Linear Algebra
Homework 1: Prerequisites

- The aim of this self-study introductory module is to review basic concepts and facts in linear algebra without which any further progress could be problematic. After having worked out this home assignment you will be able to proceed with topics that are a must for any CS master student.
- A complete solution of the home assignment amounts to 20% of the final grade.
- If you can solve all the problems without referring to the textbook or online course, just do that and enjoy your last free weeks. Otherwise, read the textbook and/or watch the online lectures and then try again to solve the problems.
- If you still do not know how to solve some problem(s), ask your classmates or me (electronically or in person during the introduction week of 11–15 Sep 2017). We encourage you to discuss the problems or questions you have with your colleagues; however, be sure to obey the academic code of honour of UCU.
- A *solution* to a problem is not just a numerical answer; a complete solution must include a concise explanation of the main steps and reasoning leading to the answer. Do not forget to examine all possible cases!
- There are 14 problems (50 points total) and one challenge problem (10 points). You do not have to solve all problems but can choose any subset whose total is 40 points (the best combination will be graded if you solve more). The challenge problem is optional, and you'll get extra points for its solution (so that your total score for this assignment can be more than 20%)
- Solutions should be submitted to the `cms` online platform (ask if you are not sure what to do). You can scan/take picture of your (readable) handwriting or use any text editor (ideally LaTeX); then upload the files to `cms`. Please name your files as "`Name-Surname-LA-HW1-#.***`", where # stands for the corresponding file number in the case of multiple files.
- The deadline for submission is 21:00 of **Fri, 15 Sep 2017**
- Reading: *David C. Lay, Linear Algebra and its Applications*; link to the corresponding parts is given in the email message
- MIT online course:
`https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/`

Problem 1 (System of linear equations; 3pt)**.** Determine all the values of k for which the matrix below is the augmented matrix of a consistent linear system.

$$(a)\quad \left(\begin{array}{cc|c} 1 & k & 4 \\ 3 & 6 & 8 \end{array}\right) \qquad (b)\quad \left(\begin{array}{cc|c} 1 & 4 & -2 \\ 3 & k & -6 \end{array}\right) \qquad (c)\quad \left(\begin{array}{cc|c} -4 & 12 & k \\ 2 & -6 & -3 \end{array}\right)$$

Problem 2 (System of linear equations; 4pt)**.** Let

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right)$$

be the augmented matrix for a linear system. Find for what values of a and b the system has

(a) a unique solution;
(b) a one-parameter solution set;
(c) a two-parameter solution set;
(d) no solution.

Problem 3 (System of linear equations; 6pt). Write a system of linear equations consisting of m equations in n unknowns with

(a) no solutions; (b) exactly one solution; (c) infinitely many solutions

for (i) $m = n = 3$; (ii) $m = 3$ and $n = 2$; (iii) $m = 2$, $n = 3$.

Problem 4 (System of linear equations; 4pt). The following are coefficient matrices of linear systems. For each system, what can you say about the number of solutions to the corresponding system **(i)** in the homogeneous case (when $b_1 = \dots = b_m = 0$) and **(ii)** for a generic RHS?

$$a)\begin{pmatrix}1 & 4\\ 2 & 1\end{pmatrix}, \qquad b)\begin{pmatrix}1 & 4 & 3\\ 2 & 1 & 0\end{pmatrix}, \qquad c)\begin{pmatrix}2 & 1\\ 1 & 4\\ 0 & 3\end{pmatrix}, \qquad d)\begin{pmatrix}1 & 4 & 3\\ 2 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}.$$

Problem 5 (System of linear equations; linear dependence; 3pt). Prove that any $n+1$ vectors in $\mathbb{R}^n$ are linearly dependent.

Hint: regard a linear combination of these vectors resulting in a zero vector as a homogeneous linear system and show that it possesses a non-trivial solution

Problem 6 (Gauss elimination; determinants; 4pt). Determine all the values of k for which the columns vectors below are linearly dependent:

$$\text{(a)} \quad \begin{pmatrix}1\\ 2\\ -1\end{pmatrix}, \begin{pmatrix}2\\ -3\\ 5\end{pmatrix}, \begin{pmatrix}6\\ k\\ 1\end{pmatrix}; \qquad \text{(b)} \quad \begin{pmatrix}-1\\ 2\\ 1\end{pmatrix}, \begin{pmatrix}2\\ -4\\ -2\end{pmatrix}, \begin{pmatrix}k\\ 3\\ -3\end{pmatrix}$$

Problem 7 (Matrix algebra; 4pt). Let $\mathbf{0}_n$ and I_n denote respectively the zero and identity matrices of size n.

(a) Is there an $n \times n$ matrix A such that $A \neq \mathbf{0}_n$ and $A^2 = \mathbf{0}_n$? Justify your answer.

(b) Is there an $n \times n$ matrix A such that $A \neq \mathbf{0}_n, I_n$ and $A^2 = A$? Justify your answer.

(c) Is there an $n \times n$ matrix A such that $A \neq I_n$ and $A^2 = I_n$? Justify your answer.

(d) Are there $n \times n$ matrices A and B such that $A \neq \mathbf{0}_n$, $B \neq \mathbf{0}_n$, $AB \neq \mathbf{0}_n$ but $BA = \mathbf{0}_n$?

Problem 8 (Determinants and cross-products; 4pt). A parallelepiped has edges from $(0;0;0)$ to $(2;1;1)$, $(1;2;1)$, and $(1;1;2)$. Find its volume and also find the area of each parallelogram face.

Hint: a cross-, or vector-product in $\mathbb{R}^3$ is handy here. Also, recall the geometric meaning of a determinant

Problem 9 (Determinants and matrix algebra; 2pt). Assume that 3×3 matrices A, B and C are as follows

$$A = \begin{pmatrix}\text{row } 1\\ \text{row } 2\\ \text{row } 3\end{pmatrix} \qquad B = \begin{pmatrix}\text{row } 1 + \text{row } 2\\ \text{row } 2 + \text{row } 3\\ \text{row } 3 + \text{row } 1\end{pmatrix} \qquad C = \begin{pmatrix}\text{row } 1 - \text{row } 2\\ \text{row } 2 - \text{row } 3\\ \text{row } 3 - \text{row } 1\end{pmatrix}$$

Given that $\det(A) = 5$, find $\det(B)$ and $\det(C)$.

Hint: use the elementary row operations to produce B from A; an alternative (and more elegant) way is to find a matrix B' such that $B = B'A$; the same for C

Problem 10 (Determinants; eigenvalues and their properties; 3pt). Using any of the methods, find all λ for which the matrix below is singular:

$$A - \lambda I = \begin{pmatrix}a-\lambda & b & c & d\\ a & b-\lambda & c & d\\ a & b & c-\lambda & d\\ a & b & c & d-\lambda\end{pmatrix}$$

Hint: one approach is to calculate the determinant and find its roots. An alternative approach is to identify the λ's looked for as eigenvalues of A. Note A is of rank 1; what conclusions on eigenvalues can you derive?

Problem 11 (Rank of a matrix; 4pt). (a) Assume that A and B are matrices such that AB is well defined. By comparing the column spaces of A and AB, show that $\operatorname{rank}(AB) \le \operatorname{rank}(A)$. Transpose to conclude that also $\operatorname{rank}(AB) \le \operatorname{rank}(B)$.

(b) Assume that A and B are non-square matrices such that both AB and BA exist. Show that at least one of AB or BA is singular.
Hint: in (b), show that at least one of AB and BA is not of full rank

Problem 12 (Trace of a matrix; 2pt). Are there $n \times n$ matrices A and B such that $AB - BA = I_n$?

Problem 13 (Bases; 3pt). For what numbers c are the following sets of vectors bases for $\mathbb{R}^3$?

(a) $(c, 1, 1)^\top$, $(1, -1, 2)^\top$, $(3, 4, -1)^\top$;

(b) $(c, 1, 1)^\top$, $(1, -1, 2)^\top$, $(-2, 2, -4)^\top$;

(c) $(c, 1, 1)^\top$, $(1, 1, 0)^\top$, $(0, 1, 2)^\top$, $(3, 0, -1)^\top$;

(d) $(c, 1, 1)^\top$, $(1, 0, 1)^\top$

Problem 14 (Bases; transition matrices; 4pt). Consider the bases $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $B' = \{\mathbf{v}_1', \mathbf{v}_2', \mathbf{v}_3'\}$ for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$$

$$\mathbf{v}_1' = \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2' = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}, \qquad \mathbf{v}_3' = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

(a) Find the transition matrix $P_{B \to B'}$ from B to B'.

(b) Compute the coordinate vector $(\mathbf{u})_B$ for $\mathbf{u} = (-5, 8, -5)^\top$.

(c) Use the transition matrix $P_{B \to B'}$ to compute the coordinate vector $(\mathbf{u})_{B'}$.

(d) Check your work by computing $(\mathbf{u})_{B'}$ directly.

Problem 15 (A challenge problem; extra 10pt). Undergraduate students like linear systems whose coefficient matrices have integer entries and whose determinant is ± 1 (do you see why?).

(a) Suggest a method generating all 3×3 matrices of this type with relatively small entries (e.g. at most 20 in absolute value). Think of efficiency of your algorithm (e.g. as compared to the brute force approach).

(b) Consider a modification of the above task where only orthogonal matrices should be generated.