Class 7 Notes: Applications of Single Variable Calculus, Multivariate Functions, Multivariate Limits

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1 Introduction

The application of single variable calculus to business problems primarily rests in three methods of application: limits, rates of change, optimization, and areas under functions. In this lecture, we will review through some fundamental principles of mathematical modeling and the general process for doing so. Later, we will profile some common types of models in different areas of business such as economics, finance, marketing, and management. Once we review through these, we will enter into the realm of multi-variate calculus by defining and exploring functions of multiple variables.

2 Introduction to Modeling in Business

2.1 Theoretical and Real World Environments

When we model the real world mathematically, we are essentially representing "things" and "concepts" in the real-world with mathematical variables. How these "things" or "concepts" change, or why they hold the values they hold, is also modeled by relating the mathematical variables to each other via equations. If we were to try and model everything in the real world that determines or affects a particular "thing", we most likely would be driven to insanity. Take for example the total revenue of a company within a year. Revenue at the end of the year is dependent on many factors. Internally, it is dependent on the type of advertisements the company runs, the level of sales activities, the portfolio of products, the distribution of the product, the inventory and production levels of the product, and so on. Externally, many other factors drive a firm's revenue, like the size of the overall market, the number of competitors in the market, macro and micro economic factors, and even factors within the natural and social environments within which the firm operates.

To try and quantify all of these factors with mathematics would be impossible, as we would need to have knowledge on what causes what, and how the total sum (or product, or combination thereof) of each of these factors would affect the revenue of a firm, as well as have the ability to directly observe, measure, and quantify each factor. This is a tall order. Put simply, working in the pure "real world" is too messy and complicated to understand. So what we do instead is opt for a *simplification* of the real world via assumption making. That is, rather than try to chase down every possible cause to an effect, we instead limit our attention to the factors that matter the most, and assume that no other factors will be used to explain the focal "thing" under study. Going back to our revenue example, we could simplify the real world by simply saying "assume this, and assume not this". So if we focused our attention to contribution to total revenue by only the product's design, then we simply ignore all of the other factors in the real world that do affect revenue.

Definition 1 *A simplifying assumption about the real world is a statement that focuses the modeler's attention on specific things and concepts in the real world while ignoring other things and concepts. Put differently, a simplifying assumption is a statement about a constructed theoretical environment.*

When we make multiple simplifying assumptions, we have essentially constructed what is called a *theoretical model*. The theoretical model is simple as a result of our assumptions regarding that environment. Because it is simple, we can more easily approach the understand of relating various "things" to each other via mathematics. When we do so, understanding how the environment functions, or what it looks like, or what it will look like, becomes must easier.

Definition 2 *A theoretical model of the real world is an environment that is fully described by a collection of simplifying assumptions.*

Intuition would tell us that this approach to understanding the real world may mislead us, and this intuition is indeed correct. When we study the mechanics of theoretical environments by way of simplifying the real world, we risk missing very important contributors to factors in the real world. We may be able to model our environment under simplifying assumptions to forecast revenue for the year, but there may also be factors that we intentionally "left out" of the model so that we can obtain a solution in the theoretical world. In other-words, if we find a solution to a decision problem in this theoretical world, it will 100% be correct in the theoretical world. However, when we go to apply that decision in the real world, because our theoretical model may not have considered important factors, then we are at risk of our "best decision" in the theoretical world perhaps not being the best decision in the real-world. When constructing our theoretical world via simplifying assumption construction, we need to keep into account a trade-off the exists:

Theorem 1 *Upon the construction of theoretical model, there is a trade-off between making the model "close" to "looking like" the real-world, but having the model very complicated and not yielding any solutions. On the other hand, if we simplify the model to allow for solutions, the simplified model may not "look like" the real-world any more, and this may result in error between the theoretical and real world.*

For example, if we are trying to understand what is the lowest cost route to send a truck from location A to location B, we can model this mathematically by simplifying the real world down. We may construct this environment with the following statements as an example:

1. There is only one truck available for delivery.
2. The truck has infinite capacity.
3. The truck has fixed fuel costs.
4. There are only three possible routes to consider.
5. There is no construction on any of the routes.
6. Each route has costs that are proportional to the distance in the same manner.
7. Each route has an exact time the truck will take to travel on the route.
8. There is nothing that will prevent the truck from delivering the product from location A to location B.

As we can see, we can "picture" this environment in our head, and focus our attention to mathematically modeling the decisions and factors within the environment. We may say that route 1 has a distance of 5 miles, route 2 has a distance of 10 miles, and route 3 has a distance of 20 miles. Given the assumptions, we know that since each route's cost is proportional to distance, that route 1 would be the best decision in this theoretical model.

In the real world, however, we may have one of our assumptions not hold true. For example, route 1 may have an accident, which would divert the truck to take an alternate route that is not planned for. If this route is 15 miles, then in fact route 2 was our best decision, not route 1. This difference between the theoretical and real world resulted in an error, despite route 1 being the "best" in the theoretical, it wouldn't be in the real-world, since the real-world had an event that was not considered in the theoretical model.

If we tried to improve the model the altering some of the assumptions to try and make the theoretical world "look like" the real world, then the theoretical model is less simple, and may require more mathematical derivation, variables, equations and relations, etc. Therefore, the modeler needs to keep in mind this trade-off when modeling the real world. We don't want the model too simple, as we will have increased levels of error between results in the theoretical world and results in the real world. But we also do not want the model too complex, as we may not be able to figure out a solution at all. We want it "just right", and this is where a little bit of art, trial and error, science, and subjective decision making of the modeler comes into play.

2.2 Types of Theoretical Models

There are different ways that we can model the real world. The first is through the construction of what is called a *conceptual model*. The idea being that we think of everything in the real world as a "concept". A concept can be abstract or tangible:

Definition 3 *A construct is an abstract concept in the real world. An observable, on the other hand, is a tangible concept that can be directly observed and measured.*

For example, stress is a concept in the real world. This is also a construct, as we cannot directly observe stress. We can only characterize it with concepts that are observable, such as blood pressure, heart rate, sweating, sound, motions of the body, etc. All of these we can directly observe with our senses. Stress is an abstract construct that is the net total of these other observed quantities. Likewise, we cannot directly observe a company's "size". We can characterize it with other observables such as it' revenue, profit, number of locations, number of employees, etc.

When we propose a theoretical model by means of a conceptual model, we are focusing our attention to a handful of concepts (both constructs and observables) and relating them in a cause an effect manner. We may say that stress is caused by smoking and bad eating habits, and that bad eating habits are caused by the home environment and genetic makeup. This causal model is not mathematical, but *conceptual*, by drawing circles to represent the concepts and arrows to represent the connections between the concepts in terms of "what" causes "what".

Conceptual models are often built within a *theoretical lens*. The concepts and causal associations are justified using *hypo-deductive reasoning*, where "simplifying assumptions" are often referred to as "axioms" and "propositions". The connections are justified via this form of reasoning by stating facts (which are observations) that support why A causes B. The axioms, and other conceptual models, are then used to support a new conceptual model. For example, the Resource-Based View of the firm states that companies out perform other companies when they identify resources that are VRIN (valuable, rare, imperfectly imitable, and non-substitutable). If we propose a conceptual model that relates the firm's technological assets to their performance, then we can justify this relationship hypo-deductively by showing how the technological assets are VRIN.

Conceptual models are commonly given *empirical justification*, that is, they are tested to see if they "hold in reality", by ways of data collection, analysis, and testing. Usually, these models also serve to help the modeler design an *empirical model*, which is a model of the real world that uses data and observations directly from the real world to support and understand causal associations and characterizations of concepts in the real world.

Definition 4 *An empirical model is a mathematical equation that is found by a combination of using a conceptual model and collecting data directly from the real world to relate one quantity to another quantity. That is, exact formulas and equations are found by using data.*

Empirical models are models that are very "close" to the real world, since we are calibrating the equations based on direct observations of the real world. The trouble that empirical modelers often have, however, is that data often represents the past, and not the present or the future. Therefore, while an empirical model may be used to explain something that happened in the past, it is often difficult to use these to explain something in the future in the real world. Furthermore, empirical models do not often take into account assumptions about the real world, and tend to be "data-driven". While complexities in the real world can be captured by these types of models, we still encounter the trade-off of simplicity vs. error (or complexity vs. error).

Another approach to modeling the real world is directly based on what we discussed earlier, namely through the use of simplifying assumptions. Rather than use data or conceptual models and axiom-based theoretical frameworks to model the real world, we can simply create our own "theoretical environment" by proposing axioms about the real world, and using mathematics, instead of data, to derive results about the theoretical world in the form of equations, which could eventually be implemented in the real world.

Definition 5 *An analytical model is a theoretical model that is built based on simplifying assumptions about the real world by leveraging mathematical variables, equations, functions, sets, and other tools from mathematics. An analytical model answers questions about decision making in the form of mathematical equations.*

The focus of our course will rest primarily within analytical models. In future courses, you will be exposed to conceptual and empirical models. These are not the only types of models we can create, however. We can combine these models together to create other models of the real world. These tend to be highly specialized types of models such as simulation and agent-based models. Typically, however, a model is one of the three types we have discussed above. We will now elaborate more on the fundamentals of an analytical model.

2.3 Fundamentals of Analytical Model Construction

There are a few components to modeling the real world with an analytical model. Not all analytical models are the same, but most do have the following components:

1. Constants
2. Decision Variables
3. Objectives and Objective Functions
4. Constraints
5. Solutions

2.4 Constants

A constant is a mathematical variable that represents a fixed or known quantity in the environment. For example, if we are trying to determine the best truck route to take, some constants the we may include in the model could be cost per route, distance of each route, and expected time of delivery. We use constants to express quantities in the environment that are either known to the decision maker a head of time, or, is fixed and does not change. We had seen earlier with the Economic Ordering Quantity (EOQ) in previous lectures that some common constants were the fixed ordering cost, variable per unit ordering cost, inventory holding cost per unit per time, and the demand rate per unit of time. Mathematically, we represent theses with variables. However, even though they are represented as variables in a model, we need to "think" about these being numbers, and treating them no differently. For example, if we have the cost function $C(Q) = \frac{K\lambda}{Q} + \frac{Qh}{2} + c\lambda$, then the only "unknown" in the equation is Q. We would treat this equation no differently than say $C(A) = \frac{500}{Q} + \frac{5Q}{2} + 10$.

2.5 Decision Variables

In the context of a business model of the real world, our mathematical variables represent decisions of the decision maker. These are often the "unknown" quantities in our set of equations that make up the model for which we would like to solve. In the EOQ model, the Q was the unknown quantity for which needs to be solved. When we model the real world, we typically model the decision alternatives (that is, the possible choices for the decision make to choose from) as an individual model. For example, if we are trying to determine which route to choose out of 3 possible routes, we can model this with three different *binary variables*, $x_{\text{Route 1}}, x_{\text{Route 2}}, x_{\text{Route 3}}$. If the variable $x_{\text{Route 1}} = 0$, we interpret this as "do not take route 1". If $x_{\text{Route 1}} = 1$ then we interpret this as "take route 1". The goal for a decision maker is to figure out which values will be assigned to which decision variables.

2.6 Objectives and Objective Functions

The decision variables in a model represent the decisions that the decision maker is going implement. However, once the decision maker chooses a decision, we need a way to understand what happens to the environment as a result of the decision maker making this decision. In the case of business environments, the "consequence" can be anything from a cost, profit, risk, or sequence of events that occur as a result of the decision. Decision makers need a way to determine how to decide. The "how" typically comes as a result of the manager's or firm's *strategy*. Some firms may want to compete by growing, and so they may want to make decisions so as to maximize revenue. Others may want to be more efficient, and make decisions by minimizing their costs. That is, not only do we need to mathematically characterize decisions via variables, but we also need to know *how* decisions are to be made.

Definition 6 *An objective is a criteria that is used to judge how "good" or "bad" a decision is.*

Once we know what the objective of the decision maker is, we need to mathematically characterize the objective. For example, in the EOQ model, the decision is the number of units Q to order every T weeks (technically, there are two decisions in this model, Q and T, but based on our assumptions, we were able to reduce this to only one decision, namely Q). How do we know if $Q = 5$ is better or worse than $Q = 10$? This depends on the objective set by the decision maker. If the decision maker has the objective of cost, then we need a way to "score" the Q in terms of it's cost. On the other hand, if the objective were revenue, then we need a way to "score" the Q in terms of it's revenue. The way that we can characterize an objective mathematically is through the use of an *objective function*:

Definition 7 *An objective function is a mathematical function that has the decision variable as input and the output represents the "score" of the decision.*

In the EOQ model, we used the function $C(Q) = \frac{K\lambda}{Q} + \frac{Qh}{2} + c\lambda$ to represent the cost of ordering Q units. This function is the objective function of the model, and it scores how "good" or "bad" the decision to order Q units is. When a decision maker is trying to determine the "best decision", sometimes "best" means "minimum", other times it means "maximum". For example, we may want to determine which truck route will get render the "minimum" time, while a different objective may be to "maximize" profit.

2.7 Constraints

When we model decisions, we do so by representing them with mathematical variables. However, variables can be assigned to any value by default. If I asked you what is the maximum profit a manufacturer can make if they sell Q units for p dollars per unit, the answer is obvious: infinity! Obviously, this is not realistic, since in the real world, the manufacturer may be *constrained* by the number of units they can manufacture in a day (this is often called a capacity constraint by the way).

In order to convert our assumptions about the theoretical environment representative and meaningful mathematics, we need to reflect the restrictions on the possible values that can be assigned to the variable by either explicitly defining a set, or, by providing an equation that restricts the possible decision. For example, in our truck example, if we have three possible decision variables, $x_{\text{Route 1}}, x_{\text{Route 2}}, x_{\text{Route 3}}$, each of which are binary, we need to ensure that a constraint is placed that only allows binary values to be assigned to the variables. Furthermore, if we say that one of our assumptions is that the truck can only be assigned to a single route, then obviously the decision $x_{\text{Route 1}} = 1, x_{\text{Route 2}} = 0, x_{\text{Route 3}} = 1$ would "not work" in the real world, despite it being "mathematically correct".

Therefore, we need to specify an equation that must be true about any of the possible solutions that reflects the constraint that only one route can be used. For example, the equation $x_{\text{Route 1}} + x_{\text{Route 2}} + x_{\text{Route 3}} = 1$ would accomplish this, since the sum of the solution $x_{\text{Route 1}} = 1, x_{\text{Route 2}} = 0, x_{\text{Route 3}} = 1$ would be 2, and that would *break the constraint*. Therefore, when we model constraints that are present in the real world, we typically do so by providing equations of equality or inequality. For example, in the EOQ model, the decision $Q = -1$ does not make sense, and hence we would put the constraint that $Q \geq 0$.

2.8 Solutions

With the constants, decision variables, and constraints specified, the last step is to leverage the objective function to determine which decision, which makes all of our constraint equations true, would be the "best" decision. The constraints essentially refine the possible values that can be assigned to the decision variables to a special type of set:

Definition 8 *A solution to an analytical model is a collection of numbers that have been assigned to the decision variables. A feasible solution is a solution such that all of the constraint equations are true when the decision variables are substituted in the equations with the values in the solution. The set of all solutions that are feasible is called the set of feasible solutions. The subset of feasible solutions that give the "best" possible score as measured by the objective function is called the set of optimal solutions.*

In summary, the goal of analytical modeling is to model the decisions and preferences of managers, as well as how those managers score those decisions. One we have a completely specified analytical model, we try to *solve* the model by finding the collection of optimal solutions (sometimes this is infinite, sometimes this is only a single solutions, sometimes this is the empty set). How do we find these solutions? We will review through this when we reach our optimization lecture. Sometimes this can be done analytically, as we will see in some of the applications below, by finding an exact equation for the "best" solution. Other times, however, we need to solve for it approximately by running an algorithm such as Newtons Method, Gradient Descent, or take a heuristic or meta-heuristic approach.

3 Business Applications in Single Variable Calculus

3.1 Economics

3.1.1 Marginal Costs and Revenues

A common application of calculus in economics is to determine the optimal market price so that the firm maximizes it's profits. Profit is defined as Total Revenue (which is price per unit times the number of units sold) less Total Cost. Hence, the profit function would be $\pi(Q) = TR(Q) - TC(Q)$. Our "decision" is the number of units to manufacture, Q. The objective function is the above profit function. We only have one constraint in this model, namely that $Q \geq 0$. To solve the model, we can do this analytically by taking the derivative and finding the first order conditions:

$$\begin{aligned} \frac{d\pi}{dQ} = \frac{d}{dQ}[TR(Q) - TC(Q)] = 0 \\ TR'(Q) - TC'(Q) = 0 \\ TR'(Q) = TC'(Q) \end{aligned}$$

The second order conditions, in order to maximize, would be:

$$\begin{aligned} \frac{d^2\pi}{dQ} = \frac{d}{dQ}[TR'(Q) - TC'(Q)] < 0 \\ TR''(Q) < TC''(Q) \end{aligned}$$

We call $MR(Q) = TR'(Q)$ and $MC(Q) = TC'(Q)$ the marginal revenue and marginal cost, and they represent the *change* in revenue and cost as a result of a change in quantity, respectively. The first order conditions help characterize the optimal solution in this model, namely that the marginal costs must equal the marginal revenue. The second order conditions further characterize the optimal solution, namely that the change in the marginal revenue must be less than the change in the marginal cost. In other words, the optimal quantity to manufacture is the quantity such that the cost to manufacture an additional unit is the same as the revenue received to manufacture an additional unit, and, that the change in the marginal cost exceeds the change in the marginal revenue. This should be intuitive. If the changes in the marginal cost are greater than the marginal revenue, and they are equal, then we will see a decrease in profit if we manufacture the additional unit (or one less unit) from the point where the marginal costs are the same as the marginal revenues.

3.1.2 Consumer and Producer Surplus

It should come at no surprise that the overall market demand for a product is dependent on the product's price. The same goes for overall market supply. If the price is low, more consumers will demand it, since they can afford it. However, when the price is low, producers may not be willing to manufacture as much given the low revenues they receive in return. If the price is higher, manufacturers are willing to produce more (and hence the supply of the product would be higher), but, as a result, there will be less consumers that are willing to pay for the product since the price is higher, and so once again, total revenue may be at risk. The relationship between the price of a product and total market demand is often referred to as a *demand curve*, while the relationship between the price of a product and total market supply is often referred to as a *supply curve*.

There are many ways to model the relationship between price and demand and price and supply, and many of these models rest on various attributes about the market, the product, the general economy, and the competitive environment, among many other internal and external factors. To simplify the modeling approach, we often assume that demand and supply curves follow a *linear function* (although, as we discussed earlier, this may be too simple, and we may need to use a different function to model the relationship). If we assume that both curves are linear, then we would have:

$$
\begin{aligned}
D(p) &= a - bp \\
S(p) &= c + dp
\end{aligned}
$$

We can find the "market price" by finding the the *market equilibrium*, which is the point where the supply is balanced with the demand:

$$
\begin{aligned}
D(p) &= S(p) \\
a - bp &= c + dp \\
a - c &= p(d + b) \\
p &= \frac{a - c}{d + b}
\end{aligned}
$$

At the market price, both consumers and producers benefit. Consumers are saving money from the price being too high, and producers are saving money by the price being too low. For

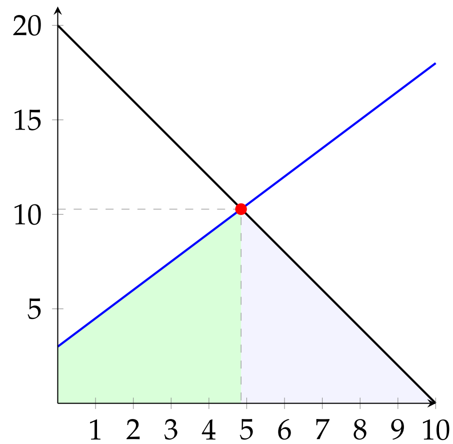

Figure 1: Consumer surplus (the area in blue) and Producer surplus (the area in green) as well as the market price and quantity equilibrium (the red point)

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example, suppose $p_1 > p^*$, where p^* is the market price. Then if the total demand does not change (which we know it does) and it remains at $D(p) = D(p^*)$ for $p^* < p < p_{max}$, where p_{max} is the price such that $D(p_{max}) = 0$, then consumers benefit with the price being at p^*. If the price were at $p \in [p^*, p_{max}]$, then they would be spending an extra $D(p^*)(p - p^*)$. Notice, however, that the value of $D(p)$ changes with different values of p.

So to find the true total savings the consumer enjoys by the price being set at the market price, we can split the interval $[p^*, p_{max}]$ into n equal slices, with length $\Delta p = \frac{p_{max}-p^*}{n}$, and assume demand does not change within each subintervals, which allows us to compute the approximate total savings the customer enjoys with the price being set at market equilibrium: $D(p^* + \Delta p)\Delta p + D(p^* + 2\Delta p)\Delta p + \cdots + D(p^* + n\Delta p)\Delta p$. If we take the limit as $n \to \infty$, then we will get the true total savings the customer enjoys: True Savings $= \lim_{n\to\infty} \sum_{i=1}^{n} D(p^* + \Delta p)\Delta p$. However, notice that this is just the definition of the integral! Hence, we have shown that:

$$\begin{aligned}\text{Customer Saving} &= \lim_{n\to\infty} \sum_{i=1}^{n} D(p^* + i\Delta p)\Delta p \\ &= \int_{p^*}^{p_{max}} D(p)dp\end{aligned}$$

In economics, this "savings" is also known as *consumer surplus*, and as we mentioned, it represents the total amount of money the consumer can save as a result of the market price being less than higher prices on the the demand curve. Likewise, by similar reasoning, the producer also "saves money" as a result of producing at a higher price. If we were to assume for the moment that supply did not change with price and was held constant at Q from p_{min} (the minimum price on the supply curve) to the equilibrium price p^*, then the producer would "save" $Q(p^* - p_{min})$. We can by a similar reasoning that the producer themselves also have a surplus, which we call *producer surplus* (which some argue is roughly close to profit):

$$\text{Producer Surplus} = \int_{p_{min}}^{p^*} S(p)dp$$

When the consumer and producer surplus are added together, we obtain what is called *social surplus*:

$$\begin{aligned}\text{Social Surplus} = f(x) &= \text{Producer Surplus} + \text{Consumer Surplus}\\ &= \int_{p_{min}}^{p^*} S(p)dp + \int_{p^*}^{p_{max}} D(p)dp\end{aligned}$$

We can maximize the social surplus by optimizing based on $p*$ and finding first and second order conditions:

$$\begin{aligned}\frac{df}{dp^*} &= \frac{d}{dp^*}\left[\int_{p_{min}}^{p^*} S(p)dp + \int_{p^*}^{p_{max}} D(p)dp\right]\\ &= \frac{d}{dp^*}\left[\int_{p_{min}}^{p^*} S(p)dp\right] + \frac{d}{dp^*}\left[\int_{p^*}^{p_{max}} D(p)dp\right]\\ &= \frac{d}{dp^*}\left[\int_{p_{min}}^{p^*} S(p)dp\right] - \frac{d}{dp^*}\left[\int_{p_{max}}^{p^*} D(p)dp\right]\\ &= S(p^*) - D(p^*) = 0\\ & S(p^*) = D(p^*)\end{aligned}$$

Likewise, the second order conditions state:

$$\begin{aligned}\frac{d}{dp^*}\left[S(p^*) - D(p^*)\right] &< 0\\ \frac{d}{dp^*}\left[S(p^*)\right] - \frac{d}{dp^*}\left[D(p^*)\right] &< 0\\ \frac{d}{dp^*}\left[S(p^*)\right] &< \frac{d}{dp^*}\left[D(p^*)\right]\end{aligned}$$

Therefore, social surplus is maximized when the market is in equilibrium (when supply is equal to demand), and when the rate of change in demand is larger than the rate of change in supply with respect to price.

3.2 Finance

3.2.1 Continuously Compounded Interest

Recall from an earlier lecture that compound interest for an initial investment of A dollars, compounded n times per year, for t years, at a rate of r per year, can be found by the following equation:

$P(t) = A(1 + \frac{r}{n})^{tn}$

We said that there are some applications where *continuous compounding*, meaning we compound an infinite number of times within a year, is more applicable in practice. This occurs when $n \to \infty$. How can we compute this limit, however? The answer lies in the definition of the exponential number as a limit, which can be shown to be $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$. We can see that reworking the previous limit, and setting $m = \frac{n}{r}$ we have:

$$P(t) = \lim_{n\to\infty} A(1 + \frac{r}{n})^{tn}$$
$$P(t) = \lim_{n\to\infty} A(1 + \frac{1}{m})^{t(mr)}$$
$$P(t) = A[\lim_{n\to\infty}(1 + \frac{1}{m})^{m}]^{rt}$$
$$P(t) = Ae^{rt}$$

3.3 Marketing

There are many different applications of single variable calculus in marketing. One of these applications is that of competition of products in market segments. A popular model to better understand this is the Hotelling Model. This model takes into account firm location and tried to determine optimal pricing of a product. The idea is as follows. All customers are located on a line in the interval $[0, 1]$. Firm 0 is located at location 0 and Firm 1 is located at location 1. Firm 0 would like to set it's price p_0. Firm 1 has already set it's price at p_1, and firm 0 knows what this price is. All customers exist on this line, and it is assumed that they are uniformly distributed in terms of their location on the interval $[0, 1]$. The two firms sell the same product.

It is assumed that c is the unit cost of the product and t is the cost of transportation per unit distance squared. For example, if a customer travels a distance of d, then the cost to the consumer to travel that distance is td^2. Each customer can either buy the product (1) or not buy the product (0). Consider a customer who is located at location $x \in [0, 1]$. Then they are x units of distance from firm 0 and $1 - x$ units of distance from firm 1. If we consider the transportation costs that this consumer will expend, then they will spend tx^2 to reach firm 0 and $t(1 - x)^2$ to reach firm 1.

Let s represent the gross consumer surplus (that is, the consumer's willingness to pay for the product), and p is the price of the product. Then the benefit received by the customer (which is referred to as the utility), would be the willingness to pay less the price less the costs of transportation. That is:

$$\text{Utility} = \text{Willingness to Buy} - \text{Price} - \text{Transportation Cost}$$

Hence, depending on where the customer is located, they may receive more utility by purchasing from one firm over the other. However, there does exist a customer located on the line at $x = x^*$ that is *indifferent*, which means that the customer receives no greater utility from firm 1 or firm 0, as the utilities would be the same. We can find where this customer is located by setting equal the utility of the customer for purchasing the product from each firm:

$$\begin{aligned}
U_{x^*}(0) &= U_{x^*}(1)\\
s_{x^*} - p_0 - t(x^*)^2 &= s_{x^*} - p_1 - t(1-x^*)^2\\
p_0 + t(x^*)^2 &= p_1 + t(1-x^*)^2\\
p_0 + t(x^*)^2 &= p_1 + t(1 - 2x^* + (x^*)^2)\\
p_0 + t(x^*)^2 &= p_1 + t - 2tx^* + t(x^*)^2)\\
p_0 &= p_1 + t - 2tx^*\\
2tx^* &= p_1 - p_0 + t\\
x^* &= \frac{p_1 - p_0 + t}{2t}
\end{aligned}$$

We therefore know that any customer located at $x < \frac{p_1 - p_0 + t}{2t}$ will buy from firm 0 (since the utility for firm 0 is higher when the customer is located to the left of the indifferent customer). Likewise, any customer located at $x > \frac{p_1 - p_0 + t}{2t}$ will buy from firm 1. This means, if $I_i(x)$ represents a function such that $I_i(x) = 1$ if customer x buys from firm i and $I_i(x) = 0$ if they do not buy from firm i, then $I_0(x) = 1$ for $x \in [0, \frac{p_1 - p_0 + t}{2t})$ and $I_0(x) = 0$ for $x \in (\frac{p_1 - p_0 + t}{2t}, 1]$. Likewise, $I_1(x) = 0$ for $x \in [0, \frac{p_1 - p_0 + t}{2t})$ and $I_1(x) = 1$ for $x \in (\frac{p_1 - p_0 + t}{2t}, 1]$. We can find the total number of customer that purchase from firm 0 by finding the integral (recall our discussion on "counting" on "uncountable sets", and that we can add a number an "uncountably infinite number of times" by finding integrals!):

$$\begin{aligned}
\int_0^1 I_0(x)dx &= \int_0^{x^*} I_0(x)dx + \int_{x^*}^1 I_0(x)dx\\
&= \int_0^{x^*} 1dx + \int_{x^*}^1 0dx\\
&= \int_0^{x^*} 1dx\\
&= (x^* - 0) = x^*
\end{aligned}$$

Likewise, we can find the total number of customers that purchase from firm 1 by:

$$
\begin{aligned}
\int_0^1 I_1(x)dx &= \int_0^{x^*} I_1(x)dx + \int_{x^*}^1 I_1(x)dx \\
&= \int_0^{x^*} 0dx + \int_{x^*}^1 1dx \\
&= \int_{x^*}^1 1dx \\
&= (1 - x^*) = 1 - x^*
\end{aligned}
$$

Therefore, the total demand for firm 0 if they set their price at p_0 would be:

$$
D(p_0) = x^* = \frac{p_1 - p_0 + t}{2t}
$$

With this information, we know can construct a *profit function* for Firm 0. Recall in our discussion from earlier that when we model the real world environment, we need a way to "score" the decision the decision maker chooses. Here, the decision is the price for the firm to set, namely p_0. In this instance, we are "scoring" how "good" or "bad" the decision to set the product at price p_0 is by constructing the corresponding profit the firm will receive by setting the price at p_0. In other words, we can define the *objective function* of our model as:

$$
\begin{aligned}
\Pi(p_0) &= (p_0 - c)D(p_0) \\
&= (p_0 - c)\frac{p_1 - p_0 + t}{2t}
\end{aligned}
$$

Hence, the first order conditions would be:

$$
\begin{aligned}
\frac{d}{dp_0}\Pi(p_0) &= 0 \\
\frac{d}{dp_0}\left[(p_0 - c)\frac{p_1 - p_0 + t}{2t}\right] &= 0 \\
\frac{d}{dp_0}\left[(p_0 - c)\right]\frac{p_1 - p_0 + t}{2t} + (p_0 - c)\frac{d}{dp_0}\left[\frac{p_1 - p_0 + t}{2t}\right] &= 0 \\
\frac{p_1 - p_0 + t}{2t} - (p_0 - c)\frac{1}{2t} &= 0 \\
p_1 - 2p_0 + t + c &= 0 \\
p_0 &= \frac{p_1 + t + c}{2}
\end{aligned}
$$

For the second order conditions, we have:

$$
\begin{aligned}
\frac{d^2}{dp_0^2}\Pi(p_0) &< 0 \\
\frac{d}{dp_0}\left[\frac{d}{dp_0}\left[(p_0 - c)\frac{p_1 - p_0 + t}{2t}\right]\right] &< 0 \\
\frac{d}{dp_0}\left[\frac{p_1 - p_0 + t}{2t} - (p_0 - c)\frac{1}{2t}\right] &< 0 \\
-\frac{1}{2t} - \frac{1}{2t} &< 0 \\
-\frac{1}{t} &< 0
\end{aligned}
$$

We therefore know that firm 0's optimal price, knowing the price that firm 1 sets, would be half of the sum of the price, cost, and transportation cost. As the transportation cost increases, the optimal policy is then to also increase the price. Intuitively this says that as the cost of transportation increases, the customers that are closer to firm 0 will be unwilling to spend the extra cost in transportation in order to purchase from firm 1, and hence would prefer firm 0. In order for take advantage of this situation, the firm can set it's price higher without risking the loss of the customer to a competitor, since the higher transportation cost would outweigh the benefits received from firm 1. Similar logic holds for the price of the competitor and the cost.

3.4 Supply Chain and Logistics

3.4.1 Capacity Management

A supply chain is a collection of organizations that mine, manufacture, assemble, and deliver a product from raw materials source to end consumer. Part of supply chain management is determining which facilities to open and operate, as well as where, how, and when to open and operate them. A common problem that supply chain planners face is that of *capacity* which is the total number of units that the supply chain as a whole can manufacture. In order to understand how often the supply chain manager should open a new facility, as well as at what capacity, they need to understand the fundamentals of market demand, costs to open and operate the facility, as well as take into account the time value of money.

Suppose that a supply chain manager would like to open a new facility with capacity y every x years. Suppose the cost to open and operate the facility with a capacity of y is $f(y) = ky^{\alpha}$. When $\alpha < 1$, then the company experience *economy of scale*, since their cost of building and operating new facilities is reduced with higher capacity. This is an effect that occurs in the real world quite often as a result of gaining experience. In other words, we assume that the more experience the firm has (with experience measured as capacity), the lower the cost it will be to open and operate a new facility. Last, suppose that D represents the annual increase in demand and does not change for years.

As we mentioned, the firm would like to open and operate a new facility every x years (and this is what it needs to determine). In order to "score" the decision to build every x years, the firm will compute the *net present value* of the entire plan assuming it builds out to infinity. The firm assumes that inflation is compounded continuously, which means that in t years at an annual inflation rate of t, a dollar today will be worth e^{-rt}. Furthermore, if the firm is building a new facility every x years, this means that the market demand has increased by xD (x years times D units per year), and it will hence need to build a new facility with capacity xD to keep up with this demand increase.

The cost for the firm to build the facility in year 0 would therefore be $C_0(x) = f(xD)$. However, in year x. the value of the dollar has decreased, and so while the cost in year x will be $f(xD)$, given that devaluation of the dollar has occurred, we actually have the real cost to build the next facility at $e^{-rx}f(xD)$, since x years have passed since the last build. At this point, the total cost of the plan would be $C_1(x) = f(xD) + e^{-rx}f(xD)$. Likewise, since another x years pass the subsequent time the firm builds the facility, the cost to build this new facility would be $e^{-r2x}f(xD)$, since a total of $2x$ years have passed since the first build, and demand has increased by xD for the subsequent x years since the previous x years. So at this point, the total cost would be $C_2(x) = f(xD) + e^{-rx}f(xD) + e^{-2rx}f(xD)$.

As we continue this pattern into infinity, we can clearly see that the total cost of the plan to build every x years would be $C(x) = f(xD) + e^{-rx}f(xD) + e^{-2rx}f(xD) + \cdots = \sum\limits_{i=0}^{\infty} e^{-irx}f(xD)$. Notice that we can rework this summation to $C(x) = f(xD)\sum_{i=0}^{\infty}(e^{-rx})^i$. If we let $a = e^{-rt}$, then we have $C(x) = f(xD)\sum_{i=0}^{\infty} a^i$. Recall that when $|a| < 1$, then we have a *geometric series*, which we know will converge to $\frac{1}{1-a}$ for $|a| < 1$. We can clearly see that if $rx > 1$, which it should be, then $e^{-rx} < 1$. Hence, we have :

$$\begin{aligned}
C(x) &= f(xD) + e^{-rx}f(xD) + e^{-2rx}f(xD) + \dots \\
&= f(xD)\sum_{i=0}^{\infty}(e^{-rx})^i \\
&= f(xD)\frac{1}{1-e^{-rx}}
\end{aligned}$$

Since the cost to build a facility of capacity y is $f(y) = ky^{\alpha}$, we have:

$$\begin{aligned}
C(x) &= f(xD)\frac{1}{1-e^{-rx}} \\
C(x) &= kD^{\alpha}\frac{x^{\alpha}}{1-e^{-rx}}
\end{aligned}$$

Now let us try to find the optimal solution. Let us start by finding the first order conditions:

$$
\begin{aligned}
\frac{d}{dx}[C(x)] &= 0 \\
\frac{d}{dx}\left[kD^{\alpha}\frac{x^{\alpha}}{1-e^{-rx}}\right] &= 0 \\
kD^{\alpha}\frac{d}{dx}\left[\frac{x^{\alpha}}{1-e^{-rx}}\right] &= 0 \\
\frac{d}{dx}\left[\frac{x^{\alpha}}{1-e^{-rx}}\right] &= 0 \\
\frac{d}{dx}\left[x^{\alpha}(1-e^{-rx})^{-1}\right] &= 0 \\
\frac{d}{dx}\left[x^{\alpha}\right](1-e^{-rx})^{-1} + x^{\alpha}\frac{d}{dx}\left[(1-e^{-rx})^{-1}\right] &= 0 \\
\alpha x^{\alpha-1}(1-e^{-rx})^{-1} + x^{\alpha}(-1)(1-e^{-rx})^{-2}(-e^{-rx})(-r) &= 0 \\
\alpha x^{\alpha-1}(1-e^{-rx}) - x^{\alpha}r(e^{-rx}) &= 0 \\
\alpha x^{-1}(1-e^{-rx}) &= r(e^{-rx}) \\
\alpha &= \frac{xr(e^{-rx})}{(1-e^{-rx})} \\
\alpha &= \frac{xr}{((e^{rx})-1)}
\end{aligned}
$$

We have encountered a problem where we cannot find an explicit formula for a solution. In these situations, we try to instead understand additional properties of the optimal solution. For example, we may be curious as to how the economies of scale parameter α affects the optimal solution by studying it's derivative. We may do the same with the rate parameter r. In addition, we may observe some properties of the optimal solution by looking at the second order conditions as well. There are a "bag of tricks" so to speak when studying the optimal solution. Not all models have an explicit solution, and hence, we have to work extra hard to better understand it's properties. When we do this,it is said that we are *characterizing the optimal solution*, that is, we are trying to understand what is the optimal solution by means of other variables rather than using a direct equation for it, since as this illustrates, we do not always have an explicit equation for it.

3.5 Production and Operations

One aspect of production and operations is trying to understand the *optimal inventory policy*. Firms hold inventory for both tactical and strategic reasons. How much the firm should hold is often determined on many factors. However, we can simplify the environment many ways to gain some intuition to understand how inventory systems work, and how to best make decisions within them (subject to error with the real world, of course). We have already seen an example of an inventory model known as the Economic Ordering Quantity model (EOQ). Here, I will illustrate an alternative model, namely the *newsvendor model*.

In the newsvendor model, we assume that demand is *random*. That is, unlike EOQ where we know what the inventory level will be at a given moment in time, in the newsvendor model, we do not. Demand is commonly modeled as a *random variable*. Recall from statistics and probability that a random variable is a mathematical variable where we do not know what value will be assigned to the variable, but we do know the *possible* values that could be assigned, as well as the probabilities of those values being assigned to the variable. Random variables can be determined based on the *probability distribution*, which characterizes the chances that the variable will be assigned to certain values. There are thousands of distributions (and you will learn a common tool set of them to use in your statistics course). For our purposes, we will just simply say the demand is distributed by the density function $f(x)$ and the cumulative distribution function $F(x)$. In probability theory, if X is a *continuous distribution function*, then $\int\limits_{-\infty}^{\infty} f(x)dx = 1$. Likewise, $F(x) = \int\limits_{-\infty}^{x} f(t)dt$. It should come at no surprise that $F'(x) = f(x)$. Likewise, we have the equality $\int\limits_{-\infty}^{a} f(x)dx = 1 - \int\limits_{a}^{\infty} f(x)dx$. In addition, when we have an equation $g(X)$ that is defined in terms of the random variable (for example, something like $g(X) = aX - b$), we cannot work directly with this function since we do not know what the value of X will be. One way that we can work with functions that are defined in terms of random variables is to find the *expected value* of the function, where is defined, respectively, as $E[g(X)] = \int_{-\infty}^{\infty} g(t)f(t)dt$. This essentially converts all the possible values of $g(X)$ to a single number $E[g(X)]$. Furthermore, the expected value is a linear operator, which means that $E[aX + Y] = aE[X] + E[Y]$, for $a \in \mathbb{R}$.

Given all of this, we can construct a *stochastic model* of inventory (stochastic is just a fancy word for "use probability theory in the model"). Suppose that within a single time period (a minute, an hour, a day, a week, etc), the firm purchases Q products for c dollars per unit and sells them for p dollars per unit. Let D be a random variable, distributed by $f(x)$ and $F(x)$ which is the total demand for the product for a single period of time. The goal of the firm is to place an order for Q units so as to minimize their overall costs. What type of costs will the firm encounter in this model? This depends on the how many units are actually demanded D and the number of units in stock Q.

Since D is random, we do not know it's value. However, we can analyze the costs based on two possible cases. If it turns out that $D < Q$, then the firm has ordered in excess of $Q - D$ units. If we assume that the firm cannot sell these units within this time frame, then the firm is at a total loss $c(Q - D)$ dollars. On the other hand, if $D > Q$, then the firm has ordered too little, and could have sold an additional $D - Q$ units. This means they could have made an extra $(p - c)(D - Q)$ dollars. This type of cost is referred to as an *opportunity cost*, and it represents the cost of making the decision Q instead of making an alternative decision (namely, the exact value of D, which is impossible to do, since D is unknown at the time that the decision is made). Therefore, the two costs are dependent on the particular case of whether or not $D < Q$.

We can construct a single equation to describe the objective function of this decision by leveraging the *max* function, where $\max(a, b)$ returns the maximum of the values a, b. When $D < Q$, we want the total over ordering cost to be $c(Q - D)$ and the under ordering cost of $(p - c)(D - Q)$ to be 0 (since in this case, $(p - c)(D - Q) < 0$). On the other hand, if $D > Q$, we want the total over ordering cost to be 0, since $c(Q - D) < 0$ and the total under ordering cost to be $(p - c)(D - Q)$. For the total under ordering cost, regardless of situation, we can represent the total cost as $\max((p - c)(D - Q), 0)$. We can see that if $D < Q$, then $\max((p - c)(D - Q), 0) = 0$ and if $D > Q$, then $\max((p - c)(D - Q), 0) = (p - c)(D - Q)$. Likewise, for the total over ordering cost, regardless of situation, we can represent the total cost as $\max(c(Q-D),0)$. If $D < Q$, then $\max(c(Q-D),0) = c(Q-D)$ and if $D > Q$, then $\max(c(Q-D),0) = 0$. Putting this all together, we have the following function that describes the total cost of ordering Q units:

$$C(Q) = \max(c(Q-D),0) + \max((p-c)(D-Q),0)$$

We mentioned, however, that D is a random variable, and so we cannot just simply find the derivative of this function to minimize it, since we do not know the value of D. However, we can instead convert the function to something that we can work with by minimizing the *expected value* of this function $E[C(Q)]$, which, using the linear operation rules, reduces to:

$$\begin{aligned}
E[C(Q)] &= E[\max(c(Q-D),0) + \max((p-c)(D-Q),0)] \\
&= E[\max(c(Q-D),0)] + E[\max((p-c)(D-Q),0)] \\
&= \int_{-\infty}^{\infty} \max(c(Q-x),0)f(x)dx + \int_{-\infty}^{\infty} \max((p-c)(x-Q),0)f(x)dx
\end{aligned}$$

For the first integral, we see that we can break it up into two pieces:

$$\int_{-\infty}^{\infty} \max(c(Q-x),0)f(x)dx = \int_{-\infty}^{Q} \max(c(Q-x),0)f(x)dx + \int_{Q}^{\infty} \max(c(Q-x),0)f(x)dx$$

For values of $x < Q$, we see that the integral reduces to

$$\int_{-\infty}^{Q} c(Q-x)f(x)dx$$

, since the function $\max(c(Q-x),0) = c(Q-x)$ when $x < Q$. By a similar reasoning, we have the second part of the integral reduce to:

$$\int_{Q}^{\infty} \max(c(Q-x),0)f(x)dx = \int_{Q}^{\infty} (0)f(x)dx = 0$$

.

Following a reasoning along these same lines, we can reduce the objective function to:

$$
\begin{aligned}
E[C(Q)] &= \int_{-\infty}^{\infty} \max(c(Q-x),0)f(x)dx + \int_{-\infty}^{\infty} \max((p-c)(x-Q),0)f(x)dx \\
&= \int_{-\infty}^{Q} \max(c(Q-x),0)f(x)dx + \int_{Q}^{\infty} \max(c(Q-x),0)f(x)dx \\
&+ \int_{-\infty}^{Q} \max((p-c)(x-D),0)f(x)dx + \int_{Q}^{\infty} \max((p-c)(x-D),0)f(x)dx \\
&= \int_{-\infty}^{Q} c(Q-x)f(x)dx + \int_{Q}^{\infty} 0f(x)dx \\
&+ \int_{-\infty}^{Q} (0)f(x)dx + \int_{Q}^{\infty} (p-c)(x-Q)f(x)dx \\
&= \int_{-\infty}^{Q} c(Q-x)f(x)dx + \int_{Q}^{\infty} (p-c)(x-Q)f(x)dx \\
&= c\int_{-\infty}^{Q} Qf(x)dx - c\int_{-\infty}^{Q} xf(x)dx + (p-c)\int_{Q}^{\infty} xf(x)dx - (p-c)\int_{Q}^{\infty} Qf(x)dx
\end{aligned}
$$

Now we are in a position to find the first order conditions for Q. Before doing so, let us review an important identity that we have not discussed yet. There is a rule in Calculus known as *Leibniz Integral Rule*, and it states that given a function of two variables, $f(x,t)$, that:

$$
\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}f(x,t)dt
$$

We will cover *partial derivatives* in our next lecture. For now, consider a partial derivative to be the same as a regular derivative, where we assume that any other variable in the function is assumed to be a constant. For example, if we have the multi-variable function $f(x,y) = xy$, then $\frac{\partial}{\partial x}[x^2y^2] = y^2\frac{\partial}{\partial x}x^2 = y^2(2x)$, since we assume in this case that all other variables in the equation (in this instance, y), is a constant.

Taking all of this into account, when we find the first order conditions $\frac{d}{dQ}E[C(Q)] = 0$, then we have:

$$
\begin{aligned}
\frac{d}{dQ}E[C(Q)] &= \frac{d}{dQ}\left[c\int_{-\infty}^{Q} Qf(x)dx - c\int_{-\infty}^{Q} xf(x)dx + (p-c)\int_{Q}^{\infty} xf(x)dx - (p-c)\int_{Q}^{\infty} Qf(x)dx\right] \\
&= \frac{d}{dQ}\left[c\int_{-\infty}^{Q} Qf(x)dx\right] - c\frac{d}{dQ}\left[\int_{-\infty}^{Q} xf(x)dx\right] \\
&+ \frac{d}{dQ}\left[(p-c)\int_{Q}^{\infty} xf(x)dx\right] - \frac{d}{dQ}\left[(p-c)\int_{Q}^{\infty} Qf(x)dx\right] \\
&= \frac{d}{dQ}Q\left[c\int_{-\infty}^{Q} f(x)dx\right] - c\frac{d}{dQ}\left[\int_{-\infty}^{Q} xf(x)dx\right] \\
&+ \frac{d}{dQ}\left[(p-c)\int_{Q}^{\infty} xf(x)dx\right] - \frac{d}{dQ}Q\left[(p-c)\int_{Q}^{\infty} f(x)dx\right] \\
&= \frac{d}{dQ}Q\left[cF(Q)\right] - c\frac{d}{dQ}\left[\int_{-\infty}^{Q} xf(x)dx\right] + \frac{d}{dQ}\left[(p-c)\int_{Q}^{\infty} xf(x)dx\right] - \frac{d}{dQ}Q\left[(p-c)(1-F(Q))\right]
\end{aligned}
$$

For the first term, using the Fundamental Theorem of Calculus and the product rule, we have $\frac{d}{dQ}QcF(Q) = cF(Q) + cQf(Q)$, since $F'(x) = f(x)$.

Working on the second term $\frac{d}{dQ}\left[\int_{-\infty}^{Q} xf(x)dx\right] = \lim_{s\to-\infty}\frac{d}{dQ}\left[\int_{s}^{Q} yf(y)dy\right]$, we can identify that $x = Q$, $t = y$, $a(x) = s$, $b(x) = Q$, $f(x,t) = yf(y)$, $f(x,b(x)) = yf(Q)$, $f(x,a(x)) = yf(s)$, and make the substitutions:

$$
\begin{aligned}
\lim_{s\to-\infty}\frac{d}{dQ}\int_{s}^{Q} yf(y)dy &= \lim_{s\to-\infty}\left[\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt\right] \\
&= \lim_{s\to-\infty}\left[f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)dt\right] \\
&= \lim_{s\to-\infty}\left[Qf(Q)\frac{d}{dQ}Q - sf(s)\frac{d}{dQ}s + \int_{s}^{Q}\frac{\partial}{\partial Q}yf(y)dy\right] \\
&= \lim_{s\to-\infty}[Qf(Q)] \\
&= Qf(Q)
\end{aligned}
$$

Working on the third term $\frac{d}{dQ}\left[\int_{Q}^{\infty} xf(x)dx\right] = \lim_{s\to\infty}\frac{d}{dQ}\left[\int_{Q}^{s} yf(y)dy\right]$, we can identify that $x = Q$, $t = y$, $a(x) = Q$, $b(x) = s$, $f(x,t) = yf(y)$, $f(x,b(x)) = sf(s)$, $f(x,a(x)) = Qf(Q)$, and make the substitutions:

$$
\begin{aligned}
\lim_{s\to\infty}\frac{d}{dQ}\left[\int_{Q}^{s} yf(y)dy\right] &= \lim_{s\to\infty}\left[\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt\right] \\
&= \lim_{s\to\infty}\left[f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)dt\right] \\
&= \lim_{s\to\infty}\left[sf(s)\frac{d}{dQ}s - Qf(Q)\frac{d}{dQ}Q + \int_{s}^{Q}\frac{\partial}{\partial Q}yf(y)dy\right] \\
&= \lim_{s\to\infty} -Qf(Q) \\
&= -Qf(Q)
\end{aligned}
$$

Working on the last term, we have:

$$
\begin{aligned}
\frac{d}{dQ}(p-c)Q(1-F(Q)) &= \frac{d}{dQ}[(p-c)Q - (p-c)QF(Q)] \\
&= (p-c) - [(p-c)F(Q) + (p-c)Qf(Q)] \\
&= (p-c) - (p-c)F(Q) - (p-c)Qf(Q)
\end{aligned}
$$

Putting all of these integrals back into the equation, we have:

$$\begin{aligned}
\frac{d}{dQ}E[C(Q)] &= \frac{d}{dQ}Q\left[cF(Q)\right] - c\frac{d}{dQ}\left[\int_{-\infty}^{Q} xf(x)dx\right] + \frac{d}{dQ}\left[(p-c)\int_{Q}^{\infty} xf(x)dx\right] - \frac{d}{dQ}Q\left[(p-c)(1-F(Q))\right] \\
&= cF(Q) + cQf(Q) - cQf(Q) - (p-c)Qf(Q) - (p-c)(1-F(Q)-Qf(Q)) \\
&= cF(Q) - (p-c)Qf(Q) - (p-c) + (p-c)F(Q) + (p-c)Qf(Q) \\
&= cF(Q) - (p-c) + (p-c)F(Q) \\
&= (p-c+c)F(Q) - (p-c) \\
&= pF(Q) - (p-c)
\end{aligned}$$

Hence, the optimal ordering quantity Q^* will satisfy the relationship:

$$\begin{aligned}
pF(Q^*) - (p-c) &= 0 \\
F(Q^*) &= \frac{p-c}{p}
\end{aligned}$$

In probability theory, $F(x) = P(D < x)$. In the context of inventory, this would mean that $F(Q) = P(D < Q)$ represents the probability of being in stock. This is often referred to as a *service level*. Therefore, we can interpret the optimal ordering quantity that minimizes the cost as the level of inventory such that the probability of remaining in stock is exactly equal to the percentage of revenue that is profit. This is often referred to as the *critical fractal* in inventory theory.

Following the second order conditions, we have:

$$\begin{aligned}
\frac{d^2}{dQ^2}[C(Q)] &= \frac{d}{dQ}\left[\frac{d}{dQ}C(Q)\right] > 0 \\
&= \frac{d}{dQ}\left[pF(Q^*) - (p-c)\right] > 0 \\
&= pf(Q^*) > 0
\end{aligned}$$

By definition of a probability density function, we know that $f(x) > 0$ for all x, and likewise, we know that $p > 0$, hence, these conditions are satisfied by the definition of the density function itself.

4 Multivariable Functions

4.1 Definition of a Multi-Variable Function

So far, up to this point, we have only explored Calculus and applications from a single variable perspective. In applications, this would correspond to the decision maker having only a single decision to make. We explored some of these decisions before:

1. Quantity of a single product for a firm to manufacture.

2. The price to set for a product so as to maximize surplus.

3. The number of times to compound an investment, and how this impacts total value as this increases to infinity.

4. The price to set a product when given the price of a competitor taking into account the customer's distance from each respective firm.

5. The number of years to wait in between the construction of new facilities.

6. The number of products to order and store in inventory when demand is random (or stochastic).

All of these applications only involved a single decision. In practice, however, managers rarely are faced with single decisions, and are in fact faced with multiple decisions. Hence, when a manager is trying to determine how to choose among alternatives for each decision, they need to take into account the total cost, profit, risk, service level, or more generally, objective, rather than the individual costs with making such decisions. Each decision that the manager makes may be traded off with another decision. For example, a manager may seek to increase production in one product that has more promising demand levels, but may not be taking away resources that can be used to produce a different product with lower demand but higher per-unit profit. In order to understand the manager's optimal decision making in these complex environments, we need a new tool.

In order to mathematically model the total cost and constraints of multiple decisions, and how they impact the ability to make other decisions, we need to introduce the concept of a *multi-variable function*. Earlier, when we introduced single variable functions, we defined these as mappings (or assignments) from one set of numbers to another set of numbers. What if we were to map, however, not a number, but a combination of numbers in one set to a single number in another? For example, suppose we have the sets $A = \{(1,0),(5,2),(4,0)\}$ and $B = \{4,6,2\}$. We can map elements in A (despite them not being single numbers) to elements in B, for example, like $(1,0) \to 6$, $(5,2) \to 2$, and $(1,0) \to 4$. Then we would have effectively defined a function $f : A \to B$, since it meets all the criteria for it being a function (every element in the domain is assigned to only one element in the codomain).

Definition 9 *A function $f : A \to B$ is said to be a function of n variables if $A = A_1 \times A_2 \times \cdots \times A_n \to B$. In other words, we are defining points in n dimensions, where a single point is defined as the n-tuple $(x_1, x_2, \ldots, x_n), x_i \in A_i$.*

Let us restrict our attention for the moment in time to working with only two variables. Functions in multiple variables can not only be defined explicitly, as we had done so above, but also according to mathematical equations. For example, we may write $f(x_1, x_2) = 3x_1 + 5x_2$. If we want to find how the function f maps a point $(1,1)$ to a number, then we would do something similar to what we do in regular algebra, namely, replace each occurrence of x_1 with 1 and each occurrence of x_2 with 1: $f(1,1) = 3(1) + 5(1) = 8$. Likewise, $f(5,2) = 3(5) + 5(2) = 25$.

Definition 10 *Given an equation for a function $f(x_1, x_2, \ldots, x_n)$, in order to find the number that the point $(x_1, \ldots, x_n)$ is mapped to, we replace every occurrence in the formula for the function with the numbers provided, separated by commas, in the function.*

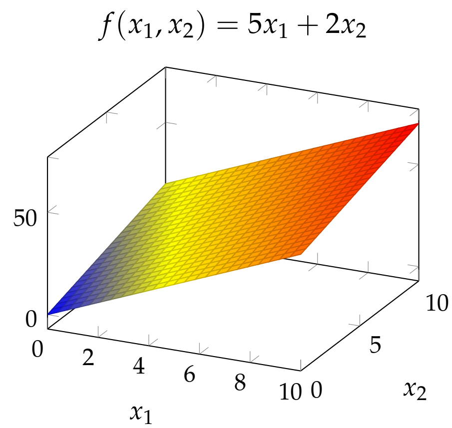

Figure 2: Plot of $f(x_1, x_2) = 5x_1 + 2x_2$

Functions of multiple variables are very difficult to visualize. A few ways exist however. If we have a function of two variables, then we commonly look at *contour plots* and three-dimensional plots. Once we surpass 2 dimensions, however, things get more complicated. One way to visualize the function is to assume the other variables are held constant and then to plot the dependent variable against only one of the independent variables. Changes in the other independent variables will subsequent change the plot, and so this is yet another way to visualize the function. For our purposes, we will not be concerned with visualization, but rather, more of the mathematical properties of multivariable functions, as well as the applications of them.

4.2 Applying Multi-Variable Functions

Let us motivate the concept of these functions with an example. Suppose that a manufacturer produces two different products. The first product sells for a profit of \$5 per product, while the second product sells for a profit of \$2 per product. If we let x_1 denote the number of products manufactured and sold for product 1, and x_2 denote the number of products manufactured and sold for product 2, then we can create a multi-variable function that would express the total profit of the firm. The total profit from product 1 would be $5x_1$ while the total profit from product 2 would be $2x_2$. Hence, the total profit of the firm can be expressed by the function $f(x_1, x_2) = 5x_1 + 2x_2$.

This function is plotted in Figure 2. Notice that if we fix a value for x_2, and "move along" the axis for x_1 (from the lower left corner to the front right), we notice that the function increases at a faster rate than fixing x_1 and moving along x_2. We can see how other functions are plotted as well. Consider the function $f(x,y) = .0208e^{(-0.54(\frac{(x-5)^2}{4}+\frac{(y-10)^2}{16}-0.075(x-5)(y-10)))}$, which is shown in Figure 3. Notice that we can visualize this function by looking at it from multiple perspectives. First, we notice in Figure 3 the 3-D plot, where the bottom axis moving from the left to the bottom right is the x axis, the left axis moving up and to the right is the y axis, and the line moving straight up is the z ($f(x,y)$) axis. We can see that lines are drawn along the function which are parallel to the x axis and other ones parallel to the y axis. We can further visualize these by looking at this plot on the $x - f(x,y)$ plane (x as horizontal axis, $f(x,y)$ as the vertical axis). We can do this by fixing different values of y and plotting the corresponding plot, since if we fix y, we turn $f(x,y)$ into a single-variable function. We can see this in Figure 4. These

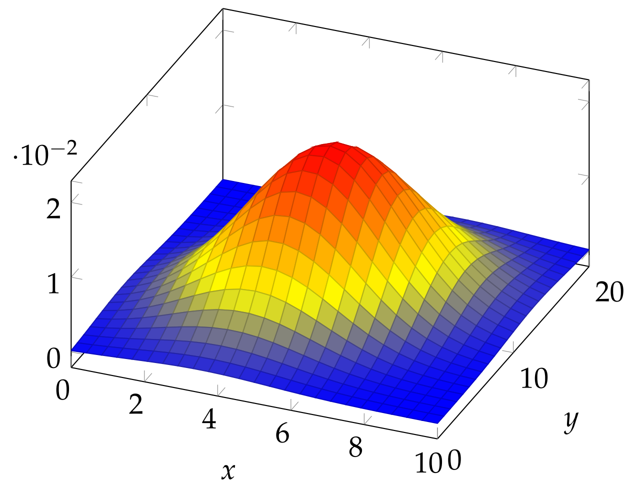

Figure 3: Plot of $f(x,y) = .0208e^{(-0.54(\frac{(x-5)^2}{4}+\frac{(y-10)^2}{16}-0.075(x-5)(y-10)))}$

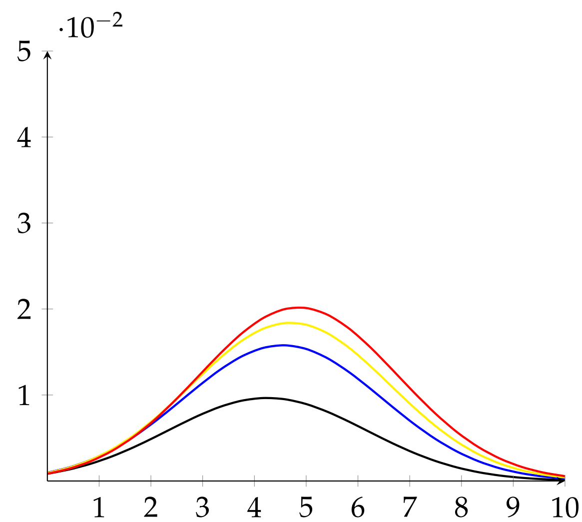

Figure 4: Cross section $x - f(x,y)$ plot of $f(x,y)$ when $y = 5$ (black), $y = 7$ (blue), $y = 8$ (yellow), and $y = 9$ (red)

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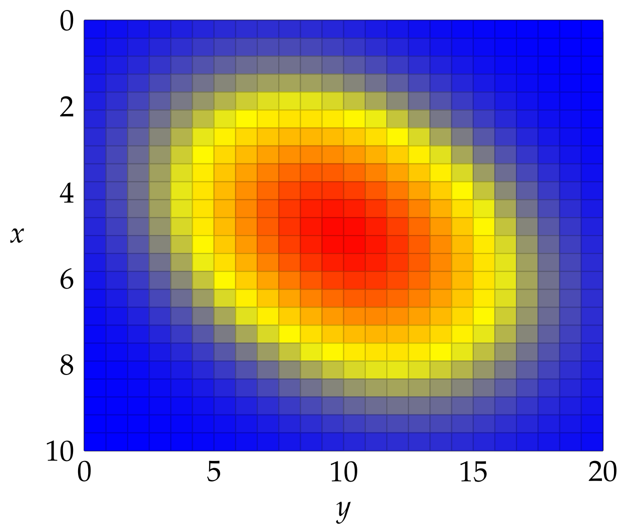

Figure 5: Plot of $f(x,y) = .0208e^{(-0.54(\frac{(x-5)^2}{4}+\frac{(y-10)^2}{16}-0.075(x-5)(y-10)))}$. This is the contour plot (the cross section $x-y$ plot) of the function, where red indicates larger numbers of $f(x,y)$ while blue represents numbers that are closer to 0. .

are called *cross sections* of the plot. That is, we can fix a value or x or y and plot $f(x,y)$ against only x or y. Another useful plot is a *contour plot*. This type of plot helps us visualize the different values of $f(x,y)$ by looking at the $x-y$ plane. Typically, we represent the values of the function by assigning different colors to lower and higher values for $f(x,y)$. These are common in weather radar maps with x being the longitudinal axis, y being the latitudinal axis, and the color indicating high (red), medium (yellow), or low (green) severity of rain. This same concept can be applied to visualizing functions of multiple variables.

Recall the newsvendor model, which had the optimal solution as $F(Q) = \frac{p-c}{p}$. If we think about this as a function of p and c (price and cost), we can observe the probability of being in stock when given different values of p and c, assuming the firm orders the optimal quantity Q^*. This is shown in plot 6. We can also see the contour version of this in Figure 7.

5 Multivariable Limits

In single variable calculus, we defined the limit of a function $f(x)$ as x approaches the value of a by looking distances. We said that $\lim_{x\to a} f(x) = L$ if we can bring the value of $f(x)$ really "close" to the value of L if we brought the value of x really "close" to the value of a, from either the left or the right. We mentioned that limits only exist when we observe the behavior of $f(x)$ approaching L by letting x approach a from any direction. In single variable calculus, since the points x and a are defined by only one dimension (that is, their points are defined on a single line with a single number), there were only two ways to approach a: from the left or from the right.

In multivariable calculus, since a single point in the domain of a function is defined by n numbers (in the $n-$dimensional hyper-plane), this means we can approach some point a in an infinite number of directions. Take for example the point $(4,4)$ in the $x-y-$plane. We can approach this point by fixing $y=4$ and move $x \to 4$, so one path may be for example $(0,4) \to$

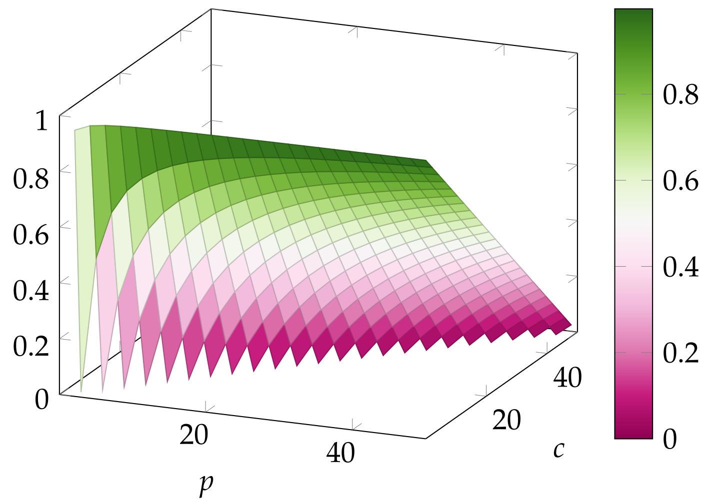

Figure 6: Plot of $f(p,c) = \frac{p-c}{p}$, which is the probability that a firm will remain in stock if it orders the optimal quantity. .

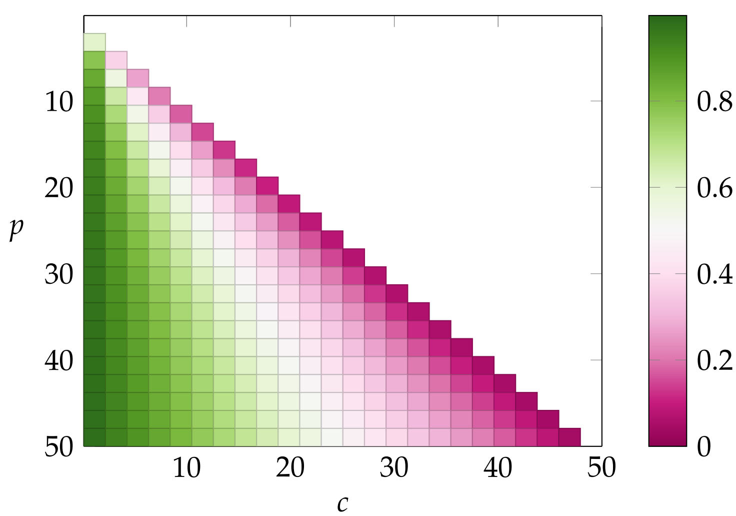

Figure 7: Plot of $f(p,c) = \frac{p-c}{p}$, which is the probability that a firm will remain in stock if it orders the optimal quantity. .

$(1,4) \to (2,4) \to \dots$. Another one may be $(8,4) \to (7,4) \to (6,4) \to \dots$. We can also approach it by fixing x and moving y: $(4,8) \to (4,7) \to (4,6) \to \dots$ or $(4,0) \to (4,1) \to (4,2) \to \dots$. We can also approach it on a given line, like $y = x$, so we may have $(0,0) \to (1,1) \to (2,2) \to \dots$, or $(7,7) \to (6,6) \to (5,5) \to \dots$. Since there are an infinite number of lines that go through the point $(4,4)$, there are an infinite number of unique paths to approach the point. This is distinct from the 1-dimensional case, where there were only two ways, namely left or right. We can motivate a conceptual definition of the limit for multivariable functions with this idea in mind:

Definition 11 *Let $f(x_1,\dots,x_n)$ be function $f : A_1 \times \dot{\times} A_n \to \mathbb{R}$, let $L \in \mathbb{R}$, $x \in A_1 \times \cdots \times A_n$, $a \in A_1 \times \cdots \times A_n$. Then we say that $f(x_1,\dots,x_n)$ approaches L, as x approach a, if we can make $f(x_1,\dots,x_n)$ really close to L by moving x to a along any possible path in the $n-$dimensional hyperplane. In such an instance, we write* $\lim\limits_{(x_1,\dots,x_n)\to(a_1,\dots,a_n)} f(x_1,\dots,x_n) = L$

How can we prove, however, that $f(x_1,\dots,x_n)$ will approach L for all paths when there are an infinite number of paths? This, obviously, is impossible, and so, we need a more refined definition that will permit us to be able to show that limits of multivariable functions exist. Recall in single variable calculus we defined limits more precisely using the concept of distances. We said that if we can choose an arbitrary distance $\epsilon > 0$, not matter how small, but still positive, between $f(x)$ and L, then we can find another distance $\delta > 0$ such that when the distance between x and a is less than δ, we will have the distance between $f(x)$ and L less than $epsilon$. Intuitively this says that as we take x very close to a, then $f(x)$ will be very close to L, regardless of direction. We can generalize this approach for multivariable functions.

The problem now, however, is how can we characterize the distance between two points, when the points are defined by two or more numbers rather than 1? Finding the distance between the points $x = 4$ and $x = 5$ is easy: $|5-4| = |1| = 1$. But how can we find the distance between say $(1,1)$ and $(5,6)$? This is not as obvious. We will turn to geometry for some help in being able to define distances between such points. Observe Figure 8, where we have the two points in question in red. We would like to find the distance between them. Notice that if we form a right triangle between the points $(1,1), (5,1)$, and $(5,6)$, then we can leverage trigonometry to find the distance between $(1,1)$ and $(5,6)$. The distance between $(1,1)$ and $(5,1)$ is easy to find, since they are on a straight line. This would be $|5-1| = |4| = 4$. The distance between $(5,1)$ and $(5,6)$ is also easy to find, since these two points also are on a straight line. We see that this distance would be $|6-1| = |5| = 5$. So the length of the bottom side of the triangle would be 4 while the length of the right side would be 5. Recall from trigonometry that when we have a right triangle ABC, where A is the length of the bottom side and B is the length of the right side, then the length of the last side is $C = \sqrt{A^2 + B^2}$, since we know that $C^2 = A^2 + B^2$ is always true when the triangle is a right triangle (that is, two sides are perpendicular and form a 90 degree angle).

Using this idea, we can see that the length of the last side would be $\sqrt{5^2 + 4^2}$. But notice how we arrived here. Recall that $4 = |5-1|$ was the distance between the x-values of our two points, and $5 = |6-1|$ was the distance between the $y-values$ of our two points. So the length of the last side would be expressed as $\sqrt{|5-1|^2 + |6-1|^2}$. Notice that this side of the triangle is a path between the two points that is the shortest distance (we state this without proof, you can go to a geometry textbook to prove this). So, one way that we can define the distance between two multidimensional points is by finding the length of the last side of the triangle. Generalizing this

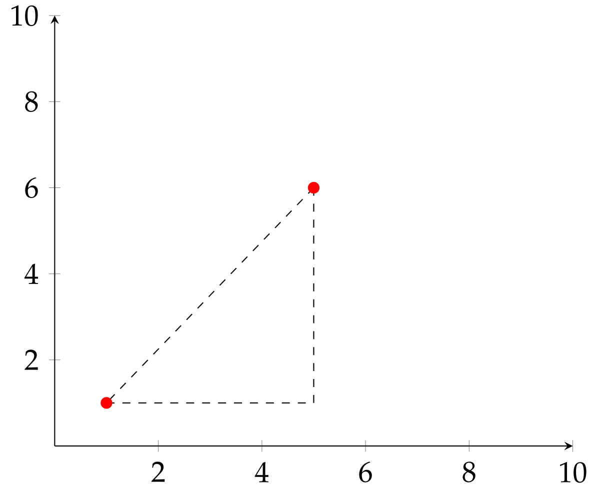

Figure 8: Finding the distance between the two red points.)

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idea for two-dimensions, we can see that given any points $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2})$ and $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2})$, that the distance between them is $||x - y|| = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$.

Definition 12 *Given two points $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ in $n-$dimensional space, we define the Euclidean Distance, denoted $||\boldsymbol{x} - \boldsymbol{y}||$, between them as:*

$$||\boldsymbol{x} - \boldsymbol{y}|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

When we defined the set of all points such that $|x - a| < \delta$, we were in essence defining an interval, namely the set of all points x such that $a - \delta < x < a + \delta$. In two or more dimensions, defining intervals is not as easy to work with, since we will be missing more points. Hence, we need to generalize the concept of an interval to higher-dimensional space. We can do this through the use of distances. Just like how we defined an interval as $|x - a| < \delta$, we can generalize this to defining the set $||\boldsymbol{x} - \boldsymbol{a}|| < \delta$. That is, we can consider all points whose *Euclidean Distance* from $\boldsymbol{a}$ is less than δ. In the two-dimensional case, we may be misled to think that this will lead to a square shape in the plane. However, we would be mistaken. It actually is a circle, and more generally, is called en ϵ-ball:

Definition 13 *An ϵ-ball in $n -$ dimensional space relative to a point $\boldsymbol{a} \in \mathbb{R}^n$ is defined as the set:*

$$B_\delta(a) = \{\boldsymbol{x} = (x_1, \ldots, x_n) | \ \ ||\boldsymbol{x} - \boldsymbol{a}|| < \delta\}$$

We can see an illustration of the set of points $B_3((4,4)) = \{(x_1, x_2) | \ \ ||(x_1, x_2) - (4,4)|| < 3\}$ in Figure 9. Notice that the blue points are outside of the set, while the red points are in the set. The blue point $(6, 8)$ is clearly out of the set, since $||(6, 8) - (4, 4)|| = \sqrt{(6-4)^2 + (8-4)^2} = 4.47 > 3$. Notice that the blue point on the circle is outside of the set. Why is this? Because epsilon balls are the multi-dimensional version of open intervals. Notice the definition, we need the distance between two points $||\boldsymbol{x} - \boldsymbol{a}||$ to be strictly less than δ, not exactly equal to it. When it is exactly equal to it, then, by definition, it is outside of the set, since we are defining the set to be all

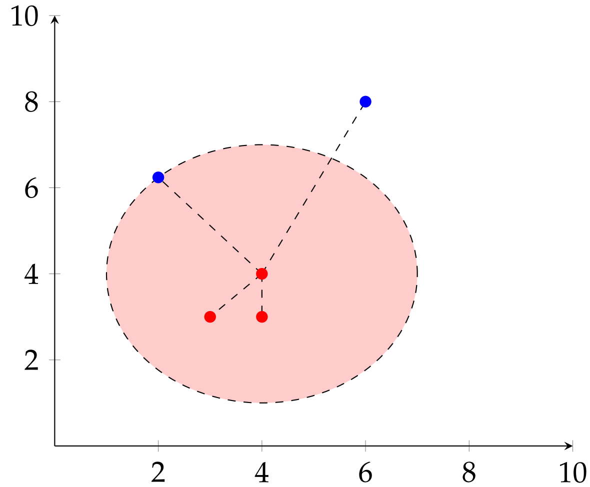

Figure 9: Finding the distance between the two red points.)

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points strictly less than, not less than or equal to! Notice that the point $(3,3)$ is in this set, since $||(3,3)-(4,4)|| = \sqrt{(3-4)^2+(3-4)^2} = \sqrt{2} < 3$. With our newfound generalization of "open intervals" in $n-$ dimensional space, we can leverage the definition of ϵ - balls to define concepts in high-dimensional spaces such as limits and continuity. We will begin by more precisely defining multi-variable limits:

Definition 14 *Let $a = (a_1, \ldots, a_n)$ be a point in n-dimensional space. Let $f(x_1, \ldots, x_n)$ be a function that maps points in n-dimensional space to $\mathbb{R}$. Let $L \in \mathbb{R}$. Then we say $\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = \lim_{(x_1,\ldots,x_n)\to(a_1,\ldots,a_n)} f(x_1, \ldots, x_n) = L$ if $\forall \epsilon > 0$, there exists a δ, such then when $(x_1, \ldots, x_n) \in B_\delta((a_1, \ldots, a_n))$, then $|f(x_1, \ldots, x_n) - L)| < \epsilon$.*

Essentially, limits are defined intuitively as follows. If we can choose an arbitrary distance between $f((x_1, \ldots, x_n))$ and L, then there must exist an open ball around the point $(a_1, \ldots, a_n)$ of radius δ such then for all values of $(x_1, \ldots, x_n)$ in this ball, the distance between the corresponding value $f(x_1, \ldots, x_n)$ and L will be strictly less than ϵ.

Let us look at a few examples. Consider the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$. Let us approach the point $(0,0)$ to see where this goes. If we approach $(0,0)$ from the x-axis (which is when $y=0$), then we have $f(x,0) = \frac{x^2-(0)^2}{x^2+(0^2)} = 1$. If we approach that point along the y-axis, which is when $x=0$, then we see that $f(0,y) = \frac{(0)^2-y^2}{(0)^2+y^2} = -1$. Notice how we approached the point from two different paths and obtained two different limits. Therefore, the limit cannot exist at $(0,0)$.

Consider the function $f(x,y) = \frac{2x^3-y^3}{x^2+y^2}$. We can show that $\lim_{(x,y)\to(0,0)} \frac{2x^3-y^3}{x^2+y^2} = 0$. First, we need to find a δ such that when $\sqrt{|x^2+y^2|} = \sqrt{x^2+y^2} < \delta$, that $\left|\frac{2x^3-y^3}{x^2+y^2} - 0\right| < \epsilon$ for any given $\epsilon > 0$. Just like we did in single variable calculus, we can work backwards:

$$\begin{aligned}\left|\frac{2x^3-y^3}{x^2+y^2}-0\right| &< \epsilon \\ \frac{|2x^3-y^3|}{|x^2+y^2|} &< \epsilon\end{aligned}$$

First, we will work on the numerator. We can see that:

$$\begin{aligned}|2x^3-y^3| &\leq 2|x|^3+2|y|^3 \\ &= 2|x||x|^2+2|y||y|^2 \\ &< 2|(x^2+y^2)^{\frac{1}{2}}||x|^2+2|(x^2+y^2)^{\frac{1}{2}}||y|^2 \\ &= 2|(x^2+y^2)^{\frac{1}{2}}|(|x|^2+|y|^2)\end{aligned}$$

Taking this into account, we see that:

$$\begin{aligned}\frac{|2x^3-y^3|}{|x^2+y^2|} &< \frac{2|(x^2+y^2)^{\frac{1}{2}}|(|x|^2+|y|^2)}{x^2+y^2} \\ &< 2|(x^2+y^2)^{\frac{1}{2}}|\end{aligned}$$

Therefore, we can solve for a δ:

$$\begin{aligned}2|(x^2+y^2)^{\frac{1}{2}}| &< \epsilon \\ |(x^2+y^2)^{\frac{1}{2}}| &< \frac{\epsilon}{2}\end{aligned}$$

So, let $\delta = \frac{\epsilon}{2}$. Then we have:

$$\begin{aligned}\left|\frac{2x^3-y^3}{x^2+y^2}-0\right| &= \frac{|2x^3-y^3|}{|x^2+y^2|} \\ &< 2|(x^2+y^2)^{\frac{1}{2}}| \\ &< 2\delta \\ &= 2\frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

So, we have shown that given any maximum distance $\epsilon > 0$ between $f(x,y)$ and 0, we can find a corresponding ball, with radius $\delta = \frac{\epsilon}{2}$, centered at $(0,0)$, such that for all points (x,y) in this ball, the distance between $f(x,y)$ and 0 is less than ϵ.

Given that the concept of *continuity* in single variable calculus was defined as a function $f(x)$ such that for any point a in the function's domain, if $\lim_{x\to a} f(x) = f(a)$, we said the function was continuous. We can construct a simliar definition for multivariable functions:

Definition 15 *A function $f(x_1, x_2, \ldots, f_n)$ is said to be continuous if*

$$\forall a \in A_1 \times \cdots \times A_n, \lim_{(x_1,\ldots,x_n)\to(a_1,\ldots,a_n)} f(x_1, \ldots, x_n) = f(a_1, \ldots, a_n)$$

Just like with the case of single variable calculus, it turns out that multi-variate polynomials, which are any functions that are summations of products of variables and numbers, are continuous. Rational functions are also continuous over points where they are defined.