4 a) SVD theorem:

Every $m \times n$ A with $\operatorname{rank}(A) = r$ can be rewritten as $A = \underline{U \Sigma V^T}$ where

$U = (u_1 \dots u_r \mid u_{r+1} \dots u_m)$

$V = (v_1 \dots v_r \mid v_{r+1} \dots v_m)$

① $\Sigma = \sigma$s (singular values) on main diagonal and 0 elsewhere

② v_j are evcs of $A^T A$ with EVs σ_j^2 $A^T A v_j = \sigma_j^2 v_j$

③ $u_j = A v_j / \|A v_j\| = A v_j / \sigma_j$ for $j = 1, \dots, r$ is an ONB orthonormal basis for the range of A

④ $u_1 \dots u_m$ is an ONB for $\mathbb{R}^m$

⑤ $A = \sigma_1 u_1 v_1^T + \dots + \sigma_r u_r v_r^T$

$u_1 \dots u_r$ – left singular vectors of A

$v_1 \dots v_r$ – right singular vectors of A

$\cancel{A v_j = \sigma_j u_j}$ $\cancel{A^T u_j = \sigma v_j}$ $\cancel{A A^T u_j = \sigma_j^2 u_j}$

So if we replace $\sigma_i u_i v_j^T$ with pq^T

$A = p_1 q_1^T + p_2 q_2^T + \dots$ we will have matrix written as a sum of ~~sum~~ r summands pq^T of rank 1

~~1) proof~~ why rank equals 1?

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \dots \\ u_n \end{pmatrix} (v_1, v_2, v_3, \dots, v_n) = \begin{pmatrix} u_1 \cdot v \\ u_2 \cdot v \\ u_3 \cdot v \\ \dots \\ u_n \cdot v \end{pmatrix}$$

$\Rightarrow$ All rows are linearly dependent therefore rank equals 1.

Uniqueness – ~~if~~ any $A = p \cdot q$ can be written using $a = \frac{p}{x}$ and $b = q \cdot x$ where $x \in \mathbb{R} \setminus \{0\}$. So, representation is not unique.

b) rank(A) = 1 because all rows are vector $(1,-2,3,-4)$ multiplied by $1, -2, 3$ and -4 correspondingly.

$$A = (1,-2,3,-4)^T \cdot (1,-2,3,-4)$$

rank(B) = 2 because all rows are linear combinations of two linearly independent row-vectors $(1,2,3,4)$ and $(1,1,1,1)$.

$\langle (1,2,3,4) \cdot (1,1,1,1) \rangle = 10 \neq 0 \Rightarrow$ linearly indep.

$$B = (1,2,3,4)^T \cdot (1,1,1,1) + (0,1,2,3)^T \cdot (1,1,1,1)$$

2 a) $AA^Tx = \lambda x$

$$A^TAA^Tx = A^T\lambda x = \lambda A^Tx \Rightarrow A^TA(A^Tx) = \lambda(A^Tx) \Rightarrow$$

A^TA has the same evs, but different eves. If x is evc for AA^T corresponding to ev λ then A^Tx is eigenvector for A^TA, standing for the same eigenvalue λ.

b) I $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$

Number of non 0 eigenvalues is equal to the rank of a matrix. So, in our case two eigenvalues are zero and third one is 5 (by the trace rule).

$\sigma_1 = \sqrt{5} \quad \sigma_2 = 0 \quad \sigma_3 = 0$

II The same as in I by proof in a) because ~~to 1 2~~ II matrix is transposed ~~of~~ I matrix.

III
$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 5 \\ 5 & 13 & 13 \\ 5 & 13 & 13 \end{pmatrix}$$

$\lambda_1 = 0$ by inspection

$\lambda_2 = 14 - \sqrt{194}$

$\lambda_3 = 14 + \sqrt{194}$ by Wolfram

$\sigma_1 = \sqrt{14 + \sqrt{194}}$ $\sigma_2 = \sqrt{14 - \sqrt{194}}$ $\sigma_3 = 0$

3 a) $A = U \Sigma V^T$

$A^T = (U \Sigma V^T)^T = (\Sigma V^T)^T \cdot U^T = V \cdot \Sigma^T \cdot U^T$

b) $A = uv^T$

Let $p = \frac{u}{\|u\|}$ $q = \frac{v}{\|v\|}$ $\sigma = \|u\| \cdot \|v\|$

A is of rank 1 and therefore has only one non-zero eigenvalue which is equal to $v^T u$ (by the trace rule).

$\lambda = v^T u = (\|u\| \cdot \|v\|)^2 \Rightarrow \sigma = \|u\| \cdot \|v\|$ is a singular value of A.

$A = \sigma \cdot p \cdot q^T$ is the SVD (by the SVD theorem).

Also we can write the SVD for A in a matrix form. For this we need to create an orthogonal matrix based on p (using Gramm - Shmidt or any other orthogonalization algorithm). Let's call it U. On the other hand, we can do the same for q (matrix name will be V). And finally $\Sigma = \begin{pmatrix} \sigma & \\ & 0 \end{pmatrix}$.

So $A = U \cdot \Sigma V^T$ will be the SVD for an A.

② b) IV $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}$

$\lambda_1 = \frac{1}{2}(9+\sqrt{17})$ $\quad \lambda_2 = \frac{1}{2}(9-\sqrt{17})$ by wolfram

$\sigma_1 = \sqrt{\frac{1}{2}(9+\sqrt{17})}$ $\quad \sigma_2 = \sqrt{\frac{1}{2}(9-\sqrt{17})}$

③ c) I $A = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}$

$$A = \begin{pmatrix} 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} = \underset{\sigma}{\sqrt{5}} \cdot \underset{u}{\begin{pmatrix} 1 \end{pmatrix}} \cdot \underset{v^T}{\begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}}$$

$$A = \begin{pmatrix} 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{5} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & 2/\sqrt{5} \\ \frac{2}{\sqrt{5}} & 0 & \frac{-1}{\sqrt{5}} \end{pmatrix}^T$$

II Matrix is transposed previous one. So,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & 2/\sqrt{5} \\ \frac{2}{\sqrt{5}} & 0 & \frac{-1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \sqrt{5} \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \end{pmatrix}^T$$

III

$$A = \begin{pmatrix} 2 & 1 & -2 \\ -2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T \cdot \begin{pmatrix} 2 & 1 & -2 \end{pmatrix} = \underset{\sigma}{3\sqrt{2}} \cdot \underset{u}{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}^T} \underset{v^T}{\begin{pmatrix} 2/3 & 1/3 & -2/ \end{pmatrix}}$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 3\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{1}{3} & 0 & \frac{-4}{\sqrt{18}} \\ \frac{-2}{3} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} \end{pmatrix}^T$$

5 a) The largest value of $\|Ax\|$ with $\|x\| \le 1$ is obtained for $x = v_1$ (a first vector from right singular vectors).

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \end{pmatrix}$$

$\|A \cdot v_1\| = \sigma_1 = \alpha$

b) $$\sigma_1 \cdot u_1 \cdot v_1^T = 5 \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{25}{2} & \frac{25}{2} & 0 \\ \frac{25}{2} & \frac{25}{2} & 0 \end{pmatrix}$$

8 $$\frac{1}{10} \begin{pmatrix} 10 & 0 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 10 & 6 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$$ $\lambda_1 = 16$, $\lambda_2 = 4$, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$, $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$

$$A^T A = PDP^{-1} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 16 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$S = \sqrt{A^T A} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

$$Q = A \cdot S^{-1} = \begin{pmatrix} 10 & 6 \\ 0 & 8 \end{pmatrix} \cdot \frac{1}{\sqrt{10}} \cdot \begin{pmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{pmatrix} = \begin{pmatrix} 24 & 8 \\ -8 & 24 \end{pmatrix} \cdot \frac{1}{8\sqrt{10}}$$

$$A = \begin{pmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

(9) b)

$$AA^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

$\lambda_1 = 4 \quad u_1 = (1,1,2)^T \cdot \frac{1}{\sqrt{6}}$

$\lambda_2 = 1 \quad u_2 = (-1,-1,1) \cdot \frac{1}{\sqrt{3}}$

$\lambda_3 = 0 \quad u_3 = (-1,1,0) \cdot \frac{1}{\sqrt{2}}$

$\sigma_1 = 2 \quad \sigma_2 = 1 \quad \sigma_3 = 0$

$$A^T A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$v_1 = (2,1,1)^T \cdot \frac{1}{\sqrt{6}}$

$v_2 = (-1,1,1)^T \cdot \frac{1}{\sqrt{3}}$

$v_3 = (0,-1,1)^T \cdot \frac{1}{\sqrt{2}}$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$

$$A^+ = V \cdot \Sigma^+ U^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$x^+ = A^+ b = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

a) $A = (1,1,1,1)$

$$(1 \ 1 \ 1 \ 1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (4) \qquad \lambda_1 = 4 \quad \sigma_1 = 2 \quad \sigma_{234} = 0$$

$((4) - \lambda I) u_1 = 0 \Rightarrow u_1 = 1$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$v_1 = (1,1,1,1)^T \cdot \frac{1}{2}$

$v_2 = (1,0,-1,0) \cdot \frac{1}{\sqrt{2}}$

$v_3 = (0,1,0,-1) \cdot \frac{1}{\sqrt{2}}$

$v_4 = (1,-1,1,-1) \cdot \frac{1}{2}$

$$A = (1) \cdot (2 \ 0 \ 0 \ 0) \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{pmatrix}}_{V}$$

$$A^+ = V \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} (1)^T = \left(\frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \right)$$

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$BB^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \lambda_{1,2} = 1 \quad u_1 = (1,0)^T$$
$$\sigma_{1,2} = 1 \quad u_2 = (0,1)^T$$

$$B^T B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{aligned} v_1 &= (0,1,0)^T \\ v_2 &= (1,0,0)^T \\ v_3 &= (0,0,1)^T \end{aligned}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^+ = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$C \cdot C^T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad \lambda_1 = 2 \quad \lambda_2 = 0 \quad u_1 = (1,0)^T$$
$$\sigma_1 = \sqrt{2} \quad \sigma_2 = 0 \quad u_2 = (0,1)^T$$

$$C^T C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad v_1 = (1,1)^T \cdot \frac{1}{\sqrt{2}} \quad v_2 = (-1,1)^T \cdot \frac{1}{\sqrt{2}}$$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$C^+ = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}$$

4 $\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} = A$

a) $\operatorname{rank}(A) = 2 \Rightarrow A$ has 2 positive singular values and 3 overall.

A^TA

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{pmatrix} \quad \begin{matrix} \lambda_1 = 25 \\ \lambda_2 = 9 \\ \lambda_3 = 0 \end{matrix}$$ by wolfram

$\sigma_1 = 5 \quad \sigma_2 = 3 \quad \sigma_3 = 0$

b) $$\begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{pmatrix}\begin{pmatrix} v_2 \end{pmatrix} = \lambda_2 v_2$$

$$\begin{pmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{pmatrix} v_1 = 0$$

$$\begin{pmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} \frac{1}{\sqrt{18}} \\ \frac{-1}{\sqrt{18}} \\ \frac{4}{\sqrt{18}} \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$$

$$v_3 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \cdot \frac{1}{3}$$

c) $AA^T u_j = \sigma_j^2 u_j$

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix} \quad \begin{matrix} \lambda_1 = 25 \\ \lambda_2 = 9 \end{matrix}$$

$$\begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix} u_1 = 0 \qquad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \left(\frac{1}{\sqrt{2}}\right)$$

$$\begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} u_2 = 0 \qquad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \left(\frac{1}{\sqrt{2}}\right)$$

d) $$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{1}{3} \end{pmatrix}^T$$

$B = |\vec{x_1} - \vec{\mu_1}| \cdots | \vec{x_n} - \vec{\mu_n}|$

$S = \frac{1}{n-1} BB^T \qquad T = tr(S)$

10) $A = \begin{pmatrix} 19 & 22 & 6 & 3 & 2 & 20 \\ 12 & 6 & 9 & 15 & 13 & 5 \end{pmatrix} \quad \begin{matrix} 12 \\ 10 \end{matrix}$

$B = \begin{pmatrix} 7 & 10 & -6 & -9 & -10 & 8 \\ 2 & -4 & -1 & 5 & 3 & -5 \end{pmatrix}$

$S_{11} = \frac{1}{5}(49 + 100 + 36 + 81 + 100 + 64) = \frac{430}{5} = 86$

$S_{21} = \frac{1}{5}(14 - 40 + 6 - 45 - 30 - 40) = -27$

$S_{22} = \frac{1}{5}(4 + 16 + 1 + 25 + 9 + 25) = 16$

$\begin{pmatrix} 86 & -27 \\ -27 & 16 \end{pmatrix} \qquad \lambda_1 = 95.2 \qquad v_1 = (-2.93, 1)$

$\lambda_2 = 6.8 \qquad v_2 = (0.34, 1)$

$\overleftarrow{v_1} = (-0.95, 0.32)$ ← principal

$\overleftarrow{v_2} = (0.32, 0.95)$ components

$y_1 = (a_1 x_1 + a_2 x_2) = v_1 \cdot A$

How much is explained = ~~$\frac{86}{86+16} = 0.84$~~

$\frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{95.2}{95.2 + 6.8} = 0.93$

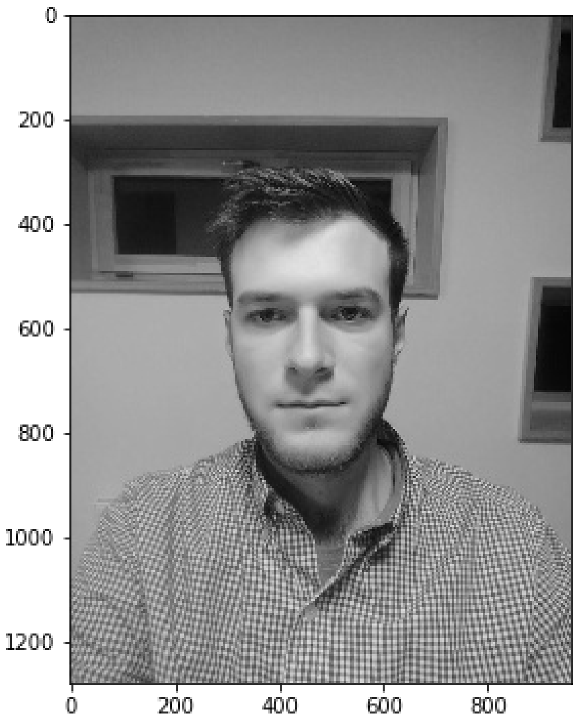
In [44]:

```
%matplotlib inline
import matplotlib.pyplot as plt
import numpy as np
import time
from numpy import linalg as LA

from PIL import Image
```

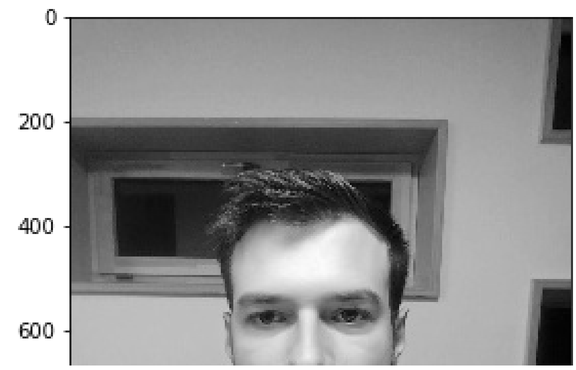
In [45]:

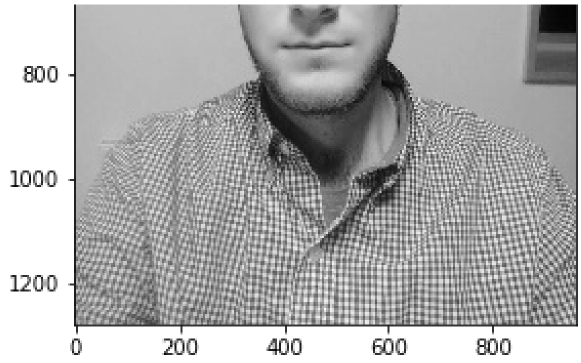
```
img = Image.open('myface.jpg')
imggray = img.convert('LA')
plt.figure(figsize=(9, 6))
plt.imshow(imggray);
```



In [46]:

```
imgmat = np.array(list(imggray.getdata(band=0)), float)
imgmat.shape = (imggray.size[1], imggray.size[0])
imgmat = np.matrix(imgmat)
plt.figure(figsize=(9,6))
plt.imshow(imgmat, cmap='gray');
```



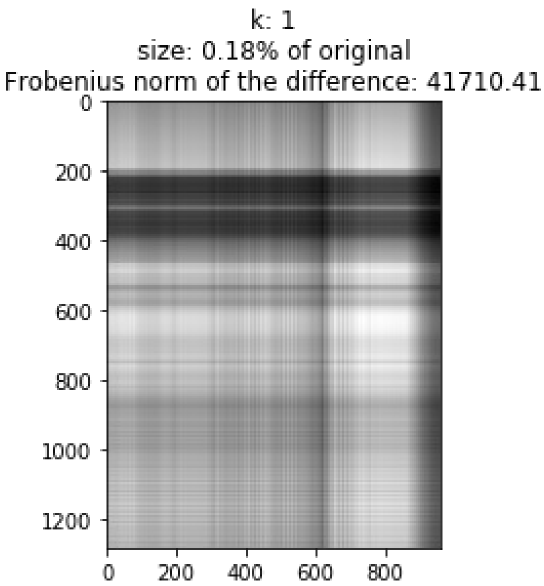


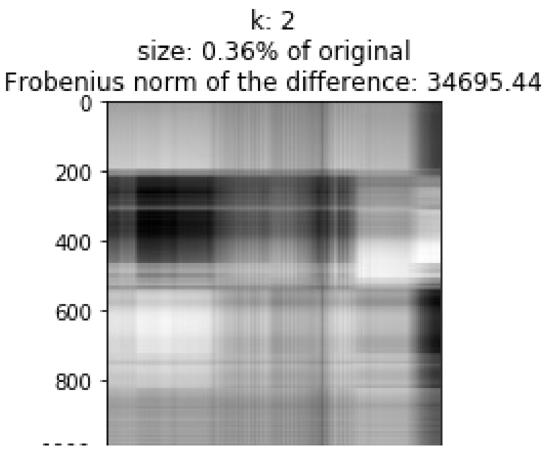
In [47]:

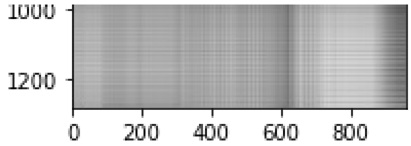
```
U, sigma, V = np.linalg.svd(imgmat)
```

In [48]:

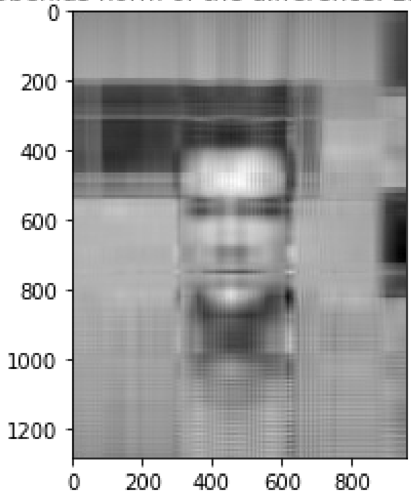
```
height, width = reconstimg.shape
for i in [1, 2, 5, 10, 50]:
    reconstimg = np.matrix(U[:, :i]) * np.diag(sigma[:i]) * np.matrix(V[:i,
:])
    diff = imgmat - reconstimg
    plt.imshow(reconstimg, cmap='gray')
    size = 100 * (i + i * height + i * width) / (height * width)
    title = "k: {}\nsize: {:.2f}% of original\nFrobenius norm of the differ
ence: {:.2f}".format(i, size, LA.norm(diff, 'fro'))
    plt.title(title)
    plt.show()
```



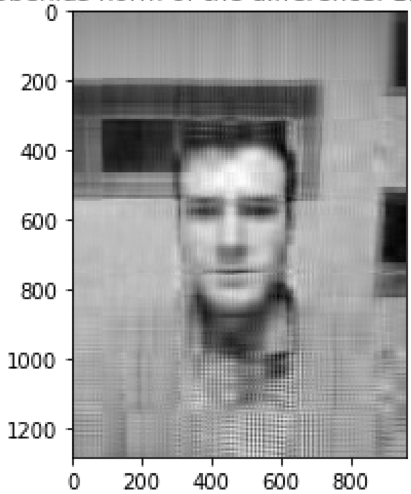




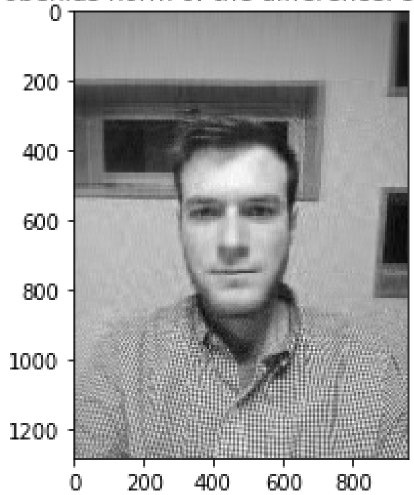
k: 5
size: 0.91% of original
Frobenius norm of the difference: 23883.70



k: 10
size: 1.82% of original
Frobenius norm of the difference: 19460.34



k: 50
size: 9.12% of original
Frobenius norm of the difference: 9659.04



- Bullets are left so that an original html file would not be cropped in the middle of the text below.
  With bigger k size increases, but not linearly. As for me, the image for k = 50 is not as good as original one for a human eye, but size of the reduced image is incredibly small. Also, with k = 5 I can recognize that this is a selfie of a person, while with k = 10 I can recognize myself on the photo.

R Notebook

11

```r
library("MASS")
set.seed(1)
```

```r
N <- 100
m <- c(1, 2)
sigma1 <- 4
sigma2 <- 9
rho <- 1 / 3

sigmas <- matrix(c(sigma1 * sigma1, sigma1 * sigma2 * rho,
                   sigma1 * sigma2 * rho, sigma2 * sigma2),
                 2)
data <- mvrnorm(N, mu = m, Sigma = sigmas)

cov_mat <- cov(data)
cat("Covariance matrix: ", cov_mat, "\n")
```

```
## Covariance matrix:  14.40551 9.442583 9.442583 65.38518
```

```r
ev <- eigen(cov_mat)
cat("Eigenvalues: ", ev$values, "\n")
```

```
## Eigenvalues:  67.07795 12.71274
```

```r
cat("Eigenvectors: ", ev$vectors, "\n")
```

```
## Eigenvectors:  0.1764569 0.9843084 -0.9843084 0.1764569
```

```r
cat("Variance along the first principal component: ", ev$values[1], "\n")
```

```
## Variance along the first principal component:  67.07795
```

```r
cat("Fraction of the total variance included: ", ev$values[1] / sum(ev$values), "\n")
```

```
## Fraction of the total variance included:  0.8406739
```

12

```r
u <- t(matrix(c(1, -2, 3, -4, 5, -6, 7, -8, 9, -10), 1, 10))
i <- diag(10)
sigma <- i + u %*% t(u)
mu <- rep(0, 10)
data <- mvrnorm(N, mu = mu, Sigma = sigma)

cov_mat <- cov(data)

ev <- eigen(cov_mat)
print("Eigenvalues: ")
```

```
## [1] "Eigenvalues: "
```

```r
ev$values
```

```
## [1] 412.9874133   1.5950829   1.5517687   1.3483115   1.1678590
## [6]   1.0312633   0.8904705   0.8654795   0.8263157   0.6295435
```

```r
print("Eigenvectors: ")
```

```
## [1] "Eigenvectors: "
```

```r
ev$vectors
```

```
##              [,1]        [,2]         [,3]        [,4]        [,5]
##  [1,]  0.04779379 -0.43768504  0.209992291  0.21860893  0.74984273
##  [2,] -0.10606308 -0.27449033 -0.358007873  0.09074627  0.05181748
##  [3,]  0.14575882 -0.03652334  0.072197479 -0.11753884  0.30987909
##  [4,] -0.20537391 -0.08573843 -0.680357963 -0.26923057  0.16354565
##  [5,]  0.24992769  0.12779980  0.445193524 -0.39695624  0.08863926
##  [6,] -0.30827696 -0.40632557  0.148776163 -0.09115236  0.02826133
##  [7,]  0.36359979 -0.49208782  0.003889823  0.49605849 -0.42260239
##  [8,] -0.40263983 -0.28882595  0.130667584 -0.26095799 -0.10743826
##  [9,]  0.46192956  0.20906353 -0.328505149  0.11515653  0.29598541
## [10,] -0.50778068  0.41676153  0.119573266  0.60362044  0.16087218
##              [,6]        [,7]        [,8]        [,9]       [,10]
##  [1,]  0.20323603 -0.20205896  0.14690253  0.16474825  0.14540211
##  [2,] -0.02848595 -0.35254130  0.05595251 -0.70535598 -0.38543682
##  [3,]  0.20248567  0.74636438 -0.30882693 -0.27598710 -0.30457236
##  [4,] -0.01797034  0.28571216  0.46416817  0.29975323 -0.03305722
##  [5,] -0.27822591 -0.06291959  0.55639167 -0.04960370 -0.40518411
##  [6,] -0.75165430  0.21620870 -0.12799826 -0.11119309  0.26104635
##  [7,] -0.03903602  0.23675317  0.22083177  0.19732941 -0.23596907
##  [8,]  0.12528511 -0.18888469 -0.34766168  0.42756168 -0.55017566
##  [9,] -0.48225213 -0.17845201 -0.37352299  0.26521045 -0.24466560
## [10,] -0.15613827  0.14804020  0.17132664  0.04900182 -0.29306023
```

```r
cat("Variance along the first principal component: ", ev$values[1], "\n")
```

```
## Variance along the first principal component:  412.9874
```

```r
cat("Fraction of the total variance included: ", ev$values[1] / sum(ev$values), "\n")
```

```
## Fraction of the total variance included:  0.9765754
```

```r
cat("Variance along the second principal component: ", ev$values[2], "\n")
```

```
## Variance along the second principal component:  1.595083
```

```r
cat("Fraction of the total variance included: ", ev$values[2] / sum(ev$values), "\n")
```

```
## Fraction of the total variance included:  0.003771831
```

```r
cat("Variance along the third principal component: ", ev$values[3], "\n")
```

```
## Variance along the third principal component:  1.551769
```

```r
cat("Fraction of the total variance included: ", ev$values[3] / sum(ev$values), "\n")
```

```
## Fraction of the total variance included:  0.003669408
```