Class 2 Notes: Introduction to Python, Introduction to Limits

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1 Introduction

In this lecture, we will discuss the fundamentals of limits. We will explore the precise mathematical definitions of limits, and show how to compute them from an algebraic perspective, where applicable. In addition, you will learn the fundamentals of Python. Given that the text book I have listed in this course for the Python programming language, I have decided not to summarize the fundamentals of the Python Programming Language. Rather, I expect you to quickly browse, and practice, the first 3 chapters of our required Python book. There is nothing more that I can to the value of this book, which is why I have decided to let you read these chapters on your own. We will review through the fundamentals of Python in our virtual video. However, I still expect you to read through the first three chapters of the Python text book.

2 Fundamentals of Limits of Sequences

We introduced in the previous lecture the idea of "approximating", by taking sequences of numbers, and trying to guess where the sequence of the numbers "go". This is a fundamental idea in Calculus known as *limits*. All of calculus is built on the idea of "what happens to one thing when we change or move another thing"? In order to understand limits, a good starting point is to understand sequences. We will only cover the fundamentals of sequences here, and return back to them in a later lecture.

A sequence $(n_1, n_2, n_3, \dots)$ is a collection of numbers that is in a bijection with $\mathbb{N}$. Typically, sequences are built by using the previous value and constructing a new one, or, by using an equation that can compute the ith value in the sequence when given a value for i. For example, take the Fibonacci sequence. This sequence is built by adding the prior two numbers. So, if F_n represents the number in the nth spot in the sequence, then we can find the value of this by adding the prior two values: $F_n = F_{n-1} + F_{n-2}$. If $F_0 = 1$ and $F_1 = 1$, then we can use that equation to compute F_2: $F_2 = F_1 + F_0 = 1 + 1 = 2$. We then can compute F_3: $F_3 = F_2 + F_1 = 2 + 1 = 3$. The resulting sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$ (Fun Fact: Fibonacci Sequences are often used to help predict the value of stocks).

Definition 1 *A sequence $(n_1, n_2, n_3, \dots)$ is a collection of numbers that is in a bijection with $\mathbb{N}$, or, a subset of $\mathbb{N}$. The natural numbers in the domain $\mathbb{N}$ indicate the position in the sequence.*

A common application of sequences is to see "where they go". Intuition would tell us that if we take a sequence to infinity, then the numbers in the sequence should also go to infinity. We have learned however that intuition often fails us in mathematics. Consider the sequence $1, 1, 1, 1, 1, 1, 1, 1, \dots = \{s_n = 1 | n \in \mathbb{N}\}$. As we can see, all values in the sequence are just simply set equal to 1. So if we take $n \to \infty$, then the sequence will just converge to 1. We can write this as $\lim_{n\to\infty} s_n = 1$. The left hand side of this says "take the sequence, defined by the formula s_n for the nth term in the sequence, to infinity. The sequence converges to the number 1".

Not all terms in the sequence will have the same value as it's limit, however. Take the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots = \{s_n = \left(\frac{1}{2}\right)^n | n \in \mathbb{N}\}$. Notice that if we take this sequence to infinity, we get closer and closer to the value of 0. Yet, it is impossible to have any term in the sequence set equal to 0 ($\frac{1}{2}$ is a positive number that is multiplied by itself, another positive number, and it is impossible to obtain 0 by multiplying two positive numbers). Yet, we can clearly see from the sequence that $\lim_{n\to\infty} s_n = 0$. One way to interpret this is that there exists no real number that will allow our sequence to exactly equal number. However, the only time that the sequence will equal 0 is in the set of *extended real numbers*, which is $\mathbb{R} \cup \{\infty\}$, and it only does so at a single point, namely ∞.

Sometimes, the limit of a sequence converges to infinity or negative infinity. Take the sequence $s_n = n$. We can clearly see that this sequence, $0, 1, 2, 3, 4, 5, \dots$ goes to infinity. In this instance, we write $\lim_{n\to\infty} s_n = \infty$. We can see that the opposite is true too. Consider $s_n = -n$. We can see that we get the sequence $-1, -2, -3, -4, -5, \dots$. In this instance, the sequence converges to negative infinity, and we write $\lim_{n\to\infty} s_n = -\infty$.

Not all sequences converge, be it to a number or infinity. Sometimes, sequences just do not "land" on a point. Take for example the sequence
$1, -1, 1, -1, 1, -1, 1, -1, 1, \dots = \{s_n = (-1)^n | n \in \mathbb{N}\}$. In this case, we say that the limit does not exist, and we sometimes write $\lim_{n\to\infty} s_n = D.N.E.$ If a limit of a sequence does exist, then we write $\lim_{n\to\infty} s_n = L$, where $L \in \mathbb{R} \cup \{-\infty, \infty\}$.

3 The Limit Laws

We can see that limits can be approximated, but can they be exactly computed algebraically? The answer is, yes, well, more along the lines of, it depends. For limits of sequences, we commonly have a few known facts about basic sequences that help us compute limits algebraically. Here are a few:

$$\lim_{n\to\infty} \frac{1}{n} = 0 \tag{1}$$

If the limits of the sequences a_n and b_n exist, that is $\lim_{n\to\infty} a_n = L_a$ and $\lim_{n\to\infty} b_n = L_b$, where $L_a, L_b \in \mathbb{R} \cup \{-\infty, \infty\}$then:

$$\lim_{n\to\infty} a = a, a \in \mathbb{R} \tag{2}$$

$$\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \tag{3}$$

$$\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n \tag{4}$$

$$\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n \tag{5}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \tag{6}$$

Let's take an example. Suppose we would like to find $\lim_{n\to\infty} \frac{5}{\frac{1}{n}-2}$. From rule (8), we know that $\lim_{n\to\infty} \frac{5}{\frac{1}{n}-2} = \frac{\lim_{n\to\infty} 5}{\lim_{n\to\infty} \frac{1}{n}-2}$. From rule (4), we know that $\lim_{n\to\infty} 5 = 5$, and from rule (5) that $\lim_{n\to\infty} \frac{1}{n} - 2 = \lim_{n\to\infty} \frac{1}{n} - \lim_{n\to\infty} 2$. We know from rule (3) that $\lim_{n\to\infty} \frac{1}{n} = 0$ and from rule (4) that $\lim_{n\to\infty} 2 = 2$. Making these substitutions, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{5}{\frac{1}{n} - 2} &= \frac{\lim_{n\to\infty} 5}{\lim_{n\to\infty} \frac{1}{n} - \lim_{n\to\infty} 2} \\
&= \frac{5}{0-2} \\
&= -\frac{5}{2} = -2.5
\end{aligned}$$

We can see that this sequence was compute using algebra and the rules put forth above. We can see how this sequence converges to -2.5 in the table below by considering the first 11 numbers in the sequence (try to compute these by hand using the equation, you should get the same results):

n	s_n
0	0.00
1	-5.00
2	-3.33
3	-3.00
4	-2.86
5	-2.78
6	-2.73
7	-2.69
8	-2.67
9	-2.65
10	-2.63

4 Limits of Functions

What if our independent variable n was not in the natural numbers, but rather in the real numbers? Unlike earlier where we had an idea of "counting" to be able to understand the values of sequences as we moved from one value in the natural numbers to another, we also can consider limits when our input variable is in an uncountable set. This is a difficult concept to grasp, but try to think about it from this perspective. Consider walking on a side walk. That is, consider the analogy of "moving" from number to number as walking on a sidewalk. Sidewalks have rectangular slabs of concrete. When we try to find limits of sequences, we are essentially "jumping" from one concrete slab to the next. On the other hand, when we find limits of functions, we are not "jumping", but rather slowly stepping, possibly on the same slab of concrete, until we move on to the next slab, covering all of the concrete in between with our feet. Thus, limits of functions work similar, the only difference being in *how* we move the independent variable to determine what happens to the dependent variable.

Definition 2 *Given a function $f(x)$ and a value of $a \in \mathbb{R}$, we say that L is the limit of the function $f(x)$ as x approaches a if we were to take a sequence of points x_i, which converge to a, and the resulting resulting sequence $f(x_i)$ converges to L. In other words, when we take values of x very close, but not equal to, the value of a, the values of $f(x)$ become very close, although not necessarily equal to, L. In this case, we write:*

$$\lim_{x \to a} f(x) = L$$

When we take limits of functions, we typically approach a number a, or $-\infty, \infty$. However, we can think about approaching these values from more than one direction. Consider the function $f(x) = x^2$, and suppose we would like to determine what happens to this function, value-wise, as we take our x and approach a number, say $a = 4$. So we would like to find $\lim_{x \to 4} x^2$. Now we can "approach" the number 4 from one of two directions. We can start at an arbitrary value for x that is less than 4, and "move" the values of x closer to the value of 4 but increasing the values. On the other hand, we can take the same approach, but starting at the right of 4 and moving to the left by decreasing the values. Let's pick a few points. Suppose we pick the points $3, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9$. In these movements, we are starting at an arbitrary number (1) and "moving" to the right by increasing it. As we move, we can see how the $f(x)$ moves:

x	$f(x)$
3.00	9.00
3.10	9.61
3.20	10.24
3.30	10.89
3.40	11.56
3.50	12.25
3.60	12.96
3.70	13.69
3.80	14.44
3.90	15.21

We notice that as we "move" the value of x closer to the value of 4, the value of $f(x)$ is increasing, and moving closer to 16. We can observe this same outcome but by moving from the right to the left, using the numbers $4.9, 4.8, 4.7, 4.6, 4.5, 4.4, 4.3, 4.2, 4.1$:

x	$f(x)$
4.90	24.01
4.80	23.04
4.70	22.09
4.60	21.16
4.50	20.25
4.40	19.36
4.30	18.49
4.20	17.64
4.10	16.81

We again notice that as we move closer to the value of 4, our values of $f(x)$ move closer to the value of 16. We therefore would say that $\lim_{x\to 4} x^2 = 16$. In general, limits of functions follow the same rules as limits of sequences, so we will not repeat them here. However, there is another handy rule about limits that we can use:
If $f(x)$ is a polynomial or rational function with the number a in the domain of the function, then $\lim_{x\to a} f(x) = f(a)$.

For example, suppose we wanted to find $\lim_{x\to 10} \frac{x-4}{(x-9)(x-8)}$, then since the number 10 is in the domain (that is, the value of $f(10)$ is defined), we would have: $\lim_{x\to 10} \frac{x-4}{(x-9)(x-8)} = \frac{10-4}{(10-9)(10-8)} = \frac{6}{(1)(2)} = 3$. If, on the other hand, we were trying to find $\lim_{x\to 8} \frac{x-4}{(x-9)(x-8)}$, we would need to think more carefully about figuring our the limit (I will leave it as an exercise for you to figure out what it is.)

Limits of functions can only exist if we obtain the same limit of the function if we approach it from the left as we do the right. Here is a classic example of a limit of a function that does not exist: $\lim_{x\to 0} \frac{1}{x}$. If we approach this number from the left, we obtain the following values (just pick a few arbitrary x values moving from left to right (that is, increasing the values), approaching 0):

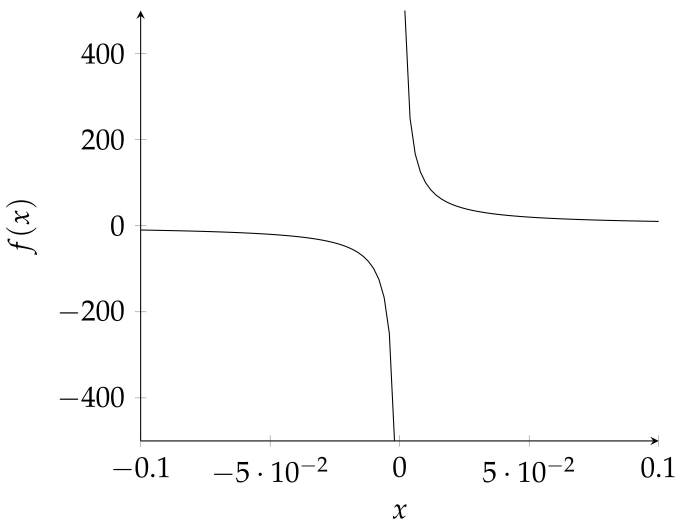

Figure 1: The function $\frac{1}{x}$. We can see that as we approach it from the left, it goes to $-\infty$, but approaching it from the right brings us to ∞, which means the limit of $f(x)$ at $a = 0$ does not exist for this function.

x	$f(x)$
-0.10	-10.00
-0.01	-100.00
-0.001	-1000.00
-0.0001	-10000.00
-0.00001	-100000.00

We notice that as $x \to 0$, $f(x) \to -\infty$. We refer to this as a *left-sided limit*, and denote it as $\lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$.

Definition 3 *Given a function $f(x)$ and a value $a \in \mathbb{R}$, if we select a sequence of values x_i such that $x_i < a \forall i$ in the sequence, but the sequence gets closer to a as we increase i (and so, x_i would be increasing towards a), then we say that L is the left-hand limit of $f(x)$ as $x \to a$ from the left if the values of $f(x_i)$ approach L. In this instance, we write:*

$$\lim_{x \to a^-} f(x) = L$$

If we do the same thing but in the opposite direction, that is, move from right to left towards zero, we can see where the function moves:

x	$f(x)$
0.10	10.00
0.01	100.00
0.001	1000.00
0.0001	10000.00
0.00001	100000.00

In this case, we notice that as $x \to 0$ but from the right, $f(x) \to \infty$. We refer to this as a *right-sided limit*, and denote it as $\lim\limits_{x \to 0^+} \frac{1}{x} = \infty$. More generally we have:

Definition 4 *Given a function $f(x)$ and a value $a \in \mathbb{R}$, if we select a sequence of values x_i such that $x_i > a \forall i$ in the sequence, but the sequence gets closer to a as we increase i (and so, x_i would be decreasing towards a), then we say that L is the right-hand limit of $f(x)$ as $x \to a$ from the right if the values of $f(x_i)$ approach L. In this instance, we write:*

$$\lim_{x \to a^+} f(x) = L$$

Obviously we can see from this example that $\lim\limits_{x \to 0^-} \frac{1}{x} \neq \lim\limits_{x \to 0^+} \frac{1}{x}$, and in this case, we say that the limit does not exist, denoted by $\lim\limits_{x \to 0} \frac{1}{x} = D.N.E$. Generally, we have:

Definition 5 *Given a function $f(x)$, we say that the limit exists at a point a if and only if it's right-hand limit equals it's left hand limit. In order words, $\lim_{x \to a} f(x) = L$ if and only if:*

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

5 The Limit Laws for Functions

Limits of functions are slightly difference than limits for sequences. In limits of sequences, we try to determine where the sequence is "going" as we increase the natural numbers, starting at 0, and going up to ∞. In functions, however, we saw that we can take limits as the input variable get "arbitrarily close to" a number. We saw that limits can be computed by writing a few numbers close to a, on both sides of a, and see where $f(x)$ is "going". However, just list with limits of sequences, we also have limit laws for functions that can be leveraged:

Theorem 1 *If the limits of the functions $f(x)$ and $g(x)$ as x approaches a exist (that is, their right limits equal their left limits), and if $c \in \mathbb{R}$ is a constant, then:*

$$\lim_{x \to a} c = c \tag{7}$$

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \tag{8}$$

$$\lim_{x \to a} f(x) - g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \tag{9}$$

$$\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \tag{10}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}, if \lim_{x \to a} g(x) \neq 0 \tag{11}$$

If we apply rule (10) a total of n times, then we actually get the following rule:

Theorem 2

$$\lim_{x\to a}\left[f(x)^n\right] = \left[\lim_{x\to a} f(x)\right]^n \tag{12}$$

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} \tag{13}$$

Another handy theorem that helps us easily reduce down our work when computing limits is the direct substitution property. As it turns out, the limits of polynomial and rational functions (in their domain at least) can be easily found by simply substituting in the value that x is approaching into the function itself:

Theorem 3 *Let $f(x)$ be a function and suppose that $x \to a$. If $f(x)$ is a rational function or a polynomial, and if a is in the domain of $f(x)$, then*

$$\lim_{x\to a} f(x) = f(a)$$

Some functions are incredibly difficult to find the limit of. Take, for example, the function $x^2 \sin\frac{1}{x}$. There is no limit rule to directly compute this. However, there is a theorem that makes it very easy to do so:

Theorem 4 *Let $f(x) \leq g(x) \leq h(x)$ near the value of a, although not necessarily at a. Then, if the limits of the individual functions as $x \to a$ exist, the following is true:*

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \tag{14}$$

$$\textit{If } \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L, \textit{ then } \lim_{x\to a} g(x) = L. \tag{15}$$

Going back to our function from before, we can see that since $-1 \leq \sin\frac{1}{x} \leq 1$, then we know that $-x^2 \leq x^2\sin\frac{1}{x} \leq x^2$. Since $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0} -x^2 = 0$, we must have $\lim_{x\to 0} x^2 \sin\frac{1}{x}$. Hence, we can apply the theorem to easily compute limits by leveraging the limits of upper and lower bounds of functions.

6 The Mathematical Definition of a Limit

When we defined the limit above, we did not give a formalized mathematical definition for a limit. In fact, our definition is a very intuitive one, but not a very rigorous one. This is problematic, since we need to understand what limits actually mean mathematically. Limits can not only be described from our intuitive perspective, but also a mathematical one.

Let's start with the basic idea for a limit of any function. First, we need a way to mathematically characterize the function "getting close but possibly not every touching" a number L. This characterization can be measured mathematically through the concept of differences. Recall that a difference between two numbers is a measure of "how close" the two numbers are to each other. Therefore, we can characterize "how close" the function $f(x)$ is to the number L by looking at the absolute value of the difference: $|f(x) - L|$. Why absolute value? Because we do not care about the direction of the distance, we only care about the magnitude of the distance to measure how "close" $f(x)$ is to L.

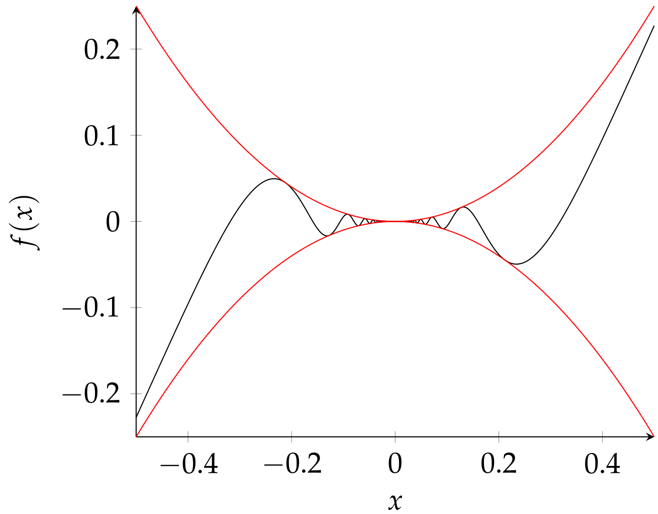

Figure 2: The function $f(x) = x^2 \sin \frac{1}{x}$ (the black curve) is less than the function $h(x) = x^2$ (the top red curve) for all $x \in \mathbb{R}$. Likewise, we can see that $g(x) = -x^2$ (the bottom red curve is less than $f(x)$ for all $x \in \mathbb{R}$. We can see easily that since $g(x)$ and $h(x)$ are polynomials, that $\lim_{x\to 0} g(x) = g(0) = 0$ and $\lim_{x\to 0} h(x) = h(0) = 0$. We see visually that $f(x)$ is "squeezed" between $g(x)$ and $h(x)$, and hence, as we approach 0, the value for $f(x)$ will also approach 0.

Now how do we characterize how "close" x is to the value of a? Given the logic above, we can also characterize how "close" x is to a by finding the absolute difference $|x - a|$. Therefore, when we say "bring x close to x", we are essentially bringing the quantity $|x - a|$ close to, but not touching, zero. When we bring $|x - a|$ close to 0, and if $\lim\limits_{x\to a} f(x) = L$, then we should see $|f(x) - L|$ get close to 0 as well.

There is another way to characterize this process of one quantity "getting close" to another quantity which results in the function quantity "getting close" to another quantity. Suppose we pick an absolute maximum distance between $f(x)$ and L, which we will call ϵ (that is, let $\epsilon = |f(x^*) - L|$) for some value x^*. The goal here is to find all the points x such that $|f(x) - L| < \epsilon$. Since we know that $\lim\limits_{x\to a} f(x) = L$, then these points must lie within an interval around a, with a in the center of the interval. To characterize this, we can consider a "maximum distance" from a, which we will denote as δ, such that $|f(x) - L| < \epsilon$ when x is within this maximum distance. That is, we can use a given maximum distance ϵ between the y-values to find a corresponding maximum distance δ between the x-values. To characterize x being "within this maximum distance", we know the distance from x to a must be less than δ, or $|x - a| < \delta$. So, we know that $x - a < \delta$ when $x > a$ and that $a - x < \delta$ when $x < a$. So we would have $x < a + \delta$ and $a - \delta < x$. Hence, we know that x must be in the interval $[a - \delta, a + \delta]$ in order for $|f(x) - L| < \epsilon$.

That is, if we know that $\lim\limits_{x\to a} f(x) = L$, then we can pick a maximum distance $\epsilon > 0$, as close to 0 as we like, and as such, we should be able to find a $\delta > 0$ such that when our x is within a distance of δ from a (or, when $|x - a| < \delta$), then $|f(x) - L| < \epsilon$. If we can find a $\delta > 0$ when given any value $\epsilon > 0$, then this means that we can take the distance as close to 0 as possible, and still be able to find points such that their functional values will be close to L. This is the way we can characterize this process of limiting, which we summarize here:

Definition 6 *Suppose we have a function $f(x)$. Given any value $\epsilon > 0$, if we can find a $\delta > 0$ such that when $|x-a| < \delta$ that $|f(x) - L| < \epsilon$, then we say that the limit of $f(x)$ as x approaches a is equal to L. Put differently, if given a value of $\epsilon > 0$, we can find a value $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all values of x in the interval $[a-\delta, a+\delta]$. If this is true for all values of $\epsilon > 0$, then we write $\lim_{x\to a} f(x) = L$.*

Example 1 *Consider the function $f(x) = x^2$. We know that since this is a polynomial, that $\lim_{x\to 2} x^2 = f(2) = 4$. To prove that this is true using the mathematical definition of a limit, we do the following. We need to show that given any value $\epsilon > 0$, we can find a value $\delta > 0$ such that when $x \in (a-\delta, a+\delta)$ (or, if $|x-a| < \delta$), that $|f(x) - L| < \epsilon$. We can find an expression for this δ by "working backwards", and then showing that the statement is true. To work backwards, we start with the expression $|f(x) - L| < \epsilon$:*

$$\begin{aligned} |f(x) - L| &< \epsilon \\ |x^2 - 4| &< \epsilon \\ |(x+2)(x-2)| &< \epsilon \\ |(x+2)||(x-2)| &< \epsilon \end{aligned}$$

Now since x will be restricted in the values it can take on (recall, we must have $|x-2| < \delta$), then there exists a number k such that $|x+2| < k$. Since we are trying to show this for all $\epsilon > 0$, it is reasonable to assume that the choice of ϵ is small enough to render $|x-2| < 1$. That is, we can bring ϵ "close enough" to 0 to make that statement $|x-2| < 1$ a true statement for ϵ smaller than the chosen ϵ. This means that $-1 < x-2 < 1$, or $-1+2 < x < 1+2$, or $1 < x < 3$, or $1+2 < x+2 < 3+2$, or $3 < x+2 < 5$. So we must have $|x+2| < 5$. Since $|x+2| < 5$ and $|x-2| < \delta$, we have:

$$|(x+2)||(x-2)| < 5\delta$$

So, if we set $\epsilon = 5\delta$, then we have $\delta = \frac{\epsilon}{5}$. We must ensure, however, given our new constraint placed on x, that both statements hold true: $|x-3| < 1$ and $|x-3| < \delta \to |x-3| < \frac{\epsilon}{5}$. To ensure both of these hold true simultaneously, we will set $\delta = \min\{1, \frac{\epsilon}{5}\}$.

Now we are ready to show that $\lim_{x\to 2} x^2 = 4$. Let $\epsilon > 0$. If we let $\delta = \min\{1, \frac{\epsilon}{5}\}$ then $|x-2| < 1$, and so $-1 < x-2 < 1 \to -1+2 < x < 1+2 \to 1+2 < x+2 < 3+2 \to 3 < x+2 < 5$. Hence, $|x+2| < 5$. For ϵ small enough, we must have $\delta = \min\{1, \frac{\epsilon}{5}\} = \frac{\epsilon}{5}$ So, we must have:

$$\begin{aligned} |f(x) - L| &= |x^2 - 4| \\ &= |x-2||x+2| \\ &< \delta 5 \\ &< \frac{\epsilon}{5} 5 \\ &= \epsilon \end{aligned}$$

Therefore, we have shown that when given a sufficiently small $\epsilon > 0$, we can find a δ (namely, $\delta = \frac{\epsilon}{5}$) such that $|f(x) - L| < \epsilon$ when $|x-a| < \delta$. Hence, we have proven that $\lim_{x\to 2} x^2 = 4$

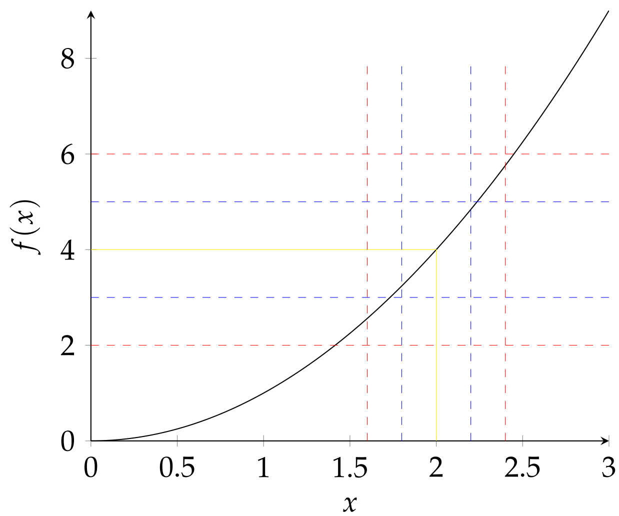

Figure 3: Computing $\lim_{x\to 2} x^2$. Notice that if we select $\epsilon = 2$ (the red lines), we can find a δ, namely $\delta = \frac{\epsilon}{5} = \frac{2}{5} = 0.4$. Notice that when $2 - .4 < x < 2 + .4$, that $4 - 2 < f(x) < 4 + 2$. If we reduce the distance down to $\epsilon = 1$ (the blue lines), we can find $\delta = \frac{\epsilon}{5} = \frac{1}{5} = 0.2$. Notice that when $2 - .2 < x < 2 + .2$, that $4 - 1 < f(x) < 4 + 5$. If we continue with this process by selecting smaller distances from 4, then we can see that we find a δ that will define an interval $2 - \delta < x < 2 + \delta$, and we will have $4 - \epsilon < f(x) < 4 + \epsilon$. This means, that as we take ϵ close to 0 (but never touching it), δ will also get close to 0.

Recall that we said that a limit only exists if the right and left limits equal each other. This would imply that we also need to mathematically characterize left and right hand limits. We can do so in a similar manner as before:

Definition 7 *Suppose we have a function $f(x)$. Given any value $\epsilon > 0$, if we can find a $\delta > 0$ such that when $\delta - a < x < a$ that $|f(x) - L| < \epsilon$, then we say that the left-hand limit of $f(x)$ as x approaches a is equal to L. Put differently, if given a value of $\epsilon > 0$, we can find a value $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all values of x in the interval $(a - \delta, a)$. If this is true for all values of $\epsilon > 0$, then we write $\lim\limits_{x\to a^-} f(x) = L$.*

Definition 8 *Suppose we have a function $f(x)$. Given any value $\epsilon > 0$, if we can find a $\delta > 0$ such that when $a < x < a + \delta$ that $|f(x) - L| < \epsilon$, then we say that the right-hand limit of $f(x)$ as x approaches a is equal to L. Put differently, if given a value of $\epsilon > 0$, we can find a value $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all values of x in the interval $(a, a + \delta)$. If this is true for all values of $\epsilon > 0$, then we write $\lim\limits_{x\to a^+} f(x) = L$.*

So what does our definition look like when a limit does not exist? If we look at Figure 4, we can see that the function $f(x) = \begin{cases} 3x^2 & x < 1.5 \\ 8x & x > 1.5 \end{cases}$ does not have a limit at $x = 1.5$. This is obvious, since $\lim\limits_{x\to 1.5^-} f(x) = 6.75 \neq 12 = \lim\limits_{x\to 1.5^+} f(x)$. With respect to our definition, notice the red lines. We can select a value of ϵ, but notice that we cannot find a corresponding δ such that when $|x - 1.5| < \delta$ that $|f(x) - L| < \epsilon$. If we select $L = 12$, then for any value of ϵ, $|f(x) - 12| > \epsilon$ when $x < 1.5$ (that is, the values of $f(x)$ fall below the bottom horizontal red line). On the other hand, if we set $L = 6.75$, then we can see that we cannot find a δ such that when $|x - 1.5| < \delta$,

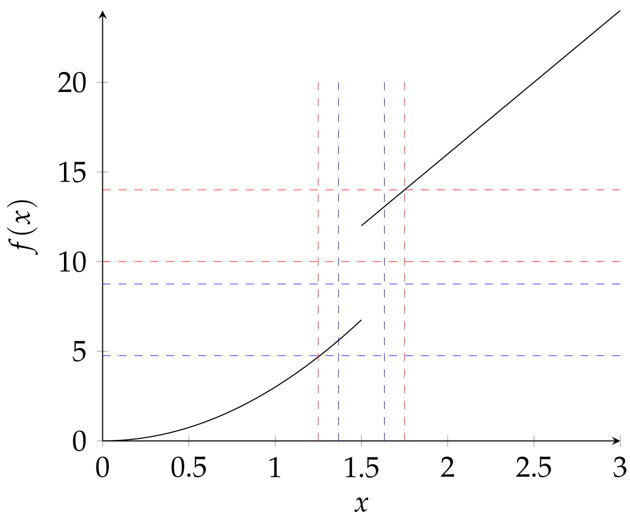

Figure 4: The function $f(x) = \begin{cases} 3x^2 & x < 1.5 \\ 8x & x > 1.5 \end{cases}$

that $|f(x) - L| < \epsilon$. Notice the reason why. We notice that $|f(x) - 6.75| > \epsilon$ when $x > 1.5$ (that is, the values of $f(x)$ fall above the top horizontal blue line). In fact, it can be shown that for any value $\epsilon > 0$, we cannot find a value δ such that when $|x - 1.5| < \delta$, $|f(x) - L| < \epsilon$ for any chosen L. This means, the limit does not exist (however, it can be shown that the left and right limits do exist).

7 Continuous Functions

An all important topic in Calculus and generally in mathematics is the notion of a continuous function. Continuous functions allow us to conduct mathematics in an easy way. Not only this, but continuous functions have a lot of "nice" properties that often make our analysis very easy. Now that we have defined the concept of a limit, we can define a continuous function:

Definition 9 *A function $f(x)$ is said to be continuous at a point a if* $\lim_{x \to a} = f(a)$.

Definition 10 *Let $f(x)$ be a function defined as $f : A \to B$. Then $f(x)$ is said to be a continuous function if* $\lim\limits_{x \to a} f(x) = f(a), \forall a \in A$.

Continuous functions have nice properties that allow us to simplify some of our mathematical analysis. Some fundamental laws of continuous functions are as follows:

Theorem 5 *If f and g are continuous function at a, and k is a constant, then the following function are* *also continuous at a:*

$$f+g \tag{16}$$
$$f-g \tag{17}$$
$$kf \tag{18}$$
$$fg \tag{19}$$
$$\frac{f}{g}, \text{ only if } g(a) \neq 0 \tag{20}$$

Earlier we stated that for polynomial and rational function, that $\lim_{x\to a} f(x) = f(a)$. This would mean that

Theorem 6 *Every polynomial is continuous. In addition, every rational function $f(x) = \frac{g(x)}{h(x)}$ is continuous at every point except for those where $h(x) = 0$.*

Theorem 7 *The following types of functions are continuous in their respective domains:*

1. *Polynomials*
2. *Rational Functions*
3. *Root Functions*
4. *Exponential functions*
5. *Logarithmic Functions*

In addition, another very useful theorem is that of the continuity of composition of functions.

Theorem 8 *If f is continuous at b, and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$, that is: $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)$*

This theorem leads us to the following theorem:

Theorem 9 *If g is continuous at a and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a.*

These theorems lead us to a very important theorem that we can leverage to identify minimum and maximum values of a function algorithmically:

Theorem 10 *(The Intermediate Value Theorem) Suppose that f is continuous at all point in the interval $[a,b]$, and let N be any number between $f(a)$ and $f(b)$, where $f(x) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$.*

Example 2 *Let $f(x) = x^2 - 4$. We can see that $f(1) = 1^2 - 4 = -3$ and $f(3) = 3^2 - 4 = 9 - 4 = 5$. Notice that since $f(1) \neq f(3)$, and since $f(1) < 0$ while $f(3) > 0$, and since $f(x)$ is continuous, then according to the intermediate value theorem, there must exist a number $c \in (1,3)$ such that $f(c) = 0$. That is, the root of this function must be somewhere between $(1,3)$. If we had chose points bigger than 1 and smaller than 3, then we can approximate the value of c.*

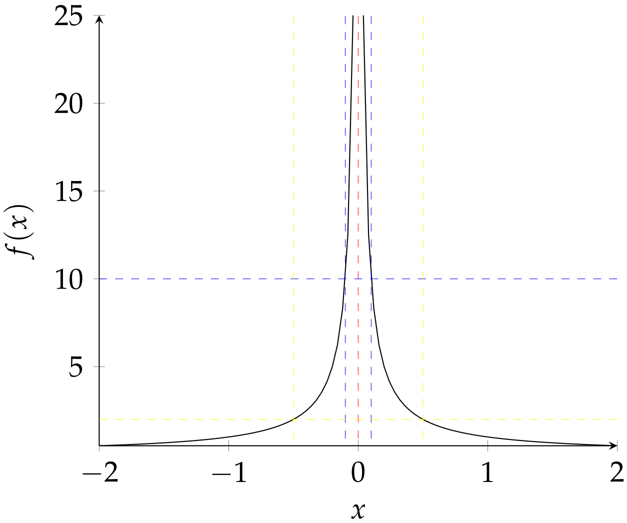

Figure 5: The function $f(x) = \left|\frac{1}{x}\right|$. Notice that when $y = 2$, we can set $\delta = 0.5$. When $|x| < 0.5$, or when $-0.5 < x < 0.5$, $f(x) > 2$ (the yellow lines). Likewise, when $y = 10$, we see that $\delta = 0.1$ so that when $-0.1 < x < 0.1$, $f(x) > 10$. Continuing in this fashion, we see that for any y, we can set $\delta = \frac{1}{y}$, and $f(x) > y$ when $-\delta < x < \delta$.

8 Limits of Functions Involving Infinity

So far, we have defined limits as the independent variable approaches a number a. Likewise, we have only looked at limits that are finite (that is, in cases where $L \in \mathbb{R}$. However, there are many applications of limits that involve understanding what happens at infinity, or, our limits tending to converge at infinity. In such instances, we need a refined definition and methodology for handling and computing limits.

Definition 11 *Let $f(x)$ be function. Then $\lim\limits_{x \to a} f(x) = \infty$ if $f(x)$ can be made arbitrarily large and close to ∞ by making x arbitrarily close to the value of a.*

Example 3 *Consider the function $f(x) = \left|\frac{1}{x}\right|$. We can see from Figure 5 that $\lim\limits_{x \to 0} \left|\frac{1}{x}\right| = \infty$*

We can define this type of limit more formally:

Definition 12 *Let $f(x)$ be a function and $a \in \mathbb{R}$. We say that the limit of $f(x)$ as x approaches a is ∞ if for any $y \in (0, \infty)$, there exists a $\delta > 0$ such that $f(x) > y \; \forall x \in (a - \delta, a + \delta)$. If this is true, then we write $\lim\limits_{x \to a} f(x) = \infty$.*

Example 4 *Consider again the function $f(x) = \left|\frac{1}{x}\right|$. To show that $\lim_{x \to 0} \left|\frac{1}{x}\right| = \infty$ we need to show that given any value $y > 0$, there exists a $\delta > 0$ such that $f(x) > y$ for all x such that $0 - \delta < x < 0 + \delta$. Just as before, we can work backwards:*

$$
\begin{aligned}
f(x) &> y \\
\left|\frac{1}{x}\right| &> y \\
|x| &< \frac{1}{y}
\end{aligned}
$$

Let $\delta = \frac{1}{y}$. Assume that $|x - 0| < \delta$. Then $\frac{1}{|x|} > \frac{1}{\delta}$. Using this, we have:

$$
\begin{aligned}
f(x) &= \frac{1}{|x|} \\
&> \frac{1}{\delta} \\
&= \frac{1}{\frac{1}{y}} \\
&= y
\end{aligned}
$$

Sometimes, we have functions $f(x)$ that converge to a number L when the value of x is taken to be an arbitrarily large number. For example, the function $f(x) = \frac{1}{x}$ converges to a number when x is taken to a very large number near ∞. Figure 6 illustrates that as x becomes larger and larger, $\frac{1}{x}$ becomes smaller and smaller, becoming arbitrarily close to the value of 0. We can make this type of definition of a limit more precise with the following mathematical definition:

Definition 13 *Let $f(x)$ be a function. Suppose we take the value of x to be arbitrarily large. Then we say that the limit of $f(x)$ as x approaches infinity is equal to L if for all $\epsilon > 0$, there exists a number $N \in (0, \infty)$ such that $|f(x) - L| < \epsilon \forall x > N$.*

Example 5 *We can show that $\lim_{x \to \infty}$ by trying to find a value of N, when given a value of ϵ, such that $|f(x) - L| < \epsilon$ when $x > N$. To do so, we can work backwards:*

$$
\begin{aligned}
|f(x) - L| &< \epsilon \\
|\frac{1}{x} - 0| &< \epsilon \\
\frac{1}{|x|} &< \epsilon \\
|x| &> \frac{1}{\epsilon}
\end{aligned}
$$

Let $N = \frac{1}{\epsilon}$. If $x > N$, then we have $x > \frac{1}{\epsilon}$, which means $\frac{1}{x} < \epsilon$. Since $N > 0$, then $x > 0$, and so

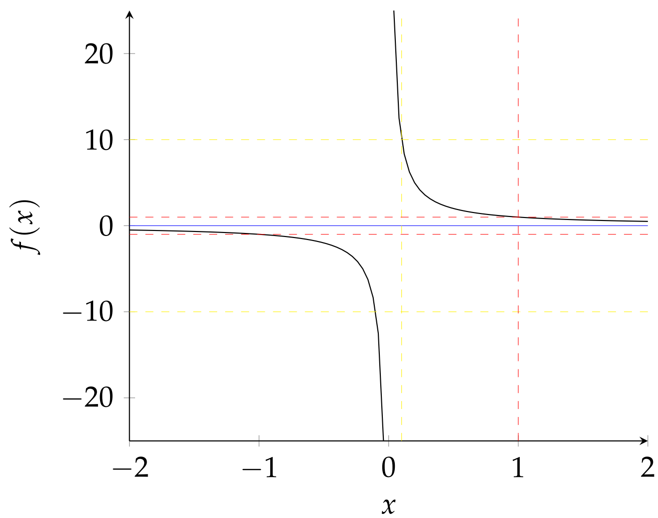

Figure 6: The function $f(x) = \frac{1}{x}$. Notice that as x increases to a very large number, $f(x)$ gets very close to 0

$|x| = x$. *Using this, we have:*

$$\begin{aligned} |f(x) - L| &= |\frac{1}{x} - 0| \\ &= \frac{1}{|x|} \\ &= \frac{1}{x} \\ &< \epsilon \end{aligned}$$

The previous results lead us to the following theorem, which can be proven by similar types of derivations:

Theorem 11 *Let $r > 0$ be a rational number. Then we have:*

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$