6-5-2019 Virtual Notes

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Sequences

$1, 1, 2, 3, 5, 8, 13, \ldots$

$\{a_n\}$

$(1, 1, 2, 3, 5, 8, 13, \ldots) \in \mathbb{R} \times \mathbb{R} \times \cdots$

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \; \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 1 & 2 & 3 & 5 & 8 & 13 & \cdots \end{array}$$

$\mathbb{R}^\infty = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{R}\}$

or $f : \mathbb{N} \longrightarrow \mathbb{R}$

Closed form

$a_n = f(n)$

$a_n = n^2 = \{1, 4, 9, 16, 25, \ldots\}$

$a_n = \dfrac{n(n+1)}{2} = \{1, 3, 6, 10, \ldots\}$

Recursive Form

$a_n = f(a_i, a_j, \ldots)$

$F_{n+2} = F_{n+1} + F_n$

$F_0 = F_1 = 1$

Convergence

$a_n = \frac{1}{n} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\} \to 0$

$n \to \infty$

L iff $a_n \to L$ as $n \to \infty$ $\qquad \lim_{n\to\infty} a_n = L$

$L \in \mathbb{R}$

Formal Def:

$\lim_{n\to\infty} a_n = L$, then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. when $n > N$,

$|a_n - L| < \varepsilon$

$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \to \quad \left|\frac{1}{n} - 0\right| < \varepsilon$

$\to \frac{1}{n} < \varepsilon$

$\to \frac{1}{\varepsilon} < n \to \frac{1}{N} > \frac{1}{n}$

Let $N = 1/\varepsilon$, then if $n > N$, then

$|a_n - L| = \left|\frac{1}{n} - 0\right| = \frac{1}{n}$

$< \frac{1}{N} = \frac{1}{(1/\varepsilon)} = \varepsilon$

$\lim_{n\to\infty} a_n = \infty$ iff $\forall M \in \mathbb{R}^+$, $\exists N \in \mathbb{N}$, s.t., when $n > N$, $a_n > M$

a_n "tends to ∞".

show $\lim_{n\to\infty} n^2 = \infty$,

$a_n > M$

$n^2 > M$

$n > \sqrt{M}$

Let $N = \sqrt{M}$, then

$n > N$

$n^2 > N^2$

$$a_n = n^2 > N^2 = (\sqrt{M})^2 = M$$

Limit Laws

$$\lim_{n\to\infty} \frac{3(n^2+1)}{n^3 - 3n} \to \frac{\left(3\frac{\cancel{n^2}}{n^3} + \frac{3}{n^3}\right)}{\frac{\cancel{n^3}}{\cancel{n^3}} - \frac{3\cancel{n}}{n^{3}}}$$

$$\to \frac{\overset{\to 0}{3/n} + \overset{\to 0}{3/n^3}}{1 - \underset{\downarrow\, 0}{3/n^3}} \text{ as } n\to\infty, \quad = \frac{0}{1} = 0$$

Properties

$\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

$a_n = (-1)^n \frac{1}{n} = \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\}$

$|a_n| = |(-1)^n \frac{1}{n}| = \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad \lim_{n\to\infty} (-1)^n \frac{1}{n} = 0$

$a_n = r^n$

if $r = 2$,

$a_n = \{2, 4, 8, 16, \ldots\} \to \infty$

$r = 1$

$a_n = \{1^1, 1^2, 1^3, \ldots\} = \{1, 1, 1, \ldots\} \to 1$

as $n \to \infty$

$-1 < r < 1$

Thm: if $r \in (-1, 1)$, then $\{r^n\} \to L$

Increasing/Decreasing Seq:

increasing iff $a_n < a_{n+1}$ | $a_n = n^2$

decreasing iff $a_n > a_{n+1}$ | $a_n = \frac{1}{n}$

$$a_{n+1} > a_n$$

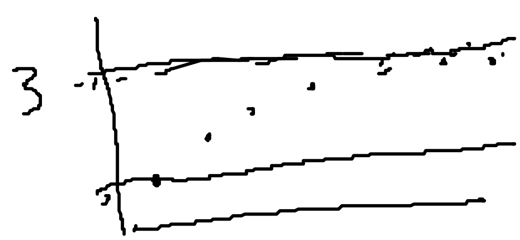
$$(n+1)^2 = n^2 + \underbrace{2n+2} > n^2$$

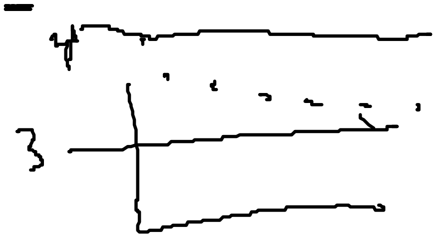
Monotonic = $\nearrow$ or $\searrow$

Boundedness

Bounded above iff $\exists M \in \mathbb{R}$ s.t. $a_n \le M, \forall n \ge 1$

$$a_n = \frac{3n-1}{n} = 3 - \frac{1}{n} \le 3$$





Bounded Below:

$\exists m \in \mathbb{R} \ni a_n \ge m \ \forall n \ge 1$ $\quad a_n = \frac{3n+1}{n} = 3 + (\frac{1}{n}) \ge 3$

BA $\wedge$ BB $\Rightarrow$ Seq is bounded seq.

Thm:

If a_n monotonic & bounded, then it converges

Cauchy Sequence

a_n is cauchy seq iff

$|a_{n+1} - a_n| \to 0$ as $n \to \infty$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_{n+1} - a_n| < \varepsilon$

$n > N$

$a_n = \frac{1}{n}$ $\quad \frac{1}{n+1} - \frac{1}{n} = \frac{n - n - 1}{n(n+1)} = -\frac{1}{n(n+1)} \xrightarrow[n \to \infty]{} 0$

Thm:

If a_n is cauchy, then it is convergent.

Series!

$$1+r+r^2+r^3+\cdots+r^n = \sum_{i=0}^{n} r^i = f(n)$$

$$1+2+3+4+\cdots+99+100+\cdots+(n-1)+n = \sum_{i=1}^{n} i$$

Closed Form

$$S_n = 1+r+r^2+\cdots+r^n$$

$$rS_n = r+r^2+r^3+\cdots+r^{n+1}$$

$$S_n - rS_n = 1 - r^{n+1} \Rightarrow S_n(1-r) = 1-r^{n+1}$$

$$\Rightarrow S_n = \frac{1-r^{n+1}}{1-r} = \sum_{i=0}^{n} r^i$$

$$\sum_{i=1}^{n} i = 1+2+3+4+\cdots+n-2+n-1+n$$

$$\underbrace{(1+n)}_{n+1}+\underbrace{(2+(n-1))}_{n+1}+(3+(n-2))+\cdots+\underbrace{\left(\frac{n}{2}+\left(n-\left(\frac{n}{2}-1\right)\right)\right)}_{n+1}$$

$$=(n+1)\left(\frac{n}{2}\right)=\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{n(n+1)}{2} = \infty$$

$$\sum_{i=0}^{\infty} r^i = \lim_{n\to\infty}\sum_{i=0}^{n} r^i = \lim_{n\to\infty}\frac{1-r^{n+1}}{1-r} = \frac{1}{1-r}$$

if $r \in (-1,1]$, $\Rightarrow |r|<1$

Convergence

$$\sum_{i=1}^{\infty}\frac{1}{i(i+1)}$$

$$a_1 = \frac{1}{1(1+1)} = \frac{1}{2}$$

$$a_2 = S_1 + S_2 = \frac{1}{2} + \frac{1}{2(2+1)} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6}$$

$$a_3 = S_1 + S_2 + S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{4}{6} + \frac{1}{12} = \frac{9}{12}$$

Partial Sums

$$\sum_{i=1}^{\infty}\frac{1}{i(i+1)} = \lim_{i\to\infty} a_i$$

$$\sum_{i=1}^{\infty}\frac{1}{i(i+1)} = \sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+1}\right)$$

$$\frac{\cancel{i+1}}{i(i+1)} - \frac{\cancel{i}}{(i+1)i}$$

$$\lim_{n\to\infty} = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \frac{1}{3} - \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n-1}} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} = 1$$

$S_n \to S$

$S_N = \sum_{i=1}^{n} a_n$ Converges if $S_n \to S$

If not, then S_n is divergent

<u>Thm</u>: If $\sum_{i=1}^{\infty} a_n$ is Convergent, then $\lim_{n\to\infty} a_n = 0$

$\sum_{n=1}^{\infty} \frac{1}{n} \neq S$ ← Divergent

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is divergent

$\sum_{i=1}^{\infty} i^2$ $\quad i^2 \to \infty$ as $i \to \infty$

$\therefore$ divergent

$\sum_{i=1}^{\infty} \frac{1}{3 - 1/i}$ $\quad \lim_{i\to\infty} \frac{1}{3-1/i} = 1/3 \neq 0$

$\therefore$ Divergent.

$\sum c a_n = c \sum a_n$

$\sum (a_n + b_n) = \sum a_n + \sum b_n$

$\sum (a_n - b_n) = \sum a_n - \sum b_n$

Ratio Test

$$\sum_{i=1}^{\infty} a_i \longrightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} <1 & \text{converge} \\ >1 & \text{diverge} \\ =1 & ? \end{cases}$$

$$\sum_{i=1}^{\infty} x^n = \frac{1}{1-x} \Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|$$

$$= \lim_{n\to\infty} |x| = |x| < 1$$

Harmonic

$$\sum_{i=1}^{\infty} \frac{1}{i} \Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{1+1/n}\right|$$

$$= |1| = 1$$

inconclusive

$$\sum_{i=0}^{\infty} \frac{x^n}{n!} \Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)!}{x^n/(n)!}\right| \qquad \boxed{n! = n(n-1)!}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{x\, n!}{(n+1)\, n!}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0$$

$\forall x \in \mathbb{R}$

$a_n(x)$, where $\sum a_n(x)$ is convergent is called the radius of convergence

Power Series

$a_n(x)$

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

"Centered at a"

(1) $x=a$

(2) $\forall x \in \mathbb{R}$

(3) R s.t. Converge $\forall x \in (a-R, a+R)$

Radius of convergence

$\frac{1}{1+x^2} =$

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-x^2)^{n+1}}{(-x^2)^n}\right|$$

$$= \lim_{n\to\infty}\left|-x^2\right| = \left|-x^2\right| < 1$$

$$= x^2 < 1$$

$x \in (-1, 1)$

$R = 1$

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f'(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$$

$f^{(0)}(x) = f(x)$

Maclaurin $a=0$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

$$e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$f(0) = e^0 = 1$

$f'(0) = e^0 = 1$

$f''(0) = e^0 = 1$

$\vdots$

$f^{(n)}(0) = e^0 = 1$

$$\lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty}\left|\frac{x\,n!}{(n+1)\,n!}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1$$

$x \in \mathbb{R}$

$f(x) = e^{-3x}$

$f^{(n)}(x) = (-3)^n$

$f^{(0)}(0) = e^{-3(0)} = 1$

$f'(x) = -3e^{-3x}$ $\quad f^{(1)}(0) = -3e^0 = -3$

$f''(x) = 9e^{-3x}$ $\quad f^{(2)}(0) = 9e^0 = 9$

$f^{(3)}(0) = -27e^0 = -27$

$$f(x) = e^{-3x} = \sum_{i=1}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{i=1}^{\infty} \frac{(-3)^n}{n!}x^n$$

$a \neq 0$: Taylor Series
$a = 0$: MacLaurin Series

$f(x) = \ln(x) = \cancel{\sum_{i=1}^{\infty} \frac{\ldots^n}{n!}(x-a)^n}$

$f^{(0)}(0) = f(0) = \ln(0)$

Problem!, So let $a = 1$

$a = 1$

$$= \sum_{i=1}^{\infty} \frac{(-1)^n (n-1)!}{n!}(x-1)^n$$

$f^{(0)}(1) = f(1) = \ln(1) = 0$

$f^{(1)}(x) = \frac{1}{x} = x^{-1}$

$$= \sum_{i=1}^{\infty} \frac{(-1)^n}{n}(x-1)^n$$

$f^{(2)}(x) = -1x^{-2} = \frac{-1}{x^2}$

$f^{(3)}(x) = 2x^{-3}$

$f^{(4)}(x) = -6x^{-4}$

$f^{(n)}(x) = (-1)^n (n-1)!\,x^{-n}$

0	0
1	1
-1	2
2	3
-6	4
24	

$$\lim_{n\to 0}\left|\frac{(-1)^{n+1}(x-1)^{n+1}/(n+1)}{(-1)^n(x-1)^n/n}\right|$$

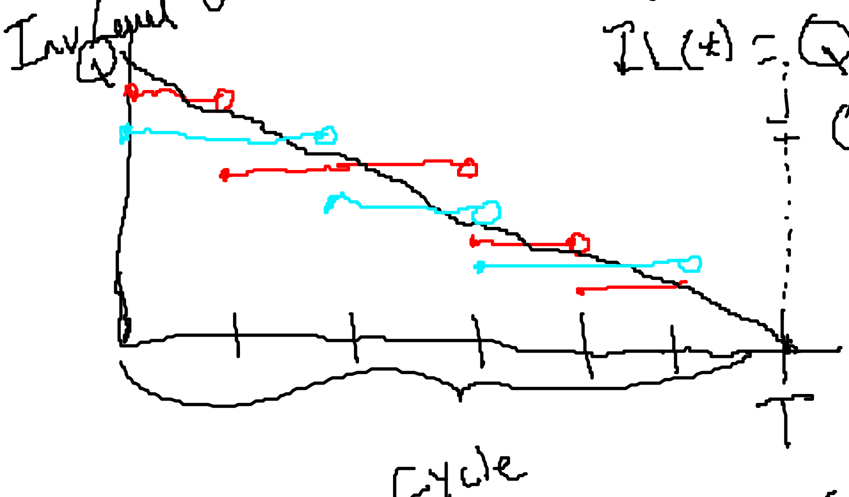
$$\lim_{n\to\infty}\left|(-1)(x-1)\frac{n}{n+1}\right| = |x-1| < 1 \Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$$

Integration

λ - demand rate $\qquad D = \lambda t$

$IL(t) = Q - D(t)$

$= Q - \lambda t$



$c(Q) = \dfrac{C_c(Q)}{T}$

$C_c(Q) = FC + VC + IHC$

$= K + Qc +$

K = fixed

c = Variable

h = inv holding / unit-week

$hIL(1)(1) +$
$hIL(2)(1) +$
$hIL(3)(1) +$
$hIL(4)(1) +$
$hIL(5)(1)$

$hIL(1/2)(1/2) +$
$\vdots$
$hIL(9/2)(1/2) +$
$hIL(5)(1/2)$

$[0, T]$ $\quad$ n rectangles. $\quad \dfrac{T-0}{n} = \dfrac{T}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} hIL\left(i\cdot\frac{T}{n}\right)\frac{T}{n} = \int_0^T hIL(x)\,dx$$

"Total Area under $IL(t)$ from $t=0$ to $t=T$"

$$\sum_{i=1}^{n} h\left(Q-\lambda\left(i\cdot\frac{T}{n}\right)\right)\frac{T}{n} = QhT - \sum \lambda h i\frac{T^2}{n^2}$$

$$= hTQ - \sum_{i=1}^{n}\lambda h i\frac{T^2}{n^2}$$

$$= hTQ - \frac{\lambda h T^2}{n^2}\sum_{i=1}^{n} i$$

$$= hTQ - \frac{\lambda h T^2}{n^2}\,\frac{n(n+1)}{2}$$

$$= \text{''} \qquad \text{''} - \frac{\lambda h T^2}{2}\left(1+\frac{1}{n}\right)$$

$$\to hTQ - \frac{\lambda h T^2}{2}$$

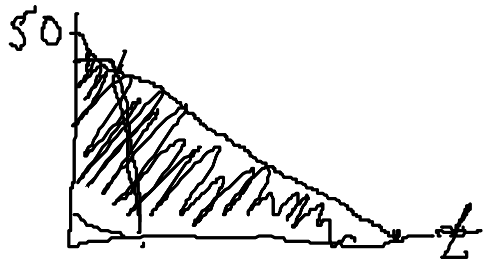
$$IL(T) = 0 = Q - \lambda T \to Q = \lambda T \to T = \frac{Q}{\lambda}$$

$$C_c(Q) = K + cQ + hTQ - \frac{\lambda h T^2}{2}$$

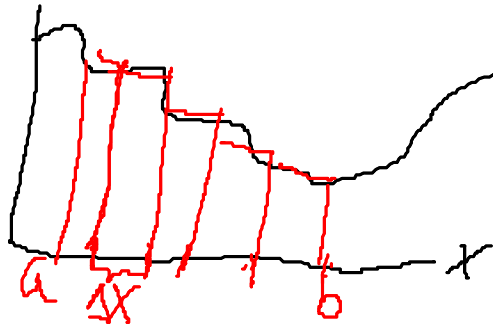
$$C(Q) = \frac{C_c(Q)}{T} = \frac{K}{T} + \frac{cQ}{T} + hQ - \frac{\lambda h T}{2}$$

$$= \frac{K\lambda}{Q} + \frac{c\cancel{Q}\lambda}{\cancel{Q}} + \frac{2hQ}{2} - \frac{h\cancel{\lambda}}{2}\,\frac{Q}{\cancel{\lambda}}$$

$$= \frac{K\lambda}{Q} + c\lambda + \boxed{\frac{hQ}{2}}$$



Def: $f(x)$



$$\Delta x = \frac{b-a}{n}$$

Definite Integral

x_i^*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\int_a^b x\, f(t)\, dt$$

Area under $f(x)$ between a and b

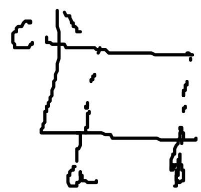
$\forall \varepsilon > 0, \exists N$, s.t. $\forall n > N$,

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon$$

Actual Area   Approx

Integration Rules

$$\int_a^b c\,dx = c(b-a)$$



$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

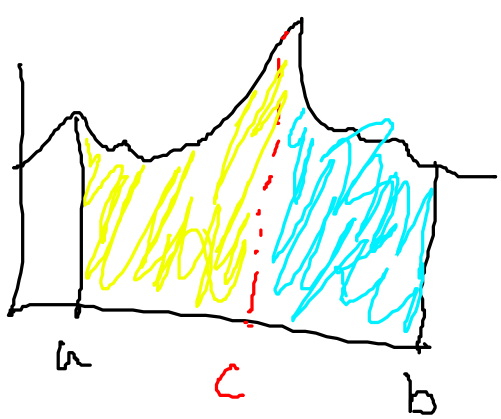
$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_0^2 \sqrt{4-x^2}+x\,dx$$

$$\int_0^2 \sqrt{4-x^2}\,dx + \int_0^2 x\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\pi + 2$$



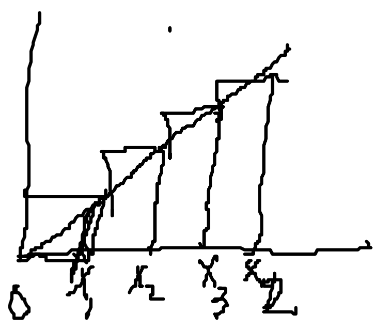
$f(x) \geq 0,$

$a \leq x \leq b, \quad \int_a^b f(x)\,dx \geq 0$

$f(x) \geq g(x)$

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

$$\int_0^2 x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$



$$= \lim_{n\to\infty}\sum_{i=1}^{n} x_i^*\left(\frac{2-0}{n}\right)$$

$$\Delta x = \frac{b-a}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} i\left(\frac{2}{n}\right)\left(\frac{2}{n}\right)$$

$$\frac{2}{n},\ 2\left(\frac{2}{n}\right)$$

$$= 4\lim_{n\to\infty}\frac{1}{n^2}\left(\frac{n^2+n}{2}\right)$$

$$= 2\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 2$$

The Fundamental Theorem of Calculus

$F'(x) = f(x)$ $\quad$ $f(x) = x^2,\ F(x) = \frac{1}{3}x^3$

(1) If $g(x) = \int_a^x f(t)\,dt$, $g'(x) = f(x)$

$x \in [a,b]$

(2) $\int_a^b f(x)\,dx = F(b) - F(a)$ $\quad$ $F'(x) = f(x)$

$$\int_0^2 x\,dx = F(2) - F(0)$$
$$= \frac{1}{2}(2)^2 - \frac{1}{2}(0)^2$$
$$= 2$$

$f(x) = x^n$

$F(x) = \frac{1}{n+1}x^{n+1}$

If $g(x) = x$,
$G(x) = \frac{1}{2}x^2$

$f(x) = \frac{1}{x}$

$F(x) = \ln|x|$

$$\int_1^2 \frac{1}{x}\,dx = \ln|x|\Big|_1^2$$
$$= \ln|2| - \ln|1|$$
$$= \ln(2)$$

$$\int_a^b f(x)\,dx \in \mathbb{R} \cup \{\pm\infty\}$$

"Definite Integral"

$F(x) = \int f(x)\,dx$

$F'(x) = f(x)$

"Indefinite Integral"

$f(x) = x^2$ $\boxed{F(x) = \frac{1}{3}x^3 + C}$

If given boundary condition $F(a)=b$, then $\int f(x)\,dx$ is a single function. Else, it's a family of functions

$F(x) = \frac{1}{3}x^3 + 5$

$F(0) = 3$

$\int cf(x)\,dx = c\int f(x)\,dx$, $\int f(x) \mp g(x)\,dx = \int f(x)\,dx \mp \int g(x)\,dx$

$\int k\,dx = kx + C$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, $\int \frac{1}{x}\,dx = \ln|x| + C$

$\int e^x\,dx = e^x + C$, $\int a^x\,dx = \frac{a^x}{\ln a} + C$

$$\int 5x^2 - 2x + e^x \, dx = \int 5x^2 dx - \int 2x \, dx + \int e^x dx$$

$$= 5\int x^2 dx - 2\int x dx + \int e^x dx$$

$$= 5\left(\frac{1}{3}x^3\right) - 2\left(\frac{1}{2}x^2\right) + e^x + \underbrace{C_1 + C_2 + C_3}$$

$$= \frac{5}{3}x^3 - x^2 + e^x + C$$

$$\underline{F(x) = \frac{5}{3}x^3 - x^2 + e^x + C}, \quad \frac{dF}{dx} = \frac{d}{dx}\left[\frac{5}{3}x^3 - x^2 + e^x + C\right]$$

$$= \frac{d}{dx}\left[\frac{5}{3}x^3\right] - \frac{d}{dx}[x^2] + \frac{d}{dx}e^x + \frac{d}{dx}[C]$$

$$F(0) = 3$$

$$F(0) = \cancel{\frac{5}{3}(0)^3} - \cancel{(0)^2} + e^0 + C = 3$$

$$= 5x^2 - 2x + e^x$$

$$C + 1 = 3$$

$$C = 2$$

$$\underline{F(x) = \frac{5}{3}x^3 - x^2 + e^x + 2}$$

$$\int e^{-2x} dx$$