1 a) $\begin{pmatrix} 1 & k & 4 \\ 3 & 6 & 8 \end{pmatrix} \sim \begin{pmatrix} 3 & 6 & 8 \\ 0 & 3k-6 & 4 \end{pmatrix}$

$3k - 6 \neq 0$

if $k \neq 2 \Rightarrow$ ~~[illegible]~~ one solution.

3 pts

b) $\begin{pmatrix} 1 & 4 & -2 \\ 3 & k & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -2 \\ 0 & k-12 & 0 \end{pmatrix}$

if $\underset{k=12}{k-12=0}$ then $x_2 \in \mathbb{R} \rightarrow$ infinitely many sol.

if $k-12 \neq 0 \Rightarrow k \neq 12$ then $x_2 = 0$ one sol.

c) ~~[illegible]~~

$\begin{pmatrix} -4 & 12 & k \\ 2 & -6 & -3 \end{pmatrix} \sim \begin{pmatrix} -4 & 12 & k \\ -4 & 12 & 6 \end{pmatrix}$

$k = 6 \rightarrow$ inf. many sol.

$k \neq 6 \rightarrow 0$ solutions

2 $\begin{pmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{pmatrix} \sim \begin{pmatrix} a & 0 & b & 2 \\ 0 & a & 2 & b \\ 0 & 0 & 2-b & 2-b \end{pmatrix}$

4 pts

a) $b=2$, $a \neq 0$ $\begin{pmatrix} a & 0 & 2 & 2 \\ 0 & a & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} x_1 = \dfrac{2-2t}{a} \\ x_2 = \dfrac{2-2t}{a} \\ x_3 = t \end{cases}$

b) $b=2$, $a=0$ $\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\begin{cases} x_1 = s \\ x_2 = t \\ x_3 = 1 \end{cases}$

c) $b \neq 2$, $a \neq 0$ unique solution

d) $b \neq 2$, $a = 0$ no solutions

③ (I) $m=n=3$

a) no sol.

$$\begin{pmatrix} 0 & 0 & 3 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

b) ex. one sol.

$$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 5 & 8 \end{pmatrix}$$

c) inf. many sol.

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

5 pts

(II) $m=3 \quad n=2$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(III) $m=2 \quad n=3$

$$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 7 \end{pmatrix}$$

There is no such matrix because amount of equations should not be less than ~~be equal to~~ amount of variables.

$$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

④ a) $\begin{pmatrix} 1 & 4 & a \\ 2 & 1 & b \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & a \\ 0 & 7 & 2a-b \end{pmatrix}$

4 pts

- $2a-b=0 \Rightarrow x_2=0 \quad x_1=a \Rightarrow$ one sol.
- $2a-b \neq 0 \Rightarrow$ one sol.

b) $\begin{pmatrix} 1 & 4 & 3 & a \\ 2 & 1 & 0 & b \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 & a \\ 0 & 7 & 6 & 2a-b \end{pmatrix}$

infinitely many sol.

$x_3 = \dfrac{2a-b}{6} - \dfrac{7}{6}x_2 \qquad x_2$ – free variable

c) $\begin{pmatrix} 2 & 1 & a \\ 1 & 4 & b \\ 0 & 3 & c \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & a \\ 0 & 7 & 2b-a \\ 0 & 3 & c \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & a \\ 0 & 7 & 2b-a \\ 0 & 0 & 6b-3a-7c \end{pmatrix}$

rank of $A = 2$

rank of $\bar{A} = \begin{cases} 3 & \text{if } 6b-3a-7c \neq 0 \Rightarrow \text{no sol. because } \operatorname{rank}(A) < \operatorname{rank}(\bar{A}) \\ 2 & \text{if } 6b-3-7c=0 \Rightarrow \text{one sol. (trivial)} \end{cases}$

d) $\begin{pmatrix} 1 & 4 & 3 & a \\ 2 & 1 & 0 & b \\ 1 & 1 & 1 & c \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & c \\ 0 & 1 & 2 & 2c-b \\ 0 & 3 & 2 & a-c \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & c \\ 0 & 1 & 2 & 2c-b \\ 0 & 0 & 4 & 7c-3b-a \end{pmatrix}$

if $7c-3b-a \neq 0 \Rightarrow$ no free variables $\Rightarrow$ one sol.

if $7c-3b-a = 0 \Rightarrow x_3 = 0$

if $2c-b \neq 0 \Rightarrow$ no free var. $\Rightarrow$ one sol

if $2c-b = 0 \Rightarrow x_2 = 0$

if $c \neq 0 \Rightarrow$ no free var. $\Rightarrow$ one sol.

if $c = 0 \Rightarrow$ one trivial sol.

6 a) $\begin{pmatrix} 1 & 2 & 6 \\ 2 & -3 & k \\ -1 & 5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 6 \\ 0 & 7 & 7 \\ 0 & 7 & k+2 \end{pmatrix}$

4 pts

rows should be linearly dependent

so $k+2 = 7 \Rightarrow k = 5$

b) $\begin{pmatrix} -1 & 2 & k \\ 2 & -4 & 3 \\ 1 & -2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -3 \\ 0 & 0 & 9 \\ 0 & 0 & k-3 \end{pmatrix}$ No matter what k we will choose, $(0,0,9)$ and $(0,0,k-3)$ are linearly dependent.

7 a) Yes. This is a mathix with such a rule of elements placement:

If non zero element is in a (i,j) position then j raw should be filled with zeroes and i column should be filled with zeroes. Also all diagonal elements should be zeroes.

Example $\begin{pmatrix} 0 & d & c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b & a & 0 \end{pmatrix}$

4 pts

b) Yes. All elements should be equal to $\frac{1}{n}$.

c) Yes. All elements of side diagonal shoul be equal to 1 and all others to 0.

d) Yes. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{B}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This may be extended to nxn case by ~~0' rows and 0' columns~~ adding proper amount of rows and columns filled with zeroes.

4 pts

8) $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 2\cdot 3 - 1\cdot 1 + 1\cdot(-1) = 4 = V$

$\begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = i\cdot(-1) - j\cdot(1) + k\cdot(3) = -1i - 1j + 3k = x$

$|x| = \sqrt{1+1+9} = \sqrt{11} = S_{\boxed{x}}$

$\begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = i\cdot(1) + j\cdot(3) + k\cdot(1) = 1i + 3j + 1k = y$

$\|y\| = \sqrt{11} = S_{\boxed{y}}$

$\begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = i\cdot(3) + j\cdot(1) + k\cdot(-1) = 3i + 1j - 1k = z$

$|z| = \sqrt{11} = S_{\boxed{z}}$

1 pt

9) $A = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad B = \begin{pmatrix} a+b \\ b+c \\ c+a \end{pmatrix} \quad C = \begin{pmatrix} a-b \\ b-c \\ c-a \end{pmatrix}$

$\underline{\det B} = \det\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \det\begin{pmatrix} b \\ c \\ a \end{pmatrix} = \det A - \det\begin{pmatrix} b \\ a \\ c \end{pmatrix} = \det A + \det\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underline{10}$

$\underline{\det C} = \det\begin{pmatrix} a-b \\ b-c \\ c-a \end{pmatrix} = \det\begin{pmatrix} a-b \\ b-c \\ 0 \end{pmatrix} = \underline{0}$

sum of all rows

(10) $\det(A-\lambda I) = \det A - \det \lambda I = \cancel{0 - \lambda^4 = 0}$

~~Because rank(A) = 1~~

1 pt

$\det A = 0$ because its rows are ~~because~~ linearly dependent

$\det \lambda I = \lambda^4$ because its determinant of ~~...~~ diagonal matrix

$\det(A-\lambda I) = 0$ because it's singular

$0 = 0 - \lambda^4 \quad \lambda^4 = 0 \Rightarrow \boxed{\lambda = 0}$

(12) Trace of $(AB - BA)$ is 0.

2 pts

Trace of I_n is n.

They are not equal → there are no such matrices.

(13) a) $\begin{vmatrix} c & 1 & 3 \\ 1 & -1 & 4 \\ 1 & 2 & -1 \end{vmatrix} = c\cdot(-7) - 1\cdot(-5) + 3\cdot(3) = -7c + 14 \neq 0$

$c \neq 2$

3 pts

b) $\begin{vmatrix} c & 1 & -2 \\ 1 & -1 & 2 \\ 1 & 2 & -4 \end{vmatrix} = c \cdot \begin{vmatrix} -1 & 2 \\ 2 & -4 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & -4 \end{vmatrix} + (-2) \cdot \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 0c \neq 0$

$c \in \emptyset$

c) These vectors are linearly dependent because their ~~amount~~ count is bigger than count of base vectors for $\mathbb{R}^3$. $c \in \emptyset$

d) with combinations of these vectors it's impossible to create all given points from $\mathbb{R}^3$.

$c \in \emptyset$