Z. Angew. Math. Phys. (2018) 69:150
https://doi.org/10.1007/s00033-018-1042-6

Zeitschrift für angewandte Mathematik und Physik ZAMP

Analytical solutions to a nonlinear diffusion–advection equation

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Abstract. In this paper, we construct some exact and analytical solutions to a nonlinear diffusion and advection model (Pudasaini in Eng Geol 202: 62–73, 2016) using the Lie symmetry, travelling wave, generalized separation of variables, and boundary layer methods. The model in consideration can be viewed as an extension of viscous Burgers equation, but it describes significantly different physical phenomenon. The nonlinearity in the model is associated with the quadratic diffusion and advection fluxes which are described by the sub-diffusive and sub-advective fluid flow in general porous media and debris material. We also observe that different methods consistently produce similar analytical solutions. This highlights the intrinsic characteristics of the flow of fluid in porous material. The nonlinear diffusion and advection is characterized by a gradually thinning tail that stretches to the rear of the fluid and the evolution of forward advecting frontal bore head, in contrast to the classical linear diffusion and advection. Additionally, we compare solutions for the linear and nonlinear diffusion and advection models highlighting the similarities and differences. The analytical solutions constructed in this paper and the existing high-resolution numerical solution presented previously for the nonlinear diffusion and advection model independently support each other. This implies that the exact and analytical solutions constructed here are physically meaningful and can potentially be applied to calculate the complex nonlinear re-distribution of fluid in porous landscape, and debris and porous materials.

Mathematics Subject Classification. 35K55, 76R50, 76S05, 76T99, 22E60, 22E70, 86A05.

Keywords. Analytical solutions, Exact solutions, Nonlinear diffusion–advection, Flow in porous media.

1. Introduction

Fluid flow through porous media is of great theoretical and practical interest in science and technology. It has application in oil exploration, environment-related problems, geophysical and industrial problems such as the flow of liquid or gas through soil and rock (e.g. shell oil extraction), clay, gravel and sand, or through sponge and foam [3,7,8,10,12,32]. Proper understanding of these fluid flows through porous media plays an important role in industrial applications, geotechnical engineering, engineering geology, subsurface hydrology, and natural hazard related phenomena [1,3,4,37,40]. As the fluid flows through the porous and debris media are coupled with the stability of the slope, the subsurface hydrology, and the transportation of chemical substances in porous landscape, they are important in geophysics and engineering. An effective analysis of slope stability, landslide initiation, debris and avalanche deposition morphologies, including seepage of fluid through relatively stationary porous matrix and consolidation, require more accurate and advanced knowledge of fluid flows, which may help to effectively plan early warning strategies in catastrophic landslides, failures in reservoir dams and embankments in geo-disaster-prone areas [15,21,25,28], and deposition processes of subsequent mass flows [24,36,39,42–44].

Recently, Pudasaini [28] derived a novel sub-diffusion and sub-advection model equation for fluid flows through inclined debris materials and porous landscapes with stationary solid matrix and presented some basic analytical solutions for the reduced equation. In this paper, we advance further by constructing several analytical and exact solutions to the full model. We show that a number of important physical phenomena are captured by the new solutions. As in Pudasaini [28], our new results reveal that solutions

Published online: 12 November 2018

Birkhäuser

to the sub-diffusive and sub-advective fluid flow in porous media and porous landscape are fundamentally different from the classical diffusive fluid flow or diffusion of heat, tracer particles and pollutant in fluid. We use the Lie symmetry method, travelling wave, generalized separation of variables, and boundary layer method to construct new analytic solutions. These solutions are in line with the high-resolution numerical solutions (presented in Pudasaini, [28]) and hence provide insights into the intrinsic nature of fluid flow in porous media represented by the considered model. We also discuss special features of the new solutions in detail.

As exact, analytical, and numerical solutions disclose many new and essential physics, the solutions derived and discussed in this paper may find applications in environmental, engineering and industrial fluid flows through general porous media, natural slopes, embankments (e.g. of hydro-electric power reservoirs and dams), and debris deposits. Moreover, analytical and exact solutions to the nonlinear model equation are necessary to gauge the accuracy of numerical solution methods [16–18,28,30]. Furthermore, the results presented here may have applications to the problems related to hydrogeology and environmental pollution remediation.

2. The nonlinear diffusion–advection model

2.1. The model and the underlying physics

The nonlinear diffusion and advection model for fluid flow through the stationary porous landscape and debris material derived by Pudasaini [28] is:

$$\frac{\partial H}{\partial t} - D\frac{\partial^2 H^\alpha}{\partial x^2} + C\frac{\partial H^\alpha}{\partial x} = 0, \tag{1}$$

where $\alpha = 2$, and D and C, respectively, are called the sub-diffusion and sub-advection coefficients (or simply diffusion and advection coefficients). These parameters and expressions emerge analytically in the model development in Pudasaini [28]. Equation (1) can be used to obtain time and spatial evolution of the fluid flow (H) through the porous media, or the porous landscape, and debris material.

Special features of the nonlinear diffusion and advection model (1) have been extensively discussed in Pudasaini [28]. The diffusion and advection coefficients, D and C, respectively, are proportional to the density ratio and the slope induced gravity force components $\cos\zeta$ and $\sin\zeta$, and inversely proportional to the generalized drag coefficient C_{DG}. Moreover, the generalized drag models fluid flows through both the densely and loosely packed porous media. Furthermore, this model (1) has several important physical aspects. First, in the derivation of (1) a relationship between the flow velocity and the pressure gradient can be obtained without extra closure, which is in contrast to the derivation of classical porous media equation where extra closure was needed. Second, the flux exponent $\alpha = 2$ emerges systematically from the underlying mixture model as it characteristically corresponds to the fluid flow through porous and debris material as derived from the two-phase mass flow model [29]. However, in the classical porous media equation (which corresponds to (1) with $C = 0$) α is not usually known, but used as a fit parameter. The model (1) covers both sub-diffusion and sub-advection phenomena that can be applied to both flows in horizontal and inclined configurations. It is derived from a completely new perspective of a mixture model [29] for a more general setting. This model can also be seen as an extension of the sub-diffusion equation that emerges from Darcy's law together with Dupuit's hypothesis [3,11]. Moreover, the sub-diffusion and sub-advection coefficients contain several physical parameters (for example, the generalized drag C_{DG} contains several physical parameters) of the solid matrix and the fluid, whereas a classical Darcy flow contains only the hydraulic conductivity. Also Dupuit-type classical model is a particular case of model (1) (see Pudasaini [28] for details). Third, the advection C is associated with the slope; in particular $C = 0$ is equivalent to $\zeta = 0$. This means that when the debris material is not inclined, there is no gravity to pull the fluid down and hence the fluid can only diffuse, but does not advect. As slope increases (relatively), diffusion decreases and advection increases, which is intuitively clear. Lastly, larger values of C_{DG} imply more denseness of the debris material with low permeability. This results in hindered diffusion and advection processes [29]. For increasingly dense material, the structures of D and C show that the diffusion and advection processes become slow, which is in line with the physics of fluid flow in porous media. We expect that these four aspects can be observed in nature which may present new insights into understanding of basic structure of fluid flow in the porous and debris material, natural landscape and lateral embankments. For these reasons, here, we construct several exact and analytical solutions to the model (1) with $\alpha = 2$. The solution techniques include, Lie symmetry, travelling wave, generalized separation of variables, and boundary layer methods. These different methods produce very similar solutions, which are in agreement with the high-resolution numerical solutions in Pudasaini [28].

2.2. Analysis of the model

The sub-diffusion and sub-advection model (1) can be viewed as a nonlinear diffusion–advection (or, convection) equation with two functions $\tilde{D}(H) = 2DH$ and $\tilde{C}(H) = CH^2$ which play the role of the variable and nonlinear diffusion and advection coefficients:

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[\tilde{D}(H)\frac{\partial H}{\partial x}\right] - \frac{d\tilde{C}}{\mathrm{d}H}\frac{\partial H}{\partial x}. \tag{2}$$

The coefficients $\tilde{D}(H)$ and $\tilde{C}(H)$ are often unknown and experimentally measured functions for particular porous medium (e.g. soil) which characterize the underlying hydrology [26]. In our case, both $\tilde{D}(H)$ and $\tilde{C}(H)$ are explicitly known non-negative functions [28] and $d\tilde{C}/dH$ is an increasing (non-decreasing) function of H. Hence, we do not need extra assumptions on the form of $\tilde{D}(H)$ and $\tilde{C}(H)$. From the physical point of view, the model (1) is a sub-diffusion and sub-advection model. However, from the mathematical point of view, its equivalent form (2) is a nonlinear diffusion and advection equation with nonlinear diffusion and advection coefficients. Hence, we will switch between the terminologies "sub-diffusion and sub-advection", and "nonlinear diffusion and advection" whatever suits the context.

Equation (2) can also be written as:

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[2DH\frac{\partial H}{\partial x}\right] - 2CH\frac{\partial H}{\partial x}. \tag{3}$$

This shows that

$$q = -2DH\frac{\partial H}{\partial x} + CH^2 = -\frac{\partial}{\partial x}\left(DH^2\right) + CH^2, \tag{4}$$

constitutes the explicit Buckingham–Darcy-type flux-law [13] for unsaturated flow in porous media. The presence of DH^2 and CH^2 explains the nonlinear diffusion and advection fluxes that defines the nonlinear diffusion and advection equation. It is important to note that with substitutions $x = (D/C)\hat{x}, t = (D/2C^2)\hat{t}$, the factors $2D$ and $2C$ in (3) virtually disappear. In other words, Eq. (3) reduces to

$$\frac{\partial H}{\partial \hat{t}} = \frac{\partial}{\partial \hat{x}}\left[H\frac{\partial H}{\partial \hat{x}}\right] - H\frac{\partial H}{\partial \hat{x}}. \tag{5}$$

As we will see later in Sect. 4.1, this will substantially help in constructing analytical solutions with Lie symmetry method. We will either keep the factors $2D$ and $2C$ as in (3), or set them equal to some other constants (in particular, to unity) according to the convenience. In what follows, without loss of generality, we drop the hats from t and x.

3. Exact analytical solutions to reduced equations

For $\alpha = 1$, exact solutions can be constructed for (1). For either $C = 0$ (diffusion only), or $D = 0$ (advection only), general exact solutions can be constructed. However, if $\alpha \neq 1$, it is mathematically challenging (even for $\alpha = 2$, our particular interest) to construct exact solutions, when $D \neq 0, C \neq 0$. Let us briefly review some known facts about exact or analytical solutions for the reduced form of (1). Since the main focus of this paper is on construction of exact and analytical simulations but not on the comparison between the model solutions and the data, we will not elaborate on the choice of parameter values (for D, C, etc.). Suitable choice of parameter values is sufficient for our purpose.

Classical diffusion and advection: The classical linear (with respect to diffusion and advection fluxes) diffusion–advection equation, which corresponds to the case $\alpha = 1$ and $C \neq 0$, $D \neq 0$, has exact analytical solutions [6,35]. However, it is pointed out in Pudasaini [28] that such solutions are not physically meaningful for the fluid flow through porous and debris materials.

Generalized diffusion, porous media equation: For $\alpha = 2$, $C = 0$, and $D \neq 0$, (1) reduces to a porous media equation for which exact analytical solution exists and can be written in the form of the q-Gaussian [1,2,4,23]. These solutions have compact supports [9,34,37,38]. We refer to Pudasaini [28] for details.

The quadratic sub-diffusive solution and solution with classical linear diffusion model are significantly different [28]. But, in the limit, sub-diffusive solution exactly recovers the classical Gaussian solution. The solution with quadratic flux is more ellipsoidal rather than Mexican-hat type corresponding to the solution of the classical diffusion equation with linear flux. It has also been observed in Pudasaini [28] that the diffusion (or dispersion) is much slower and much less spread as compared to the same with the linear flux (the classical diffusion model) for the sub-diffusive model. The sub-diffusive solution reveals that the flow of fluid through porous and debris material should be modelled by the nonlinear (quadratic) diffusion flux rather than the classical linear diffusion model.

Burgers equation: For $\alpha = 2$, $C \neq 0$, and $D = 0$, (1) becomes Burgers equation [5,14,20,31]. It is well known that exact solutions can be constructed; for example, see Pudasaini [28].

4. Analytical solutions to nonlinear diffusion–advection model

Numerical solutions of nonlinear diffusion and advection model: Let us briefly explain some salient features of the high-resolution numerical solutions for the full model (1) presented in Pudasaini [28]. These solutions revealed a very special sub-diffusive and sub-advective flow behaviour: the flow becomes more sharp (bore-type) in the front and it also elongates in the downslope direction. This results in the asymmetric solution with the flow front continuously advecting downslope. Since the flow is sub-diffusive, the flow does not spread laterally which would be the case in the classical diffusion and advection processes. This solution is completely different from the solution for the classical diffusion–advection equation, where the entire fluid pocket would advect in the downslope direction, which at the same time also diffuses with spreading Gaussian profile. Again in contrast to the classical diffusion–advection of fluid in open environment or transport of tracer particles and other substances by transporting fluid, where the tail of the initial substance distribution also advects (with the speed of the transporting background fluid) in the downslope direction, in this solution for the viscous fluid flow through porous media, the tail (although less in intensity) always remains in its original position. This is the most likely scenario for fluid flows in the porous media. This indicates that the fluid flows through the porous media and debris material should be described by sub-diffusive and sub-advective model rather than classical diffusion–advection model. These observations point towards the need for analytical and exact solutions for the full system (1) which may play a significant role to gain physical insights of fluid flows through the porous media. This motivates the main contribution of this paper, which is construction of analytical and exact solutions of the full model. To achieve this, we apply the Lie symmetry method, travelling wave method, generalized separation of variables and boundary layer methods. Such solutions can help to validate numerical solution methods and results. These newly constructed analytical solutions are similar to those high-resolution numerical solutions in Pudasaini [28].

4.1. Solutions using symmetry Lie algebra

There is a huge literature on Lie symmetry method to solve PDEs (see, for example, [13] and references therein). The basic idea behind the Lie symmetry method is to transform the given partial differential equation into a simpler partial differential equation or an ordinary differential equation that can potentially be solved analytically. With the help of Lie symmetries, we can find analytical or numerical solutions directly after the reduction or after possibly multiple successive reductions; see, for example, Ghosh Hajra et al. [16,17] and references therein. It is known that a large number of diffusion and advection equations admit Lie symmetries [13]. Our goal here is to use the ideas from Edwards [13] to find a symmetry Lie algebra of the model (1) and use it to construct analytical solutions.

We first show that our governing equation admits a symmetry Lie algebra spanned by the following vector fields:

$$\Gamma_1 = \frac{\partial}{\partial x},\ \ \Gamma_2 = \frac{\partial}{\partial t},\ \ \Gamma_3 = -t\frac{\partial}{\partial t} + H\frac{\partial}{\partial H}. \tag{6}$$

In order to verify (6), we use the transformation introduced in Sect. 2.2, namely $x = (D/C)\hat{x}, t = (D/2C^2)\hat{t}$. It is known [13] that

$$\hat{\Gamma}_1 = \frac{\partial}{\partial \hat{x}},\ \ \hat{\Gamma}_2 = \frac{\partial}{\partial \hat{t}},\ \ \hat{\Gamma}_3 = -\hat{t}\frac{\partial}{\partial \hat{t}} + H\frac{\partial}{\partial H}, \tag{7}$$

span a Lie symmetry algebra of Eq. (5). Note that

$$\hat{\Gamma}_1 = \frac{D}{C}\Gamma_1,\ \ \hat{\Gamma}_2 = \frac{D}{2C^2}\Gamma_2,\ \ \hat{\Gamma}_3 = \Gamma_3,$$

which immediately implies that Γ_1, Γ_2 and Γ_3 are symmetries of (3) and hence it verifies our claim (6).

Recall that the calculation of optimal systems of the symmetry Lie algebra given by (6) is an abstract calculation, and it does not use the model into the consideration. This allows us to directly use the results from Edwards [13]. Hence, the optimal system of one-dimensional Lie subalgebras of the nonlinear diffusion–advection Eq. (1) are given by:

$$\Gamma_1,\ \ \Gamma_2,\ \ \Gamma_1 + \Gamma_2,\ \ \lambda\Gamma_1 + \Gamma_3, \tag{8}$$

where λ is a parameter.

4.1.1. Solutions from the symmetry $\boldsymbol{\lambda\Gamma_1 + \Gamma_3}$. Using $\lambda\Gamma_1 + \Gamma_3$, we can reduce (3) into a second-order nonlinear ODE as follows. Let ξ and F be new independent and dependent variables obtained by solving the characteristic equation of $\lambda\Gamma_1 + \Gamma_3$. More precisely,

$$\xi = x + \lambda \ln(t), \quad F(\xi) = Ht. \tag{9}$$

Using the transformation (9), the model (3) yields the following ODE

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\xi^2} = -\frac{1}{F}\left(\frac{\mathrm{d}F}{\mathrm{d}\xi}\right)^2 + \left(\frac{\lambda}{2D}\frac{1}{F} + \frac{C}{D}\right)\frac{\mathrm{d}F}{\mathrm{d}\xi} - \frac{1}{2D}. \tag{10}$$

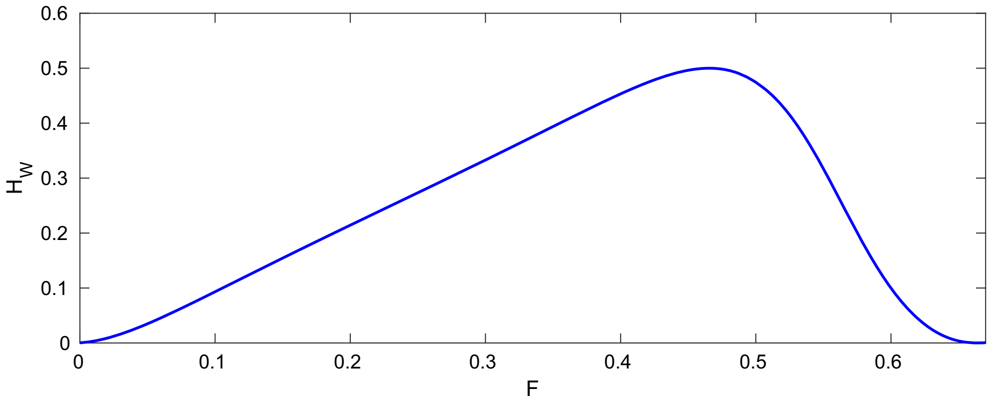

Fig. 1. Propagating front of the fluid flow in porous media as described by nonlinear diffusion and advection model and its solution (13) with Lie symmetry method

Since (10) is an autonomous ODE, it can be reduced to a first-order ODE by introducing a new dependent variable $v = v(F)$ such that

$$\frac{\mathrm{d}F}{\mathrm{d}\xi} = v(F). \tag{11}$$

This transforms (10) into

$$\frac{\mathrm{d}v}{\mathrm{d}F} = -\frac{v}{F} + \left(\frac{\lambda}{2D}\right)\frac{1}{F} - \left(\frac{1}{2D}\right)\frac{1}{v} + \frac{C}{D}. \tag{12}$$

Without loss of generality, we assume that $C = 1$ and $D = 1/2$, which can be achieved by a proper substitution as mentioned in Sect. 2.2. Equation (12) can be solved analytically for $\lambda = 1$ resulting in the following exact solution:

$$-K = \frac{WJ_1(2W) - FJ_0(2W)}{WY_1(-2W) + FY_0(-2W)}, \tag{13}$$

where $W = i\sqrt{F(F-v)}$ and K is a constant, and J and Y are the Bessel functions of the first and second kinds, respectively. Figure 1 plots the solution (13) for $K = 0.25$, where

$$H_W = \frac{WJ_1(2W) - FJ_0(2W)}{WY_1(-2W) + FY_0(-2W)} + K. \tag{14}$$

More generally, Eq. (12) is an Abel's equation of second kind for any C and D, and hence, it has an analytic solution (when we take $\lambda = 2D/C$) given by:

$$H_W = \frac{DWJ_1(W) - C^2FJ_0(W)}{DWY_1(-W) + C^2FY_0(-W)} + K, \tag{15}$$

where $W = i\sqrt{C^3F(CF-2Dv)}/D$, and K is a constant. In fact, an Abel's equation of the second kind which has the form

$$\frac{\mathrm{d}v}{\mathrm{d}F} = -\frac{v}{F} + \frac{a}{F} - \frac{ab}{2v} + b, \tag{16}$$

has an analytic solution given by

$$H_W = \frac{aWJ_1(W) - bFJ_0(W)}{aWY_1(-W) + aFY_0(-W)} + K, \tag{17}$$

where $W = i\sqrt{bF(bF-2v)/a^2}$. In particular, if we take $a = 1/C$ and $b = C/D = 2/\lambda$, then (17) reduces to (13).

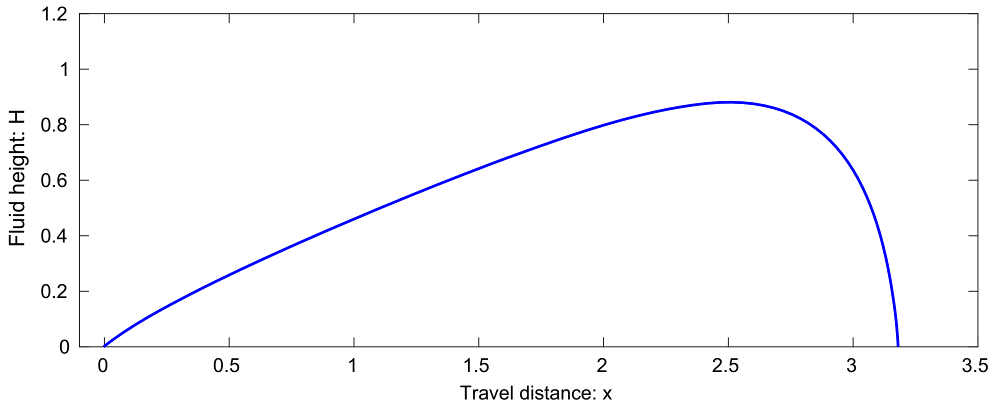

Fig. 2. Propagating front of the fluid flow in porous media as described by nonlinear diffusion and advection model and its solution with Lie symmetry method and the system of ODEs (18)

Alternatively, (10) can be reduced differently by introducing a different variable V, as $\mathrm{d}F/\mathrm{d}\xi = V$, that leads to a system of two ODEs:

$$\frac{\mathrm{d}F}{\mathrm{d}\xi} = V, \quad \frac{\mathrm{d}V}{\mathrm{d}\xi} = -\frac{V^2}{F} + \left(\frac{\lambda}{2D}\frac{1}{F} + \frac{C}{D}\right)V - \frac{1}{2D}. \tag{18}$$

We are not aware of analytical solutions of this equation, but it can be solved numerically. A solution to the advecting fluid front as described by the model (18) is shown in Fig. 2 for $C = 1.555, D = 1.0$ and $\lambda = -0.0255$, and for the initial value and slope: 0.135 and 0.75.

Both solutions presented in Figs. 1 and 2 are asymmetrically propagating fluid fronts; the front in Fig. 1 is a bit diffused, whereas the front in Fig. 2 is sharp or a bore-type, as also seen in the numerical simulation in Pudasaini [28]. Although both solutions can be physically relevant, Fig. 2 might be more realistic.

4.2. Travelling wave solutions

The travelling wave solutions of the model Eq. (1) are associated with the symmetry $\Gamma_1 + \lambda\Gamma_2$. This means we can define a travelling wave variable

$$H(\xi) = H(\omega(x - \lambda t)), \tag{19}$$

where λ is the wave speed, and ω is a stretching factor in ξ. Let us use the notation H' for $dH/\mathrm{d}\xi$. Applying this to (1) results in

$$\left[-\lambda H - 2\omega DHH' + CH^2\right]' = 0, \tag{20}$$

which after integration yields the following first-order ODE:

$$-\lambda H - 2\omega DHH' + CH^2 = K, \tag{21}$$

where K is a constant of integration. It is important to note that if the second term is $2\omega DH'$, then (21) represents the viscous Burgers equation and this is associated with (1) with $\alpha = 2$ and the diffusion term $D\partial^2 H/\partial x^2$. This implies that our model (1) can be considered as a generalization of the viscous Burgers equation. In the following, we construct travelling wave solutions.

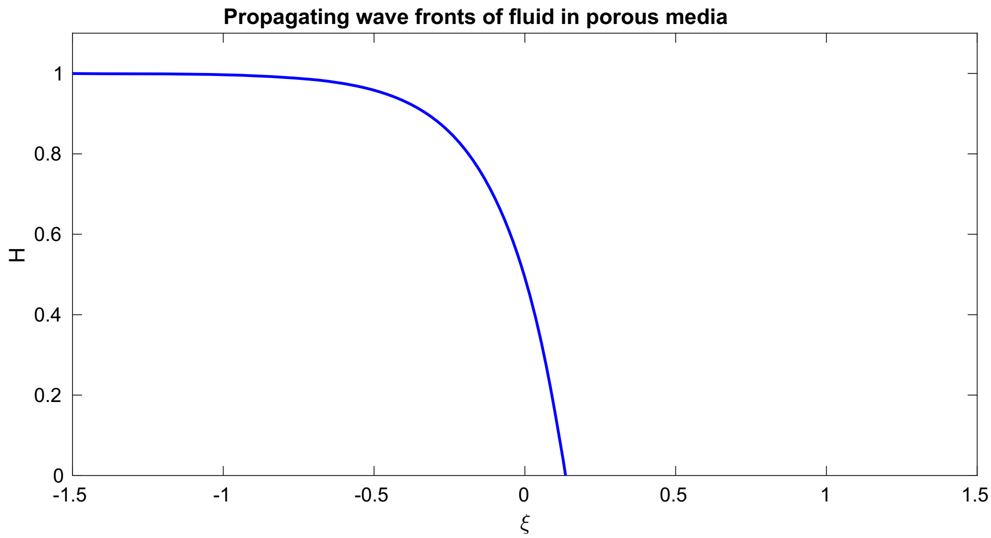

FIG. 3. Propagating front of the fluid flow in porous media as described by nonlinear advection and diffusion model and its travelling wave solution (23) with boundary at infinity

4.2.1. Boundary conditions at infinity. Let us introduce the following boundary conditions

$$\lim_{\xi\to\pm\infty} H(\xi) = H_{\pm\infty}, \ \lim_{\xi\to\pm\infty} \frac{\mathrm{d}H}{\mathrm{d}\xi} = 0, \tag{22}$$

where $H_{\pm\infty}$ are real numbers. Such boundary conditions may be applicable to some technical problems, namely either infinitely spreading flows or the flow front emerging from an infinitely extending fluid reservoir (or ground water table). Here, we will concentrate on the case $H_{+\infty} \neq H_{-\infty}$. In this case, we can express the wave speed λ and the constant K of Eq. (21) in terms of the boundary values as $\lambda = C(H_{+\infty} + H_{-\infty})$ and $K = -CH_{+\infty}H_{-\infty}$. For meaningful solutions, at least one of the $H_{\pm\infty}$ must be nonzero. However, such a typical boundary condition (e.g. the water table) is not so easy to realize. It is interesting to note that λ is proportional to the advection (wave speed) C in the diffusion–advection equation and $H_{+\infty} + H_{-\infty}$; and the constant K is identically equal to zero if one of $H_{\pm\infty}$ is zero.

If we further assume $H_{+\infty} = 0$ and $\omega > 0$, then a solution of (21) is given by

$$H(\xi) = \begin{cases} H_{-\infty}(1 - e^{C(\xi-\xi_0)/2\omega D}), & \xi < \xi_0; \\ 0, & \xi \geq \xi_0. \end{cases} \tag{23}$$

Similarly, when $H_{-\infty} = 0$ and $\omega < 0$, a solution of (21) is given by

$$H(\xi) = \begin{cases} H_{+\infty}(1 - e^{C(\xi-\xi_0)/2\omega D}), & \xi > \xi_0; \\ 0, & \xi \leq \xi_0. \end{cases} \tag{24}$$

Similar solutions for a more general diffusion–convection equation are discussed in Hayek [19]. We note that these solutions are not applicable for general initial conditions, for example, the propagating fluid fronts presented in Figs. 1 and 2. Hence, these solutions are physically less attractive. Also, these solutions have compact supports in the flow directions. Figure 3 shows this solution for parameters $(H_{-\infty}, C, D, \omega, \xi_0) = (1.0, 0.5, 0.1, 0.5, 0.135)$. The remaining case $K \neq 0$ will be discussed in the next section.

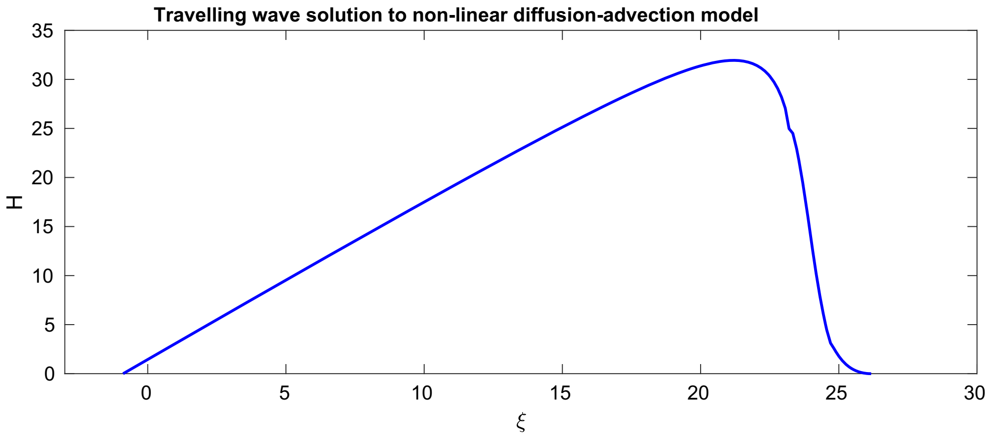

Fig. 4. Travelling wave solution (25) to nonlinear diffusion and advection equation

4.2.2. General travelling wave solutions. In what follows, for simplicity, we take $\omega = 1$. Note that Eq. (21) can be solved exactly and solutions depend on whether $\lambda^2 + 4CK$ is positive or, negative or, zero.

A. Nonlinear diffusion–advection: When $\lambda^2 + 4CK < 0$, a general solution is given by:

$$K_1 + \frac{\xi}{2D} = \frac{1}{2C}\left(\frac{2\lambda \tan^{-1}\left[\frac{2CH-\lambda}{\sqrt{-\lambda^2-4CK}}\right]}{\sqrt{-\lambda^2-4CK}} + \ln\left[-\lambda H + CH^2 - K\right]\right), \tag{25}$$

where K_1 is a constant of integration. The solution is presented in Fig. 4 for $(\lambda, C, D, K, K_1) = (1.62, 1.1, 0.299, -10.1, -10.1)$. Note that these parameter choice makes $\lambda^2 + 4CK < 0$. This solution lies in between the solutions presented in Figs. 1 and 2 corresponding to the model (13) and (18), respectively.

Equation (21) can also be written as

$$\frac{\mathrm{d}H}{\mathrm{d}\xi} = \frac{1}{2D}\left[\left(CH - \frac{K}{H}\right) - \lambda\right]. \tag{26}$$

Later we will derive solution to this equation. First, we consider only the diffusion equation.

B. Nonlinear diffusion: For nonlinear diffusion equation (i.e. $C = 0$), exact solution can be constructed in explicit form:

$$H(\xi) = -\frac{K}{\lambda} W\left[-\frac{1}{K}\exp\left(\frac{\lambda^2}{KD}\frac{\xi}{2} + \frac{\lambda^2}{K}K_1 - 1\right)\right] - \frac{K}{\lambda}, \tag{27}$$

where W is the Euler–Lambert omega (or product log) function. Solution is shown in Fig. 5 for $(\lambda, D, K, K_1) = (0.922, 0.2, 0.95, -0.50)$ and $t = 1.0$. The solution is similar to the q-Gaussian or the Barenblatt solution as discussed in Pudasaini [28]. However, (27) is simpler than the q-Gaussian solution.

It is important to note that for the usual diffusion–advection model (with linear fluxes), $C \to 0$ does not imply solution in travelling waveform, but for the nonlinear diffusion and advection model it does. Furthermore, $D \to 0$ does not imply physically relevant travelling wave solution even for the nonlinear diffusion and advection model.

C. Nonlinear advection-diffusion–elegant form: We further construct the travelling wave solution for the full nonlinear diffusion and advection model when $\lambda^2 + 4CK > 0$. In this case, (21) can be re-written by

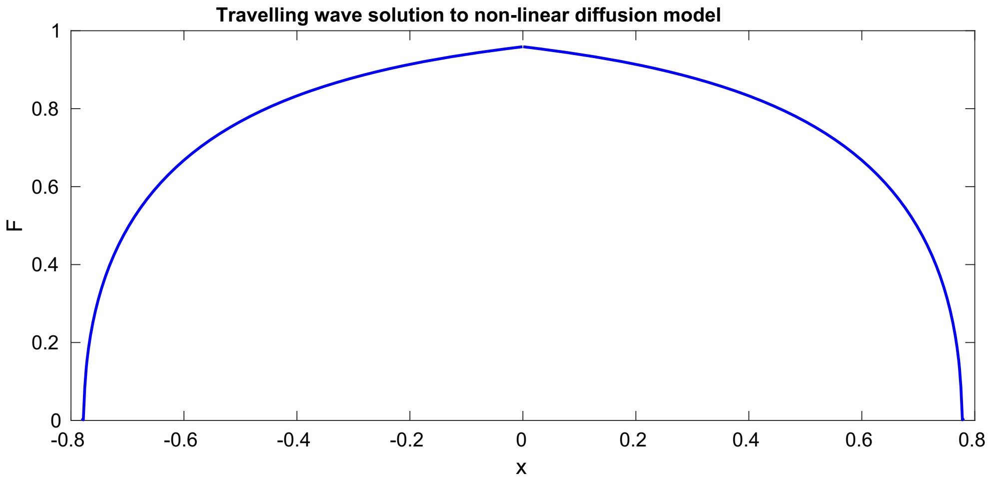

FIG. 5. Travelling wave solution (27) to nonlinear diffusion equation

factorizing the right hand side as

$$\frac{\mathrm{d}H}{\mathrm{d}\xi} = \frac{\tilde{C}}{H}\Big[(H - H_1)(H - H_2)\Big], \tag{28}$$

where

$$H_1 = \frac{\lambda + \sqrt{\lambda^2 + 4CK}}{2C}, \;\; H_2 = \frac{\lambda - \sqrt{\lambda^2 + 4CK}}{2C}, \tag{29}$$

which are independent of diffusion, and $\tilde{C} = C/2D$. By the definition of the parameters, we observe $H_1 > H_2$. Furthermore, the wave speed λ can be expressed explicitly as $\lambda = C\,(H_1 + H_2)$. The first-order ODE (28) is separable and can be solved analytically to obtain the solution:

$$\frac{(H - H_1)^{H_1}}{(H - H_2)^{H_2}} = e^{(H_1 - H_2)(\xi - \xi_0)\tilde{C}}. \tag{30}$$

Since $(H_1 - H_2)\,\tilde{C} = \frac{1}{2D}\sqrt{\lambda^2 + 4CK}$, the left hand side of (30) is independent of diffusion, while the right hand side contains both the diffusion and advection contributions. This is the most elegant exact solution in the travelling waveform for the nonlinear diffusion and advection model (1). For parameter values $(\lambda, C, D, K) = (1.22, 0.60, 0.40, -0.95)$, the solution is shown in Fig. 6 which is similar to the solutions presented in Fig. 1, model (13), and Fig. 4, model (25). Nevertheless, the solution here shows a depletion behind the front surge head which in the other figures is more straight. Such a special behaviour is attributed to the expression $\lambda^2 + 4CK > 0$. At the first glance, all these solutions are quantitatively similar; however, the detailed pictures show substantial quantitative differences.

D. Reduced nonlinear advection–diffusion: We further present solution for $\lambda^2 + 4KC = 0$. In this case, (21) can be written as:

$$\frac{\mathrm{d}H}{\mathrm{d}\xi} = \frac{C}{2DH}\left(H - \frac{\lambda}{2C}\right)^2. \tag{31}$$

This equation can be integrated and results in a general implicit solution for H:

$$\ln\left(H - \frac{\lambda}{2C}\right) = \frac{\lambda}{2C}\left(H - \frac{\lambda}{2C}\right) + \frac{C}{2D}(\xi - \xi_0). \tag{32}$$

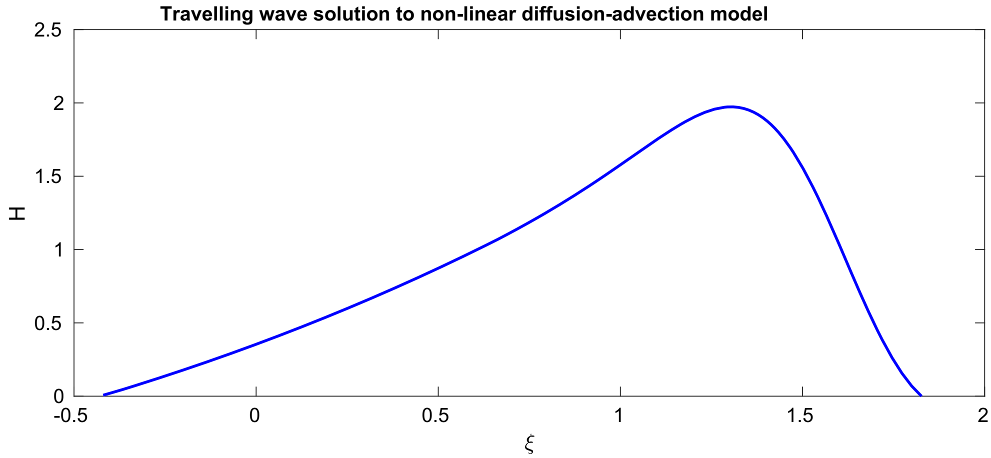

FIG. 6. Travelling wave solution (30) to the nonlinear diffusion–advection equation in the most elegant form

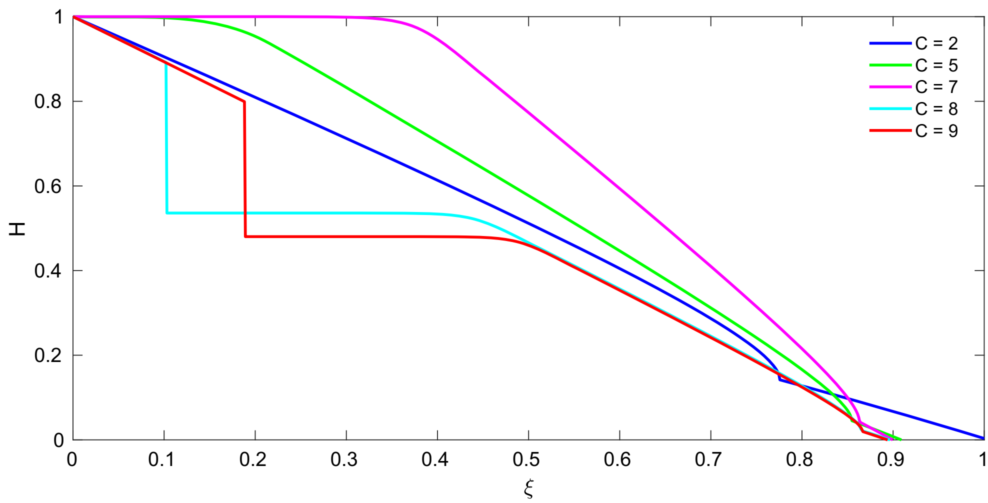

FIG. 7. Travelling wave solution (33) to the reduced nonlinear diffusion–advection equation

Equation (32) can further be transformed into an explicit form for H:

$$H = \left(\frac{2C}{\lambda}\right)\left(\left(\frac{\lambda}{2C}\right)^2 - W\left[-\frac{\lambda}{2C}\exp\left(\frac{C}{2D}(\xi-\xi_0)\right)\right]\right), \tag{33}$$

where W is the Euler–Lambert omega function.

Figure 7 shows different solutions to (33). As compared to the solutions for $\lambda^2 + 4CK \neq 0$, now the solution produces kinks and jumps that had previously been smoothed out by non-vanishing of λ^2+4CK. Note that the solution appears to be reasonable only for $W(-1, Z)$, i.e. smaller of the two real solutions represented by W, where $Z = \lambda^2 + 4CK$. For smaller advection ($C = 2$), the solution has one kink, with a parabolic-type branch to the left and straight branch to the right of the kink. As C increases ($C = 5, 7$), the kink moves to the lower right, but on the left, the solution saturates and this saturation propagates to the right. Hence, these solution curves have three branches: saturated straight lines on the left, parabolic curves on the middle, and straight lines on the right of the kinks. As we continue to increase the advection ($C = 8, 9$), the solution behaves completely differently. It shows a shock wave (sudden drop) on the left branch, which then continues as a horizontal straight line until it meets the parabolic branch. For increased value of advection ($C = 9$), similar shock wave is produced, but the shock has now been strongly pushed to the right, and to the lower horizontal water table, until it meets the parabolic branch again. Such dramatic changes in the solutions are attributed to the higher values of the advection and the removal of the smoothing contributions associated with $\lambda^2 + 4KC$. To our knowledge, this is the first observation of such a complex phenomena.

4.3. Solutions from generalized separation of variables

Here, we will solve (3) by following an approach similar to the one in [18]. This approach is not new, and it is often called the method of generalized separation of variables [27]. The basic idea is to use the method of separation of variables and reduce the PDE into a number of ODEs by making an appropriate ansatz. In our case, we will seek for a solution of (3) which is of the form:

$$H(x,t) = f(t) + g(t)h(x). \tag{34}$$

Using (34) in (3), we get

$$\begin{aligned} &f'(t) + h(x)g'(t) \\ &= 2Dg(t)\left[g(t)h'(x)^2 + (f(t) + g(t)h(x))h''(x)\right] - 2Cg(t)(f(t) + g(t)h(x))h'(x), \end{aligned} \tag{35}$$

where

$$f'(t) = \frac{\mathrm{d}f}{\mathrm{d}t},\ g'(t) = \frac{\mathrm{d}g}{\mathrm{d}t},\ h'(x) = \frac{\mathrm{d}h}{\mathrm{d}x} \text{ and } h''(x) = \frac{\mathrm{d}^2h}{\mathrm{d}x^2}.$$

We can get first-order ODEs from (35) if we assume that $h''(x) = 0$ as follows. Our assumption implies that $h(x) = ax + b$ where a and b are constants and using this in (35), we get

$$f'(t) = 2a^2Dg(t)^2 - 2aCf(t)g(t) - (ax + b)[2aCg(t)^2 + g'(t)]. \tag{36}$$

Equation (36) suggests that if we can find $g(t)$ such that

$$2aCg(t)^2 + g'(t) = 0, \tag{37}$$

then we can determine $f(t)$ by solving

$$f'(t) = 2a^2Dg(t)^2 - 2aCf(t)g(t). \tag{38}$$

Note that a general solution of (38) is given by

$$\begin{aligned} f(t) = {}& K_2 \exp\left(-\int 2aCg(t)\,\mathrm{d}t\right) \\ &+ 2a^2D \exp\left(-\int 2aCg(t)\,\mathrm{d}t\right) \int g(t)^2 \exp\left(\int 2aCg(t)\,\mathrm{d}t\right)\,\mathrm{d}t. \end{aligned}$$

Hence, a solution of (3), which is of the form (34), can be obtained by solving the first-order ODE (37). Solving (37), we get

$$g(t) = \frac{1}{2aCt + K_1}, \tag{39}$$

and hence

$$f(t) = \frac{K_2}{(2aCt + K_1)} + \frac{aD}{C(2aCt + K_1)} \ln(2aCt + K_1). \tag{40}$$

The following proposition summarizes the discussion above.

Proposition 1. *Let $g(t)$ and $f(t)$ be as in (39) and (40), respectively. Then, $H(x,t) = f(t) + (ax + b)g(t)$ solves Eq. (3).*

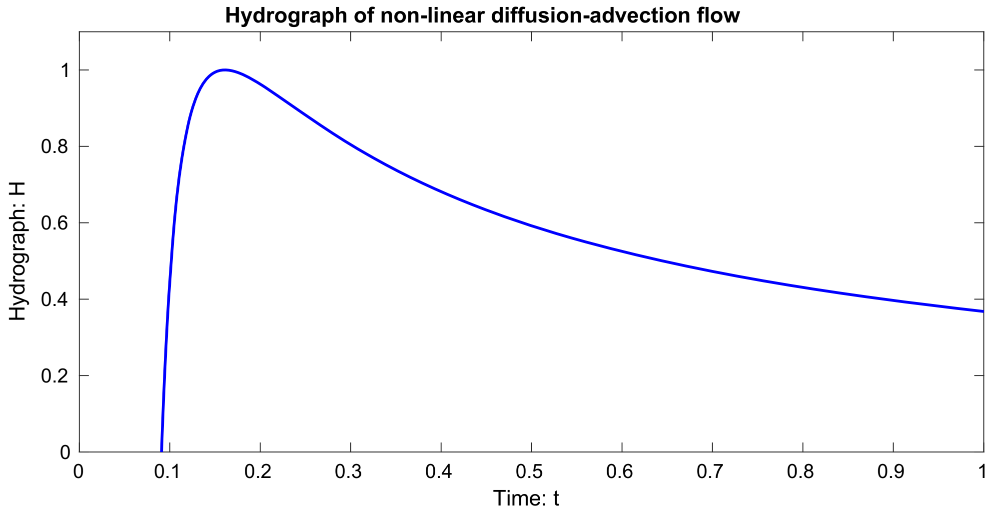

Fig. 8. Subsurface hydrograph as produced by the solution by separation of variables (41) to the nonlinear advection and diffusion model

Note that the discussion leading to Proposition 1 also shows that Eq. (3) cannot have a solution of the form $f(t)+h(x)$ or $g(t)h(x)$ unless one of these functions is a constant function. Hence, it is necessary to start with an ansatz of the form (34) in order to find interesting solutions. Moreover, if we insist $h(x)$ to be a polynomial function, then it is necessary that $h(x)$ must be of the form $h(x) = ax + b$. This can be verified using degree counting as follows. Suppose that $h(x)$ is a polynomial function of degree n. Then, the expression on the left hand side of (3) will be polynomial of degree n in x and the right hand side of (3) will be a polynomial of degree $2n-1$ in x. This means if the ansatz (34) is a solution of (3), then we must have $2n-1 = n$, in other words $n = 1$.

Now, if we take $h(x) = x + b$ and $g(t) = 1/2Ct$, we find that

$$H(x,t) = \frac{D}{2C^2}\frac{\ln t}{t} + \frac{M}{t} + \frac{x+b}{2Ct}, \tag{41}$$

is a solution of (3) which is discussed in Loubens and Ramakrishnan [22] without referring to separation of variable method. Figure 8 shows the subsurface hydrograph solution, the hydrograph in porous media, as produced by the separation of variable method applied to the nonlinear advection and diffusion model. Such subsurface hydrograph can be an observable phenomenon.

Similarly, we can verify that

$$H(x,t) = \frac{K_3}{(K_1 - 2Dt)^{1/3}} + \frac{K_2^2}{4(K_1 - 12Dt)} + \frac{K_2 x}{K_1 - 12Dt} + \frac{x^2}{K_1 - 2Dt}, \tag{42}$$

where K_1, K_2 and K_3 are constants, solves the model (3) when it is just a diffusion equation, i.e. $C = 0$. In fact, solutions of form (42) can be constructed for a general diffusion–reaction equation; we refer to Polyanin and Zhurov [27] for details. We remark that it is not possible to improve (42) to include cubic or powers of x. This can be verified using a degree counting argument as above, and it also shows that the coefficient function of x^2 of a solution of the diffusion equation, i.e. (3) with $C = 0$, which is of the form (42), must be nonzero.

Finally, we show that some travelling wave solutions of (3) can be obtained from the method of generalized separation of variables. For this, we assume $f(t) = f$, where f is a constant. Then, from (35), we get

$$\begin{aligned} &h(x)g'(t) - 2Dfg(t)h''(x) + 2Cfg(t)h'(x) \\ &= g(t)^2\left(2Dh'(x)^2 + 2Dh(x)h''(x) - 2Ch(x)h'(x)\right). \end{aligned} \tag{43}$$

It is easy to check $h(x) = e^{Cx/2D}$ solves

$$2Dh'(x)^2 + 2Dh(x)h''(x) - 2Ch(x)h'(x) = 0. \tag{44}$$

Using $h(x) = e^{Cx/2D}$ in (43), we get the following ODE for $g(t)$

$$g'(t) + \frac{fC^2}{2D}g(t) = 0, \tag{45}$$

and this equation has a general solution of the form

$$g(t) = K_1 e^{-fC^2t/2D},$$

where K_1 is a constant of integration. Hence, if we take $h(x) = e^{Cx/2D}$ and $g(t) = K_1 e^{-fC^2t/2D}$ in (34), the discussion above shows that

$$H(x,t) = f + K_1 e^{(Cx - fC^2t)/2D},$$

where f and K_1 are constants, solves Eq. (3). Note that $H(x,t)$ is a travelling wave solution obtained in the previous section with the wave variable $\xi = C(x - fCt)/2D$.

4.4. Boundary layer solutions

As indicated in Sect. 4.2, in this section we seek to use the idea that the nonlinear advection and diffusion model (1) is in fact an extended viscous Burgers equation. This helps to construct solutions to the more complex model (1) by applying the methods that have been used to solve the viscous Burgers equation [33,41]. We note that there are two important characteristics of the viscous Burgers equation. On the one hand, it is nonlinear so the characteristic lines may cross, while, on the other hand, the presence of diffusion (viscosity) keeps the solution single-valued [33]. The model (1) we are interested in also includes these aspects even with higher complexity due to the nonlinear (sub-diffusive) nature of the viscous term.

Here, we will construct a solution of the nonlinear diffusion–advection Eq. (1) using the boundary layer method. For simplicity, consider the initial condition $H(x,t) = H_0(x) = \mathcal{C}$ to the left of the origin and zero elsewhere, where $\mathcal{C} > 0$. This initial condition corresponds to the infinite water table to the left. In principle, any suitable initial condition can be chosen. Following [33], we realize that some mechanisms, such as "breaking wave", may produce a boundary layer in a vicinity of some location $x_b(t)$. On either side of this boundary layer, the solution of the nonlinear diffusion–advection model (1) can well be approximated by the sub-advection model

$$\frac{\partial H}{\partial t} + C\frac{\partial H^2}{\partial x} = 0, \tag{46}$$

which corresponds to the inviscid Burgers equation. Assume that, after some time t, the solution of (46) breaks and develops into a triple-valued function with the front at $x = \mathcal{C}t$. Hence, the solution is triple-valued in the domain $0 < x_b(t) < \mathcal{C}t$.

Consider the point $x_b(t)$ in the boundary layer. Using the definition of the new independent variable (boundary layer coordinate) $\eta = x - x_b(t)$, the model (1) reduces to [31]:

$$\frac{\partial H}{\partial t} - C\left(H - \dot{x_b}(t)\right)\frac{\partial H}{\partial \eta} = D\frac{\partial^2 H^2}{\partial \eta^2}. \tag{47}$$

The important mechanism (understanding) is that, in the boundary layer, the flow height changes rapidly with space, but its change with time is negligible, i.e. $\partial H/\partial \eta \gg \partial H/\partial t$ and hence, $\partial H/\partial t$ can be ignored. This implies we get the following expression from (47) after integration:

$$C\left(\frac{1}{2}H^2 - \dot{x_b}(t)H\right) = D\frac{\partial H^2}{\partial \eta} + K(t), \tag{48}$$

where $K(t)$ is a constant of integration with respect to η and t is only a parameter. Equation (48) is an ODE for H in the only independent variable η. The boundary conditions on the right imply that both H and its spatial gradient $\partial H/\partial \eta$ vanish as $\eta \to \infty$ implying $K(t) = 0$. Moreover, the boundary conditions to the left imply that $H \to \mathcal{C}$ and $\partial H/\partial \eta \to 0$ as $\eta \to -\infty$. Thus, $\dot{x} = \mathcal{C}/2$, where $\mathcal{C}/2$ is the wave speed. Hence, (48) yields

$$\frac{\partial H}{\partial \eta} = \frac{C}{4D}\left(H - \mathcal{C}\right). \tag{49}$$

A general solution to this equation is given by

$$H(\eta) = K\exp\left(\frac{C}{4D}\eta\right) + \mathcal{C}, \tag{50}$$

where K is a constant of integration. Without loss of generality, we assume $H = \mathcal{C}/2$ at $\eta = 0$, which implies $K = -\mathcal{C}/2$. This means the solution (50) finally takes the form:

$$H(x) = \mathcal{C}\left[1 - \frac{1}{2}\exp\left(\frac{C}{D}\left(x - x_b(t)\right)\right)\right]. \tag{51}$$

However, if we consider the viscous Burgers equation instead of (1), the corresponding Eq. (48) will be of the form:

$$C\left(\frac{1}{2}H^2 - \dot{x_b}(t)H\right) = D\frac{\partial H}{\partial \eta}, \tag{52}$$

which reduces to

$$\frac{\partial H}{\partial \eta} = \frac{CH}{2D}\left(H - \mathcal{C}\right). \tag{53}$$

This equation has a general solution of the form

$$H_{vB}(\eta) = \frac{\mathcal{C}\exp\left(\mathcal{C}K\right)}{\exp\left(\mathcal{C}K\right) - \exp\left(\frac{C}{2D}\mathcal{C}\eta\right)}, \tag{54}$$

where vB stands for viscous Burgers. Using $H = \mathcal{C}/2$ at $\eta = 0$ we get $K = i\pi/\mathcal{C}$, which in fact interestingly leads (connects) to the Euler identity: $\exp(\mathcal{C}K) = \exp(i\pi) = -1$. Finally, (54) can be written as

$$H_{vB}(x) = \frac{\mathcal{C}}{1 + \exp\left[\frac{C}{2D}\mathcal{C}\left(x - x_b(t)\right)\right]}. \tag{55}$$

From (51) and (55), it is clear that the solutions (51) to nonlinear diffusion and advection of fluid flow (1) are fundamentally different from the solutions (55) of the flow modelled by the viscous Burgers equation. These solutions are shown in Fig. 9 for the parameters $(C, D, \mathcal{C}) = (0.5, 0.1, 1.0)$. Both the solutions are uniformly valid in the boundary layer region. These solutions represent right-ward propagating "sharp step" front for nonlinear diffusion and advection model and "smooth-step", but infinitely extended front, for viscous Burgers equation with wave speed of $\mathcal{C}/2$. Evidently, H decreases from $\mathcal{C}$ (here, from 1) to 0 in a boundary layer of thickness $D/2C$. Interestingly, both solutions intersect at the middle (pivotal point) at the origin with value of about 0.5. On the left, the nonlinear advection and diffusion solution dominates, while on the right the viscous Burgers solution dominates. This happens because of the profile of the fluid as described by the nonlinear advection and diffusion process has a right-ward compact support, while the solution space of the Burgers equation is extended to infinity along the flow direction.

Significant differences between these solutions emerge from the fact that the diffusion flux in the nonlinear diffusion and advection model is quadratic, whereas in the viscous Burgers equation it is linear.

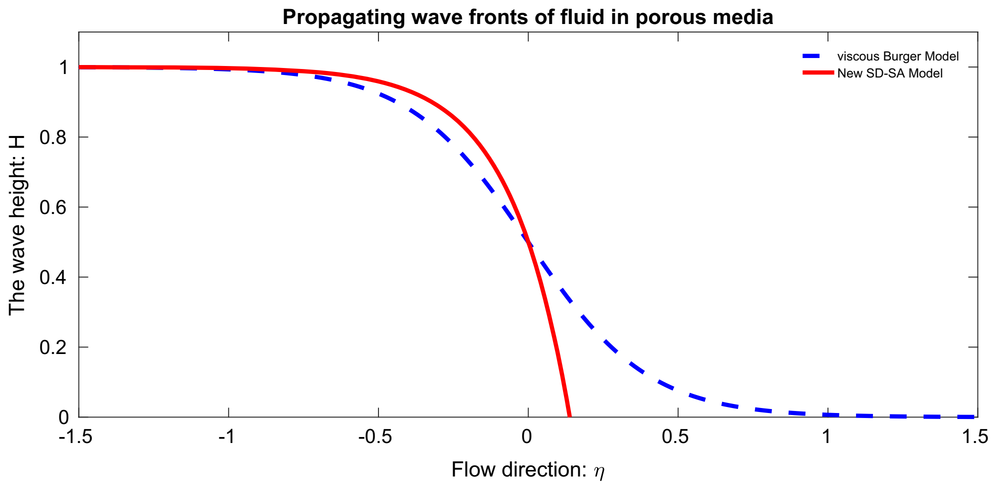

Fig. 9. Boundary layer solution (51) to the nonlinear advection and diffusion model and comparison with the solution (55) to viscous Burgers model

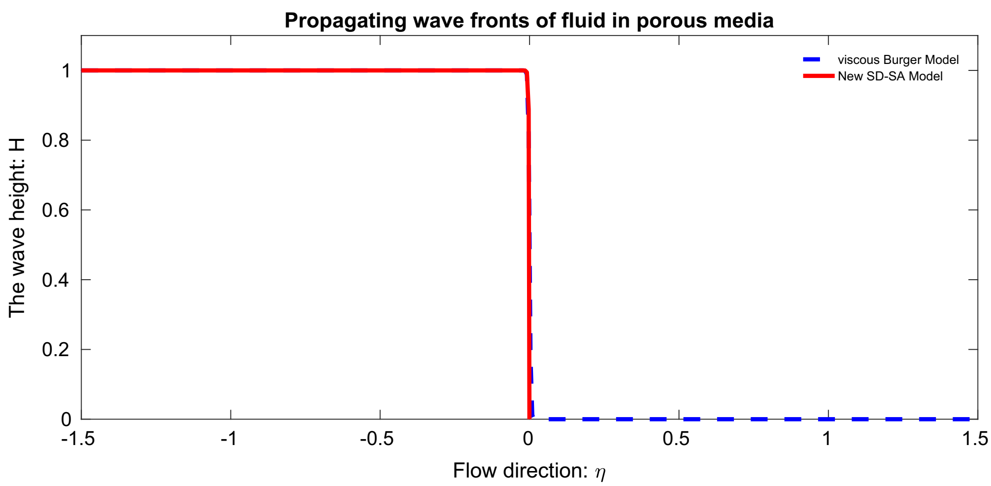

Fig. 10. In the limit of vanishing diffusion, solution (51) to the nonlinear advection and diffusion, and the solution (55) to the viscous Burgers model overlap

This quadratic flux, in fact, strongly controls the spreading of the front and restricts it in a confined domain with compact support. However, the linear diffusion flux in the viscous Burgers equation cannot control the fluid spreading and thus the fluid spreads infinitely. Hence, the diffusion term in nonlinear diffusion and advection model plays a crucial role in controlling the waveform than in viscous Burgers equation. The physical reason for these distinct behaviours is explained at the beginning of Sect. 4.

There are other important features of the new solution (51) to the nonlinear diffusion–advection model and the solution (55) to the viscous Burgers equation. Perhaps, the most appealing aspect is the following: The validity of the solutions can be checked by taking very large value of C (equivalently, very small value of D), which means advection dominates strongly, and thus in both nonlinear advection and diffusion, and Burgers equation, the diffusion can be neglected. In the limit, as $D \to 0$, both models are same, so should be the constructed solutions. Figure 10, in fact, reveals this by showing that in this vanishing

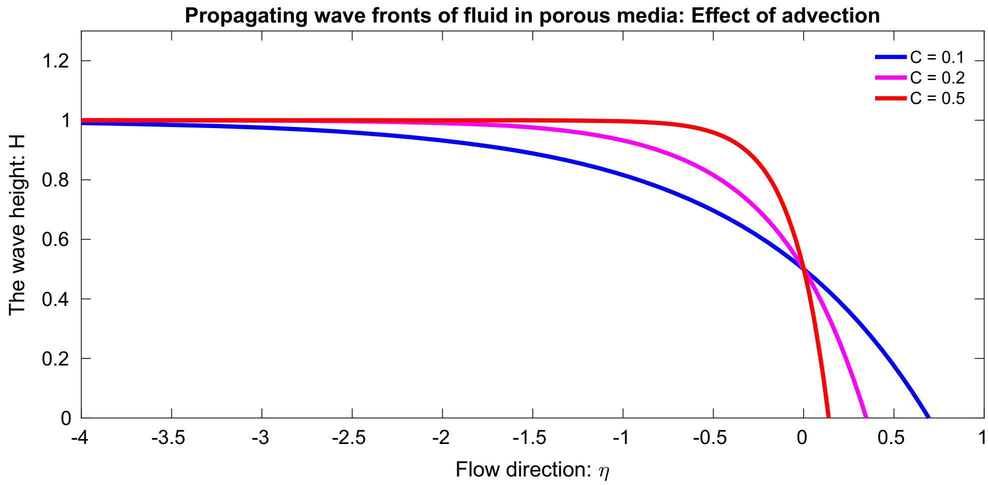

Fig. 11. Advection and diffusion effects in the boundary layer solution (51) to the nonlinear diffusion–advection model

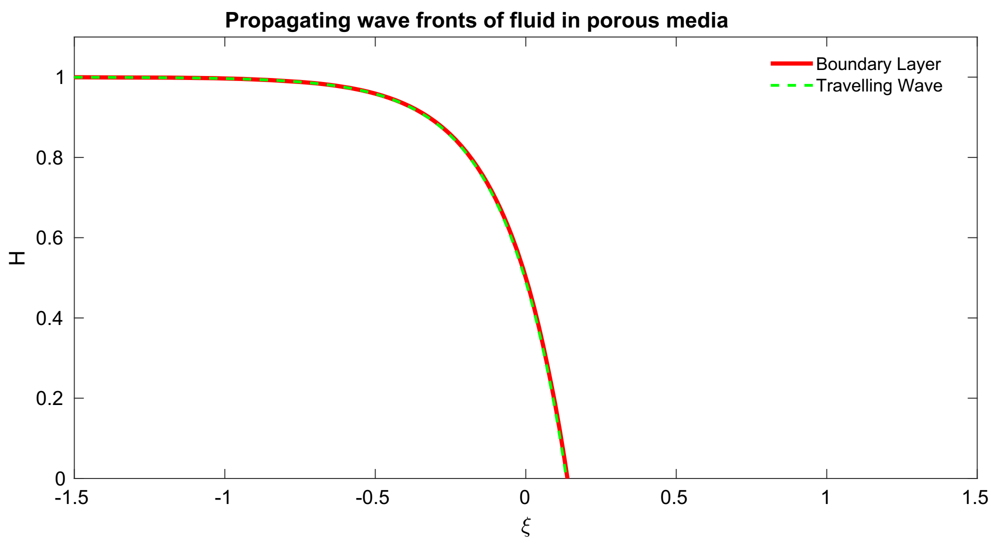

Fig. 12. Travelling wave and boundary layer solutions fit exactly.

viscous limit, both solutions are virtually same. This justifies the underlying physics captured by the two solutions.

Large value of the advection coefficient C means relatively less diffusion, so the front propagates without much deformation, keeping the bore shape, whereas smaller value of C means relatively diffusion dominates, so the front deforms substantially (according to viscous resistance). This behaviour is shown in Fig. 11. Again, all the solutions intersect at origin at the same half of the flow heights, the pivotal point.

4.5. Agreement between travelling wave and boundary layer solutions

The travelling wave solution with boundary conditions at infinity constructed in Sect. 4.2.1 and the boundary layer solutions constructed in Sect. 4.4 agree completely. This has been shown in Fig. 12.

This is very interesting, because the two solutions developed based on two completely independent solution methods are almost identical. The merging of the independently constructed solutions is great because: (*i*) both solutions prove their physical relevance and (*ii*) highlight the underlying physics of the model.

5. Summary

In this paper, we discussed how the nonlinear sub-diffusion and sub-advection model in consideration can be viewed as an extension of viscous Burgers equation but with much higher complexity describing significantly different physical phenomenon. This allowed us to effectively probe our model using existing solution techniques for viscous Burgers equation. We observed different solution methods used here produced very similar analytical solutions. This highlights the intrinsic characteristics of the flow of fluid in porous material as described by our model. These features are represented by propagating shock or bore-type fronts with backward elongated thin tails, and asymmetrical forms with compact supports for most of the solutions constructed with general boundary conditions by applying the method of Lie symmetry and travelling wave. These characteristics are similar to those produced by high-resolution shock-capturing numerical solution technique applied to the considered model. Hence, both the analytical solutions constructed here and the existing high-resolution solution presented previously independently support each other and explain the very special features of the sub-advection and sub-diffusion fluid flow in porous and debris material. The analytical solutions produced by different techniques, namely the travelling wave solutions with boundary at infinity and the boundary layer methods, produce virtually the same results. This implies that both the constructed solutions and the model equations are physically meaningful. We also observed that both the solutions to our nonlinear advection and diffusion equation and the viscous Burgers equation produce exactly the same results in the limit of vanishing viscosity. Although different solutions may represent different physical scenarios, these solutions must further be scrutinized, for example, by experimental data or observations. All this implies that the novel exact and analytical solutions obtained here are physically meaningful and can be applied to calculate the complex nonlinear re-distribution of fluid, from a given source or an initial fluid profile, in porous landscape, and debris and porous materials.

Acknowledgements

This work has been financially supported by the German Research Foundation (DFG) through the research projects, PU 386/3-1: "Development of a GIS-based Open Source Simulation Tool for Modelling General Avalanche and Debris Flows over Natural Topography" within a transnational research project, D-A-CH, and PU 386/5-1: "A novel and unified solution to multi-phase mass flows": $\mathrm{U}^{\mathrm{MultiSol}}$. Santosh Kandel's research is partially supported by the NCCR SwissMAP, funded by the Swiss National Science Foundation, by the SNF Grant No. 200020 172498/1, and by the COST Action MP1405 QSPACE, supported by COST(European Cooperation in Science and Technology).

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(Received: March 14, 2018)