Linear Algebra
Homework 4: Factorization and iteration

Problem 1 (Rank of a matrix; 2 pt). (a) Assume that A is an $m \times n$ matrix of rank r. Prove that A can be written as a sum of r summands $\mathbf{u}_j\mathbf{v}_j^\top$ of rank 1. Is such a representation unique?

(b) For the matrices below, find their ranks r and represent them as the sum of r rank-one summands.

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 4 & -6 & 8 \\ 3 & -6 & 9 & -12 \\ -4 & 8 & -12 & 16 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}.$$

Problem 2 (Singular values; 3 pt). (a) Prove that matrices A and $A^\top$ have the same non-zero singular values.

(b) Find singular values of the following matrices:

$$(i) \quad \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}; \qquad (ii) \quad \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}^\top; \qquad (iii) \quad \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}; \qquad (iv) \quad \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

Hint: Use (a) in (i) and (iii)

Problem 3 (Singular value decomposition; 4 pt). (a) If $A = U\Sigma V^\top$ is the SVD of A, what is the SVD of $A^\top$?

(b) Assume that $A = \mathbf{u}\mathbf{v}^\top$ is an $m \times n$ matrix of rank 1. Find the SVD of A.

(c) Find SVD of the following matrices:

$$(i) \quad \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}; \qquad (ii) \quad \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}^\top; \qquad (iii) \quad \begin{pmatrix} 2 & 1 & -2 \\ -2 & -1 & 2 \end{pmatrix}.$$

Problem 4 (Singular value decomposition, 4 pt). Find the singular value decomposition of the matrix

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

To this end, complete the following steps:

(a) How many singular values σ_j does A have? How many of them are non-zero? Find the nonzero singular values $\sigma_1, \dots, \sigma_r$.

(b) find the right singular vectors $A\mathbf{v}_j = \sigma_j \mathbf{u}_j$, $j = 1, \dots, r$;

(c) find the left singular vectors $A^\top \mathbf{u}_j = \sigma_j \mathbf{v}_j$, $j = 1, \dots, r$;

(d) form the unitary matrices U and V and write the singular value decomposition of A.

(Hint: you may find it easier to work with $A^\top$)

Problem 5 (Low-rank approximation; 2 pt). (a) For the matrix A in Problem 4, find a unit vector $\mathbf{x} \in \mathbb{R}^3$ for which $A\mathbf{x}$ has the maximal length α. What is α?

(b) Find the best rank one approximation for the matrix A of Problem 4 in the Frobenius norm.

Problem 6 (Low-rank approximation; 4 pt). Prove that the best rank k approximation A_k of an $m \times n$ matrix A in the Frobenius norm is given by the first k terms in the SVD of A, i.e.,

$$A_k = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^\top + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^\top.$$

Problem 7 (SVD and image compression; 5 pt). Take a jpg-picture A of yourselves of reasonable size (say 1000×1000 pixels), perform the SVD (use `Python` or `R` or any other program of your choice) and find the best rank-k approximation of A with $k = 1, 2, 5, 10, 20, 50$. For what k can one recognize the picture? Comment on how much the quality and the size increase along with k.

(Hint: one can use the Frobenius distance to the original picture as a quality measure)

Problem 8 (Polar decomposition; 3 pt). Find the positive definite square root $S = V\Sigma V^\top$ of $A^\top A$ and its polar decomposition $A = QS$:

$$A = \frac{1}{\sqrt{10}} \begin{pmatrix} 10 & 6 \\ 0 & 8 \end{pmatrix}$$

Problem 9 (Pseudoinverses and shortest solutions; 5 pt). (a) Find the SVD and the pseudoinverse $V\Sigma^+U^\top$ of the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

(b) Find the minimum-length solution $\mathbf{x}^+ = A^+\mathbf{b}$ of the equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

Problem 10 (Principal component analysis; 3 pt). (a) Convert the matrix of observation A to the zero-mean form and then construct the sample covariance matrix:

$$A = \begin{pmatrix} 19 & 22 & 6 & 3 & 2 & 20 \\ 12 & 6 & 9 & 15 & 13 & 5 \end{pmatrix}$$

(b) Find the principal components of the data in matrix A.

(c) Let x_1, x_2 denote the variables for the two-dimensional data in (a). Find the new variable $y_1 = a_1x_1 + a_2x_2$ such that y_1 has maximum possible variance over the given data. How much of the variance in the data is explained by y_1?

Problem 11 (PCA; 5 pt). (a) Simulate $N = 100$ data (x_k, y_k) from the two-dimensional Gaussian (normal) distribution $\mathcal{N}(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, \rho)$ with $\mu_1 = 1$, $\mu_2 = 2$, $\sigma_1 = 4$, $\sigma_2 = 9$, $\rho = \frac{1}{3}$.

Hint: Can you do this easily if $\rho = 0$? If $(Z_1, Z_2)^\top \sim \mathcal{N}(0, 0; 1, 1, 0)$, show that $(X, Y)^\top$ with $X = \mu_1 + \sigma_1 Z_1$ and $Y = \mu_2 + \sigma_2 \rho Z_1 + \sqrt{1 - \rho^2}\sigma_2 Z_2$ has the required distribution

(b) Form the empirical covariance matrix C for the data simulated and find its eigenvalues and eigenvectors.

(c) Perform the PCA on the data generated. Calculate the variance along the first component; what fraction of the total variance does it include?

(d) Predict how the above fraction depends on ρ and confirm your reasoning numerically.

Problem 12 (PCA in many dimensions; 8 pt)**.** This is a 10-dimensional analogue of Problem 11.

(a) Simulate $N = 100$ data $\mathbf{x}_j = (x_1^{(j)}, x_2^{(j)}, \dots, x_{10}^{(j)})^\top$ from the Gaussian (normal) distribution $\mathcal{N}(\mathbf{0}, \Sigma)$ with $\Sigma = I + \mathbf{u}\mathbf{u}^\top$, where $\mathbf{u} = (1; -2; 3; \dots; -10)^\top$

Hint: if you factorize $\Sigma = LL^\top$ with a lower-triangular L (Cholesky factorization), simulate the standard Gaussian vectors $\mathbf{z}_j = (z_1^{(j)}, z_2^{(j)}, \dots, z_{10}^{(j)})^\top$ (ie, with independent components of variance 1), then $\mathbf{x} = L\mathbf{z} + \boldsymbol{\mu}$ will follow $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$. Justify before using that!

(b) Form the empirical covariance matrix C for the data simulated and find its eigenvalues and eigenvectors.

(c) Perform the PCA on the data generated. Calculate the variance along the several first component; what fraction of the total variance does it include?

(d) Predict how the above fraction depends on the shape of the set $\{\mathbf{x} \mid \mathbf{x}^\top \Sigma \mathbf{x} = 1\}$ and confirm your reasoning numerically.

Hint: certainly you will need to use Python or MatLab libraries to solve this problem!

Problem 13 (Jacobi and Gauss–Seidel iteration scheme; 6 pt)**.** Use the Jacobi and Gauss–Seidel methods to solve the 3×3 system $A\mathbf{x} = \mathbf{b}$ with

$$A = \alpha I_3 + \mathbf{u}\mathbf{u}^\top, \qquad \mathbf{b} = \mathbf{u},$$

where $\mathbf{u} = (1, -1, 1)^\top$.

(a) For what α do the methods work?

(b) Write the corresponding iteration scheme for the Jacobi method and find the solution starting with $\mathbf{x}^{(1)} = \mathbf{0}$. How many iteration are required to achieve 0.001 accuracy?

(c) Write the corresponding iteration scheme for the Gauss–Seidel method and find the solution starting with $\mathbf{x}^{(1)} = \mathbf{0}$. How many iteration are required to achieve 0.001 accuracy?

Problem 14 (Conjugate gradient method; 6 pt)**.** Consider the quadratic function

$$f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^\top G\mathbf{x} - \mathbf{b}^\top \mathbf{x}$$

in four variables $\mathbf{x} = (x_1, x_2, x_3, x_4)$, where

$$G = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

and $\mathbf{b} = (1, 0, 2, \sqrt{5})^\top$. Apply the conjugate gradient method to this problem with $\mathbf{x}_0 = (0, 0, 0, 0)^\top$ and show that it converges in two iterations to the exact solution of $G\mathbf{x} = \mathbf{b}$.