Mathematical analysis of heat pulse signals for soil water flux determination

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[1] Soil water flux is an important parameter in studies of runoff, infiltration, groundwater recharge, and subsurface chemical transport. Heat pulse sensors have been proposed as promising tools for measuring soil water fluxes. To date, heat pulse methods have required cumbersome mathematical analyses to calculate soil water flux from the measured data. We present a new mathematical analysis showing that a simple linear relationship exists between soil water flux and the natural log of the ratio of the temperature increase downstream from the line heat source to the temperature increase upstream from the line heat source. The simplicity of this relationship makes heat pulse sensors an attractive option for measuring soil water fluxes. In theory, this method is valid for fluxes with magnitudes between 10^{-4} and 10^{-7} m s^{-1}. The range of measurable fluxes is defined by temperature measurement resolution at the lower end and by the assumptions used in the analysis at the higher end. *INDEX TERMS:* 1832 Hydrology: Groundwater transport; 1875 Hydrology: Unsaturated zone; 1878 Hydrology: Water/energy interactions; 1894 Hydrology: Instruments and techniques; *KEYWORDS:* soil water flux, groundwater flow, heat pulse probe, convection, conduction, Peclet number

1. Introduction

[2] Many soil scientists, hydrologists, and environmental scientists are interested in knowing the magnitude and direction of soil water flux (J) at a particular location. This interest arises from the major role of J in processes such as infiltration, runoff, and subsurface chemical transport. J can vary widely in time and space depending on the soil and environmental conditions. This variability makes modeling J difficult. In some cases, measuring J directly would be a more attractive option than modeling J; however, few practical techniques for measuring J in situ are available.

[3] *Byrne et al.* [1967, 1968] introduced the idea of using heat as a tracer to measure J. Their instruments measured distortion of the steady state thermal field around point and line heat sources. *Melville et al.* [1985] performed a laboratory investigation of the effectiveness of a similar device for measuring groundwater flow. Complications, primarily related to instrument calibration, limited the effectiveness of the instruments used by *Byrne et al.* [1967, 1968] and *Melville et al.* [1985].

[4] Building on the work of *Byrne et al.* [1967, 1968] and *Melville et al.* [1985], *Ren et al.* [2000] developed an improved heat pulse technique to measure J. The probe used by *Ren et al.* [2000] consisted of three stainless steel needles embedded in a waterproof epoxy body. The needles were parallel, aligned in a common plane, and separated by approximately 6 mm. The exposed length of the needles was 4 cm. The center needle contained a resistance heater, and the outer two needles contained thermocouples. The probe was embedded in a soil column through which a constant J was established. This experimental setup is depicted in Figure 1. Electrical current was passed for 15 s from an external power supply through the resistance heater in the center needle, creating a heat pulse. Heat transfer away from the center needle occurred via conduction and convection. The resulting temperature increase at the thermocouples in the two outer needles was measured and recorded by an external datalogger. The convection of heat by the flowing water resulted in a larger temperature increase downstream from the heat source than upstream from the heat source.

[5] *Ren et al.* [2000] developed an analytical solution of the appropriate heat transfer equation and showed that the solution could be used to calculate J from the difference between the measured temperature increases at the downstream and upstream needles, if the thermal properties of the soil were known. A disadvantage of the *Ren et al.* [2000] solution is that it contains an integral that requires numerical integration. *Kluitenberg and Warrick* [2001] improved the evaluation procedure by converting the equations into the well function for leaky aquifers and by using an infinite series to approximate the well function. Although the improved method eliminates the need for numerical integration, it still is inconvenient to analyze the relations among variables and to estimate J, because the infinites series is quite complicated. In this paper we analyze the heat pulse signal in a new way with the goals of clarifying the relationships between the variables in this heat transfer

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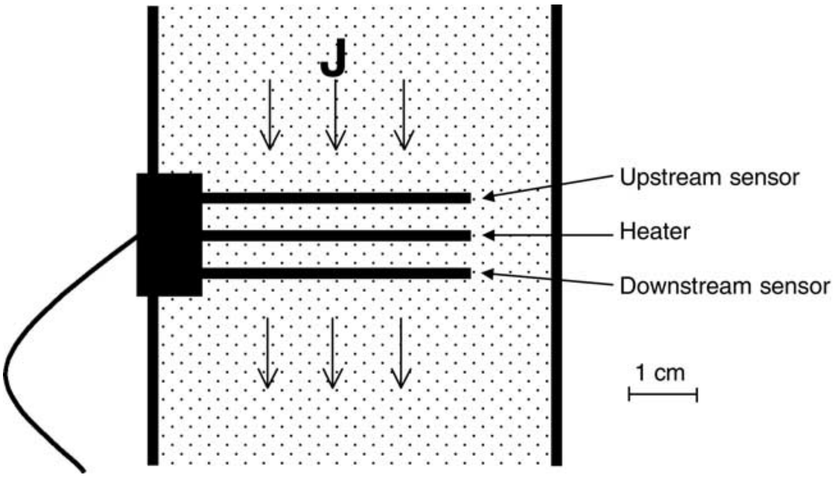

Figure 1. Sketch of a heat pulse probe embedded in a soil column through which a steady, uniform water flux (J) has been established.

problem and simplifying the procedure for calculating J from heat pulse measurements.

2. Theory

2.1. General Solution for the Heat Transfer Equation

[6] The heat transfer equation utilized in the heat pulse method of *Ren et al.* [2000] is

$$\frac{\partial T}{\partial t} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] - V\frac{\partial T}{\partial x} \tag{1}$$

where T is temperature increase (°C), t is time (s), α is the soil thermal diffusivity (m^2 s^{-1}), and x and y are space coordinates. V is defined as

$$V = \theta V_w(\rho c)_l/\rho c = J(\rho c)_l/\rho c \tag{2}$$

where θ is the soil volumetric water content (m^3 m^{-3}), V_w is the pore water velocity (m s^{-1}), ρc is the volumetric heat capacity of the soil (J m^{-3} °C^{-1}), and $(\rho c)_l$ is the volumetric heat capacity of water (J m^{-3} °C^{-1}).

[7] *Ren et al.* [2000] presented a solution for equation (1) corresponding to a heat pulse produced by an infinite line source in an infinite, homogeneous, porous media through which water is flowing uniformly

$$T(x, y, t) = \frac{q}{4\pi\lambda}\int_0^t s^{-1}\exp\left(-\frac{(x - Vs)^2 + y^2}{4\alpha s}\right)ds \qquad 0 < \mathrm{t} \le \mathrm{t}_0 \tag{3}$$

$$T(x, y, t) = \frac{q}{4\pi\lambda}\int_{t-t_0}^t s^{-1}\exp\left(-\frac{(x - Vs)^2 + y^2}{4\alpha s}\right)ds \qquad \mathrm{t} > \mathrm{t}_0 \tag{4}$$

where q is the heating power (W m^{-1}), t_0 is the heat pulse duration (s), and where $\lambda = \alpha\,\rho c$ is the thermal conductivity (W m^{-1} °C^{-1}). This solution is based on the assumption that conductive heat transfer dominates over convective heat transfer. In other words, local thermal equilibrium between the liquid and solid phases is always maintained.

[8] Correspondingly, the temperature increase at a distance x_d (m) directly downstream from the line source is

$$T_d(t) = \frac{q}{4\pi\lambda}\int_0^t s^{-1}\exp\left(-\frac{(x_d - Vs)^2}{4\alpha s}\right)\mathrm{ds} \qquad 0 < t \le t_0 \tag{5}$$

$$T_d(t) = \frac{q}{4\pi\lambda}\int_{t-t_0}^t s^{-1}\exp\left(-\frac{(x_d - Vs)^2}{4\alpha s}\right)ds \qquad \mathrm{t} > \mathrm{t}_0 \tag{6}$$

and the temperature increase at a distance x_u (m) directly upstream from the line source is

$$T_u(t) = \frac{q}{4\pi\lambda}\int_0^t s^{-1}\exp\left(-\frac{(x_u + Vs)^2}{4\alpha s}\right)\mathrm{ds} \qquad 0 < \mathrm{t} \le \mathrm{t}_0 \tag{7}$$

$$T_u(t) = \frac{q}{4\pi\lambda}\int_{t-t_0}^t s^{-1}\exp\left(-\frac{(x_d + Vs)^2}{4\alpha s}\right)ds \qquad \mathrm{t} > \mathrm{t}_0 \tag{8}$$

Figure 2 shows typical temperature increase versus time curves generated using equations (5)–(8).

2.2. Difference of Downstream and Upstream Temperature Increases

[9] *Ren et al.* [2000] chose to focus their analysis on the dimensionless temperature difference (DTD) defined by

$$DTD = \frac{4\pi\lambda(T_d - T_u)}{q} \tag{9}$$

This time-dependent temperature difference is a function of x_d, x_u, t_0, q, and the thermal properties of the soil. Figure 3a shows the heat pulse signal of Figure 1 converted to DTD. The maximum value of the DTD (MDTD) is given by

$$\mathrm{MDTD} = \int_{t_m-t_0}^{t_m} s^{-1}\left\{\exp\left[-\frac{(x_d - Vs)^2}{4\alpha s}\right] - \exp\left[-\frac{(x_u + Vs)^2}{4\alpha s}\right]\right\}ds \tag{10}$$

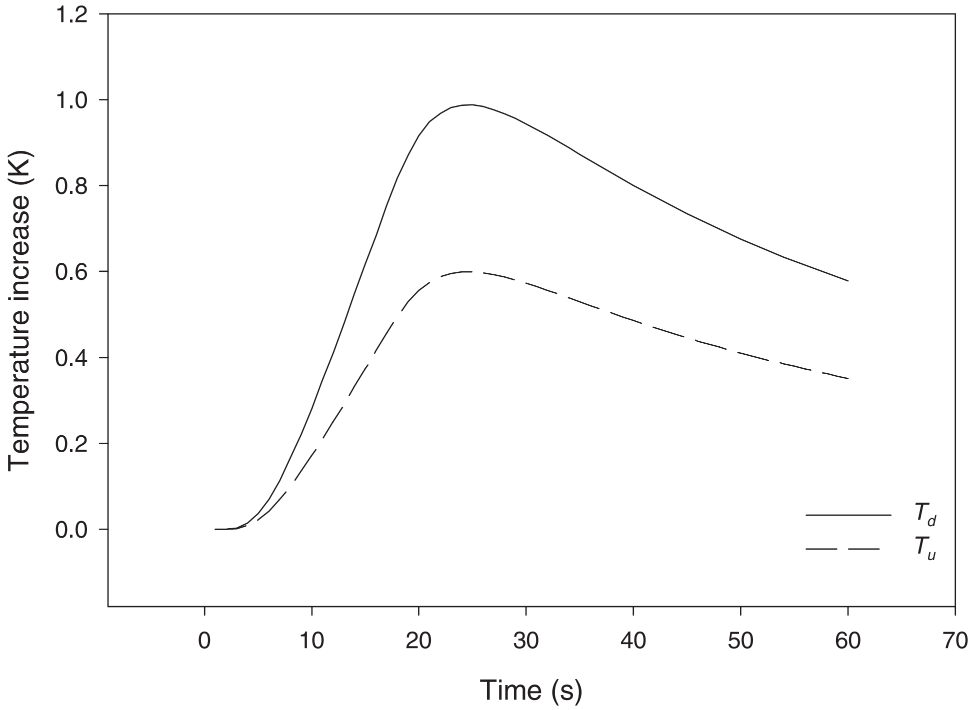

Figure 2. Theoretical heat pulse signal downstream (T_d) and upstream (T_u) of the heater for $V = 5 \times 10^{-5}$ m s^{-1}, $\alpha = 6 \times 10^{-7}$ m^2 s^{-1}, $\lambda = 1.8$ W m^{-1} K^{-1}, $x_0 = 0.006$ m, $q = 50$ W m^{-1}, and $t_0 = 15$ s.

where t_m is the time at which DTD reaches a maximum. By evaluating equation (10) for V on the order of 10^{-5} m s^{-1}, *Ren et al.* [2000] found a nearly linear relationship between V and MDTD. When the soil thermal properties are known, the V necessary to produce the measured MDTD can be found using equation (10). V can then be converted to J using equation (2).

2.3. Ratio of Downstream and Upstream Temperature Increases

[10] The solution described by equations (5)–(8) contains various integrals over time, and it is difficult to evaluate the integrals in order to deduce exact equations describing the temperature variations. By taking the partial derivative of temperature with respect to time, however, we can eliminate the integrals. Differentiating equations (5), (6), (7), and (8) using Leibnitz's rule, we find

$$\frac{\partial T_d(t)}{\partial t} = \frac{q}{4\pi\lambda t}\exp\left(-\frac{(\mathrm{x_d} - Vt)^2}{4\alpha t}\right) \qquad 0 < \mathrm{t} \le \mathrm{t_0} \qquad (11)$$

$$\frac{\partial T_d(t)}{\partial t} = \frac{q}{4\pi\lambda}\left[\frac{1}{\mathrm{t}}\exp\left(-\frac{(x_d - Vt)^2}{4\alpha t}\right) - \frac{1}{t - t_0}\exp\left(-\frac{(x_d - V(t - t_0))^2}{4\alpha(t - t_0)}\right)\right] \qquad t > t_0 \qquad (12)$$

$$\frac{\partial T_u(t)}{\partial t} = \frac{q}{4\pi\lambda t}\exp\left(-\frac{(\mathrm{x_u} - Vt)^2}{4\alpha t}\right) \qquad 0 < \mathrm{t} \le \mathrm{t_0} \qquad (13)$$

$$\frac{\partial T_u(t)}{\partial t} = \frac{q}{4\pi\lambda}\left[\frac{1}{\mathrm{t}}\exp\left(-\frac{(x_u + Vt)^2}{4\alpha t}\right) - \frac{1}{t - t_0}\exp\left(-\frac{(x_u - V(t - t_0))^2}{4\alpha(t - t_0)}\right)\right] \qquad t > t_0 \qquad (14)$$

Now dividing equation (11) by equation (13) for the heating period gives

$$\frac{\partial T_d(t)}{\partial T_u(t)} = \exp\left(\frac{1}{4\alpha t}\left[-x_d^2 + x_u^2 + 2Vt(x_u + x_d)\right]\right) \qquad 0 < t \le t_0 \qquad (15)$$

When $x_d = x_u = x_0$, equation (15) becomes

$$\frac{\partial T_d(t)}{\partial T_u(t)} = \exp\left(\frac{x_0 V}{\alpha}\right) \qquad 0 < \mathrm{t} \le \mathrm{t_0} \qquad (16)$$

Likewise, dividing equation (12) by equation (14) for the time after the heating period gives

$$\frac{\partial T_d(t)}{\partial T_u(t)} = \exp\left(\frac{-x_d^2 + x_u^2 + 2Vt(x_d + x_u)}{4at}\right) \times \frac{\frac{1}{\mathrm{t}} - \frac{1}{t-t_0}\exp\left(\frac{-x_d^2 t_0 + V^2 t t_0 (t-t_0)}{4\alpha t(t-t_0)}\right)}{\frac{1}{\mathrm{t}} - \frac{1}{t-t_0}\exp\left(\frac{-x_u^2 t_0 + V^2 t t_0 (t-t_0)}{4\alpha t(t-t_0)}\right)} \qquad t > t_0 \qquad (17)$$

When $x_d = x_u = x_0$, equation (17) becomes

$$\frac{\partial T_d(t)}{\partial T_u(t)} = \exp\left(\frac{x_0 V}{\alpha}\right) \qquad t > t_0 \qquad (18)$$

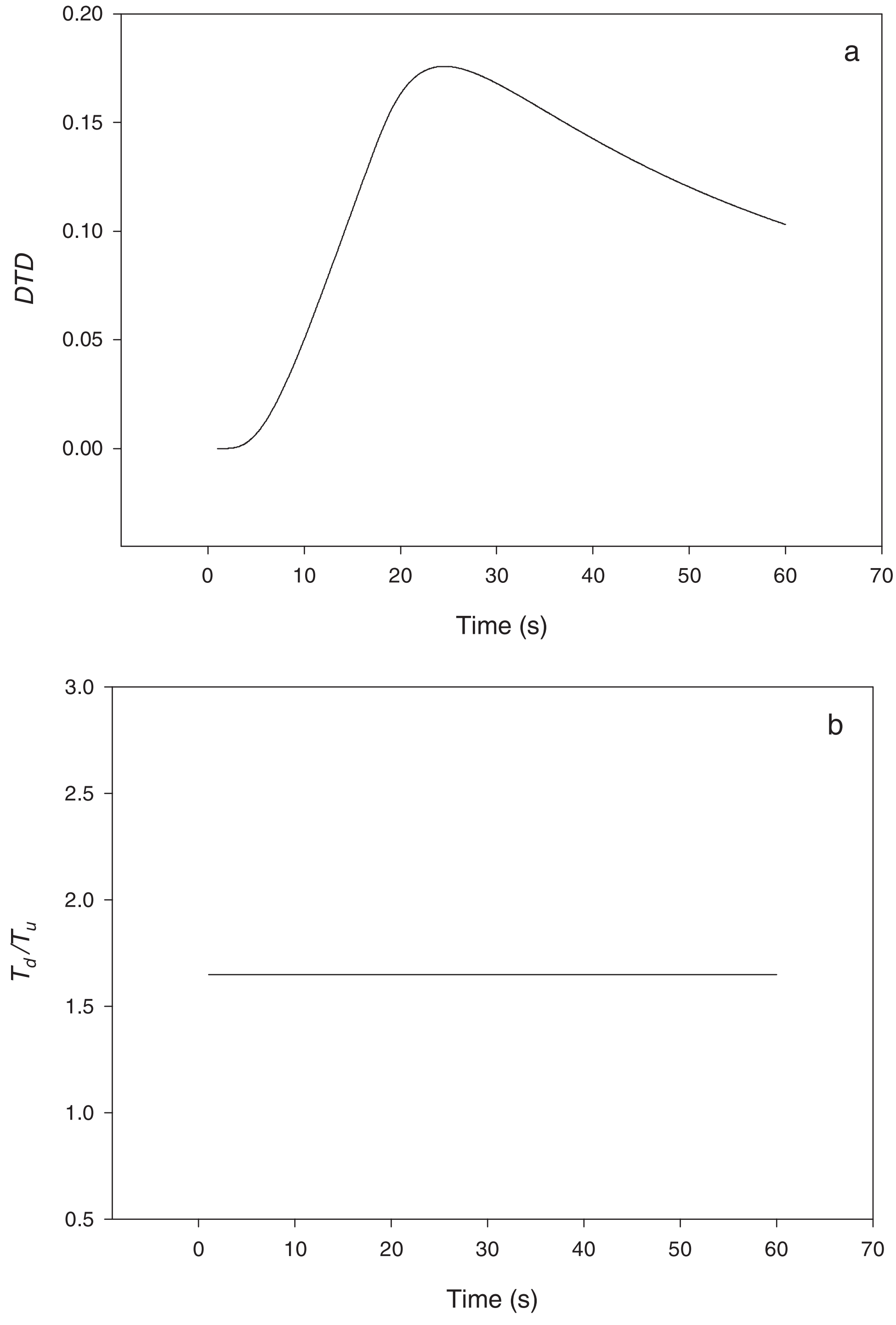

Figure 3. Heat pulse signals from Figure 1 (a) converted to DTD and (b) converted to T_d/T_u.

Equations (16) and (18) indicate that the ratio of the time derivatives of the temperature increases at the upstream and downstream locations is only a function of V, x_0, and α and is independent of time when $x_d = x_u = x_0$. Now, integrating equation (16) from 0 to t ($t \leq t_0$) and integrating equation (18) from t_0 to t ($t > t_0$) gives

$$\frac{T_d}{T_u} = \exp\left(\frac{x_0 V}{\alpha}\right) \tag{19}$$

Equation (19) demonstrates that when $x_d = x_u = x_0$ the ratio of the downstream temperature increase to the upstream temperature increase (T_d/T_u) is independent of time. (An alternative derivation of equation (19) based on the equations of *Kluitenberg and Warrick* [2001] is possible.) When $x_d = x_u = x_0$, T_d/T_u is only a function of x_0, V, and α, and, unlike MDTD, is independent of t_0, q, and λ. Figure 3b shows the heat pulse signal from Figure 1 converted to T_d/T_u.

[11] When $x_d \neq x_u$, T_d/T_u is time dependent but approaches a constant value. In fact, as $t \rightarrow \infty$ equation (17) approaches the simple limit

$$\frac{T_d}{T_u} = \exp\left[\frac{(x_d + x_u)V}{2\alpha}\right] \quad (20)$$

The ratio of derivatives on the left hand side of equation (17) has been replaced by the ratio T_d/T_u using L'hopital's rule. Logically, the x_0 in the exponential term of equation (19) is replaced by the arithmetic average of the distances x_d and x_u when $x_d \neq x_u$.

2.4. Time Corresponding to the Maximum Temperature Increase

[12] Next, we consider the time corresponding to the maximum temperature increase at the upstream and downstream positions. The temperature versus time curves produced using the method of *Ren et al.* [2000] exhibit a global maximum, and the functions of temperature with time described by equation (6) and (8) are differentiable for $t > t_0$. Therefore equations (12) and (14) should equal zero at the time corresponding to the maximum temperature increase. Let t_d be the time corresponding to the maximum temperature increase at the downstream position, and let t_u be the time corresponding to the maximum temperature increase at the upstream position. Setting equation (12) equal to zero and rearranging gives

$$\frac{t_d - t_0}{t_d} = \exp\left(\frac{-(x_d - V(t_d - t_0))^2}{4\alpha(t_d - t_0)}\right)\exp\left(\frac{(x_d - Vt_d)^2}{4at_d}\right) \quad (21)$$

Further rearrangement yields the following expression

$$V^2 = \frac{4\alpha}{t_0}\ \ln\left(\frac{t_d - t_0}{t_d}\right) + \frac{x_d^2}{(t_d - t_0)t_d} \quad (22)$$

Following the same procedure equation (14) becomes

$$V^2 = \frac{4\alpha}{t_0}\ln\left(\frac{t_u - t_0}{t_u}\right) + \frac{x_u^2}{(t_u - t_0)t_u} \quad (23)$$

Equations (22) and (23) reveal that the time corresponding to the maximum temperature increase at upstream and downstream positions is a function of t_0, V, α, and the distance from the heater. Furthermore, equations (22) and (23) must be equal; therefore $t_d = t_u$ when $x_d = x_u$.

2.5. Parameter Estimation Methods

[13] The key parameters in the heat transfer equation are V and α. The preceding analysis facilitates direct methods to estimate these parameters from heat pulse data. If α is known (by prior measurement for example) and if $x_d = x_u$, then V can be calculated from measured temperature increase data by rearranging equation (19) to get

$$V = \frac{\alpha}{x_0} ln\frac{T_d}{T_u} \quad (24)$$

Interestingly, this equation derived for pulsed heating of an infinite line source is identical to an equation derived by *Marshall* [1958] based on instantaneous heating of an infinite line source. If α is known and $x_d \neq x_u$, then V can be calculated by using the measured value of t_d in equation (22) or the measured value of t_u in equation (23). Alternatively, equation (20) may be used if the ratio T_d/T_u reaches a constant value during the measurement period.

[14] If both V and α are unknown and if $x_d = x_u = x_0$, then equation (24) may be used together with equation (22) or (23) to estimate the two parameters. For example, combining equations (22) and (24) and rearranging gives

$$\alpha^2\left(\frac{1}{x_0}\ln\frac{T_d}{T_u}\right)^2 - \alpha\frac{4}{t_0}\ln\left(\frac{t_d - t_0}{t_0}\right) - \frac{x_0^2}{(t_d - t_0)t_d} = 0 \quad (25)$$

Now, α may be determined by finding the positive root of equation (25), and the result can be used in equation (24) to estimate V. If both V and α are unknown and $x_d \neq x_u$, then α is found by setting equation (22) equal to equation (23) and rearranging to give

$$\alpha = \frac{t_0\left(\frac{x_d^2}{(t_d - t_0)t_d} - \frac{x_u^2}{(t_u - t_0)t_u}\right)}{4\ln\left(\frac{(t_u - t_0)}{(t_d - t_0)}\frac{t_u}{t_d}\right)} \quad (26)$$

The α calculated using equation (26) can be used in equation (22) or (23) to find V.

[15] Once V is calculated using one of the procedures above, equation (2) can be used to calculate J, assuming ρc is known. The thermal properties ρc and α can be measured directly under no-flow conditions using the heat pulse method of *Bristow et al.* [1994]. Alternatively, ρc can be estimated based on θ and the soil bulk density.

3. Discussion

[16] One goal of this new analysis was to clarify the relationships between the key variables in the heat pulse technique for measuring J. We will now discuss four interesting relationships revealed by this new analysis and some implications of the relationships. First, equations (19) and (20) reveal that T_d/T_u is a function of a single dimensionless number, Vx_0/α, which is the Peclet number. This number is the ratio of convective heat transfer to conductive heat transfer. Figure 4 shows MDTD and ln (T_d/T_u) as functions of Vx_0/α. The slope of the MDTD versus Vx_0/α relationship depends on the values of x_0, α, t_0, q, and λ, but the slope of ln (T_d/T_u) versus Vx_0/α should always equal one as long as conduction is the dominant mechanism of heat transfer. For computing the results in Figure 4, $x_0 = 0.006$ m, $\alpha = 6.00 \times 10^{-7}$ m^2 s^{-1}, $t_0 = 15$ s, $q = 50$ W m^{-1}, and $\lambda = 1.8$ W m^{-1} K^{-1} were used. For reference, Table 1 shows the relationship between Vx_0/α, V, and J for a typical situation where $x_0 = 0.006$ m, $\alpha = 6.00 \times 10^{-7}$ m^2 s^{-1}, $\rho c = 3.00 \times 10^6$ J m^{-3} K^{-1}, and $(\rho c)_l = 4.17 \times 10^6$ J m^{-3} K^{-1}.

[17] In formulating the general solution to equation (1) we have assumed that conduction is the dominant mechanism of heat transfer. As mentioned previously the Peclet number, Vx_0/α, is the ratio of convective heat transfer to conductive heat transfer. Clearly, for conduction to be the dominant mechanism of heat transfer, we must have $Vx_0/\alpha < 1$ (or $T_d/T_u < 2.72$) [*Melville et al.*, 1985]. This condition clarifies the theoretical upper limit on the magnitude of V measurable

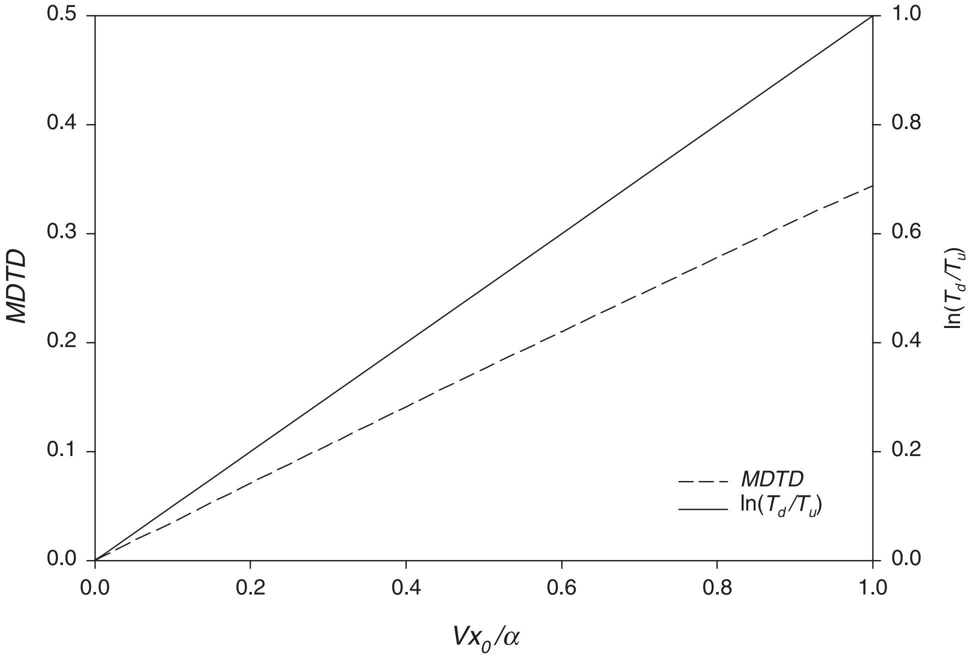

Figure 4. MDTD and ln (T_d/T_u) as functions of Vx_0/α for $\alpha = 6 \times 10^{-7}$ m^2 s^{-1}, $\lambda = 1.8$ W m^{-1} K^{-1}, $x_0 = 0.006$ m, $q = 50$ W m^{-1}, and $t_0 = 15$ s.

with a given heat pulse probe geometry. For example, a typical heat pulse probe with sensing needles spaced 0.006 m from the heater in a soil with $\alpha = 6 \times 10^{-7}$ m^2 s^{-1} would need $V < 1 \times 10^{-4}$ m s^{-1} (36 cm hr^{-1}) to avoid violating the thermal equilibrium assumption. Furthermore, equations (19) and (20) suggest minimum values of V measurable with a given heat pulse probe geometry. If the temperature measurement circuitry of the heat pulse probe in use has maximum resolution of 0.01°C, if the heating power is selected to create a 1°C temperature increase at the downstream position, and if the probe has sensing needles spaced 0.006 m from the heater and is embedded in a soil with $\alpha = 6 \times 10^{-7}$ m^2 s^{-1}, then the system is theoretically capable of measuring $V > 1 \times 10^{-6}$ m s^{-1} (0.36 cm hr^{-1}). If the temperature can be resolved to within 0.001°C, the system is theoretically capable of measuring $V > 1 \times 10^{-7}$ m s^{-1} (0.036 cm hr^{-1}). These lower limits are nearly identical to the lower limits ($V > 9 \times 10^{-7}$ m s^{-1} and $V > 9 \times 10^{-8}$ m s^{-1}) estimated for the *Ren et al.* [2000] method. In theory, increasing x_0 would further decrease the lower limit of V measurable with the heat pulse technique.

[18] The second interesting fact about the heat pulse technique revealed by this new mathematical analysis is that whenever $x_d = x_u$ the maximum temperature increases at the upstream and downstream positions occur simultaneously regardless of the magnitude of V. This somewhat counterintuitive result is predicted by equations (22) and (23). The heat pulse signal travels just as rapidly upstream as it does downstream; only the magnitude of the signal is decreased in the upstream direction.

[19] A third interesting fact revealed by this mathematical analysis is that when $x_d \neq x_u$, T_d/T_u is time dependent but approaches a constant value. In practice it is very difficult to construct heat pulse probes with exactly equal spacings from the heater to the upstream and downstream needles. Equation (20) suggests that data from a heat pulse probe with slight differences between x_d and x_u can still be analyzed in a simple manner. For example, consider a situation where x_d = 0.006 m, x_u = 0.0054 m (10% less than x_d), $t_0 = 15$ s, $\alpha = 6 \times 10^{-7}$ m^2 s^{-1}, and $V = 2.5 \times 10^{-5}$ m s^{-1}. In this case, equation (20) predicts that when t $\rightarrow \infty$, ln (T_d/T_u) $\rightarrow$ 0.238. Numerical integration of equations (5) through (8) indicates that ln (T_d/T_u) is within 10% of this value after 128 s. In other words, at times greater than ~2 min measured values of T_d/T_u could be used in equation (20) to estimate V (or J) within 10%. As V increases or as the difference between x_d and x_u decreases then the amount of time required for T_d/T_u to reach within 10% of its asymptotic value decreases sharply. This result suggests that equation (20) can be useful when slight differences between x_d and x_u are unavoidable. Two potential problems may arise when using equation (20) for long times. First, the magnitudes of T_d and T_u are decreasing with time, so the relative error in the temperature measurements may be greater at long times. Second, the solution we are using is based on the assumption of an infinite line source. This solution may not be appropriate at long times as the finite length of the heater may become important. *Kluitenberg et al.* [1993] have shown that finite heater length effects can

Table 1. Example of the Relationship Between Vx_0/α, V, and J[a]

		J	
Vx_0/α	V, m s^{-1}	m s^{-1}	cm hr^{-1}
0.001	1.00E-07	7.19E-08	0.03
0.005	5.00E-07	3.60E-07	0.13
0.01	1.00E-06	7.19E-07	0.26
0.05	5.00E-06	3.60E-06	1.29
0.1	1.00E-05	7.19E-06	2.59
0.5	5.00E-05	3.60E-05	12.95
1	1.00E-04	7.19E-05	25.90

[a]For $x_0 = 0.006$ m, $\alpha = 6 \times 10^{-7}$ m^2 s^{-1}, $\rho c = 3 \times 10^6$ J m^{-3} K^{-1}, and $(\rho c)_l = 4.17 \times 10^6$ J m^{-3} K^{-1}. Read 1.00E-07 as 1.00×10^{-7}.

be significant for $t > 2$ min in the case of zero convection. Further experimental data and theoretical analysis is necessary to evaluate the significance of these two potential problems in using equation (20) at long times.

[20] The fourth noteworthy finding of this mathematical analysis is a new insight into the relationship between MDTD and J. *Ren et al.* [2000] found a nearly linear relationship between MDTD and J for V on the order of 10^{-5} m s^{-1}, but they were unable to explicitly state the form of the relationship due to the complexity of their solution (equation (10)). Hypothetically, the relationship might look like

$$J = B \cdot MDTD = B \cdot \frac{4\pi\lambda}{q}[T_d - T_u]_{t=t_m} \tag{27}$$

where B is some unknown parameter that depends on x_0, t_0, and α in an unclear manner. Now consider the relationship between J and T_d/T_u. Substituting equation (2) into equation (19) and rearranging gives

$$J = \frac{\lambda}{x_0(\rho c)_l} ln\left(\frac{T_d}{T_u}\right) \tag{28}$$

Hence the ambiguous relationship between J and (T_d -T_u) in equation (27) can now be replaced by the simple, explicit relationship between J and T_d/T_u in equation (28). The explicit form of equation (28) may make it very useful for designing and implementing calibration procedures for this technique.

[21] The second goal of our mathematical analysis was to simplify the procedure for calculating J from heat pulse measurements. We will now explain how we have met this goal. First, consider the case where the soil thermal properties are known (from prior measurement or from models). In this case, to calculate J from heat pulse measurements using the *Ren et al.* [2000] equations requires a numerical integration routine coupled with a nonlinear regression routine. With these two routines one can solve for the J required to produce the measured MDTD. Using the *Kluitenberg and Warrick* [2001] equations requires a similar process except that the integral is replaced by an infinite series, so numerical integration is unnecessary. Due to the complexity of the infinite series, however, these calculations are still cumbersome. In contrast, J can be calculated directly by using equation (28) when $x_d = x_u$ or by using equation (20) or (22) or (23) together with equation (2) when $x_d \neq x_u$. These are all simple, explicit equations that can easily be evaluated using a hand calculator or a datalogger. Note that when equation (28) is used the only soil thermal property required to calculate J is λ.

[22] Now, consider the second case where the soil thermal properties are not known a priori. In this case it is not clear that it is possible to calculate J using the *Ren et al.* [2000] or *Kluitenberg and Warrick* [2001] equations. However, the results of our analysis show that we can gain some information from the heat pulse data even without a priori knowledge of the soil thermal properties. equation (25) can be used to calculate α, and equation (24) can then be used to calculate V when $x_d = x_u$. If $x_d \neq x_u$, then α can be calculated from equation (26), and then V can be calculated from equation (22) or (23). Once V is known, only an estimate of ρc is needed to calculate J.

4. Conclusion

[23] *Ren et al.* [2000] established a new method to estimate J from heat pulse data using MDTD. That method has some disadvantages, however. One is that the calculation requires numerical integration, which is not trivial and may induce some error. The series solution presented by *Kluitenberg and Warrick* [2001] eliminates the need for numerical integration but is still quite complicated to use. The new analysis presented in this paper greatly simplifies the determination of J from heat pulse data compared with the procedures of *Ren et al.* [2000] and *Kluitenberg and Warrick* [2001]. Using equation (28) only three parameters (x_0, λ, and $(\rho c)_l$) are needed to quickly calculate J from heat pulse data.

[24] A second disadvantage of the previous methods is that only a single data point, the MDTD, is used; therefore the measurement accuracy of that single point will directly affect the calculated J. The new analysis presented in this paper enables an average value of T_d/T_u over an appropriate time interval to be used with equation (19) or (20) to calculate V, which can be converted to J. This averaging can reduce the influence of measurement error in a single data point. If both V and α are unknown, then knowledge of t_d or t_u (when $x_d = x_u$) or both (when $x_d \neq x_u$) is required to calculate the parameters. In this case, reliance on a single measurement point cannot be avoided. The previous methods do not provide any method to calculate V if α is unknown, however, so this new analysis is still an improvement.

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