right) \to n\frac{\partial}{\partial t_n}$.



$\mathcal{T}_D(V(t_1)\cdots V(t_n))$ symmetric of $(t_1\cdots t_n)$ it will return corresponding factor due to the order of $t_1t_2t_3 + t_2t_3t_1 + \cdots$ then u can do integral by part

Causal GFs

We usually convert

$$\langle\psi_\alpha^D(0)|\hat{a}_i^\dagger(t)a_j(t')|\psi_\alpha^D(0)\rangle$$

into Casaul Green Functions

$$G = \frac{1}{\omega-\epsilon_0-\Sigma}$$

$\square$

5.7.1 No-Necessity of GML

Bascially, we can also do perturbation & variational method. When given a H_0, $\rho_0 + \delta H$ where $\delta\rho = \rho - \rho_0$, and energy δE.

Reasit adiabaticity.

route cause: $|\psi\rangle = \frac{1}{E_0}$ when write into Taylor series: $|\psi(\lambda)\rangle = |\psi_0\rangle + \lambda|\delta\psi\rangle + \ldots$ and the energy $E_0(\lambda) = E_0 + \lambda\delta\epsilon^{(1)} + \frac{\lambda^2}{2}\delta\epsilon^{(2)}$ they should be true separate.

There's no need for E_0.

In QM: Lippmann-Schwinger

$$\hat{G} = (G_0 - H)^{-1}$$

But the Lippmann-Schwinger equation we put down is

$$\hat{G}(\omega) = G_0 + G_0 V G$$

diagnal in ω.

As well,

$$G(\boldsymbol{r};\boldsymbol{r}') = G_0(\boldsymbol{r},\boldsymbol{r}') - \int \mathrm{d}\boldsymbol{r}'' G_0(\boldsymbol{r},\boldsymbol{r}'')\delta V(\boldsymbol{r}'')G(\boldsymbol{r}'',\boldsymbol{r}')$$

DOES NOT depend on E_0.

5.8 Complete operator basis set

wave function $|\Psi\rangle = \mathrm{e}^{\mathrm{i}\alpha(\ldots)} = \langle\hat{O}\rangle$

"Inner product" (orthorgonal, completness) $\mathrm{Tr}[\hat{u}^\alpha, \hat{u}^\beta] = \delta_{\alpha,\beta}$ C

$$\hat{O} = \sum \langle u^\alpha\rangle \hat{o}\hat{U}^\alpha = \sum_\alpha (\boldsymbol{u}^\alpha \cdot \boldsymbol{O})\boldsymbol{u}^\alpha$$

where $\langle\rangle_{\hat{O}} = \mathrm{tr}[u^\alpha, \hat{O}]$

$$\hat{\rho} = \sum \langle\hat{u}^\alpha\rangle\hat{u}^\alpha$$

so we come to some oeelgebraic description

$$[\rho, H] = 0$$

($\otimes$!!! SSB $[\rho, H] \neq 0$) and the project $\hat{\rho}^2 = \hat{\rho}$, with the eigenenergy for pure state condition

$$\{\rho, H\} = 2E\hat{\rho}$$

* Gerivile to thermal states

$\hat{\rho}^2 \neq \rho$, but

5.8.1 Perturbation theory

$$\rho(g) = \rho_0 + g\rho_1 + \frac{g!}{2!}\rho_2 \tag{5.16}$$

$$E = E_0 + \delta E = E_0 + gE_1 + \frac{g^2}{2!}E_2 + ... \tag{5.17}$$

$\hat{\rho} = \hat{\rho}_0 + \delta\hat{\rho}$

$$\rho^2 = \rho_0^2 + \delta\rho^2 + \{\hat{\rho}_0, \delta\hat{\rho}\} = \hat{\rho} = \hat{\rho}_0 + \delta\hat{\rho} = \rho_0 + \delta\hat{\rho}^2 + \{\hat{\rho}_0, \delta\hat{\rho}\}$$

we obtain

$$\{\hat{\rho}_0, \delta\rho\} = \delta\hat{\rho} - \delta\hat{\rho}^2$$

For the Hamiltonian

$$H = H_0 + \delta H$$

the first order term becomes

$$O(g') : [\rho_1, H_0] + [\hat{\rho}_0, \delta H] \approx 0.$$

then

$$\{\hat{\rho}_0, \hat{\rho}_1\} = \hat{\rho}_1$$

and we have

$$[\{\hat{\rho}_0, \hat{\rho}_1\}, H_0] + [\hat{\rho}_0, \delta H] = 0$$

we expand $\rho_1 = \sum \langle \hat{u}^\alpha \rangle_1 \hat{u}^a$ while $\hat{\rho}_0 = \sum_{\alpha_0} \langle u^{\alpha_0} \rangle u^\alpha$ and the trace $\mathrm{Tr}[u^\alpha(\text{the equation})] = 0$. Finally, we get

$$\langle [\delta H, \hat{u}^{\alpha_0}] \rangle + \sum_\alpha \langle \{[H_0, \hat{u}^{\alpha_0}], \hat{u}^\alpha\} \rangle_0 \langle u^\alpha \rangle_1 = 0$$

and we can write it into linear equation

$$\hat{L}\langle u \rangle_1 + \langle [\delta H, u^{\alpha_0}] \rangle = 0$$

where The $\hat{L}$ is Liouver super op. for time evo of ρ:

$$[H, \delta\rho] = \frac{\mathrm{d}}{\mathrm{d}t}\rho = \hat{L}\rho$$

More formally,

$$[[L_0]][\rho_1] + [V]_0 = 0$$

so u can see

$$\{\rho_0, \delta\rho\} = \delta\hat{\rho} - \delta\hat{\rho}^2 = \delta\hat{\rho} - \delta\hat{\rho}(\{\rho_0, \delta\rho\} + \delta\hat{\rho}^2) = \delta\hat{\rho}(1 - \{\rho_0, \delta\rho\}) - \delta\hat{\rho}^3$$

concerning the 2nd order term

$$O = \langle [\delta H, u^{\alpha_0}] \rangle_0 + \sum \delta\langle u^\alpha \rangle \langle \{u^\alpha, [H, \hat{u}^{\alpha_0}]\} \rangle_0 + \sum_{\gamma\beta} \delta\langle u^\alpha \rangle \delta\langle u^\beta \rangle \langle \{u^\gamma, [H, \{u^{\alpha_0}, u^\beta\}]\} \rangle_0 + \cdots$$

then we start from $\mathbb{1} \in coBasisset$. Eq of $\mathbb{1}$, $u^{\alpha_0} = \mathbb{1}$:

$$0 = 0 + \sum \ldots$$

called linked-cluster-theorem. keep connected term.
In a global form, we can find

$$[V]_{0,\alpha_0} + [[L]] \cdot [\delta\hat{\rho}] + \mathrm{Tr}[\hat{u}^{\alpha_0}[\delta\hat{\rho}, H]] = 0$$
$$[V] + [[L]][\delta\hat{\rho}] + [[L_2]][\delta\rho^2] + \mathrm{Tr}[u^{a_0}, [\delta\rho^3, H]]$$

then L_n can be defined as

$$L_n \cdot \delta\hat{\rho}^n = [H, \delta\rho^n]$$

then

$$[V]_0 + \sum_{n=1}^{\infty} [[L_n]][\delta\rho^n] = 0$$

then u can get δE from $\delta\rho$.

5.8.2 Example

Transversefield Ising model

$$H_{\text{TFIM}} = -J\sum_{ij} S_i^z S_j^z - h_z \sum_i S_i^z = -\delta H - H_0$$

$|\psi_0\rangle = |{\rightarrow}{\rightarrow}{\rightarrow}\cdots\rangle$ $\rho_0 : \langle S_i^x \rangle_0 = \frac{1}{2}$, $\langle SA^\alpha xx\rangle_0 = 0$, $\langle S_i^\alpha S_j^\beta \rangle_{C,0} = 0$[1]. Denote $u^{\alpha_0} = S_{i0}^y S_{i0+1}^z$

$$\langle [\delta H, n^{\alpha_0}]\rangle_0 \propto \langle S_i^x S_{i0+1}^x \rangle_0$$

we often get

$$[H_0, u^{\alpha_0}] = S_{i_e}$$
$$\{u^\alpha, [H_0 u^{\alpha_0}]\} = s_{10} y + S_{i0+1}^x Y + \langle S^x \rangle_x \langle S_x \rangle_z = 0$$

that is

$$\langle S_i^x, S_{i+1}^x \rangle \sim h/1g - g_c|^{1/8}$$

2D Heisenberg module

5.9 Perturbation

5.9.1 Review of Path Integral in QM

$$\psi(\boldsymbol{x}, t) = \int \mathrm{d}^3 x' K(\boldsymbol{x}, t; x', t_0)\psi(\boldsymbol{x}', t_0)$$

[1] Cumulant: $\langle S_i S_j \rangle_C = \langle (S_i - \langle S_i \rangle)(S_j - \langle S_j \rangle)\rangle$.

$\langle x|\psi(t)\rangle \Rightarrow K(\boldsymbol{x},t;\boldsymbol{x}',t_0) = \langle \boldsymbol{x}|\mathrm{e}^{-\mathrm{i}H(t-t_0)/\hbar}|\boldsymbol{x}'\rangle = \langle 0|\varphi(x,t)\varphi^\dagger(x',t_0)|0\rangle$
PI is based computed on GF directly.

$$\sum \mathrm{e}^{\mathrm{i}\int_{t,t}^{t,t} L/t}$$

$$\mathrm{e}^{\mathrm{i}Ht} \to \mathrm{e}^{N_i Ht/N} = E^o$$
$$\left(-\frac{\hbar^2}{2m}(\nabla')^2 + V(\boldsymbol{x}) - \mathrm{i}\hbar\frac{\partial}{\partial t}\right)K(\boldsymbol{x},t;\boldsymbol{x}'_j,t_0) = -\mathrm{i}\hbar\delta^3(\boldsymbol{x}-\boldsymbol{x}')\delta(G-Gaaa)$$

5.9.2 bass Precedures

$$L = L(x, \partial_t, xx, \cdots t)$$

- lassical route
- Fluctuations around classic route

5.9.3 Auadratic Legrangian

$$\psi(x_N, t_\sim, |X_0 t_0\rangle) = \int \mathrm{d}[x(t)] \exp\left[\frac{\mathrm{i}}{\hbar}S[x,t]\right]$$

while consider the Lagrangian

$$\mathcal{L}(x,\dot{x},t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}c(t)x^2 - c(t)x$$

and since $\delta L = 0$, SInce Euler-Lagrangian eq., we have EOM

$$m\ddot{X}_{cl} + c(t)x_{cp} + \mathrm{e}(t) = 0$$

then $x(t) = x_{cl}(t)+y(t)$, and $y(t_0) = y(t_N) = 0$. then $L(x_{cl}+y, \dot{x}_{cl}+\dot{y}, t) = L_{cl}+L'(y,\dot{y}(t),t)+\delta L(|\overset{x}{y}\rangle)$

$$L_{cl} = \frac{1}{2}m\dot{x}_{cl}^2 - \frac{1}{2}c(t)x_{cl} - e(t)x_{cl}$$
$$L' = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}c(t)y^2\delta L = m\dot{x}_{cl}\dot{y}(t) - c(t)x_{cl}y - e(t)y$$

5.9.4 Perturbation Theory by path integral

$$\dot{x}_{cl}\dot{y} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}_{cl}y) - \ddot{x}_{cl}y$$

$$\int_{t_0}^{t_N} \mathrm{d}t\delta L = m[\dot{x}_{cl}y]\Big|_{t_1}^{t_N} - \int_{t_0}^{t_N} (m\ddot{x}_{cl}(t) + c(t)x_{cl}(t) + e(t))y(t)\,\mathrm{d}t$$

then

$$\phi(x_N, t_N; x_0, t_0) = \exp\left\{\frac{\mathrm{i}}{t}S[x_{cl}(t)]\right\}\tilde{\phi}(0, t_N; 0, t_0)$$

then

$$\tilde{\phi}(0,t_N,0,t_0)=\int_{y(t_0)=d}^{y(t_N)=0} d[y(t_1)]\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}tL'\right)$$

For discretisaition on t:

$\epsilon_N=(t_N-t_0)/N$, and $t_j=t_0+\epsilon_N j$

$$\tilde{\phi}=\lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\hbar\epsilon_0}\right)^{N/2}\times\int \mathrm{d}y_1\,\mathrm{d}y_2\cdots\mathrm{d}y_{N-1}\exp\left[\frac{\mathrm{i}}{\hbar}c_N\sum\frac{1}{2}M\frac{(y_{j+1}-y_j)^2}{\epsilon_N^2}-\frac{1}{2}c_jy_j^2\right]$$

and the energy $E=\mathrm{i}\sum y_ia_{jk}y_k$, we have a $(N-1)\times(N-1)$ matrix, where $N\to\infty$

$$(a_{jk})=\frac{m}{2\hbar\epsilon_N}\begin{pmatrix}2 & -1 & \dots & \\ -1 & 2 & -1 & \\ & -1 & 2 & \dots\end{pmatrix}-\frac{\epsilon_N}{2\hbar}\begin{pmatrix}c_1 & & & \\ & \ddots & & \\ & & c_j & \\ & & & \ddots\end{pmatrix}=\lim_{N\to\infty}\left(\frac{1}{\det[a]}\right)^{1/2}$$

In the first term of a_{jk}, only the diag and two subdiag is non-zero, other elements are zero.

let $D_{N-1}=\det[a]$, then

$$D_n=\left(2-\frac{\epsilon_N^2}{m}c_N\right)D_{n-1}-D_{n-2}$$

and then u get

$$D_0=1,D_1=2-\frac{\epsilon_N^2}{m}c_1$$

to the infinite limit, $\frac{D_{n+1}-2D_n+D_{n-1}}{\epsilon_N^2}=-\frac{c_{n+1}D_n}{m}$ and when $N\to\infty,\epsilon_N\to 0$,

$$\frac{\mathrm{d}^2f(t_0,t)}{\mathrm{d}t^2}=-\frac{c(t)}{m}f(t_1,t)$$

apply the bound condition, show that

$$f(t_1,t_0)=0,\frac{\mathrm{d}}{\mathrm{d}t}f(t_0,t)=1$$

$$\begin{cases}\phi(xt;x_0t_0)=\left(\frac{m}{2\pi\mathrm{i}\hbar f(t_0,t)}\right)^{1/2}\exp\left[\frac{\mathrm{i}}{\hbar}S[X_{ct}(t)]\right],\\ \frac{\mathrm{d}^2f(t_0,t)}{\mathrm{d}t^2}=-\frac{c(t)}{m}f(t_0,t)\end{cases}$$

For Harmonic Oscillator, $c(t)=m\omega^2$, then $\ddot{f}=-\omega^2f$, so $f(t_0,t)=\sin(\omega(t-t_0))/\omega$.

If consider $\omega\sim\omega_0+gc(t)$, ...

We are getting H, ϵ_n, $|n\rangle$, then we get $\langle x|\psi(t)\rangle=\sum_n\underbrace{\langle x|n\rangle}_{\to\mathrm{e}^{-\epsilon_n\tau/\hbar}}\mathrm{e}^{\mathrm{i}\epsilon_nt/\hbar}$.

To be specifitly,

$$\langle xt|x't'\rangle=\sum_n\varphi_n^*(t)\varphi_n(t')\,\mathrm{e}^{-\mathrm{i}\epsilon_n\Delta t/\hbar}$$

5.9.5 Enclidean Path Integral

means $t \to i\tau$. Relation to partition func, and spectral repr. decomposition of GFs.

$$\langle x_f, -\mathrm{i}\tau/2|x_i, \mathrm{i}\tau/2\rangle|_{\tau+\infty} = \lim_{T\to c_0} \mathrm{e}^{-E_0 T}\varphi_0(x_f)\varphi_0^*(x_i) = \langle x_f|\mathrm{e}^{-TH}|x_i\rangle$$

5.9.6 Double well problem for PI

$$V(x) = \lambda(x-a)^2 = \lambda(x-a)^2(x+a)^2 \sim \lambda(x+a)^2(2a)^2 \sim \lambda(x-a)^2(2a)^2$$

eigenstates: $|n, +a\rangle$, $|n, -a\rangle$, $\langle n_1 + a|H|n_1' - a\rangle$

$$H' = \begin{pmatrix} H_{a-} & \Delta H \\ \Delta H & H_{-aa} \end{pmatrix}$$

1. x_{cl}: $E = 0$.
2. $t \to \mathrm{i}\tau$, the potential $V(x)$, the Lagrangian from $L = T - V$ to $T + V = T - (-V)$, a flipped potential. (Why? consider $\mathrm{e}^{\mathrm{i}\mathrm{d}tL}$), get $-\dot{x}^2 - V(x)$, but when $t \to \mathrm{i}\tau$, u will get $\mathrm{e}^{-\tau(\dot{x}^2-(-V(x)))}$, under the view of Hamiltonian

For 1: $\mathrm{e}^{\sum x_{cl}}$
For 2: $t \to \infty$, $\mathrm{e}^{\sum_{EPI} inst}$

5.10 Apply P.I. for second Quantization

Begin with the coherent state of H.O.. For Bosons,

$$[a_i, a_j^\dagger] = \delta_{ij}$$

Apply H.O. to Bosons for second quantization. The eigenstate of the annihilation / creation operator is

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \text{but it's wrong that} \quad \hat{a}^\dagger|\alpha\rangle = \alpha|\alpha\rangle$$

where the coherent state can be expanded in terms of the quantum number state

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum \frac{\alpha^n}{\sqrt{n!}} |\varphi_n\rangle$$

It has some properties

(a) Non-orthogonal

$$\langle\alpha|\alpha'\rangle = \mathrm{e}^{-|\alpha|^2/2}\,\mathrm{e}^{-|\alpha'|^2/2}\,\mathrm{e}^{\alpha^*\alpha'} \tag{5.18}$$

(b) In the quantum number state, we have the identity

$$\sum_n |\varphi_n\rangle\langle\varphi_n| = \mathbb{1},$$

While in the coherent state

$$\mathbb{1} = \frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\,\mathrm{d}\,\mathrm{Re}\,\alpha\,\mathrm{d}\,\mathrm{Im}\,\alpha \tag{5.19}$$

which is overcompleteness.

5.10.1 Coherentstate of P.I.

We want to compute

$$\langle x|\mathcal{U}|x'\rangle = \langle x|\mathcal{T}\,\mathrm{e}^{\mathrm{i}H(t-t')}|x'\rangle = \langle x|\mathrm{e}^{\mathrm{i}\mathcal{H}\Delta_t}\,\mathrm{e}^{\mathrm{i}\mathcal{H}\Delta_t}\cdots\mathrm{e}^{\mathrm{i}\mathcal{H}\Delta_t}|x'\rangle \tag{5.20}$$

where $\Delta t = \frac{t-t'}{N}\big|_{N\to\infty}$, and we can insert the identity $\mathbb{1}$ between the $\mathrm{e}^{\mathrm{i}\mathcal{H}\Delta_t}$s. Assume for the coherent state $|z\rangle$, we have

$$\hat{a}|z\rangle = z|z\rangle, \quad \hat{a}^\dagger|z\rangle = \partial_z|z\rangle$$

also for the invert

$$\langle z|\hat{a} = z\,\partial_{\bar{z}}\langle z|, \quad \langle z|\hat{a}^\dagger = z\langle z|,$$

and we can define

$$\psi(\bar{z}) = \langle z|\psi\rangle, \quad \text{and} \quad \bar{\psi}(z) = \langle\psi|z\rangle \tag{5.21}$$

Then the integral can be written as

$$I = \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2\pi\,\mathrm{i}}\,\mathrm{e}^{-z\bar{z}}|z\rangle\langle z| \tag{5.22}$$

then, after inserting the identity $\int \frac{\mathrm{d}z(t_n)\mathrm{d}\bar{z}(t_n)}{2\pi \mathrm{i}}$, we have

$$\cdots |z_{t_{n-1}}\rangle\langle z_{t_{n-1}}|\exp(\mathrm{i}\mathcal{H}(a^\dagger, a)\Delta t)|z_{t_n}|\rangle\langle z_{t_n}|\,\mathrm{e}^{\mathrm{i}\mathcal{H}}\cdots$$

where $\langle z_{t_{n-1}}|\exp(\mathrm{i}\mathcal{H}(a^\dagger, a)\Delta t)|z_{t_n}|\rangle$ is normal ordering, we can rewrite it as

$$\langle z|f(a_i^\dagger a_j^\dagger \cdots a_i a_j)|z'\rangle = f(\bar{z}_i, \bar{z}_j, \cdots z_i, z_j \cdots)$$

i.e., $a^\dagger \to \bar{z}$, $a \to z$. For Δt, the normal ordering term can be expressed as

$$\begin{aligned}
\langle z_{t_{n-1}}|1 - \frac{\mathrm{i}}{\hbar}\Delta t H(a^\dagger, a)|z_{t_n}\rangle &= \langle z_{t_{n-1}}|z_{t_n}\rangle\Big(1 - \frac{\mathrm{i}}{\hbar}\Delta\mathcal{H}(\bar{z}_{t_{n-1}}, z_{t_n})\Big) \\
&= \mathrm{e}^{\bar{z}_{t_{n-1}} z_{t_n}}\,\mathrm{e}^{-|z_{t_{n-1}}|^2/2}\,\mathrm{e}^{-|z_{t_n}|^2/2}(1 - \cdots) \\
&= \int \mathcal{D}z\mathcal{D}\bar{z}\exp\Big[\frac{\mathrm{i}}{\hbar}\int_{t_i}^{t_f}\frac{\hbar}{2\,\mathrm{i}}(z\,\partial_t\bar{z} - \bar{z}\,\partial_t z) - H(\bar{z}, z)\Big] \\
&= \ldots \exp[\frac{1}{2}(|z_i|^2 + |z_f|^2)]\bar{\psi}_f(z_f)\psi_i(\bar{z}_i)
\end{aligned} \tag{5.23}$$

5.10.2 Fermionic coherent P.I.

For the fermions, it satisfy the anti-commutation

$$\{c_i c_j^\dagger\} = \delta_{ij}, \{c_i, c_j\} = 0 \tag{5.24}$$

then for the coherent state, we have

$$c_i|\xi_i\rangle = \xi_i|\xi_i\rangle, \tag{5.25}$$

where $|\xi_i\rangle = a|0\rangle + b|1\rangle$. Then, the action can be written into the number state

$$c|\xi_i\rangle = b_0 \tag{5.26}$$

where $b = \xi_i$. We also have the anti-commutator

$$\{\xi_\alpha, \xi_\beta\} = 0 \tag{5.27}$$

We get the Grassmann-variables ξ_i, which satisfy

$$\xi_\alpha^2 = 0, \quad f(\xi_\alpha) = a{+}b\xi_\alpha, \quad \mathrm{e}^{\xi_\alpha} = 1{+}\xi_\alpha, \quad \int \mathrm{d}\xi 1 = 0, \int \mathrm{d}\xi\xi = 1, \frac{\partial}{\partial\xi}(\xi^*)\xi = \frac{\partial}{\partial\xi}{-}\xi\xi^* = -\xi^* \tag{5.28}$$

We can have the table of Gaussian $[a_\alpha, a_\beta^\dagger]{-}\xi = \delta_{\alpha\beta}$, $|\xi\rangle = \mathrm{e}^{\xi}\sum_\alpha \xi_\alpha a_\alpha^\dagger|0\rangle$, $a|\xi\rangle = \xi|\xi\rangle$
$\mathbb{1} = \ldots$

$$\int \mathcal{D}[\xi_\alpha, \xi_\beta^*]\exp\Bigg[-\sum_{\alpha\beta}\xi_\alpha^*\mathcal{H}_{\alpha\beta}\xi_\beta + \sum_\alpha(\eta_\alpha\xi_\alpha^* + \eta_\alpha^*\xi_\alpha)\Bigg] = [\det, \mathcal{H}]^{-\xi}\sum_{\alpha\beta}\eta_\alpha^*\mathcal{H}_{\alpha\beta}^{-1}\eta_\beta \tag{5.29}$$

we will discuss later: by taking $\frac{\partial}{\partial\eta}\frac{\partial}{\partial\eta^*}$ with $\eta \to 0$ and $\eta^* \to 0$ to the expression above.

5.10.3 spin-coherent state

Example : Wess-Zumino-Witten term & Haldane conjecture with S^+, S^-, and $|S,\mu\rangle$
We can start from the state

$$|0\rangle = |SS\rangle, \quad S_z|0\rangle = S|0\rangle, \quad \hat{S}^2|0\rangle = S(S+1)|0\rangle \tag{5.30}$$

where S is the magnetic number. The three-dimensional vector $\boldsymbol{n}$

$$|\boldsymbol{n}\rangle = \mathrm{e}^{\mathrm{i}\theta(\boldsymbol{n}_0\times\boldsymbol{n})\cdot\boldsymbol{S}}|SS\rangle \tag{5.31}$$

where $\boldsymbol{n}\cdot\boldsymbol{n}_0 = \cos\theta$, and $\boldsymbol{n}_0 = (0,0,1)$ is a constant vector point to north.

$$|\boldsymbol{n}\rangle = \sum_{M=-S}^{M=S} \mathcal{D}(\boldsymbol{n})_{MS}|SM\rangle \tag{5.32}$$

and the inner product

$$\langle\boldsymbol{n}_1|\boldsymbol{n}_2\rangle = \left[\frac{(1+\boldsymbol{n}_1\boldsymbol{n}_2)}{2}\right]^S \mathrm{e}^{\mathrm{i}\Phi(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_0)S} \tag{5.33}$$

The vector basis $\{\boldsymbol{n}_i\}$ can be expressed in a Bloch-sphere, while the identity

$$\mathbb{1} = \int \mathrm{d}\mu[\boldsymbol{n}]|\boldsymbol{n}\rangle\langle\boldsymbol{n}| \tag{5.34}$$

where

$$\mathrm{d}\mu(\boldsymbol{n}) = \left(\frac{2S+1}{4\pi}\right)\mathrm{d}^3n\,\delta(\boldsymbol{n}^2-1) \tag{5.35}$$

Now, start from

$$\langle\boldsymbol{n}(t)|\mathrm{e}^{\mathrm{i}\Delta t\mathcal{H}}|\boldsymbol{n}(t_{j+1})\rangle = (\langle\boldsymbol{n}(t_j)|\boldsymbol{n}(t_{j+1})\rangle + \mathrm{i}\Delta t\langle\boldsymbol{n}(t_j)|\mathcal{H}|\boldsymbol{n}(t_{j+1})\rangle)$$

and

$$\frac{\langle n(t_j)|\mathcal{H}|n(t_{j+1})\rangle}{\langle n(t_j)|n(t_{j+1})\rangle} \approxeq \langle\boldsymbol{n}(t_j)|\mathcal{H}|\boldsymbol{n}(t_j)\rangle + O(\Delta t)$$

combine them together

$$\langle\boldsymbol{n}(t_j)|\boldsymbol{n}(t_{j+1})\rangle = \mathrm{e}^{\mathrm{i}\Phi S}\left(1+\frac{\boldsymbol{n}(t_j)\cdot\boldsymbol{n}(t_{j+1})}{2}\right)^S \tag{5.36}$$

then we have

$$\mathrm{i}S[\boldsymbol{n}] = \mathrm{i}S\sum_{j=1}^{N_t}\Phi(\boldsymbol{n}_0,\boldsymbol{n}(t_j),\boldsymbol{n}(t_{j+1})) + S\sum\ln\left(1+\frac{\boldsymbol{n}\cdot\boldsymbol{n}}{2}\right) - \langle n|H|n\rangle \tag{5.37}$$

For a series of magnetic momentum $\boldsymbol{n}(t)$ along a chain, P.I. to the Hamiltonian is

$$\int \mathrm{d}n_i(t)\,\mathrm{e}^{\mathrm{i}J\sum_i\boldsymbol{S}_i\boldsymbol{S}_{i+1}}$$

and we can see the P.I. approach the phase term

$$\frac{S}{2}\underbrace{\int \mathrm{d}t\boldsymbol{n}(t)\cdot(\partial_t\boldsymbol{n}\times\partial_x\boldsymbol{n})}_{4\pi} = S$$

For $S = \frac{2n+1}{2}$, it behaves as a “crossing” (X); when $S = n$, it behaves as a gapped .

$$I_x\boldsymbol{S}_i^x\boldsymbol{S}_j^x + I_y\boldsymbol{S}_i^x\boldsymbol{S}_j^y \longrightarrow S_i^xS_j^- + S_i^-S_j^+ + S^+S^+ + S^-S^-$$

5.10.4 Review of Methodology & Source field method

(a) H-Q.M. GML

$$|\Phi\rangle = \mathcal{T}_0 \underbrace{\hat{S}|\Phi_0\rangle}_{\langle\Phi_0|\mathcal{T}_0\hat{S}|\Phi_0\rangle} \tag{5.38}$$

Then the inner product

$$\langle\Phi|\mathcal{T}\psi(1)\psi^\dagger(2)|\Phi\rangle = \underbrace{\langle\Phi|\mathcal{T}_0\hat{S}\psi(1)\psi^\dagger(2)|\Phi_0\rangle}_{\langle\Phi_0|\mathcal{T}_0\hat{S}|\Phi_0\rangle} \tag{5.39}$$

The operator $\hat{S}$ can be expressed as

$$\mathcal{S} = \mathcal{T}_0 \exp\left[-\mathrm{i}\int V(t')\,\mathrm{d}t'\right] = \mathcal{T}_0 \sum_n (\sim \int \mathrm{d}t_1 \cdots \mathrm{d}t_n) \tag{5.40}$$

(b) P.I. compute the S-matrix scattering

$$\langle\Phi_0|S|\Phi_0\rangle = \int \mathcal{D}[\psi,\psi^\dagger]\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\mathcal{L}^{\text{Gaussian}} - V\right) \tag{5.41}$$

we can map $t \to \mathrm{i}\tau$. The question is how to get

$$\langle\Phi_0|S\psi(1)\psi^\dagger(2)|\Phi_0\rangle \tag{5.42}$$

(c) $\mathrm{i}\,\partial_t\langle\Psi|\mathcal{T}_0\psi(1)\psi^\dagger(2)|\Psi\rangle = \delta[1-2] + \langle|[\psi(1),H]|\rangle$, Then we will get the derivative of Green function

$$\mathrm{i}\,\partial_t G = \delta[\,] + G_k^0 G + \underbrace{\langle[\psi(1),V],\psi^\dagger(2)\rangle}_{\text{Vertex-function}}$$

from which we can get *Schrödinger-Dyson-EOM*. Link through source field method

$$H^{\text{source}} = H + \sum_{t,j}\nu_j(t)\hat{\eta}|_{\nu_\eta(t)\to 0} \tag{5.43}$$

Consider add a source of c, and $\hat{\eta}_1 = c_i$, $\hat{\eta}_2 = c_j^\dagger$. Then we can computer the Green function by taking the functional derivative

$$\langle|c_i(t)c_j^\dagger(t')|\rangle = \frac{\delta}{\delta\nu_{\eta_1}(t),\nu_{\eta_2}(t')}\langle|\hat{S}|\rangle\Big|_{\text{source}}\Big|_{\nu\to 0} \tag{5.44}$$

we can define

$$\Gamma = \langle\hat{O}\psi(1);\psi^\dagger(2)\rangle = \frac{\delta}{\delta\nu_{\hat{O}}}\langle\psi(1);\psi^\dagger(2)\rangle\Big|_{\text{Source}}\Big|_{\nu\to 0} \tag{5.45}$$

For $H_n = \hat{n}_\uparrow\hat{n}_\downarrow$, $\Gamma = u\langle n_\downarrow(1)c_\uparrow(1);c_\uparrow^\dagger(2)\rangle$. Then the Fourier transformation $\mathcal{F}[\langle c_\uparrow(1),c_\uparrow^\dagger(2)\rangle]$.

For exact G_0, We can using the skeleton diagram to generate $G_0 + G_0G_\alpha$.

The Green basically contains

$$G[\,] = \Delta E_{(n-m)}\langle\Psi|\psi(x)\psi^\dagger(y)|\psi\rangle, \tag{5.46}$$

and $|\Psi\rangle$, (ρ), Σ.

Chapter 6 More Green's Functions

6.1 Green's function

6.1.1 GFs differential Eqs

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f + S(x)f(x) = 0, \quad \text{or} \quad \hat{\mathcal{L}} f = 0 \tag{6.1}$$

then we can construct $\frac{\mathrm{d}^2}{\mathrm{d}x^2} g = S(x)$, $f = \int \mathrm{d}x g \cdot S(x)$.

6.1.2 GF as propagators in Q.M.

From the S.E.

$$\mathrm{i}\hbar\,\partial_t \psi = H\psi$$

we get the G.F. $G = \langle x|U|x'\rangle$. To be specific

$$\left(\mathrm{i}\hbar\,\partial_t + \frac{\hbar^2}{2m}\nabla^2\right) G_{t,x;t',x'} = \delta(t-t')\delta(x-x') \tag{6.2}$$

which is the EOM, and $G_{t,x;t',x'}$ is the P.I.

6.1.3 One particle GF & real frequency/real time

Consider the indicate label λ, the GF

$$G_{\lambda\lambda'} = -\mathrm{i}\langle\phi|\mathcal{T}\psi_\lambda(t)\psi^\dagger_{\lambda'}(t')|\phi\rangle \tag{6.3}$$

where

$$\mathcal{T}\hat{\psi}_\lambda(t)\hat{\psi}^\dagger_{\lambda'} = \begin{cases} \psi_\lambda(t)\psi(t'), & t > t' \\ \pm\psi^\dagger_\lambda(t')\psi_\lambda(t), & t < t' \end{cases} \tag{6.4}$$

and the Bosons for $+$ and Fermions for $-$. To combine,

$$\mathcal{T}\hat{\psi}_\lambda(t)\hat{\psi}^\dagger_{\lambda'} = \theta(t-t')\psi_\lambda(t)\psi^\dagger_{\lambda'}(t') \pm \theta(t'-t)\psi^\dagger_{\lambda'}(t')\psi_\lambda(t) \tag{6.5}$$

when we take the time derivative,

$$\begin{aligned} \mathrm{i}\,\partial_t G &= \delta(t-t')(\psi_\lambda\psi^\dagger_{\lambda'} \mp \psi^\dagger_{\lambda'}\psi_\lambda) + \langle\mathcal{T}[\psi,\mathrm{i}t],\psi_{\lambda'}\rangle \\ &= \begin{cases} \delta(t-t')[\psi_\lambda,\psi^\dagger_{\lambda'}] = \delta(t-t')\delta_{\lambda\lambda'}, & \text{Bosons} \\ \delta(t-t')\{\psi_\lambda,\psi^\dagger_{\lambda'}\} = \delta(t-t')\delta_{\lambda\lambda'}, & \text{Fermions} \end{cases} + \langle\mathcal{T}[\psi,\mathrm{i}t],\psi_{\lambda'}\rangle \end{aligned} \tag{6.6}$$

Classically, the commutator is the Poisson bracket $\{\quad\} = \delta_{ij}$. For bosons and fermions, $\delta_{\lambda\lambda'}$ are all canonical operators, also exist for many-body operators.

6.1.4 Linear response problem: double time – GFs for arbitrary operators

$$G^{\mathrm{ret}}_{\hat{A},\hat{B}} = -\mathrm{i}\theta(t-t')\left\langle\langle[\hat{A}(t),\hat{B}(t')]_{\pm}\rangle\right\rangle \tag{6.7}$$

$$G^{\mathrm{adv}}_{\hat{A},\hat{B}} = +\mathrm{i}\theta(t'-t)\left\langle\langle[\hat{A}(t),\hat{B}(t')]_{\pm}\rangle\right\rangle G^{\mathrm{C}}_{\hat{A},\hat{B}} = -\mathrm{i}\left\langle\langle\mathcal{T}_{\pm}(\hat{A}(t)\hat{B}(t'))\rangle\right\rangle \tag{6.8}$$

where C = Causal = T-ordered, and

$$\langle\langle\hat{A}(t),\hat{B}(t')\rangle\rangle = \langle\Phi|(\hat{A}(t)-\left\langle\hat{A}(t')\right\rangle)(\hat{B}(t')-\left\langle\hat{B}(t')\right\rangle)|\Phi\rangle = \mathrm{Tr}[\rho\cdots]$$

and $\langle A\rangle = \langle\Phi|A|\Phi\rangle$, also for B.

6.1.5 Spectral Density of GFs

$$S_{AB}(t,t') = \frac{1}{2\pi}\langle[A(t),B(t')]_{\xi}\rangle \tag{6.9}$$

If set $t\to t'$, $G[0] = S_{AB}(t=t')$, and we can prove the quantities should evolve to

$$\langle A(t)B(t')\rangle = \langle A(t-t')B(0)\rangle$$

6.1.6 Spectral representation of GF

$$H|E_n\rangle = E_n|E_n\rangle,\ \sum_n|E_n\rangle\langle E_n| = \mathbb{1},\ \langle E_n|E_m\rangle = \delta_{nm} \tag{6.10}$$

Take the trace with the exact eigenstates

$$\begin{aligned}\mathrm{Tr}[\mathrm{e}^{-\beta H}A(t)B(t')] &= \sum_{nm}\langle E_n|\mathrm{e}^{-\beta H}A(t)|E_n\rangle\langle E_m|B(t')|E_n\rangle\\ &= \sum \mathrm{e}^{-\beta E_n}\langle E_n|A^{(t')}|E_m\rangle\langle E_m|B^{(t')}|E_m\rangle\\ &= \sum \mathrm{e}^{-\beta E_n}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(E_n-E_m)(t-t')}\langle E_n|\hat{A}|E_m\rangle\langle E_m|\hat{B}|E_n\rangle\end{aligned} \tag{6.11}$$

Then we can have

$$S_{AB}[E] = \sum_{n,m}\delta(E-(E_n-E_m))(\mathrm{e}^{\beta E}-\epsilon)\langle E_n|B|E_m\rangle\langle E_m|A|E_n\rangle\,\mathrm{e}^{-\beta E_n} \tag{6.12}$$

For the fourier transformation of the step function

$$\begin{aligned}\theta(t-t') &= \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}E\frac{\mathrm{e}^{-\mathrm{i}E(t-t')}}{E+\mathrm{i}0^{+}},\\ \theta(t'-t) &= \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}E\frac{\mathrm{e}^{-\mathrm{i}E(t-t')}}{E+\mathrm{i}0^{-}}\end{aligned}$$

Then

$$G[E] = \frac{f_e(E\cdots)}{f_2(E\cdots)} = \frac{f_1(E)}{\prod(E-E_i)} = \sum_i\frac{g_i(E)}{E-E_i+\mathrm{i}\,\mathrm{sgn}[E_i]} \tag{6.13}$$

$$f_2(E\cdots) = \sum_n(E^n a_n + E^{n-1}a_n\cdots) = \sum_{i=1}^{n}(E-E_i) \tag{6.14}$$

Then, the Hilbert transformation of the retard/advanced/canonical GF and S-matrix can be written as

$$G^{\mathrm{ret}}(E) = \int_{-\infty}^{\infty} \mathrm{d}E' \frac{S_{AB}(E')}{E - E' + \mathrm{i}0^+}, \tag{6.15}$$

$$G^{\mathrm{adv}}(E) = \int_{-\infty}^{\infty} \mathrm{d}E' \frac{S_{AB}(E')}{E - E' + \mathrm{i}0^-}, \tag{6.16}$$

$$G^{\mathrm{C}}(E) = \int_{-\infty}^{\infty} \mathrm{d}E' \frac{S_{AB}(E')}{E - E' + \mathrm{i}\,\mathrm{sgn}[E']0^+}, \tag{6.17}$$

$$S_{AB}(E) = \frac{\mathrm{i}}{2\pi}[G_{AB}(E + \mathrm{i}0^+) - G_{AB}(E - \mathrm{i}0^+)] = -\frac{1}{\pi}\,\mathrm{Im}[G^{\mathrm{ret}}_{AB}(E)]. \tag{6.18}$$

6.2 GFs continent

For arbitrary operator A and B, which could be both Fermionic or Boson

$$\mathrm{i}G^c_{AB} = -\,\mathrm{i}\big[\theta(t-t')\langle A(t)B(t')\rangle \pm \theta(t'-t)\langle B(t')A(t)\rangle\big] \tag{6.19}$$

where $\langle A(t)B(t')\rangle$ and $\langle B(t')A(t)\rangle$ is so-called the greater and lesser Green functions: $G^>$, $G^<$, and $[c, c^\dagger]_\xi = 1$. The retarded/advanced Green function are defined as

$$G^{\mathrm{ret}}_{AB} = -\,\mathrm{i}\theta(t-t')\langle [A(t), B(t')]_\xi\rangle \tag{6.20}$$

$$G^{\mathrm{adv}}_{AB} = +\,\mathrm{i}\theta(t'-t)\langle [A(t), B(t')]_{-\xi}\rangle \tag{6.21}$$

Now the solution will like

$$\frac{1}{E - E_i + \mathrm{i}\,\mathrm{sgn}(E)}$$

where $\mathrm{sgn}(E)$ can be 0^+ (retarded Green function) and 0^- (advanced green functioin). The Heaviside function satisfies the Fourier Transformation

$$\theta(t-t') = \frac{1}{2\pi}\int \mathrm{d}x \frac{\mathrm{e}^{\mathrm{i}x(t-t')}}{x + \mathrm{i}0^+}$$

If the state Φ_0 is applied to $G^<$, i.e.,

$$\langle \Phi_0 | B(t')A(t) | \Phi_0\rangle \to \mathrm{Tr}[\rho A(t)B(t')]$$

where

$$G[t, t'] = G(t-t', 0) = G(0, t-t')$$

We define

$$S_{AB}(E) = \langle A(t)B(t')\rangle \tag{6.22}$$

Then the retarded Green function becomes

$$G^{\mathrm{ret}}_{AB}(E) = \int \mathrm{d}E' \frac{S_{AB}(E)}{E - E' + i0^+} \sim \delta(E - E') \tag{6.23}$$

and we can get $S_{AB}(E)$ from the Green function

$$S_{AB}(E) = \mp\frac{1}{\pi}\,\mathrm{Im}\,G^{\mathrm{ret/adv}} \tag{6.24}$$

and $-\frac{1}{\pi}\operatorname{Im}(G[c_k, c_k^\dagger]) = A_k(\omega)$. Typically,

$$S_{AB}(t,t') = \langle A(t)B(t')\rangle, \tag{6.25}$$
$$S_{BA}(t,t') = \langle B(t')A(t)\rangle, \tag{6.26}$$

If one is measuring $k > k_F$, then $\langle c_k^\dagger, c_k\rangle = 0$, $\langle c_k, c_k^\dagger\rangle = 1$; $k < k_F$. $\langle S^+(t)S^-(t')\rangle$, $\langle S^-(t')S^+(t)\rangle$.

6.2.1 Spectral density

The commutator and anticommutator

$$S_{AB}^{(-)} = \langle\{A(t), B(t')\}\rangle,$$
$$S_{AB}^{(+)} = \langle[A(t), B(t')]\rangle,$$

If we define $\tilde{A} = A - \langle A\rangle$, $\tilde{B} = B - \langle B\rangle$, then $S_{AB}^{(+)} = S_{\tilde{A}\tilde{B}}^{(+)}$,

$$S_{AB}^{(\epsilon)}(E) = S_{\tilde{A}\tilde{B}}^{(\epsilon)}(E) + (1-\epsilon)DS(E) \tag{6.27}$$

which is so-called spectral theorem.
Expand the $\tilde{A}$, $\tilde{B}$ term

$$\langle\{(A - \langle A\rangle)(B - \langle B\rangle)\}\rangle = \langle\{A, B\}\rangle - 2\langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle$$

where we can define

$$D = -2\langle A\rangle\langle B\rangle = \frac{1}{Z}\sum_{n,m}^{E_n=E_m} \mathrm{e}^{-\beta E_n}\langle E_n|B|E_m\rangle\langle E_m|A|E_n\rangle \tag{6.28}$$

We can get the relation between S

$$S[A,B] = \frac{1}{2}(S^{(-)} + S^{(+)}), \quad S[B,A] = \frac{1}{2}(S^{(-)}S^{(+)})$$

If taking the limit

$$\lim_{E\to 0} EG^(\epsilon)(E) = (1-\epsilon)D$$

Keldysh-formula.
From the EOM of $\tilde{G}^{\pm}$

$$\mathrm{i}\,\partial_t\tilde{G}^{+,c} = \delta(t)\langle[A,B]\rangle + \Gamma^+(ee) = \delta(t)\tilde{G}^{(-)}(t=0) + \Gamma^+ \tag{6.29}$$
$$\mathrm{i}\,\partial_t G_{AB}^{-,c} = \delta(t)\langle\{A,B\}\rangle + \Gamma^- \tag{6.30}$$

The COBS are $\{\hat{u}^\alpha\}$, where$u^\alpha u^\beta = f_\gamma^{\alpha\beta}u^\gamma$. The commutator

$$\langle[u^\alpha, u^\beta] = a^{\alpha\beta}, \langle\{u^\alpha, u^\beta\}\rangle = c_\gamma^{\alpha\beta}u^\gamma\rangle$$

Since $[\langle S^+, S^z\rangle] = \langle S^x\rangle$.

6.3 Finite-T Green function & Matsubara method

Denote $\langle \quad \rangle_{\text{thermal}}$, then acting

$$\mathcal{T}\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}t H} \xleftrightarrow{t \leftrightarrow \mathrm{i}\tau} \mathrm{e}^{-\int_0^\beta H\tau \mathrm{d}\tau} \tag{6.31}$$

Starting from the thermal average

$$\langle A(t)B(t')\rangle = \langle A(t-t')B(0)\rangle \tag{6.32}$$

considering setting $t \to t - \mathrm{i}\beta$

$$Z\, \mathrm{tr}[\mathrm{e}^{-\beta H}\, \mathrm{e}^{\mathrm{i}Ht} A\, \mathrm{e}^{-\mathrm{i}Ht}\, \mathrm{e}^{+\mathrm{i}Ht'} B\, \mathrm{e}^{-\mathrm{i}Ht'}]$$

The trace $\mathrm{Tr}[AB\cdots C]$ is invariant under cyclE. The thermal average

$$\langle A(t-\mathrm{i}\beta)B(t')\rangle_{\text{thermal}} = \mathrm{Tr}[\mathrm{e}^{-\beta H}\, \mathrm{e}^{\mathrm{i}H(t-\mathrm{i}\beta)A\mathrm{e}^{-\mathrm{i}(i)}}] = \mathrm{Tr}[B(t')\, \mathrm{e}^{\mathrm{i}Ht} A\, \mathrm{e}^{-\mathrm{i}Ht}\, \mathrm{e}^{-\beta H}] = \mathrm{Tr}[\mathrm{e}^{-\beta H} B(t')A(t)] \tag{6.33}$$

Som we fund that when $t = t' = 0$

$$\langle A(\tau = \beta)B(0)\rangle_\beta = \langle A(\mathrm{i}\tau)B(0)\rangle$$

and we obtain the EOM and solution

$$-\frac{\partial}{\partial \tau}A(\tau) = [A(\tau), H], \quad A(\tau) = \mathrm{e}^{H\tau} A(0)\, \mathrm{e}^{-HE}$$

Sincee

$$\theta(\tau) = \begin{cases} 1, & \text{if } \tau > 0 \\ 0, & \text{if } \tau < 0 \end{cases}$$

Then $T_\tau\{A(\tau)B(\tau')\}$.

$$\langle A(t;)B(t')\rangle \Rightarrow \tau, G^M_{AB}(\tau,\tau') = -\langle \mathcal{T}_\tau(A(\tau, B(\tau')))\rangle = G^M_{AB}(\tau-\tau', 0)$$

$$G^M_{AB}(\tau-\tau', 0) = \epsilon G^M_{AB}(\tau - \tau' + n\beta, 0)$$

To proof that,

Proof.

$$G^M_{AB}(\tau-\tau', 0) = \mathcal{E} G^M_{AB}(\tau - \tau' + n\beta, 0) \tag{6.34}$$

then

$$\begin{aligned} G^M_{AB}(\tau-\tau'+\beta, 0) &= \mathrm{Tr}[\mathrm{e}^{-\beta H}\, \mathrm{e}^{H(\tau-\tau'+\beta)A}\, \mathrm{e}^{-H(\tau-\tau'+\beta)B}] \\ &= \mathrm{Tr}[\mathrm{e}^{-\beta H} B\, \mathrm{e}^{H(\tau-\tau')A\mathrm{e}^{-H(\tau-\tau')}}] = G^M_{AB}[B(0)A(\tau-\tau')] = \epsilon G^M_{AB}(\tau-\tau', 0) \end{aligned}$$

For boson: periodic; for fermion: antiperiodic. Then apply the Fourier transformathon, we have Matsubara freq. Then let $E \to \mathrm{i}E_n$, $E_n = 2m\pi/\beta$, or $E_n = (2n+1)\pi/\beta$, where $\tau \in (-\beta, \beta)$.

$$\int \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega t} G[\omega] \ldots \Rightarrow \sum_{n=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}E_n\tau} G[\mathrm{i}E_n](\ldots) \Rightarrow \sum_{n=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}E_n\tau} G[E_n]$$

then

$$G[E_n] = \int \mathrm{d}E' \frac{S_{AB}(E')}{\mathrm{i}E_n - E' + \mathrm{i}0^+} \tag{6.35}$$

along the contour

$$\sum_{n=-\infty}^{+\infty} f(\mathrm{i}E_n) = \oint \mathrm{d}z \frac{1}{\mathrm{e}^{\beta z} \mp 1} f(z)$$

to calculate the residue, let $\mathrm{e}^{\beta z = \pm 1}$, then $\tau \to \beta$, we have

$$\sum_n \mathrm{e}^{\mathrm{i}E_n t} G[\mathrm{i}E_n] = \int \mathrm{d}z\, \mathrm{d}z' \frac{\mathrm{e}^{\tau z} S_{AB}(E')}{(\mathrm{e}^{\beta z} \pm 1)(z - E')} = \int \mathrm{d}z\, \mathrm{d}z' f^{\mathrm{BF}}(z) \frac{S_{AB}(E')}{(z - E')}\bigg|_{\tau \to \beta} \tag{6.36}$$

SPEOM: G. □

6.4 Examples of GFs

Example 6.4.1. For fermions, the Hamiltonian

$$H = H_0 - \mu N = \sum \epsilon_k c_k^\dagger c_{k\sigma}$$

So the free fermion

$$|\Phi\rangle = \prod_{k<k_F, \epsilon_k<0} c_{k\sigma}^\dagger |0\rangle$$

then

where $n_k = \theta(k_F - k]$. THen

$$\langle \Phi | c_{k\sigma}(t) c_{k'\sigma'}^\dagger(t') | \Phi \rangle = \delta_{\sigma\sigma'} \delta_{kk'}\, \mathrm{e}^{-\mathrm{i}\epsilon_k} \langle \phi | c_{k\sigma} c_{k\sigma}^\dagger | \phi \rangle = \delta_{\sigma\sigma'} \delta_{kk'}\, \mathrm{e}^{-\mathrm{i}\epsilon_k(\tau - \tau')} (1 - n_k)$$

So, the Green function

$$G^c(k,t) = -\mathrm{i}[(1-n_k)\theta(t) - n_k\theta(-t)]\, \mathrm{e}^{-\mathrm{i}\epsilon_k t} = \begin{cases} -\mathrm{i}\theta(|k| - k_F)\, \mathrm{e}^{-\mathrm{i}\epsilon_k t}, & t > 0 : \text{particle} \\ \mathrm{i}\theta(k_F - |k|)\, \mathrm{e}^{-\mathrm{i}\epsilon_k t}, & t < 0 : \text{holes} \end{cases}$$

From the Fourier transformathon

$$G^c(k,\omega) = -\mathrm{i}\left(\frac{\theta(k_1 - k_F)}{\delta - \mathrm{i}(\omega - \epsilon_k)} - \frac{\theta(k_1 - |k|)}{\delta + \mathrm{i}(\omega - \epsilon_k)} \right) \to \frac{1}{\omega - \epsilon_k + \mathrm{i}\delta^+ \operatorname{sgn}(|k| - k_F)}$$

where $\theta(k_1 - k_F) \sim 1 - n_k$, $\theta(k_f - |k|) \to n$. i.e., the valid for any eigenstate of H

$$\mathrm{i}\, \partial_t G = \delta(t) + \epsilon G \tag{6.37}$$

Then,

$$G^c = \frac{1}{\omega - \epsilon_k + \mathrm{i}\delta^+ \operatorname{sgn}(\epsilon_k)}$$

$\prod_{k\in(k_1,k_0)} c_k^\dagger|0\rangle$

$$\frac{1-n_k}{\omega-\epsilon_k}+\frac{n_k}{\omega-\epsilon_k}\Rightarrow\frac{1}{\omega-\epsilon_k}$$

From this, we have

$$E=\sum_k \epsilon_k\langle n_k\rangle\sim\int G\epsilon\,\mathrm{d}k$$

and then, we can minimize the energy: $\delta E=0$, for S_{AB} and S_{BA}.

Example 6.4.2 (Free boson capped)**. No BEC** The Hamiltonian

$$H=\sum\omega_q b_q^\dagger b_q$$

and the Green functions

$$G[b_q,b_q^\dagger](k,\omega)=\frac{1}{\omega-\omega_q}$$

$G[\phi_q,\Phi_q]$, where $\Phi_q=b_q+b_q^\dagger$.

$$H_0=\sum_k \epsilon_k b_k^\dagger b_k$$

BEC The ground state is

$$|\Psi\rangle=(a_0^\dagger)^N|0\rangle$$

that is called the BEC. when $\epsilon_k\to 0$, $N/V=n_0\big|_{L\to\infty,N\to\infty}$. The Green function

$$G_0(k,\omega)=G(a_{k:\epsilon_k=0},a_k^\dagger)(k,\omega)=\underbrace{-\,\mathrm{i}(2\pi)^d n_0\delta(k)\delta(\omega)}_{\text{Coherent Part}}+\underbrace{\frac{1}{\omega-\epsilon_k+\mathrm{i}0^-}}_{\text{Incoherent Part}}$$

The interaction Hamiltonian

$$H_{\text{int}}=g\sum_{k,k'}a_k^\dagger a_{-k}^\dagger a_{k'}a_{-k'}$$

When acting on $|\Psi\rangle$, $k'=0$

$$H_{\text{int}}|\Psi\rangle\sim H_{\text{eff, int}}=\sum_{k\neq 0}gn_0\left(a_k^\dagger a_k+\frac{1}{2}a_k^\dagger a_{-k}^\dagger+\frac{1}{2}a_{-k}a_k\right),\quad H_{B,E}=\sum_{k\neq 0}\epsilon_k a_k^\dagger a_k$$

When acting on $\langle\Psi|>$, $k=0$

$$\langle\Psi|H_{\text{int}}$$

then $a_0|\psi_0\rangle\sim\sqrt{n_0}|\Psi_0\rangle$ In total, $H_B=H_{\text{eff, int}}+H_{B,E}$. Since Bogolinbov transformation

$$b_k=u_k a_k-v_k a_{-k}^\dagger$$

Then

$$E_k=\sqrt{\epsilon_k(c\epsilon_k+2gn_0)}\approxeq\sqrt{\frac{k^2}{2\pi}\left(\frac{k^2}{2\pi}+2gn_0\right)}\approxeq|k|\sqrt{gn_0}/m$$

which is so-called Goldstone mode. The Green function

$$\mathrm{i}\,\partial_t G=\delta+\epsilon_{k_0}G+\langle[a_{k_{0'}},_{H_{\text{inf}}}],a_{k'}^\dagger\rangle\to\langle g\sum_{k,k'}a_k^\dagger a_k a_k\delta_{k,k'}\rangle\ldots$$

... Bascially we have $H_0|\psi_0\rangle$, by adding δH, we have

$$G_0+\delta G\propto\delta H/\propto H_0\delta\psi$$

6.5 Free spin wave / magnos

* Most accurate to know the Hamiltonian of magnetic field: Applied a strong field

$$H_0 = -B \sum_j S_j^z$$

The ground state

$$|\Psi\rangle = |\uparrow\uparrow\cdots\rangle$$

Then

$$\delta H = \sum T_{ij}^{+}(S_i^+ S_j^- + S_i^- S_j^+) + \sum J_{ij}^z S_i^z S_j^z$$

Since $[S_i^+, S_j^-] = \delta_{ij} 2\langle S_i^z\rangle$, $\{S_i^+, S_i^- = 1\}$, $\{S_i^+, S_j^-\} \neq 0$, $[S_i^+, S_i^z] = S_i^+$ The correlation function

$$\mathrm{i}\,\partial_x[S^+, S_{-j}^-] = \delta_{ij} 2\langle S_i^z\rangle \delta(t) + \langle \underbrace{[S^+, H_0]}_{B\langle S_i^+, S_f^-\rangle}, S_f^-\rangle + \langle [S^+, \delta H^+], S^-\rangle$$
$$+ J^2 \langle S_j^z\rangle\langle S_i^+, S_f^-\rangle + J^z\langle S_i^z\rangle\langle S_j^+, S_f^-\rangle + \langle [S^+, \delta H^+], S^-\rangle + J_{ij}^+\langle 2S_i^z; S_j^+; S_f^-\rangle$$

Then do the Foruier Transform

$$\omega G[S_i^+, S_f^-] \approxeq 2\langle S^z\rangle + B\langle S_i^+, S_f^-\rangle + \sum_{\langle ij\rangle} J_{ij} 2\langle S_i^z\rangle\langle S_j^+, S_f^-\rangle$$

Acting ∂_t, summing over possible spin moments. Finally, this can be

$$\omega G[S^+, S^-](k, \omega) = 2\langle S^z\rangle + BG + 2\langle S^z\rangle J\tilde{\epsilon}_k$$

$$G \approxeq \frac{2\langle S^z\rangle}{\omega - B - 2J^+\epsilon_k}$$

Another:

$$H_0 = \sum_{\langle ij\rangle} J_{ij}^z S_i^z S_j^z, \quad \delta H = \sum J_{ij}^+ (S_i^+ S_j^- + \text{H.c.})$$

If $J^z < 0$, then FM; If $^z > 0$: then AFM.

$$\left|\begin{smallmatrix}\uparrow\uparrow\\ \uparrow\uparrow\end{smallmatrix}\right\rangle, \quad \left|\begin{smallmatrix}\uparrow\downarrow\\ \downarrow\uparrow\end{smallmatrix}\right\rangle$$

are all the classic type: SSB.

For Ruanfum SSB

$$H = \sum J\boldsymbol{S}\cdot\boldsymbol{S}, \quad [\rho, H] \neq 0$$

$$\omega G = 2\langle S^z\rangle + \sum_{\langle ij\rangle} J_{ij}^z \langle S_j^z\rangle G[S_i^+, S_f^-] + \sum_{\langle ij\rangle} J_{ij}^z\langle S_i^z\rangle G[S_j^+, S_f^-] + \sum J^+ 2\langle S_i^z\rangle G[S_j^+, S_f^-]$$

For FM:

$$\omega G = 2\langle S^z\rangle + 4J^z\langle S^z\rangle G + \langle S^z\rangle J^z \epsilon_k G + \langle S^z\rangle J^2\epsilon_k G$$

then

$$\omega G = 2\langle S^z\rangle + (4J^z\langle S^z\rangle + (\langle S^z\rangle J^z + \langle S^z\rangle J^z)\epsilon_k)F$$

then the Green function is

$$G = \frac{2\langle S^z\rangle}{\omega - 4J^+\langle S^z\rangle} + \epsilon_k J^z(1 + J^z/J^+)$$

so-called renormalization of (effective) mass of magon.

For the anti-ferron $\left|\begin{smallmatrix}\uparrow\downarrow\\ \downarrow\uparrow\end{smallmatrix}\right\rangle$

$$\begin{pmatrix} G[S_A^+ S_A^-] & G[S_B^+ S_A^-] \\ G[S_A^+ S_B^-] & G[S_B^+ S_B^-] \end{pmatrix}$$

Assume $k - k' = nQ$, $Q = (\pi, \pi)$.

$$\begin{pmatrix} G[k,k] & Gk+Q,k \\ G[k,k+Q] & G[k+Q,k+Q] \end{pmatrix}$$

$$\omega[G] = \begin{pmatrix} 2\langle S^z\rangle_A & \\ & 2\langle S^z\rangle_B \end{pmatrix} + \begin{pmatrix} J^z\epsilon_k & J^z\epsilon_k \\ -J^z\epsilon_k & -J^z\epsilon_k \end{pmatrix}$$

where $\epsilon_{k+Q} = -\epsilon_k$. Eventually we will find that

$$\sqrt{\epsilon_k(\epsilon_k + J^z)} \propto k$$

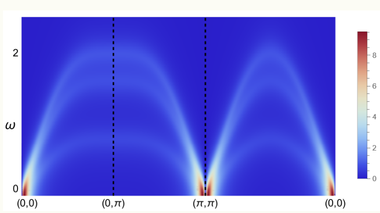
AFM: Spin-wave. In the k-space, the $\cos k$ will flipped. For FM: $\epsilon_k \propto k^2$.

Only consider the very simple ground state $|\Psi\rangle_0$.

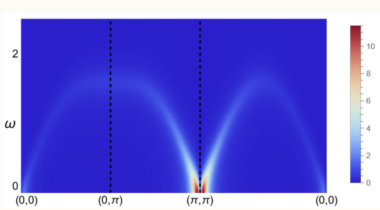
$G_{kk} \sim G_{AA}$, $k \in$ First Brillouin Zone

6.8 EXAMPLES OF GFS · 277 ·

1. Gapless points: The dispersion relation Eq.(6.8.1) becomes gapless when $J_{xy} \geq J_z$. The gapless points are $Q = (0,0)$ and (π,π).



FIGURES 6.3: AFM spin wave DSF from a single sublattice G_{AA} at different magnetizations, from top to bottom: $m_z = 0.4, 0.2, 0.1$. For $m_z = 0.1$, the strength is amplified by a factor of 2.



FIGURES 6.4: AFM spin wave DSF summed over all sublattices $S_{AA} + S_{BB} - S_{AB} - S_{BA}$. (note: it should all be plus sign. for some reason, I find it only fit experiment with minus sign for $S_{AB} = S_{BA}$.)

6.8.5 The two-spin problem by GFs

As a third example of an application, we wish to treat a model system with interactions, whose partition function can still be calculated exactly, so that all the interesting correlation functions are known in principle. This thus opens up the possibility of comparing the results of the Green's function method with the exact solutions.

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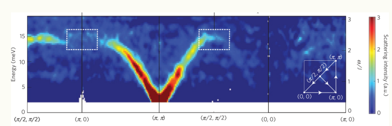
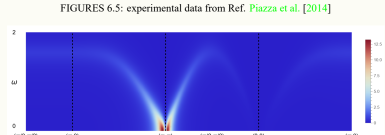


FIGURES 6.5: experimental data from Ref. Piazza et al. [2014]



FIGURES 6.6: AFM spin wave DSF summed over all sublattices $S_{AA}+S_{BB}-S_{AB}-S_{BA}$ for direct comparison with 6.4.

The model system in question consists of two spins of magnitudes

$$S_1 = S_2 = \frac{1}{2},$$

which are coupled to each other via an exchange interaction J and are presumed to be acted upon by a homogeneous magnetic field. We describe them in terms of the correspondingly simplified Heisenberg model (2.221):

$$H = -J\left(S_1^+ S_2^- + S_1^- S_2^+ + 2S_1^z S_2^z\right) - b\left(S_1^z + S_2^z\right),$$

where

$$b = \frac{1}{\hbar} g_J \mu_B B_0.$$

The limitation to $S_1 = S_2 = 1/2$ allows some simplifications:

$$S_i^{\mp} S_i^{\pm} = \frac{\hbar^2}{2} \mp \hbar S_i^z$$
$$S_i^{\pm} S_i^z = -S_i^z S_i^{\pm} = \mp\frac{\hbar}{2} S_i^{\pm}$$
$$\left(S_i^+\right)^2 = 0; \quad (S_i^z)^2 = \frac{\hbar^2}{4}$$

For our further discussion, we require several commutators:

$$\left[S_1^-, H\right]_- = -J\left[S_1^-, S_1^+\right]_- S_2^- - 2J\left[S_1^-, S_1^z\right]_- S_2^z - b\left[S_1^-, S_1^z\right]_- =$$
$$= 2\hbar J\left(S_1^z S_2^- - S_1^- S_2^z\right) - \hbar b S_1^-$$

Roton: liquid He. Roton get deeper during the salivation process.

6.6 Quasi-Particle Concept

$$\mathrm{i}\,\partial_t G_{AB} = \delta(t)\langle[A,B]\rangle_\xi(t=0) + \langle[A,H],B\rangle = \delta(t)\cdots + \epsilon_k G_0 + \Gamma \tag{6.38}$$

where $\theta(t-t')AB = \theta(t'-t)BA$. $\langle[A,H],B\rangle$ is the so-called vertex function $\Gamma = \sum G$, $H = H_0 + V$. Then, we have the Dyson equation

$$G_{k\sigma}(E) = G^0_{k\sigma} + G_k(E)^0\Sigma(k,E)G_{k\sigma}(E) \tag{6.39}$$

and we can solve the Green function formally as

$$G_{k\sigma}(E) = \frac{1}{E-\epsilon_k+\Sigma}, \quad \text{where} \quad G^0_k = \frac{1}{E-\epsilon_k} \tag{6.40}$$

6.6.1 Self energy

We assume that Σ is the self energy.

$$\Sigma = \mathrm{Re}(k,E) + \mathrm{i}\,\mathrm{Im}(k,E)$$

we consider the relation between the advanced and retard Green function

$$(G^{\text{adv}})^* = G^{\text{ret}}.$$

In particular,

$$G^{\text{ret}}_{k\sigma}(E) = \frac{(E-\epsilon_k+R)+\mathrm{i}\mathbb{1}}{(E-\epsilon_k+R)^2+\mathrm{i}\mathbb{1}^2} \tag{6.41}$$

and the equivalent one-electron spectral density

$$S_{k\sigma}(E) = -\frac{1}{\pi}\frac{\mathbb{1}}{(E-\epsilon_k+R)^2+I^2} \tag{6.42}$$

Case A $I=0$ Denote $I\to -0^+$. The δ-function becomes

$$\delta(E-E_0) = \frac{1}{\pi}\lim_{x\to 0}\frac{x}{(E-E_0)^2+x^2}$$

Then the equivalent one-electron spectral density becomes

$$S_k(E) = \delta(E-\epsilon_k+R) \tag{6.43}$$

To get the solution, let the δ-function to be 1

$$E-\epsilon_k+R=0, \Rightarrow E_i(k)$$

then, we have

$$\delta(E-\epsilon_k+R) = S(E) = \sum \alpha_i(k)\delta(E-E_i(k)), \quad \text{and} \quad \alpha_i(k) = \left|1-\frac{\partial}{\partial E}k(E)\right|^{-1}$$

where make use of

$$\delta[f(x)] = \sum_i \frac{1}{f'(x_i)}\delta(x-x_i)$$

Case B $\quad I \neq 0$

$$
\begin{aligned}
&\langle S^+|S^-\rangle \to \langle \psi(t')|\psi(t)\rangle \xrightarrow{t-t'\to\infty} \text{Const} \to 0; \\
&\langle c|c^\dagger\rangle
\end{aligned}
$$

means $|\psi\rangle = S^-|\Psi_0\rangle$ is no longer eigenstate. Particle will decay.

$$|I| \ll |\epsilon_k + R|$$

Since

$$F = \epsilon_k + R = F(E_i) + (E - E_i)\frac{\partial F}{\partial E}\bigg|_{E=E_i} + \cdots = E_i + (E - E_i)\frac{\partial F}{\partial E}\bigg|_{E=E_i} + \cdots$$

Then, the element of the density satisfies

$$S^{(i)} \approxeq \alpha_i \, \mathrm{e}^{-\mathrm{i}E_i(t-t')} \tag{6.44}$$

and we call the lifetime of quasi-particles $\mathrm{e}^{-|\alpha\mathbb{1}||t-t'|} \to \mathrm{e}^{-|t-t'|/\tau}$, where the lifetime $\tau = 1/|\alpha I|$. The Real part $\mathrm{Re}[\Sigma]$ and the imaginary part $\mathrm{Im}[\Sigma]$ are *not independent* from each other.

$$\epsilon_k \approxeq T_0 + \frac{k^2}{2m} \tag{6.45}$$

$$E_i = T_0 + \frac{k^2}{2m^*} \tag{6.46}$$

then, we have

$$E_i = T_0 + \frac{m}{m^*}(\epsilon_k - T_0),$$

and the fraction of m and m^*

$$\frac{m}{m^*} = \frac{\partial E_i(k)}{\partial \epsilon_k} = 1 + \frac{\partial R}{\partial \epsilon_k} = 1 + \frac{\partial R}{\partial E_i}\frac{\partial E_i}{\partial k} = \frac{1 - \left(\frac{\partial R}{\partial E_i}\right)_{\epsilon_k}}{1 + \left(\frac{\partial R}{\partial \epsilon_k}\right)_{E_i}}$$

Chapter 7 Landau Fermi Liquid Theory

To begin with Superfluid Helium on YouTube.
Stick stand straightly in a cup of liquid He $p \propto \frac{\partial S}{\partial n}\frac{\partial n}{\partial T}$, i.e., entropy per particle, $\frac{\partial n}{\partial T} \equiv \nabla n$ is the density graident. Ref: Volovik, Universe in a drop of He.
LFLT paradigim of many-theory. 1900, Onnes.

(a) Adiabatic: quasi-particle concept weight, effective mass

(b) Variational $\xrightarrow{\langle n\rangle \to \delta\langle n\rangle}$ functional

(c) Quantum-statistical: Landau parameters

(d) Effective theory

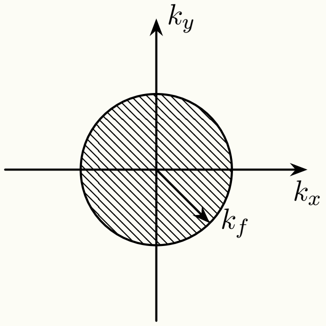
(e) Collective: mode/excitation, zero sound; first sound $\partial n/\partial\mu$.

(f) $J^{-1}(\epsilon) \propto (\epsilon^2 + \pi^2 T^2)$

(g) Degenerate Fermi gas of Sommerfeld

$$\epsilon_k = \frac{k^2}{2m} - \mu$$

let $k_f^2/2m = \mu$, then $n(\boldsymbol{k}) = \begin{cases} 1, & |k| < k_f \\ 0, & |k| > k_f \end{cases}$



The density of states at Fermi energy (or Fermi surface along the circle in the Figure above)

$$N(0) = 2\frac{(4\pi)p^2}{(2\pi\hbar)^3}\frac{\mathrm{d}p}{\mathrm{d}\epsilon_p}\bigg|_{p=p_f} = \frac{mp_f}{\pi^2\hbar^3} \tag{7.1}$$

$$N(E) = \int_{\epsilon_k=0}^{\epsilon_k=\epsilon} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^2} n(\boldsymbol{k}) = \int \mathrm{d}\epsilon\rho(\epsilon)n(\epsilon) \tag{7.2}$$

where $\rho(\epsilon)$ is the density of state. The total energy

$$E = \int \mathrm{d}\epsilon\rho(\epsilon)\epsilon n(\epsilon) \tag{7.3}$$

Similarly, the heat capacity

$$C_v = \frac{\mathrm{d}E}{\mathrm{d}T} = \frac{\pi^3}{3} N(0) k_0^2 \Big|_{C_v, T \to 0} = \gamma \tag{7.4}$$

$\partial C_v / \partial T \mid_{T \to 0}$. The eexception $\chi_c = \partial n / \partial \mu$.

(h) magnetic susceptibility.

When a magnetic field $\boldsymbol{B}$ is applied, we will get an energy diffenerce from spin-up and spin-down $\mu(N_\uparrow - N_\downarrow)\mu_B B$. The magnetic susceptibility

$$\chi = \mu_F (N_\uparrow - N_\downarrow)/B = \mu_F^2 N(0) \tag{7.5}$$

$\mu_F = \mu_B = \frac{e\hbar}{2m}$. $\mathcal{E}(\epsilon, k, \cdots)$, $\mu_B(\epsilon, k, \cdots)$. The Wilson ratio

$$W = \frac{\chi}{\gamma} = 3 \left(\frac{\mu_F}{\pi k_B} \right)^2 \tag{7.6}$$

$v_i = \frac{k^2}{2m} \approx \frac{2k_F \delta k}{2m}$, $m^* = (2.8) m_{(3\mathrm{He})}$, and the effect magnetic moment $(g^*)^2 = 3.3 (g^2)_{^3\mathrm{He}}$. $\epsilon(k) \sim m^*$.

The low energy excition of unperturbed Hamiltonian.

(a) Quasi-particle $c^\dagger_{p_0 \sigma_0}$:

$$n_{p\sigma} = \begin{cases} 1, & (p < p_f), \text{or} p = p_0, \sigma = \sigma_0 \\ 0, & \text{otherwise} \end{cases}$$

(b) Quasi-hole $c_{k_0 \sigma_0}$:

$$n_{p\sigma} = \begin{cases} 1, & p < p_f \text{except} p = p_0, \sigma = \sigma 0 \\ 0, & \text{otherwise} \end{cases}$$

(c) Particle-hole pair (create one and annihilate one at the same time).

On the fermi surface,

$$[H, n_{pf,\sigma}] = 0 \quad (p_F \in \text{FS})$$

If $p - p_F = \delta p$, then the function will linear to δp. When taking the limit

$$[H, n_{pF,\sigma}] \propto \delta p \big|_{\delta p \to 0} \to 0$$

Then, the expectation

$$\langle \Psi_0 | [H, n_p, \sigma] | \Psi_0 \propto \delta p |_{\delta \to p} \rangle$$

The residual scattering remains on the Fermi surface, which is forward scatting. The two particles

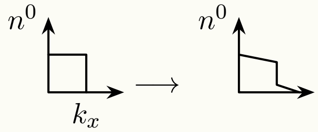
$$(\boldsymbol{p}_1, \boldsymbol{p}_2) \to (\boldsymbol{p}_1 - \boldsymbol{q}, \boldsymbol{p}_2 + \boldsymbol{q}), \quad \text{and} \quad q = 0$$

form the restrict scattering phase space, $\mathrm{Mod}(\boldsymbol{q}) = 0$. It is a narrow window, i.e., thine shell, near the fermi surface.

7.1 Change-neutral Fermi Liquid (With short range interaction: $\sim 1/\mathrm{r}^6$)

$$\delta n_{p\sigma} = n_{p\sigma} - n_{p\sigma}^{(0)}$$

from equilibrium static



Then the functional equation

$$E = E_0 + \sum_{p\sigma} (E_{p\sigma}^{(0)-\mu})\delta n_{p\sigma} + \frac{1}{2}\sum_{pp'\sigma\sigma'} f_{p\sigma,p'\sigma'}\delta n_{p\sigma}\delta n_{p'\sigma'} + \cdots \tag{7.7}$$

means that $\delta n_{p\sigma}(E_{q\sigma}^{(0)},\mu)E_{p\sigma}^{(0)}(\delta n_{p\sigma},\cdots)f(\delta n_p, E_p^{(0)},\cdots)$.

$$\left.\frac{\delta E}{\delta n_{p\sigma}}\right|_{\delta n_{p'\sigma'=0}} = \epsilon_{p\sigma}^{(0)} = E_{p\sigma}^{(0)} - \mu, \quad v_f = \left.\frac{\mathrm{d}\epsilon_p}{\mathrm{d}p}\right|_{p=p_f} = \frac{p_f}{m^*}, \quad \text{and} \quad N^*(0) = \frac{m^* p_F}{\pi^2\hbar^3} \tag{7.8}$$

Then, we can also the landau parameters as the second order perturbation

$$f_{p\sigma,p'\sigma'} = \left.\frac{\delta^2 E}{\delta n_{p\sigma}, n_{p'\sigma'}^2}\right|_{\delta n_{p''\sigma''}=0} \tag{7.9}$$

From the functional equation, we can have

$$\frac{\delta E}{\delta n_{p\sigma}} = \epsilon_{p\sigma} = \epsilon_c \epsilon_{p\sigma}^{(0)} + \sum_{p'\sigma'} f_{p\sigma,p'\sigma'}\delta n_{p'\sigma'}$$

then the entropy

$$S \approxeq -k_B \sum_{p,\sigma}[n_{p\sigma}\ln n_{p\sigma} + (1 - n_{p\sigma}\ln(1 - n_{p\sigma}))]$$

Just mathematically, no coherent term

$$\langle n_p n_{p'}\rangle_C = \langle n_p n_{p'}\rangle - \langle n_p\rangle\langle n_{p'}\rangle$$

the really defiened $S = -\rho\ln\rho = f(\langle n\rangle, \langle n^2\rangle, \langle n^3\rangle, \cdots)$.
Combine the two quantities

$$\begin{aligned} F(\{n_{p\sigma}\}) = E_0(\mu) + \sum_{p\sigma}\epsilon_{p\sigma}^{(0)}\delta n_{p\sigma} + \frac{1}{2}\sum_{pp'\sigma\sigma'} f_{pp'\sigma\sigma'}\delta n_{p\sigma}\delta n_{p'\sigma'} \\ + k_B T \sum [n_{p\sigma}\ln n_{p\sigma} + (1 - n_{p\sigma})\ln(1 - n_{p\sigma})], \end{aligned} \tag{7.10}$$

and $\delta F = 0$. The (Helmbotz) Free energy is

$$F = E - ST$$

where

$$E = \int \mathrm{d}k n_k \epsilon_k, \quad N = \int \mathrm{d}k n_k, \quad \text{and} \quad E_0 = \int \mathrm{d}k \theta(|k - k_F|) \epsilon_k$$

Since $\delta n = n_k - \theta(k - k_f)$, and $n^0 = \theta$, then

$$\delta E = \int \mathrm{d}k \epsilon_k \delta n_k$$

to do the variation δn_k

$$\delta n = \underbrace{\nu^{(0)}(k)}_{\to \delta p_f} + \sum \underbrace{\frac{\partial n^0}{\partial k}}_{\delta(k-k_F)\sim\partial_k\theta} \nu_k^{(1)} + \frac{1}{2}\frac{\partial^2 n^0}{\partial k_i \, \partial k_j} \nu^{(2)}_{k_i,k_j} + \cdots$$

7.2 Landau Fermi Liquid Theory

The energy can be written as

$$E = E_0 + \sum \epsilon^{(0)}_{p\sigma} \delta n_{p\sigma} + \frac{1}{2} \sum_{pp'\sigma\sigma'} f_{p\sigma p'\sigma'} \delta n_{p\sigma} \delta n_{p'\sigma'} \tag{7.11}$$

where $\epsilon^{(0)}_{p\sigma}$ is the functional derivation

$$\epsilon^{(0)}_{p\sigma} = \left.\frac{\delta E}{\delta n_{p\sigma}}\right|_{\text{1st order}}, \tag{7.12}$$

with $|\boldsymbol{p}| = |p_F|$ and the heat capacity $C_v \propto N^*(i)$. The Fermi velocity is defined of the Fermi surface

$$v_F = \left.\frac{\mathrm{d}\epsilon^{(0)}}{\mathrm{d}p}\right|_{P_F} = \frac{p_F}{m^*} \tag{7.13}$$

In 2nd order theory, we get the actual physical particle energy

$$\left.\frac{\delta E}{\delta n_{p\sigma}}\right|_{\text{2st order}} = \epsilon_{\rho\sigma} = \delta \epsilon^{(0)}_{\rho\sigma} + \underbrace{\sum_{p\sigma} f_{p\sigma p'\sigma'} \delta n_{p'\sigma'}}_{\text{Renormalization part}} \tag{7.14}$$

The free energy for the Fermi liquid

$$F = E - TS[\{\delta n_{p\sigma}\}] \tag{7.15}$$

Example 7.2.1. Consider the Hamiltonian

$$\mathcal{H} = \sum E_p n_{p\sigma} + \frac{1}{2} \sum V(q) c^\dagger_{p-q\sigma} c^\dagger_{p'+q\sigma'} c_{p'\sigma'} c_{p\sigma}$$

Denote $\langle \Psi | \mathcal{H} | \Psi \rangle$, where $|\Psi\rangle = |n_{p_1\sigma_1} n_{p_2\sigma_2} \cdots \rangle$. Then, the energy

$$E = \sum E_p n_{p\sigma} + \frac{\lambda}{2} \sum V(q) \langle \Psi | \cdots | \Psi \rangle$$

we can pair

(a) $c^\dagger_{p'+q\sigma'}$ with $c_{p'\sigma'}$, then $p'+q$ with p': $q=0$.

(b) $c^\dagger_{p-q\sigma}$ with $c_{p\sigma}$, then $p-q$ with p: $q=0$.

(c) $c^\dagger_{p'+q\sigma'}$ with $c_{p\sigma}$, then $p'+q=p$.

(d) $c^\dagger_{p-q\sigma}$ with $c_{p'\sigma'}$, then $p-q=p'$.

Then we will get

$$(\delta_{q=0}-\delta_{p-q,p'}\delta_{\sigma\sigma'})n_{p\sigma}n_{p'\sigma'}$$

the energy will become

$$E=\sum E_p n_{p\sigma}+\frac{\lambda}{2}\sum(V(0)-V(p-p')\delta_{\sigma\sigma'})n_{p\sigma}n_{p\sigma'}$$

with

$$f_{p\sigma p'\sigma'=\lambda(V(0)-V(p-p')\delta_{\sigma\sigma'})}$$

which is so-called the Fermion-Renormalization group. Ref: Shanker, RMP; or Polchinski on arXiv, EFT & Fermi Surface.

7.2.1 Feedback effects of interaction

$$\delta\epsilon^0_{p\sigma}=\begin{cases}-\delta\mu, & \text{isotropic enlarge / compression of Fermi surface}\\ -\sigma\mu_F B, & \text{magnetic field,}\\ \frac{1}{2}m(\boldsymbol{v}-\boldsymbol{u})^2, & \text{shift in } \boldsymbol{v}\text{, in } \boldsymbol{u}\text{, the Galilean boost.}\end{cases} \tag{7.16}$$

The actual physical particle energy

$$\delta\epsilon_{p\sigma}=\delta\epsilon^0_{p\sigma}+\sum f_{p\sigma p'\sigma'}\delta n_{p'\sigma'} \tag{7.17}$$

where

$$n_{p\sigma}=f(\epsilon^0_{p\sigma}+\delta\epsilon_{p\sigma})\underset{T\to\sigma}{\approxeq}f(\epsilon^0_p)+f'(\epsilon^{(0)}_p)\delta\epsilon_{p\sigma}=\underbrace{\theta(-\epsilon^0_p)}_{n^0_{p\sigma}}+\underbrace{[-\delta(\epsilon^0_p),\delta\epsilon_{p\sigma}]}_{\delta n_{p\sigma}}$$

i.e., $n_{p\sigma}\to n^0_{p\sigma}$, and $\delta n_{p\sigma}\to\delta n_{p'\sigma'}$. So, the actual physical particle energy can be written as

$$\delta\epsilon_{p\sigma}=\delta\epsilon^0_{p\sigma}+\sum_{p'}f_{pp'}(-\delta(\epsilon^0_{p'}))\cdot\delta\epsilon_{p'}$$

where

$$f_{pp'}=\sum_l f_l P_l(\cos\theta),\quad \delta\epsilon^{(0)}_p=v_l Y_{lm}(\theta,\phi),\quad \delta_{p\sigma}=\sum_l t_l Y_{lm}$$

then, we obtain

$$t_l=v_l-F^s_l t_l=\frac{v_l}{1+F^s_l}$$

We shall only consider the leading term, i.e. effective mass per coleman

$$v_1\propto\delta\epsilon^0_p\approxeq-m\boldsymbol{v}_F\cdot\boldsymbol{u}=-m\cos\theta p_F\propto P_1(\cos\theta)$$

Then, we have

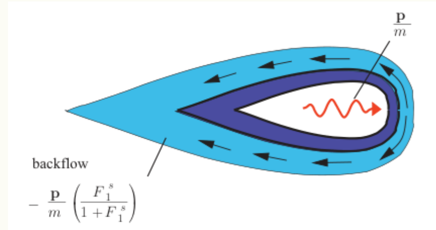
$$t_1=\frac{v_1}{1+F^S_1}$$

There will have a displacement of the fermi surface.
The effective mass satisfies

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}p} = \frac{p_F}{m^*} = \frac{p_F}{m} + \frac{F_1^s p_F}{m^*}$$

then, it can be expressed as

$$m^* = m(1 + F_1^s)$$



V.s. the band effect mass,

$$m^* = m\left(1 + \frac{1}{3}F_1^s\right)$$

7.2.2 Equilibrium Properties

We specific head

$$\delta S = \frac{1}{TV}\sum(\epsilon_p - \mu)\delta n_p, \tag{7.18}$$

$$C_V = S_1 = \frac{m^* p_F}{3\hbar^2} k_B^2 T. \tag{7.19}$$

E.Fradin. We can write

$$\delta n_\sigma(T, \mu, \epsilon_p) = \frac{\partial n}{\partial \epsilon}[-(\epsilon_p - \mu)\delta T + \delta\cancel{\epsilon_p - \delta\mu}]\ \frac{\delta S}{\delta T} \tag{7.20}$$

Then, we have

$$\delta S = -\frac{1}{V}\sum_p \frac{\partial n}{\partial \epsilon}(\epsilon_p - \mu^2)\delta T = -\frac{1}{V}\underbrace{\int p^2 \frac{\mathrm{d}p}{\mathrm{d}\epsilon}}_{3\mathrm{D}}\mathrm{d}\epsilon_p \left(\frac{\partial n}{\partial \epsilon}\right)_p (\epsilon_p - \mu)^2 \delta T$$

$$= -k_B^2 N^*(0)\int \mathrm{d}x \frac{\partial}{\partial x}\left(\frac{1}{\mathrm{e}^x + 1}\right) x^2 \delta T \tag{7.21}$$

The Fermi energy for Landau Fermi liquid $E_F \propto T_F$, then

$$T_F = \frac{p_F^2}{2m^* k_B} = \frac{\epsilon_F}{k_B} \tag{7.22}$$

Then,

$$\mu'(n, T) = -(\frac{\partial F}{\partial n})_T = \mu(n, 0) - \frac{T_1^2}{4}k_s\left(\frac{1}{3} + \frac{n}{m^*}\frac{\partial m^*}{\partial n}\right)\frac{T^2}{T_F} \tag{7.23}$$

The compressibility, $\delta\epsilon_p^{(0)=-\delta\mu}$. Then,

$$\kappa = -\frac{1}{V}\frac{\partial V}{\partial p} = \frac{1}{n^2}\frac{\partial n}{\partial \mu} \tag{7.24}$$

which is so-called the charge susceplibility. Then,

$$\delta n_p = \frac{\partial n}{\partial \epsilon_p}(\delta\epsilon_p - \delta\mu),$$

where $\delta E_p = F_0^S \delta n_{p_F}$. Substitute it

$$\delta n_p = -N(0)(F_D^s \delta n - \delta\mu) \tag{7.25}$$

Then, we can obtain the relation between δn and $\delta\mu$

$$\frac{\delta n}{\delta\mu} = \frac{\partial n}{\partial\mu} = \frac{N(0)}{1+F_0^s}. \tag{7.26}$$

The Fermi sphere get smaller

$$\delta\epsilon_p^{(0)} = -\sigma\mu_F\beta \begin{cases} \delta\mu_h = +\frac{B}{2}, \\ \delta\mu_l = -\frac{B}{2}, \end{cases} \quad , \qquad \frac{\partial S_Z}{\partial B} = \frac{\delta n_\uparrow - \delta n_\downarrow}{B}, \qquad \chi = \frac{\hbar^2}{2}\gamma^2 N(0)\frac{1}{1+p_0^s}.$$

7.2.3 Thermodynamic stability of Fermi liquid

$$n_{p\sigma}^0 = \theta(p_F(0) - |\boldsymbol{p}|) \tag{7.27}$$

Since the Gibbs free energy

$$G = E - \mu n \tag{7.28}$$

then

$$(E-\mu_n) - (E-\mu_n)_0 = \overbrace{\frac{1}{V}\sum(\epsilon_p^0 - \mu)\delta n_p}^{\propto\, \delta p_F(\theta)} + \frac{1}{2V^2}\sum f_{p\gamma'}\delta n_p \delta n_{p'} \tag{7.29}$$

and

$$\delta n_p = n_p - n_p^0 = \delta p_F \delta(p_F - |\boldsymbol{p}|) - \frac{1}{2}(\delta p_F)^2 \frac{\partial}{\partial p}\left((p_F - p)\right) \tag{7.30}$$

where

$$\delta p_F(\theta) = p_F(\theta) - p_F^0(\theta) \longrightarrow v_F \delta p_F(\theta) = \sum_{l=0}^{\infty} v_l P_l(\cos\theta)$$

The 1st term

$$\delta - \mu\delta n = \sum \frac{N(0)}{(2l+1)\delta}\left[(v_{l\uparrow}+v_{l\downarrow})^2\left(1+\frac{F_l^s}{2l+1}\right) + (v_{l\uparrow}-v_{l\downarrow})^2\left(1+\frac{Fl^A}{2l+1}\right)\right].$$

We should find the Fermi-stability condition

$$\delta E - \mu\delta n > 0, \quad \text{and} \quad 1 + \frac{F_l^{S,A}}{2l+1} \geq 0, \tag{7.31}$$

and the Pometanchuk inequility (transition)

$$F_l^{S,A} \geq -(2l+1). \tag{7.32}$$

If p-inequility, violated, then $\delta p_F = v_{l_0}$, which will change the Fermi surface shape.

7.2.4 Non-Equilibrium Properties

Steady state

7.2.5 Changed Fermi liquid: Landau-Silin theory

$\delta n_p(x)$:

$$\nabla^2\phi_p = \frac{e}{\epsilon^0}\sum_{p'}\delta n_p(x) = \frac{e}{\epsilon^0}\delta n(x) \tag{7.33}$$

We can have

$$\epsilon_{p\sigma}(x) = \epsilon_p^{(0)} + e\phi_p + \sum_{p'\sigma'} f_{p\sigma p'\sigma'}\delta n_{p'}(x) \tag{7.34}$$

Then, take the Fourier transform

$$\delta\epsilon_{p\sigma}(q) = e\phi_p(q) + \sum_{p'\sigma'} f_{p\sigma p'\sigma'}\underbrace{\delta n_{p'\sigma'}(\epsilon)}_{\delta n(\delta\epsilon_p)} = \sum_{p'\sigma'}\left(\frac{e^2}{\epsilon_0 q^2} + f_{p\sigma p'\sigma'}\right)\delta n_{p'\sigma'}(q) \tag{7.35}$$

where

$$\tilde{f}_{p\sigma p'\sigma'}(q) = \frac{e^2}{\epsilon_0 q^2} + f_{p\sigma p'\sigma'} = \frac{N^*(0)}{1+\left(\frac{e^2 N^*(0)}{\epsilon_0 q^2} + F_0^s\right)} \equiv \chi_c(q) \propto \frac{\partial n}{\partial \mu}$$

where we can approximately write

$$1 + \left(\frac{e^2 N^*(0)}{\epsilon_0 q^2} + F_0^s\right) \propto 1 + \frac{\kappa^2}{q^2}$$

with $q \to 0$. Then,

$$\delta\mu = \delta\mu\cdot\delta(x)9, \quad \text{the Foruier Transform} \quad \delta n(x-x_0) = \mathcal{F}(\delta n(q)) \to \delta n(x)\delta n(x_0) \sim -\kappa\,\mathrm{e}^{-\kappa|x-x_0|}$$

where

$$\kappa \approxeq \frac{e^2 N^*(0)}{\epsilon_0}\frac{1}{2}\left(1+F_0^S\right)$$

The above is the so-called Thomas-Fermi screening.
Consider the quasi-particle length scale

$$l \gg \frac{\hbar}{\Delta p} \sim \frac{\hbar v_F}{k_B T} \tag{7.36}$$

$\delta n_p(x)$ is the Wigner distribution function, where $p \to \langle p\rangle$ and $x \to \langle x\rangle$. The wavepacket of δn_p

$$W(\boldsymbol{r}_1,\sigma_1;\boldsymbol{r}_1,\sigma_2) = \int \mathrm{d}\boldsymbol{p}_1^3\,\mathrm{d}\boldsymbol{p}_2^3\,\mathrm{e}^{\mathrm{i}/\hbar(\boldsymbol{p}_1\boldsymbol{r}_1-\boldsymbol{p}_2\boldsymbol{r}_2)}\langle a^\dagger_{p_2\sigma_2}(t)a_{p_1\sigma_1}(t)\rangle \tag{7.37}$$

The wave function of q and p: $U^\dagger c^\dagger_{p\sigma}U$. Ref: Fradkin's.

7.2.6 Kinetic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta n_{p\sigma}(\boldsymbol{r},t) = 0 \tag{7.38}$$

where

$$[\delta n_{p=p_F}, H] = 0, \quad \text{and} \quad [\delta n_{p\neq p_F}, H] \neq 0$$

Then, q.p. will decay $\to$ q.p. collision, cause the quasi-particle lifetime

$$p_1 \leftrightarrow p_2 : p_3,\ p_4.$$

If we take the collisionless limit, then only achievable of $T = 0$. Then term $I(\delta n_{p'})$, the collision integral, will describe the process of quasi-particle lifetime.
The derivatives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\delta n_{\boldsymbol{p}}(\boldsymbol{r},t) &= \frac{\partial}{\partial t}\delta n + \frac{\partial}{\partial \boldsymbol{r}}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\delta n_{\boldsymbol{p}}(\boldsymbol{r},t)\right) + \frac{\partial}{\partial \boldsymbol{p}}\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\delta n_{\boldsymbol{p}}(\boldsymbol{r},t), \\
\frac{\mathrm{d}\boldsymbol{r}_p}{\mathrm{d}t} &= \boldsymbol{v}_{\boldsymbol{p}} = \frac{\partial \epsilon_{\boldsymbol{p}}(\boldsymbol{r},t)}{\partial \boldsymbol{p}}, \\
\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \boldsymbol{f}_{\boldsymbol{p}}(\boldsymbol{r},t) = -\frac{\partial}{\partial \boldsymbol{r}}\epsilon_p(\boldsymbol{r},t)
\end{aligned}$$

Then, the Landau's kinetic equation becomes

$$\frac{\partial}{\partial t}\delta n_{\boldsymbol{p}}(\boldsymbol{r},t) - \{\epsilon_{\boldsymbol{p}}(\boldsymbol{r},t), \delta n_{\boldsymbol{p}}(\boldsymbol{r},t)\} = \begin{cases} 0, & T \to 0, \\ T(\delta n_{p'}), & T \text{ is finite} \end{cases} \tag{7.39}$$

where the Poisson brackets

$$\{\epsilon_p, \delta n_p\}_{\mathrm{PB}} = \frac{\partial}{\partial \boldsymbol{r}}\epsilon_p \cdot \frac{\partial}{\partial \boldsymbol{p}}\delta n_p - \frac{\partial \epsilon_p}{\partial \boldsymbol{p}} \cdot \frac{\partial \delta n_p}{\partial \boldsymbol{r}}$$

The external potential $U(\boldsymbol{r},t)$ satisfies

$$\int \mathrm{d}\boldsymbol{r} U(\boldsymbol{r},t)\delta n_{\boldsymbol{p}}(\boldsymbol{r},t)$$

Then, the derivative becomes

$$\frac{\partial \epsilon_{\boldsymbol{r}}}{\partial \boldsymbol{r}} = \frac{\partial U}{\partial \boldsymbol{r}} + \int \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi\hbar)^3} f_{pp'}\frac{\partial}{\partial \boldsymbol{r}}\delta n_p(\boldsymbol{r},t)$$

To linearize this equation: keep to the 1st order derivative in $\boldsymbol{r}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta n = \frac{\partial}{\partial t}\delta n + \frac{\partial}{\partial \boldsymbol{r}}(\boldsymbol{v}_p \cdot \delta n) + \frac{\partial}{\partial p} - \frac{\partial \epsilon}{\partial \boldsymbol{r}} \cdot \delta n = 0$$

where $\epsilon = \epsilon^0 + \delta\epsilon$, and the term

$$\frac{\partial \epsilon}{\partial \boldsymbol{r}} \cdot \delta n \sim \frac{\partial \delta\epsilon}{\partial \boldsymbol{r}} \cdot \frac{\partial}{\partial \boldsymbol{p}}\delta \boldsymbol{n}$$

where $\frac{\partial}{\partial \epsilon}\frac{\partial \epsilon}{\partial p}\delta$ the first derivative act on δn. Then $\frac{\mathrm{d}}{\mathrm{d}t}\delta n$ satisfies

$$\frac{\partial}{\partial t}\delta n + \boldsymbol{v}_p\frac{\partial}{\partial \boldsymbol{r}}\delta n - \frac{\partial}{\partial \epsilon}n_p^0\delta\epsilon_p = \begin{cases} 0, \\ T(\delta n) \end{cases}$$

Apply the Foruier transformation to $t\,\mathrm{e}^x \Rightarrow \omega,\ q$

$$(\omega - \boldsymbol{q}, \boldsymbol{v}_{\boldsymbol{p}})\delta n_{\boldsymbol{p}}(\boldsymbol{q},\omega) - \boldsymbol{q}, \boldsymbol{v}_{\boldsymbol{p}}\frac{\partial n^0}{\partial \epsilon_p}\delta\epsilon_p(\boldsymbol{q},\omega) = \Big\{ 0,$$

The conservation laws

$$n(\boldsymbol{r},t)=\int \mathrm{d}^3\boldsymbol{p}n_p(\boldsymbol{r},t)$$

Since $\int \mathrm{d}^3\boldsymbol{p}[t,n_{p'}]=0$, $\int \mathrm{d}p\frac{\partial}{\partial p}[f_p,n_p]=0$, and the continuity equation

$$\frac{\partial}{\partial t}n(\boldsymbol{r},t)+\nabla\cdot\boldsymbol{j}=0$$

then we have

$$\boldsymbol{j}=\sum_{p,\sigma}\boldsymbol{v}_{\boldsymbol{p}}\cdot n_{\boldsymbol{p},\sigma}=\sum\nabla_p\epsilon_p\cdot n_p \tag{7.40}$$

The energy conservation law (thermal transport $\Delta U(\boldsymbol{r},t)$)

$$\frac{\partial}{\partial t}(E-Un)-\nabla\boldsymbol{j}_F=-\boldsymbol{j}\cdot\nabla\boldsymbol{u} \tag{7.41}$$

Then, we have

$$\boldsymbol{j}_E=\int \mathrm{d}p\nabla_r\epsilon_p(\boldsymbol{r},t)(\epsilon_p-U)n_{p\sigma}$$

and also the graident energy.

7.3 Collective modes: zero sound

$$\delta n_p(\boldsymbol{r},t)=\sum_q\delta n_p(q,\omega)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)}$$

and the potential

$$U(\boldsymbol{r},t)=U\,\mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)}$$

The frequency satisfies

$$(\omega-\boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{p}})\delta n_p(q,\omega)+\frac{\partial n_p^0}{\partial\epsilon_p}\boldsymbol{q}\cdot\boldsymbol{v}_p(u+\int \mathrm{d}p'f_{pp'}\delta n_{p'}(q,\omega))=0 \tag{7.42}$$

where

$$\delta n_{\boldsymbol{p}}=-\frac{\partial n_p}{\partial\epsilon_p}v_{\boldsymbol{p}},\quad\text{and}\quad v_{\boldsymbol{p}}=\sum_{l=0}^{\infty}v_lP_l(\cos\theta)$$

Substitute them

$$v_p+\frac{\boldsymbol{q}\cdot\boldsymbol{v}_p}{\omega-\boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{p}}}\int \mathrm{d}p'f_{pp'}\frac{\partial n^0}{\partial\epsilon_{\boldsymbol{p}}}=\frac{\boldsymbol{q}\cdot\boldsymbol{v}_p}{\omega-\boldsymbol{q}\cdot\boldsymbol{U}_{\boldsymbol{p}}}U$$

Then, we have the single frequency

$$S=\frac{\omega}{qv_F},\quad\text{and}\quad\Omega_{ll'}(S)=\frac{1}{2}\int_{-1}^{1}\mathrm{d}xP_l(x)\frac{x}{x-S}P_{l'}(x)$$

To diagonal l

$$\frac{v_l}{2l+1}+\sum_{l'=0}^{\infty}\Omega_{ll'}F_{l'}^s\frac{v_{l'}}{2l'+1}=-\Omega-l_0U$$

take $l' = l = 0$

$$\Omega_{00} = 1 + \frac{S}{2}\ln\left|\frac{S-1}{S+1}\right| + \mathrm{i}\frac{\pi}{2}S\theta(1-|S|).$$

When S is small, we will omit the imaginary part

$$v_0(s) = -\frac{\Omega_{00}(s)U}{1+F_0^S\Omega_{00}(S)}$$

Take $0 \leq S < 1$ (ıt sound damped), then

$$V_0(S) \to \delta n_p(x,t) \sim \delta n_p^{S_0}(x,t) \propto \mathrm{e}^{-S},$$

which is the so-salled $p-n$-continuity. When $S > 1$,

$$1 + F_0^{(S)}\Omega_{00}(S_0) = 0, \quad \text{zero sound}$$

means that the Fermit liquid self is resonatiny with external driving force, corresponding to an collective eigenmode, and there is none dissipative in the collisonless limit.
The frequency $S_0 = \frac{\omega_0}{qv_F}$, i.e., the disspresion relation of the collective eigenmode.

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Lecture #1 Homework #1 [2025-09-02]

Problem 1.1. Prove the following equations

$$[\hat{d}_n, \hat{d}_m^\dagger]_\mp = \langle n|m\rangle = \delta_{nm},$$
$$\hat{d}_n|n\rangle = |0\rangle.$$

Solution. *Proof.* Reviewing the definitions of $\hat{d}_n^\dagger$ and $\hat{d}_n$

$$\hat{d}_n = \int \mathrm{d}x \varphi_n^*(x)\hat{\psi}(x), \quad \text{and} \quad \hat{d}_n^\dagger = \int \mathrm{d}x \varphi_n(x)\hat{\psi}^\dagger(x),$$

then substitute them into the (anti)commutator

$$\begin{aligned}[\hat{d}_n, \hat{d}_m^\dagger]_\mp &= \left(\int \mathrm{d}x\varphi_n^*(x)\hat{\psi}(x)\right)^\dagger \left(\int \mathrm{d}x'\varphi_m(x')\hat{\psi}(x')\right) \mp \left(\int \mathrm{d}x'\varphi_m(x')\hat{\psi}(x')\right)\left(\int \mathrm{d}x\varphi_n^*(x)\hat{\psi}(x)\right)^\dagger \\ &= \int \mathrm{d}x \int \mathrm{d}x' \varphi_n^*(x)\varphi_m(x')[\hat{\psi}^\dagger(x)\hat{\psi}(x') \mp \hat{\psi}(x')\hat{\psi}^\dagger(x)] \\ &= \int \mathrm{d}x \int \mathrm{d}x' \varphi_n^*(x)\varphi_m(x')[\hat{\psi}^\dagger(x), \hat{\psi}(x')]_\mp.\end{aligned}$$

Since the (anti)commutator of the field operator is

$$[\hat{\psi}^\dagger(x), \hat{\psi}(x')]_\mp = \delta(x - x'),$$

integrate over x', and then recognize the inner product expression. Since the functions φ_n, φ_m are orthonormal, we arrive at

$$[\hat{d}_n, \hat{d}_m^\dagger]_\mp = \int \mathrm{d}x \int \mathrm{d}x' \, \varphi_n^*(x) \, \varphi_m(x')\delta(x - x') = \int \mathrm{d}x\varphi_n^*(x)\varphi_m(x) = \langle\varphi_n|\varphi_m\rangle = \delta_{nm}.$$

The integration above used the *Dirac measure* of the δ-function. □

Proof. The one-particle ket $|n\rangle$ can be expanded in the position-spin kets using the identity

$$|n\rangle = \int \mathrm{d}x|x\rangle\langle x|n\rangle = \int \mathrm{d}x\varphi_n(x)|x\rangle = \int \mathrm{d}x\varphi_n(x)\hat{\psi}^\dagger(x)|0\rangle.$$

Since $\hat{\psi}(x)|0\rangle = 0$, that is *the annihilation operator kills the vacuum*, then we have

$$\hat{\psi}(x)\hat{\psi}^\dagger(x')|0\rangle = [\hat{\psi}(x), \hat{\psi}^\dagger(x')]_\mp|0\rangle = \delta(x - x')|0\rangle,$$

then substitute the relation above, the expression of $\hat{d}_n$ and the expansion of $|n\rangle$ into $\hat{d}_n|n\rangle$

$$\begin{aligned}\hat{d}_n|n\rangle &= \int \mathrm{d}x\varphi_n^*(x)\hat{\psi}(x) \int \mathrm{d}x'\varphi_n(x')\hat{\psi}^\dagger(x')|0\rangle = \int \mathrm{d}x \int \mathrm{d}x' \, \varphi_n^*(x) \, \varphi_n(x')\delta(x - x') \, |0\rangle \\ &= \int \mathrm{d}x\varphi_n^*(x)\varphi_n(x)|0\rangle = \int \mathrm{d}x|\varphi_n(x)|^2|0\rangle = |0\rangle.\end{aligned}$$

The integration above used *the sifting property of the δ-function*, and the normalization of $\varphi_n(x)$. □

Problem 1.2. Prove the inverse relations (3.6.4)[1] i.e., the inverse relations of

$$\hat{\psi}(\boldsymbol{x}) = \sum_n \varphi_n(\boldsymbol{x})\hat{d}_n, \quad \text{and} \quad \hat{\psi}^\dagger(\boldsymbol{x}) = \sum_n \varphi_n^*(\boldsymbol{x})\hat{d}_n^\dagger,$$

meaning expressing $\hat{d}_n$ and $\hat{d}_n^\dagger$ by $\hat{\psi}(x)$.

Solution. To express $\hat{d}_n$ (and $\hat{d}_n^\dagger$), we can use the orthogonality $\int \mathrm{d}x\varphi_m^*(\boldsymbol{x})\varphi_n(\boldsymbol{x}) = \delta_{mn}$ to eliminate itself on the right side. Left multiply both sides of $\varphi_m^*(\boldsymbol{x})$ by $\hat{\psi}(\boldsymbol{x})$, and then integrate over x

$$\int \mathrm{d}x\varphi_m^*(\boldsymbol{x})\hat{\psi}(\boldsymbol{x}) = \int \mathrm{d}x\varphi_m^*(\boldsymbol{x})\left(\sum_n \varphi_n(\boldsymbol{x})\hat{d}_n\right).$$

Since the integral and sum actions are exchangeable in this case, then we arrive at

$$\int \mathrm{d}x\varphi_m^*(\boldsymbol{x})\hat{\psi}(\boldsymbol{x}) = \sum_n \hat{d}_n \int \mathrm{d}x\varphi_m^*(\boldsymbol{x})\varphi_n(\boldsymbol{x}) = \sum_n \delta_{mn}\hat{d}_n = \hat{d}_m.$$

Similarly, for the creation operator $\hat{d}_n^\dagger$, repeat the actions above to $\hat{\psi}(\boldsymbol{x})^\dagger$ as following

$$\int \mathrm{d}x\varphi_m(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})^\dagger = \sum_n \hat{d}_n^\dagger \int \mathrm{d}x\varphi_m(\boldsymbol{x})\varphi_n(\boldsymbol{x})^* = \sum_n \delta_{mn}\hat{d}_n^\dagger = \hat{d}_m^\dagger.$$

To summarize, the inverse relations of $\hat{\psi}(\boldsymbol{x})$ and $\hat{\psi}(\boldsymbol{x})^\dagger$ for mode n are given by

$$\hat{d}_n = \int \mathrm{d}x\varphi_n^*(\boldsymbol{x})\hat{\psi}(\boldsymbol{x}), \quad \text{and} \quad \hat{d}_n^\dagger = \int \mathrm{d}x\varphi_n(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})^\dagger.$$

Problem 1.3. Let $|n\rangle = |\boldsymbol{p}\tau\rangle$ be a momentum-spin ket so that $\langle \boldsymbol{x}|\boldsymbol{p}\tau\rangle = \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\delta_{\sigma\tau}$.[2]

$$\langle \boldsymbol{x}|\boldsymbol{p}\sigma'\rangle = \delta_{\sigma\sigma'}\langle \boldsymbol{r}|\boldsymbol{p}\rangle \quad \text{with} \quad \langle \boldsymbol{r}|\boldsymbol{p}\rangle = \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}.$$

Show that the (anti)commutation relation in (3.6.2) then reads

$$[\hat{d}_{\boldsymbol{p}\tau}, \hat{d}^\dagger_{\boldsymbol{p}',\tau'}]_\mp = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{\tau\tau'},$$

and that the expansion (3.6.4) of the field operators in terms of the $\hat{d}$-operators is

$$\hat{\psi}(\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\,\hat{d}_{\boldsymbol{p}\sigma}, \quad \text{and} \quad \hat{\psi}^\dagger(\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\,\hat{d}^\dagger_{\boldsymbol{p}\sigma}.$$

Solution. *Proof.* When $|n\rangle = |\boldsymbol{p}\tau\rangle$, $\varphi_n(x)$ becomes $\varphi_{\boldsymbol{p}\tau}(\boldsymbol{x}) = \langle \boldsymbol{x}|\boldsymbol{p}\tau\rangle = \delta_{\sigma\tau}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}$. Using the combined coordinate $\boldsymbol{x} \equiv (\boldsymbol{r}, \sigma)$ that provides a complete description of a particle's state: continuous spatial position $\boldsymbol{r}$ and discrete spin projection quantum number σ. The expression of the operator $\hat{d}_{\boldsymbol{p}\tau}$ in terms of $\hat{\psi}(\boldsymbol{r}, \sigma)$ should be *not only integrate over* $\boldsymbol{r}$*, but also sum over spin* σ. That is

$$\hat{d}_{\boldsymbol{p}\tau} = \int \mathrm{d}^3\boldsymbol{x}\varphi^*_{\boldsymbol{p}\tau}(\boldsymbol{x})\hat{\psi}(\boldsymbol{x}) = \int \mathrm{d}^3\boldsymbol{r}\sum_\sigma \delta_{\sigma\tau}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\hat{\psi}(\boldsymbol{r},\sigma) = \int \mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\hat{\psi}(\boldsymbol{r},\tau),$$

$$\hat{d}^\dagger_{\boldsymbol{p}\tau} = \int \mathrm{d}^3\boldsymbol{x}\varphi_{\boldsymbol{p}\tau}(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})^\dagger = \int \mathrm{d}^3\boldsymbol{r}\sum_\sigma \delta_{\sigma\tau}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\hat{\psi}(\boldsymbol{r},\sigma)^\dagger = \int \mathrm{d}^3\boldsymbol{r}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\hat{\psi}(\boldsymbol{r},\tau)^\dagger.$$

[1] Stefanucci and Leeuwen [see 5, eq. (1.60)]
[2] [see 5, eq. (1.10)]

Promoting $[\hat{\psi}^\dagger(x), \hat{\psi}(x')]_\mp = \delta(x-x')$ to "4D" (means 3D coordinate r + 1D spin σ (or the variable τ))

$$[\hat{\psi}^\dagger(\boldsymbol{x}), \hat{\psi}(\boldsymbol{x}')]_\mp = \delta^3(\boldsymbol{r}-\boldsymbol{r}')\delta_{\tau\tau'},$$

we get a "4D" δ-result. Then substitute $\hat{d}_{\boldsymbol{p}\tau}$ and $\hat{d}^\dagger_{\boldsymbol{p}\tau}$ into the commutator

$$\begin{aligned}
[\hat{d}_{\boldsymbol{p}\tau}, \hat{d}^\dagger_{\boldsymbol{p}',\tau'}]_\mp &= \int \mathrm{d}^3\boldsymbol{r} \int \mathrm{d}^3\boldsymbol{r}'\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{r}-\boldsymbol{p}'\cdot\boldsymbol{r}')}[\hat{\psi}(\boldsymbol{r},\tau), \hat{\psi}(\boldsymbol{r}',\tau')^\dagger]_\mp \\
&= \int \mathrm{d}^3\boldsymbol{r} \int \mathrm{d}^3\boldsymbol{r}'\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{r}-\boldsymbol{p}'\cdot\boldsymbol{r}')}\delta^3(\boldsymbol{r}-\boldsymbol{r}')\delta_{\tau\tau'} \\
&= \int \mathrm{d}^3\boldsymbol{r}\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{r}-\boldsymbol{p}'\cdot\boldsymbol{r})}\delta_{\tau\tau'} = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{\tau\tau'}.
\end{aligned}$$

The last line of the above derivation used the normalization of the plane waves, with a factor of $(2\pi)^3$. □

Proof. Similar to Problem 1.1, we need to left multiply both sides by $\varphi_{p\tau}(\boldsymbol{x})$, sum over τ, and integrate over $\boldsymbol{p}$ to $\hat{d}_{\boldsymbol{p}\tau}(\boldsymbol{x}')$ (In order to arrive at $\hat{\psi}(\boldsymbol{x})$, a normalization factor $1/(2\pi)^3$ is applied)

$$\sum_\tau \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\varphi_{p\tau}(\boldsymbol{x})\hat{d}_{\boldsymbol{p}\sigma} = \sum_\tau \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\varphi_{p\tau}(\boldsymbol{x}) \int \mathrm{d}^3\boldsymbol{x}'\; \varphi^*_{\boldsymbol{p}\tau}(\boldsymbol{x}')\; \hat{\psi}(\boldsymbol{x}').$$

We arrive at the identity of $\varphi_{p\tau}(\boldsymbol{x})$. Now, calculate it.

$$\begin{aligned}
\text{Identity} &= \sum_\tau \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\varphi_{p\tau}(\boldsymbol{x})\varphi^*_{\boldsymbol{p}\tau}(\boldsymbol{x}') = \sum_\tau \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}(\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\delta_{\sigma\tau})(\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}'}\delta_{\sigma'\tau}) \\
&= \sum_\tau \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\delta_{\sigma\tau}\delta_{\sigma'\tau} = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\delta_{\sigma\sigma'} = \delta^3(\boldsymbol{r}-\boldsymbol{r}')\delta_{\sigma\sigma'},
\end{aligned}$$

When handling $\delta_{\sigma\tau}\delta_{\sigma'\tau}$, we can consider choosing all of the τs (points) that equal to σ and σ' at the same time, it is equal to find filtering all the points on the axis that satisfy $\sigma = \sigma'$, that is $\sum_\tau \delta_{\sigma\tau}\delta_{\sigma'\tau} = \delta_{\sigma\sigma'}$.

Now, return to the main topic. Substituting the identity and expand $\int \mathrm{d}^3\boldsymbol{x}$

$$\sum_\tau \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\varphi_{p\tau}(\boldsymbol{x})\hat{d}_{\boldsymbol{p}\sigma} = \sum_{\sigma'} \int \mathrm{d}^3\boldsymbol{r}'\hat{\psi}(\boldsymbol{r}',\sigma')\delta^3(\boldsymbol{r}'-\boldsymbol{r})\delta_{\sigma\sigma'} = \hat{\psi}(\boldsymbol{r},\sigma) = \hat{\psi}(\boldsymbol{x}),$$

that is

$$\hat{\psi}(\boldsymbol{x}) = \sum_\tau \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}(\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\delta_{\sigma\tau})\hat{d}_{\boldsymbol{p}\sigma} = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\hat{d}_{p\sigma}.$$

Similarly to $\hat{\psi}^\dagger(\boldsymbol{x})$. To summarize

$$\hat{\psi}(\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\hat{d}_{\boldsymbol{p}\sigma}, \quad \text{and} \quad \hat{\psi}^\dagger(\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}}\hat{d}^\dagger_{\boldsymbol{p}\sigma}.$$

□

Lecture #2 Homework #2 [2025-09-09]

Problem 2.1. Prove

$$\hat{H}_{\text{int}} = \frac{1}{2}\int \mathrm{d}x \int \mathrm{d}x' v(x,x')\hat{n}(x)\hat{n}(x') - \frac{1}{2}\int \mathrm{d}x v(x,x)\hat{n}(x),$$

from $\hat{H}_{\text{int}}|x_1 \dots x_n\rangle = \left(\frac{1}{2}\sum_{i\neq j} v(x_i,x_j)\right)|x_1 \dots x_N\rangle$ as hinted in class, beginnning with inserting $\int \mathrm{d}x \sum_i \delta(x-x_i)$ to the original, first-quantization expression and make use of Eq. (3.8.5).

$$\begin{aligned}\hat{n}(x)\underbrace{\hat{\psi}^\dagger(x_N)\hat{\psi}^\dagger(x_{N-1})\dots\hat{\psi}^\dagger(x_1)|0\rangle}_{|x_1\dots x_N\rangle} &= [\hat{n}(x),\hat{\psi}^\dagger(x_N)]_-\hat{\psi}(x_{N-1})\dots\hat{\psi}^\dagger(x_1)|0\rangle \\ &+ \hat{\psi}^\dagger(x_N)[\hat{n}(x),\hat{\psi}^\dagger(x_{N-1})]_-\dots\hat{\psi}^\dagger(x_1)|0\rangle \\ &\vdots \\ &+ \hat{\psi}^\dagger(x_N)\hat{\psi}^\dagger(x_{N-1})\dots[\hat{n}(x),\hat{\psi}^\dagger(x_1)]_-|0\rangle \\ &= \underbrace{\left(\sum_{i=1}^{N}\delta(x-x_i)\right)}_{\substack{\text{density of } N \text{ particles}\\ \text{in } x_1,\dots,x_N}}|x_1\dots x_N\rangle.\end{aligned} \tag{3.8.5}$$

Solution. *Proof.* Rewrite the first-quantized interaction $\frac{1}{2}\sum_{i\neq j} v(x_i,x_j)$ first: separate it into two terms

$$\sum_{i\neq j} v(x_i,x_j) = \sum_{i,j} v(x_i,x_j) - \sum_i v(x_i,x_i),$$

then inserting $\int \mathrm{d}x \sum_i \delta(x-x_i)$ to the two terms above

$$\begin{aligned}\sum_{i,j} v(x_i,x_j) &= \int \mathrm{d}x \int \mathrm{d}x' v(x,x') \sum_i \delta(x-x_i)\sum_j \delta(x'-x_j), \\ \sum_i v(x_i,x_i) &= \int \mathrm{d}x v(x,x)\sum_i \delta(x-x_i).\end{aligned}$$

To simplify, define $|\mathcal{X}\rangle = |x_1\dots x_N\rangle = \hat{\psi}^\dagger(x_N)\dots\hat{\psi}^\dagger(x_1)|0\rangle$.

1. To generate the first term of $\hat{H}_{\text{int}}$, acting $\hat{n}(x)$ and $\hat{n}(x')$ on $|\mathcal{X}\rangle$, and use the eigenvalue of (3.8.5)

   $$\hat{n}(x')\hat{n}(x)|\mathcal{X}\rangle = \sum_i \delta(x-x_i)\sum_j \delta(x'-x_j)|\mathcal{X}\rangle,$$

   then multiply $v(x,x')$ and integrate over x and x' and using the sifting property of δ-function

   $$\begin{aligned}\int \mathrm{d}x \int \mathrm{d}x' v(x,x')\hat{n}(x)\hat{n}(x')|\mathcal{X}\rangle &= \int \mathrm{d}x \int \mathrm{d}x' v(x,x')\sum_i \delta(x-x_i)\sum_j \delta(x'-x_j)|\mathcal{X}\rangle \\ &= \sum_{i,j} v(x_i,x_j)|\mathcal{X}\rangle,\end{aligned}$$

   match the first term of $\hat{H}_{\text{int}}$.

2. Concerning the second term of $\hat{H}_{\text{int}}$, acting $\hat{n}(x)$ on $|\mathcal{X}\rangle$, then multiply $v(x,x)$ and integrate over x

$$\int \mathrm{d}x v(x,x)\hat{n}(x)|\mathcal{X}\rangle = \int \mathrm{d}x v(x,x)\sum_i \delta(x-x_i)|\mathcal{X}\rangle = \sum_i v(x_i,x_i)|\mathcal{X}\rangle.$$

In summary,

$$\begin{aligned}\left(\frac{1}{2}\int \mathrm{d}x\,\mathrm{d}x' v(x,x')\hat{n}(x)\hat{n}(x') - \frac{1}{2}\int \mathrm{d}x v(x,x)\hat{n}(x)\right)|\mathcal{X}\rangle &= \frac{1}{2}\left(\sum_{i,j} v(x_i,x_j) - \sum_i v(x_i,x_i)\right)|\mathcal{X}\rangle \\ &= \frac{1}{2}\sum_{i\neq j} v(x_i,x_j)|\mathcal{X}\rangle.\end{aligned}$$

Then we arrive at

$$\hat{H}_{\text{int}} = \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}x' v(x,x')\hat{n}(x)\hat{n}(x') - \frac{1}{2}\int \mathrm{d}x v(x,x)\hat{n}(x).$$

□

Problem 2.2 (Exercise 1.4 of [5]). Let $\hat{n}_n \equiv \hat{d}_n^\dagger \hat{d}_n$ be the occupation operator for particles with quantum number n, see (1.74). Prove that in the fermionic case

$$\hat{n}_n^2 = \hat{n}_n.$$

and hence that the eigenvalues of $\hat{n}_n$ are either 0 or 1 — that is, it is not possible to create two fermions in the same state $|n\rangle$. This is a direct consequence of the Pauli exclusion principle.

Solution. *Proof.* As is known to all, for fermions, the creation and annihilation operators satisfy the following relations

$$\begin{cases} \{\hat{d}_n, \hat{d}_n^\dagger\} = \hat{d}_n\hat{d}_n^\dagger + \hat{d}_n^\dagger\hat{d}_n = 1, \\ \{\hat{d}_n, \hat{d}_n\} = 0 \Rightarrow \hat{d}_n\hat{d}_n = 0, \\ \{\hat{d}_n^\dagger, \hat{d}_n^\dagger\} = 0 \Rightarrow \hat{d}_n^\dagger\hat{d}_n^\dagger = 0. \end{cases}$$

Now calculating $\hat{n}_n^2$

$$\hat{n}_n^2 = \hat{d}_n^\dagger\hat{d}_n\hat{d}_n^\dagger\hat{d}_n = \hat{d}_n^\dagger(1-\hat{d}_n^\dagger\hat{d}_n)\hat{d}_n = \hat{d}_n^\dagger\hat{d}_n - \hat{d}_n^\dagger\hat{d}_n^\dagger\hat{d}_n\hat{d}_n = \hat{d}_n^\dagger\hat{d}_n = \hat{n}.$$

□

Proof. The eigenfunctions for $\hat{n}_n$ and $\hat{n}_n^2$ are

$$\hat{n}_n|\psi\rangle = \lambda|\psi\rangle, \quad \text{and} \quad \hat{n}_n^2|\psi\rangle = \lambda^2|\psi\rangle.$$

Since $\hat{n}_n^2 = \hat{n}_n$, we have

$$\hat{n}_n|\psi\rangle = \lambda^2|\psi\rangle.$$

Thus, $\lambda = \lambda^2$ and $\lambda = 0$ or 1, means it is impossible to have more than one fermion in the same quantum state $|n\rangle$, which is the Pauli exclusion principle. □

Problem 2.3 (Exercise 1.5 of [5]). Prove that the total number of particle operators $\hat{N} = \int \mathrm{d}x \hat{\psi}^\dagger(x)\hat{\psi}(x)$ can also be written as $\hat{N} = \sum_n \hat{d}_n^\dagger \hat{d}_n$ for any orthonormal basis $|n\rangle$. Calculate the action of $\hat{N}$ on a generic ket $|\Psi_N\rangle$ with N particles ($|\Psi_N\rangle \in H_N$) and prove that

$$\hat{N}|\Psi_N\rangle = N|\Psi_N\rangle.$$

Solution. *Proof.* Recalling the definitions of the field operators

$$\hat{\psi}(x) = \sum_n \langle x|n\rangle \hat{d}_n = \sum_n \phi_n(x)\hat{d}_n,$$
$$\hat{\psi}(x)^\dagger = \sum_n \langle n|x\rangle \hat{d}_n^\dagger = \sum_n \phi_n^*(x)\hat{d}_n^\dagger.$$

then substitute them into $\hat{N}$ and use the orthonormality of $\{|n\rangle\}$

$$\hat{N} = \int \mathrm{d}x \hat{\psi}^\dagger(x)\hat{\psi}(x) = \int \mathrm{d}x \sum_m \phi_m^*(x)\hat{d}_m^\dagger \sum_n \phi_n(x)\hat{d}_n^\dagger$$
$$= \sum_{m,n} \hat{d}_m^\dagger \hat{d}_n \int \mathrm{d}x \phi_m^*(x)\phi_n(x) = \sum_{m,n} \hat{d}_m^\dagger \hat{d}_n \delta_{m,n} = \sum_n \hat{d}_n^\dagger \hat{d}_n,$$

means that $\hat{N}$ can be written as $\hat{N} = \sum_n \hat{d}_n^\dagger \hat{d}_n$ for any orthonormal basis $|n\rangle$. □

Proof. When $\hat{N}$ act on $|\Psi_N\rangle$

$$\hat{N}|\Psi_N\rangle = \int \mathrm{d}x \hat{\psi}(x)^\dagger \hat{\psi}(x)|x_1, x_2, \ldots, x_N\rangle = \int \mathrm{d}x \sum_{i=1}^N \delta(x - x_i)|x_1, x_2, \ldots, x_N\rangle = N|x_1, x_2, \ldots, x_N\rangle,$$

So, we proved that $\hat{N}|\Psi_N\rangle = N|\Psi_N\rangle$. □

The total Hamiltonian of a system of interacting identical particles is the sum of $\hat{H}_0$ and $\hat{H}_{\text{int}}$. Expressing $\hat{H}$ in terms of the $\hat{d}$-operators as $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} = \sum_{ij} h_{ij}\hat{d}_i^\dagger \hat{d}_j + \frac{1}{2}\sum_{ij,n} v_{ijmn}\hat{d}_i^\dagger \hat{d}_j^\dagger \hat{d}_m \hat{d}_n$.

Problem 2.4 (Exercise 1.6 of [5]). Prove that $\hat{N}$ commutes with $\hat{H}_0$ and $\hat{H}_{\text{int}}$ — that is,

$$[\hat{N}, \hat{H}_0]_- = [\hat{N}, \hat{H}_{\text{int}}]_- = 0.$$

This means that the eigenkets of $\hat{H}$ can be chosen as kets with a fixed number of particles.

Solution. *Proof.* Since the annihilation operator $\hat{d}_m$ destorys a particle, so it decreases the number by 1: $[\hat{N}, \hat{d}_m] = -\hat{d}_m$; in contrast, the creation operator $\hat{d}_m^\dagger$ creates a particle, so $[\hat{N}, \hat{d}_m^\dagger] = \hat{d}_m^\dagger$. Then

$$[\hat{N}, \hat{H}_0] = \hat{N}\hat{H}_0 - \hat{H}_0\hat{N} = \sum_{ij} h_{ij}[\hat{N}, \hat{d}_i^\dagger \hat{d}_j] = \sum_{ij} h_{ij}\hat{d}_i^\dagger[\hat{N}, \hat{d}_j] + \sum_{ij} h_{ij}[\hat{N}, \hat{d}_i^\dagger]\hat{d}_j$$
$$= \hat{d}_i^\dagger(-\hat{d}_j) + (\hat{d}_i^\dagger)\hat{d}_j = 0,$$

similarly, since

$$[\hat{N}, \hat{d}_i^\dagger \hat{d}_j^\dagger \hat{d}_m \hat{d}_n] = (\hat{d}_i^\dagger)\hat{d}_j^\dagger \hat{d}_m \hat{d}_n + \hat{d}_i^\dagger(\hat{d}_j^\dagger)\hat{d}_m \hat{d}_n + \hat{d}_i^\dagger \hat{d}_j^\dagger(-\hat{d}_m)\hat{d}_n + \hat{d}_i^\dagger \hat{d}_j^\dagger \hat{d}_m(-\hat{d}_n) = 0,$$

we arrive at $[\hat{N}, \hat{H}_{\text{int}}] = 0$. □

Problem 2.5 (Exercise 1.7 of [5]). Let $n = s\sigma$ and $\sigma =\uparrow,\downarrow$ be the spin projection for fermions of spin $\frac{1}{2}$. We consider the operators

$$\hat{S}_s^z \equiv \frac{1}{2}(\hat{n}_{s\uparrow} - \hat{n}_{s\downarrow}),\ \hat{S}_s^+ \equiv \hat{d}_{s\uparrow}^\dagger \hat{d}_{s\downarrow},\ \hat{S}_s^- \equiv \hat{d}_{s\downarrow}^\dagger \hat{d}_{s\uparrow} = (\hat{S}_s^+)^\dagger.$$

Using the anticommutation relations,

$$\boxed{\{\hat{d}_{s\sigma}, \hat{d}_{s'\sigma'}^\dagger\} = \delta_{ss'}\delta_{\sigma\sigma'},\ \{\hat{d}_{s\sigma}, \hat{d}_{s'\sigma'}\} = 0,\ \{\hat{d}_{s\sigma}^\dagger, \hat{d}_{s'\sigma'}^\dagger\} = 0.}$$

prove that the action of the above operators on the kets $|s\sigma\rangle \equiv \hat{d}_{s\sigma}^\dagger|0\rangle$ is

$$\hat{S}_s^z|s\uparrow\rangle = \frac{1}{2}|s\uparrow\rangle,\ \hat{S}_s^+|s\uparrow\rangle = |\emptyset\rangle,\ \hat{S}_s^-|s\uparrow\rangle = |s\downarrow\rangle,$$

and

$$\hat{S}_s^z|s\downarrow\rangle = -\frac{1}{2}|s\downarrow\rangle,\ \hat{S}_s^+|s\downarrow\rangle = |s\uparrow\rangle,\ \hat{S}_s^-|s\downarrow\rangle = |\emptyset\rangle,$$

To what operator do $\hat{S}_s^z$, $\hat{S}_s^+$, $\hat{S}_s^-$ correspond?

Solution. Using the occupation operator $\hat{n}_{s\sigma} = \hat{d}_{s\sigma}^\dagger \hat{d}_{s\sigma}$ and the relation $\hat{d}_{s\sigma}|0\rangle = 0$, $\hat{d}_{s\sigma}^\dagger = |s\sigma\rangle$.

1. $\hat{S}_s^z|s\uparrow\rangle = \frac{1}{2}(\hat{d}_{s\uparrow}^\dagger\hat{d}_{s\uparrow}\hat{d}_{s\uparrow}^\dagger - \hat{d}_{s\downarrow}^\dagger\hat{d}_{s\downarrow}\hat{d}_{s\uparrow}^\dagger)|0\rangle = \frac{1}{2}[\hat{d}_{s\uparrow}^\dagger(1-\hat{d}_{s\uparrow}^\dagger\hat{d}_{s\uparrow}) - (-\hat{d}_{s\downarrow}^\dagger\hat{d}_{s\uparrow}^\dagger\hat{d}_{s\downarrow})]|0\rangle = \frac{1}{2}|s\uparrow\rangle$. □
2. $\hat{S}_s^+|0\rangle = \hat{d}_{s\uparrow}^\dagger\hat{d}_{s\downarrow}\hat{d}_{s\uparrow}^\dagger|0\rangle = -\hat{d}_{s\uparrow}^\dagger\hat{d}_{s\uparrow}^\dagger\hat{d}_{s\downarrow}|0\rangle = 0$, so $\hat{S}_s^+|s\uparrow\rangle = |\emptyset\rangle$. □
3. $\hat{S}_s^-|s\uparrow\rangle = \hat{d}_{s\downarrow}^\dagger\hat{d}_{s\uparrow}\hat{d}_{s\uparrow}^\dagger|0\rangle = \hat{d}_{s\downarrow}^\dagger(1-\hat{d}_{s\uparrow}^\dagger\hat{d}_{s\uparrow})|0\rangle = |s\downarrow\rangle$. □
4. $\hat{S}_s^z|s\downarrow\rangle = \frac{1}{2}(\hat{d}_{s\uparrow}^\dagger\hat{d}_{s\uparrow}\hat{d}_{s\downarrow}^\dagger - \hat{d}_{s\downarrow}^\dagger\hat{d}_{s\downarrow}\hat{d}_{s\downarrow}^\dagger)|0\rangle = \frac{1}{2}[-\hat{d}_{s\uparrow}^\dagger\hat{d}_{s\downarrow}^\dagger\hat{d}_{s\uparrow} - \hat{d}_{s\downarrow}^\dagger(1-\hat{d}_{s\downarrow}^\dagger\hat{d}_{s\downarrow})]|0\rangle = -\frac{1}{2}|s\downarrow\rangle$. □
5. $\hat{S}_s^+|s\downarrow\rangle = \hat{d}_{s\uparrow}^\dagger\hat{d}_{s\downarrow}\hat{d}_{s\downarrow}^\dagger|0\rangle = \hat{d}_{s\uparrow}^\dagger(1-\hat{d}_{s\downarrow}^\dagger\hat{d}_{s\downarrow})|0\rangle = |s\uparrow\rangle$. □
6. $\hat{S}_s^-|0\rangle = \hat{d}_{s\downarrow}^\dagger\hat{d}_{s\uparrow}\hat{d}_{s\downarrow}^\dagger|0\rangle = -\hat{d}_{s\downarrow}^\dagger\hat{d}_{s\downarrow}^\dagger\hat{d}_{s\uparrow}|0\rangle = 0$, so $\hat{S}_s^-|s\downarrow\rangle = |\emptyset\rangle$. □

The operators $\hat{S}_s^z$, $\hat{S}_s^+$, and $\hat{S}_s^-$ correspond to the spin operators

- Since $\hat{S}_s^z$ will not change the state, just "returns" different eigenvalues, so it measures the z-component of spin.
- Since $\hat{S}_s^+$ annihilates $|\uparrow\rangle$ and raises $|\downarrow\rangle$ to $|\uparrow\rangle$, so it is the raising operator.
- Since $\hat{S}_s^-$ lowers $|\uparrow\rangle$ and annihilates $|\downarrow\rangle$, so it is the lowering operator.

Problem 2.6 (Exercise 1.8 of [5]). Let us define the spin operators along the x and y directions as

$$\hat{S}_s^x \equiv \frac{1}{2}(\hat{S}_s^+ + \hat{S}_s^-),\ \hat{S}_y^x \equiv \frac{1}{2\,\mathrm{i}}(\hat{S}_s^+ - \hat{S}_s^-),$$

and the spin operator $\hat{S}_s^z$ along the z direction as in (1.91). Prove that these operators can also be written as

$$\hat{S}_s^j = \frac{1}{2}\sum_{\sigma\sigma'} \hat{d}_{s\sigma}^\dagger \sigma_{\sigma\sigma'}^j \hat{d}_{s\sigma'},\ j = x, y, z$$

with

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the Pauli matrices. Using the anticommutation relations, verify that

$$[\hat{S}_s^i, \hat{S}_{s'}^j]_- = \mathrm{i}\delta_{ss'} \sum_{k=x,y,z} \epsilon_{ijk} \hat{S}_s^k,$$

where ϵ_{ijk} is the Levi-Civita tensor.

Solution. *Proof.* Verify every directions: x, y, z respectively.

For $j = x$

$$\begin{aligned}\frac{1}{2}\sum_{\sigma,\sigma'} \hat{d}_{s\sigma}^\dagger \sigma_{\sigma\sigma'}^x \hat{d}_{s\sigma'} &= \frac{1}{2}\Big[\hat{d}_{s\uparrow}^\dagger(0)\hat{d}_{s\uparrow} + \hat{d}_{s\uparrow}^\dagger(1)\hat{d}_{s\downarrow} + \hat{d}_{s\downarrow}^\dagger(1)\hat{d}_{s\uparrow} + \hat{d}_{s\downarrow}^\dagger(0)\hat{d}_{s\downarrow}\Big] \\ &= \frac{1}{2}(\hat{d}_{s\uparrow}^\dagger \hat{d}_{s\downarrow} + \hat{d}_{s\downarrow}^\dagger \hat{d}_{s\uparrow}) = \hat{S}_s^x,\end{aligned}$$

similarly, for $j = y$ and $j = z$

$$\begin{aligned}\frac{1}{2}\sum_{\sigma,\sigma'} \hat{d}_{s\sigma}^\dagger \sigma_{\sigma\sigma'}^y \hat{d}_{s\sigma'} &= \frac{1}{2}\Big[\hat{d}_{s\uparrow}^\dagger(0)\hat{d}_{s\uparrow} + \hat{d}_{s\uparrow}^\dagger(-\mathrm{i})\hat{d}_{s\downarrow} + \hat{d}_{s\downarrow}^\dagger(\mathrm{i})\hat{d}_{s\uparrow} + \hat{d}_{s\downarrow}^\dagger(0)\hat{d}_{s\downarrow}\Big] = \hat{S}_s^y, \\ \frac{1}{2}\sum_{\sigma,\sigma'} \hat{d}_{s\sigma}^\dagger \sigma_{\sigma\sigma'}^z \hat{d}_{s\sigma'} &= \frac{1}{2}\Big[\hat{d}_{s\uparrow}^\dagger(1)\hat{d}_{s\uparrow} + \hat{d}_{s\uparrow}^\dagger(0)\hat{d}_{s\downarrow} + \hat{d}_{s\downarrow}^\dagger(0)\hat{d}_{s\uparrow} + \hat{d}_{s\downarrow}^\dagger(-1)\hat{d}_{s\downarrow}\Big] = \hat{S}_s^z.\end{aligned}$$

Then, $\hat{S}_s^j = \frac{1}{2}\sum_{\sigma\sigma'} \hat{d}_{s\sigma}^\dagger \sigma_{\sigma\sigma'}^j \hat{d}_{s\sigma'},\ j = x, y, z$ is valid. □

Consider the simple situation $s' = s$, that is

$$[\hat{S}_s^i, \hat{S}_s^j] = \mathrm{i}\sum_k \epsilon_{ijk} \hat{S}_s^k.$$

Write $\hat{S}_s^i$, $\hat{S}_s^j$ in terms of the representation just proved

$$\hat{S}_s^i = \frac{1}{2}\sum_{\alpha\beta} \hat{d}_{s\alpha}^\dagger \sigma_{\alpha\beta}^i \hat{d}_{s\beta}, \quad \hat{S}_s^j = \frac{1}{2}\sum_{\gamma\delta} \hat{d}_{s\gamma}^\dagger \sigma_{\gamma\delta}^j \hat{d}_{s\delta}.$$

and substitute them into the commutator

$$[\hat{S}_s^i, \hat{S}_s^j] = \hat{S}_s^i\hat{S}_s^j - \hat{S}_s^j\hat{S}_s^i = \frac{1}{4}\sum_{\alpha\beta\gamma\delta}\Big[\hat{d}_{s\alpha}^\dagger \sigma_{\alpha\beta}^i \hat{d}_{s\beta}\hat{d}_{s\gamma}^\dagger \sigma_{\gamma\delta}^j \hat{d}_{s\delta} - \hat{d}_{s\gamma}^\dagger \sigma_{\gamma\delta}^j \hat{d}_{s\delta}\hat{d}_{s\alpha}^\dagger \sigma_{\alpha\beta}^i \hat{d}_{s\beta}\Big].$$

Consider the product $\hat{d}_{s\beta}\hat{d}_{s\gamma}^\dagger = \delta_{\beta\gamma} - \hat{d}_{s\gamma}^\dagger\hat{d}_{s\beta}$ and the relation $\delta_{\beta\gamma}\sigma_{\gamma\delta}^j = \sigma_{\beta\delta}^j$, the first term gets

$$\begin{aligned}\hat{d}_{s\alpha}^\dagger \sigma_{\alpha\beta}^i \hat{d}_{s\beta}\hat{d}_{s\gamma}^\dagger \sigma_{\gamma\delta}^j \hat{d}_{s\delta} &= \hat{d}_{s\alpha}^\dagger \sigma_{\alpha\beta}^i[\delta_{\beta\gamma} - \hat{d}_{s\gamma}^\dagger\hat{d}_{s\beta}]\sigma_{\gamma\delta}^j \hat{d}_{s\delta} = \hat{d}_{s\alpha}^\dagger \sigma_{\alpha\beta}^i \delta_{\beta\gamma}\sigma_{\gamma\delta}^j \hat{d}_{s\delta} - \hat{d}_{s\alpha}^\dagger \sigma_{\alpha\beta}^i \hat{d}_{s\gamma}^\dagger \hat{d}_{s\beta}\sigma_{\gamma\delta}^j \hat{d}_{s\delta} \\ &= \hat{d}_{s\alpha}^\dagger \sigma_{\alpha\beta}^i \sigma_{\beta\delta}^j \hat{d}_{s\delta} - \hat{d}_{s\alpha}^\dagger \sigma_{\alpha\beta}^i \hat{d}_{s\gamma}^\dagger \hat{d}_{s\beta}\sigma_{\gamma\delta}^j \hat{d}_{s\delta},\end{aligned}$$

similarly, for the other term in the commutator

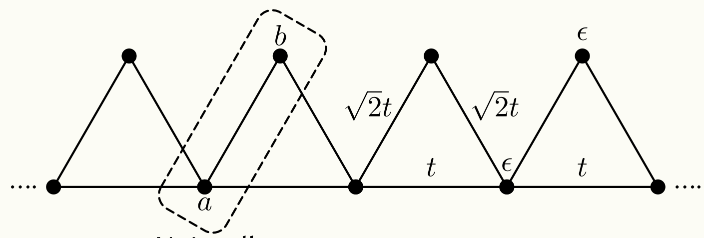
$$\hat{d}^\dagger_{s\gamma}\sigma^j_{\gamma\delta}\hat{d}_{s\delta}\hat{d}^\dagger_{s\alpha}\sigma^i_{\alpha\beta}\hat{d}_{s\beta} = \hat{d}^\dagger_{s\gamma}\sigma^j_{\gamma\delta}[\delta_{\delta\alpha} - \hat{d}^\dagger_{s\alpha}\hat{d}_{s\delta}]\sigma^i_{\alpha\beta}\hat{d}_{s\beta} = \hat{d}^\dagger_{s\gamma}\sigma^j_{\gamma\delta}\sigma^i_{\delta\beta}\hat{d}_{s\beta} - \hat{d}^\dagger_{s\gamma}\sigma^j_{\gamma\delta}\hat{d}^\dagger_{s\alpha}\hat{d}_{s\delta}\sigma^i_{\alpha\beta}\hat{d}_{s\beta}.$$

The terms with four operators cancel, since the spin operators are bilinear in fermions and the commutation should be close to bilinear.

For the situation $s \neq s'$, the operators commute because they act on different sites. So a δ-function will be inserted. Therefore

$$\begin{aligned}[\hat{S}^i_s, \hat{S}^j_s] &= \frac{1}{4}\sum_{\alpha\beta}\hat{d}^\dagger_{s\alpha}[\sigma^i,\sigma^j]_{\alpha\beta}\hat{d}_{s\beta} = \frac{1}{4}\sum_{\alpha\beta}\hat{d}^\dagger_{s\alpha}\left(2i\sum_k \epsilon_{ijk}\sigma^k\right)_{\alpha\beta}\hat{d}_{s\beta} \\ &= \frac{\mathrm{i}}{2}\sum_k \epsilon_{ijk}\sum_{\alpha\beta}\hat{d}^\dagger_{s\alpha}\sigma^k_{\alpha\beta}\hat{d}_{s\beta} = i\sum_k \epsilon_{ijk}\hat{S}^k_s.\end{aligned}$$

Problem 2.7 (Exercise 2.2 of [5]). Consider the one-dimensional crystal below,



with marices $h = \begin{pmatrix} \epsilon & \sqrt{2}t \\ \sqrt{2}t & \epsilon \end{pmatrix}$ and $T = \begin{pmatrix} t & \sqrt{2}t \\ 0 & 0 \end{pmatrix}$. Show that for $t > 0$ the two bands are

$$\begin{aligned}\epsilon_{k1} &= \epsilon - 2t, \\ \epsilon_{k2} &= \epsilon + 2t + 2t\cos k,\end{aligned}$$

with $k \in (-\pi, \pi)$. The first band is therefore perfectly flat. If we have half an electron per unit cell, then the ground state is highly degenerate. For instance, the states obtained by occupying each k-level of the flat band with an electron of either spin up or down all have the same energy. This degeneracy is lifted by the electron-electron interaction and the ground state truns out to be the one in which all electrons have parallel spins. The crystal is then a ferromagnet. The ferromagnetism in flat-band crystals was proposed by Mielke and Tasaki [11,12,13] and is usually called flat-band ferromagnetism.

Solution. For a periodic system, the Bloch Hamiltonian $H(k)$ is

$$\mathcal{H}(k) = h + T\,\mathrm{e}^{\mathrm{i}k} + T^\dagger\,\mathrm{e}^{-\mathrm{i}k} = \begin{pmatrix} \epsilon + 2t\cos k & \sqrt{2}t(1+\mathrm{e}^{\mathrm{i}k}) \\ \sqrt{2}t(1+\mathrm{e}^{-ik}) & \epsilon \end{pmatrix}.$$

To find the eigenvalues ϵ_k, let $\det(\mathcal{H}(k) - \epsilon_k\mathbb{1}) = 0$

$$\begin{vmatrix} \epsilon + 2t\cos k - \epsilon_k & \sqrt{2}t(1+\mathrm{e}^{\mathrm{i}k}) \\ \sqrt{2}t(1+e^{-ik}) & \epsilon - \epsilon_k \end{vmatrix} = 0,$$

and let $\Delta = \epsilon_k - \epsilon$ for simplicity. Then we get $\Delta_1 = 2t\cos k + 2t$, $\Delta_2 = t\cos k - t(\cos k + 2) = -2t$. Thus, the two bands are

$$\epsilon_{k1} = \epsilon - 2t, \quad \epsilon_{k2} = \epsilon + 2t + 2t\cos k.$$

Lecture #3 Homework #3 [2025-09-16]

Problem 3.1. Compute the Schrieffer-Wolff transformation for the *single impurity Anderson model (SIAM)*. The single impurity Anderson Hamiltonian in second quantized notation

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{\sigma} \epsilon_d d^\dagger_\sigma d_\sigma + \sum_{k\sigma} V_k (c^\dagger_{k\sigma} d_\sigma + d^\dagger_\sigma c_{k\sigma}) + U n_{d\uparrow} n_{d\downarrow}.$$

The Hamiltonian has one off-diagonal term, which we call $\mathcal{H}_v$ and diagonal terms, which together we call $\mathcal{H}_0$

$$\begin{aligned} \mathcal{H}_0 &= \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{\sigma} \epsilon_d d^\dagger_\sigma d_\sigma + U n_{d\uparrow} n_{d\downarrow}, \\ \mathcal{H}_v &= \sum_{k\sigma} V_k (c^\dagger_{k\sigma} d_\sigma + d^\dagger_\sigma c_{k\sigma}). \end{aligned}$$

The **generator** of SWT for SIAM is given by

$$\mathcal{S} = \sum_{k\sigma} (A_k + B_k n_{d\bar{\sigma}}) V_k (c^\dagger_{k\sigma} d_\sigma - d^\dagger_\sigma c_{k\sigma}),$$

where A_k and B_k are coefficients to be determined.

Remark. Denote

$$\Psi^\dagger_k = \begin{pmatrix} c^\dagger_{k\uparrow} \\ c^\dagger_{k\downarrow} \end{pmatrix} \quad \Psi_k = \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix} \quad \Psi^\dagger_d = \begin{pmatrix} d^\dagger_\uparrow \\ d^\dagger_\downarrow \end{pmatrix} \quad \Psi_d = \begin{pmatrix} d_\uparrow \\ d_\downarrow \end{pmatrix}$$

$$\begin{aligned} 4(\Psi^\dagger_k S \Psi_{k'})(\Psi^\dagger_d S \Psi_d) &= \Psi^\dagger_k \Psi_{k'} (\sigma.\sigma) \Psi^\dagger_d \Psi_d = \Psi^\dagger_k \Psi_{k'} (\sigma.\sigma) \Psi^\dagger_d \Psi_d \\ &= (\Psi^\dagger_k \sigma_z \Psi_k)(\Psi^\dagger_d \sigma_z \Psi_d) + \frac{1}{2}\Big[(\Psi^\dagger_k \sigma^+ \Psi_k)(\psi^\dagger_d \sigma^- \Psi_d) + (\Psi^\dagger_k \sigma^- \Psi_k)(\psi^\dagger_d \sigma^+ \Psi_d)\Big], \end{aligned}$$

then compute and identify each of these three terms in the last line. You will be getting other extra terms besides the Kondo exchange term.

Solution. Apply Schrieffer-Wolff transformation to $\mathcal{H}$ to second order. Using BCH, we have

$$\tilde{\mathcal{H}} = \mathrm{e}^{\hat{\mathcal{S}}} \mathcal{H}\, \mathrm{e}^{-\mathcal{S}} = \mathcal{H}_0 + \mathcal{H}_v + [\mathcal{S}, \mathcal{H}_0] + [\mathcal{S}, \mathcal{H}_v] + \frac{1}{2}([\mathcal{S}, [\mathcal{S}, \mathcal{H}_0]] + [\mathcal{S}, [\mathcal{S}, \mathcal{H}_v]]) + \dots$$

To ensure the Hamiltonian is diagonal to first order in $\mathcal{H}_v$, the generator $\mathcal{S}$ should satisfy

$$\mathcal{H}_v + [\mathcal{S}, \mathcal{H}_0] = 0, \quad [\mathcal{S}, \mathcal{H}_0] = -\mathcal{H}_v.$$

Since $S \sim \mathcal{H}_v \sim \mathcal{O}(V^1)$, the transformed Hamiltonian is

$$\tilde{\mathcal{H}} = \mathcal{H}_0 + [\mathcal{S}, \mathcal{H}_v] + \frac{1}{2}[\mathcal{S}, -\mathcal{H}_v] + \mathcal{O}(V^3) = \mathcal{H}_0 + \frac{1}{2}[\mathcal{S}, \mathcal{H}_v] + \mathcal{O}(V^3).$$

Since the **generator** of SWT for SIAM is given by

$$\mathcal{S} = \sum_{k\sigma} (A_k + B_k n_{d\bar{\sigma}}) V_k (c^\dagger_{k\sigma} d_\sigma - d^\dagger_\sigma c_{k\sigma}).$$

Next, we need to use the identity $[\mathcal{S}, \mathcal{H}_0] = -\mathcal{H}_v$ to deterimine the coefficients A_k and B_k for the generator $\mathcal{S}$, and then calculate the commutator $[\mathcal{S}, \mathcal{H}_v]$.

1. Determine A_k and B_k.

We need to calculate $[\mathcal{S}, \mathcal{H}_0]$ first. To distinguish the variables to be summed in the commutator, we need to express $\mathcal{H}_0$ as

$$\mathcal{H}_0 = \underbrace{\sum_{k'\sigma'} \epsilon_{k'} c^\dagger_{k'\sigma'} c_{k'\sigma'}}_{\text{Bath term}} + \underbrace{\sum_{\sigma'} \epsilon_d d^\dagger_{\sigma'} d_{\sigma'}}_{\text{Impurity level term}} + \underbrace{U n_{d\uparrow} n_{d\downarrow}}_{\text{Interaction term}},$$

then calculate every term accordingly.

i. Bath term.

The main part of the commutator is

$$[c^\dagger_{k\sigma} d_\sigma - d^\dagger_\sigma c_{k\sigma}, c^\dagger_{k'\sigma'} c_{k'\sigma'}] = [c^\dagger_{k\sigma} d_\sigma, c^\dagger_{k'\sigma'} c_{k'\sigma'}] - [d^\dagger_\sigma c_{k\sigma}, c^\dagger_{k'\sigma'} c_{k'\sigma'}],$$

and it can be expanded with the identity of the commutator

$$\cancel{c^\dagger_{k\sigma}[d_\sigma, c^\dagger_{k'\sigma'} c_{k'\sigma'}]} + [c^\dagger_{k\sigma}, c^\dagger_{k'\sigma'} c_{k'\sigma'}] d_\sigma - d_\sigma [c_{k\sigma}, c^\dagger_{k'\sigma'} c_{k'\sigma'}] - \cancel{[d_\sigma, c^\dagger_{k'\sigma'} c_{k'\sigma'}]} c_{k\sigma},$$

where c and d are commutate. Since they are fermions, we need to generate anticommutators from the four terms above to obtain the δ-terms

- ⋆ $[c^\dagger_{k\sigma}, c^\dagger_{k'\sigma'} c_{k'\sigma'}] d_\sigma = \left(\cancel{\{c^\dagger_{k\sigma}, c^\dagger_{k'\sigma'}\}} c_{k'\sigma'} - c^\dagger_{k'\sigma'} \{c^\dagger_{k\sigma}, c_{k'\sigma'}\}\right) d_\sigma = -\delta_{kk'} \delta_{\sigma\sigma'} c^\dagger_{k'\sigma'} d_\sigma$
- ⋆ $d_\sigma [c_{k\sigma}, c^\dagger_{k'\sigma'} c_{k'\sigma'}] = d_\sigma \left(\{c_{k\sigma}, c^\dagger_{k'\sigma'}\} c_{k'\sigma'} - c^\dagger_{k'\sigma'} \cancel{\{c_{k\sigma} c_{k'\sigma'}\}}\right) = \delta_{kk'} \delta_{\alpha\alpha'} d_\sigma c_{k'\sigma'}$

After summing, the Bath term will give a factor of

$$\sum_{k'\sigma'} \epsilon_{k'} \left(-\delta_{kk'} \delta_{\sigma\sigma'} c^\dagger_{k'\sigma'} d_\sigma - \delta_{kk'} \delta_{\alpha\alpha'} d_\sigma c_{k'\sigma'}\right) = -\epsilon_k \left(c^\dagger_{k\sigma} d_\sigma + d^\dagger_\sigma c_{k\sigma}\right).$$

ii. Impurity level term

Since $[n_{d\bar\sigma}, d_\sigma] = 0$, we do not consider the $B_k n_{d\bar\sigma}$ factor. Similarly, the commutator with $d^\dagger_{\sigma'} d_{\sigma'}$

$$[c^\dagger_{k\sigma} d_\sigma - d^\dagger_\sigma c_{k\sigma}, d^\dagger_{\sigma'} d_{\sigma'}] = \delta_{\sigma\sigma'} \left(c^\dagger_{k\sigma} d_{\sigma'} + d^\dagger_{\sigma'} c_{k\sigma}\right),$$

and summing over σ' gives the factor

$$\epsilon_d \left(c^\dagger_{k\sigma} d_\sigma + d_{\sigma^\dagger} c_{k\sigma}\right).$$

iii. Interaction term

It is trivial that the commutator depends on the spin.

- ⋆ $\sigma = \uparrow$, $n_{d\bar\sigma} = n_{d\downarrow}$, the commutator gives a factor of

$$[n_{d\downarrow}(c^\dagger_{k\uparrow} d_\uparrow - d^\dagger_\uparrow c_{k\uparrow}), n_{d\uparrow} n_{d\downarrow}] = c^\dagger_{k\uparrow} d_\uparrow n_{d\downarrow} - (-d^\dagger_\uparrow n_{d\downarrow} c_{k\uparrow}) = n_{d\downarrow} \left(c^\dagger_{k\uparrow} d_\uparrow + d^\dagger_\uparrow c_{k\uparrow}\right).$$

- ⋆ $\sigma = \uparrow$, by symmetry, just reverse the sign of σ, we get a factor

$$n_{d\uparrow} \left(c^\dagger_{k\downarrow} d_\downarrow + d^\dagger_\downarrow c_{k\downarrow}\right).$$

To summarize, the interaction term will bring the factor

$$[c^\dagger_{k\sigma} d_\sigma - d^\dagger_\sigma c_{k\sigma}, U n_{d\uparrow} n_{d\downarrow}] = U n_{d\bar\sigma} \left(c^\dagger_{k\sigma} d_\sigma + d^\dagger_\sigma c_{k\sigma}\right).$$

Now, we arrive at the result of the commutator $[\mathcal{S}, \mathcal{H}_0]$

$$[\mathcal{S}, \mathcal{H}_0] = \sum_{k\sigma}(A_k + B_k n_{d\bar{\sigma}})V_k(-\epsilon_k + \epsilon_d + U n_{d\bar{\sigma}})(c^\dagger_{k\sigma}d_\sigma + d^\dagger_\sigma c_{k\sigma}) = -\mathcal{H}_v.$$

Compare the expression of $\mathcal{H}_v$, we have

$$(A_k + B_k n_{d\bar{\sigma}})(-\epsilon_k + \epsilon_d + U n_{d\bar{\sigma}}) = -1.$$

Since the sign of σ can be $\uparrow$ or $\downarrow$, let $n_{d\bar{\sigma}} = 0$ and 1, we obtain the coefficients

$$A_k = \frac{1}{\epsilon_k - \epsilon_d}, \quad \text{and} \quad B_k = \frac{U}{(\epsilon_k - \epsilon_d)(\epsilon_k - \epsilon_d - U)}.$$

2. Second term of the transformed Hamiltonian
   Then, we need to get $[\mathcal{S}, \mathcal{H}_v]$, the second term of $\tilde{\mathcal{H}}$. Similar to the method of obtaining $[\mathcal{S}, \mathcal{H}_0]$, the result is

$$\begin{aligned}[\mathcal{S}, \mathcal{H}_v] = &\sum_{k\sigma} V_k^2 (A_k d^\dagger_\sigma d_\sigma + B_k n_{d\bar{\sigma}} d^\dagger_\sigma + \text{H.c.}) \\ &+ \sum_{kk'\sigma} V_k V_{k'} (A_k c^\dagger_{k\sigma} C_{k'\sigma'} + B_k d^\dagger_{\bar{\sigma}} c_{k'\bar{\sigma}} c^\dagger_{k\sigma} d_\sigma + B_k c^\dagger_{k'\bar{\sigma}} d_{\bar{\sigma}} c^\dagger_{k\sigma} d_\sigma + B_k c^\dagger_{k\sigma} c_{k'\sigma'} n_{d\bar{\sigma}} + \text{H.c.}),\end{aligned}$$

   where A_k and B_k have been already deterimined.

Eventually, we finished the SWT for SIAM

$$\tilde{\mathcal{H}} = \mathcal{H}_0 + \frac{1}{2}[\mathcal{S}, \mathcal{H}_v] + \mathcal{O}(V^3),$$

where the corresponding terms/coefficients are obtained in the previous step.

Lecture #4 Homework #4 [2025-09-23]

Problem 4.1 (Exercise 5.1 of Coleman). A particle with $S = \frac{1}{2}$ is placed in a large magnetic field $\boldsymbol{B} = (B_1 \cos\omega t, B_1 \sin\omega_t, B_0)$, where $B_0 \gg B_1$.

(a) Treating the oscillating part of the Hamiltonian as the interaction, write down the Schrödinger equation in the interaction representation.

(b) Find $\hat{U}(t) = \mathcal{T}\exp[-\mathrm{i}\int_{-\infty}^{t} \mathcal{H}_{\mathsf{int}}(t')\,\mathrm{d}t']$ by whatever method proves most convenient.

(c) If the particle starts at time $t = 0$ in the state $S_z = -\frac{1}{2}$, what is the probability it is in this state at time t?

Solution.

(a) The Hamiltonian is

$$\hat{H}(t) = -\gamma \boldsymbol{S}\cdot\mathbf{B}(t) = -\gamma B_0 S_z - \gamma B_1(S_x\cos\omega t + S_y \sin\omega t),$$

with $\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\sigma}$. Denote $\hat{H}_0 = -\gamma B_0 S_z$ and $\hat{H}_1(t) = -\gamma B_1(S_x\cos\omega t + S_y\sin\omega t)$, we obtain

$$\hat{H}_1(t) = -\frac{\gamma B_1}{2}\Big(S_+\,\mathrm{e}^{-\mathrm{i}\omega t} + S_-\,\mathrm{e}^{\mathrm{i}\omega t}\Big).$$

where $S_\pm = S_x \pm iS_y$. In the interaction picture, the interaction Hamiltonian is expressed as

$$\hat{H}_{1,I}(t) = \mathrm{e}^{\mathrm{i}\hat{H}_0 t/\hbar}\hat{H}_1(t)\,\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\hbar},$$

with the BCH theory, using $e^{\mathrm{i}\hat{H}_0 t/\hbar}S_\pm\,\mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\hbar} = S_\pm\,\mathrm{e}^{\mp\mathrm{i}\omega_0 t}$, where $\omega_0 = \gamma B_0$, we obtain

$$\hat{H}_{1,I}(t) = -\frac{\gamma B_1}{2}\Big(S_+\,\mathrm{e}^{-\mathrm{i}(\omega+\omega_0)t} + S_-\,\mathrm{e}^{\mathrm{i}(\omega+\omega_0)t}\Big).$$

So, the Schrödinger equation in the interaction representation is

$$\mathrm{i}\hbar\frac{\mathrm{d}|\psi_I(t)\rangle}{\mathrm{d}t} = -\frac{\gamma B_1}{2}\Big(S_+\,\mathrm{e}^{-\mathrm{i}(\omega+\omega_0)t} + S_-\,\mathrm{e}^{\mathrm{i}(\omega+\omega_0)t}\Big)|\psi_I(t)\rangle,$$

(b) Let $\Delta = \omega + \omega_0$, $\Omega = -\frac{\gamma B_1}{2}$. Then

$$\hat{H}_{1,I}(t) = \Omega\Big(S_+\,\mathrm{e}^{-\mathrm{i}\Delta t} + S_-\,\mathrm{e}^{\mathrm{i}\Delta t}\Big) = 2\Omega\,\mathrm{e}^{-\mathrm{i}\Delta t S_z}S_x\,\mathrm{e}^{\mathrm{i}\Delta t S_z}.$$

The time-evolution operator is

$$\hat{U}(t) = \mathcal{T}\exp\Big[-\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{t}\hat{H}_{1,I}(t')\,\mathrm{d}t'\Big].$$

Using the identity for time-ordered exponentials of this form, we find

$$\hat{U}(t) = \mathrm{e}^{-\mathrm{i}\Delta t S_z}\exp\Big[-\frac{\mathrm{i}}{\hbar}(2\Omega S_x - \Delta S_z)t\Big] = \mathrm{e}^{-\mathrm{i}(\omega+\omega_0)tS_z}\exp\Big[\frac{\mathrm{i}}{\hbar}t(\gamma B_1 S_x + (\omega+\omega_0)S_z)\Big].$$

(c) Assume in the interaction picture, $|\psi_I(0)\rangle = |{\downarrow}\rangle$. The amplitude to remain in $|{\downarrow}\rangle$ is

$$A(t) = \langle{\downarrow}|\hat{U}(t)|{\downarrow}\rangle.$$

Let $\Omega_R = \gamma B_1$, $\tilde{\Delta} = \omega + \omega_0$, and define the effective Hamiltonian according to (b)

$$H_{\text{eff}} = -\Omega_R S_x - \tilde{\Delta} S_z,$$

then

$$\hat{U}(t) = \mathrm{e}^{-i\tilde{\Delta}tS_z}\,\mathrm{e}^{-\mathrm{i}H_{\text{eff}}t/\hbar}.$$

Since $|{\downarrow}\rangle$ is an eigenstate of S_z with eigenvalue $-\hbar/2$,

$$e^{-\mathrm{i}\tilde{\Delta}tS_z}|{\downarrow}\rangle = \mathrm{e}^{\mathrm{i}\tilde{\Delta}t/2}|{\downarrow}\rangle,$$

so

$$A(t) = \mathrm{e}^{\mathrm{i}\tilde{\Delta}t/2}\langle{\downarrow}|\mathrm{e}^{-\mathrm{i}H_{\text{eff}}t/\hbar}|{\downarrow}\rangle, \quad \text{and} \quad H_{\text{eff}} = -\frac{1}{2}\Big(\Omega_R\sigma_x + \tilde{\Delta}\sigma_z\Big).$$

The eigenvalues of H_{eff} are $\pm\frac{\lambda}{2}$, where $\lambda = \sqrt{\Omega_R^2 + \tilde{\Delta}^2}$. So

$$A(t) = \mathrm{e}^{\mathrm{i}\tilde{\Delta}t/2}\left[\cos\left(\frac{\lambda t}{2}\right) - \mathrm{i}\frac{\tilde{\Delta}}{\lambda}\sin\left(\frac{\lambda t}{2}\right)\right],$$

and we obtain the survival probability is

$$P(t) = |A(t)|^2 = 1 - \frac{(\gamma B_1)^2}{(\gamma B_1)^2 + (\omega + \gamma B_0)^2}\sin^2\left(\frac{1}{2}\sqrt{(\gamma B_1)^2 + (\omega + \gamma B_0)^2}t\right).$$

Problem 4.2 (Exercise 3.1.1 of Nolting). For the non-interacting electron gas (H_e) and for the non-interacting phonon gas (H_p),

$$H_e = \sum_{k,\sigma} \epsilon(k) a^\dagger_{k\sigma} a_{k\sigma}; \quad \text{and} \quad H_p = \sum_{q,r} \hbar\omega_r(q)\left(b^\dagger_{qr} b_{qr} + \frac{1}{2}\right),$$

compute the time dependence of the annihilation operators $a_{k\sigma}(t)$, $b_{qr}(t)$ in the Heisenberg picture.

Solution. We just need to apply EOM to $a_{k\sigma}$ and b_{qr}.

(a) Annihilation operator $a_{k\sigma}(t)$
Calculate the commutator $[H_e, a_{k\sigma}]$ first

$$\begin{aligned}[H_e, a_{k\sigma}] &= \sum_{k',\sigma'} \epsilon(k')[a^\dagger_{k'\sigma'} a_{k'\sigma'}, a_{k\sigma}] \\ &= \sum_{k',\sigma'} \epsilon(k')\left(a^\dagger_{k'\sigma'} \cancel{[a_{k'\sigma'}, a_{k\sigma}]} + [a^\dagger_{k'\sigma'}, a_{k\sigma}] a_{k'\sigma'}\right) = -\epsilon(k) a_{k\sigma}.\end{aligned}$$

Then the EOM becomes

$$\frac{\mathrm{d}a_{k\sigma}(t)}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[-\epsilon(k) a_{k\sigma}(t)].$$

It's trivial that

$$a_{k\sigma}(t) = a_{k\sigma}(0)\,\mathrm{e}^{-\mathrm{i}\epsilon(k)t/\hbar}.$$

(b) Similarly, the commutator

$$[H_p, b_{qr}] = -\hbar\omega_r(q) b_{qr},$$

and the EOM becomes

$$\frac{\mathrm{d}b_{qr}(t)}{\mathrm{d}t} = -\,\mathrm{i}\omega_r(q) b_{qr}(t),$$

and it's trivial that

$$b_{qr}(t) = b_{qr}(0)\,\mathrm{e}^{-\mathrm{i}\omega_r(q)t}.$$

Problem 4.3. Consider an arbitrary **time-dependent** operator in Schrödinger picture $\hat{A}_S(t)$ then re-express $\hat{A}_S(t)$ in Heisenberg picture as $\hat{A}_H(t)$ and in Dirac (interaction) picture as $\hat{A}_D(t)$.

1. Show that $\langle\psi_H|A_H(t)|\psi_H\rangle \overset{!}{=} \langle\psi_S(t)|A_S|\psi_S(t)\rangle$;
2. Derive the Heisenberg equation-of-motion for $\hat{A}_H(t)$ which should read;
3. Similarly, derive the equation-of-motion in Dirac (interaction) picture for $\hat{A}_D(t)$.

Solution.

(a) In Heisenberg picture, $\psi_H = \psi_S(0)$, so we have

$$\langle\psi_H|\hat{A}_H(t)|\psi_H\rangle = \langle\psi_S(0)|\hat{U}^\dagger(t,0)\hat{A}_S(t)\hat{U}(t,0)|\psi_S(0)\rangle.$$

Since $|\psi_S(t)\rangle = \hat{U}(t,0)|\psi_S(0)\rangle$, so

$$\langle\psi_S(t)|\hat{A}_S(t)|\psi_S(t)\rangle = \langle\psi_S(0)|\hat{U}^\dagger(t,0)\hat{A}_S(t)\hat{U}(t,0)|\psi_S(0)\rangle,$$

which is exactly $\langle\psi_H|\hat{A}_H(t)|\psi_H\rangle$. So

$$\langle\psi_H|A_H(t)|\psi_H\rangle = \langle\psi_S(t)|A_S|\psi_S(t)\rangle.$$

(b) Since the operator in the Heisenberg picture can be expressed as

$$\hat{A}_H = \hat{U}^\dagger(t,0)\hat{A}_S(t)\hat{U}(t,0),$$

and its differentiate

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_H(t) = \frac{\partial\hat{U}^\dagger}{\partial t}\hat{A}_S(t)U + \hat{U}^\dagger\frac{\partial\hat{A}_S(t)}{\partial t}\hat{U} + \hat{U}^\dagger\hat{A}_S(t)\frac{\partial\hat{U}}{\partial t}.$$

Substitute the equation that $\hat{U}$ satisfies under the Heisenberg picture

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{U} = \hat{H}_S\hat{U},$$

we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_H &= \left(-\frac{1}{\mathrm{i}\hbar}\hat{U}^\dagger H_S\right)A_S U + \hat{U}^\dagger\frac{\partial\hat{A}_S(t)}{\partial t}\hat{U} + \hat{U}^\dagger\hat{A}_S\left(\frac{1}{\mathrm{i}\hbar}H_S\hat{U}\right)\\ &= \frac{1}{\mathrm{i}\hbar}[-\hat{U}^\dagger H_S\hat{A}_S\hat{U} + \hat{U}^\dagger\hat{A}_S H_S\hat{U}] + \hat{U}^\dagger\dot{A}_S\hat{U},\end{aligned}$$

where $-\hat{U}^\dagger H_S\hat{A}_S\hat{U} = -H_H\hat{A}_H$, $\hat{U}^\dagger\hat{A}_S H_S\hat{U} = \hat{A}_H H_H$, and $\hat{U}^\dagger\dot{A}_S\hat{U} = \left(\frac{\partial A_S}{\partial t}\right)_H$. So we obtain the Heisenberg equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}A_H(t) = \frac{1}{\mathrm{i}\hbar}[\hat{A}_H(t), H_H(t)] + \left(\frac{\partial A_S}{\partial t}\right)_H.$$

(c) Similarly from

$$A_I(t) = \hat{U}_0^\dagger(t,0)\hat{A}_S(t)\hat{U}_0(t,0),$$

we have the differentiate

$$\frac{\mathrm{d}\hat{A}_I}{\mathrm{d}t} = \frac{\partial\hat{U}_0^\dagger}{\partial t}A_S\hat{U}_0 + \hat{U}_0^\dagger\frac{\partial\hat{A}_S}{\partial t}\hat{U}_0 + \hat{U}_0^\dagger\hat{A}_S\frac{\partial\hat{U}_0}{\partial t}.$$

Substitute the equation that $\hat{U}$ satisfies

$$\mathrm{i}\hbar\frac{\partial\hat{U}_0}{\partial t} = H_{0,S}\hat{U}_0,$$

we have

$$\frac{\mathrm{d}\hat{A}_I}{\mathrm{d}t} = -\frac{1}{\mathrm{i}\hbar}\left[\hat{U}_0^\dagger H_{0,s}\hat{A}_s\hat{U}_0 + \hat{U}_0^\dagger\dot{A}_S\hat{U}_0 + \frac{1}{\mathrm{i}\hbar}\hat{U}_0^\dagger A_S H_{0,s}\hat{U}_0\right].$$

Since $H_{0,I}(t) = \hat{U}_0^\dagger H_{0,S}\hat{U}_0$ and $\hat{A}_I = \hat{U}_0^\dagger\hat{A}_S\hat{I}_0$, we obtain the Dirac picture equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}A_I(t) = \frac{1}{\mathrm{i}\hbar}[A_I(t), H_{0,I}(t)] + \left(\frac{\partial\hat{A}_S}{\partial t}\right)_I.$$

Lecture #5 Homework #5 [2025-10-09]

Problem 5.1. Verify the key step for proving the GML theorem:

$$\left(\sum_{j=1}^{n}\frac{\partial}{\partial t_j}\right)\longrightarrow n\frac{\partial}{\partial t_n}$$

explicitly. Consider $n=2$, and do the following

(a) Carry out the integration-by-path for each t_j.

(b) Show that for all j's the final results are the same, i.e., independent on j.

Solution.

(a) Denote

$$F(t_1,t_2)=\langle 0|\mathcal{T}[\phi_{\mathsf{I}}(t_1)\phi_{\mathsf{I}}(t_2)U]|0\rangle,$$

where U is the interaction picture evolution operator and $\mathcal{T}$ is the time-ordering operator. To simplify, define the unordered pieces

$$A(t_1,t_2)=\langle 0|\phi_{\mathsf{I}}(t_1)\phi_{\mathsf{I}}(t_2)U|0\rangle,\quad \text{and}\quad B(t_1,t_2)=\langle 0|\phi_{\mathsf{I}}(t_2)\phi_{\mathsf{I}}(t_1)U|0\rangle.$$

Then, by definition of $\mathcal{T}$, we have

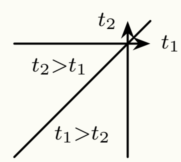
$$F(t_1,t_2)=\theta(t_1-t_2)A(t_1,t_2)+\theta(t_2-t_1)B(t_1,t_2),$$

where θ is the Heaviside function. Now act $(\partial_{t_1}+\partial_{t_1})$ on F

$$\begin{aligned}\frac{\partial F}{\partial t_1}+\frac{\partial F}{\partial t_2}&=\theta(t_1-t_2)\left(\frac{\partial A}{\partial t_1}+\frac{\partial A}{\partial t_2}\right)+\theta(t_2-t_1)\left(\frac{\partial B}{\partial t_1}+\frac{\partial B}{\partial t_2}\right)+\delta(t_1-t_2)(A-B)\\&=\theta(t_1-t_2)(\partial_{t_1}A+\partial_{t_2}A)+\theta(t_2-t_1)(\partial_{t_1}B+\partial_{t_2}B),\end{aligned}$$

where the last term vanishes since $(A-B)|_{t_1=t_2}=0$. Then, consider the integration

$$\begin{aligned}I&=\int_{-\infty}^{0}\mathrm{d}t_1\int_{-\infty}^{0}\mathrm{d}t_2(\partial_{t_1}+\partial_{t_2})F(t_1,t_2)\\&=\underbrace{\int_{-\infty}^{0}\mathrm{d}t_2\int_{t_2}^{0}\mathrm{d}t_1(\partial_{t_1}A+\partial_{t_2}A)}_{\text{Region 1: } t_1>t_2}+\underbrace{\int_{-\infty}^{0}\mathrm{d}t_1\int_{t_2}^{0}\mathrm{d}t_2(\partial_{t_1}B+\partial_{t_2}B)}_{\text{Region 2: } t_2>t_1}.\end{aligned}$$



Consider Region I

$$\begin{aligned}I_1&=\int_{-\infty}^{0}\mathrm{d}t_2\int_{t_2}^{0}\mathrm{d}t_1\,\partial_{t_1}A(t_1,t_2)+\int_{-\infty}^{0}\mathrm{d}t_2\int_{t_2}^{0}\mathrm{d}t_1\,\partial_{t_2}A(t_1,t_2)\\&\xlongequal{\text{Leibniz integral rule}}\int_{-\infty}^{0}\mathrm{d}t_2[A(0,t_2)-A(t_2,t_2)]+\int_{-\infty}^{0}\mathrm{d}t_2\left[A(t_2,t_2)-\frac{\mathrm{d}}{\mathrm{d}t_2}\left(\int_0^{t_2}A(t_1,t_2)\,\mathrm{d}t_1\right)\right]\\&=\int_{-\infty}^{0}\mathrm{d}t_2A(0,t_2)-\left[\int_0^{0}A(t_1,0)\,\mathrm{d}t_1-\int_0^{-\infty}A(t_1,-\infty)\,\mathrm{d}t_1\right]=\int_{-\infty}^{0}\mathrm{d}t_2A(0,t_2).\end{aligned}$$

The last term vanishes due to the boundary condition. Similarly, for Region II, the integration can be written as

$$I = \int_{-\infty}^{0} \mathrm{d}t_2 A(0,t_2) + \int_{-\infty}^{0} \mathrm{d}t_1 B(0,t_1) = \int_{-\infty}^{0} \mathrm{d}t_2 A(0,t_2) + \int_{-\infty}^{0} \mathrm{d}t_1 A(t_1,0) = 2\int_{-\infty}^{0} \mathrm{d}t_1 A(0,t_1).$$

In the last step, both the integral region and the variables are switched. Since

$$F(t_1,0) = \theta(t_1-0)A(t_1,0) + \theta(0-t_1)B(t_1,0) = A(0,t_1),$$

integral over t_2 to $\partial_{t_2}F(t_1,t_2)$

$$\int_{-\infty}^{0} \mathrm{d}t_2\, \partial_{t_2}F(t_1,t_2) = F(t_1,0) - F(t_1,-\infty) = F(t_1,0) = A(0,t_1).$$

So, we have

$$I = 2\int_{-\infty}^{0} \mathrm{d}t_1 \int_{-\infty}^{0} \mathrm{d}t_2\, \partial_{t_2}F(t_1,t_2) = \int_{-\infty}^{0} \mathrm{d}t_1 \int_{-\infty}^{0} \mathrm{d}t_2 (\partial_{t_1} + \partial_{t_2})F(t_1,t_2).$$

Then we obtain

$$\partial_{t_1} + \partial_{t_2} = 2\,\partial_{t_2}, \quad \text{i.e.,} \quad \left(\sum_{j=1}^{2} \partial_{t_j}\right) \longrightarrow 2\,\partial_{t_2}.$$

(b) Denote $T = \frac{1}{2}(t_1+t_2)$, $\tau = t_1 - t_2$. Holding τ fixed,

$$\partial_{t_1}\big|_{\tau} = \frac{\partial}{\partial T}\frac{\partial T}{\partial t_1} = \frac{1}{2}\frac{\partial}{\partial T}, \quad \text{and} \quad \partial_{t_2}\big|_{\tau} = \frac{\partial}{\partial T}\frac{\partial T}{\partial t_2} = \frac{1}{2}\frac{\partial}{\partial T}$$

So, we have $2\frac{\partial}{\partial t_1}|_{\tau} = 2\frac{\partial}{\partial t_2}|_{\tau} = \partial_T$, hence the result is independent of which t_j is chosen.

Problem 5.2 (The two-spin problem). Consider two spin-$1/2$'s, $\hat{s}_a = \hat{s}_b = \frac{1}{2}$, which are coupled to each other via an exchange interaction J and are presumed to be acted upon by a homogeneous magnetic field. We describe them in terms of the correspondingly simplified Heisenberg model

$$\mathcal{H}_0 = \left(\frac{J_\perp}{2}(\hat{s}_a^{++}\hat{s}_b^{-} + \hat{s}_a^{-}\hat{s}_b^{++}) + J_z\hat{s}_z^z\hat{s}_b^z\right) - h_z(\hat{s}_a^z + \hat{s}_b^z).$$

The parameter choice is to ensure that when $J_\perp = J_z$ the first term reduces to the Heisenberg interaction. For this problem, we take $J_\perp = 0$, which is called the Ising limit, $b = 0$ and also assume $J_z > 0$ (for this problem, the sign does not make much difference, but it will be important when we turn on $J_\perp$). Next, consider a transverse field as the perturbation

$$\delta\mathcal{H} = -h_x(\hat{s}_a^x + \hat{s}_b^x).$$

Consider the two-spin COBS, $U_{g2} = \{\hat{I}, \hat{S}^\alpha, \hat{\eta}, \hat{B}_S^\alpha, \hat{B}_A^\alpha, \hat{D}^\alpha\}$ defined as

$$\begin{aligned}
\hat{S}_{ab}^{\alpha} &= \hat{s}_a^\alpha + \hat{s}_b^\alpha,\ \hat{\eta}_{ab}^\alpha = \hat{s}_a^\alpha - \hat{s}_b^\alpha,\\
\hat{B}_{A,ab}^{\gamma} &= 2\epsilon_{\alpha\beta\gamma}\hat{s}_a^\alpha\hat{s}_b^\beta,\\
\hat{B}_{S,ab}^{\gamma} &= 2\epsilon_{\alpha\beta\gamma}^2\hat{s}_a^\alpha\hat{s}_b^\beta,\\
\hat{D}_{ab}^{\alpha} &= 2\hat{s}_a^\alpha\hat{s}_b^\alpha,
\end{aligned}$$

where the Einstein summation notation over dummy indices is assumed.

(a) Determine the exact ground state(s) of $\mathcal{H}_0$, specify the degeneracy, if any, by diagonalization, i.e., rewrite $\mathcal{H}_0$ as a 4×4 matrix.

(b) Parameterize the degenerate ground state subspace; compute expectation values of COBS elements for a parameterized state; try to use symmetry as a guide, only 5 out of the 15 operators have none-zero values; for two-body operators, use the cumulant expression.

(c) Determine the exact ground state energy and the ground state wavefunction for $\mathcal{H} = \mathcal{H}_0 + \delta\mathcal{H}$, assuming $J_z > 4h_x > 0$; also compute expectation values for the COBS elements; carry out the Taylor expansion for $\langle \hat{S}^x \rangle$ to the first order, to determine the transverse field susceptibility $\chi_x = \frac{\partial \langle \hat{S}^x \rangle}{\partial h_x}\Big|_{h_x \to 0}$; determine the ground state in the same limit $|\Psi\rangle\big|_{h_x \to 0^+}$.

(d) Try to use the perturbative VES Eq. (5.6.10)

$$0 = \langle [\delta\mathcal{H}, \hat{u}^{\alpha_0}] \rangle_0 + \sum_{\alpha} \delta\langle \hat{u}^{\alpha} \rangle \langle \{ \hat{u}^{\alpha}, [\mathcal{H}, \hat{u}^{\alpha_0}] \} \rangle_0 + \sum_{\gamma\beta} \langle \delta\hat{u}^{\gamma} \rangle \langle \delta\hat{u}^{\beta} \rangle \langle \{ \hat{u}^{\gamma}, [\mathcal{H}, \{ \hat{u}^{\alpha_0}, \hat{u}^{\beta} \}] \} \rangle_0 + \dots, \tag{5.6.10}$$

to do the computation, starting from a parameterized state, and express the transverse field susceptibility χ_x as a function of COBS expectation values; determine the parameterized state with the largest χ_x and compare with the diagonalization results you obtained in the previous question.

(e) You should find that the fully entangled triplet state has the largest χ_x, try to discuss the reason based on your calculation results.

Solution. Since we take $J_\perp = 0$, and omit the perturbation term. The Hamiltonian becomes

$$\mathcal{H}_0 = J_z \hat{s}_a^z \hat{s}_b^z.$$

(a) Recall $\hat{s}^z = \frac{1}{2}\sigma^z$, then $\hat{s}_a^z \hat{s}_b^z = \frac{1}{4}\sigma_a^z \sigma_b^z$. Taking the basis

$$\{ |\uparrow\uparrow\rangle, \ |\uparrow\downarrow\rangle, \ |\downarrow\uparrow\rangle, \ |\downarrow\downarrow\rangle \}.$$

Therefore, the eigenenergies, i.e., the diagonal matrix elements, are

$$E(\uparrow\uparrow) = \frac{J_z}{4}, \quad E(\uparrow\downarrow) = -\frac{J_z}{4}, \quad E(\downarrow\uparrow) = -\frac{J_z}{4}, \quad E(\downarrow\downarrow) = \frac{J_z}{4}.$$

So, $\mathcal{H}_0$ can be written into a 4×4 diagonal matrix

$$\mathcal{H}_0 = \begin{pmatrix} \frac{J_z}{4} & & & \\ & -\frac{J_z}{4} & & \\ & & -\frac{J_z}{4} & \\ & & & \frac{J_z}{4} \end{pmatrix}.$$

The ground states are $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$. Since the two lowest eigenvalues are equal, the degeneracy is 2.

(b) Parameterize a normalized general state in the ground subspace as

$$|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|\uparrow\downarrow\rangle + \mathrm{e}^{\mathrm{i}\phi}\sin\frac{\theta}{2}|\downarrow\uparrow\rangle.$$

The general is normalized. The non-zero values of the 5 COBS elements are listed as follows

$$\begin{aligned}
\hat{\eta}^z &= \hat{s}_a^z - \hat{s}_b^z, & \langle\hat{\eta}^z\rangle &= |\cos(\theta/2)|^2 - |\,\mathrm{e}^{\mathrm{i}\theta}\sin(\theta/2)|^2 = \cos\theta,\\
\hat{D}^x &= 2\hat{s}_a^x\hat{s}_b^x, & \langle\hat{D}^x\rangle &= \mathrm{Re}\{[\cos(\theta/2)]^*\,\mathrm{e}^{\mathrm{i}\theta}\sin(\theta/2)\} = \tfrac{1}{2}\sin\theta\cos\phi,\\
\hat{D}^y &= 2\hat{s}_a^y\hat{s}_b^y, & \langle\hat{D}^y\rangle &= \langle\hat{D}^x\rangle = \tfrac{1}{2}\sin\theta\cos\phi,\\
\hat{D}^z &= 2\hat{s}_a^z\hat{s}_b^z, & \langle\hat{D}^z\rangle &= -\tfrac{1}{2},\\
\hat{B}_A^z &= 2(\hat{s}_a^x\hat{s}_b^y - \hat{s}_a^y\hat{s}_b^x), & \langle\hat{B}_A^z\rangle &= -2\,\mathrm{Im}\{[\cos(\theta/2)]^*\,\mathrm{e}^{\mathrm{i}\theta}\sin(\theta/2)\} = -\sin\theta\sin\phi.
\end{aligned}$$

(c) The total Hamiltonian is

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_0 + \delta\mathcal{H} = J_z\hat{s}_a^z\hat{s}_b^z - h_x(\hat{s}_a^x + \hat{s}_b^x) = J_z\hat{s}_a^z\hat{s}_b^z - h_x\hat{S}_x\\
&= \begin{pmatrix} \frac{J_z}{4} & -\frac{h_x}{2} & -\frac{h_x}{2} & 0\\ -\frac{h_x}{2} & -\frac{J_z}{4} & 0 & -\frac{h_x}{2}\\ -\frac{h_x}{2} & 0 & -\frac{J_z}{4} & -\frac{h_x}{2}\\ 0 & -\frac{h_x}{2} & -\frac{h_x}{2} & \frac{J_z}{4}\end{pmatrix} \sim \begin{pmatrix} -\frac{1}{4}\sqrt{J_z^2+16h_x^2} & & & \\ & -\frac{J_z}{4} & & \\ & & +\frac{J_z}{4} & \\ & & & +\frac{1}{4}\sqrt{J_z^2+16h_x^2}\end{pmatrix}.
\end{aligned}$$

Therefore, the exact ground state energy is $E_0 = -\frac{1}{4}\sqrt{J_z^2+16h_x^2}$. The exact ground eigenvector (wavefunction) can be expressed as

$$|\Psi_0\rangle = |\uparrow\uparrow\rangle + \left(\frac{J_z}{4h_x} - \frac{\sqrt{J_z^2+16h_x^2}}{4h_x}\right)(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) + |\downarrow\downarrow\rangle.$$

The expectation values are

$$\begin{aligned}
\langle\hat{S}_x\rangle &= \tfrac{h_x}{\sqrt{(J_z/4)^2+h_x^2}}, & \langle\hat{S}^y\rangle &= \langle\hat{S}^z\rangle = 0, & \langle\hat{\eta}\rangle &= 0,\\
\langle\hat{B}_A^\alpha\rangle &= \langle\hat{B}_s^\alpha\rangle = 0, & \langle\hat{D}^x\rangle &= \langle\hat{D}^y\rangle = \tfrac{1}{2}, & \langle\hat{D}^z\rangle &= \tfrac{2\lambda^2-1}{2(1+2\lambda^2)},
\end{aligned}$$

where $\lambda = \frac{\sqrt{(J_z/4)^2+h_x^2}-J_z/4}{\sqrt{2}h_x}$. Under the limitation of $h_x \to 0$, the ground energy can be expanded as

$$E_1 \approxeq -\frac{J_z}{4} - \frac{2h_x^2}{J_z} + \mathcal{O}(h_x^4),$$

and the expectation of $\langle\hat{S}^x\rangle$ under this limitation becomes

$$\lim_{h_x\to 0}\langle\hat{S}^x\rangle = \frac{4h_x}{J_z},$$

and the transverse field susceptibility $\chi_x = 4/J_z$, the ground state

$$\lim_{h_x\to 0}|\Psi\rangle = \frac{1}{\sqrt{2}}|\uparrow\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow\rangle,$$

where the normalization factor is $\frac{1}{\sqrt{2}}$.

(d) We only consider the top 2 orders of the VES equation. Choose $\hat{u}^{\alpha_0} = \hat{S}^x$,then

$$\langle[\delta\mathcal{H}, \hat{S}^x]\rangle_0 = -h_x\langle[\hat{S}^x, \hat{S}^x]\rangle_0 = 0,$$

and the VES equation becomes

$$\sum_\alpha \delta\langle\hat{u}^\alpha\rangle\langle\{\hat{u}^\alpha, [\mathcal{H}_0, \hat{u}^{\alpha_0}]\}\rangle_0 = 0.$$

i. Compute $[\mathcal{H}_0, \hat{S}^x]$.

$$[\mathcal{H}_0, \hat{S}^x] = J_z[\hat{S}_a^z\hat{S}_b^z, \hat{S}_a^x + \hat{S}_b^x] \xrightarrow[{[\hat{S}_b^z,\hat{S}_b^x]=i\hat{S}_b^y}]{[\hat{S}_a^z,\hat{S}_a^x]=i\hat{S}_a^y} \mathrm{i}J_z(\hat{S}_a^y\hat{S}_b^z + \hat{S}_a^z\hat{S}_b^y).$$

ii. Choose the relevant Ooerator $\hat{u}^\alpha = \hat{B}_A^y$.
From symmetry analysis, the operator is expressed as $\hat{B}_A^y = 2(\hat{S}_a^z\hat{S}_b^x - \hat{S}_a^x\hat{S}_b^z)$. Then

$$\langle\{\hat{B}_A^y, [\mathcal{H}_0, \hat{S}^x]\}\rangle_0 = -\mathrm{i}J_z\langle\hat{D}^x\rangle_0, \quad \text{and} \quad \delta\langle\hat{B}_A^y\rangle = -2\delta\langle\hat{S}^x\rangle,$$

where $\hat{D}^x = 2\hat{S}_a^x\hat{S}_b^x$.

Substitute them into the VES equation

$$0 = 0 + \delta\langle\hat{B}_A^y\rangle\cdot(-\mathrm{i}J_z\langle\hat{D}^x\rangle_0) = (-2\delta\langle\hat{S}^x\rangle)(-iJ_z\langle\hat{D}^x\rangle_0), \quad \langle\hat{S}^x\rangle \approxeq \delta\langle\hat{S}^x\rangle = \frac{2h_x}{J_z}(1 + 2\langle\hat{D}^x\rangle_0).$$

Then, the transverse susceptibility

$$\chi_x = \frac{\partial\hat{S}^x}{\partial h_x} = \frac{2}{J_z}(1 + 2\langle\hat{D}^x\rangle_0) = \frac{2}{J_z}\left(1 + 2\times\frac{1}{2}\right) = \frac{4}{J_z},$$

which is the same as the previous question.

(e) The triplet state $|1, 0\rangle$ maximizes χ_x because it achieves the largest $\langle\hat{D}^x\rangle_0 = 1/2$ in the susceptibility formula $\chi_x = (2/J_z)(1 + 2\langle\hat{D}^x\rangle_0)$, yielding $\chi_x = 4/J_z$. Its symmetric structure provides optimal matrix elements for transverse field coupling.

Problem 5.3 (extra credit, I don't see the answer right way yet but I think both approaches should agree). Use the results of section 5.7.1.5 to compute the propagator of a driven harmonic oscillator discussed in section 5.8.1.1. You may change the creation / annihilation operator representation into the x/p representation if needed; i.e., consider the following Hamiltonian and perturbation,

$$\hat{H}_0 = \omega\left(b^\dagger b + \frac{1}{2}\right), \quad \hat{V}(t) = \bar{z}(t)b(t) + b^\dagger(t)z(t),$$

and prove that the propagator is

$$S[\bar{z}, z] = \left\langle 0\middle|\mathcal{T}\exp\left(-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t[\bar{z}(t)b(t) + b^\dagger(t)z(t)]\right)\middle|0\right\rangle = \exp\left[\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{d}t'\bar{z}(t)G(t-t')z(t')\right],$$

where $G(t-t') = -\mathrm{i}\theta(t-t')\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}$.

Remark. Seems all contributions directly step from the factor $\exp\{\frac{\mathrm{i}}{\hbar}S[x_cl(t), \dot{x}(t), t]\}$. Connect the contribution to the $G(t-t')$ in the absence of the external drive $z(t)$.

Solution. The time evolution of the creation and annihilation operators in Dirac picture can be written as

$$b_{\mathsf{I}}(t) = b\,\mathrm{e}^{-\mathrm{i}\omega t}, \quad \text{and} \quad b_I^\dagger(t) = b^\dagger\,\mathrm{e}^{\mathrm{i}\omega t}.$$

The propagator can be written as a vacuum-to-vacuum amplitude in the Dirac picture

$$S[\bar{z}, z] = \left\langle 0 \middle| \mathcal{T}\exp\left(-\mathrm{i}\int_{-\infty}^{\infty} \hat{V}_{\mathsf{I}}(t)\,\mathrm{d}t\right) \middle| 0 \right\rangle,$$

where $\hat{V}_{\mathsf{I}}(t)$ is the Dirac picture perturbation

$$\hat{V}_{\mathsf{I}}(T) = \mathrm{e}^{\mathrm{i}\hat{H}_0 t}\hat{V}(t)\,\mathrm{e}^{-\mathrm{i}\hat{H}_0 t} = \bar{z}(t)b\,\mathrm{e}^{-\mathrm{i}\omega t} + b^\dagger z(t)\,\mathrm{e}^{\mathrm{i}\omega t}$$

the above expression used the following identities

$$\mathrm{e}^{\mathrm{i}\hat{H}_0 t}b\,\mathrm{e}^{-\mathrm{i}\hat{H}_0 t} \xlongequal[{[\mathrm{i}\hat{H}_0 t, b] = -\mathrm{i}\omega t b}]{\text{BCH identity}} b\,\mathrm{e}^{-\mathrm{i}\omega t}, \quad \text{and} \quad \mathrm{e}^{\mathrm{i}H_0 t}b^\dagger\,\mathrm{e}^{-\mathrm{i}\hat{H}_0 t} \xlongequal[{[\mathrm{i}\hat{H}_0 t, b^\dagger] = \mathrm{i}\omega t b^\dagger}]{\text{BCH identity}} b^\dagger\,\mathrm{e}^{\mathrm{i}\omega t}.$$

Then, we compute the vacuum expectation

$$\begin{aligned}\langle 0|\mathcal{T}[\hat{V}_{\mathsf{I}}(t_1)\hat{V}_{\mathsf{I}}(t_2)]|0\rangle &= \theta(t_1-t_2)\,\mathrm{e}^{-\mathrm{i}\omega(t_1-t_2)}\bar{z}(t_1)z(t_2) + \theta(t_2-t_1)\,\mathrm{e}^{-\mathrm{i}\omega(t_2-t_1)}\bar{z}(t_2)z(t_1) \\ &= 2\theta(t_1-t_2)\,\mathrm{e}^{-\mathrm{i}\omega(t_1-t_2)}\bar{z}(t_1)z(t_2).\end{aligned}$$

The $\mathcal{S}$-matrix can be expanded as

$$\mathcal{S} = \mathcal{T}\exp\left(-\mathrm{i}\int_{-\infty}^{\infty}\hat{V}_{\mathsf{I}}(t)\,\mathrm{d}t\right) = 1 + (-\mathrm{i})\int \mathrm{d}t_1\hat{V}_{\mathsf{I}}(t_1) + \frac{(-\mathrm{i})^2}{2}\int \mathrm{d}t_1\int \mathrm{d}t_2\mathcal{T}[\hat{V}_{\mathsf{I}}(t_1)\hat{V}_{\mathsf{I}}(t_2)] + \cdots.$$

Then, the connected part of the vacuum expectation is

$$\log\langle 0|\mathcal{S}|0\rangle = -\frac{1}{2}\int \mathrm{d}t_1\,\mathrm{d}t_2\langle 0|\mathcal{T}[\hat{V}_{\mathsf{I}}(t_1)\hat{V}_{\mathsf{I}}(t_2)]|0\rangle + \cdots = -\int \mathrm{d}t_1\,\mathrm{d}t_2\theta(t_1-t_2)\,\mathrm{e}^{-\mathrm{i}\omega(t_1-t_2)}\bar{z}(t_1)z(t_2),$$

where only the contractions between b and $b^\dagger$ are non-zero. Therefore

$$\langle 0|\mathcal{S}|0\rangle = \exp\left(-\int \mathrm{d}t\,\mathrm{d}t'\theta(t-t')\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\bar{z}(t)z(t')\right).$$

We define the retarded Green function

$$G(t-t') = -\mathrm{i}\theta(t-t')\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}.$$

Then, the propagator can be expressed as

$$\mathcal{S}[\bar{z}, z] = \exp\left[\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{d}t'\bar{z}(t)G(t-t')z(t')\right],$$

where $G(t-t') = -\mathrm{i}\theta(t-t')\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}$.

Lecture #6 Homework #6 [2025-10-16]

Problem 6.1. Derive the completeness relation Eq. (1.123)

$$\int \prod_\alpha \frac{\mathrm{d}\phi_\alpha^* \,\mathrm{d}\phi_\alpha}{2\,\mathrm{i}\pi}\,\mathrm{e}^{-\sum_\alpha \phi_\alpha^*\phi_\alpha}|\phi\rangle\langle\phi| = \mathbb{1} \tag{1.123}$$

by integration. First consider one single-particle state α and let $|n\rangle$ denote the state with n particles in α. Show that by writing in polar form $\phi = \rho\,\mathrm{e}^{\mathrm{i}\theta}$ one obtains

$$\int \frac{\mathrm{d}\phi^*\,\mathrm{d}\phi}{2\pi\,\mathrm{i}}\,\mathrm{e}^{-\phi^*\phi}|\phi\rangle\langle\phi| = \int \rho\frac{\mathrm{d}\rho\,\mathrm{d}\theta}{\pi}\,\mathrm{e}^{-\rho^2}\sum_m \frac{(\rho\,\mathrm{e}^{\mathrm{i}\theta})}{\sqrt{m!}}|m\rangle\sum_n \frac{(\rho\,\mathrm{e}^{-\mathrm{i}\theta})}{\sqrt{n!}}\langle n| = \sum_n |n\rangle\langle n|.$$

Now generalize to a set of single-particle states $\{|\alpha\rangle\}$ noting that the closure relation Eq. (1.89) may be written

$$\sum_{\{n_\alpha\}} |n_{\alpha_1} n_{\alpha_2}\dots\rangle\langle n_{\alpha_1} n_{\alpha_2}\dots| = \mathbb{1},$$

where $\{n_\alpha\}$ denotes a complete set of occupation numbers.

Solution. Expand the coherent state $|\phi\rangle$ in terms of number state $|n\rangle$

$$|\phi\rangle = \sum_{n=0}^\infty c_n|n\rangle.$$

Here, we merge the normalization factor into c_n to simplify, and it will be released at the end. Act $\hat{a}$ on the expand expression of $|\phi\rangle$, due to the property of annihilation operator

$$\hat{a}|\phi\rangle = \sum_{n=0}^\infty c_n\hat{a}|n\rangle = \sum_{n=1}^\infty c_n\sqrt{n}|n-1\rangle = \phi\sum_{n=0}^\infty c_n|n\rangle,$$

the third term summing from $n=1$ is due to $\hat{a}|0\rangle = 0$, which $n=0$ make no sence. To uniform the lower limit of n, shift the summation index

$$\hat{a}|\phi\rangle = \sum_{n=0}^\infty c_{n+1}\sqrt{n+1}|n\rangle = \phi\sum_{n=0}^\infty c_n|n\rangle,$$

then we obtain the expression of the expansion coefficient

$$c_{n+1}\sqrt{n+1} = \phi c_n, \quad \text{or} \quad c_n = \frac{\phi}{\sqrt{n}}c_{n-1} = \frac{\phi}{\sqrt{n}}\frac{\phi}{\sqrt{n-1}}c_{n-2} = \dots = \frac{\phi^n}{\sqrt{n!}}c_0,$$

thus we obtain the expansion of $|\phi\rangle$ in terms of number states

$$|\phi\rangle = c_0\sum_{n=0}^\infty \frac{\phi^n}{\sqrt{n!}}|n\rangle \xlongequal{\text{normalization}} \mathrm{e}^{-|\phi|^2/2}\sum_{n=0}^\infty \frac{\phi^n}{\sqrt{n!}}|n\rangle.$$

Writing ϕ in polar form $\phi = \rho\,\mathrm{e}^{\mathrm{i}\theta}$, we expand $\mathrm{d}\phi^*\,\mathrm{d}\phi$ via Wedge product

$$\mathrm{d}\phi^*\wedge\mathrm{d}\phi = (\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}\rho - \mathrm{i}\rho\,\mathrm{e}^{-\mathrm{i}\theta}\,\mathrm{d}\theta)\wedge(\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\rho + \mathrm{i}\rho\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta) = 2\,\mathrm{i}\rho\,\mathrm{d}\rho\wedge\mathrm{d}\theta,$$

then $\mathrm{d}\phi^*\,\mathrm{d}\phi = 2\,\mathrm{i}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$. Substitute the above expressions into the identity

$$\mathbb{1} = \int \frac{\mathrm{d}\phi^*\,\mathrm{d}\phi}{2\pi\,\mathrm{i}}\,\mathrm{e}^{-\phi^*\phi}|\phi\rangle\langle\phi| = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\infty}\frac{\rho\,\mathrm{d}\rho}{\pi}\,\mathrm{e}^{-\rho^2}\sum_{m,n=0}^{\infty}\frac{\rho^{m+n}\,\mathrm{e}^{(m-n)\mathrm{i}\theta}}{\sqrt{m!n!}}|m\rangle\langle n|.$$

Evaluate the angular integral

$$\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{e}^{(m-n)\mathrm{i}\theta} = 2\pi\delta_{mn}.$$

Evaluate the radial integral for $m = n$ (since there will be a δ_{mn} factor)

$$\int_0^{\infty}\mathrm{d}\rho\,\mathrm{e}^{-\rho^2}\rho^{2n+1} = \frac{1}{2}\Gamma(n+1) = \frac{1}{2}n!.$$

Substitute them into the integral

$$I = \sum_{m,n=0}\frac{1}{\pi}\frac{1}{\sqrt{m!n!}}(2\pi\delta_{mn})\left(\frac{1}{2}n!\right)|m\rangle\langle n| \xlongequal{\text{only } m=n \text{ survives}} \sum_n |n\rangle\langle n|.$$

When generalize to a set of complete single-particle basis $\{|\alpha\rangle\}$ and build the full Fock space with occupation numbers $\{n_\alpha\}$, i.e.,

$$|\{\phi_\alpha\}\rangle = \prod_\alpha |\phi_\alpha\rangle_\alpha,$$

the product measure over modes gives the multi-mode resolution of identity. Concretely, for each mode α introduce a complex coordinate ϕ_α and form the product integral

$$\prod_\alpha\int\frac{\mathrm{d}\phi_\alpha^*\,\mathrm{d}\phi_\alpha}{2\pi\,\mathrm{i}}\,\mathrm{e}^{-\phi_\alpha^*\phi_\alpha}\otimes_\alpha|\phi_\alpha\rangle\langle\phi_\alpha|,$$

expanded into a Fock basis, each mode's angular integral enforces equality of occupation numbers in bra $\langle|$ and ket $|\rangle$, each radial integral produces the factorial that cancels the $1/\sqrt{n!}$ factors, so the identity ends up with the sum over all occupation-number configurations

$$\sum_{\{n_\alpha\}}|n_{\alpha_1}n_{\alpha_2}\ldots\rangle\langle n_{\alpha_1}n_{\alpha_2}\ldots| = \mathbb{1}_{\text{Fock}}.$$

In other words, the modes are independent, and for each that has the completeness shown above.

Problem 6.2. Generalize the properties of Grassmann variables demonstrated in Section 1.5 for the pari of generators ξ, ξ^* to the case $2p$ generators $\{\xi_1 \ldots \xi_p \xi_1^* \ldots \xi_p^*\}$. In particular, determine the general form of a function $f(\xi_\alpha)$ and an operator $A(\xi_\alpha^*, \xi_\alpha)$, show that $\frac{\partial}{\partial \xi_\alpha}$, $\frac{\partial}{\partial \xi_\beta}$, ξ_γ, and ξ_δ^* anticommute, determine if an analogous property holds for integration, find and verify an expression for the p-dimensional δ-function $\delta^P(\boldsymbol{\xi} - \boldsymbol{\xi}')$, and generalise Eq. (1.157)

$$\begin{aligned} \langle f|g\rangle &= \int \mathrm{d}\xi^* \, \mathrm{d}\xi (1 - \xi^* \xi)(f_0^* + f_1^* \xi)(g_0 + g_1 \xi^*) \\ &= -\int \mathrm{d}\xi^* \, \mathrm{d}\xi \xi^* \xi f_0^* g_0 + \int \mathrm{d}\xi^* \, \mathrm{d}\xi \xi \xi^* f_1^* g_1 = f_0^* g_0 + f_1^* g_1 \end{aligned} \tag{1.157}$$

for $\langle f|g\rangle$.

Solution.

(a) Function $f(\boldsymbol{\xi}) \equiv f(\xi_1, \, \xi_1, \, \ldots, \, \xi_p)$ is a general element of the Grassmann algebra generated by $\{\xi_1, \, \xi_1, \, \ldots, \, \xi_p\}$. Since $\xi_\alpha^2 = 0$, any function must be "at most" linear in each variable ξ_α, and the function is a *finite polynomial*. Apply Taylor expansion to $f(\xi)$, we can obtain its general form

$$f(\xi) = f_0 + \sum_{i=1}^p f_i \xi_i + \sum_{i<j}^p f_{ij} \xi_i \xi_j + \cdots + f_{1\ldots p} \xi_1 \cdots \xi_p = \sum_{n=0}^p \frac{1}{n!} \sum_{\alpha_1 < \alpha_2 < \cdots < \alpha_n} f_{\alpha_1 \cdots \alpha_n} \xi_{\alpha_1 \cdots \alpha_n},$$

where the coefficients $f_{\{\alpha_i\}}$ are antisymmertic under interchange of any two indices. The Operator depends on $2p$ generators. Apply the same logic to the function, the operator is also a finite linear combination of monomials

$$A(\xi^*, \xi) = \sum_{m=0}^p \sum_{n=0}^p \sum_{\substack{\alpha_1 < \alpha_2 < \cdots < \alpha_n \\ \beta_1 < \beta_2 < \cdots < \beta_n}} \frac{1}{m! n!} A^{(\alpha_1, \alpha_2, \ldots, \alpha_m)}_{(\beta_1, \beta_2, \ldots, \beta_n)} \xi_{\alpha_1}^* \xi_{\alpha_2}^* \cdots \xi_{\alpha_m}^* \xi_{\beta_1} \xi_{\beta_2} \cdots \xi_{\beta_n},$$

with coefficients $A^{\{\alpha_i\}}_{\{\beta_j\}}$ antisymmetric in the upper indices and antisymmetric in the lower indices.

(b) Act the partial derivatives with respect to Grassmann variables on a Grassmann variable, or a multiple of Grassmann functions, can be expressed as

$$\partial_{\xi_\alpha}(\xi_\beta) = \delta_{\alpha\beta}, \quad \text{and} \quad \partial_{\xi_\alpha}(AB) = (\partial_{\xi_\alpha} A)B + (-1)^{\epsilon_A} A(\partial_{\xi_\alpha} B),$$

where $\epsilon_A = 0$ (even) or 1 (odd) is the Grassmann parity of A. Then, we can obtain the canonical anticommutators

$$[\xi_\alpha, \xi_\beta]_+ = 0, \quad \text{and} \quad [\partial_{\xi_\alpha}, \partial_{\xi_\beta}]_+ = 0,$$

and the mixed relation

$$[\partial_{\xi_\alpha}, \xi_\beta]_+ = \partial_{\xi_\alpha} \xi_\beta + \xi_\beta \, \partial_{\xi_\alpha} = \delta_{\alpha\beta}.$$

With respect to the integration properties

$$\int \mathrm{d}\xi_\alpha = 0, \quad \int \mathrm{d}\xi_\alpha \xi_\alpha = 1, \quad \mathrm{d}\xi_\alpha^{(*)} \, \mathrm{d}\xi_\beta^{(*)} = -\, \mathrm{d}\xi_\beta^{(*)} \, \mathrm{d}\xi_\alpha^{(*)}.$$

With the chosen ordering the standard normalization is

$$\mathrm{d}^p \boldsymbol{\xi} \equiv \mathrm{d}\xi_p \, \mathrm{d}\xi_{p-1} \cdots \mathrm{d}\xi_1, \quad \int \mathrm{d}^p \boldsymbol{\xi}_{\alpha_1} \xi_{\alpha_2} \cdots \xi_{\alpha_p} = (-1)^{\epsilon_{\xi_i}} \delta_{np}.$$

(c) Due to the integration properties and the *Dirac measure*, the p-dimensional δ-function satisfies

$$\int \mathrm{d}^p\xi f(\boldsymbol{\xi})\delta^p(\boldsymbol{\xi}-\boldsymbol{\xi}') = f(\boldsymbol{\xi}'),$$

then, the p-dimensional δ-function can be expressed as

$$\delta^p(\boldsymbol{\xi}-\boldsymbol{\xi}') \equiv \prod_{\alpha=1}^{p}(\xi_\alpha - \xi'_\alpha),$$

and it can be trivially verified by denoting $f(\boldsymbol{\xi}') = \sum_i f_i\xi'_i$.

$$\int \mathrm{d}\xi_1 f(\xi_1,\ \xi_2,\ \ldots,\ \xi_p)(\xi_1-\xi'_1)\cdots(\xi_p-\xi'_p) = f(\xi'_1,\ \xi_2,\ \ldots,\ \xi_p)(\xi_2-\xi'_2)\cdots(\xi_p-\xi'_p),$$

which it is similar for $\xi_2, \ldots, \xi_p$. And after p integrations, the RHS becomes $f(\xi'_1,\ \xi'_2,\ \ldots,\ \xi'_p) \equiv f(\boldsymbol{\xi}')$.

(d) Expand $f(\xi)$ and $g(\xi^*)$ as

$$f(\xi) = \sum_i f_i\xi_i, \quad \text{and} \quad g(\xi^*) = \sum_j g_j\xi_j^*,$$

then, the inner product can be expressed as

$$\langle f|g\rangle = \int\left(\prod_{\alpha=1}^{p}\mathrm{d}\xi_\alpha^*\,\mathrm{d}\xi_\alpha\right)\left[\prod_{\alpha=1}^{p}(1-\xi_\alpha^*\xi_\alpha)\right]f^*(\xi)g(\xi^*) \equiv \sum_i^p f_i^* g_i.$$

If $p=2$, then we will generate Eq. (1.157), i.e., $\langle f|g\rangle = f_0^* g_0 + f_1^* g_1$.

Problem 6.3. Prove the closure relation Eq. (1.123)

$$\int \prod_\alpha \frac{\mathrm{d}\phi_\alpha^* \, \mathrm{d}\phi_\alpha}{2\,\mathrm{i}\pi} \, \mathrm{e}^{-\sum_\alpha \phi_\alpha^* \phi_\alpha} |\phi\rangle\langle\phi| = \mathbb{1} \tag{1.123}$$

to its fermionic form

$$\mathbb{1} = \int \mathrm{d}\mu(\xi) \, \mathrm{e}^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} |\xi\rangle\langle\xi|$$

for Fermions using Schur's lemma. As in the Boson case, $\left[a_\alpha, \left|\phi\right\rangle\left\langle\phi\right\|\right] = \left(\xi_\alpha - \frac{\partial}{\partial \xi_\alpha'}\right)\left|\phi\right\rangle\left\langle\phi\right|\Big]$, so one must show

$$\int \prod_\alpha \mathrm{d}\xi_\alpha^* \, \mathrm{d}\xi_\alpha \, \mathrm{e}^{-\sum_\alpha \xi_\alpha^* \xi_\alpha} \left(\xi_\alpha - \frac{\partial}{\partial \xi_\alpha^*}\right) A(\xi_\alpha, \xi_\alpha^*) = 0 \tag{P6.1}$$

for any A. First, prove Eq. (P6.1) as it stands, establishing that the left-hand side of Eq. (1.126)

$$\left[a_\alpha, \int \prod_{\alpha'} \frac{\mathrm{d}\phi_{\alpha'}^* \, \mathrm{d}\phi_{\alpha'}}{2\pi\,\mathrm{i}} \, \mathrm{e}^{-\sum_{\alpha'} \phi_{\alpha'}^* \phi_{\alpha'}} |\phi\rangle\langle\phi|\right] = \int \prod_{\alpha'} \frac{\mathrm{d}\phi_{\alpha'}^* \, \mathrm{d}\phi_{\alpha'}}{2\pi\,\mathrm{i}} \, \mathrm{e}^{-\sum_{\alpha'} \phi_{\alpha'}^* \phi_{\alpha'}} |\phi\rangle\langle\phi| = 0 \tag{1.126}$$

must be proportional to unity, and evaluate the constant of proportionality by calculating the matrix element in the zero-particle state.

Then note that Eq. (P6.1) is a special case of integration by parts for Grassmann variables. Show that the general rule for integration by parts is

$$\int \prod_\alpha \mathrm{d}\xi_\alpha^* \, \mathrm{d}\xi_\alpha A(\xi_\alpha^*, \xi_\alpha) \frac{\overrightarrow{\partial}}{\partial \xi_\alpha} B(\xi_\alpha^*, \xi_\alpha) = \int \prod_\alpha \mathrm{d}\xi_\alpha^* \, \mathrm{d}\xi_\alpha A(\xi_\alpha^*, \xi_\alpha) \frac{\overleftarrow{\partial}}{\partial \xi_\alpha} B(\xi_\alpha^*, \xi_\alpha) \tag{P6.2}$$

where $\frac{\overleftarrow{\partial}}{\partial \xi_\alpha}$ acts to the left and the variable ξ_α must be anticommuted to the right to be adjacent to the derivative. Note in particular that the sign in Eq. (P6.2) is reversed from the usual relation for complex variables, and that expressions like $\int \prod_\alpha \mathrm{d}\xi_\alpha^* \, \mathrm{d}\xi_\alpha \frac{\partial}{\partial \xi_\alpha}[A(\xi_\alpha^*, \xi_\alpha)] B(\xi_\alpha^*, \xi_\alpha)$ do not reproduce the right-hand side.

Solution. We need to list some identities of the Grassmann variables.

1. $\xi^2 = 0$ 2. $\xi = \xi^*$ 3. $\xi^* \xi = -\xi \xi^*$ 4. $\int 1 \, \mathrm{d}\xi = 0$ 5. $\int \xi \, \mathrm{d}\xi = 1$ 6. $\int \left[\frac{\partial f(\theta)}{\partial \theta}\right] \mathrm{d}\theta = 0$

Then the exponential term and $A(\xi^*, \xi)$ can be expanded as

$$\mathrm{e}^{-\xi^* \xi} = 1 - \xi^* \xi, \quad \mathrm{e}^{-\xi^* \xi} \xi = \xi, \quad \text{and} \quad A(\xi^*, \xi) = A_{00} + A_{01}\xi + A_{10}\xi^* + A_{11}\xi\xi^*,$$

where the other terms consist ξ^2, ξ^{*2}, or $(\xi^*\xi)^2$ vanish. To prove Eq. (P6.1), consider

$$\int \mathrm{d}\xi^* \, \mathrm{d}\xi \, \mathrm{e}^{-\xi^* \xi} (\xi - \partial_{\xi^*}) A(\xi^*, \xi).$$

The first term of the integral can be expanded as

$$\int \mathrm{d}\xi^* \int \mathrm{d}\xi \, \mathrm{e}^{-\xi^* \xi} \xi A(\xi, \xi^*) = \int \mathrm{d}\xi^* \int \mathrm{d}\xi \xi (A_{00} + A_{01}\xi + A_{10}\xi^* + A_{11}\xi\xi^*) = A_{10},$$

where only the term containing $\xi\xi^*$ survives. Since the derivative of two Grassmann terms satisfies

$$\partial_{\xi^*}[\mathrm{e}^{-\xi^* \xi} A(\xi, \xi^*)] = \partial_{\xi^*}(\mathrm{e}^{-\xi^* \xi}) A(\xi, \xi^*) + \mathrm{e}^{-\xi^* \xi} \, \partial_{\xi^*} A(\xi, \xi^*).$$

Then the second term of the integral can be expanded as

$$\int \mathrm{d}\xi^* \int \mathrm{d}\xi\, \mathrm{e}^{-\xi^*\xi}\, \partial_{\xi^*} A(\xi,\xi^*) = \cancel{\int \mathrm{d}\xi^* \int \mathrm{d}\xi\, \partial_{\xi^*}[\mathrm{e}^{-\xi^*\xi} A(\xi,\xi^*)]} - \int \mathrm{d}\xi^* \int \mathrm{d}\xi\, \partial_{\xi^*}(\mathrm{e}^{-\xi^*\xi}) A(\xi,\xi^*)$$
$$= -\int \mathrm{d}\xi^* \int \mathrm{d}\xi\, \partial_{\xi^*}(1-\xi^*\xi) A(\xi,\xi^*) = \int \mathrm{d}\xi^* \int \mathrm{d}\xi \xi (A_{00} + A_{01}\xi + A_{10}\xi^* + A_{11}\xi\xi^*) = A_{10}.$$

Combining the two terms, we have

$$\int \mathrm{d}\xi^*\, \mathrm{d}\xi\, \mathrm{e}^{-\xi^*\xi}(\xi - \partial_{\xi^*}) A(\xi^*,\xi) = A_{10} - A_{10} = 0.$$

Concerning multi-mode: for a fixed label β, consider the total derivative $\partial_{\xi^*_\beta}$ to $\mathrm{e}^{-\sum_\alpha \xi^*_\alpha \xi_\alpha} A(\xi_\alpha, \xi^*_\alpha)$. The same algebra gives

$$\frac{\partial}{\partial \xi^*_\beta}\, \mathrm{e}^{-\sum_\alpha \xi^*_\alpha \xi_\alpha} A(\xi_\alpha,\xi^*_\alpha) = -\xi_\beta\, \mathrm{e}^{-\sum_\alpha \xi^*_\alpha \xi_\alpha} + \mathrm{e}^{-\sum_\alpha \xi^*_\alpha \xi_\alpha}\, \partial_{\xi^*_\beta} A(\xi,\xi^*),$$

and integrating over $\prod_\alpha \mathrm{d}\xi^*_\alpha\, \mathrm{d}\xi_\alpha$ yields zero. Thus, the multi-mode identity Eq. (P6.1) follows. Then, we shall generate Eq. (1.126). Consider the single mode first, we calculate the "kernel" commutator $[\hat{a}, |\xi\rangle\langle\xi|]$.

(a) $a|\xi\rangle$. Since $|\xi\rangle$ can be expanded as

$$|\xi\rangle = \mathrm{e}^{a^\dagger \xi}|0\rangle = (1 + a^\dagger \xi)|0\rangle.$$

Left act a on it, then we have

$$a|\xi\rangle = a|0\rangle + aa^\dagger \xi|0\rangle = 0 + (1 + a^\dagger a)\xi|0\rangle = \xi|0\rangle.$$

If we multiply ξ to $|\xi\rangle$

$$\xi|\xi\rangle = \xi(1 + a^\dagger\xi)|0\rangle = \xi|0\rangle,$$

the ξ^2 term vanishes. So, we get the expression $a|\xi\rangle\langle\xi| = \xi|\xi\rangle\langle\xi|$.

(b) $\langle\xi|a$. Since $\langle\xi|$ can be expanded as

$$\langle\xi| = \mathrm{e}^{a\xi^*}\langle 0| = \langle 0|(1 + a\xi^*).$$

Right act a on it, then we have

$$\langle\xi|a = \langle 0|(1 + a\xi^*)a = \langle 0|a + \langle 0|a\xi^* a = \langle 0|a,$$

where $\langle 0|a = (a^\dagger|0\rangle)^\dagger = \langle 1|$, and aa forms a zero operator, i.e., $aa = 0$ (for Fermions). If we right act ∂_{ξ^*} on $\langle\xi|$

$$\langle\xi|\overleftarrow{\partial}_{\xi^*} = (1 + a\xi^*)\langle 0|\overleftarrow{\partial}_{\xi^*} = \langle 0|a,$$

So, we get the expression $\langle\xi|a = \langle\xi|\overleftarrow{\partial}_{\xi^*}$, and we have

$$(|\xi\rangle\langle\xi|)\overleftarrow{\partial}_{\xi^*} = (|\xi\rangle\overleftarrow{\partial}_{\xi^*})\langle\xi| + |\xi\rangle(\langle\xi|\overleftarrow{\partial}_{\xi^*}) = ((1 + a^\dagger\xi)|0\rangle\overleftarrow{\partial}_{\xi^*})\langle\xi| + |\xi\rangle(\langle\xi|\overleftarrow{\partial}_{\xi^*}) = |\xi\rangle(\langle\xi|\overleftarrow{\partial}_{\xi^*}).$$

(c) Prove the relation between left and right derivatives. Denote

$$\mathrm{d}\mu(\xi) \equiv \prod_{\alpha} \mathrm{d}\xi_\alpha^* \xi_\alpha.$$

Due to the identities of the derivative of Grassmann variables

$$\frac{\partial}{\partial\xi}(AB) = \left(\frac{\partial A}{\partial \xi}\right)B + (-1)^{\epsilon_A} A\left(\frac{\partial B}{\partial\xi}\right),$$

where ϵ_A is the parity of A. Integral over the total derivative, and using the partial integration formula of Grassmann variables

$$0 \equiv \int \mathrm{d}\mu \overrightarrow{\partial}_\xi (AB)\xi = \int \mathrm{d}\mu(\overrightarrow{\partial}_\xi A)B + (-1)^{\epsilon_A}\int \mathrm{d}\mu A(\overrightarrow{\partial}_\xi B),$$
$$\int \mathrm{d}\mu A(\overrightarrow{\partial}_\xi B) = -(-1)^{\epsilon_A}\int \mathrm{d}\mu(\overrightarrow{\partial}_\xi A)B.$$

Due to the property of the Grassmann variable, the left-acting derivative to A can be expanded as

$$A\overleftarrow{\partial}_\xi = -(-1)^{\epsilon_A}\overrightarrow{\partial}_\xi A,$$

by combining the two expressions above, we can obtain

$$\int \mathrm{d}\mu A(\overrightarrow{\partial}_\xi B) = \int \mathrm{d}\mu A\overleftarrow{\partial}_\xi B.$$

Also for multi-mode, which we proved Eq. (P6.2).

Then, the "kernel" commutator becomes

$$[\hat{a}, |\xi\rangle\langle\xi|] = a|\xi\rangle\langle\xi| - |\xi\rangle\langle\xi|a = \xi|\xi\rangle\langle\xi| - (|\xi\rangle\langle\xi|)\overleftarrow{\partial}_{\xi^*} = (\xi - \overrightarrow{\partial}_{\xi^*})|\xi\rangle\langle\xi|,$$

and the commutator in Eq. (1.126) can be written as

$$\left[a, \int \mathrm{d}\mu\, \mathrm{e}^{\xi^*\xi}|\xi\rangle\langle\xi|\right] = \int \mathrm{d}\mu\, \mathrm{e}^{\xi^*\xi}(\xi - \overrightarrow{\partial}_{\xi^*})|\xi\rangle\langle\xi| \overset{\text{Eq. (P6.1)}}{=\!=\!=} 0,$$

and it also satisfies multi-mode, which we proved Eq. (1.126). Due to Schur's lemma, the operator $\mathcal{O} \equiv \int \mathrm{d}\mu\, \mathrm{e}^{\xi^*\xi}|\xi\rangle\langle\xi|$ commutes to a, then it is proportional to unity, i.e., $\mathcal{O} = C\mathbb{1}$. To determine C, calculate the vacuum expectation of $\mathcal{O}$

$$C \equiv \langle 0|\mathcal{O}|0\rangle = \int \mathrm{d}\mu(\xi)\, \mathrm{e}^{-\sum_\alpha \xi_\alpha^*\xi}\langle 0|\xi\rangle\langle\xi|0\rangle.$$

Since the expansion of $|\xi\rangle$, $\langle\xi|$, and $\mathrm{e}^{-\xi^*\xi}$, we have

$$\langle 0|\xi\rangle = \langle\xi|0\rangle = 1, \quad \text{and} \quad \int \mathrm{d}\xi^*\, \mathrm{d}\xi\, \mathrm{e}^{-\xi^*\xi} = \int \mathrm{d}\xi^*\, \mathrm{d}\xi(1 - \xi^*\xi) = 1.$$

Therefore

$$C = 1, \quad \mathcal{O} = \mathbb{1}, \quad \text{and} \quad \int \mathrm{d}\mu(\xi)\, \mathrm{e}^{-\sum_\alpha \xi_\alpha^*\xi}|\xi\rangle\langle\xi| = \mathbb{1},$$

which we proved the fermionic form for Eq. (1.123).

Lecture #7 Homework #7 [2025-10-22]

Problem 7.1. Investigate a simple model Hamiltonian for BCS superconductivity,

$$\mathcal{H}_k = t(k)(a^\dagger_{k\uparrow}a_{k\uparrow} + a^\dagger_{-k\downarrow}a_{-k\downarrow}) - \Delta(a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow} + a_{-k\downarrow}a_{k\uparrow}).$$

(a) Find the energy eigenvalues of $\mathcal{H}_k$.

(b) Give the corresponding eigenstates.

(c) Give the possible excitation energies of the system.

(d) Identify the excitation gap, i.e. the minimum of the excitation energies. Consider the simplest 1D case: assuming $t(k) = -2t\cos k - \mu$.

Solution.

(a) We consider the two big terms of the Hamiltonian, respectively. Working in the 4D Fock space, there are 4 identities

1. $|0\rangle \equiv |0_{k\uparrow}, 0_{-k\downarrow}\rangle$ 2. $|k\uparrow\rangle \equiv a^\dagger_{k\uparrow}|0\rangle$ 3. $|-k\downarrow\rangle \equiv a^\dagger_{-k\downarrow}|0\rangle$ 4. $|k\uparrow, -k\downarrow\rangle \equiv a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}|0\rangle$

Now, we shall compute how each of the 4 operators in $\mathcal{H}_k$ acts on the basis.

i. Number term

$$\begin{aligned} a^\dagger_{k\uparrow}a_{k\uparrow}|k\uparrow\rangle &= |k\uparrow\rangle, & a^\dagger_{k\uparrow}a_{k\uparrow}\{|0\rangle,\ |-k\downarrow\rangle,\ |k\uparrow,-k\downarrow\rangle\} &= \{0\},\\ a^\dagger_{-k\downarrow}a_{-k\downarrow}|k\uparrow\rangle &= |-k\downarrow\rangle, & a^\dagger_{-k\downarrow}a_{-k\downarrow}\{|0\rangle,\ |-k\downarrow\rangle,\ |k\uparrow,-k\downarrow\rangle\} &= \{0\}.\end{aligned}$$

ii. Pairing term

$$\begin{aligned} a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}|0\rangle &= |k\uparrow,-k\downarrow\rangle, & a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}\{|k\uparrow\rangle,\ |-k\downarrow\rangle,\ |k\uparrow,-k\downarrow\rangle\} &= \{0\},\\ a_{-k\downarrow}a_{k\uparrow}|k\uparrow,-k\downarrow\rangle &= |0\rangle, & a_{-k\downarrow}a_{k\uparrow}\{|0\rangle,\ |k\uparrow\rangle,\ |k\downarrow\rangle\} &= \{0\}.\end{aligned}$$

The number term acts diagonally, giving

State	$\lvert 0\rangle$	$\lvert k\uparrow\rangle$	$\lvert -k\downarrow\rangle$	$\lvert k\uparrow,-k\downarrow\rangle$
Energy	0	$t(k)$	$t(k)$	$2t(k)$

Pairing term couples only $|0\rangle$ and $|k\uparrow,-k\downarrow\rangle$

$$\begin{aligned} &-\Delta(a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow} + a_{-k\downarrow}a_{-k\downarrow})|0\rangle = -\Delta|k\uparrow,-k\downarrow\rangle,\\ &-\Delta(a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow} + a_{-k\downarrow}a_{-k\downarrow})\Delta|k\uparrow,-k\downarrow\rangle = -|0\rangle.\end{aligned}$$

Since some of the actions act on, or create / annihilate the number of particles are odd, and other actions are even, so we can separate the Hamiltonian into two independent blocks

$$\mathcal{H}_k = \begin{cases} \{|k\uparrow\rangle,\ |-k\downarrow\rangle\}, & \text{Odd sector,}\\ \{|0\rangle,\ |k\uparrow,\ -k\downarrow\rangle\}, & \text{Even sector,}\end{cases}$$

where

$$\begin{aligned} \mathcal{H}_{\text{odd}} &= \begin{pmatrix} \langle k\uparrow|\mathcal{H}_k|k\uparrow\rangle & \langle k\uparrow|\mathcal{H}_k|-k\downarrow\rangle \\ \langle -k\downarrow|\mathcal{H}_k|k\uparrow\rangle & \langle -k\downarrow|\mathcal{H}_k|-k\downarrow\rangle\end{pmatrix} &= \begin{pmatrix} t(k) & 0\\ 0 & t(k)\end{pmatrix},\\ \mathcal{H}_{\text{even}} &= \begin{pmatrix} \langle 0|\mathcal{H}_k|0\rangle & \langle 0|\mathcal{H}_k|k\uparrow,-k\downarrow\rangle \\ \langle k\uparrow,-k\downarrow|\mathcal{H}_k|0\rangle & \langle k\uparrow,-k\downarrow|\mathcal{H}_k|k\uparrow,-k\downarrow\rangle\end{pmatrix} &= \begin{pmatrix} 0 & -\Delta\\ -\Delta & 2t(k)\end{pmatrix}.\end{aligned}$$

It is trivial that in the odd sector, both states have energy $E_{\text{odd}} = t(k)$, corresponding to 2nd degeneracy. In the even sector, we need to solve the eigenvalue equation

$$\det |\mathcal{H}_{\text{even}} - \lambda| = 0.$$

Hence, we have the eigenvalues for the even sector

$$\lambda_1 = t(k) + \sqrt{t^2(k) + \Delta^2}, \quad \lambda_2 = t(k) - \sqrt{t^2(k) + \Delta^2}.$$

In summary, the eigenvalues of $\mathcal{H}_k$ are

$$E_{\text{odd}} = t(k), \quad E_{\text{ground}} = t(k) - \sqrt{t^2(k) + \Delta^2}, \text{ (Degeneracy} = 2) \quad E_{\text{excited}} = t(k) + \sqrt{t^2(k) + \Delta^2}.$$

(b) Since in the odd sector, the Hamiltonian is diagonal, so it is trivial that the eigenstates in the odd sector are

$$\psi_{\text{odd 1}} = |k\uparrow\rangle, \quad \psi_{\text{odd 2}} = |-k\downarrow\rangle.$$

Concerning the even sector. Denote $E_k = \sqrt{t^2(k) + \Delta^2}$ for simplification. Since in the 4D Fock space, the basis should be orthonormal. So, assume the two eigenstates in the even sector to be

$$\psi_{\text{ground}} = a_1|0\rangle + b_1|k\uparrow, -k\downarrow\rangle, \quad \psi_{\text{excited}} = a_2|0\rangle + b_2|k\uparrow, -k\downarrow\rangle.$$

Since the orthonormality $\langle \psi_{\text{even 1}} | \psi_{\text{even 2}} \rangle = 0$, then we can derive

$$\frac{a_1}{b_1} = -\frac{b_2}{a_2}.$$

Let $a_1 = a_k$, $b_1 = b_k$. Then, the two eigenstates in the even sector can be expressed as

$$\psi_{\text{ground}} = a_k|0\rangle + b_k|k\uparrow, -k\downarrow\rangle, \quad \psi_{\text{excited}} = b_k|0\rangle - a_k|k\uparrow, -k\downarrow\rangle.$$

where a_k and b_k should satisfies the normalization $a_k^2 + b_k^2 = 1$. To obtain a_k and b_k, substitute ψ_{ground} into the eigenequation

$$\begin{pmatrix} 0 & -\Delta \\ -\Delta & 2t(k) \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} = E_{\text{ground}} \begin{pmatrix} a_k \\ b_k \end{pmatrix}.$$

Then we will obtain

$$a_k^2 = \frac{1}{2}\left(1 + \frac{t(k)}{E_k}\right), \quad b_k^2 = \frac{1}{2}\left(1 - \frac{t(k)}{E_k}\right).$$

So, the eigenstates in the even sector are

$$\psi_{\text{ground}} = \sqrt{\frac{1}{2}\left(1 + \frac{t(k)}{E_k}\right)}|0\rangle + \sqrt{\frac{1}{2}\left(1 - \frac{t(k)}{E_k}\right)}|k\uparrow, -k\downarrow\rangle,$$
$$\psi_{\text{excited}} = \sqrt{\frac{1}{2}\left(1 - \frac{t(k)}{E_k}\right)}|0\rangle - \sqrt{\frac{1}{2}\left(1 + \frac{t(k)}{E_k}\right)}|k\uparrow, -k\downarrow\rangle.$$

(c) Since the energy eigenvalues of $\mathcal{H}_k$ are

$$\{t(k),\ t(k),\ t(k) - E_k,\ t_k + E_k\}.$$

It is trivial that the ground energy is $E_0 = t(k) - E_k$, and other there energies corresponding states are all excited states. So, the excitation energies of the system are

$$\Delta E_{\text{1qp}} = t(k) - (t(k) - E_k) = E_k, \quad \Delta E_{\text{2qp}} = (t(k) + E_k) - (t(k) - E_k) = 2E_k.$$

(Note: "qp" means "quasi-particle").

(d) It is trivial that the minimum of the excitation energies is

$$\Delta E_{\min} = \Delta E_{1qp} = E_k = \sqrt{t^2(k) + \Delta^2}.$$

Since $t(k) = -2t\cos k - \mu$, so the minimum occurs when $t(k) = 0$, i.e.,

$$k = \cos^{-1}\left(-\frac{\mu}{2t}\right) = \pi - \cos^{-1}\left(\frac{\mu}{2t}\right).$$

Then, the minimum excitation energy in the simplest 1D case is

$$\Delta E_{\min} = \Delta.$$

Problem 7.2.

(a) Show that the excitations of a BCS superconductor are generated from Exercise 7.1 by the operators

$$\rho_{k\uparrow}^\dagger = u_k a_{k\uparrow}^\dagger - v_k a_{-k\downarrow}; \quad \rho_{-k\downarrow}^\dagger = u_k a_{-k\downarrow}^\dagger + v_k a_{k\uparrow}.$$

The coefficients u_k and v_k are defined as

$$u_k^2 = \frac{1}{2}\left\{1 + \frac{t(k)}{[t^2(k)+\Delta^2]^{1/2}}\right\}, \quad v_k^2 = 1 - u_k^2.$$

(b) Prove that these operators are purely Fermionic operators.

(c) Compute the commutator

$$[\mathcal{H}^*, \rho_{k\uparrow}^\dagger]_-.$$

How is this result to be interpreted?

(d) Formulate and solve the equation of motion of the retarded Green's function

$$\hat{G}_{k\uparrow}^{\text{ret}}(E) = \langle\langle \rho_{k\uparrow}; \rho_{k\uparrow}^\dagger \rangle\rangle_E^{\text{ret}}.$$

(e) Compute Green's functions of the original fermions a_ks and $a_k^\dagger$s, consider all non-zero combinations. You can express them via linear transformations between ρ_ks and a_ks.

Solution.

(a) *Proof.* Obviously, the given coefficients u_k and v_k is just a_k and b_k that obtained in Exercise 7.1. List $\rho_{k\uparrow}$, $\rho_{-k\downarrow}$ and their conjugates first

$$\begin{aligned}
\rho_{k\uparrow}^\dagger &= u_k a_{k\uparrow}^\dagger - v_k a_{-k\downarrow}, \quad \rho_{-k\downarrow}^\dagger = u_k a_{-k\downarrow}^\dagger + v_k a_{k\uparrow}, \\
\rho_{k\uparrow} &= u_k a_{k\uparrow} - v_k a_{-k\downarrow}^\dagger, \quad \rho_{-k\downarrow} = u_k a_{-k\downarrow} + v_k a_{k\uparrow}^\dagger.
\end{aligned}$$

from which we can obtain the inverse transformations

$$\begin{aligned}
a_{k\uparrow}^\dagger &= u_k \rho_{k\uparrow}^\dagger + v_k \rho_{-k\downarrow}, \quad a_{-k\downarrow}^\dagger = u_k \rho_{-k\downarrow}^\dagger - v_k \rho_{k\uparrow}, \\
a_{k\uparrow} &= u_k \rho_{k\uparrow} + v_k \rho_{-k\downarrow}^\dagger, \quad a_{-k\downarrow} = u_k \rho_{-k\downarrow} - v_k \rho_{k\uparrow}^\dagger.
\end{aligned}$$

Then, using the identities that can be easily obtained

$$u_k^2 - v_k^2 = \frac{t(k)}{E_k}, \quad 2u_k v_k = \frac{\Delta}{E_k}, \quad u_k^2 + v_k^2 = 1, \quad E_k = \sqrt{t^2(k)+\Delta^2},$$

and the operator identities of the fermions $a_{k\uparrow}$, $a_{-k\uparrow}$ and their conjugates to substitute them into $\mathcal{H}_k$, we have

$$\mathcal{H}_k = E_k(\rho_{k\uparrow}^\dagger \rho_{k\uparrow} + \rho_{-k\uparrow}^\dagger \rho_{-k\uparrow}) + (t(k) - E_k) \equiv \mathcal{H}^*,$$

where the constant term $t(k) - E_k$ is just an energy shift and doesn't affect the excitation spectrum. Thus, the operators $\rho_{k\uparrow}^\dagger$ and $\rho_{k\downarrow}^\dagger$ (and their conjugates) create create excitations with energy E_k above the BCS ground state. □

(b) *Proof.* Since the fact that Fermionic operators $a_{k\uparrow}, a_{-k\downarrow}$ satisfy the anticommutation relations of, then calculate the same anticommutation relations between the operators $\rho_{k\uparrow}$, $\rho_{-k\downarrow}$, and their conjugates

i. $\{\rho_{k\uparrow}, \rho^\dagger_{k\uparrow}\}$

$$\begin{aligned}\rho_{k\uparrow}\rho^\dagger_{k\uparrow} &= u_k^2 a_{k\uparrow}a^\dagger_{k\uparrow} - u_kv_ka_{k\uparrow}a_{-k\downarrow} - u_kv_ka^\dagger_{-k\downarrow}a^\dagger_{k\uparrow} + v_k^2a^\dagger_{-k\downarrow}a_{-k\downarrow},\\ \rho^\dagger_{k\uparrow}\rho_{k\uparrow} &= u_k^2a^\dagger_{k\uparrow}a_{k\uparrow} - u_kv_ka^\dagger_{k\uparrow}a^\dagger_{-k\downarrow} - u_kv_ka_{-k\downarrow}a_{k\uparrow} + v_k^2a_{-k\downarrow}a^\dagger_{-k\downarrow}.\end{aligned}$$

Adding and using fermionic anticommutators

$$\{a_{k\uparrow}, a^\dagger_{k\uparrow}\} = 1, \quad \{a_{-k\downarrow}, a^\dagger_{-k\downarrow}\} = 1,$$

where the cross terms vanish. We get

$$\{\rho_{k\uparrow}, \rho^\dagger_{k\uparrow}\} = u_k^2 + v_k^2 = 1.$$

ii. $\{\rho_{-k\downarrow}, \rho^\dagger_{-k\downarrow}\}$

$$\begin{aligned}\rho_{-k\downarrow}\rho^\dagger_{-k\downarrow} &= u_k^2a_{-k\downarrow}a^\dagger_{-k\downarrow} + u_kv_ka_{-k\downarrow}a_{k\uparrow} + u_kv_ka^\dagger_{k\uparrow}a^\dagger_{-k\downarrow} + v_k^2a^\dagger_{k\uparrow}a_{k\uparrow},\\ \rho^\dagger_{-k\downarrow}\rho_{-k\downarrow} &= u_k^2a^\dagger_{-k\downarrow}a_{-k\downarrow} + u_kv_ka^\dagger_{-k\downarrow}a^\dagger_{k\uparrow} + u_kv_ka_{k\uparrow}a_{-k\downarrow} + v_k^2a_{k\uparrow}a^\dagger_{k\uparrow}.\end{aligned}$$

Adding and using anticommutators

$$\{\rho_{-k\downarrow}, \rho^\dagger_{-k\downarrow}\} = u_k^2 + v_k^2 = 1.$$

iii. $\{\rho_{k\uparrow}, \rho^\dagger_{-k\downarrow}\}$

$$\begin{aligned}\rho_{k\uparrow}\rho^\dagger_{-k\downarrow} &= u_k^2a_{k\uparrow}a^\dagger_{-k\downarrow} + u_kv_ka_{k\uparrow}a_{k\uparrow} - u_kv_ka^\dagger_{-k\downarrow}a^\dagger_{-k\downarrow} - v_k^2a^\dagger_{-k\downarrow}a_{k\uparrow},\\ \rho^\dagger_{-k\downarrow}\rho_{k\uparrow} &= u_k^2a^\dagger_{-k\downarrow}a_{k\uparrow} + u_kv_ka_{k\uparrow}a_{k\uparrow} - u_kv_ka^\dagger_{-k\downarrow}a^\dagger_{-k\downarrow} - v_k^2a_{k\uparrow}a^\dagger_{-k\downarrow}.\end{aligned}$$

All terms vanish due to anticommutation

$$\{\rho_{k\uparrow}, \rho^\dagger_{-k\downarrow}\} = 0.$$

iv. $\{\rho_{k\uparrow}, \rho_{-k\downarrow}\}$

$$\begin{aligned}\rho_{k\uparrow}\rho_{-k\downarrow} &= u_k^2a_{k\uparrow}a_{-k\downarrow} + u_kv_ka_{k\uparrow}a^\dagger_{k\uparrow} - u_kv_ka^\dagger_{-k\downarrow}a_{-k\downarrow} - v_k^2a^\dagger_{-k\downarrow}a^\dagger_{k\uparrow},\\ \rho_{-k\downarrow}\rho_{k\uparrow} &= u_k^2a_{-k\downarrow}a_{k\uparrow} + u_kv_ka^\dagger_{k\uparrow}a_{k\uparrow} - u_kv_ka_{-k\downarrow}a^\dagger_{-k\downarrow} - v_k^2a^\dagger_{k\uparrow}a^\dagger_{-k\downarrow}.\end{aligned}$$

Adding and simplifying

$$\{\rho_{k\uparrow}, \rho_{-k\downarrow}\} = u_kv_k - u_kv_k = 0.$$

v. For other anticommutators, by similar computations or Hermitian conjugation

$$\{\rho^\dagger_{k\uparrow}, \rho^\dagger_{-k\downarrow}\} = 0, \quad \{\rho_{k\uparrow}, \rho_{k\uparrow}\} = 0, \quad \text{etc.}$$

In summary, the operators $\rho_{k\sigma}$, $\rho^\dagger_{k\sigma}$, and their conjugates satisfy

$$\{\rho_{k\sigma}, \rho^\dagger_{k'\sigma'}\} = \delta_{kk'}\delta_{\sigma\sigma'}, \quad \{\rho_{k\sigma}, \rho_{k'\sigma'}\} = 0, \quad \{\rho^\dagger_{k\sigma}, \rho^\dagger_{k'\sigma'}\} = 0.$$

Thus, they are purely Fermionic operators. □

(c) i. The first term

$$[\rho^\dagger_{k'\uparrow}\rho_{k'\uparrow},\rho^\dagger_{k\uparrow}]=\rho^\dagger_{k'\uparrow}\{\rho_{k'\uparrow},\rho^\dagger_{k\uparrow}\}-\{\rho^\dagger_{k'\uparrow},\rho^\dagger_{k\uparrow}\}\rho_{k'\uparrow}=\delta_{kk'}\rho^\dagger_{k'\uparrow}\cdot 1-0=\delta_{kk'}\rho^\dagger_{k\uparrow}.$$

ii. The second term

$$[\rho^\dagger_{-k'\downarrow}\rho_{-k'\downarrow},\rho^\dagger_{k\uparrow}]=\rho^\dagger_{-k'\downarrow}\{\rho_{-k'\downarrow},\rho^\dagger_{k\uparrow}\}-\{\rho^\dagger_{-k'\downarrow},\rho^\dagger_{k\uparrow}\}\rho_{-k'\downarrow}=0.$$

Thus, only the term with $k'=k$ contributes

$$[\mathcal{H},\rho^\dagger_{k\uparrow}]=E_k\rho^\dagger_{k\uparrow}.$$

This result interrupts that $\rho^\dagger_{k\uparrow}$ is an eigenoperator of the Hamiltonian $\mathcal{H}^*$ with eigenvalue E_k. Equivalently, $\rho^\dagger_{k\uparrow}$ to an eigenstate of $\mathcal{H}$ creates an excitation of energy E_k measured from the BCS ground state, confirming that $\rho^\dagger_{k\uparrow}$ is the quasiparticle creation operator.

(d) The retarded Green's function for the quasiparticle is formulated as

$$\hat{G}^{\text{ret}}_{k\uparrow}(E)=\langle\langle\rho_{k\uparrow};\rho^\dagger_{k\uparrow}\rangle\rangle_E=-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}Et}\theta(t)\langle\{\rho_{k\uparrow}(t),\rho^\dagger_{k\uparrow}(0)\}\rangle.$$

where we used the "God-Given" unit to omit the $\hbar$, and use the Heaviside step function to complete the integral range to the whole time-axis. Apply the equation of motion for the retarded Green's function

$$E\hat{G}^{\text{ret}}_{k\uparrow}(E)=\langle\{\rho_{k\uparrow},\rho^\dagger_{k\uparrow}\}\rangle+\langle\langle[\rho_{k\uparrow},\mathcal{H}^*];\rho^\dagger_{k\uparrow}\rangle\rangle=1+E_k\langle\langle\rho_{k\uparrow};\rho^\dagger_{k\uparrow}\rangle\rangle_E=1+E_k\hat{G}^{\text{ret}}_{k\uparrow}(E),$$

where $\{\rho_{k\uparrow},\rho^\dagger_{k\uparrow}\}=1$. Rearrange the equation of motion, we have

$$(E-E_k)\hat{G}^{\text{ret}}_{k\uparrow}(E)=\mathbb{1},\quad \hat{G}^{\text{ret}}_{k\uparrow}(E)=\frac{1}{E-E_k+\mathrm{i}0^=+}$$

where the $\mathrm{i}0^+$ enforces retarded boundary conditions.

(e) We have already calculated the retarded Green's function

$$\hat{G}^{\text{ret}}_{k\uparrow}(E)=\langle\langle\rho_{k\uparrow};\rho^\dagger_{k\uparrow}\rangle\rangle^{\text{ret}}_E.$$

So, similarly, we can replace $\rho_{k\uparrow}$ and $\rho_{k\uparrow}$ with $a_{k\uparrow}$, $a_{-k\downarrow}$ and their conjugates. By substituting the inverse transformations obtained in (a)

$$\begin{aligned}a^\dagger_{k\uparrow}&=u_k\rho^\dagger_{k\uparrow}+v_k\rho_{-k\downarrow},\quad a^\dagger_{-k\downarrow}=u_k\rho^\dagger_{-k\downarrow}-v_k\rho_{k\uparrow},\\ a_{k\uparrow}&=u_k\rho_{k\uparrow}+v_k\rho^\dagger_{-k\downarrow},\quad a_{-k\downarrow}=u_k\rho_{-k\downarrow}-v_k\rho^\dagger_{k\uparrow}.\end{aligned}$$

we can obtain the following Green functions

$$\begin{aligned}
G^{\text{ret}}_{\uparrow\uparrow}(k,E)&=\langle\langle a_{k\uparrow};a^\dagger_{k\uparrow}\rangle\rangle &&=\tfrac{u_k^2}{E-E_k+\mathrm{i}0^+}+\tfrac{v_k^2}{E+E_k+\mathrm{i}0^+}, &&\text{Normal Green's function,}\\
G^{\text{ret}}_{\downarrow\downarrow}(k,E)&=\langle\langle a_{k\downarrow};a^\dagger_{k\downarrow}\rangle\rangle &&=\tfrac{u_k^2}{E-E_k+\mathrm{i}0^+}+\tfrac{v_k^2}{E+E_k+\mathrm{i}0^+}, &&\text{Normal Green's function,}\\
G^{\text{ret}}_{\uparrow\downarrow}(k,E)&=\langle\langle a_{k\uparrow};a_{-k\downarrow}\rangle\rangle &&=-\tfrac{\Delta}{E^2-E_k^2+\mathrm{i}0^+}, &&\text{Anomalous Green's function,}\\
\tilde{G}^{\text{ret}}_{\uparrow\downarrow}(k,E)&=\langle\langle a^\dagger_{k\uparrow};a^\dagger_{-k\downarrow}\rangle\rangle &&=\tfrac{\Delta}{E^2-E_k^2+\mathrm{i}0^+}, &&\text{Anomalous Green's function,}
\end{aligned}$$

and other combinations vanish due to symmetry.

Lecture #8 Homework #8 [2025-10-28]

Problem 8.1 (Read material Sec. 6.10 and finish the following exercise.)**.** Consider the Haldane model. The Bloch Hamiltonian is given by

$$\mathcal{H}(\boldsymbol{k}) = \hat{d}_x(\boldsymbol{k})\hat{\sigma}_x + \hat{d}_y(\boldsymbol{k})\hat{\sigma}_y + \hat{d}_z(\boldsymbol{k})\hat{\sigma}_z,$$

where $\sigma_{x,y,z}$ are Pauli matrices, and the vector $\hat{\boldsymbol{d}}(\boldsymbol{k})$ has components

$$d_x(\boldsymbol{k}) = t_1 \sum_{i=1}^{3} \cos(\boldsymbol{k}\cdot\boldsymbol{a}_i), \quad d_y(\boldsymbol{k}) = t_1 \sum_{i=1}^{3} \sin(\boldsymbol{k}\cdot\boldsymbol{a}_i), \quad d_z(\boldsymbol{k}) = M - 2t_2 \sin\phi \sum_{i=1}^{3} \sin(\boldsymbol{k}\cdot\boldsymbol{b}_i),$$

where t_1 is the real NN hopping amplitude, t_2 is the NNN hopping amplitude with complex phase ϕ, M is the sublattice mass (Semenoff mass), and $\boldsymbol{a}_i$, $\boldsymbol{b}_i$ are lattice vectors for NN and NNN hoppings, respectively. The parameter space is spanned by (M,ϕ). We formally obtain the Green's function as

$$\mathbf{G}[\omega,k] = [\omega\mathbb{1} - \mathcal{H}(k)]^{-1} = \operatorname{adj}(\omega\mathbb{1} - \mathcal{H}(k)) \det[\omega\mathbb{1} - \mathcal{H}(k)]^{-1} = \begin{pmatrix} \frac{\omega+d_z}{\omega^2-\mathbf{d}^2} & \frac{d_x+\mathrm{i}d_y}{\omega^2-\mathbf{d}^2} \\ \frac{d_x-\mathrm{i}d_y}{\omega^2-\mathbf{d}^2} & \frac{\omega-d_z}{\omega^2-\mathbf{d}^2} \end{pmatrix}.$$

Try to write the above Green function into self-energy form in the following different ways.

(a) Focusing on a diagonal component, say $\frac{\omega+d_z}{\omega^2-\mathbf{d}^2}$, write it into a self-energy form.

Remark. Consider $G^0 = (\omega - d_z)^{-1}$ and note $\mathbf{d}^2 = d_x^2 + d_y^2 + d_z^2$. Discuss the "mass renormalization" and "lifetime". Discuss how to interpret them.

Solution. Since $\mathbf{d}^2 = d_x^2 + d_y^2 + d_z^2$, then, the denominator of the diagonal element of the Green function can be expressed as

$$\omega^2 - \mathbf{d}^2 = (\omega - d_z)(\omega + d_z) - (d_x^2 + d_y^2).$$

So, the numerator and denominator can be divided by $(\omega + d_z)$. Yields,

$$G_{11}(\omega,\mathbf{k}) = \frac{\omega + d_z}{\omega^2 - \mathbf{d}^2} = \frac{1}{(\omega - d_z) - \frac{d_x^2+d_y^2}{\omega+d_z}}.$$

It can be written as the usual "bare inverse minus self-energy" form

$$G_{11}(\omega,\mathbf{k}) = \frac{1}{G_{11}^0(\omega,\mathbf{k})^{-1} - \Sigma_{11}(\omega,\mathbf{k})},$$

where

$$G_{11}^0(\omega,\mathbf{k})^{-1} \equiv \omega - d_z(\mathbf{k}), \quad \text{and} \quad \Sigma_{11}(\omega,\mathbf{k}) = \frac{d_x(\mathbf{k})^2 + d_y(\mathbf{k})^2}{\omega + d_z(\mathbf{k})}.$$

show that the off-diagonal kinetic terms d_x and d_y appear as an energy-dependent self-energy for the diagonal component.

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Physics interpretation This exact representation shows how inter-sublattice hopping (d_x, d_y) modifies the bare propagator $(\omega - d_z)^{-1}$ through an energy-dependent self-energy.

(a) **Mass Renormalization.** The real part of Σ shifts the quasiparticle pole from d_z to the band energy $|\mathbf{d}| = \sqrt{d_x^2 + d_y^2 + d_z^2}$. Near $\omega \approx d_z$, we have the approximations

$$\Sigma \approx \frac{d_x^2 + d_y^2}{2d_z}, \quad \text{and} \quad d_z^{\text{eff}} \approx d_z + \frac{d_x^2 + d_y^2}{2d_z},$$

showing how inter-sublattice hopping renormalizes the sublattice mass.

(b) **Lifetime.** The imaginary part $-\operatorname{Im}\Sigma = \pi(d_x^2 + d_y^2)\delta(\omega + d_z)$ is non-zero only at $\omega = -d_z$. Since the band energies are $\pm|\mathbf{d}|$, this causes no broadening at the quasiparticle poles, which interprets that the clean, noninteracting system exhibits infinite lifetime.

Lecture #9 Homework #9 [2025-11-04]

Problem 9.1. Calculate the Landau parameters to leading order in $\lambda_{1,2}$ for a Fermi liquid with the contact interactions

(a) $V(x-x') = \lambda_1 \delta^{(3)}(x-x')$.

(b) $V(x-x') = -\lambda_2 \nabla^2 \delta^{(3)}(x-x')$ (so that $V(q) = \lambda_1 q^2$ in Fourier space).

(c) Taking the results of (a) and (b) literally, sketch the regions of the λ_1, λ_2 phase diagram where the Fermi surface becomes unstable.

Solution.

(a)

Problem 9.2. Test your understanding of Landau's mass renormalization formula by generalizing it to include the effect of a magnetization. Suppose we introduce a second vector potential into (6.86)

$$A(\theta) \underset{|\boldsymbol{k}|\to 0}{\sim} \int_{k_F - k\cos\theta}^{k_F} \mathrm{d}q \frac{2\,\mathrm{i}\pi a^2}{\omega - v_F(|\boldsymbol{k}+\boldsymbol{q}| - q)} = \frac{2\,\mathrm{i}\pi a^2 k\cos\theta}{\omega - v_F k\cos\theta}. \tag{6.86}$$

that couples to the spin current, writing

$$\mathcal{H}[\mathbf{A}_N, \mathbf{W}] = \sum_\sigma \int \mathrm{d}^3 x \frac{1}{2m} \psi_\sigma^\dagger(x)[(-\mathrm{i}\hbar\nabla - \mathbf{A}_N - \sigma\mathbf{W})^2]\psi_\sigma(x) + \hat{V}$$

Whereas $\mathbf{A}_N$ couples to the current of particles, $\mathbf{W}$ couples to the (z component of the) spin current. Assume that V conserves spin current.

(a) By comparing the bare shift of the energies

$$\delta\epsilon_{\mathbf{p}\sigma}^{(0)} = -\frac{\mathbf{p}}{m} \cdot (\mathbf{A}_N + \sigma\mathbf{W})$$

with the shift that result from interaction feedback,

$$\delta\epsilon_{\mathbf{p}\sigma} = -\frac{\mathbf{p}}{m^*} \cdot \mathbf{A}_N - \sigma \frac{\mathbf{p}}{m_s^*} \cdot \mathbf{W},$$

show that there are two different mass renormalizations,

$$\frac{m}{m^*} = \frac{1}{1+F_1^s},$$
$$\frac{m}{m_s^*} = \frac{1}{1+F_1^a}.$$

(b) Show that, when the Fermi liquid is polarized, the masses of the "up" and "down" quasiparticles are now different, and given by

$$\frac{1}{m_\sigma^*} = \frac{1}{m}\left[\frac{1}{1+F_1^s} + \frac{M}{1+F_1^a}\right], \quad (\sigma = \uparrow, \downarrow)$$

where the magnetization $M = n_\uparrow - n_\downarrow$ is the difference of "up" and "down" densities.

Solution.

(a)