Problem Set 3: Matrix Algebra

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For all problems, ensure that you show your work. Ensure you use the definitions, and present an argument, if needed, to prove your answer is correct. For example, if I ask you to find a relation that maps the even numbers in $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ to it's subsequent odd number in $B = \{1, 2, 3, 4, 5, 6, 7\}$, your answer would look like this:

We know the relation is defined with the first element being an even number in A and the second element being the next odd number in B. Let $E = \{x | x \in A \text{ and } x \text{ is even}\}$ *and let* $D = \{y | y \in B \text{ and } y \text{ is odd}\}$. *Then* $E = \{2, 4, 6\}$ *and* $D = \{1, 3, 5, 7\}$. *So, the relation that maps the even to the next odd in these sets would be* $R = \{(x, y) | x \in E \wedge y \in D \wedge y = x + 1\} = \{(2, 3), (4, 5), (6, 7)\}$.

1. In game theory, a reaction function is a function that tells the firm the best decision it should make based on the decision of another firm. Suppose that firm 1's profit function is $\pi(q_1) = pq_1 - cq_1$ and firm 2's profit function is $\pi(q_2) = pq_2 - cq_2$. If price is determined by the market, then $p = a - bQ$, where $Q = q_1 + q_2$. Firm 1's optimal profit would then be:

$$\begin{aligned}
\frac{\partial \pi_1}{\partial q_1}[pq_1 - cq_1] &= \frac{\partial \pi_1}{\partial q_1}[(a - bQ)q_1 - cq_1] \\
&= \frac{\partial \pi_1}{\partial q_1}[(a - b(q_1 + q_2))q_1 - cq_1] \\
&= \frac{\partial \pi_1}{\partial q_1}[aq_1 - bq_1^2 - q_2q_1 - cq_1] \\
&= a - 2bq_1 - q_2 - c = 0 \\
q_1 &= \frac{a - q_2 - c}{2}
\end{aligned}$$

   By symmetry, $q_2 = \frac{a - q_1 - c}{2}$

2. We therefore have the following system of linear equations:

$$\begin{aligned}
q_1 + \frac{1}{2}q_2 &= \frac{a - c}{2} \\
\frac{1}{2}q_1 + q_2 &= \frac{a - c}{2}
\end{aligned}$$

   Represent this system of linear equations in Matrix Notation.

3. From the previous problem, Use Gauss-Jordan Elimination to solve this system of equations for q_1 and q_2. This model is called a Cournot Competition Model. The solution of this

system is what is called the *Nash Equilibrium*, and it represents a firm's minimum best possible profit they can earn assuming the competitor also takes a strategy at it's minimum best possible strategy.

4. Using the answer from the previous problem, find an equation for the optimal profit levels of each firm in terms of the q_1 and q_2 respectively. Interpret the result. Who makes more profit and why?

5. Determine the basis for the vector space that is spanned by the vectors $\left\{ \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 19 \\ 30 \end{bmatrix} \right\}$.

   Determine, and explain, the dimension of this vector space.

6. Show that the set of all vectors $\left\{ s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$ is a vector space.

7. Given the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 6 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, find a formula for the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that describes the column space of A.

8. Consider the Fibonacci Sequence $F_{n+2} = F_{n+1} + F_n$, $F_0 = F_1 = 1$. If we set $F_{n+1} = F_{n+1}$, we obtain the system of equations

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n \\ F_{n+1} &= F_{n+1} \end{aligned}$$

   Represent this system in matrix notation.

9. From the previous problem, show that $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where A is the matrix found in the previous problem.

10. Find the Eigenvalues and Eigenvectors for the matrix A in the previous problem. Use these to find a closed form equation for the nth term in the Fibonacci Sequence.