

# TTK4135 Optimization and Control

## Helicopter Lab Report

716120, 723987  
Group 1

April 28, 2015

## **Abstract**

A movable arm capable of lateral and longitudinal motion by applied thrust from an attached dual rotor pitch head assembly is stabilized by pitch and elevation angle PID controllers, and the feasibility of controlling the arm by employing in an open loop a sequence of pitch and elevation angle setpoints is investigated. The input sequences are obtained by minimizing a quadratic objective function constrained by the system dynamics and with linear and non-linear constraints according to the given travel and elevation trajectory. Enhanced performance is achieved by adding a linear quadratic controller to update the optimal input sequence with state feedback.

# Contents

<b>Abstract</b>	<b>2</b>
<b>Contents</b>	<b>3</b>
<b>1 Introduction</b>	<b>4</b>
<b>2 Problem Description</b>	<b>5</b>
<b>3 Repetition/Introduction to Simulink/QuaRC</b>	<b>8</b>
3.1 PID-(re)tuning . . . . .	8
3.2 Model parameter estimation . . . . .	8
3.3 Results and discussion . . . . .	9
<b>4 Optimal Control of Pitch/Travel without Feedback</b>	<b>12</b>
4.1 State space model . . . . .	12
4.2 Discretization . . . . .	12
4.3 Optimal trajectory . . . . .	13
4.4 Results and discussion . . . . .	14
<b>5 Optimal Control of Pitch/Travel with Feedback (LQ)</b>	<b>15</b>
5.1 Discrete LQR . . . . .	15
5.2 Results and discussion . . . . .	15
5.2.1 MPC discussion . . . . .	16
<b>6 Optimal Control of Pitch/Travel and Elevation with and without Feedback</b>	<b>18</b>
6.1 State space model . . . . .	18
6.2 Discretization . . . . .	18
6.3 Optimization problem with nonlinear constraints . . . . .	18
6.4 Discrete LQR . . . . .	19
6.5 Optional: Additional constraints . . . . .	20
6.6 Results and discussion . . . . .	20
<b>7 Discussion</b>	<b>24</b>
<b>8 Conclusion</b>	<b>25</b>
<b>A MATLAB Code</b>	<b>26</b>
A.1 System identification . . . . .	26
<b>B Simulink Diagrams</b>	<b>28</b>
<b>Bibliography</b>	<b>30</b>

# **1 Introduction**

Your introduction should contain an overview of the work you were assigned, as well as a few sentences putting the work into a larger perspective. You should also give a quick description of how the report is organized (as is done below).

## 2 Problem Description

The lab setup consists of a movable arm equipped with two rotors. The movable arm is hinged to a fixed point, allowing for both lateral and longitudinal motion. The arm is also fitted with a counterweight which effectively slows the dynamics down considerably, as well as lower the amount of rotor thrust needed. The two rotors are fixed to a pitch head assembly hinged to the movable arm. This allows the rotor thrust direction to be indirectly controlled by the differential thrust applied.

From first principles analysis we can derive simple differential equations to describe the system dynamics about the equilibrium:

$$\ddot{p} = K_1 V_d, \quad K_1 = \frac{K_f l_h}{J_p}, \quad (1a)$$

$$\ddot{\lambda} = -K_2 p, \quad K_3 = \frac{K_f l_a}{J_e} \quad (1b)$$

$$\ddot{e} = K_3 V_s - \frac{T_g}{J_e}, \quad K_2 = \frac{K_p l_a}{J_t} \quad (1c)$$

Note simplifications and limitations:

- The time derivative of travel rate is a linear function of pitch only. This small angle approximation does not really hold, as the intended operating range of pitch is as much as 40 degrees.
- By simple inspection of the lab setup it is clear that the pitch head assembly is hinged slightly above its center of mass. The resulting restoring force, as well as the hinge joint dampening, is not directly included in this model.
- The rotor thrust is assumed to be proportional to the voltage applied to the motor. This is a simplification. Generally, rotor angular velocity is proportional to the voltage applied, and thrust is proportional to the square of the angular velocity.

To stabilize the plant, adding the pitch PD controller

$$V_d = K_{pp}(p_c - p) - K_{pd}\dot{p}, \quad K_{pp}, K_{pd} > 0$$

and the elevation PID controller

$$V_s = K_{ei} \int (e_c - e) dt + K_{ep}(e_c - e) - K_{ed}\dot{e}, \quad K_{ei}, K_{ep}, K_{ed} > 0$$

yields the model equations

$$\ddot{e} + K_3 K_{ed} \dot{e} + K_3 K_{ep} e = K_3 K_{ep} e_c \quad (2a)$$

$$\ddot{p} + K_1 K_{pd} \dot{p} + K_1 K_{pp} p = K_1 K_{pp} p_c \quad (2b)$$

$$\ddot{\lambda} = -K_2 p \quad (2c)$$

Table 1: Parameters and values.

Symbol	Parameter	Value	Unit
$l_a$	Distance from elevation axis to helicopter body	0.63	m
$l_h$	Distance from pitch axis to motor	0.18	m
$K_f$	Force constant motor	0.25	N/V
$J_e$	Moment of inertia for elevation	0.83	kg m <sup>2</sup>
$J_t$	Moment of inertia for travel	0.83	kg m <sup>2</sup>
$J_p$	Moment of inertia for pitch	0.034	kg m <sup>2</sup>
$m_h$	Mass of helicopter	1.05	kg
$m_w$	Balance weight	1.87	kg
$m_g$	Effective mass of the helicopter	0.05	kg
$K_p$	Force to lift the helicopter from the ground	0.49	N

Table 2: Variables

Symbol	variable
$p$	Pitch
$p_c$	Pitch setpoint
$\lambda$	Travel
$\lambda_c$	Travel rate setpoint
$e$	Elevation
$e_c$	Elevation setpoint
$V_f$	Voltage input, front motor
$V_b$	Voltage input, back motor
$V_d$	Voltage difference, $V_f - V_b$
$V_s$	Voltage sum, $V_f + V_b$
$K_{pp}, K_{pd}, K_{ep}, K_{ei}, K_{ed}$	Controller gains
$T_g$	Torque exerted by gravity

where it is assumed the elevation integral term counteracts the constant disturbance  $-\frac{T_g}{J_e}$  and cancel out.

However, in order to achieve a more accurate model we can utilize statistical procedures to best identify the parameters of the first and second order systems. This eliminates errors present in the measurement of the lab setup, and gives us the best model with the given number of states. This gray-box system identification also allows us to verify that the proposed number of states, derived by first principles, yields a model whose performance matches that of the actual system.

### 3 Repetition/Introduction to Simulink/QuaRC

#### 3.1 PID-(re)tuning

The pre-tuned PID showed unsatisfactory performance and was re-tuned to better serve as the stable plant for the rest of the assignment. In particular, the pre-tuned pitch controller was tuned in an aggressive fashion, impacting elevation control significantly. Figure 1 shows the coupling of pitch and elevation control before and after tuning. The two controllers were tuned independent of each other, in a manual fashion.

Table 3: Controller gains comparison

Gain	Original	Improved
$K_{pp}$	93.2	14.0
$K_{pd}$	13.2	2.5
$K_{ei}$	2.3	2.3
$K_{ep}$	7.0	15.0
$K_{ed}$	10.0	13.0

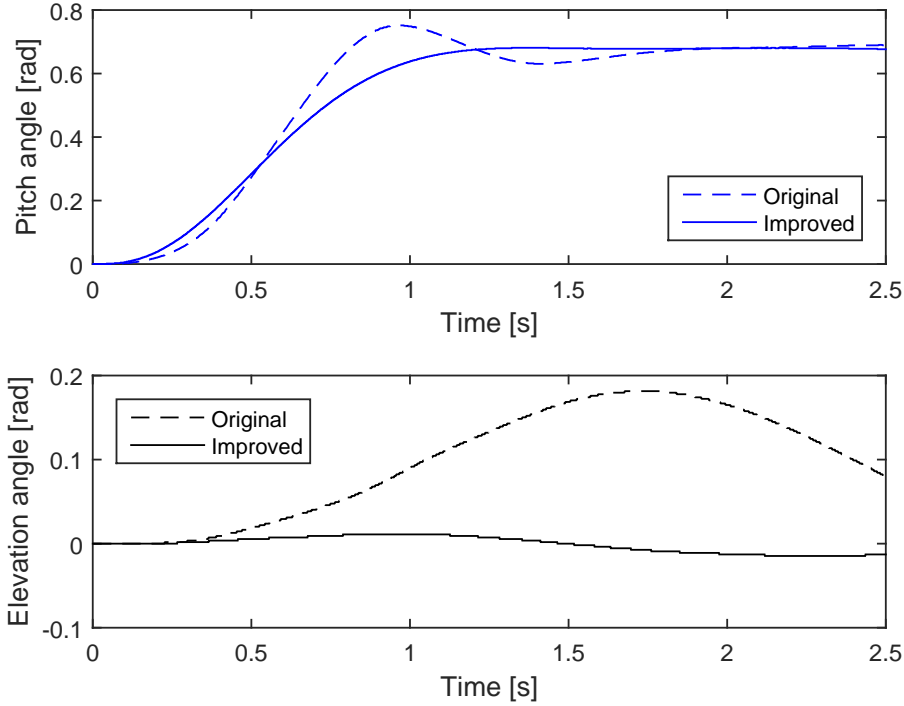


Figure 1: Pitch and elevation response to pitch step input, with original and improved pitch and elevation controller tuning.

#### 3.2 Model parameter estimation

The derived model in (1) showed considerably different dynamics than the observed ones. The system identification toolbox in Matlab was therefore used to compute the parameters which



fitted the recorded step responses. This was done by estimating the follow three step responses based on known input and recorded system behavior:

- Pitch setpoint to pitch
- Elevation setpoint to elevation
- Pitch to travel rate

Figure 2 shows the pitch step response of the derived model compared with the newly estimated model, and the measured step response. The estimated model is clearly a better match than the derived model. Optimal input sequences calculated based on the derived model would need extensive help from feedback in order to yield usable performance. The elevation step responses of the derived and estimated model, shown in figure 3, and travel rate response in figure 4, displays similar results.

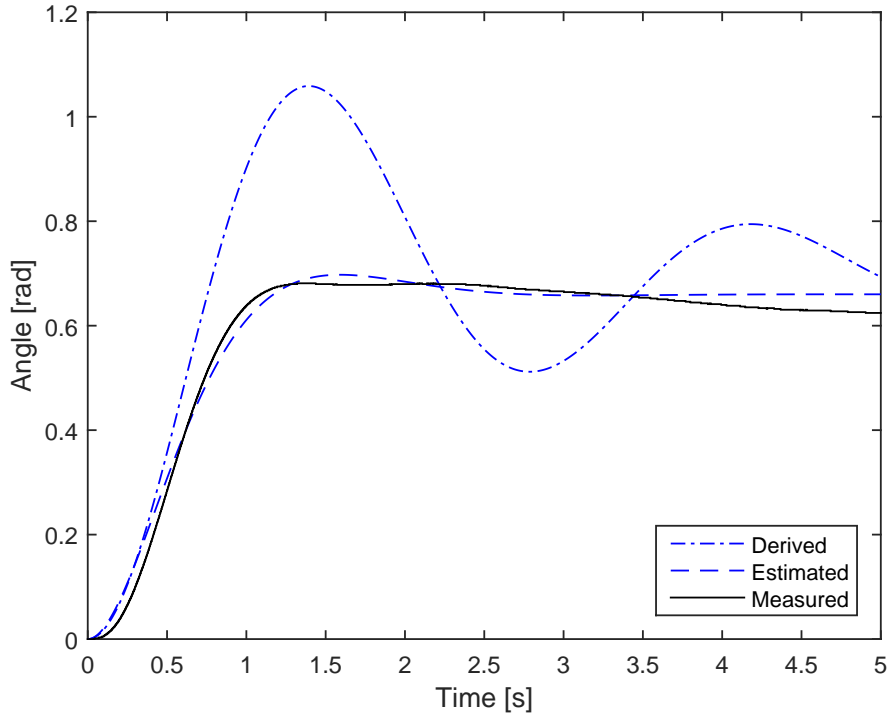


Figure 2: Pitch step responses of the derived and estimated model, compared with the measured step response.

### 3.3 Results and discussion

The following transfer functions were estimated from measured step responses:

$$\frac{p}{p_c}(s) = \frac{6.74}{s^2 + 3.60s + 7.13} \quad (3)$$

$$\frac{e}{e_c}(s) = \frac{3.13}{s^2 + 2.45s + 3.03} \quad (4)$$

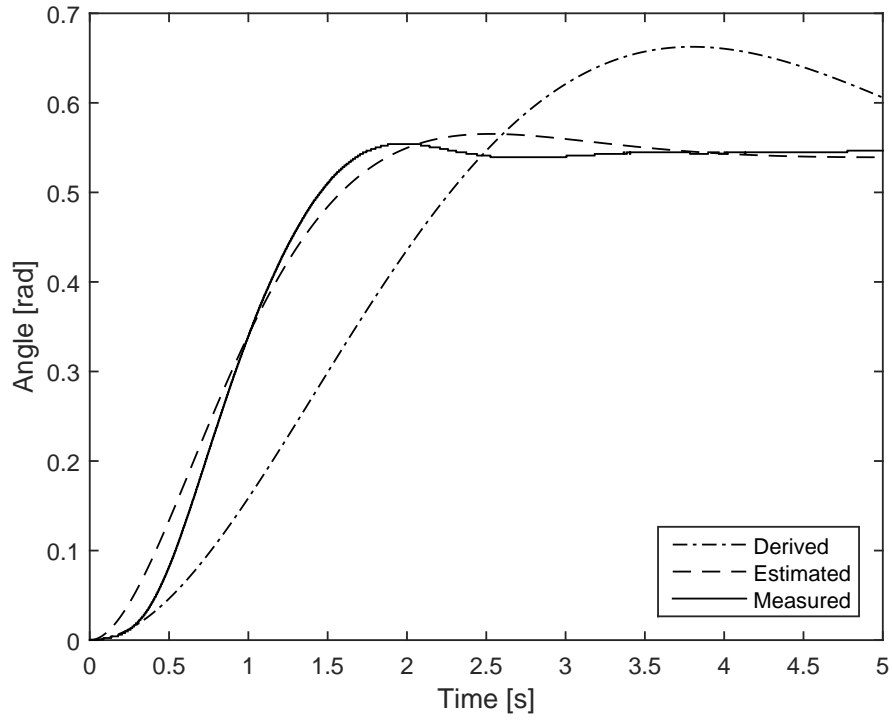


Figure 3: Elevation step responses of the derived and estimated model, compared with the measured step response.

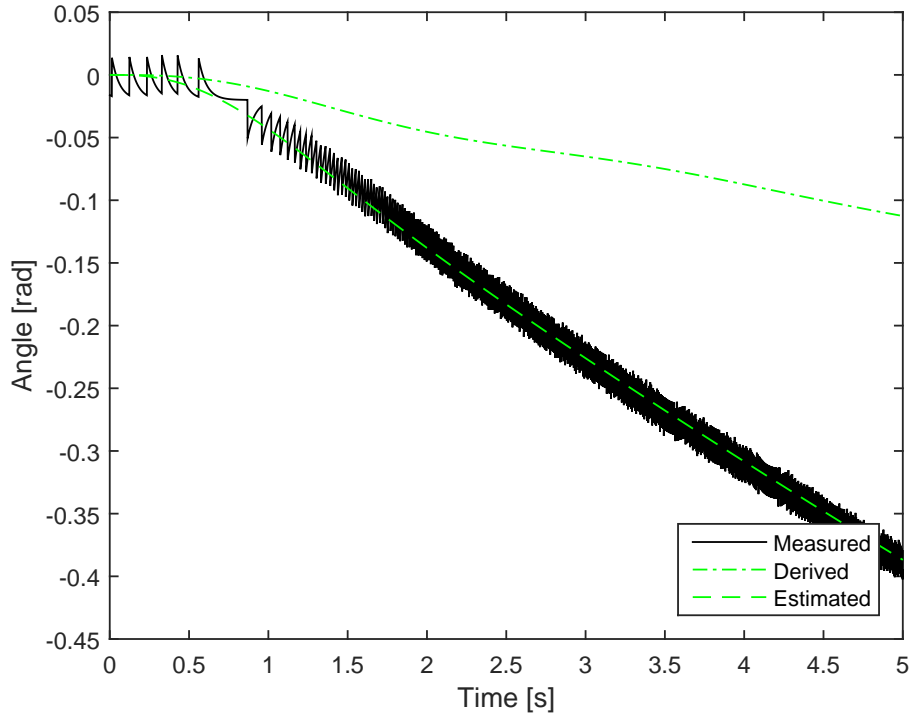


Figure 4: Travel rate response of commanded step in pitch. Derived and estimated model compared with the measured response.

$$\frac{\lambda}{p}(s) = \frac{-0.29}{s + 0.05} \quad (5)$$

The calculated response of the travel rate in figure 4 is almost identical to the recorded step response. The pitch and elevation models are satisfactory, but the dynamics are clearly not matching the actual dynamics to the same degree as with travel rate. This should be expected, as the small angle approximation used is bound to yield errors, especially in the pitch model. Both the off-axis hinged pitch head and the non-linearity of rotor thrust are also unaccounted for when settling on the number of states. A third state for both pitch and elevation might have helped capture some of the dynamics, but as we want to minimize the number of states, the results are found to be well within acceptable levels.

## 4 Optimal Control of Pitch/Travel without Feedback

### 4.1 State space model

From the model equations in (2) we get the continuous state space equation

$$\begin{bmatrix} \dot{\lambda} \\ \ddot{\lambda} \\ \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -K_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -K_1 K_{pp} & -K_1 K_{pd} \end{bmatrix} \begin{bmatrix} \lambda \\ \dot{\lambda} \\ p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ K_1 K_{pp} \end{bmatrix} p_c$$

or, alternatively

$$\begin{bmatrix} \dot{\lambda} \\ \ddot{\lambda} \\ \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -0.0663 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -5.3095 & -0.9481 \end{bmatrix} \begin{bmatrix} \lambda \\ \dot{\lambda} \\ p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5.3095 \end{bmatrix} p_c$$

However, in an effort to achieve a more accurate model, alternative state space equations are developed from estimated transfer functions based on measured step responses as discussed in 3.3. The model

$$\begin{bmatrix} \dot{\lambda} \\ \ddot{\lambda} \\ \dot{p} \\ \ddot{p} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.03 & -0.39 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -7.13 & -3.6 \end{bmatrix}}_{A_c} \begin{bmatrix} \lambda \\ \dot{\lambda} \\ p \\ \dot{p} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 6.74 \end{bmatrix}}_{B_c} p_c \quad (6)$$

is constructed from (3) and (4), as well as some preliminary testing. The latter lead to a slight increase in the amount of change in travel rate as a function of pitch.

The reason for the increase in pitch's effect on travel rate is due to the fact that the pitch step response was run with a step size of 20 degrees. The usual operating range (when aiming for somewhat aggressive maneuvers) is round 30 to 45 degrees. This amount of pitch angle will lead to a significant loss of downwards thrust and sequentially elevation. The elevation controller will in turn attempt to compensate by increasing the overall thrust by a large amount, but since the pitch head is far from equilibrium, a substantial amount of that thrust will be affecting the travel rate. This effect overshadows that of the small angle approximation.

### 4.2 Discretization

Let  $x = [\lambda \quad \dot{\lambda} \quad p \quad \dot{p}]^\top$ ,  $u = p_c$ . Using forward Euler with a time step  $\Delta t = 0.25$  we are able to obtain an approximate discretization of (6):

$$x_{k+1} = x_k + \Delta t \dot{x}_k \quad (7a)$$

$$= x_k + \Delta t (A_c x_k + B_c u_k) \quad (7b)$$

$$= (I + \Delta t A_c) x_k + (\Delta t B_c) u_k \quad (7c)$$

$$= A x_k + B u_k \quad (7d)$$

where  $x_k = x(k\Delta t) \in \mathbb{R}^{n_x}$ ,  $u_k = u(k\Delta t) \in \mathbb{R}^{n_u}$ , and

$$A = \begin{bmatrix} 1 & 0.25 & 0 & 0 \\ 0 & 0.9925 & -0.0975 & 0 \\ 0 & 0 & 1 & 0.25 \\ 0 & 0 & -1.7825 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.685 \end{bmatrix}. \quad (8)$$

### 4.3 Optimal trajectory

We calculate the trajectory from  $x_0 = [\lambda_0 \ 0 \ 0 \ 0]^\top$  to  $x_f = [\lambda_f \ 0 \ 0 \ 0]^\top$  minimizing the objective function

$$\phi = \sum_{i=1}^N (\lambda_i - \lambda_f)^2 + r p_{c_i}^2, \quad r \geq 0, \quad (9)$$

where we let  $\lambda_0 = 0$ ,  $\lambda_f = \pi$ .

The parameter  $r$  weights the relative importance of low input expenditure, in this case set-point for the pitch angle, versus a rapid convergence of the travel trajectory to  $\lambda_f$ . Equivalently we can define (9) in terms of the full state and input variables.

$$\phi = \frac{1}{2} \sum_{i=0}^{N-1} (x_{i+1} - x_f)^\top Q (x_{i+1} - x_f) + u_i^\top R u_i, \quad (10)$$

where

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = r.$$

The system dynamics (7) subjects (10) to the linear equality constraints

$$\underbrace{\left[ \begin{array}{ccc|ccc} I & & & & -B & \\ -A & \ddots & & & & \\ & \ddots & \ddots & & & \\ & & -A & I & & -B \end{array} \right]}_{A_{eq} \in \mathbb{R}^{Nn_x \times N(n_x+n_u)}} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{z \in \mathbb{R}^{N(n_x+n_u) \times 1}} = \underbrace{\begin{bmatrix} -Ax_f \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B_{eq} \in \mathbb{R}^{Nn_x \times 1}}. \quad (11)$$

To express (10) in terms of the optimization variable  $z$  we define the matrix  $G \in \mathbb{R}^{N(n_x+n_u) \times N(n_x+n_u)}$ :

$$G = \begin{bmatrix} Q & & & & \\ & \ddots & & & \\ & & Q & & \\ & & & R & \\ & & & & \ddots \\ & & & & & R \end{bmatrix}.$$

With lower and upper bounds imposed on the pitch state and controller set-point, the QP-problem can then be stated:

$$\min_z \frac{1}{2} z^\top G z \quad (12a)$$

subject to

$$A_{eq} z = B_{eq}, \quad (12b)$$

$$p^{\text{low}} \leq p_k \leq p^{\text{high}}, \quad k \in \{1, \dots, N\}. \quad (12c)$$

#### 4.4 Results and discussion

(12) is solved using MATLAB's `quadprog`.  $r = 0.1$  is chosen to achieve a relatively rapid convergence rate in with the effect of maximizing the pitch between the lower and higher bounds. With rapid convergence rate in mind the pitch bounds were set to  $\pm \frac{45\pi}{180}$ . The optimal input sequence  $u^*$  is applied to the plant in an open loop with results shown in figure 5, with the measured trajectory compared to the calculated optimal trajectory  $x^*$ .

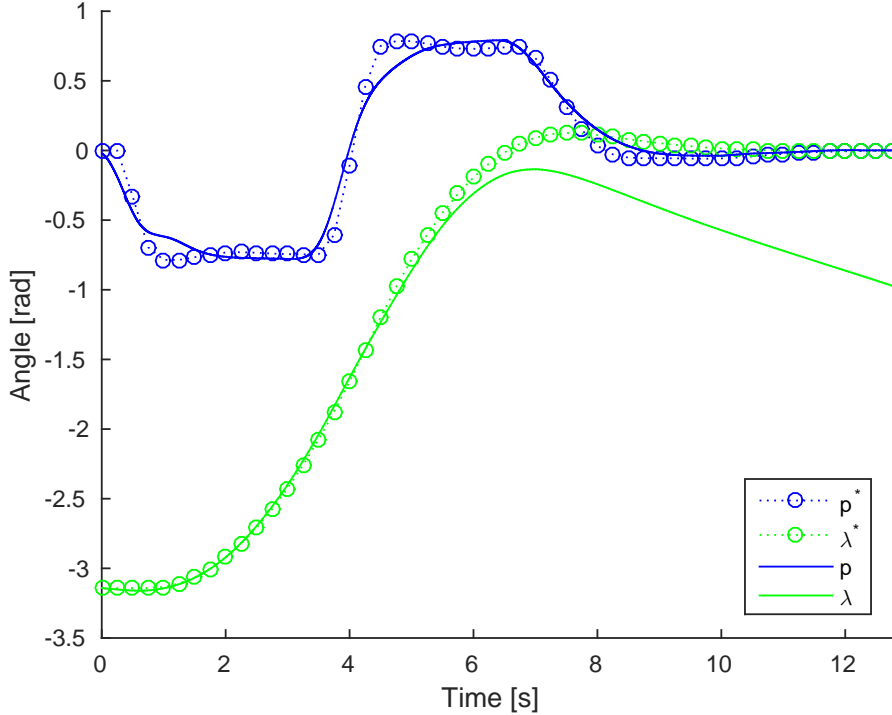


Figure 5: Optimal vs. measured trajectory and input sequence.

The measured trajectory of pitch coincides with the calculated optimal trajectory. The lack of substantial deviation is also a testimonial to the pitch model. Unlike pitch, travel is not controlled by an inner controller, and its deviation from the optimal trajectory should therefore be expected when using open loop.

## 5 Optimal Control of Pitch/Travel with Feedback (LQ)

### 5.1 Discrete LQR

To eliminate the discrepancy between the optimal and measured trajectory observed in figure 5, we can update the optimal trajectory for every time step with a state feedback term weighted by a suitable gain matrix  $K$ :

$$u_k = u_k^* - K(x_k - x_k^*),$$

or, alternatively

$$\Delta u_k = -K \Delta x_k, \tag{13}$$

where

$$\begin{aligned} \Delta x_k &= x_k - x_k^*, \\ \Delta u_k &= u_k - u_k^*. \end{aligned}$$

It can be shown<sup>1</sup> that the controller (13) is the optimal solution minimizing the quadratic objective function

$$J = \sum_{i=0}^{\infty} \Delta x_{i+1}^\top \tilde{Q} \Delta x_{i+1} + \Delta u_i^\top \tilde{R} \Delta u_i,$$

subject to the system dynamics (7), where

$$K = (R + B^\top P B)^{-1} B^\top P A, \tag{14}$$

and  $P$  is the unique positive definite solution to the discrete time algebraic Riccati equation. (14) is used as the state feedback gain, and the resulting Linear Quadratic controller is implemented, with weighing matrices  $\tilde{Q}$  and  $\tilde{R}$  chosen to penalize deviations in states and input for a satisfactory results.

### 5.2 Results and discussion

The state penalty matrix  $\tilde{Q}$  and input penalty  $\tilde{R}$

$$\tilde{Q} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{R} = 0.1.$$

are chosen with travel accuracy in mind. Deviation in travel as well as travel rate is penalized fairly hard. Input deviation is necessary for corrections in travel and travel rate, and a low penalty is therefore chosen.

Using MATLAB's `dlqr` we obtain the LQ state feedback gain  $K$ , and the controller is applied with results shown in figure 6.

---

<sup>1</sup>Its optimality is derived in numerous books, e.g. Kwakernaak and Sivan (1972).

The measured trajectories are similar to the ones without feedback, shown in figure 5, except for the removal of travel drift over time. The travel rate model is clearly not perfect as the measured trajectory seemingly wants to deviate from the optimal path, but the discrepancies are small and the state feedback controller deals with them nicely.

### 5.2.1 MPC discussion

An alternative approach to optimal trajectories with state feedback would be to use an MPC controller. An MPC controller simply computes the optimal trajectory from the current state to the target state given a finite window of time. Only the first input of the optimal input sequence is used. This is then repeated at the next iteration. Solving an optimization problem at each time iteration is clearly computationally expensive, and even QP problems of this relatively small size would force the iteration period to increase substantially. An MPC controller with a time step on the order of several seconds is unlikely to yield better performance than a predetermined trajectory with short time steps combined with state feedback control, due to the fast dynamics of our system. Deviation from optimal trajectory is also likely to be caused by either an incomplete model or unmodeled external disturbances, and an MPC controller will not withstand either very well.



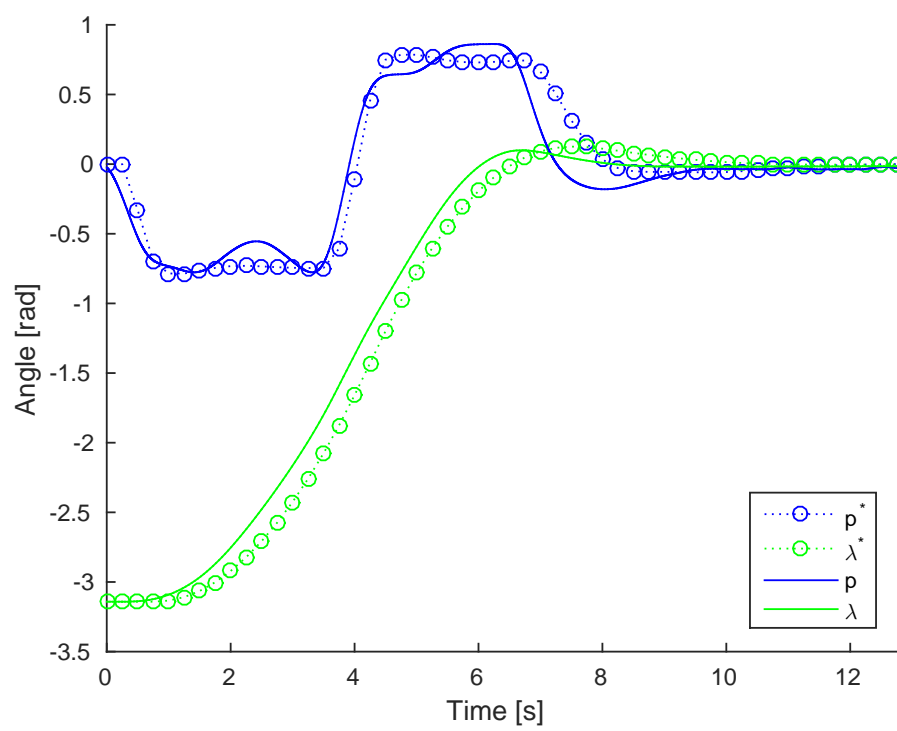


Figure 6: LQR

## 6 Optimal Control of Pitch/Travel and Elevation with and without Feedback

The

### 6.1 State space model

We wish to calculate an optimal trajectory in two dimension, adding a constraint to the elevation. Adding  $e$  and  $\dot{e}$  to the previous state space model (6) and using (4), we get

$$\begin{bmatrix} \dot{\lambda} \\ \ddot{\lambda} \\ \dot{p} \\ \ddot{p} \\ \dot{e} \\ \ddot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.03 & -0.39 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -7.13 & -3.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -3.03 & -2.44 \end{bmatrix}}_{A_c} \begin{bmatrix} \lambda \\ \dot{\lambda} \\ p \\ \dot{p} \\ e \\ \dot{e} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 6.74 & 0 \\ 0 & 0 \\ 0 & 3.13 \end{bmatrix}}_{B_c} \begin{bmatrix} p_c \\ e_c \end{bmatrix} \quad (15)$$

The added states are clearly decoupled from the rest of the system. This is due to the fact that the model is linearized around the equilibrium. At small pitch angles this simplification does not cause any large discrepancies. However, when the pitch angle is large, an increase in elevation rate is clearly accompanied by an increase in travel rate.

### 6.2 Discretization

Now let  $x = [\lambda \ \dot{\lambda} \ p \ \dot{p} \ e \ \dot{e}]^\top$ ,  $u = [p_c \ e_c]^\top$ . Again, using approximate discretization via Euler we obtain a discrete state space model

$$x_{k+1} = (I + \Delta t A_c)x_k + (\Delta t B_c)u_k. \quad (16a)$$

$$= Ax_k + Bu_k, \quad (16b)$$

where

$$A = \begin{bmatrix} 1 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0.9925 & -0.0975 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.25 & 0 & 0 \\ 0 & 0 & -1.7825 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.25 \\ 0 & 0 & 0 & 0 & -0.7575 & 0.39 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1.685 & 0 \\ 0 & 0 \\ 0 & 0.7825 \end{bmatrix}.$$

### 6.3 Optimization problem with nonlinear constraints

We calculate an optimal trajectory from  $x_0 = [\lambda_0 \ 0 \ 0 \ 0 \ 0 \ 0]^\top$  to  $x_f = [\lambda_f \ 0 \ 0 \ 0 \ 0 \ 0]^\top$  minimizing the objective function

$$\phi = \sum_{i=1}^N (\lambda_i - \lambda_f)^2 + r_1 p_{c_i}^2 + r_2 e_{c_i}^2, \quad r_1, r_2 \geq 0,$$

or alternatively

$$\phi = \sum_{i=0}^{N-1} (x_{i+1} - x_f)^\top Q (x_{i+1} - x_f) + u_i^\top R u_i, \quad (17)$$

where

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}. \quad (18)$$

The second weighting parameter  $r_2$  is added as we impose an inequality constraint on the elevation for every time step:

$$c(x_k) = \alpha \exp(-\beta (\lambda_k - \lambda_t)^2) - e_k \leq 0, \quad k = \{1, \dots, N\}, \quad (19)$$

where we let  $\alpha = 0.2$ ,  $\beta = 20$ ,  $\lambda_t = \frac{2\pi}{3}$ .

The objective function (17) is subject to the system dynamics (16a) and thus imposed to linear equality constraints identically defined to that of (11). Similarly to (12) we define the optimization variable  $z$  and the matrix  $G$ , and the resulting optimization problem can be stated:

$$\min_z \quad z^\top G z \quad (20a)$$

subject to

$$A_{eq} z = B_{eq}, \quad (20b)$$

$$c(x_k) \leq 0, \quad k = \{1, \dots, N\}, \quad (20c)$$

$$p^{\text{low}} \leq p_k \leq p^{\text{high}}, \quad k = \{1, \dots, N\}. \quad (20d)$$

## 6.4 Discrete LQR

In addition to running the optimal input sequence  $u^*$  in an open loop, a discrete LQ-controller is applied, with the weighing matrices

$$Q = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (21)$$

## 6.5 Optional: Additional constraints

Although the calculated input yielded satisfactory performance, the model has shortcomings. Specifically, when a low decent in elevation is commanded the rotor blades almost come to a complete stop, during which control of pitch is severely reduced. As this is impossible to model with a linear model, additional bounds are imposed on elevation rate. This will reduce the effect of the unmodeled coupling of elevation to the rest of the system. In a further attempt to keep the system within the linear area the travel rate is also bounded. The bounds are

$$\begin{aligned} |\dot{e}_k| &\leq 0.05, \\ |\dot{\lambda}_k| &\leq 0.5. \end{aligned}$$

## 6.6 Results and discussion

Because of the non-linearity of (19) it is no longer viable to use a QP-solver, and (20) is solved using MATLAB's `fmincon` which uses an active set method. The pitch bounds are also tightened from 45 to 25 degrees in order to lower the effect of elevation rate and travel rate coupling at large pitch angles.

The optimal input sequence  $u^*$  is applied to the plant in an open loop with results shown in figure 7. The system follows the optimal trajectory except for the usual discrepancy in travel present when using open loop control. We further employ an LQ-controller similar to in 5.1 to guide the trajectory to the set-point and eliminate steady state deviations. The result, shown in figure 8, is without any large discrepancies between optimal and measured trajectory. However, the constraint bound imposed on elevation is violated, as shown in figure 10. The reason for this is probably a combination of issues. The optimal path is calculated with a fairly large time step, in order to lower the computational time required. This leads to an optimal path that when interpolated will violate the constraint. Also, as the increase in elevation will largely happen when the pitch angle is at its maximum, and the resulting increase in elevation will be lower than if the increase were to be commanded at zero pitch. State feedback is also unable to deal with this problem completely, as any aggressive elevation correction is bound to happen at the cost of pitch, and therefore travel rate control. Lastly, the trajectory is corrected with respect to the optimal path in time, while the bound on elevation is defined with respect to travel. This means that a deviation in travel, as visible in figure 8, effectively will shift the optimal elevation path with respect to travel. This lag is visible in figure 10.

The mentioned effects contributing to constraint violation are attempted minimized by the addition of extra constraints on elevation rate and travel rate, as mentioned in 6.5. The resulting optimal path reflects this, and the slope of the optimal elevation path is significantly reduced. This yields an overall less aggressive system, with decreased violation of the elevation constraint. The added constraint on travel rate also means that the system uses the entire time slot to reach the desired end state.

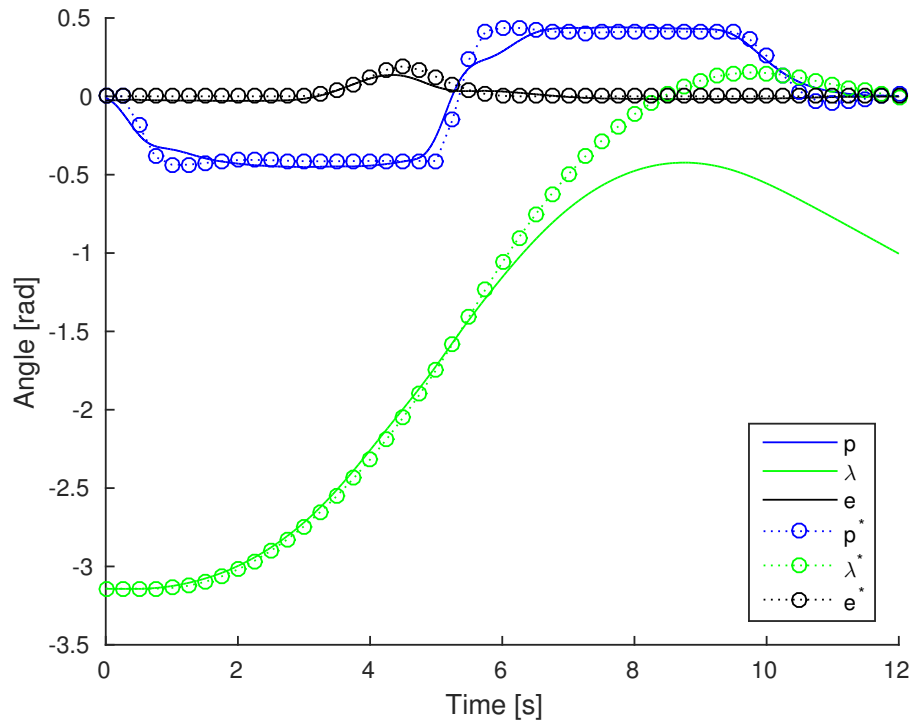


Figure 7: Open Loop 25.

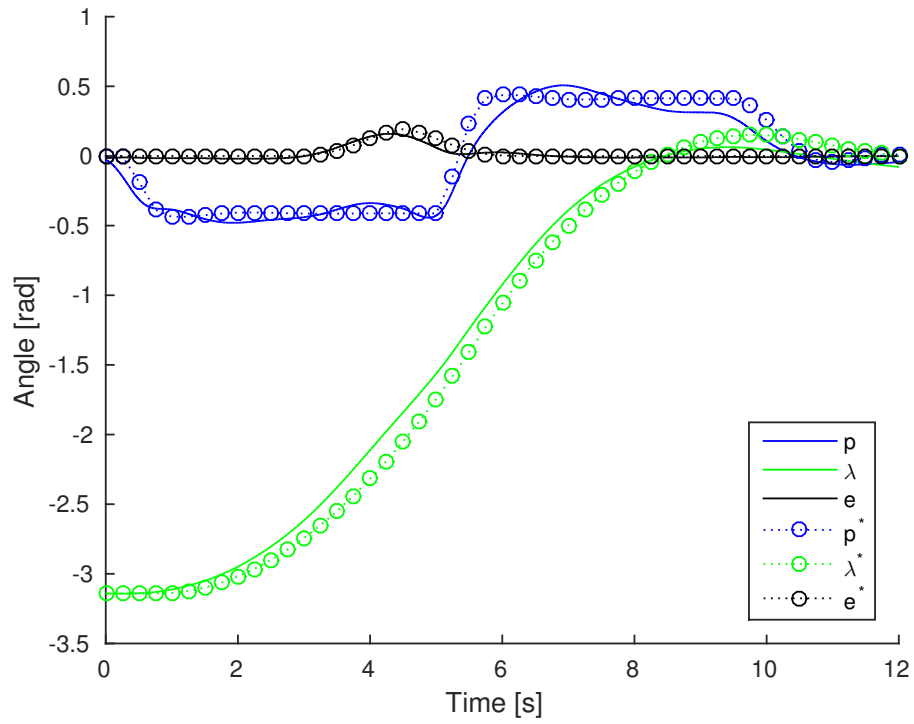


Figure 8: Closed Loop 25.

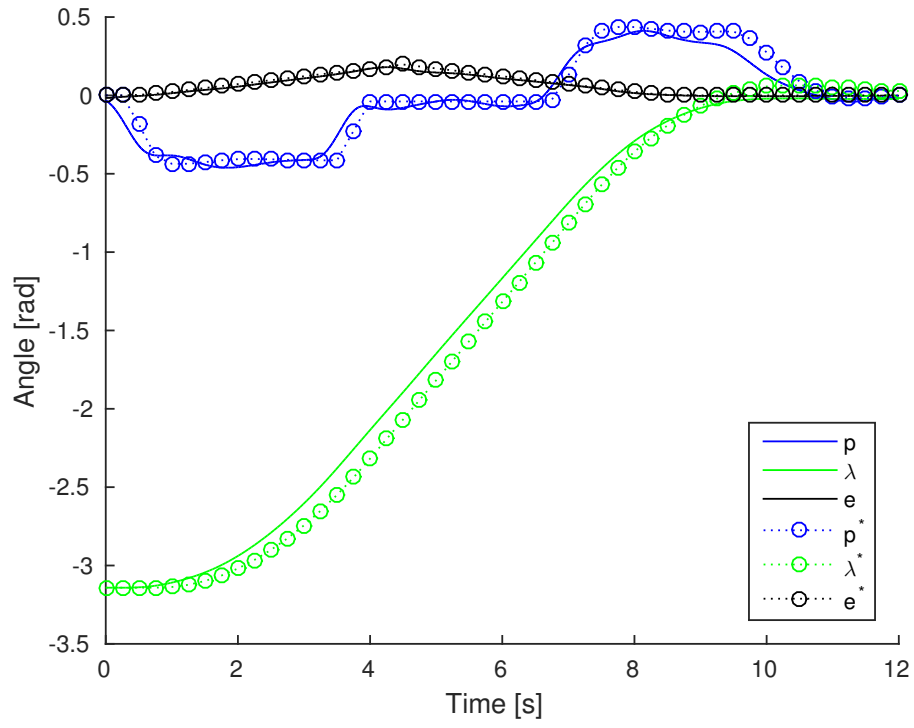


Figure 9: Closed Loop 25, extra constraints.

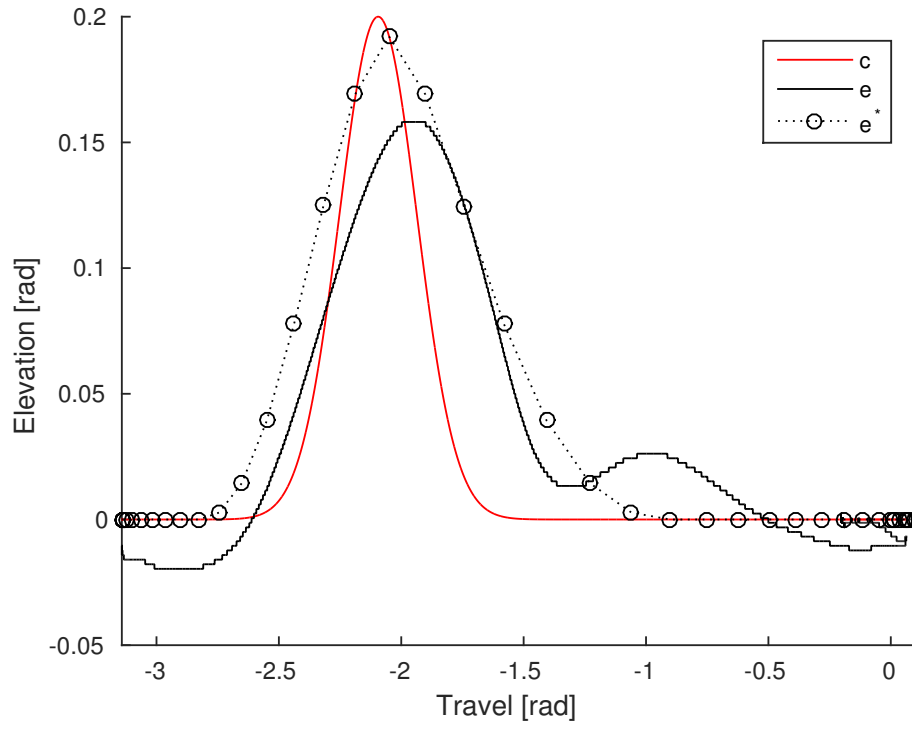


Figure 10: Closed Loop 25 - constraint hill.

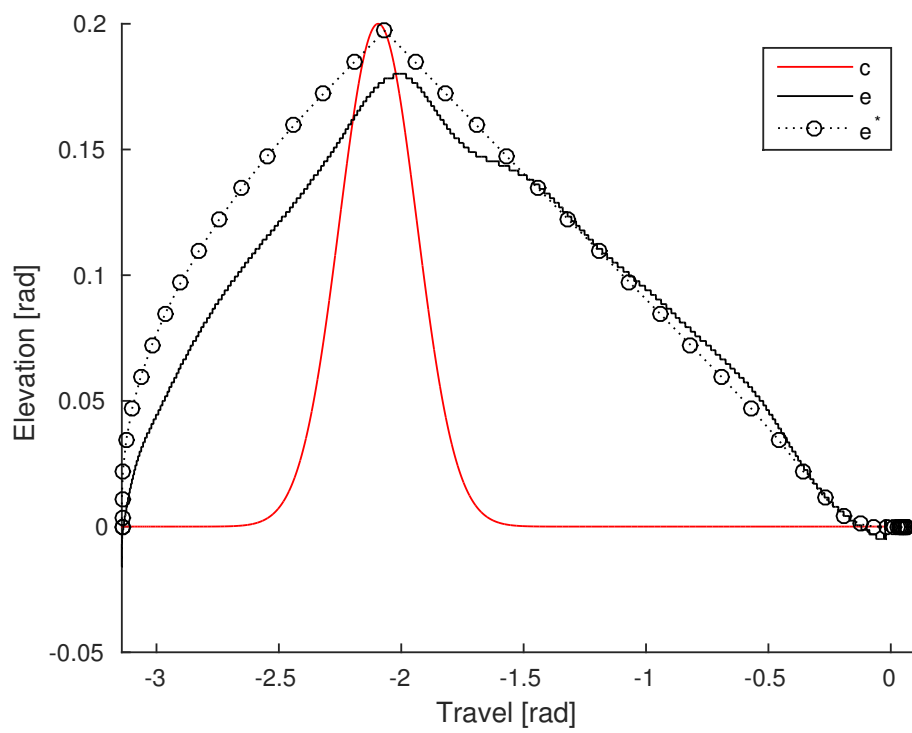


Figure 11: Closed Loop 25, extra constraints - constraint hill.

## 7 Discussion

The optimal paths computed have shown to be quite compatible with the actual system. This is reflected in the displayed data. The reason for discrepancies have also been relatively easy to identify. This is probably due to mainly two related facts. The controllers of pitch and elevation were tuned in a way that gave the least nonlinear coupling between them. This greatly improves the changes of a linear model being sufficient. Secondly, the statistically system identification yields a model with an accuracy far greater than one based on indirect measurements. Optimal paths based on an incomplete system model will probably yield suboptimal performance at best.

As shown in both figure 5 and 7 the need for feedback control is present. A linear quadratic controller is a computationally inexpensive way of achieving this. The feedback importance is also reduced by having an accurate system model, and only light corrections were necessary.

The effect of model shortcomings as discussed in 3.3, 4.1, 6.1, 6.5 and 6.6 are attempted minimized by adding extra constraints to the optimization problem. This forces the system to operate inside the linear area, yielding slower but more accurate performance.



## 8 Conclusion

Optimal operation of the movable arm is achieved by minimizing a quadratic cost function, and precomputed input sequences are supplemented by state feedback. The linear model is shown to yield satisfactory performance, and enhanced performance is achieved by adding constraints designed to minimize nonlinear effects.

## A MATLAB Code

### A.1 System identification

```
1
2 clc
3 close all
4
5 %%% Pitch %%%
6 % h = pitch/pitch_ref
7
8 load pitchStep40deg
9 u_pitch = 40*pi/180 * ones(3000,1);
10 u_pitch(1) = 0;
11 y_pitch = pitchStep40deg.signals.values(1:3000);
12 pitch_data = iddata(y_pitch, u_pitch, 0.001);
13 pitch_time = 0.001:0.001:3;
14
15 opt = tfestOptions('InitialCondition', 'zero');
16 pitch_sys = tfest(pitch_data, 2, 0,opt); % poles, zeroes
17
18 subplot(131);
19 plot(pitch_time, y_pitch/(40*pi/180), 'r'); % scale for easy comparison with step()
20 hold on
21 step(pitch_sys);
22 hold off
23 title('Pitch');
24
25
26 %%% Elevation %%%
27 % h = elevation/elevation_ref
28
29 load elevStep30deg
30 u_elev = 30*pi/180 * ones(7000,1);
31 u_elev(1) = 0;
32 y_elev = elevStep30deg.signals.values(1:7000) + 16.8*pi/180; % unbiased the step
33 elev_data = iddata(y_elev, u_elev, 0.001);
34 elev_time = 0.001:0.001:7;
35
36 opt = tfestOptions('InitialCondition', 'zero');
37 elev_sys = tfest(elev_data, 2, 0,opt); % poles, zeroes
38
39 subplot(132);
40 plot(elev_time, y_elev/(30*pi/180), 'r'); % scale for easy comparison with step()
41 hold on
42 step(elev_sys);
43 hold off
44 title('Elevation');
45
46
47
48 %%% TravelRate %%%
49 % h = travelRate/pitch
50
51 load travelRateStep20deg
52 travelRate_time = 0.001:0.001:8;
53 u_travelRate = 20*pi/180 * step(pitch_sys,travelRate_time);
54 y_travelRate = travelRateStep20deg.signals.values(1:8000) - 0.02; % unbiased
55 travelRate_data = iddata(y_travelRate, u_travelRate, 0.001);
56
```

```

57 opt = tfestOptions('InitMethod','iv','InitialCondition','zero');
58 travelRate_sys = tfest(travelRate_data, 1, 0, opt); % poles, zeroes
59
60 subplot(133);
61 plot(travelRate_time, y_travelRate/(20*pi/180), 'r'); % scale for easy comparison with step()
62 hold on
63 step(travelRate_sys*pitch_sys, travelRate_time);
64 hold off
65 title('Travel rate');

```

## B Simulink Diagrams

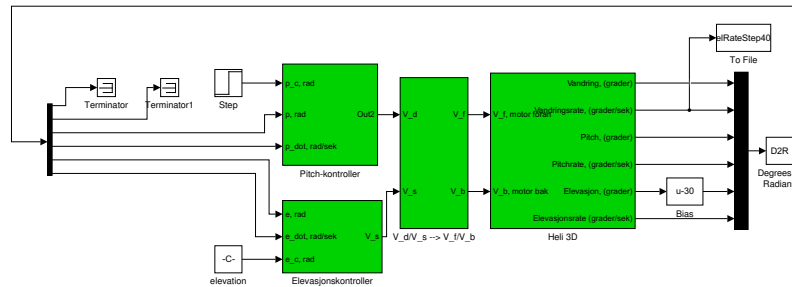


Figure 12: Simulink model used with section 3.

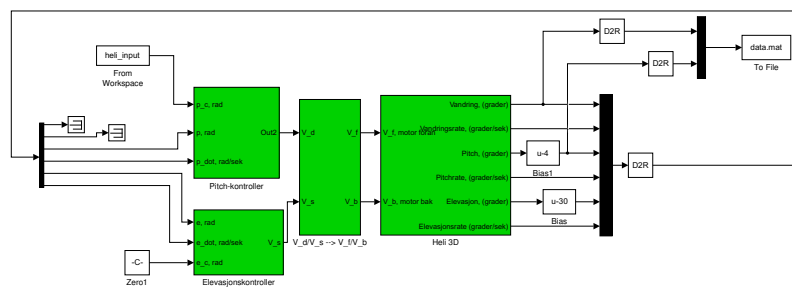


Figure 13: Simulink model used with section 4.



## Bibliography

Kwakernaak, H. and Sivan, R. (1972). *Linear Optimal Control Systems. First Edition.* Wiley-Interscience.