Linear Algebra
Homework 3: Eigenvalues and eigenvectors

Problem 1 (Oblique projectors; 5pt). Assume that $\mathbb{R}^n$ is represented as the *direct* (but not necessarily *orthogonal*) sum $M_1 \dotplus M_2$ of two its subspaces M_1 and M_2. In particular, every $\mathbf{x} \in \mathbb{R}^n$ can be represented in a unique way as $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ with some $\mathbf{x}_j \in M_j$, and the mapping $P_j : \mathbf{x} \mapsto \mathbf{x}_j$ is called the *(oblique) projector onto M_j parallel to M_{3-j}*. It is easy to show that P_j satisfy the properties $P_1 + P_2 = I_n$, $P_j^2 = P_j$ and $P_1P_2 = P_2P_1 = 0$.

(a) Show that any matrix P satisfying the relation $P^2 = P$ is a projector onto some subspace L parallel to M, and identify these L and M.

(b) Show that the projector P is an orthogonal projector if and only if the matrix P is symmetric.

(c) Assume that two $n \times n$ matrices satisfy the following conditions: $P_1 + P_2 = I_n$ and $P_1P_2 = 0$. Prove that P_1 and P_2 are projectors and that $P_2P_1 = 0$.

Problem 2 (Projectors; 5pt). (a) Find a matrix of oblique projector in $\mathbb{R}^3$ onto the subspace $U = \mathrm{ls}\{(1,0,1)^\top\}$ parallel to the subspace $W = \mathrm{ls}\{(1,1,0)^\top, (0,1,1)^\top\}$.

(b) Find a projection matrix of $\mathbb{R}^3$ onto the subspace $U = \mathrm{ls}\{(1,2,1)^\top, (1,0,-1)^\top\}$ parallel to the subspace $W = \mathrm{ls}\{(1,0,1)^\top\}$.

(c) Is it possible to fill in the missing entries in the matrix

$$A = \begin{pmatrix} 1 & * & 0 \\ 0 & \frac{1}{2} & * \\ * & * & * \end{pmatrix}$$

to get a matrix of an orthogonal projection in $\mathbb{R}^3$? If so, find the subspace U of $\mathbb{R}^3$ such that A is an orthogonal projection onto U.

Hint: Do you see why only one of (a) or (b) needs to be worked out in detail?

Problem 3 (Eigenvalues; 3 pt). Find all the eigenvalues of the following matrices by **inspection**:

(a) $\begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, (e) $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Hint: Look for constant row/column sums, diagonal entries with zeros in the corresponding row or column otherwise, use eigenvalue sum/product rules, try subtracting λI for "tempting" candidates for λ etc

Problem 4 (Eigenvalues and eigenvectors; 4 pt). For the matrix A in each part below, find the eigenvalues and eigenvectors of A, A^2, A^{100}, A^{-1} and e^{tA}:

(a) $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$, (b) $\begin{pmatrix} 4 & 0 & -1 \\ 0 & -1 & 4 \\ 0 & 2 & 1 \end{pmatrix}$

Problem 5 (Difference equation; 4 pt). A *generalized Fibonacci* sequence g_n is defined through the relations $g_0 = 0$, $g_1 = 1$, $g_2 = 2$ and $g_{n+3} = 3g_{n+2} - g_{n+1} - g_n$ for $n \in \mathbb{Z}_+$. Find the formula for the n^{th} generalized Fibonacci number g_n. To this end,

(a) represent the difference equation in the form $\mathbf{x}_{n+1} = A\mathbf{x}_n$ for a suitable 3×3 matrix A, find its eigenvalues and eigenvectors and then diagonalize A;

(b) find a general solution $\mathbf{x}_n$ and then the one satisfying the initial condition $\mathbf{x}_0 = (0, 1, 2)^\top$.

Problem 6 (Jordan form; 4 pt). For each of the following matrices A, find P so that $P^{-1}AP$ is in the Jordan form (ie, either diagonal or a Jordan block), and write this Jordan form:

$$\text{(a)} \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}, \qquad \text{(b)} \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}, \qquad \text{(c)} \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}, \qquad \text{(d)} \begin{pmatrix} -5 & 2 \\ -\frac{1}{2} & -3 \end{pmatrix}.$$

Hint: You do not have to calculate $P^{-1}AP$; you can just write it out!

Problem 7 (Eigenvalues and eigenvectors; 2pt). What are the eigenvalues and eigenvectors of the linear transformation A defined on $\mathbb{C}^3$ via $A(x_1, x_2, x_3) = (x_3, x_1, x_2)$?

Problem 8 (Eigenvalues and eigenvectors; 2pt). Let $\mathbf{a}$ be a non-zero vector in $\mathbb{R}^3$ and A a linear transformation of $\mathbb{R}^3$ given by $A\mathbf{x} = \mathbf{x} \times a$, where "$\times$" denotes the cross-product (i.e., vector product). Find the eigenvalues and eigenvectors of A.

Hint: do not use coordinates; use the geometric meaning of the cross product instead

Problem 9 (Eigenvalues and eigenvectors; 4pt). (a) Assume that $\mathbf{u}$ and $\mathbf{v}$ are two non-zero (column) vectors of $\mathbb{R}^n$. Find eigenvalues and eigenvectors of the matrix $A := \mathbf{u}\mathbf{v}^\top$.

(b) Assume that a and b are two non-equal numbers. Find eigenvalues and eigenvectors of the $n \times n$ matrix

$$\begin{pmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{pmatrix}$$

Hint: think geometrically! What does the matrix A do? In (b), what happens if $a = b$?

Problem 10 (Commuting matrices; 4pt). (a) Assume that $n \times n$ matrices A and B commute, i.e., $AB = BA$. Show that any eigenvector of A corresponding to a nonzero eigenvalue is also an eigenvector of B.

(b) Assume that A commutes with B and B commutes with C. Is it true that A and C must commute?

Problem 11 (Symmetric matrices; 5pt). Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$:

$$(a) \quad \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}; \qquad (b) \quad \begin{pmatrix} -2 & 0 & -36 \\ 0 & 3 & 0 \\ -36 & 0 & -23 \end{pmatrix}; \qquad (c) \quad \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

Problem 12 (Symmetric matrices; 4pt). (a) Find all values of a and b for which there exists a 3×3 symmetric matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 7$ and the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

(b) Reconsider the problem if $\lambda_3 = 3$.

(c) Find all symmetric matrices A satisfying the conditions of parts (a) and (b).

Problem 13 (Symmetric matrices; 4pt). A 3×3 symmetric matrix A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = \lambda$ and the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

(a) Find all possible values of a, b, and λ.

(b) Find the corresponding matrices A.

Problem 14 (Quadratic forms; 4 pt). Find an orthogonal change of variables that eliminates the cross product terms in the quadratic form Q, and express Q in terms of the new variables:

(a) $Q(x_1, x_2) = 2x_1^2 + 2x_2^2 - 2x_1x_2$;

(b) $Q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$

Problem 15 (Quadratic forms; 2 pt). Find all values of k for which the quadratic form

$$5x_1^2 + x_2^2 + kx_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$$

is positive definite.

Problem 16 (Quadratic forms; 4 pt). Consider the matrix A and the vector $\mathbf{v}_1$, where

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Show that $\mathbf{v}_1$ is an eigenvector of A, and find its corresponding eigenvalue. Then find the other two eigenvalues and eigenvectors and orthogonally diagonalize A.

(b) Is the quadratic form $Q(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$ positive definite, negative definite or indefinite? Find the principal axes of Q and the corresponding transition matrix.