Analytical solutions for the propagation of heat pulses with temperature gradient length-dependent diffusion coefficient

Article *in* Plasma Physics and Controlled Fusion · August 2002

DOI: 10.1088/0741-3335/44/8/301

2 authors:

F. Imbeaux
Atomic Energy and Alternative Energies Commission

Xavier Garbet
Atomic Energy and Alternative Energies Commission

Analytical solutions for the propagation of heat pulses with temperature gradient length dependent diffusion coefficient

F. Imbeaux, X. Garbet

Association EURATOM-CEA, CEA/DSM/DRFC
CEA-Cadarache, 13108 Saint-Paul-lez-Durance (France)

Published in : Plasma Phys. Contr. Fusion 44 (2002) 1425

e-mails : imbeaux@drfc.cad.cea.fr, garbet@drfc.cad.cea.fr

Is it possible to identify unambiguously the $\chi(T, \quad T)$ dependence from modulated heating experiments ?

Analytical solutions for the heat pulse propagation allow :

- faster comparison with experiment than numerical simulation (transport code)
- highlight characteristic features of the solutions (parametric dependences, ...)
- identify the relevant transport model formulation (best fit of experimental results)

Solving the time-averaged and linearised heat transport equation for models of the type $\chi = \chi_0 T^{\mu} G\left(\frac{T}{T}\right)$

- Assumptions : flat density profile, negligible electron-ion coupling, no source/sink term in the region of integration.
- Small temperature modulation ⇨ linearisation of the problem $T = \bar{T} + \tilde{T}$

  time-averaged modulated
- Time Fourier transform of the linearised transport equation ⇨ amplitude and phase of $\tilde{T}$
- ***Reformulation of the heat transport equation as a function of temperature and temperature gradient length related variables***

$$F = -(\mu+1)\ln\left(\frac{T}{T_0}\right) \qquad P = \quad F = -(\mu+1)\frac{T}{T}$$

(usual configuration space variables not convenient for temperature gradient length dependent models)

Results

$$\chi = \chi_0 T^{\mu} G\left(\frac{T'}{T}\right)$$

Slab geometry

- Reformulation can be carried out without specifying the function *G*
- Exact solutions are found in the case $G = \left(\frac{-\partial_x T}{T}\right)^{\alpha}$
- Approximate solutions are found in the case $G = \left(\frac{-\partial_x T}{T}\right)^{\alpha} \left(\frac{-\partial_x T}{T} - \kappa\right)^{\beta}$ (to the first order in $\partial_x T / T - \kappa$), valid for strongly stiff profiles

Cylindrical geometry

- Reformulation can be carried out in the case $G = \left(\frac{-\partial_\rho T}{T}\right)^{\alpha}$
- Werner-Kramers-Brillouin solutions are found in this case

Detailed results in slab geometry

$$\chi = \chi_0 T^{\mu} \left(\frac{-\partial_x T}{T} \right)^{\alpha}$$

- Heat transport equation $\frac{3}{2} n \partial_t T - \partial_x \left(n \chi \partial_x T \right) = 0$
- Reformulation (change of variables)

$$F = -(\mu + 1) \ln \left(\frac{T}{T_0} \right) \qquad P = -(\mu + 1) \frac{\partial_x T}{T}$$

- Time-averaged profile (exact solution)

$$\bar{T} = \left(\frac{1}{\alpha + 1} \right)^{\frac{\alpha+1}{\mu+1}} \bar{\phi}^{\frac{1}{\mu+1}} (D - x)^{\frac{\alpha+1}{\mu+1}}$$

$$\bar{\phi} = \left(\frac{\bar{T}}{T_0} \right)^{\mu+1} \bar{P}^{\alpha+1} = cst$$

$$\bar{P} = \frac{\alpha + 1}{D - x}$$

- Modulated profile (exact solution)

$$\tilde{F}(\bar{P}) = A_1 \bar{P}^{\frac{1}{2}} C_{\nu}(\lambda \bar{P}^{\frac{1}{2\nu}}) + A_2 \bar{P}^{\frac{1}{2}} D_{\nu}(\lambda \bar{P}^{\frac{1}{2\nu}})$$

$$\lambda = \pm 2 \nu e^{i\frac{\pi}{4}} \sqrt{\Omega(\alpha + 1)}\, \bar{\phi}^{\frac{1}{2(\mu+1)}}$$

$$\Omega = \frac{\frac{3}{2} \omega (\mu + 1)^{\alpha}}{\chi_0 T_0^{\mu} \bar{\phi}}$$

$$\nu = -\frac{\mu + 1}{\mu + \alpha + 2}$$

C_{ν} and D_{ν} are two Bessel functions

Detailed results in slab geometry

$$\chi = \chi_0 T^{\mu} \left(\frac{-\partial_x T}{T} \right)^{\alpha} \left(\frac{-\partial_x T}{T} - \kappa \right)^{\beta}$$

- Heat transport equation $\frac{3}{2} n \partial_t T - \partial_x \left(n \chi \partial_x T \right) = 0$
- Reformulation (change of variables)

$$F = -(\mu + 1) \ln \left(\frac{T}{T_0} \right) \qquad P = -(\mu + 1) \frac{\partial_x T}{T}$$

- Time-averaged profile (exact implicit solution)

$$-(\alpha + 1) \frac{1}{\bar{P}} + \frac{\beta}{\kappa} \ln \left(\frac{\bar{P} - \kappa}{\bar{P}} \right) = x + cste$$

- Modulated profile (solution to first order in $\bar{P} - \kappa$)

$$\tilde{F}(\bar{P}) = A_1 (\bar{P} - \kappa)^{\frac{1}{2}} C_{\nu} \left(\lambda (\bar{P} - \kappa)^{\frac{1}{2\nu}} \right) + A_2 (\bar{P} - \kappa)^{\frac{1}{2}} D_{\nu} \left(\lambda (\bar{P} - \kappa)^{\frac{1}{2\nu}} \right) \qquad \bar{\phi} = \left(\frac{\bar{T}}{T_0} \right)^{\mu + 1} \bar{P}^{\alpha + 1} = cst$$

$$\lambda = \pm 2 \nu e^{i \frac{\pi}{4}} \sqrt{\Omega \beta \kappa}^{\,-\frac{2\mu + \alpha + 3}{2(\mu + 1)}} \bar{\phi}^{\frac{1}{2(\mu + 1)}} \qquad \Omega = \frac{\frac{3}{2} \omega (\mu + 1)^{\alpha}}{\chi_0 T_0^{\mu} \bar{\phi}}$$

$$\nu = \frac{\mu + 1}{\mu - \beta + 1}$$

C_{ν} and D_{ν} are two Bessel functions

Detailed results in cylindrical geometry $\chi = \chi_0 T^{\mu} \left(\frac{-\partial_\rho T}{T} \right)^{\alpha}$

- Results can be used for circular shaped plasmas (e.g. Tore Supra tokamak)
- Heat transport equation $\frac{3}{2} n\rho \partial_t T - \partial_\rho \left(n\rho\chi\partial_\rho T \right) = 0$
- Reformulation (change of variables)

$$F = -(\mu + 1)\ln\left(\frac{T}{T_0}\right) \qquad P' = -(\mu + 1)\frac{\partial_{x'} T}{T} \qquad x' = \frac{\alpha + 1}{\alpha}\rho^{\frac{\alpha}{\alpha+1}}$$

- Time-averaged profile (exact solution)

$$\bar{T} = \left(\frac{1}{\alpha + 1}\right)^{\frac{\alpha+1}{\mu+1}} \bar{\phi}'^{\frac{1}{\mu+1}} (D' - x')^{\frac{\alpha+1}{\mu+1}}$$

$$\bar{\phi}' = \frac{P_H}{4\pi^2 R_0 n} \frac{1}{T_0^{\mu+1}} \frac{(\mu+1)^{\alpha+1}}{\chi_0}$$

$$\bar{P}' = \frac{\alpha + 1}{D' - x'}$$

- Modulated profile (WKB solution)

$$\tilde{T} = -\frac{1}{\mu + 1}\bar{T}(D'\bar{P}' - 1)^{-\frac{\alpha+2}{4\alpha}} \bar{P}'^{\frac{\gamma}{2}} \left(A_1 e^{iS} + A_2 e^{-iS} \right)$$

$$\gamma = \frac{4\mu\alpha + \alpha^2 + 5\alpha + 2\mu + 2}{2\alpha(\mu + 1)}$$

$$\Omega' = \frac{\frac{3}{2}\omega(\mu+1)^{\alpha}}{\chi_0 T_0^{\mu} \bar{\phi}'}$$

$$S = \frac{1+i}{\sqrt{2}} \sqrt{\Omega'(\alpha + 1)}\,(-\alpha)^{\frac{\alpha+2}{2\alpha}} \bar{\phi}'^{\frac{1}{2(\mu+1)}} \frac{\bar{P}'^{1-\gamma}}{1-\gamma} \times {}_2F_1\left(1-\gamma, -\frac{\alpha+2}{2\alpha}; 2-\gamma; D'\bar{P}'\right)$$

Frequency dependence (cylindrical geometry $\chi = \chi_0 T^{\mu} \left(\frac{-\partial_\rho T}{T} \right)^{\alpha}$)

- The slope of the temperature phase profile is proportionnal to the square root of the frequency, for any model parameters.
- The temperature amplitude profile verifies :

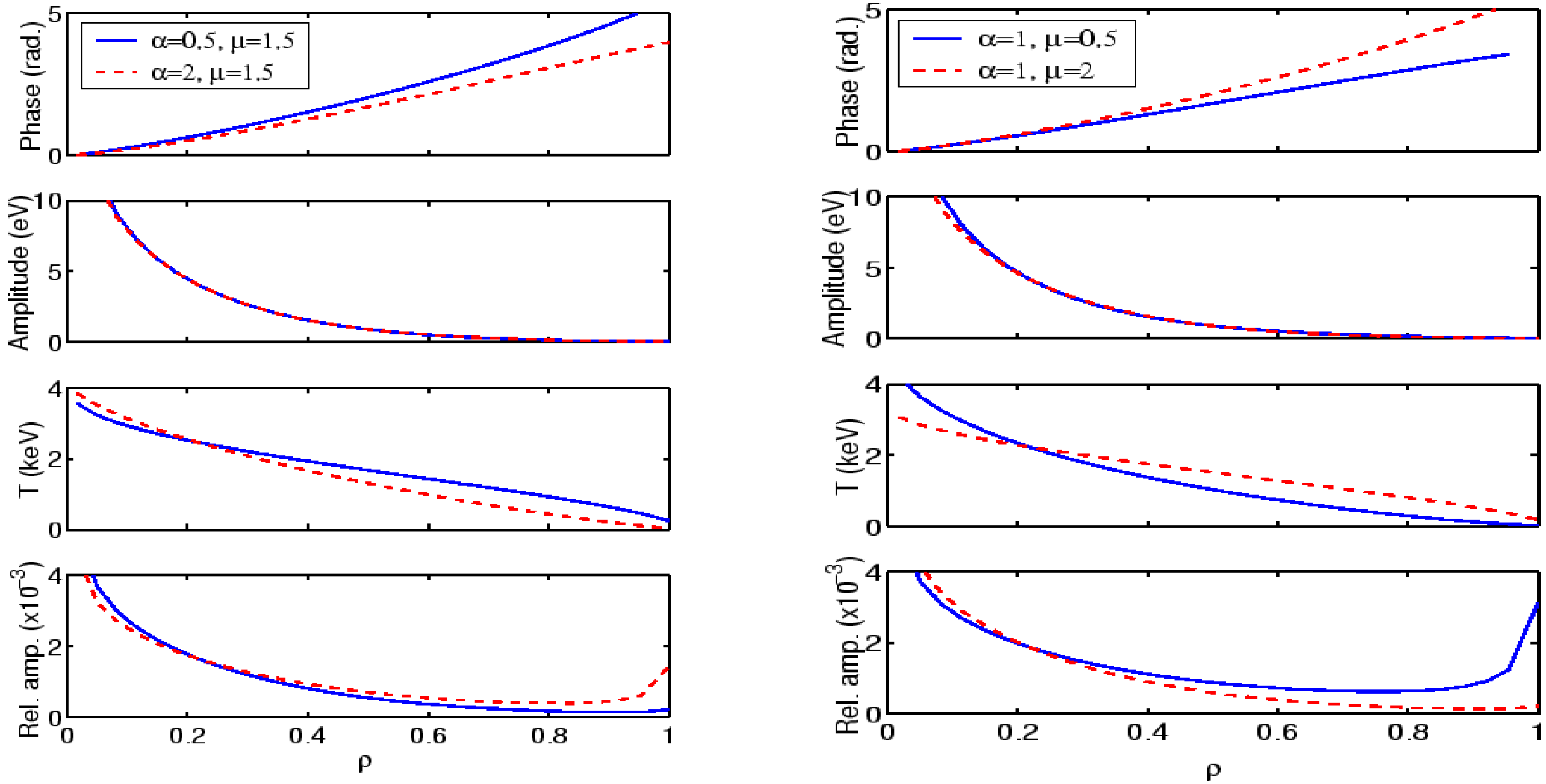
$$\frac{\log\left(\left(|\tilde{T}|(\rho_2,\omega_2)|\tilde{T}|(\rho_1,\omega_1)\right)/\left(|\tilde{T}|(\rho_2,\omega_1)|\tilde{T}|(\rho_1,\omega_2)\right)\right)}{\log\left(\left(|\tilde{T}|(\rho_2,\omega_3)|\tilde{T}|(\rho_1,\omega_1)\right)/\left(|\tilde{T}|(\rho_2,\omega_1)|\tilde{T}|(\rho_1,\omega_3)\right)\right)} = \frac{\sqrt{\omega_2}-\sqrt{\omega_1}}{\sqrt{\omega_3}-\sqrt{\omega_1}}$$

Can these relations help in determining the transport model ?

- They do not involve α nor μ ⇨ cannot be used to determine them
- Numerical simulations show that a model of the class $\chi = \chi_0 T^{\mu} \left(\frac{-\partial_\rho T}{T} - \kappa \right)^{\beta}$ deviates by only 4 % from the amplitude criterion, which is likely within the experimental errorbars.
- Significant deviation from the above properties may indicate dominant convective transport.

Dependence on α and μ (cylindrical geometry $\chi = \chi_0 T^{\mu} \left(\frac{\partial_\rho T}{T} \right)^{\alpha}$)

Power modulated at $\rho = 0$



- Almost no difference on the absolute temperature amplitude profile (renormalised)
- Significant variations on the other profiles, but no spectacular change

Identification of α and μ from a modulation experiment

(cylindrical geometry $\chi = \chi_0 T^{\mu}\left(\frac{-\partial_\rho T}{T}\right)^{\alpha}$)

- ***Analytical solution used to best fit an experimental data set*** (here provided by an Astra simulation of central ECRH heating, 10 % of the power modulated)

- Distance to the experimental data must involve both modulated and time-averaged profiles :

$$N = \left\| \left|\overline{P'} - \overline{P'}_{\mathrm{exp}}\right| + \left|\log\left|\tilde{T}\right| - \log\left|\tilde{T}_{\mathrm{exp}}\right|\right| + \left|\varphi_{\tilde{T}} - \varphi_{\tilde{T}\,\mathrm{exp}}\right| \right\|$$

- Bi-parametric best fit on α and μ (Nelder-Mead algorithm). The value of χ_0 is constrained by the relation (global power balance + heat flux conservation) :

$$\chi_0 = \frac{P_H}{4\pi^2 R_0 n}\,\frac{1}{\overline{T}^{\mu+1}}\left(\frac{\mu+1}{\overline{P'}}\right)^{\alpha+1}$$

- The uniformity of χ_0 evaluated using this relation has to be verified a posteriori.

- Fitting procedure yielding bad accuracy and/or non constant χ_0 ⇨ dataset not consistent with a model of the type $\chi = \chi_0 T^{\mu}\left(\frac{-\partial_\rho T}{T}\right)^{\alpha}$

Identification of α and μ from a modulation experiment

(cylindrical geometry $\chi = 0.2T^{2}\left(\frac{-\partial_{\rho} T}{T}\right)$)

Identification results as function of the initial guess

Initial guess [α μ]	Solution α	Solution μ	Solution χ_0 ($m^{2+\alpha} keV^{-\mu} s^{-1}$)
[1 2] (exact values in the simulation)	0.97	2.03	0.20
[1.5 1.5]	0.93	2.00	0.21
[2 1]	1.10	2.10	0.17
[0.5 2.5]	1.14	2.12	0.17

Table 1 : First column : initial value of the couple [α, μ]. Second to fourth columns : solution of the identification procedure (using norm N).

Identification results as function of the noise to signal ratio

Noise relative amplitude	Maximum deviation observed on α or μ
± 5 %	± 10 %
± 10 %	± 20 %
± 15 %	± 25 %
± 20 %	± 66 %

Table 2 : Maximum deviation observed on α or μ, as a function of the noise relative amplitude, using norm N and [1.5 1.5] as initial guess for [α μ].

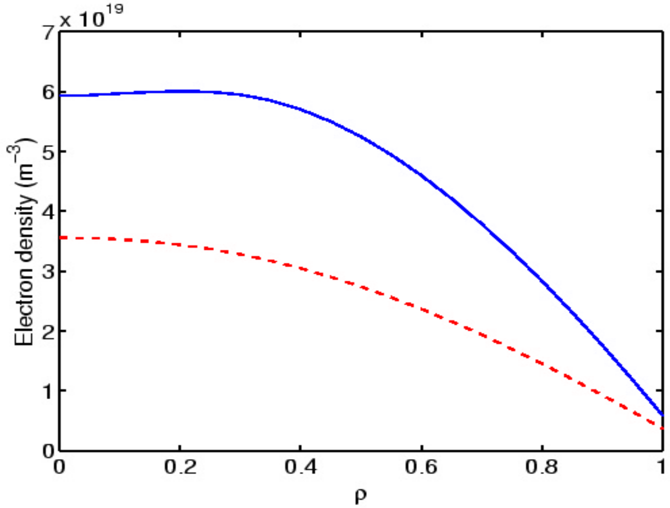
Identification of α and μ from a modulation experiment

(cylindrical geometry $\chi = 0.2T^2\left(\frac{-\partial_\rho T}{T}\right)$)

Identification results with non-uniform density profile

	Low density				High density			
Region of analysis	$\frac{\Delta n}{n}$	α	μ	χ_0 ($m^{2+\alpha}keV^{-\mu}s^{-1}$)	$\frac{\Delta n}{n}$	α	μ	χ_0 ($m^{2+\alpha}keV^{-\mu}s^{-1}$)
$0.2 < \rho < 0.6$	37 %	1.31	2.94	0.08	26 %	0.98	2.48	0.15
$0.2 < \rho < 0.5$	23 %	1.23	2.85	0.09	14 %	0.97	2.40	0.16
$0.2 < \rho < 0.4$	13 %	1.00	2.73	0.11	5 %	0.91	2.36	0.17

Table 3 : Results of the identification procedure for the low and high density cases (see fig. 5). The identification has been carried out by minimising the norm N over various radial domain (first column). The relative variation of the density $\frac{\Delta n}{n}$ *inside each domain is indicated for each case (low and high density).*



The presence of a density gradient leads to an overestimation of μ

Identification of α and μ from a modulation experiment

(cylindrical geometry $\chi = 0.2T^{2}\left(\frac{-\partial_\rho T}{T}\right)$)

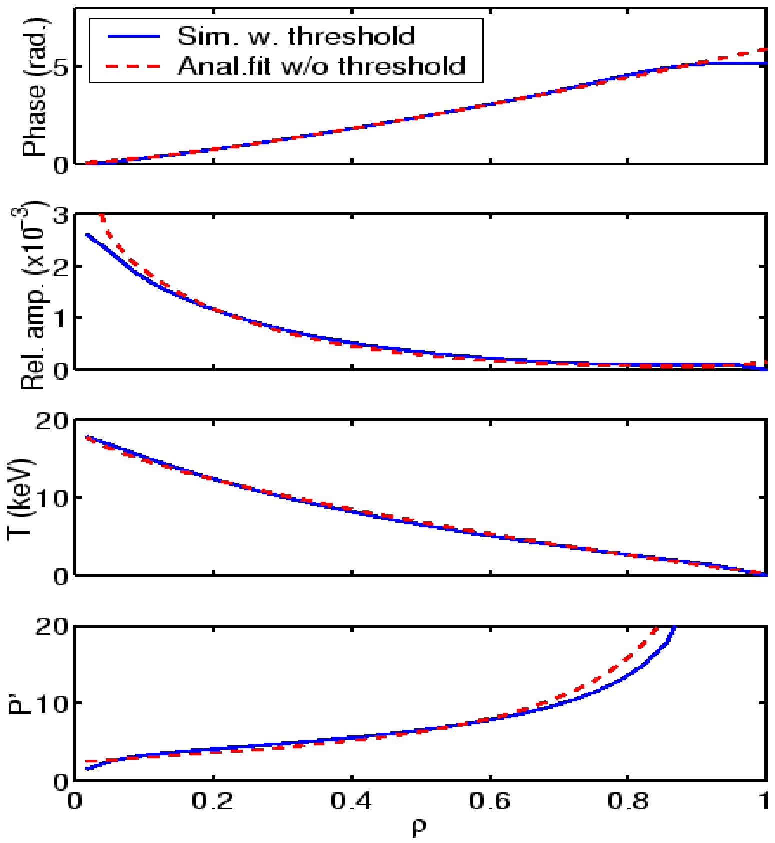
Identification results with non-zero coupling between electrons and ions

Steady heating power (MW)	P_{e-i} (MW)	Solution α	Solution μ	Solution χ_0 ($m^{2+\alpha}keV^{-\mu}s^{-1}$)
0.4	0.37	0.46	0.32	0.71
1	0.98	0.90	0.47	0.49
1.5	1.48	0.98	0.74	0.51
2	1.82	1.20	1.25	0.34
3	1.81	1.13	1.62	0.31
4	1.77	1.18	1.80	0.24
6	1.71	1.12	1.86	0.24
8	1.66	1.10	1.87	0.24

Table 4 : First column : steady heating power. Second column : total electron-ion power exchange. Third to fifth columns : solution of the identification procedure (using norm N).

The identification method can still be used if the steady heating power is at least twice larger than the electron-ion exchanged power

Identification procedure applied in a case with critical gradient length (cylindrical geometry $\chi = 0.1T^{3/2}\left(\frac{\partial_\rho T}{T} - 2\right)$)

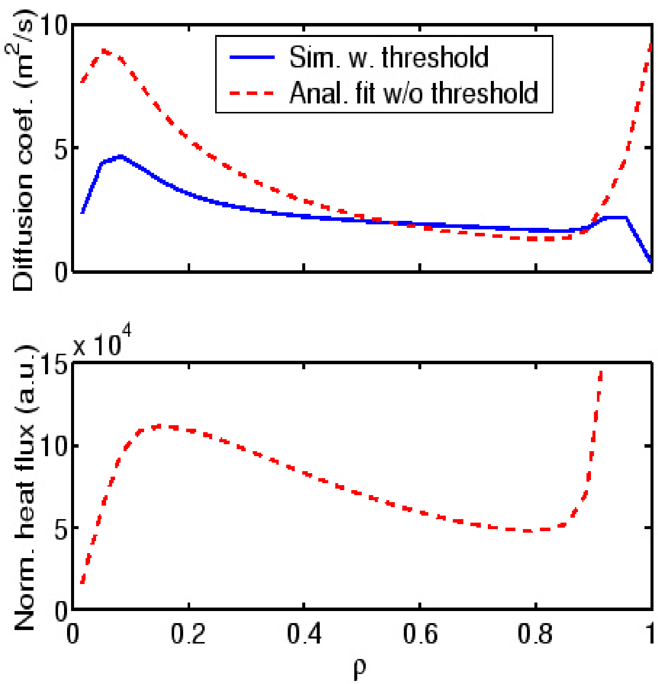


The identification procedure yields the following result :

$$\chi = 6.2\times10^{-3}T^{1.8}\left(\frac{-\partial_\rho T}{T}\right)^{2.05}$$

with rather good accuracy, except concerning the $\overline{P'}$ profile.

Identification procedure applied in a case with critical gradient length (cylindrical geometry $\chi = 0.1 T^{3/2}\left(\frac{\partial_\rho T}{T} - 2\right)$)



$$\bar{\phi}' = \left(\frac{\bar{T}}{T_0}\right)^{\mu+1} \bar{P}'^{\alpha+1}$$ should be uniform

Nevertheless, the solution

$$\chi = 6.2 \times 10^{-3} T^{1.8} \left(\frac{\partial_\rho T}{T}\right)^{2.05}$$

is not consistent with respect to the order of magnitude of χ and to the uniformity of the normalised heat flux.

the fitting procedure allows to detect that the model without threshold is not relevant to the dataset

Conclusions and perspectives

- ***Linearised transport equation reformulated and solved*** in slab geometry for models without and with critical gradient length, in cylindrical geometry without critical gradient length.
- The method may be generalised to obtain semi-analytical solutions for arbitrary geometries and density profiles.
- ***Procedure developped to identify the exponents of models*** without threshold, which best fit an experimental dataset, in cylindrical geometry.
- Can be used to assess wether a specific formulation of the diffusion coefficient is compatible with an experimental dataset.
- ***Next step : apply these solutions and methods to a real modulated heating experiment.*** Modulated ECRH + steady ICRH in the Fast Wave Electron Heating scheme is a promising scenario (central power deposition, weak electron-ion coupling).