Appendix: Improving the Generalization Performance of Multi-class SVM via Angular Regularization

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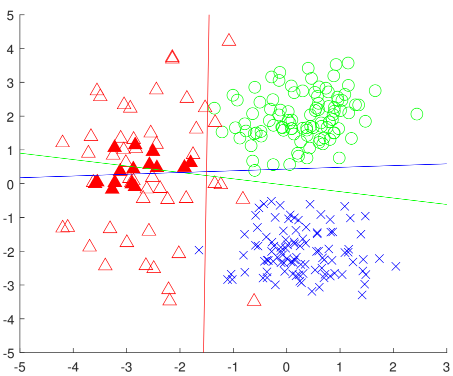
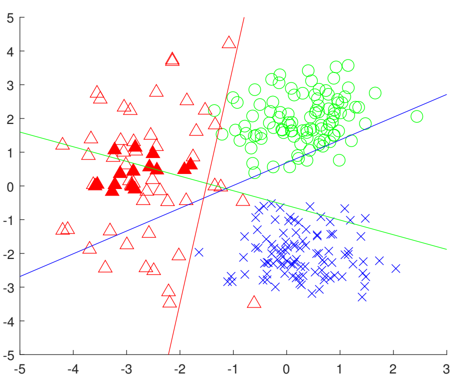
1 Supplementary for intuitive figure

Recall the intuitive figure Fig. 1 (in main paper), it is better to run an example of how the angular regularization technique taking effects on multi-class SVM. Here, we have a try on randomized toy data to verify its correctness.



(a)



(b)

Figure 1: (a) Without angular regularization; (b) with angular regularization. The toy data is generated on a 2D-plane with 3 categories. Remarking that the solid red triangles are the training samples while the void ones denote test samples.

2 Proof of Lemma 1

To prove Theorem 1 (in main paper), the following lemma is needed.

Lemma 1. *Let the weight vector* $\mathbf{w}_k$ *of hyperplane* k *be decomposed into* $\mathbf{w}_k = \mathbf{x}_k + l_k\mathbf{e}_k$, *where* $\mathbf{x}_k = \sum_{j=1,j\neq k}^{K} \alpha_j \mathbf{w}_j$ *lies in the subspace* L *spanned by* $\{\mathbf{w}_1, \dots, \mathbf{w}_K\} \backslash \{\mathbf{w}_k\}$, $\mathbf{e}_k$ *is in the orthogonal complement of* L, $\|\mathbf{e}_k\| = 1$, $\mathbf{e}_k \cdot \mathbf{w}_k > 0$, l_k *is a scalar. Then the gradient of* $\widehat{\mathcal{R}}(\mathbf{W})$ *w.r.t* $\mathbf{w}_k$ *is* $p_k\mathbf{x}_k + q_k\mathbf{e}_k$, *where* p_k *is a positive scalar .*

Proof To prove Lemma 1, the following lemma is needed.

Lemma 2. *[Xie* et al.*, 2015a] Let the weight vector* $\mathbf{w}_k$ *of hyperplane* k *be decomposed into* $\mathbf{w}_k = \mathbf{x}_k + l_k\mathbf{e}_k$, *where* $\mathbf{x}_k = \sum_{j=1,j\neq k}^{K} \alpha_j \mathbf{w}_j$ *lies in the subspace* L *spanned by* $\{\mathbf{w}_1, \dots, \mathbf{w}_K\} \backslash \{\mathbf{w}_k\}$, $\mathbf{e}_k$ *is in the orthogonal complement of* L, $\|\mathbf{e}_k\| = 1$, $\mathbf{e}_k \cdot \mathbf{w}_k > 0$, l_k *is a scalar. Then* $\det(\mathbf{W}^\top\mathbf{W}) = \det(\mathbf{W}_{-k}^\top\mathbf{W}_{-k})(l_k\mathbf{e}_k \cdot \mathbf{w}_k)$, *where* $\mathbf{W}_{-i} = [\mathbf{w}_1, \dots, \mathbf{w}_{k-1}, \mathbf{w}_{k+1}, \dots, \mathbf{w}_K]$ *with* $\mathbf{w}_k$ *excluded.*

According to the chain rule, the gradient of $\mathcal{R}(\mathcal{W})$ w.r.t $\mathbf{w}_k$ can be written as

$$\begin{aligned}\frac{\partial \mathcal{R}(\mathcal{W})}{\partial \mathbf{w}_k} =& g'(\mathrm{tr}(\mathbf{W}^\top\mathbf{W}))\frac{\partial \mathrm{tr}(\mathbf{W}^\top\mathbf{W})}{\partial \mathbf{w}_k} \\ & - \frac{1}{K}g'(\det(\mathbf{W}^\top\mathbf{W}))\frac{\partial \det(\mathbf{W}^\top\mathbf{W})}{\partial \mathbf{w}_k},\end{aligned}$$

where $g(x) = \log(x)$. It is easy to check that $g(x)$ is an increasing function and $g'(x) = 1/x$. As assumed earlier, the weight vectors in $\mathbf{W}$ are linearly independent and hence $\mathrm{tr}(\mathbf{W}^\top\mathbf{W}) > 0$ and $\det(\mathbf{W}^\top\mathbf{W}) > 0$.

According to Lemma 2, we have

$$\frac{\partial \det(\mathbf{W}^\top\mathbf{W})}{\partial \mathbf{w}_k} = \det(\mathbf{W}_{-k}^\top\mathbf{W}_{-k})l_k\mathbf{e}_k,$$

where $\det(\mathbf{W}_{-k}^\top\mathbf{W}_{-k}) > 0$ and $l_k > 0$ (knowing from $\det(\mathbf{W}^\top\mathbf{W}) = \det(\mathbf{W}_{-k}^\top\mathbf{W}_{-k})l_k\mathbf{e}_k \cdot \mathbf{w}_k > 0$ and $\mathbf{e}_k \cdot \mathbf{w}_k > 0$). Besides, we have

$$\frac{\partial \mathrm{tr}(\mathbf{W}^\top\mathbf{W})}{\partial \mathbf{w}_k} = 2\mathbf{w}_k,$$

Substitute above equations into the gradient of $\widehat{\mathcal{R}}(\mathbf{W})$

$$\frac{\partial \widehat{\mathcal{R}}(\mathbf{W})}{\partial \mathbf{w}_k} = \frac{2\mathbf{w}_k}{\operatorname{tr}(\mathbf{W}^\top \mathbf{W})} - \frac{1}{K}\frac{\det(\mathbf{W}_{-k}^\top \mathbf{W}_{-k}) l_k \mathbf{e}_k}{\det(\mathbf{W}^\top \mathbf{W})}.$$

With the weight vector $\mathbf{w}_k$ decomposition, we have

$$\begin{aligned}\frac{\partial \widehat{\mathcal{R}}(\mathbf{W})}{\partial \mathbf{w}_k} &= \frac{2}{\operatorname{tr}(\mathbf{W}^\top \mathbf{W})}\mathbf{x}_k \\ &\quad + [\frac{2}{\operatorname{tr}(\mathbf{W}^\top \mathbf{W})} - \frac{\det(\mathbf{W}_{-k}^\top \mathbf{W}_{-k})}{K\det(\mathbf{W}^\top \mathbf{W})}] l_k \mathbf{e}_k \\ &= p_k \mathbf{x}_k + q_k \mathbf{e}_k,\end{aligned}$$

where $p_k = \frac{2}{\operatorname{tr}(\mathbf{W}^\top \mathbf{W})} > 0$ and $q_k = [\frac{2}{\operatorname{tr}(\mathbf{W}^\top \mathbf{W})} - \frac{\det(\mathbf{W}_{-k}^\top \mathbf{W}_{-k})}{K\det(\mathbf{W}^\top \mathbf{W})}] l_k$. □

3 Supplement to the proof of Theorem 1

In the main paper, we give a brief proof of Theorem 1 and ignore the time stamp t for simplicity. Our aim is to declare that the minimum angle $\mathcal{R}(\mathbf{W})$ will decrease alongside with the negative gradient direction of the regularizer $\widehat{\mathcal{R}}(\mathbf{W})$. We use its cosine similarity to measure the minimum angle (denotes by *):

$$s_{min}(\mathbf{W}^{(t)}) = \cos(-\mathcal{R}(\mathbf{W}^{(t)})) = \cos(\theta_{i*j*}). \quad (1)$$

Equally, we first analysis $\widehat{\mathcal{R}}(\mathbf{W})$'s behaviour on a trivial angle θ_{ij}:

$$s_{ij}(\mathbf{W}^{(t)}) = \cos(\theta_{ij}). \quad (2)$$

Some notations need to be introduced. Let $V = \{(i,j) | 1 \leq i,j \leq K, i \neq j, \mathbf{w}_i^{(t)} \cdot \mathbf{w}_j^{(t)} = 0\}$, $N = \{(i,j) | 1 \leq i,j \leq K, i \neq j, \mathbf{w}_i^{(t)} \cdot \mathbf{w}_j^{(t)} \neq 0\}$, where $\mathbf{w}_i^{(t)}$ is the i-th column of $\mathbf{W}_t$. Let $x_{ij}^{(t)} = \mathbf{w}_i^{(t)} \cdot \mathbf{w}_j^{(t)}$, $y_{ij}^{(t)} = \|\mathbf{w}_i^{(t)}\|_2 \cdot \|\mathbf{w}_j^{(t)}\|_2$, $x_{ij}^{(t+1)} = \mathbf{w}_i^{(t+1)} \cdot \mathbf{w}_j^{(t+1)}$, $y_{ij}^{(t+1)} = \|\mathbf{w}_i^{(t+1)}\|_2 \cdot \|\mathbf{w}_j^{(t+1)}\|_2$. Following the gradient direction of $\widehat{\mathcal{R}}(\mathbf{W})$ from Lemma 1, we have

$$\begin{aligned}\mathbf{w}_i^{(t+1)} &= \mathbf{w}_i^{(t)} - \eta(p_i \mathbf{x}_i + q_i \mathbf{e}_i) \\ \mathbf{w}_j^{(t+1)} &= \mathbf{w}_j^{(t)} - \eta(p_j \mathbf{x}_j + q_j \mathbf{e}_j)\end{aligned},$$

and acquire some important equations

$$\begin{aligned}\mathbf{e}_i \cdot \mathbf{w}_j^{(t)} = 0, &\quad \mathbf{e}_j \cdot \mathbf{w}_i^{(t)} = 0 \\ \mathbf{e}_i \cdot \mathbf{x}_i = 0, &\quad \mathbf{e}_j \cdot \mathbf{x}_j = 0 \\ \mathbf{e}_i \cdot \mathbf{x}_j = \alpha_i \mathbf{e}_i \cdot \mathbf{w}_i^{(t)}, &\quad \mathbf{e}_j \cdot \mathbf{x}_i = \alpha_j \mathbf{e}_j \cdot \mathbf{w}_j^{(t)}\end{aligned}.$$

Thus, we have:

$$\begin{aligned}x_{ij}^{(t+1)} &= [\mathbf{w}_i^{(t)} - \eta(p_i \mathbf{x}_i + q_i \mathbf{e}_i)][\mathbf{w}_j^{(t)} - \eta(p_j \mathbf{x}_j + q_j \mathbf{e}_j)] \\ &= \mathbf{w}_i^{(t)} \cdot \mathbf{w}_j^{(t)} - \eta(p_j \mathbf{w}_i^{(t)} \cdot \mathbf{x}_j + p_i \mathbf{x}_i \cdot \mathbf{w}_j^{(t)}) \\ &\quad + \eta^2(p_i \mathbf{x}_i + q_i \mathbf{e}_i)(p_j \mathbf{x}_j + q_j \mathbf{e}_j) \\ &= \mathbf{w}_i^{(t)} \cdot \mathbf{w}_j^{(t)} - \eta a + \eta^2 b\end{aligned},$$

where $a = p_j \mathbf{w}_i^{(t)} \cdot \mathbf{x}_j + p_i \mathbf{x}_i \cdot \mathbf{w}_j^{(t)}$;

$$\begin{aligned}y_{ij}^{(t+1)} &= \sqrt{(\mathbf{w}_i^{(t)} - \eta(p_i \mathbf{x}_i + q_i \mathbf{e}_i))^2}\sqrt{(\mathbf{w}_j^{(t)} - \eta(p_j \mathbf{x}_j + q_j \mathbf{e}_j))^2} \\ &= \sqrt{\mathbf{w}_i^{(t)} \cdot \mathbf{w}_i^{(t)} - 2\eta(p_i \mathbf{x}_i \cdot \mathbf{x}_i + q_i l_i) + \eta^2(p_i^2 \mathbf{x}_i \cdot \mathbf{x}_i + q_i^2)} \\ &\quad \cdot \sqrt{\mathbf{w}_j^{(t)} \cdot \mathbf{w}_j^{(t)} - 2\eta(p_j \mathbf{x}_j \cdot \mathbf{x}_j + q_j l_j) + \eta^2(p_j^2 \mathbf{x}_j \cdot \mathbf{x}_j + q_j^2)}, \\ &= \sqrt{\mathbf{w}_i^{(t)} \cdot \mathbf{w}_i^{(t)} - 2\eta c + \eta^2 d}\sqrt{\mathbf{w}_j^{(t)} \cdot \mathbf{w}_j^{(t)} - 2\eta e + \eta^2 f}\end{aligned}$$

where $c = p_i \mathbf{x}_i \cdot \mathbf{x}_i + q_i l_i$ and $e = p_i \mathbf{x}_i \cdot \mathbf{x}_i + q_i l_i$.

Thereby, we can represent the cosine similarity s_{ij} as

$$s_{ij}(\mathbf{W}^{(t+1)}) = \frac{x_{ij}^{(t+1)}}{y_{ij}^{(t+1)}}, \qquad s_{ij}(\mathbf{W}^{(t)}) = \frac{x_{ij}^{(t)}}{y_{ij}^{(t)}}$$

The following lemma 3 and lemma 4 are needed for proving Theorem 1.

Lemma 3. *$\forall (i,j) \in V$, we have $s_{ij}(\mathbf{W}^{(t+1)}) - s_{ij}(\mathbf{W}^{(t)}) = o(\eta)$, where $\lim_{\eta \to 0} \frac{o(\eta)}{\eta} = 0$.*

Proof For $(i,j) \in V$, $\mathbf{w}_i^{(t)} \cdot \mathbf{w}_j^{(t)} = 0$, thereby $x_{ij}^t = 0$ and

$$s_{ij}(\mathbf{W}^{(t+1)}) - s_{ij}(\mathbf{W}^{(t)}) = x_{ij}^{(t+1)} / y_{ij}^{(t+1)} - 0.$$

Simply we have $x_{ij}^{(t+1)} = -\eta a + \eta^2 b$, where

$$\begin{aligned}a &= p_j \mathbf{w}_i^{(t)} \cdot \mathbf{x}_j + p_i \mathbf{x}_i \cdot \mathbf{w}_j^{(t)} \\ &= p_j \mathbf{w}_i^{(t)} \cdot (\mathbf{w}_j^{(t)} - l_j \mathbf{e}_j) + p_i(\mathbf{w}_i^{(t)} - l_i \mathbf{e}_i) \cdot \mathbf{w}_j^{(t)} \\ &= p_j \mathbf{w}_i^{(t)} \cdot \mathbf{w}_j^{(t)} + p_i \mathbf{w}_i^{(t)} \cdot \mathbf{w}_j^{(t)} \\ &= (p_j + p_i)\mathbf{w}_i^{(t)} \cdot \mathbf{w}_j^{(t)} \\ &= 0\end{aligned}$$

Thus we can derive $x_{ij}^{(t+1)} = \eta^2 b$.

Next we consider $1/y_{ij}^{(t+1)} = \frac{1}{\sqrt{\mathbf{w}_i^{(t)} \cdot \mathbf{w}_i^{(t)} - 2\eta \cdot c + \eta^2 \cdot d}} \cdot \frac{1}{\sqrt{\mathbf{w}_j^{(t)} \cdot \mathbf{w}_j^{(t)} - 2\eta \cdot e + \eta^2 \cdot f}}$, we take $\|\mathbf{w}_i^{(t)}\|_2$ out and discuss the first term in denominator. Using the Taylor expansion of $\frac{1}{\sqrt{1+x}}$ at $x = 0$, we obtain that

$$\begin{aligned}&\frac{1}{\|\mathbf{w}_i^{(t)}\|_2 \sqrt{1 + \frac{-2\eta \cdot c + \eta^2 \cdot d}{\mathbf{w}_i^{(t)} \cdot \mathbf{w}_i^{(t)}}}} \\ &\quad = \frac{1}{\|\mathbf{w}_i^{(t)}\|_2}[1 - \frac{1}{2}(\frac{-2\eta \cdot c + \eta^2 \cdot d}{\mathbf{w}_i^{(t)} \cdot \mathbf{w}_i^{(t)}}) + o(\frac{-2\eta \cdot c + \eta^2 \cdot d}{\mathbf{w}_i^{(t)} \cdot \mathbf{w}_i^{(t)}})] \\ &\quad = \frac{1}{\|\mathbf{w}_i^{(t)}\|_2}[1 + \frac{\eta \cdot c}{\mathbf{w}_i^{(t)} \cdot \mathbf{w}_i^{(t)}} + o(\eta)]\end{aligned}$$

Thereby, we have

$$\begin{aligned}&1/y_{ij}^{(t+1)} \\ &\quad = \frac{1}{\|\mathbf{w}_i^{(t)}\|_2}[1 + \frac{\eta \cdot c}{\mathbf{w}_i^{(t)} \cdot \mathbf{w}_i^{(t)}} + o(\eta)] \\ &\qquad \cdot \frac{1}{\|\mathbf{w}_j^{(t)}\|_2}[1 + \frac{\eta \cdot e}{\mathbf{w}_j^{(t)} \cdot \mathbf{w}_j^{(t)}} + o(\eta)] \\ &\quad = \frac{1}{\|\mathbf{w}_i^{(t)}\|_2 \|\mathbf{w}_j^{(t)}\|_2}[1 + (\frac{c}{\mathbf{w}_i^{(t)} \cdot \mathbf{w}_i^{(t)}} + \frac{e}{\mathbf{w}_j^{(t)} \cdot \mathbf{w}_j^{(t)}})\eta + o(\eta)]\end{aligned}$$

Now we prove $\lim_{\eta\to 0}\frac{s_{ij}(\mathbf{W}^{(t+1)})-s_{ij}(\mathbf{W}^{(t)})}{\eta}=0$. Consider $x_{ij}^{(t+1)}/y_{ij}^{(t+1)}=\frac{\eta^2 b}{\|\mathbf{w}_i^{(t)}\|\|\mathbf{w}_j^{(t)}\|}[1+(\frac{c}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}+\frac{e}{\mathbf{w}_j^{(t)}\cdot\mathbf{w}_j^{(t)}})\eta+o(\eta)]=o(\eta)$, then $\lim_{\eta\to 0}\frac{x_{ij}^{(t+1)}/y_{ij}^{(t+1)}}{\eta}=0$ and $\lim_{\eta\to 0}\frac{x_{ij}^{(t+1)}}{y_{ij}^{(t+1)}}=0$. Thereby we have

$$\begin{aligned}\lim_{\eta\to 0}\frac{s_{ij}(\mathbf{W}^{(t+1)})-s_{ij}(\mathbf{W}^{(t)})}{\eta}&=\lim_{\eta\to 0}\frac{x_{ij}^{(t+1)}/y_{ij}^{(t+1)}-0}{\eta}\\&=0\end{aligned}.$$

The proof completes. □

Lemma 4. *$\forall(i,j)\in N$, $\exists\psi_{ij}<0$, such that $s_{ij}(\mathbf{W}^{(t+1)})/s_{ij}(\mathbf{W}^{(t)})=1+\psi_{ij}\eta+o(\eta)$, where $\lim_{\eta\to 0}\frac{o(\eta)}{\eta}=0$.*

Proof For $(i,j)\in N$, $\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}\neq 0$, thereby

$$s_{ij}(\mathbf{W}^{(t+1)})/s_{ij}(\mathbf{W}^{(t)})=\frac{x_{ij}^{(t+1)}/y_{ij}^{(t+1)}}{x_{ij}^{(t)}/y_{ij}^{(t)}}$$

According to the definition of $x_{ij}^{(t+1)}/y_{ij}^{(t+1)}$, we have

$$x_{ij}^{(t+1)}/y_{ij}^{(t+1)}=\frac{x_{ij}^{(t)}}{y_{ij}^{(t)}}\frac{1+\frac{-\eta a+\eta^2 b}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}}}{\sqrt{1+\frac{-2\eta\cdot c+\eta^2\cdot d}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}}\sqrt{1+\frac{-2\eta\cdot e+\eta^2\cdot f}{\mathbf{w}_j^{(t)}\cdot\mathbf{w}_j^{(t)}}}}$$

Easily $1+\frac{-\eta a+\eta^2 b}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}}=1-\frac{a}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}}\eta+o(\eta)$. Recall the analysis in previous proof, $\frac{1}{\sqrt{1+\frac{-2\eta c+\eta^2 d}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}}\sqrt{1+\frac{-2\eta e+\eta^2 f}{\mathbf{w}_j^{(t)}\cdot\mathbf{w}_j^{(t)}}}}=1+(\frac{c}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}+\frac{e}{\mathbf{w}_j^{(t)}\cdot\mathbf{w}_j^{(t)}})\eta+o(\eta)$. Substituting the above equations to $x_{ij}^{(t+1)}/y_{ij}^{(t+1)}$, we can obtain that

$$\begin{aligned}&x_{ij}^{(t+1)}/y_{ij}^{(t+1)}\\&=\frac{x_{ij}^{(t)}}{y_{ij}^{(t)}}[1-\frac{a}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}}\eta+o(\eta)]\\&\quad\cdot[1+(\frac{c}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}+\frac{e}{\mathbf{w}_j^{(t)}\cdot\mathbf{w}_j^{(t)}})\eta+o(\eta)],\\&=\frac{x_{ij}^{(t)}}{y_{ij}^{(t)}}[1-(\frac{a}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}}-\frac{c}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}-\frac{e}{\mathbf{w}_j^{(t)}\cdot\mathbf{w}_j^{(t)}})\eta\\&\quad+o(\eta)]\end{aligned}$$

Thereby,

$$\begin{aligned}s_{ij}(\mathbf{W}^{(t+1)})/s_{ij}(\mathbf{W}^{(t)})&=\frac{x_{ij}^{(t+1)}/y_{ij}^{(t+1)}}{x_{ij}^{(t)}/y_{ij}^{(t)}}\\&=1-(\frac{a}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}}-\frac{c}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}-\frac{e}{\mathbf{w}_j^{(t)}\mathbf{w}_j^{(t)}})\eta+o(\eta)\end{aligned}.$$

Let $\psi=-[\frac{a}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}}-\frac{c}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}-\frac{e}{\mathbf{w}_j^{(t)}\cdot\mathbf{w}_j^{(t)}}]$ and then we prove $\psi<0$. Consider the first term

$$\begin{aligned}\frac{a}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}}&=\frac{(p_j+p_i)\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}}\\&=p_j+p_i\end{aligned}.$$

Actually, we have $p_i=p_j=\frac{2}{\mathrm{tr}(\mathbf{W}^\top\mathbf{W})}>0$ and it indicates that $\frac{a}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_j^{(t)}}>0$.

Before moving further, we look into the decomposition $\mathbf{w}_k=\mathbf{x}_k+l_k\mathbf{e}_k$ firstly. If we square both side of the equation,

$$\mathbf{w}_k\cdot\mathbf{w}_k=\mathbf{x}_k\cdot\mathbf{x}_k+l_k^2,$$

notice $\mathbf{w}_k\cdot\mathbf{w}_k$ is square of the norm of $\mathbf{w}_k$. With simple algebraic geometry, we have $\mathbf{x}_k\cdot\mathbf{x}_k\geq 0$ and $\mathbf{w}_k\mathbf{w}_k\geq l_k^2$. Consider the last two terms

$$\begin{aligned}\frac{c}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}&=\frac{p_i\mathbf{x}_i\cdot\mathbf{x}_i+q_il_i}{\mathbf{w}_i^{(t)}\mathbf{w}_i^{(t)}}>0,\\\frac{d}{\mathbf{w}_j^{(t)}\cdot\mathbf{w}_j^{(t)}}&=\frac{p_j\mathbf{x}_j\cdot\mathbf{x}_j+q_jl_j}{\mathbf{w}_j^{(t)}\cdot\mathbf{w}_j^{(t)}}>0.\end{aligned}$$

Substitute above equation and we have

$$\begin{aligned}\frac{c}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}&=\frac{p_i(\mathbf{x}_i\cdot\mathbf{x}_i)+q_il_i}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}\\&=\frac{p_i(\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}-l_i^2)+q_il_i}{\mathbf{w}_i^{(t)}\cdot\mathbf{w}_i^{(t)}}.\\&=p_i+\frac{q_il_i-p_il_i^2}{\|\mathbf{w}_i^{(t)}\|^2}\end{aligned}$$

and

$$\frac{d}{\mathbf{w}_j^{(t)}\cdot\mathbf{w}_j^{(t)}}=p_j+\frac{q_jl_j-p_jl_j^2}{\|\mathbf{w}_j^{(t)}\|^2}$$

Thereby, we have

$$\begin{aligned}\psi&=-[(p_i+p_j)-(p_i+\frac{q_il_i-p_il_i^2}{\|\mathbf{w}_i^{(t)}\|^2})-(p_j+\frac{q_jl_j-p_jl_j^2}{\|\mathbf{w}_j^{(t)}\|^2})]\\&=-[\frac{p_il_i^2-q_il_i}{\|\mathbf{w}_i^{(t)}\|^2}+\frac{p_jl_j^2-q_jl_j}{\|\mathbf{w}_j^{(t)}\|^2}]\end{aligned}$$

If $\frac{p_il_i^2-q_il_i}{\|\mathbf{w}_i^{(t)}\|^2}>0$ and $\frac{p_jl_j^2-q_jl_j}{\|\mathbf{w}_j^{(t)}\|^2}>0$, we can immediately draw the conclusion that $\psi<0$.

Obviously, we only need to discuss one case. From Lemma 2, we have $p_i=\frac{2}{\mathrm{tr}(\mathbf{W}^\top\mathbf{W})}>0$ and $q_i=[\frac{2}{\mathrm{tr}(\mathbf{W}^\top\mathbf{W})}-\frac{\det(\mathbf{W}_{-i}^\top\mathbf{W}_{-i})}{K\det(\mathbf{W}^\top\mathbf{W})}]l_i$. Let $\Omega_i=\frac{\det(\mathbf{W}_{-i}^\top\mathbf{W}_{-i})}{K\det(\mathbf{W}_i^\top\mathbf{W}_i)}$, we have

$$q_i=(p_i-\Omega_i)l_i,$$

$$\begin{aligned}\Omega_i&=\frac{\det(\mathbf{W}_{-i}^\top\mathbf{W}_{-i})}{K\cdot\det(\mathbf{W}_{-i}^\top\mathbf{W}_{-i})(l_i\mathbf{e}_i\cdot\mathbf{w}_i^{(t)})}.\\&=\frac{1}{Kl_i\mathbf{e}_i\cdot\mathbf{w}_i^{(t)}}\end{aligned}$$

Thereby,

$$\begin{aligned}\frac{p_i l_i^2 - q_i l_i}{\|\mathbf{w}_i^{(t)}\|^2} &= \frac{p_i l_i^2 - (p_i - \Omega_i) l_i \cdot l_i}{\|\mathbf{w}_i^{(t)}\|^2} \\ &= \frac{\Omega_i l_i^2}{\|\mathbf{w}_i^{(t)}\|^2} \\ &= \frac{l_i}{K(\mathbf{e}_i \cdot \mathbf{w}_i^{(t)})\|\mathbf{w}_i^{(t)}\|^2}\end{aligned},$$

where K denotes the number of hyperplanes, $l_i > 0$ and $(\mathbf{e}_i \cdot \mathbf{w}_i^{(t)}) > 0$ (discussed in Lemma 2). Thus,

$$\frac{p_i l_i^2 - q_i l_i}{\|\mathbf{w}_i^{(t)}\|^2} > 0.$$

The proof completes. □

Given these two lemmas, we can prove Theorem 1 now.

Proof For the cosine similarity $s(\mathbf{W}^{(t)})$ between hyperplanes $\mathbf{w}_i$ and $\mathbf{w}_j$,

A. if $s_{ij}(\mathbf{W}^{(t)}) \in V$,

$$\begin{aligned}\lim_{\eta \to 0} \frac{s_{ij}(\mathbf{W}^{(t+1)}) - s_{ij}(\mathbf{W}^{(t)})}{\eta} &= \lim_{\eta \to 0} \frac{o(\eta)}{\eta} \\ &= 0\end{aligned}.$$

If $s_{min}(\mathbf{W}^{(t)}) = 0$, we have

$$\lim_{\eta \to 0} \frac{s_{min}(\mathbf{W}^{(t+1)}) - s_{min}(\mathbf{W}^{(t)})}{\eta} = 0. \quad (3)$$

B. if $s_{ij}(\mathbf{W}^{(t)}) \in N$,

$$\begin{aligned}&\lim_{\eta \to 0} \frac{s_{ij}(\mathbf{W}^{(t+1)}) - s_{ij}(\mathbf{W}^{(t)})}{\eta} \\ &= \lim_{\eta \to 0} \frac{\frac{s_{ij}(\mathbf{W}^{(t+1)})}{s_{ij}(\mathbf{W}^{(t)})} - 1}{\eta} \cdot s_{ij}(\mathbf{W}^{(t)}) \\ &= \psi_{ij} \cdot s_{ij}(\mathbf{W}^{(t)})\end{aligned}.$$

Since the minimal angle $\theta_{i*j*} \in [0, \frac{\pi}{2})$, we have $s_{min}(\mathbf{W}^{(t)}) > 0$. Then

$$\lim_{\eta \to 0} \frac{s_{min}(\mathbf{W}^{(t+1)}) - s_{min}(\mathbf{W}^{(t)})}{\eta} < 0$$

So $\exists \kappa > 0$, such that $\forall \eta \in (0, \kappa)$, we have $\frac{s_{min}(\mathbf{W}^{(t+1)}) - s_{min}(\mathbf{W}^{(t)})}{\eta} < 0$. That is $s_{min}(\mathbf{W}^{(t+1)}) - s_{min}(\mathbf{W}^{(t)}) < 0$.

Combine both **A.** and **B.**, we have:

$$s_{min}(\mathbf{W}^{(t+1)}) - s_{min}(\mathbf{W}^{(t)}) \leq 0.$$

Using the $\cos(\cdot)$'s monotonicity in $[0, \frac{\pi}{2}]$ and $s_{min}(\mathbf{W}^{(t)}) = \cos(-\mathcal{R}(\mathbf{W}^{(t)}))$, we have

$$\mathcal{R}(\mathbf{W}^{(t+1)}) \leq \mathcal{R}(\mathbf{W}^{(t)}), \quad (4)$$

where $\mathbf{W}^{(t+1)} = \mathbf{W}^{(t)} - \eta \nabla \widehat{\mathcal{R}}(\mathbf{W}^{(t)})$.

The proof completes. □

4 Proof of Theorem 2

4.1 Preliminary

We first present the problem setup before further discussion.

- Task: Large Margin Machine for Multi-class Learning
- Input: $S = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_m, y_m)\}$ is a set of m training examples, where $\mathbf{x}_i$ is drawn from a domain $\mathcal{X} \subseteq \mathbb{R}^D$ and each label y_i is an integer from $\mathcal{Y} = \{1, \dots, K\}$.
- Distribution: $\mathbb{D}$ represents true distribution of sample S and $\widehat{\mathbb{D}}$ is the empirical distribution.
- Hypothesis set: Parameterizing with hyperplanes matrix $\mathbf{W}$, a family of hypotheses is defined

$$H = \{h | h(\mathbf{x}, y) = \mathbf{w}_y^\top \mathbf{x},\ \mathbf{x} \in \mathcal{X},\ y \in \mathcal{Y}\}, \quad (5)$$

which maps $\mathcal{X} \times \mathcal{Y}$ to $\mathbb{R}$. For each hypothesis $h \in H$, its multi-class margin for the input-output pair $(\mathbf{x}, y)$ is

$$\rho_h(\mathbf{x}, y) = h(\mathbf{x}, y) - \max_{r \neq y} h(\mathbf{x}, r). \quad (6)$$

- Loss set: The hinge loss function is

$$\mathcal{A} = \{l | l(\mathbf{x}, y) = \Phi_p(\rho_h(\mathbf{x}, y))\}, \quad (7)$$

where $\Phi_p(x) = \max(0, 1 - x/p)$ is a p-margin hinge loss taking values in $[0, +\infty)$. Typically, p is set to 1.
- Error: The generalization error of a hypothesis h is

$$L(h) = E_{(\mathbf{x}, y) \sim \mathbb{D}}[\Phi_p(\rho_h(\mathbf{x}, y))]. \quad (8)$$

And the training error of a hypothesis h is

$$\widehat{L}(h) = E_{(\mathbf{x}, y) \sim \widehat{\mathbb{D}}}[\Phi_p(\rho_h(\mathbf{x}, y))]. \quad (9)$$

Moreover, we need extra bounds in error analysis.

- Let input vector $\mathbf{x}_i \in \mathbb{R}^n$ bounded with $\|\mathbf{x}_i\|_2 \leq C_1$.
- Let each hyperplane $\mathbf{w}_r \in \mathbb{R}^n$ w.r.t class r bounded with $\|\mathbf{w}_r\| \leq C_2$. The minimal pairwise angle is $-\mathcal{R}(\mathbf{W}) = \min_{i \neq j} \theta_{ij}$, denoting by θ_{min} for short.

The Rademacher complexity $R_m(\mathcal{A})$ of the loss function set $\mathcal{A}$ is defined as $R_m(\mathcal{A}) = \mathbb{E}[\sup_{l \in \mathcal{A}} \frac{1}{m} \sum_{i=1}^m \sigma_i \cdot l(\mathbf{x}_i, y_i)]$, where σ_i is uniform over $\{-1, 1\}$ and $(\mathbf{x}_i, y_i)_{i=1}^m$ are i.i.d samples drawn from $\mathbb{D}$. The following analysis is based on Bartlett and Mendelson; Percy [2002; 2015]'s Corollary.

Lemma 5. *Fix $p > 0$. With probability at least $1 - \delta$*

$$L(\hat{h}) - L(h^*) \leq 4R_m(\mathcal{A}) + B\sqrt{\frac{2\log(2/\delta)}{m}} \quad (10)$$

for $B \geq \sup_{\mathbf{x}, y, l} |l(\mathbf{x}, y)|$.

Regarding to two situations H^0 and H^1, further sequently boundings on $R_m(\mathcal{A})$ and B complete the proof.

4.2 A1. Upper bound the term $R_m(\mathcal{A})$

Firstly, we should find an upper bound of the Rademacher complexity of the hypothesis set H, that is

$$R_m(\mathcal{A}) = \mathbb{E}[\sup_{l \in \mathcal{A}} \frac{1}{m} \sum_{i=1}^m \sigma_i \cdot l(\mathbf{x}_i, y_i)]. \quad (11)$$

However, the hypothesis set H has no direct connection with the loss function set (no matter $0 \cdot 1$ or hinge loss), defined on the multi-class margin. Let $\widetilde{H}$ be the family of hypothesis mapping $\mathcal{X} \times \mathcal{Y}$ to $\mathbb{R}$ defined by $\widetilde{H} = \{z = (\mathbf{x}, y) \mapsto \rho_h(\mathbf{x}, y) : \mathbf{x} \in \mathcal{X},\ y \in \mathcal{Y},\ h \in H\}$. Then we expand the intractable Rademacher complexity term as

$$R_m(\mathcal{A}) = R_n(l \circ \widetilde{H}). \tag{12}$$

To employ the conclusion from [Bartlett and Mendelson, 2002], we rewrite the Rademacher complexity $R_m(\mathcal{A})$ in Eq.(11)'s definiton in absolute form

$$R_m^{||}(\mathcal{A}) = \mathbb{E}[\sup_{l \in \mathcal{A}} \frac{2}{m} \sum_{i=1}^{m} |\sigma_i \cdot l(\mathbf{x}_i, y_i)|]. \tag{13}$$

Considering previous definition with the loss function $l(\cdot) \geq 0$ in Eq.(11), we have

$$R_m(\mathcal{A}) \leq \tfrac{1}{2} R_m^{||}(\mathcal{A}). \tag{14}$$

From Eq.(12), it is natural that

$$R_m^{||}(\mathcal{A}) = R_m^{||}(l \circ \widetilde{H}). \tag{15}$$

Bounding $R_m^{||}(\mathcal{A})$ equals bounding $R_m^{||}(\widetilde{H})$ through discussing the L-Lipschitz property on two kinds of loss function, and the analysis of $R_m^{||}(\widetilde{H})$ is put in advance for brevity. It requires us to extend the proof of Theorem 1 in Cortes *et al.* [2013]'s work.

Let $H_{\mathcal{X}}$ denotes a set of functions defined over $\mathcal{X}$ and derived from H as follows: $H_{\mathcal{X}} = \{\mathbf{x} \mapsto h(\mathbf{x}, y) : y \in \mathcal{Y},\ h \in H\}$. For any fixed $y \in \mathcal{Y}$, $H_{\mathcal{X}}$ only takes $\mathbf{x}$ into consideration and its empirical Rademacher complexity becomes an upper bound of $\widetilde{H}$, which is given in Lemma 6 (proof is given in Sec. 5).

Lemma 6. *With the definition of $\rho_h(\mathbf{x}, y)$ and $\widetilde{H}$, the empirical Rademacher complexity:*

$$R_m^{||}(\widetilde{H}) \leq K^2 R_m^{||}(H_{\mathcal{X}}). \tag{16}$$

Next, we bound $R_m^{||}(H_{\mathcal{X}})$ [Xie *et al.*, 2015b]. For any $y \in \mathcal{Y}$ we have following key steps

$$\begin{aligned}
R_m^{||}(H_{\mathcal{X}}) &= \mathbb{E}[\ \sup\nolimits_{h \in H_{\mathcal{X}}} \tfrac{2}{m} |\sum\nolimits_{i=1}^{m} \sigma_i \mathbf{w}_y^\top \mathbf{x}_i|\ ] \\
&\leq \tfrac{2C_2}{m} \mathbb{E}[\ \|\sum\nolimits_{i=1}^{m} \sigma_i \mathbf{x}_i\|_2\ ] \quad (\|\mathbf{w}_{\mathrm{r}}\| \leq \mathrm{C}_2) \\
&= \tfrac{2C_2}{m} \mathbb{E}_{\mathbb{D}}[\ \mathbb{E}_{\sigma}[\ \|\sum\nolimits_{i=1}^{n} \sigma_i \mathbf{x}_i\|_2 \mid \mathbf{x}_i \sim \mathbb{D}\ ]\ ] \\
&\quad \text{(the definition of Rademacher complexity,} \\
&\quad \text{expanding Expectation)} \\
&= \tfrac{2C_2}{m} \mathbb{E}_{\mathbb{D}}[\ \sqrt{\mathbb{E}_{\sigma}[\ \sum\nolimits_{i=1}^{n} \sigma_i^2 (\mathbf{x}_i)^2 \mid \mathbf{x}_i \sim \mathbb{D}\ ]}\ ] \\
&\quad (\forall\ i \neq j,\ \sigma_i \perp \sigma_j) \\
&= \tfrac{2C_2}{m} \mathbb{E}_{\mathbb{D}}[\ \sqrt{\sum\nolimits_{i=1}^{m} (\mathbf{x}_i)^2}\ ] \\
&\quad (\mathbf{x}_i \text{ are i.i.d samples from } \mathbb{D},\ \text{same to } \mathbf{x}_{\mathrm{i}}) \\
&\leq \tfrac{2C_1 C_2}{\sqrt{m}} \quad (\|\mathbf{x}_i\| \leq C_1)
\end{aligned}$$

Plug above results into Lemma 6,

$$R_m^{||}(\widetilde{H}) \leq \tfrac{2K^2 C_1 C_2}{\sqrt{m}}. \tag{17}$$

Ultimately, we discuss the Lipschitz connectivity of the $0 \cdot 1$ loss and p-margin hinge loss. It's easy to find that both of them are differentiable almost everywhere, that is, differentiable at every point outside a set of Lebesgue measure zero (e.x. $x = 0$ for $\mathbb{I}(\cdot)$, $x = p$ for $\Phi_p(\cdot)$). We have Lipschitz constants

$$\mathcal{L}_{0 \cdot 1} = 1, \qquad \mathcal{L}_{hing} = \tfrac{1}{p}. \tag{18}$$

Let $l'(\cdot) = l(\cdot) - l(0)$, then $l'(0) = 0$ and l' is also $\mathcal{L}$-Lipschitz. Then

$$\begin{aligned}
R_m^{||}(l \circ \widetilde{H}) &= R_m^{||}(l' \circ \widetilde{H} + l(0)) \\
&\leq R_m^{||}(l' \circ \widetilde{H}) + \tfrac{\|l(0)\|_\infty}{\sqrt{m}} \\
&\leq 2 \cdot \mathcal{L} R_m^{||}(\widetilde{H}) + \tfrac{\|l(0)\|_\infty}{\sqrt{m}}
\end{aligned} \tag{19}$$

The last two inequations are derived from Theorem 12.5 and Theorem 12.4 in Bartlett and Mendelson [2002]' work. Note that the $l(0)$ is required to be uniformly bounded function by Bartlett's work. For the $0 \cdot 1$ loss function, $\mathbb{I}(0)$ can be assigned any value t while maintains its measurable property. For the p-margin hinge loss, $\Phi_p(0) = \max(0, 1 - 0/p) = 1$. Without lose of generality, we set $t = 1$ and $l(0)$ is uniformly bounded within $[1-\epsilon, 1+\epsilon]$ for all ϵ. So we have $\|l(0)\|_\infty = 1$ with $\epsilon \to 0$ for both of them. This is different from previous work [Xie *et al.*, 2015b].

So far, we find an upper bound for $4R_m(\mathcal{A})$ in RHS of Eq.(10) through combing Eq.(14, 15, 19, 17) sequently.

4.3 A2. Find the upper bound B

Finding the maximum of $|l(\mathbf{x}, y)|$ completes our prove. Given the formulation of p-margin hinge loss as $l(\cdot) = \Phi_p(\cdot)$ and its $\frac{1}{p}$-Lipschitz property, we have

$$\begin{aligned}
& |\Phi_p(\mathbf{z}) - \Phi_p(\mathbf{0})| \leq \tfrac{1}{p} |\mathbf{z} - \mathbf{0}| \\
\Rightarrow\ & |\Phi_p(\mathbf{z}) - 1| \leq \tfrac{1}{p} |\mathbf{z}| \\
\Rightarrow\ & |\Phi_p(\mathbf{z})| \leq \tfrac{1}{p} \|\mathbf{z}\| + 1
\end{aligned}.$$

With the margin condition $\rho_h(\mathbf{x}, y) \leq p$ and $p \geq 0$ we let $z = \rho_h(\mathbf{x}, y)$:

$$|l(\mathbf{x}, y)| = |\Phi_p(\rho_h(\mathbf{x}, y))| \leq \tfrac{1}{p} |\rho_h(\mathbf{x}, y)| + 1, \tag{20}$$

where the only remaining issue is to bound $|\Phi_p(\rho_h(\mathbf{x}, y))|$. Due to the different formulation of cost function, this proof is different from [Xie *et al.*, 2015b].

With the hyperplanes set $\mathbf{W}$, we define an adjunction one $\mathbf{W}_y = \{\mathbf{w}_y \ldots \mathbf{w}_y \ldots \mathbf{w}_y\}$ for a fixed $y \in \mathcal{Y}$. Let $\mathbf{w}_R^\top \mathbf{x} = \max_{r \in \mathcal{Y} \setminus y} \mathbf{w}_r^\top \mathbf{x}$, $\mathbf{w}_{r*}^\top \mathbf{x} = \{\mathbf{w}_r^\top \mathbf{x} \mid r \in \mathcal{Y} \setminus \{y, R\}\}$ and we have

$$\mathbf{w}_R^\top \mathbf{x} > all(\mathbf{w}_{r*}^\top \mathbf{x}).$$

Moreover, we define $\mathbf{w}_{r+}^\top \mathbf{x} = \{\mathbf{w}_r^\top \mathbf{x} \mid \mathbf{w}_r^\top \mathbf{x} \geq 0, r \in \mathcal{Y} \setminus \{y, R\}\}$ and $\mathbf{w}_{r-}^\top \mathbf{x}$ reversely, the inequalities hold

$$\mathbf{w}_R^\top \mathbf{x} > all(\mathbf{w}_{r+}^\top \mathbf{x}) \quad , \quad \mathbf{w}_R^\top \mathbf{x} > all(\mathbf{w}_{r-}^\top \mathbf{x}).$$

Let $g(\cdot)$ be an auxiliary function as

$$g(z) = |\mathbf{w}_y^\top \mathbf{x} - z|. \tag{21}$$

From the analysis in Sec.6, we have that condition A1c, A2c and B1a all share an important property

$$\mathbf{w}_R^\top \mathbf{x} \le \mathbf{w}_y^\top \mathbf{x} \quad \Leftrightarrow \quad \max_{r\in\mathcal{Y}\setminus y} \mathbf{w}_r^\top \mathbf{x} \ \le \ \mathbf{w}_y^\top \mathbf{x},$$

which indicates that the labeled input data $(\mathbf{x}, y)$ has the biggest activation other than wrong labels $\mathcal{Y}\setminus y$ in weighted matrix $\mathbf{W}$. The analysis is given in the extreme situation from the worst $\mathbf{W}$ to perfect $\mathbf{W}$.

Proposition 1. *If $g(\mathbf{w}_R^\top \mathbf{x}) < g(any(\mathbf{w}_{r*}^\top \mathbf{x}))$, then we can put the alongside*

$$\begin{array}{r}
|\mathbf{w}_y^\top \mathbf{x} - \mathbf{w}_R^\top \mathbf{x}| < |\mathbf{w}_y^\top \mathbf{x} - any(\mathbf{w}_{r+}^\top \mathbf{x})| \\
|\mathbf{w}_y^\top \mathbf{x} - \mathbf{w}_R^\top \mathbf{x}| < |\mathbf{w}_y^\top \mathbf{x} - any(\mathbf{w}_{r-}^\top \mathbf{x})| \\
|\mathbf{w}_y^\top \mathbf{x} - \mathbf{w}_R^\top \mathbf{x}| = |\mathbf{w}_y^\top \mathbf{x} - \mathbf{w}_R^\top \mathbf{x}| \\
0 = |\mathbf{w}_y^\top \mathbf{x} - \mathbf{w}_y^\top \mathbf{x}|
\end{array}.$$

We take square on both sides, add them all and have

$$\begin{array}{l}
|\mathbf{w}_y^\top \mathbf{x} - \mathbf{w}_R^\top \mathbf{x}|^2 \le \frac{1}{K}\sum_{r\in\mathcal{Y}} |\mathbf{w}_y^\top \mathbf{x} - \mathbf{w}_r^\top \mathbf{x}|^2 \\
\Rightarrow |\mathbf{w}_y^\top \mathbf{x} - \mathbf{w}_R^\top \mathbf{x}|^2 \le \frac{1}{K}\|\mathbf{W}^\top \mathbf{x} - \mathbf{W}_y^\top \mathbf{x}\|_2^2
\end{array}.$$

Building further on above proposition, we should bound B by two separated situations.

B1. For hypothesis set $H^0 = \{h | h(\mathbf{x}, y) = \mathbf{w}_y^\top \cdot \mathbf{x},\ \mathbf{w}_y^\top \mathbf{x} < \mathbf{w}_R^\top \mathbf{x}\}$, then $|\rho_h(\mathbf{x}, y)|$ has a natural upper bound by expanding $\rho_h(\mathbf{x}, y)$'s definition on hypothesis set H

$$\begin{aligned}
|\rho_h(\mathbf{x}, y)| &= |\mathbf{w}_y^\top \mathbf{x} - \max_{r\in\mathcal{Y}\setminus y} \mathbf{w}_r^\top \mathbf{x}| \\
&\le |\mathbf{w}_y^\top \mathbf{x}| + |\mathbf{w}_R^\top \mathbf{x}| \\
&\le \|\mathbf{w}_y\| \cdot \|\mathbf{x}\| + \|\mathbf{w}_R\| \cdot \|\mathbf{x}\| \\
&\le 2C_1C_2
\end{aligned} \tag{22}$$

B2. For hypothesis set $H^1 = \{h | h(\mathbf{x}, y) = \mathbf{w}_y^\top \cdot \mathbf{x},\ \mathbf{w}_y^\top \mathbf{x} \ge \mathbf{w}_R^\top \mathbf{x}\}$, $|\rho_h(\mathbf{x}, y)|$ has a tighter upper bound

$$\begin{aligned}
|\rho_h(\mathbf{x}, y)|^2 &= |\mathbf{w}_y^\top \mathbf{x} - \max_{r\in\mathcal{Y}\setminus y} \mathbf{w}_r^\top \mathbf{x}|^2 \\
&\le \tfrac{1}{K}\|\mathbf{W}^\top \mathbf{x} - \mathbf{W}_y^\top \mathbf{x}\|_2^2 \\
&\le \tfrac{1}{K}\|(\mathbf{W} - \mathbf{W}_y)^\top\|_{op}^2 \|\mathbf{x}\|_2^2 \\
&= \tfrac{1}{K}\|\mathbf{W} - \mathbf{W}_y\|_{op}^2 \|\mathbf{x}\|_2^2
\end{aligned} \tag{23}$$

Following the property of $\|\cdot\|_{op}$ (operator norm)

$$\|\mathbf{W} - \mathbf{W}_y\|_{op}^2 \le \|\mathbf{W}\|_{op}^2 + \|\mathbf{W}_y\|_{op}^2. \tag{24}$$

Next we can make use of the lower bound of θ_{ij} between weights $\mathbf{w}_i$ and $\mathbf{w}_j$ ($i \ne j$), which is θ_{min}, to get the bound of $\|\mathbf{W}\|_{op}$ (see in Xie *et al.* [2015b]). Here only gives main steps:

$$\begin{aligned}
\|\mathbf{W}\|_{op}^2 &= \sup_{\|\mathbf{u}\|_2=1} \|\mathbf{W}\mathbf{u}\|_2^2 \\
&= \sup_{\|\mathbf{u}\|_2=1} (\mathbf{u}^\top \mathbf{W}^\top \mathbf{W}\mathbf{u}) \\
&= \sup_{\|\mathbf{u}\|_2=2} \sum_{p=1}^{K}\sum_{q=1}^{K} \mathbf{u}_p\mathbf{u}_q\mathbf{w}_p \cdot \mathbf{w}_q \\
&\le \sup_{\|\mathbf{u}\|_2=2} \sum_{p=1}^{K}\sum_{q=1}^{K} |\mathbf{u}_p||\mathbf{u}_q|\|\mathbf{w}_p\|\|\mathbf{w}_q\|\cos(\theta_{pq}) \\
&\le C_2^2 \sup_{\|\mathbf{u}\|_2=2} \sum_{p=1}^{K}\sum_{q=1}^{K} |\mathbf{u}_p||\mathbf{u}_q|\cos(\theta_{pq}) \\
&\qquad (\|\mathbf{w}_r\| \le C_2) \\
&\le C_2^2 \sup_{\|\mathbf{u}\|_2=2}[\ \sum_{p=1}^{K}\sum_{q=1}^{K} |\mathbf{u}_p||\mathbf{u}_q|\cos(\theta_{min}) \cdot \\
&\qquad \mathbb{I}(p \ne q) + \sum_{p=1}^{K} |\mathbf{u}_p|^2\ ]
\end{aligned},$$

Define $\mathbf{u}' = [|\mathbf{u}_1|, \ldots, |\mathbf{u}_{K^\mathbf{u}}|]^\top$, $Q \in \mathbb{R}^{K^\mathbf{u}\times K^\mathbf{u}} : Q_{pq} = \cos\theta_{pq}$ for $p \ne q$ and $Q_{pp} = 1$, then we have $\|\mathbf{u}'\|_2 = \|\mathbf{u}\|$ and

$$\begin{aligned}
\|\mathbf{W}\|_{op}^2 &\le C_2^2 \sup_{\|\mathbf{u}\|_2=2} [\mathbf{u}'^\top Q\mathbf{u}'] \\
&\le C_2^2 \sup_{\|\mathbf{u}\|_2=2} [\lambda_1(Q)\|\mathbf{u}'\|_2^2] \\
&\le C_2^2\lambda_1(Q)
\end{aligned},$$

where $\lambda_1(Q)$ denotes the largest eigenvalue of Q and we can derive $\lambda_1(Q) = (K-1)\cos\theta_{min} + 1$. Thus

$$\|\mathbf{W}\|_{op}^2 \le ((K-1)\cos\theta_{min} + 1)C_2^2. \tag{25}$$

Since columns in $\mathbf{W}_y$ are same, we get

$$\|\mathbf{W}_y\|_{op}^2 \le KC_2^2. \tag{26}$$

Put above equations together, we have the upper bound:

$$\begin{aligned}
|\rho_h(\mathbf{x}, y)| &\le \sqrt{\tfrac{1}{K}(\|\mathbf{W}\|_{op}^2 + \|\mathbf{W}_y\|_{op}^2)\|\mathbf{x}\|_2^2} \\
&\le \sqrt{\tfrac{1}{K}((K-1)\cos\theta_{min} + K + 1)C_2^2\|\mathbf{x}\|_2^2}. \\
&\le \sqrt{(1 - \tfrac{1}{K})(\cos\theta_{min} + \tfrac{K+1}{K-1})}C_1C_2
\end{aligned} \tag{27}$$

From the above **B1** and **B2**, if the hypothesis set $H \rightsquigarrow H^0$ (such as $W_r = 0$ at the start of the training), then

$$|l(\mathbf{x}, y)| \le \tfrac{2}{p}C_1C_2 + 1. \tag{28}$$

Otherwise the hypothesis set $H \rightsquigarrow H^1$ (while training), then

$$|l(\mathbf{x}, y)| \le \tfrac{1}{p}\sqrt{(1 - \tfrac{1}{K})(\cos(-\mathcal{R}(\mathbf{W})) + \tfrac{K+1}{K-1})}C_1C_2 + 1. \tag{29}$$

4.4 A3. Combination

Finally, combining **A1**'s results and Eq.(20, 28, 29) sequently in **A2**, we acquire desired results in Theorem 2. $\square$

5 Proof of Lemma 6

Proof For any fixed $y \in \mathcal{Y}$ and any $i \in [1, m]$, define ϵ_i as $2(\mathbb{I}(y = y_i)) - 1$. Naturally $\epsilon_i \in \{-1, +1\}$, the Rademacher random variables σ_i and $\sigma_i \epsilon_i$ follow the same distribution. Thus

$$
\begin{aligned}
R^{\|}_m(\widetilde{H}) &= \mathbb{E}[\sup_{h \in H} \tfrac{2}{m} \textstyle\sum_{i=1}^{m} |\sigma_i \rho_h(\mathbf{x}_i, y_i)|] \\
&= \tfrac{2}{m} \mathbb{E}[\sup_{h \in H} \textstyle\sum_{i=1}^{m} \sum_{y \in \mathcal{Y}} |\sigma_i \rho_h(\mathbf{x}_i, y) \mathbb{I}(y = y_i)|] \\
&\le \tfrac{2}{m} \textstyle\sum_{y \in \mathcal{Y}} \mathbb{E}[\sup_{h \in H} \sum_{i=1}^{m} |\sigma_i \rho_h(\mathbf{x}_i, y) \mathbb{I}(y = y_i)|] \\
&\qquad (\mathrm{sub-additivity\ of\ sup}) \\
&= \tfrac{2}{m} \textstyle\sum_{y \in \mathcal{Y}} \mathbb{E}[\sup_{h \in H} \sum_{i=1}^{m} |\sigma_i \rho_h(\mathbf{x}_i, y) \cdot \\
&\qquad\qquad (\tfrac{2 \cdot \mathbb{I}(y = y_i) - 1}{2} + \tfrac{1}{2})|] \\
&\le \tfrac{1}{m} \textstyle\sum_{y \in \mathcal{Y}} \mathbb{E}[\sup_{h \in H} \sum_{i=1}^{m} |\sigma_i \epsilon_i \rho_h(\mathbf{x}_i, y)|] \\
&\quad + \tfrac{1}{m} \textstyle\sum_{y \in \mathcal{Y}} \mathbb{E}[\sup_{h \in H} \sum_{i=1}^{m} |\sigma_i \rho_h(\mathbf{x}_i, y)|] \\
&\qquad (\mathrm{substituting\ } \epsilon \mathrm{\ and\ } |\mathrm{a}+\mathrm{b}| \le |\mathrm{a}| + |\mathrm{b}|) \\
&= \tfrac{2}{m} \textstyle\sum_{y \in \mathcal{Y}} \mathbb{E}[\sup_{h \in H} \sum_{i=1}^{m} |\sigma_i \rho_h(\mathbf{x}_i, y)|] \\
&\qquad (\mathrm{merging\ the\ variables\ from\ same\ distribution})
\end{aligned}
$$

Let $H_{\mathcal{X}}^{(\backslash y)} = \{\max(h_1, \ldots) : h_j \in H_{\mathcal{X}}, j \in [1, \ldots, k] \backslash y\}$, and we have $R^{\|}_m(H_{\mathcal{X}}^{(\backslash y)}) \le (k-1) R^{\|}_m(H_{\mathcal{X}})$ from Ledoux and Talagrand [2013]'s conclusion similar to the empirical case.

Next, we can rewrite $\rho_h(\mathbf{x}_i, y_i)$ explicitly

$$
\begin{aligned}
R^{\|}_m(\widetilde{H}) &\le \tfrac{2}{m} \textstyle\sum_{y \in \mathcal{Y}} \mathbb{E}[\sup_{h \in H} \sum_{i=1}^{m} |\sigma_i (h(\mathbf{x}_i, y) \\
&\qquad\qquad - \max_{r \ne y} h(\mathbf{x}_i, r))|] \\
&\le \textstyle\sum_{y \in \mathcal{Y}} [\, \tfrac{2}{m} \mathbb{E}[\sup_{h \in H} \sum_{i=1}^{m} |\sigma_i h(\mathbf{x}_i, y)|] \\
&\quad + \tfrac{2}{m} \mathbb{E}[\sup_{h \in H} \textstyle\sum_{i=1}^{m} | - \sigma_i \max_{r \ne y} h(\mathbf{x}_i, r)|] \,] \\
&= \textstyle\sum_{y \in \mathcal{Y}} [\, \tfrac{2}{m} \mathbb{E}[\sup_{h \in H} \sum_{i=1}^{m} |\sigma_i h(\mathbf{x}_i, y)|] \\
&\quad + \tfrac{2}{m} \mathbb{E}[\sup_{h \in H} \textstyle\sum_{i=1}^{m} |\sigma_i \max_{r \ne y} h(\mathbf{x}_i, r)|] \,] \\
&= \textstyle\sum_{y \in \mathcal{Y}} [\, \tfrac{2}{m} \mathbb{E}[\sup_{h \in H_{\mathcal{X}}} \sum_{i=1}^{m} |\sigma_i h(\mathbf{x}_i)|] \\
&\quad + \tfrac{2}{m} \mathbb{E}[\sup_{h \in H_{\mathcal{X}}^{\backslash y}} \textstyle\sum_{i=1}^{m} |\sigma_i \max_{r \ne y} h(\mathbf{x}_i)|] \,] \\
&\le k[\, \tfrac{2k}{m} \mathbb{E}[\sup_{h \in H_{\mathcal{X}}} \textstyle\sum_{i=1}^{m} |\sigma_i h(\mathbf{x}_i)|] \,] \\
&= k^2 R^{\|}_m(H_{\mathcal{X}})
\end{aligned}
$$

That concludes the proof. □

6 Analysis of the function $g(\cdot)$

Depending on the positive and negative property of $\mathbf{w}_y^\top \mathbf{x}$, our results are divided into several situations. For the sake of simplifying, some shorthands are defined as

$$
\begin{aligned}
&R \triangleq g(\mathbf{w}_R^\top \mathbf{x}), \\
&r+ \triangleq g(any(\mathbf{w}_{r+}^\top \mathbf{x})), \quad r- \triangleq g(any(\mathbf{w}_{r-}^\top \mathbf{x}))
\end{aligned}
$$

If A: $\mathbf{w}_R^\top \mathbf{x} \ge -p$ with margin $p > 0$:

1.	$0 < \mathbf{w}_y^\top \mathbf{x} \le \mathbf{w}_R^\top \mathbf{x} + p$	
	a) $2\mathbf{w}_y^\top \mathbf{x} \le \mathbf{w}_R^\top \mathbf{x}$	$R > r+, \quad R ? r-$
	b) $\mathbf{w}_y^\top \mathbf{x} < \mathbf{w}_R^\top \mathbf{x} < 2\mathbf{w}_y^\top \mathbf{x}$	$R ? r+, \quad R > r-$
	c) $-p \le \mathbf{w}_R^\top \mathbf{x} < \mathbf{w}_y^\top \mathbf{x}$	$R < g(any(\mathbf{w}_{r*}^\top \mathbf{x}))$
2.	$-p < \mathbf{w}_y^\top \mathbf{x} \le 0 \le \mathbf{w}_R^\top \mathbf{x} + p$	
	a) $\mathbf{w}_y^\top \mathbf{x} < 0 \le \mathbf{w}_R^\top \mathbf{x}$	$R > r+, \quad R ? r-$
	b) $\mathbf{w}_y^\top \mathbf{x} \le \mathbf{w}_R^\top \mathbf{x} < 0$	$\{\mathbf{w}_{r+}^\top \mathbf{x}\} = \emptyset, R ? r-$
	c) $-p \le \mathbf{w}_R^\top \mathbf{x} < \mathbf{w}_y^\top \mathbf{x}$	$\{\mathbf{w}_{r+}^\top \mathbf{x}\} = \emptyset, R < r-$
3.	$\mathbf{w}_y^\top \mathbf{x} \le -p < 0 \le \mathbf{w}_R^\top \mathbf{x} + p$	
	a) $\mathbf{w}_R^\top \mathbf{x} \ge -p$	$\{\mathbf{w}_{r+}^\top \mathbf{x}\} = \emptyset, R ? r-$

If B: $\mathbf{w}_R^\top \mathbf{x} < -p$ with margin $p > 0$:

1.	$\mathbf{w}_y^\top \mathbf{x} < 0$, because $\mathbf{w}_y^\top \mathbf{x} \le \mathbf{w}_R^\top \mathbf{x} + p < 0$	
	a) $\mathbf{w}_R^\top \mathbf{x} \le \mathbf{w}_y^\top \mathbf{x}$	$\{\mathbf{w}_{r+}^\top \mathbf{x}\} = \emptyset, R ? r-$
	b) $\mathbf{w}_y^\top \mathbf{x} < \mathbf{w}_R^\top \mathbf{x} < -p$	$\{\mathbf{w}_{r+}^\top \mathbf{x}\} = \emptyset, R ? r-$

The Fig. 3 below gives the reference sketch used to analyse above conditions.

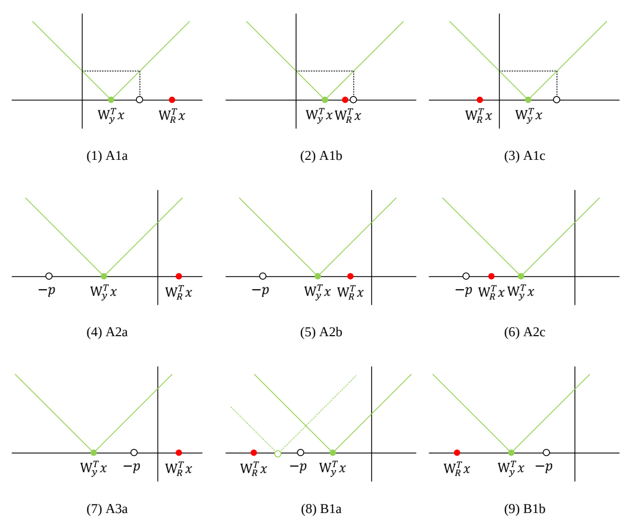


Figure 2: Reference sketch

7 Calculation of the gradient

The primal objective function is defined as

$$
\begin{aligned}
P_i(\mathbf{W}) = &\frac{\lambda}{2} \|\mathbf{W}\|_2^2 + L(\mathbf{W}; (\mathbf{x}_i, \mathbf{y}_i)) \\
&+ \frac{\beta}{2} \log \mathrm{tr}(\mathbf{W}^\top \mathbf{W}) - \frac{\beta \cdot \mathbb{I}}{2K} \log |\mathbf{W}^\top \mathbf{W}|.
\end{aligned}
$$

Using SGD to solve this problem, we have to calculate the gradient of $P_i(\mathbf{W})$. Though the $\mathbf{W}$ is a $n \times k$ (n denotes the feature vectors' dimension and k is the number of classes) matrix, the results of $P_i(\mathbf{W})$ applied with a matrix variable is a scoring $\in \mathbb{R}$. To our knowledge, there is a matrices expression of $\frac{\partial P_i(\mathbf{W})}{\partial \mathbf{W}}$ for a typical formulation.

To make it clearly, we define

$$\begin{aligned}\frac{\partial P_i(\mathbf{W})}{\partial \mathbf{W}} &= \frac{\lambda}{2}\frac{\partial \|\mathbf{W}\|_2^2}{\partial \mathbf{W}} + \frac{\partial L(\mathbf{W};(\mathbf{x}_i, y_i))}{\partial \mathbf{W}} \\ &+ \frac{\beta}{2}\frac{\partial \log \mathrm{tr}(\mathbf{W}^\top \mathbf{W})}{\partial \mathbf{W}} - \frac{\beta \cdot \mathbb{I}}{2K}\frac{\partial \log |\mathbf{W}^\top \mathbf{W}|}{\partial \mathbf{W}} \\ &= \frac{\lambda}{2}\nabla_{L2} + \nabla_{Loss} + \frac{\beta}{2}\nabla_{logtr} - \frac{\beta \cdot \mathbb{I}}{2K}\nabla_{logdet}\end{aligned} \quad . \tag{30}$$

Then, we do the derivation partly.

(1) Firstly, we have

$$\begin{aligned}\nabla_{L2} &= \tfrac{\partial [\|\mathbf{W}\|_2^2]_{1\times 1}}{\partial [\mathbf{W}]_{n\times k}} \\ &= \tfrac{\partial [\mathbf{1}^\top \mathbf{W}^\top \mathbf{W} \mathbf{1}]_{1\times 1}}{\partial [\mathbf{W}]_{n\times k}} \\ &= \mathbf{W}(\mathbf{1}\mathbf{1}^\top + \mathbf{1}\mathbf{1}^\top) \\ &= [\mathbf{W}]_{n\times k}[(\mathbf{1}\mathbf{1}^\top + \mathbf{1}\mathbf{1}^\top)]_{k\times k} \\ &= \mathbf{W} \cdot 2\mathbf{I} \\ &= 2\mathbf{W}\end{aligned} \tag{31}$$

(2) Secondly, the ∇_{Loss} needs the updating rules with subgradient and

$$\nabla_{Loss} = \frac{1}{m}\sum_{i=1}^{m} \max(0, 1 + \mathbf{w}_{r_i}^\top \mathbf{x}_i - \mathbf{w}_{y_i}^\top \mathbf{x}_i). \tag{32}$$

- If hingloss Eq.(32) equals zero:

$$\nabla_{Loss}^j = 0,$$

- If hingloss Eq.(32) is above zero:

$$\nabla_{Loss}^j = \begin{cases} [-\mathbf{x}_i]_{n\times 1}, & if\ j = y_i \\ [\mathbf{x}_i]_{n\times 1}, & if\ j = R_i \\ 0, & \text{otherwise} \end{cases}$$

And all the $[\nabla_{Loss}^1 \ldots \nabla_{Loss}^j \ldots \nabla_{Loss}^k]$ make the ∇_{Loss}.

(3) Thirdly, we have

$$\begin{aligned}\nabla_{logtr} &= \frac{\partial [\log \mathrm{tr}(\mathbf{W}^\top \mathbf{W})]_{1\times 1}}{\partial [\mathbf{W}]_{n\times k}} \\ &= \frac{1}{\mathrm{tr}(\mathbf{W}^\top \mathbf{W})} \cdot \frac{\partial [\mathrm{tr}(\mathbf{W}^\top \mathbf{W})]_{1\times 1}}{\partial [\mathbf{W}]_{n\times k}} \\ &= \frac{2\mathbf{W}}{\mathrm{tr}(\mathbf{W}^\top \mathbf{W})}\end{aligned} \tag{33}$$

(4) Fourthly, the ∇_{logdet} term's results are divided based on the gradient direction we pursuit as

$$\nabla_{logdet} = \frac{\partial [\log |\mathbf{W}^\top \mathbf{W}|]_{1\times 1}}{\partial [\mathbf{W}]_{n\times k}}. \tag{34}$$

If we regard each coordinate's partial direction $\mathbf{W}(i,j)$ as a whole to replace the gradient direction, we have

$$\begin{aligned}&\nabla_{logdet}(i,j) \\ &= \tfrac{\partial [\log |\mathbf{W}^\top \mathbf{W}|]_{1\times 1}}{\partial [W_{ij}]_{n\times 1}} \\ &= tr(\tfrac{\partial \log |\mathbf{W}^\top \mathbf{W}|}{\partial \mathbf{W}^\top \mathbf{W}} \tfrac{\partial (\mathbf{W}^\top \mathbf{W})^\top}{\partial W_{ij}}) \\ &= tr(((\mathbf{W}^\top \mathbf{W})^\top)^{-1} \tfrac{\partial \mathbf{W}^\top \mathbf{W}}{\partial W_{ij}}) \\ &= tr((\mathbf{W}^\top \mathbf{W})^{-1} \tfrac{\partial \mathbf{W}^\top \mathbf{W}}{\partial W_{ij}}) \\ &= tr((\mathbf{W}^\top \mathbf{W})^{-1} (\tfrac{\partial \mathbf{W}^\top}{\partial W_{ij}} \mathbf{W} + \mathbf{W}^\top \tfrac{\partial \mathbf{W}}{\partial W_{ij}})) \\ &= \sum\nolimits_{p=1}^{k} (\mathbf{W}^\top \mathbf{W})_{pi}^{-1} \mathbf{W}_{jp} \\ &\quad\quad + \sum\nolimits_{p=1}^{k} (\mathbf{W}^\top \mathbf{W})_{jp}^{-1} \mathbf{W}_{pi}^\top \\ &= 2 \cdot \sum\nolimits_{p=1}^{k} (\mathbf{W}^\top \mathbf{W})_{jp}^{-1} \mathbf{W}_{pi}^\top \\ &= 2((\mathbf{W}^\top \mathbf{W})^{-1} \mathbf{W}^\top)_{ji}\end{aligned} \quad , \tag{35}$$

where $W_{ij} \in \mathbb{R}^{n\times k}$ which requires $n \le k$ which is common in ordinary dataset and $\mathbf{W}^+$ denotes the pseudo inverse of $\mathbf{W}$. Thus we have the gradient

$$\nabla_{logdet} = 2\mathbf{W}(\mathbf{W}^\top \mathbf{W})^{-1}, \tag{36}$$

where the symmetric matrix $\mathbf{W}^\top \mathbf{W}$ is invertible.

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