5 $g_0 = 0 \quad g_1 = 1 \quad g_2 = 2 \qquad g_{n+3} = 3g_{n+2} - g_{n+1} - g_n \quad n \in \mathbb{Z}_+$

a) $x_{n+1} = Ax$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$\underline{\lambda_1} = 1$ by sum of rows

$tr = 3 \quad det = -1 \Rightarrow \underline{\lambda_2} = \sqrt{2} + 1 \quad \underline{\lambda_3} = -\sqrt{2} + 1$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -\sqrt{2}-1 & 1 & 0 \\ 0 & -\sqrt{2}-1 & 1 \\ -1 & -1 & 2-\sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \cancel{\begin{pmatrix} -1 & -1 & 2-\sqrt{2} \\ 0 & -\sqrt{2}-1 & 1 \\ 0 & \sqrt{2}+2 & -\sqrt{2} \end{pmatrix}} \sim \cancel{\begin{pmatrix} -1 & -1 & 2-\sqrt{2} \\ 0 & -\sqrt{2}-1 & 1 \\ 0 & \sqrt{2}-1 & 0 \end{pmatrix}}$$

$$\begin{cases} x_2 = (\sqrt{2}+1)x_1 \\ x_3 = (\sqrt{2}+1)x_2 = (3+2\sqrt{2})x_1 \\ x_1 + x_2 + (\sqrt{2}-2)x_3 = 0 \end{cases}$$

$3\sqrt{2} + 4 - 6 - 4\sqrt{2}$

$x_1 + (3+2\sqrt{2})(\sqrt{2}-2)x_1 + (\sqrt{2}+1)x_1 = 0$

$(1 + 1 + \sqrt{2} - 2 - \sqrt{2})x_1 = 0 \Rightarrow 0x_1 = 0 \qquad x_1 = 1$

$x_2 = \sqrt{2} + 1$

$x_3 = 3 + 2\sqrt{2}$

$$v_2 = \begin{pmatrix} 1 \\ \sqrt{2}+1 \\ 3+2\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{2}-1 & 1 & 0 \\ 0 & \sqrt{2}-1 & 1 \\ -1 & -1 & \sqrt{2}+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_1 \\ x_3 \end{pmatrix} = 0 \quad \begin{cases} x_2 = (1-\sqrt{2})x_1 \\ x_3 = (1-\sqrt{2})x_2 = (3-2\sqrt{2})x_1 \end{cases} \text{take } x_1 = 1$$

$$v_3 = \begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 3-2\sqrt{2} \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}+1 & 0 \\ 0 & 0 & -\sqrt{2}+1 \end{pmatrix}$$

b) $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sqrt{2}+1 & 1-\sqrt{2} \\ 1 & 3+2\sqrt{2} & 3-2\sqrt{2} \end{pmatrix}$

$A = PDP^{-1}$ $\qquad x^{n+1} = Ax^{n}$

$x_{n+1} = A^{n+1}x_0 \Rightarrow x_n = A^n x_0 = PD^nP^{-1}x_0$

$P^{-1} = \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1-\sqrt{2}}{4} & \frac{-2+\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1+\sqrt{2}}{4} & \frac{-2-\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix}$

$$x_n = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \sqrt{2}+1 & 1-\sqrt{2} \\ 1 & 3+2\sqrt{2} & 3-2\sqrt{2} \end{pmatrix} \begin{pmatrix} 1^n & 0 & 0 \\ 0 & (\sqrt{2}+1)^n & 0 \\ 0 & 0 & (-\sqrt{2}+1)^n \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1-\sqrt{2}}{4} & \frac{-2+\sqrt{2}}{4} & \frac{1}{4} \\ \frac{1+\sqrt{2}}{4} & \frac{-2-\sqrt{2}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

~~There is no reason to calculate first ~~row~~ column because in the end ~~they~~ it'll be multiplied by 0 from x_0. Also $\lambda_3^n \xrightarrow{n\to\infty} 0$ so 3rd column in D will be 0. => we need~~

③ a) $\begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$ → Markov matrix $\Rightarrow \lambda_1 = 1$

$tr(A) = \frac{1}{2} = \cdot \quad \lambda_2 = -\frac{1}{2}$

b) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ sum of elements in all rows = 0

$\Rightarrow \lambda_1 = 0$

$tr(A) = 2 \Rightarrow \lambda_2 = 2$

c) $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ sum of el in all columns $= 4 \Rightarrow \lambda_1 = 4$

$tr(A) = 5 \Rightarrow \lambda_2 = 1$

d) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ sum of el in each row $= 3 \Rightarrow \lambda_1 = 3$

if we will subtract $(-1)I$ from A

then all ~~first two~~ rows will be the same $\Rightarrow \det(A+I) = 0 \Rightarrow \lambda_2 = -1 \quad tr(A) = 1 \Rightarrow \lambda_3 = -1$

e) $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ 5 is diagonal element and the only one in 2nd row/column not equal to zero. So if we will subtract $5I$ from A we will obtain 0 row/col $\Rightarrow$ 5 is e.v. If we will subtract $(-1)I$ then first and third row will be equal $\Rightarrow$ $\det(A+I) = 0 \Rightarrow -1$ is eigenvalue

$\lambda_1 = 5 \quad \lambda_2 = -1 \quad \lambda_3 = 2$ (by trace).

④ a) $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ => by sum in column $\lambda_1 = 4$
by trace $\lambda_2 = 1$

$(A - \lambda I)x = 0$

$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} x = 0 \quad \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} x = 0 \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x = 0 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Eigenvectors are the same for all powers of a matrix.
$\det(A) = 4 \to A$ is invertible
~~For~~ Eigenvalues of different matrices:

A^2	$\lambda_1 = 4^2$	$\lambda_2 = 1^2$
A^{100}	$\lambda_1 = 4^{100}$	$\lambda_2 = 4^2$
A^{-1}	$\lambda_1 = \frac{1}{4}$	$\lambda_2 = \frac{1}{1}$ because A is invertible
e^{tA}	$\lambda_1 = e^{4t}$	$\lambda_2 = e^{t}$

We can rewrite matrix A in form $A = PDP^{-1}$ and then if we want to calculate any power n of A we can calculate PD^nP^{-1} where D is diagonal ~~mat~~ matrix with all λs on diagonal or a Jordan form if not all EVs are distinct.
So eigenvalues of A^n will be λ^n.
In our case all λs are distinct, so D will be diagonal.

(4) б) sum of el in each row = 3 => $\lambda_1 = 3$
If we will subtract $(-3)I$ them second and 3rd rows will be the same => $\det(A+3I)=0$
$\lambda_2 = -3$. By trace $\lambda_3 = 4$
$\det(A) = -36$ => A is invertible

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -5 & 4 \\ 0 & 2 & -3 \end{pmatrix} x = 0 \qquad v_1 = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \text{ or } v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -4 & 4 \\ 0 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \text{ or } v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 7 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 7 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} t \\ -14t \\ 7t \end{pmatrix} \text{ or } v_3 = \begin{pmatrix} 1 \\ -14 \\ 7 \end{pmatrix}$$

Algorithm is the same as in a).

A^2	$\lambda_1 = 9$	$\lambda_2 = 9$	$\lambda_3 = 16$
A^{100}	$\lambda_1 = 3^{100}$	$\lambda_2 = (-3)^{100}$	$\lambda_3 = 4^{100}$
A^{-1}	$\lambda_1 = \frac{1}{3}$	$\lambda_2 = -\frac{1}{3}$	$\lambda_3 = \frac{1}{4}$
e^{At}	$\lambda_1 = e^{3t}$	$\lambda_2 = e^{-3t}$	$\lambda_3 = e^{4t}$

(6) a) Matrix is the same as in (4.a) so I'll take answer from there.

$$P = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \quad \lambda_1 = 4 \quad \lambda_2 = 1 \quad J = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

b) $\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$

$\det(A - \lambda I) = 0 \quad -(3-\lambda)(1+\lambda) - 2 = 0$

$\lambda^2 - 2\lambda - 5 = 0$

$\lambda_{1,2} = 1 \pm \sqrt{6}$

$$\begin{pmatrix} 2-\sqrt{6} & 2 \\ 1 & -2-\sqrt{6} \end{pmatrix} x = 0 \qquad (2-\sqrt{6})(2+\sqrt{6}) + 2 = 0$$

$$\cancel{\begin{pmatrix} 3-\sqrt{6} & -\sqrt{6} \\ 1 & -2-\sqrt{6} \end{pmatrix} x = 0} \qquad v_1 = \begin{pmatrix} 2+\sqrt{6} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2+\sqrt{6} & 2 \\ 1 & -2+\sqrt{6} \end{pmatrix} x = 0 \qquad v_2 = \begin{pmatrix} 2-\sqrt{6} \\ 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2+\sqrt{6} & 2-\sqrt{6} \\ 1 & 1 \end{pmatrix} \qquad J = \begin{pmatrix} 1+\sqrt{6} & 0 \\ 0 & 1-\sqrt{6} \end{pmatrix}$$

c) $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ Sum by rows $= 4 \Rightarrow \lambda_1 = 4$

$tr = 3 \Rightarrow \lambda_2 = -1$

$$\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix} x = 0 \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} x = 0 \quad v_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 3 \\ 1 & -2 \end{pmatrix} \qquad J = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$$

d) $\begin{pmatrix} -5 & 2 \\ -\frac{1}{2} & -3 \end{pmatrix}$ $\left.\begin{matrix} tr(A) = -8 \\ det(A) = 16 \end{matrix}\right\} \Rightarrow \lambda_1 = \lambda_2 = -4$

$\begin{pmatrix} -1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix} v_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$(A - \lambda I) v_2 = v_1$

$\begin{pmatrix} -1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix} v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\begin{cases} x_1 = 2x_2 - 2 \\ x_2 \in \mathbb{R} \end{cases}$ $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad J = \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix}$

7 $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}$

$\lambda_1 = 1$ By sum of row elements

$det \left| \begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} \right| = 0$ $\quad \underline{-\lambda^3 = -1 \quad \lambda^2 = 1}$ $\quad -\lambda_1\lambda_2\lambda_3 = -1$

$\lambda_1 + \lambda_2 + \lambda_3 = 0$

$\begin{cases} \lambda_2 \cdot \lambda_3 = 1 \\ \lambda_2 + \lambda_3 = -1 \end{cases}$ $\begin{cases} \lambda_2 = -1 - \lambda_3 \\ \lambda_3^3 + \lambda_3^2 + 1 = 0 \end{cases}$

$\lambda_3^2 + \lambda_3 + 1 = 0$

$D = 1 - 4 = (\sqrt{3})^2$ $\quad \lambda_{3_{1,2}} = \frac{-1 \pm i\sqrt{3}}{2}$

$\lambda_3 = \frac{-1 - i\sqrt{3}}{2}$ $\quad \lambda_2 = \frac{-1 + i\sqrt{3}}{2}$

$V_2 = \begin{pmatrix} \frac{-1 - i\sqrt{3}}{2} \\ \frac{-1 + i\sqrt{3}}{2} \\ 1 \end{pmatrix}$ $\quad V_3 = \begin{pmatrix} \frac{-1 + i\sqrt{3}}{2} \\ \frac{-1 - i\sqrt{3}}{2} \\ 1 \end{pmatrix}$

1) a) $\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$ $tr = 7$ $det = 6$

$\lambda_1 = 1$ $\lambda_2 = 6$

$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} v_1 = 0$ $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} v_2 = 0$ $v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix} = P$ $P^{-1} = P^T$

$P^{-1}AP = D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} = P^{-1}$

b) $\begin{pmatrix} -2 & 0 & -36 \\ 0 & 3 & 0 \\ -36 & 0 & -23 \end{pmatrix}$ $\lambda_1 = 3$ by inspection

$tr(A) = -22$ $det(A) = -3750$

$\lambda_2 = 25$ $\lambda_3 = -50$

$\begin{pmatrix} -5 & 0 & -36 \\ 0 & 0 & 0 \\ -36 & 0 & -26 \end{pmatrix} v_1 = 0$ $v_1 = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$

$\begin{pmatrix} -27 & 0 & -36 \\ 0 & -22 & 0 \\ -36 & 0 & -48 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 4 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{pmatrix}$ $v_2 = \begin{pmatrix} 4t \\ 0 \\ -3t \end{pmatrix}$

$\begin{pmatrix} 48 & 0 & -36 \\ 0 & 53 & 0 \\ -36 & 0 & 27 \end{pmatrix}$ $v_3 = \begin{pmatrix} 3t \\ 0 \\ 4t \end{pmatrix}$ $P = \begin{pmatrix} 0 & 4/5 & 3/5 \\ 1 & 0 & 0 \\ 0 & -3/5 & 4/5 \end{pmatrix}$

$D = P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{pmatrix}$

$P^{-1} = P^T = \begin{pmatrix} 0 & 1 & 0 \\ 4/5 & 0 & -3/5 \\ 3/5 & 0 & 4/5 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$ $\lambda_1 = -3$ by inspection

$tr = 0$

$\det = 1(-6) + 4(12) + 2(6) = 54$

$\lambda_2 = -3 \quad \lambda_3 = 6$

$\begin{pmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{pmatrix} \sim \begin{pmatrix} \cancel{1} & \cancel{-1} & \cancel{-4} \\ \cancel{-1} & \cancel{1} & \cancel{4} \end{pmatrix} \quad V_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$?

$\cancel{\begin{pmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{pmatrix} V_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}$

$\begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad V_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\cancel{\begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix} * \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$ V_2 should be orthogonal to V_1 and V_3

Let's take $V_2 = \begin{pmatrix} 2 \\ -2 \\ 8 \end{pmatrix}$

All eVCs are normalized in order to make P orthogonal.

$P = \begin{pmatrix} 1/\sqrt{2} & 2/\sqrt{72} & 2/3 \\ 1/\sqrt{2} & -2/\sqrt{72} & -2/3 \\ 0 & 8/\sqrt{72} & 1/3 \end{pmatrix} \qquad P^{-1}AP = D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$P^{-1} = P^T$

Because P should be orthogonal.

(12) a) $\lambda_1 = -1 \quad \lambda_2 = 3 \quad \lambda_3 = 7$

If matrix is symmetric then all eigenvectors are mutually orthogonal. In this case all evs are distinct $\to$ all eigenvectors are orthogonal.

$$\left.\begin{array}{l} (v_1, v_3) = 0 \Rightarrow 1b = 0 \Rightarrow b = 0 \\ (v_2, v_3) = 0 \Rightarrow 1a = 0 \Rightarrow a = 0 \end{array}\right\} \Rightarrow v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

It's impossible because eigenvector can't be 0 vector. $\to$ There is no symmetric matrices for this conditions.

b) $\lambda_1 = -1 \quad \lambda_2 = 3 \quad \lambda_3 = 3$

A is symmetric $\Rightarrow$ all eigenvectors of distinct evs are orthogonal. $\Rightarrow$

$(v_1, v_3) = 0 \Rightarrow b = 0,\ a \in \mathbb{R} \setminus \{0\}$

c) for b)

$$A = \begin{pmatrix} 0 & 1 & a \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{a} & -\frac{1}{2a} & -\frac{1}{2a} \end{pmatrix}$$

$$w_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix} \qquad w_2 = \frac{v_2}{\|v_2\|} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \quad w_3 = \frac{v_3}{\|v_3\|} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$P = (w_1\, w_2\, w_3) = \begin{pmatrix} 0 & 1/3 & 1 \\ 1/2 & 1/3 & 0 \\ 1/2 & 1/3 & 0 \end{pmatrix} \quad P^{-1} = P^T \Rightarrow$$

$$A = \begin{pmatrix} 0 & 1/3 & 1 \\ 1/2 & 1/3 & 0 \\ 1/2 & 1/3 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \end{pmatrix}$$

(13) $\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = \lambda$

a) Let $\lambda \neq 1 \quad \lambda \neq 2$

All evcs should be orthogonal because matrix is symmetric.

$$\left.\begin{aligned}(a,b,0)\begin{pmatrix}1\\0\\1\end{pmatrix} = 0 &\Rightarrow a = 0\\ (0,b,0)\begin{pmatrix}1\\1\\1\end{pmatrix} = 0 &\Rightarrow b = 0\end{aligned}\right\} v_3 = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

it is not evc $\rightarrow$ there is no symm matrix satisfying this conditions.

Let $\lambda = 1$

$$\begin{pmatrix}1 & 1 & -1\end{pmatrix}\begin{pmatrix}a\\b\\0\end{pmatrix} = 0 \Rightarrow a+b=0 \quad a=-b \quad a \neq 0$$

$$A = PDP^{-1} = \underbrace{\begin{pmatrix}1 & 1 & a\\0 & 1 & -a\\1 & -1 & 0\end{pmatrix}}_{P}\begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{pmatrix}\cdot P^{-1}$$

Let $\lambda = 2$

$$\begin{pmatrix}1\\0\\1\end{pmatrix}^T\begin{pmatrix}a\\b\\0\end{pmatrix} = 0 \Rightarrow a=0 \quad b \in \mathbb{R}\setminus\{0\}$$

$A = PDP^{-1} =$

$$\omega_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \qquad \omega_2 = \frac{v_2}{\|v_2\|} = \begin{pmatrix} 1/3 \\ 1/3 \\ -1/3 \end{pmatrix} \quad \omega_3 = \frac{v_3}{\|v_3\|} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$P = (\omega_1, \omega_2, \omega_3) \qquad P^{-1} = P^T$$

$$A = \begin{pmatrix} 1/2 & 1/3 & 1/\sqrt{2} \\ 0 & 1/3 & -1/\sqrt{2} \\ 1/2 & -1/3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & -1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

(15) A quadratic form is called positive definite when all its EVs are positive (>0). On the other hand it's also positive definite when all its ~~m~~ principal minors in top let corner are positive (>0).

$$\begin{pmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & K \end{pmatrix} \qquad \begin{aligned} &|5| > 0 \\ &\begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} = 5 - 4 = 1 > 0 \end{aligned}$$

$$\begin{vmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & K \end{vmatrix} = 5\cdot(K-1) - 2(2K-1) - 1(-2+1) = \\ = 5K - 5 - 4K + 2 + 2 - 1 = K - 2 > 0 \Rightarrow$$

$\underline{\underline{K > 2}}$

9 $A = uv^T$

$$\begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_m \end{pmatrix} (v_1 \ v_2 \ \dots \ v_n) = \begin{pmatrix} u_1 \cdot v \\ u_2 \cdot v \\ u_3 \cdot v \\ u_n \cdot v \end{pmatrix} \Rightarrow$$

All of this rows are linearly dependent and rank (A) = 1 (Because we can eliminate all rows except first).