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Integration w/ Power Series

Taylor Series

$$\int e^{-2x}\,dx$$

$$f(x) = e^{-2x}$$

$$f^{(0)}(0) = e^{0} = 1 \qquad 0$$

$$f^{(1)}(0) = -2 \qquad 1$$

$u = -2x$

$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}$$

$$= -2e^{-2x}$$

$$f^{(2)}(0) = 4 \qquad 2$$

$$f^{(3)}(0) = -8 \qquad 3$$

$$f^{(n)}(0) = (-2)^{n}$$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^{i}$$

$$= \boxed{\sum_{i=0}^{\infty} \frac{(-2)^{i}}{i!} x^{i}}$$

$$\int e^{-2x}\,dx = \int \sum_{i=0}^{\infty} \frac{(-2)^{i}}{i!} x^{i}\,dx$$

$$= \sum_{i=0}^{\infty} \int \frac{(-2)^i}{i!} x^i \, dx = \sum_{i=0}^{\infty} \frac{(-2)^i}{i!} \int x^i \, dx$$

$$= \sum_{i=0}^{\infty} \frac{(-2)^i}{i!} \left(\frac{1}{i+1} \right) x^{i+1}$$

$$= \sum_{i=0}^{\infty} \frac{(-2)^i}{(i+1)!} x^{i+1}$$

$$j = i+1$$

$$= -\frac{1}{2} \sum_{i=0}^{\infty} \frac{(-2)^{i+1}}{(i+1)!} x^{i+1}$$

$$= \left(-\frac{1}{2} \sum_{j=1}^{\infty} \frac{(-2)^j}{(j!)} x^j - \frac{1}{2} \right) + \frac{1}{2}$$

$$= -\frac{1}{2} \underbrace{\sum_{j=0}^{\infty} \frac{(-2)^j}{j!} x^j} + \frac{1}{2}$$

$$= -\frac{1}{2} e^{-2x} + \frac{1}{2} + C$$

Substitution

$f(x) = f(g(x))$, $\int \overline{f(g(x))\, g'(x)}\, dx = \int f(u)\, du$

$u = g(x)$

$\int e^{-2x} dx \to \boxed{u = -2x}$, $\frac{du}{dx} = -2 \to du = -2\,dx$

$\to dx = -\frac{1}{2} du$

$= \int e^{u}\left(-\frac{1}{2} du\right)$

$= -\frac{1}{2} \int e^{u} du$

$= -\frac{1}{2} e^{u} + C = -\frac{1}{2} e^{-2x} + C$

$\int \frac{1}{u} du = \ln|u|$

$\int \frac{1}{6-2x} dx$

$u = 6-2x \to du = -2dx$

$dx = -\frac{1}{2} du$

$= \int \frac{1}{u}\left(-\frac{1}{2} du\right)$

$= -\frac{1}{2} \int \frac{1}{u} du = -\frac{1}{2} \ln|u| + C = -\frac{1}{2} \ln|6-2x| + C$

$$\int \frac{1}{u} du$$

$$\int \frac{1}{6-2x^2} dx \qquad u = 6-2x^2 \qquad 2x^2 = 6-u$$

$$du = -4x\,dx$$

$$= \int \frac{1}{u}\left(-\frac{1}{4x}du\right) \qquad \underbrace{dx} = -\frac{1}{4x}du \qquad x = \pm\sqrt{\frac{6-u}{2}}$$

$$= -\frac{1}{4}\int \frac{1}{ux} du$$

$$\int \frac{x}{\sqrt{(1-2x^2)}} dx \rightarrow \begin{array}{l} u = 1-2x^2 \\ du = -4x\,dx \\ -\frac{1}{4}du = x\,dx \end{array}$$

$$\int \frac{1}{\sqrt{u}}\left(-\frac{1}{4}du\right) \rightarrow -\frac{1}{4}\int u^{-1/2} du$$

$$= -\frac{1}{4}\left(\frac{1}{-1/2+1} u^{-1/2+1}\right) + C \rightarrow$$

$$= -\frac{1}{4}\left(2u^{1/2}\right) + C$$

$$= -\frac{1}{2}\left(\sqrt{1-2x^2}\right) + C$$

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$$\int_0^1 e^{-\frac{1}{4}x}\,dx \rightarrow \quad u = -\tfrac{1}{4}x$$

$$du = -\tfrac{1}{4}\,dx \rightarrow dx = -4\,du$$

$$= -4\int_0^{-1/4} e^u\,du = -4e^u\Big|_0^{-1/4} = -4\left[e^{-1/4} - e^0\right]$$

$$= -4e^{-1/4} - 4$$

$$-4e^{(-\frac{1}{4}x)}\Big|_0^1 = -4e^{-\frac{1}{4}(1)} - 4.$$

$$\int x e^{-x^2}\,dx \qquad u = -x^2$$

$$du = -2x\,dx$$

$$-\tfrac{1}{2}\,du = x\,dx$$

$$= \int e^u\left(-\tfrac{1}{2}\,du\right)$$

$$= -\tfrac{1}{2}\int e^u\,du = -\tfrac{1}{2}e^u + C = -\tfrac{1}{2}e^{-x^2} + C$$

$$\int e^{x^2}\,dx \qquad u = x^2$$

$$du = 2x\,dx \rightarrow dx = \tfrac{1}{2x}\,du$$

$$\rightarrow \int e^u \cdot \tfrac{1}{2x}\,du \rightarrow \tfrac{1}{2}\int e^u \cdot \tfrac{1}{\sqrt{u}}\,du$$

Integration by Parts

$$\int \frac{d}{dx}[f(x)g(x)] = \int f'(x)\, g(x) + f(x)\, g'(x)\, dx$$

$$f(x)\, g(x) = \int f'(x)\, g(x)\, dx + \int f(x)\, g'(x)\, dx$$

$$\int \overset{u}{f(x)}\, g'(x)\, dx = f(x)\, \overset{v}{g(x)} - \int f'(x)\, g(x)\, dx$$

$$\int u\, dv = uv - \int v\, du \qquad f'(x) = \frac{du}{dx}$$

$$\int t^2 e^t\, dt$$

$u = t^2 \qquad \int dv = \int e^t\, dt$

$du = 2t\, dt \qquad v = e^t$

$$\int t^2 e^t\, dt = \int u\, dv = uv - \int v\, du = t^2 e^t - 2\int e^t\, t\, dt$$

$u = t \quad dv = e^t dt \qquad\qquad t^2 e^t - 2\left[te^t - \int e^t dt\right]$

$du = dt \quad v = e^t \qquad\qquad = t^2 e^t - 2te^t + 2e^t + C$

What is wrong

$$\int e^{t^2}\,dt = te^{t^2} - \int 2t e^{t^2}\,dt = te^{t^2} - e^{t^2} = (t-1)e^{t^2}$$

$u = e^{t^2}$ $\quad dv = dt$

$du = 2te^{t^2}\,dt$ $\quad v = t$

$u = t^2$

$du = 2t\,dt$

$\int e^u\,du$

$- \quad -e \quad -$

$$\int \ln x\,dx = x\ln(x) - \int x\left(\tfrac{1}{x}\right)dx = x\ln(x) - x + C$$

$u = \ln x$ $\quad dv = dx$

$du = \frac{1}{x}\,dx$ $\quad v = x$

Im

Improper Integration

$$\int_a^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_a^n f(x)\,dx$$

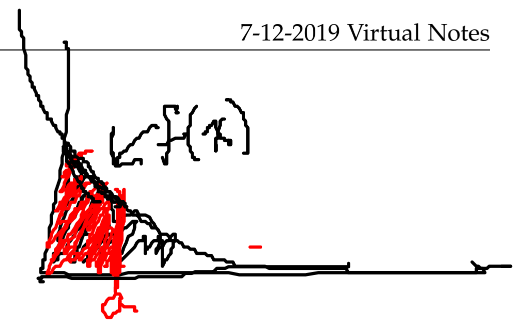
$$\int_{-\infty}^{a} f(x)\,dx = \lim_{n\to-\infty} \int_n^a f(x)\,dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

F

$f(x) = \lambda e^{-\lambda x}$

Exponential Dist



$X \quad [0,\infty) \quad P(X<a) = \int_0^a f(x)\,dx = F(a) - F(0)$

$$\int_0^\infty \lambda e^{-\lambda x}\,dx = 1$$

$$= \lim_{t\to\infty} \boxed{\int_0^t \lambda e^{-\lambda x}\,dx} = \quad u = -\lambda x, \quad -du = \lambda\,dx$$

$$= \int -e^u\,du = -e^u = -e^{-\lambda x}\Big|_0^t$$

$$= -e^{-\lambda t} + e^{-\lambda(0)} = 1 - e^{-\lambda t}$$

$$\Rightarrow \int_0^\infty \lambda e^{-\lambda x}\,dx = \lim_{t\to\infty} \boxed{\int_0^t \lambda e^{-\lambda x}\,dx} = \lim_{t\to\infty} 1 - e^{-\lambda t}$$

$$= 1$$

PDF

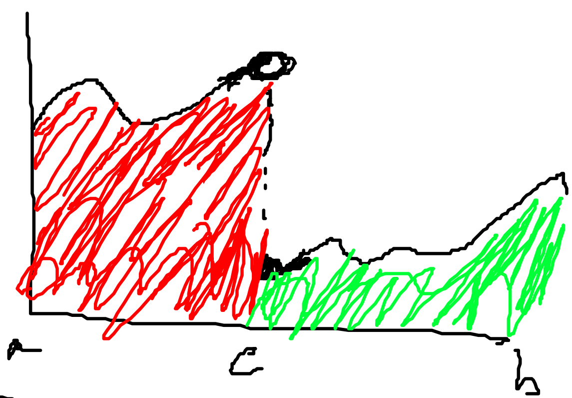
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{-2+1}x^{-2+1}\Big|_1^t\right]$$

$$= \lim_{t\to\infty}\left[-x^{-1}\Big|_1^t\right] = \left[-\frac{1}{t}+1\right]$$

$$= 1$$



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Newsvendor Model

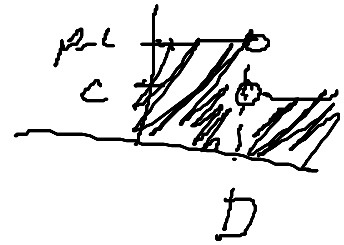
$D \sim N(\mu, \sigma)$ $f(x)$, $F(x)$

Q, c, p, D

$D < Q$ $D > Q$

$E[(Q-D)c$ $(D-Q)(p-c)]$

$$E\left[c\max\{0,Q-D\}+(p-c)\max\{D-Q,0\}\right]$$



$$E[g(x)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

$$= cE[\max\{0,Q-D\}] + (p-c)E[\max\{D-Q,0\}]$$

$$E[\max\{0,Q-D\}] = \int_{-\infty}^{\infty} \max\{0,Q-x\}f(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

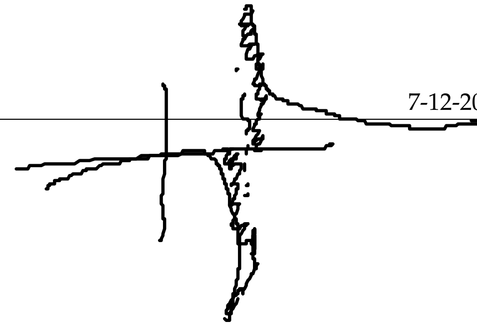
$$= \int_{-\infty}^{Q} \max\{0,Q-x\}f(x)\,dx = \int_{-\infty}^{Q}(Q-x)f(x)\,dx$$

$$+\int_{Q}^{\infty} \cancel{\max\{0,Q-x\}f(x)\,dx}$$

$$E[\max\{0,D-Q\}] = \int_{Q}^{\infty}(x-Q)f(x)\,dx$$

$$C(Q) = c\int_{-\infty}^{Q}(Q-x)f(x)\,dx + (p-c)\int_{Q}^{\infty}(x-Q)f(x)\,dx$$

$$\frac{d}{dx}\int^{x} f(x,t) = \int_a^x \frac{d}{dx} f(x,t)\,dt$$



$$\int_0^2 \frac{1}{x-1}\,dx$$

$u = x-1$

$du = dx$

$$\int \frac{1}{u}\,du = \ln|u| + C \Big|_0^2$$

$$\int_0^1 \frac{1}{x-1}\,dx + \int_1^2 \frac{1}{x-1}\,dx$$

$$\lim_{t\to 1} \int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1} \ln|x-1|$$

$$= \lim_{t\to 1} \ln|x-1| - 0$$

$-\infty$