① a) (I) $\frac{4}{7} \neq \frac{-1}{-3} \neq \frac{2}{4}$ => planes are not parallel

(II) $\frac{1}{3} = \frac{-4}{-12} = \frac{-9}{-3}$ => planes are parallel

b) (I) $(3,-1,1)(1,0,2) = 3+0+2 = 5 \neq 0$ => planes are not perpendicular

(II) $(1,-2,3)(-2,5,4) = -2-10+12 = 0$ => planes are perpendicular

② a) $W: x-2y-3z=0$

normal vector $(1;-2;-3)$

Orthogonal complement for a plane is a line which will have the guiding vector equal to normal vector of the plane.

$$W^{\perp} = \begin{cases} x = 1t \\ y = -2t \\ z = -3t \end{cases}$$

b) $$W: \begin{cases} x = 2t \\ y = -5t \\ z = 4t \end{cases}$$

Orthogonal complement for a line is a plane which will have the normal vector equal to guiding vector of a line.

$W^{\perp}: 2x - 5y + 4z = 0$

c) $x+y+z=0$ ~~This planes~~

$x-y+z=0$ The intersection of the two planes is line or empty set if they are parallel.

$\frac{1}{1} \neq \frac{1}{-1} \neq \frac{1}{1}$ => planes are not parallel ant the intersection is a line.

guiding vector: $\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2i - 2k \Rightarrow (2,0,-2)$

The same as in (b): $W^{\perp}: 2x + 0y - 2z = 0$

2a) W : $x-2y-3z=0$

normal vector. $(1;-2;-3)$

line which lays on this vector and goes through the origin

$$W^{\perp} \begin{cases} x = 1t \\ y = -2t \\ z = -3t \end{cases}$$

(b)

$$W = \begin{cases} x = 2t \\ y = -5t \\ z = 4t \end{cases} \qquad 2x - 5y + 4z = t$$

~~one of normal vectors to the plane is~~

~~$(1;2;2)$. W Proof:~~

~~$(2;-5;4)(1;2;2) = 2-10+8=0$~~

~~So $W^{\perp} = \begin{cases} x = 1t \\ y = 2t \\ z = 2t \end{cases}$~~ ~~$x+2y+2z=t$~~

(c) $x+y+z=0$

$x-y+z=0$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = i \cdot 2 + j \cdot 0 + k \cdot 2$$

~~$W^{\perp} = \begin{cases} x = 2t \\ y = 0 \\ z = 2t \end{cases}$~~

③ a) $P=(1,1,0)$ $\quad \frac{x-1}{2}=\frac{y-2}{1}=\frac{z+1}{2}$

$\bar{s}=(2,1,2)$ guidance vector of a line

$X=(1,2,-1)$ point on the line

$\cancel{\overline{XP}=}$ $\overline{PX}=(0;1;-1)$

$$\bar{n}=\begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 0 & 1 & -1 \end{vmatrix} = -3i+2j+2k$$ orthogonal vector to $\bar{s}$ and $\overline{PX}$

$$d=\frac{|\bar{n}|}{|\bar{s}|}=\frac{\sqrt{9+4+4}}{\sqrt{4+1+4}}=\frac{\sqrt{17}}{3}$$

④ a) ~~####~~

$u=(u_1,u_2,u_3)^T$ $v=(v_1,v_2,v_3)^T$ $w=(w_1,w_2,w_3)^T$

$w_1=\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}$ $w_2=-\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}$ $w_3=\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$

$$A=\begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}=u_1\cdot\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}-u_2\cdot\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}+u_3\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}=$$

$=u_1w_1+u_2w^2+u_3w_3=w^Tu=0$ because matrix A has two identical rows.

~~The~~ Proof is the same for w^Tv ~~with~~ ~~but~~ with $A=\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$.

b) $u_1,u_2,\dots,u_{n-1}$ are linearly independent in $\mathbb{R}^n$

~~Let $A=(u_1^T,u_2^T,\dots,u_{n-1}^T)$~~

(Let $A=\begin{pmatrix} u_{11} & u_{21} & \dots & u_{n-1\,1} \\ u_{12} & u_{22} & \dots & u_{n-1\,2} \\ u_{13} & u_{23} & \dots & u_{n-1\,3} \\ \dots & & & \dots \\ u_{1n} & u_{2n} & \dots & u_{n-1\,n} \end{pmatrix}$

(E

(: To find an ortogonal vector $\vec{x}$ to the subspace spanned by A we need to find the solution

b) of $A\vec{x}=\vec{0}$. It will be ~~the solution~~ such a vector because

1): for $k=\overline{1..(n-1)}$ $u_k\cdot x=\vec{0}$ which means that vector $\overline{x}$

2) A is ortogonal to all u_k vectors, thus it is ortogonal to the supspace spanned by $u_1,\dots,u_{n-1}$

3) A because $u_1,\dots,u_{n-1}$ form

upf

mat

(5) a) Let $Q_1 = X$ and $Q_2 = Y$

X is orthogonal $\Rightarrow X^{-1} = X^T$

is X^{-1} orthogonal ?

$(A^{-1})^{-1} = A$ and $(A^T)^T = A$

X is ortogonal $\Rightarrow (X^T)^T = (X^{-1})^T \Rightarrow$

$(X^{-1})^T = (X^{-1})^{-1} \Rightarrow X^{-1}$ is orthogonal ∎

X- orthogonal and Y orthogonal

is XY orthogonal ?

Lets prove that $(AB)^T = A^T B^T$

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\left.\begin{aligned} ((AB)^T)_{ij} &= \sum_{k=1}^{n} b_{ki} a_{jk} = (AB)_{ji} \\ (B^T A^T)_{ij} &= \sum_{k=1}^{n} a_{jk} b_{ki} = (AB)_{ji} \end{aligned}\right\} \Rightarrow (AB)^T = B^T A^T$$

$$XY \cdot (XY)^T = XYY^TX^T = X \cdot I \cdot X^T = X \cdot X^T = I \Rightarrow XY(XY)^T = I \Rightarrow$$

$(XY)^T = (XY)^{-1} \Rightarrow XY$ - orthogonal.

b) Let A be orthogonal and upper triangular, then:

$$\left.\begin{aligned} &1)\ A^{-1} \text{ is also upper triangular} \\ &2)\ A^T \text{ is lower triangular} \\ &3)\ A^{-1} = A^T \text{ because } A \text{ is orthogonal} \end{aligned}\right\} \Rightarrow$$

upper triang. matrix equal to lower triang. matrix so it has elements only on diagonal

⑥ $A = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 1 \\ s \end{pmatrix}$ $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\cancel{A^T A x =}$ $\cancel{A^T x}$

$A^T A x = A^T b$

$A^T A x = \begin{pmatrix} 71 \\ 95 \end{pmatrix}$ - here and further all matrix operations were calculated using Python because people are error prone and slow in computations.

$A^T b = \begin{pmatrix} 3+4s \\ 2+5s \end{pmatrix}$

$\begin{cases} 3+4s = 71 \\ 2+5s = 95 \end{cases}$ $\begin{cases} s = \frac{68}{4} \\ s = \frac{93}{5} \end{cases}$ => there is no such s that x is the LS solution to the linear system above

⑦ a)

$$\begin{array}{cc} x & y \end{array}$$
$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 3 & 1 \\ 3 & 2 \end{pmatrix} = A$$

$y = kx + b$

$\bar{x} = 2$ $\bar{y} = 1$

$$k = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{1}{6}$$

$$b = \bar{y} - k\bar{x} = 1 - \frac{1}{6} \cdot 2 = \frac{1}{3}$$

$$y = \frac{1}{6}x + \frac{2}{3}$$

b)

x	y
2	0
3	-10
5	-48
6	-76

$y = ax^2 + bx + c$

$4a + 2b + c = 0$

$9a + 3b + c = -10$

$25a + 5b + c = -48$

$-76 = 36a + 6b + c$

$$\begin{pmatrix} 4 & 2 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \\ 36 & 6 & 1 \end{pmatrix} = A \quad \begin{pmatrix} 0 \\ -10 \\ -48 \\ -76 \end{pmatrix} = b$$

$A^T A x = A^T b$ => $\cancel{x = (A^T)^{-1} A^T A x = \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}}$ $(A^T A)^{-1} A^T b = x = \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}$

$y = -3x^2 + 5x + 2$

c)

x	y
-1	-14
0	-5
1	-4
2	1
3	22

$y = ax^3 + bx^2 + cx + d$

$$A = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \end{pmatrix} \quad b = \begin{pmatrix} -14 \\ -5 \\ -4 \\ 1 \\ 22 \end{pmatrix}$$

$$A^T A X = A^T b \Rightarrow X = (A^T A)^{-1} A^T b = \begin{pmatrix} 2 \\ -4 \\ 3 \\ 5 \end{pmatrix}$$

$$\underline{y = 2x^3 - 4x^2 + 3x + 5}$$

9) a) $v_1 = (1,3)$

$v_2 = (2,-2)$

$u_1 = v_1 \qquad \|u_1\| = \sqrt{10}$

$u_2 = v_2 - \text{proj}_{u_1} v_2 = \left(\frac{12}{5}; -\frac{4}{5}\right) \qquad \|u_2\| =$

$\text{proj}_{u_1} v_2 = \frac{(1,3)(2,-2)}{(1,3)(1,3)} \cdot (1,3) = \left(-\frac{2}{5}; -\frac{6}{5}\right)$

$e_1 = \frac{u_1}{\|u_1\|} = \frac{(1,3)}{\sqrt{10}} = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) \qquad \cancel{e_2 = \frac{u_2}{\|u_2\|} = \left(\frac{3}{5\sqrt{10}}; \frac{-1}{5\sqrt{10}}\right)} \quad e_2 = \left(\frac{3}{5\sqrt{10}}; \frac{-1}{5\sqrt{10}}\right)$

b) $u_1 = (1,0,1) \quad u_2 = (1,3,-2) \quad u_3 = (0,2,1)$

$v_1 = u_1 \qquad \|v_1\| = \sqrt{2}$

$v_2 = u_2 - \text{proj}_{v_1}(u_2) = \left(\frac{3}{2}; 3; \frac{-3}{2}\right) \qquad \|u_2\| = \frac{3\sqrt{6}}{2}$

$\text{proj}_{v_1}(u_2) = \frac{(1;0,1)(1,3,-2)}{(1,0,1)(1,0,1)}(1,0,1) = \left(\frac{-1}{2}; 0; \frac{-1}{2}\right)$

$\text{proj}_{v_1}(u_3) = \frac{(0,2,1)(1,0,1)}{(1,0,1)(1,0,1)}(1,0,1) = \left(\frac{1}{2}; 0; \frac{1}{2}\right)$

$\text{proj}_{v_2}(u_3) = \frac{(0,2,1)\left(\frac{3}{2}; 3; \frac{-3}{2}\right)}{\left(\frac{3}{2}; 3; \frac{-3}{2}\right)\left(\frac{3}{2}; 3; \frac{-3}{2}\right)}\left(\frac{3}{2}; 3; \frac{-3}{2}\right) = \left(\frac{1}{2}; 1; \frac{-1}{2}\right)$

$v_3 = u_3 - \text{proj}_{v_1}(u_3) - \text{proj}_{v_2}(u_3) = (-1; 1; 1) \quad \|v_3\| = \sqrt{3}$

$e_1 = \left(\frac{1}{\sqrt{2}}; 0; \frac{1}{\sqrt{2}}\right) \quad e_2 = \left(\frac{1}{\sqrt{6}}; \frac{2}{\sqrt{6}}; -\frac{1}{\sqrt{6}}\right) \quad e_3 = \left(-\frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}\right)$

(10) a) $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$

$u_1 = a_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad e_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

$u_2 = (-1, 3) - (1, 2) = (-2, 1)$ $\quad e_2 = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ $\quad \langle e_1, a_1 \rangle = \frac{5}{\sqrt{5}}$

$Q = [e_1, e_2] = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$ $\quad R = \begin{pmatrix} \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{5} \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{pmatrix} = A$

$u_1 = a_1 \quad \|a_1\| = \sqrt{2}$

$u_2 = a_2 - proj_{u_1} a_2 = (-1; 1; 1) \quad \|u_2\| = \sqrt{3}$

$proj_{u_1} a_2 = \frac{(2,1,4)(1,0,1)}{(1,0,1)(1,0,1)}(1,0,1) = (3,0,3)$

$e_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ $\quad e_2 = \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

because of
the orthogonality

$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$ $\quad \cancel{R = Q^{-1} A}$ $\quad A = QR \quad R = Q^{-1}A = Q^T A$

$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 3\sqrt{2} \\ 0 & \frac{2}{\sqrt{3}} \end{pmatrix}$

10 c) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} = A$

$u_1 = a_1 \qquad \|u_1\| = \sqrt{5} \qquad e_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

$u_2 = (0,1,0) - \frac{(0,1,0)(1,0,2)}{(1,0,2)(1,0,2)}(1,0,2) = (0,1,0) \quad \|u_2\| = 1 \quad e_1 = (0,1,0)^T$

$u_3 = (2,1,1) - \frac{(2,1,1)(0,1,0)}{(0,1,0)(0,1,0)}(0,1,0) - \frac{(2,1,1)(1,0,2)}{(1,0,2)(1,0,2)}(1,0,2) = \begin{pmatrix} \frac{6}{5} \\ 0 \\ \frac{-3}{5} \end{pmatrix}$

$\|u_3\| = \frac{3}{\sqrt{5}} \qquad e_3 = \left(\frac{2}{\sqrt{5}}, 0, \frac{-1}{\sqrt{5}}\right)^T$

$Q = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}$

$\langle e_1, a_2 \rangle = 0$

$\langle e_1, a_3 \rangle = \frac{4}{\sqrt{5}}$

$\langle e_2, a_3 \rangle = 1$

$R = \begin{pmatrix} \sqrt{5} & 0 & \frac{4}{\sqrt{5}} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{3}{\sqrt{5}} \end{pmatrix}$

(8) $u_1 = \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ \dots \\ u_{1n} \end{pmatrix}$ $u_2 = \begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \\ \dots \\ u_{2n} \end{pmatrix}$ $a_1 = u_1$ $a_2 = u_1 + \varepsilon u_2$

$A = (a_1 \ \ a_2)$

$$A^T A = \begin{pmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_1 \rangle \\ \langle u_1, u_1 \rangle & \langle u_1, u_1 \rangle + \langle u_2, u_2 \rangle \end{pmatrix}$$

$$(A^T A)^{-1} = \begin{pmatrix} \dfrac{\langle u_1, u_1 \rangle + \langle u_2, u_2 \rangle}{\langle u_1, u_1 \rangle \cdot \langle u_2, u_2 \rangle} & \dfrac{-1}{\langle u_2, u_2 \rangle} \\ \dfrac{-1}{\langle u_2, u_2 \rangle} & \dfrac{1}{\langle u_2, u_2 \rangle} \end{pmatrix}$$

$$\cancel{(A^T A)^{-1} \cdot A^T = \begin{pmatrix} \dfrac{\langle u_1, u_1 \rangle + \langle u_2, u_2 \rangle}{\langle u_1, u_1 \rangle \cdot \langle u_2, u_2 \rangle} \cdot (u_1 + \varepsilon u_2) - \dfrac{u_1}{\langle u_2, u_2 \rangle} \\ - \dfrac{u_1 + \varepsilon u_2}{\langle u_2, u_2 \rangle} \quad \dfrac{u_1}{} \end{pmatrix}}$$

$$\cancel{(A^T A)^{-1} \cdot A^T = \left(\dfrac{\langle u_1, u_1 \rangle + \langle u_2, u_2 \rangle}{\langle u_1, u_1 \rangle \cdot \langle u_2, u_2 \rangle} \right.}$$