Problem Set 2: Derivatives, Sequences, Series, and Integrals

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For all problems, ensure that you show your work. Ensure you use the definitions, and present an argument, if needed, to prove your answer is correct. For example, if I ask you to find a relation that maps the even numbers in $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ to it's subsequent odd number in $B = \{1, 2, 3, 4, 5, 6, 7\}$, your answer would look like this:

We know the relation is defined with the first element being an even number in A and the second element being the next odd number in B. Let $E = \{x | x \in A \text{ and } x \text{ is even}\}$ *and let* $D = \{y | y \in B \text{ and } y \text{ is odd}\}$. *Then* $E = \{2, 4, 6\}$ *and* $D = \{1, 3, 5, 7\}$. *So, the relation that maps the even to the next odd in these sets would be* $R = \{(x, y) | x \in E \wedge y \in D \wedge y = x + 1\} = \{(2, 3), (4, 5), (6, 7)\}$.

1. **Explain why the derivative of the function $f(x) = |x|$ does not exist at $x = 0$.**

   Recall that a derivative is defined as the limit $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. This means that the derivative can only exist at a point x if the left and right limits exist. Notice that if we take $\lim_{\Delta x \to 0^-} \frac{|0+\Delta x| - |0|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x}$ from the left, then $\Delta x < 0$, and so $|\Delta x| = -\Delta x$. This means that $\lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$. If we find the right limit, then $\Delta x > 0$, and so $|\Delta x| = \Delta x$. This means that $\lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1$. Since $-1 \neq 1$, we conclude that the limit cannot exist at $x = 0$.

2. **Let $f(x) = \frac{1}{\sqrt{2\sigma^2 \pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, and assume that σ, μ and π are numbers (i.e. constants). Find, and prove, the maximum of this function using first and second order conditions.**

   We first find the first order conditions. We can make some of our math easier to work with if we set $\beta = \frac{1}{\sqrt{2\sigma^2 \pi}}$ and $\alpha = -\frac{(x-\mu)^2}{2\sigma^2}$. Using the chain rule, we have:

$$
\begin{aligned}
\frac{d}{dx} f(x) &= \frac{d}{d\alpha} \beta e^{\alpha} \frac{d\alpha}{dx} \\
&= \beta e^{\alpha} \left(\frac{-2(x-\mu)}{2\sigma^2} \right)
\end{aligned}
$$

The first condition is that this is equal to zero, so, solving for this, we have:

$$\begin{aligned}\frac{df}{dx} &= 0\\ \beta e^{\alpha}\left(\frac{-2(x-\mu)}{2\sigma^2}\right) &= 0\\ \frac{-2(x-\mu)}{2\sigma^2} &= 0\\ -2(x-\mu) &= 0\\ x &= \mu\end{aligned}$$

Now we find the second order conditions:

$$\begin{aligned}\frac{d^2f}{dx^2} &= \frac{d}{dx}\left[\frac{df}{dx}\right]\\ &= \frac{d}{dx}\left[\beta e^{\alpha}\left(\frac{-2(x-\mu)}{2\sigma^2}\right)\right]\\ &= \frac{d}{dx}[\beta e^{\alpha}]\left[\left(\frac{-2(x-\mu)}{2\sigma^2}\right)\right] + \beta e^{\alpha}\frac{d}{dx}\left[\left(\frac{-2(x-\mu)}{2\sigma^2}\right)\right]\\ &= \frac{d}{dx}[\beta e^{\alpha}]\left[\left(\frac{-2(x-\mu)}{2\sigma^2}\right)\right] - \beta e^{\alpha}\left(\frac{1}{\sigma^2}\right)\end{aligned}$$

When $x=\mu$, we see the second derivative is then equal to $-\beta\frac{e^{\alpha}}{\sigma^2}$. Since all terms in this are positive, then the overall term is negative, which means that at the point $x=\mu$, $f'(x)=0$ and $f''(x)<0$, implying that there is a maximum at $x=\mu$.

3. **Find an equation for the nth derivative of the function $f(x)=\frac{1}{x}$.** Working out the first few derivatives for $f(x)=x^{-1}$, we have

$$\begin{aligned}f^{(1)}(x) &= -x^{-2}\\ f^{(2)}(x) &= -(-2)x^{-3} = 2x^{-3}\\ f^{(3)}(x) &= 2(-3)x^{-4} = -(2)(3)x^{-4}\\ f^{(4)}(x) &= -(2)(3)(-4)x^{-5} = (2)(3)(4)x^{-5}\end{aligned}$$

We can see that a general pattern begins to emerge. First, we have the negative sign alternative every turn, with it being negative when n is odd and positive when n is even. The first part of the general expression would then be $(-1)^n$. Next, we notice that the coefficient is $(1)(2)(3)\dots(n)$. By definition, we know that this is the same as $n!$. Last, we notice that for n, the exponent on x is $-(n+1)$. Hence, we have: $f^{(n)}(x)=(-1)^n n! x^{-(n+1)}=(-1)^n\frac{n!}{x^{n+1}}$

4. **The Cobb-Douglas Production function is defined as $P=bL^{\alpha}K^{1-\alpha}$, where P represent production levels, L represents the amount of labor in person-hours, and K represents the amount of capital invested, α and b are parameters in the model. Use implicit differentiation to find an expression for the rate of change of capital with respect to labor**

(Hint: it is okay to have a rate of change in your expression, but you should be able to find an explicit expression for the rate of change of capital with respect to labor).

In order to do implicit differentiation, we start with the equation, take the derivative with respect to labor of both sides, and last solve for the expression $\frac{dK}{dL}$:

$$\begin{aligned}
P &= bL^{\alpha}K^{1-\alpha} \\
\frac{d}{dL}[P] &= \frac{d}{dL}[bL^{\alpha}K^{1-\alpha}] \\
\frac{dP}{dL} &= bK^{1-\alpha}\frac{d}{dL}[L^{\alpha}] + bL^{\alpha}\frac{d}{dL}[K^{1-\alpha}] \\
\frac{dP}{dL} &= \alpha bK^{1-\alpha}L^{\alpha-1} + (1-\alpha)bL^{\alpha}K^{-\alpha}\frac{dK}{dL} \\
\frac{dP}{dL} &= \alpha bK^{1-\alpha}L^{\alpha-1} + (1-\alpha)bL^{\alpha}K^{-\alpha}\frac{dK}{dL} \\
(1-\alpha)bL^{\alpha}K^{-\alpha}\frac{dK}{dL} &= \alpha bK^{1-\alpha}L^{\alpha-1} - \frac{dP}{dL} \\
\frac{dK}{dL} &= \frac{1}{(1-\alpha)bL^{\alpha}K^{-\alpha}}\left(\alpha bK^{1-\alpha}L^{\alpha-1} - \frac{dP}{dL}\right)
\end{aligned}$$

5. **Firms often try to maximize their profits as one of their primary objectives. Generally, profit is defined as total revenue minus total cost, or $\pi(Q) = TR(Q) - TC(Q)$, where Q is the total number of units produced. Use the first and second order conditions to characterize the firm's optimal production policy. That is, why conditions need to be true so that the firm is running at an optimal level of profit?**

   First, in order to find the optimal profit, we take the derivative with respect to quantity:

$$\begin{aligned}
\frac{d}{dQ}\pi(Q) &= \frac{d}{dQ}[TR(Q) - TC(Q)] \\
&= \frac{d}{dQ}[TR(Q) - TC(Q)] \\
&= TR'(Q) - TC'(Q)
\end{aligned}$$

   The first order conditions are then:

$$\begin{aligned}
\frac{d\pi}{dQ} &= 0 \\
TR'(Q) - TC'(Q) &= 0 \\
TR'(Q) &= TC'(Q)
\end{aligned}$$

   In practical terms, the derivative of the total revenue represents the rate of change in revenue as we change ordering quantity. This is called *marginal revenue*, and it represents the *change in revenue* when we observe a *change in quantity*. That is, how much *additional* revenue the firm will bring in when it increases the quantity of production or sales. A simliar interpretation can be given for costs, and hence the derivative of total cost would represent what is called *marginal cost*, or, the change in cost if we were to experience a change in quantity.

The first order conditions essentially state that the firm's optimal profit levels occur when the marginal revenue is equal to the marginal costs, or, $MR(Q) = MC(Q)$. In other words, the optimal quantity for the firm to produce is when the change in total revenue is equal to the change in total cost. This essentially means, that if we were to produce less than this quantity, it is possible that our revenues will increase faster than increases in our cost, but we still can obtain additional benefit if we produce more, since the marginal revenue would still be greater than marginal cost. If we were to produce beyond this point, than our marginal cost will exceed our marginal revenue, and we obtain a *point of diminishing return*, or in otherwords, a point where we obtain no benefit from producing more, since the extra cost to produce the additional unit will exceed the additional revenue in doing so. As such, it would be optimal to manufacture at the point where MR(Q) = MC(Q).

Likewise, for the second order condition, we see that : $TR''(Q) - TC''(Q) < 0$ in order for the profit at the point Q such that $MR(Q) = MC(Q)$ to be at a maximum. Hence, we need $MR'(Q) < MC'(Q)$. Put differently, we need the change in the marginal revenue to be less than changes in the marginal costs. Put differently, at the point of optimally, the marginal costs will begin to change in excess of changes in the marginal revenue. We can think about this as the "acceleration of revenue increase" being less than the "acceleration of cost increase", which means, again, at the point of optimally, if we exceed this point, we begin to observe diminishing returns, as the benefits received from additional production is no longer present.

6. **Let $f(x) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, and assume that σ, μ and π are numbers (i.e. constants). Find the Taylor series for this function.**

   Finding the Taylor Series for this can be very tricky. However, if we make some substitutions, we can drastically reduce the amount of our work down. Let $b = \frac{1}{\sqrt{2\sigma^2\pi}}$, $y = x - \mu$, $a = \frac{y^2}{2\sigma^2}$, $g = \frac{1}{\sigma^2}$, $\frac{da}{dy} = gy$ and $k = be^{-a}$. First, we can see that $\frac{dy}{dx} = 1$, and as such, our analysis working in y will be exactly the same as if we were to work in x, which helps simplify much of our analysis down. In addition, we can see that $\frac{dk}{dy} = -be^{-a}gy = -kgy$. Keeping all of this in mind, we can find the Taylor Series by first finding an equation for the coefficients. We will center the coefficients at $x = \mu$, or equivalently at $y = 0$. Doing this will greatly simplify our coefficients down. Let us work on the first few:

$$
\begin{aligned}
f^{(0)}(y) &= f(y) \\
&= be^{-a} \\
&= k \\
f^{(1)}(y) &= \frac{d}{dy}[k] \\
&= -kgy \\
f^{(2)}(y) &= \frac{d}{dy}[-kgy] \\
&= -kg\frac{d}{dy}[y] - yg\frac{d}{dy}[k] \\
&= -kg - yg(-kgy) \\
&= -kg + ky^2g^2 \\
&= k(y^2g^2 - g) \\
f^{(3)}(y) &= \frac{d}{dy}[k(y^2g^2 - g)] \\
&= \frac{d}{dy}[k](y^2g^2 - g)] + k\frac{d}{dy}[(y^2g^2 - g)] \\
&= (-kgy)(y^2g^2 - g) + k(2yg^2) \\
&= k(-y^3g^3 + g^2y + 2yg^2) \\
&= k(-y^3g^3 + (1+2)yg^2) \\
f^{(4)}(y) &= \frac{d}{dy}[k(-y^3g^3 + (1+2)yg^2)] \\
&= \frac{d}{dy}[k](-y^3g^3 + (1+2)yg^2) + k\frac{d}{dy}[(-y^3g^3 + (1+2)yg^2)] \\
&= (-kgy)(-y^3g^3 + (1+2)yg^2) + k(-3y^2g^3 + (1+2)g^2) \\
&= k(y^4g^4 - (1+2)y^2g^3) + k(-3y^2g^3 + (1+2)g^2) \\
&= k(y^4g^4 - (1+2)y^2g^3 - 3y^2g^3 + (1+2)g^2) \\
&= k(y^4g^4 - (1+2+3)y^2g^3 + (1+2)g^2) \\
f^{(5)}(y) &= \frac{d}{dy}[k(y^4g^4 - (1+2+3)y^2g^3 + (1+2)g^2)] \\
&= \frac{d}{dy}[k](y^4g^4 - (1+2+3)y^2g^3 + (1+2)g^2) + k\frac{d}{dy}[(y^4g^4 - (1+2+3)y^2g^3 + (1+2)g^2)] \\
&= (-kgy)(y^4g^4 - (1+2+3)y^2g^3 + (1+2)g^2) + k(4y^3g^4 - 2(1+2+3)yg^3) \\
&= k(-y^5g^5 + (1+2+3)y^3g^4 - (1+2)g^3y) + k(4y^3g^4 - 2(1+2+3)yg^3) \\
&= k(-y^5g^5 + (1+2+3)y^3g^4 - (1+2)g^3y + 4y^3g^4 - 2(1+2+3)yg^3) \\
&= k(-y^5g^5 + (1+2+3+4)y^3g^4 - [(1+2) + 2(1+2+3)]yg^3) \\
&= k(-y^5g^5 + (\sum_{i=1}^{4} i)y^3g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]yg^3)
\end{aligned}
$$

$$
\begin{aligned}
f^{(6)}(y) &= \frac{d}{dy}[k(-y^5g^5 + (\sum_{i=1}^{4} i)y^3g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]yg^3)] \\
&= \frac{d}{dy}[k](-y^5g^5 + (\sum_{i=1}^{4} i)y^3g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]yg^3)] + k\frac{d}{dy}[(-y^5g^5 + (\sum_{i=1}^{4} i)y^3g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]yg^3)] \\
&= (-kgy)(-y^5g^5 + (\sum_{i=1}^{4} i)y^3g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]yg^3)] + k[(-5y^4g^5 + 3(\sum_{i=1}^{4} i)y^2g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]g^3)] \\
&= k(y^6g^6 - (\sum_{i=1}^{4} i)y^4g^5 + [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]y^2g^4)] + k(-5y^4g^5 + 3(\sum_{i=1}^{4} i)y^2g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]g^3) \\
&= k(y^6g^6 - (\sum_{i=1}^{5} i)y^4g^5 + [(\sum_{i=1}^{3} i \sum_{j=1}^{i+1} j)]y^2g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]g^3) \\
f^{(7)}(y) &= \frac{d}{dy}[k(y^6g^6 - (\sum_{i=1}^{5} i)y^4g^5 + [(\sum_{i=1}^{3} i \sum_{j=1}^{i+1} j)]y^2g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]g^3)] \\
&= \frac{d}{dy}[k](y^6g^6 - (\sum_{i=1}^{5} i)y^4g^5 + [(\sum_{i=1}^{3} i \sum_{j=1}^{i+1} j)]y^2g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]g^3)] + k\frac{d}{dy}[(y^6g^6 - (\sum_{i=1}^{5} i)y^4g^5 \\
&+ [(\sum_{i=1}^{3} i \sum_{j=1}^{i+1} j)]y^2g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]g^3)] \\
&= (-kgy)(y^6g^6 - (\sum_{i=1}^{5} i)y^4g^5 + [(\sum_{i=1}^{3} i \sum_{j=1}^{i+1} j)]y^2g^4 - [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]g^3)] + k[(6y^5g^6 - 4(\sum_{i=1}^{5} i)y^3g^5 \\
&+ 2[(\sum_{i=1}^{3} i \sum_{j=1}^{i+1} j)]yg^4] \\
&= k(-y^7g^7 + (\sum_{i=1}^{5} i)y^5g^6 - [(\sum_{i=1}^{3} i \sum_{j=1}^{i+1} j)]y^3g^5 + [(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)]g^4y) + k[(6y^5g^6 - 4(\sum_{i=1}^{5} i)y^3g^5 \\
&+ 2[(\sum_{i=1}^{3} i \sum_{j=1}^{i+1} j)]yg^4] \\
&= k(-y^7g^7 + (\sum_{i=1}^{6} i)y^5g^6 - [(\sum_{i=1}^{4} i \sum_{j=1}^{i+1} j)]y^3g^5 + ([(\sum_{i=1}^{2} i \sum_{j=1}^{i+1} j)] + 2[(\sum_{i=1}^{3} i \sum_{j=1}^{i+1} j)])g^4y \\
&= k(-y^7g^7 + (\sum_{i=1}^{6} i)y^5g^6 - [(\sum_{i=1}^{4} i \sum_{j=1}^{i+1} j)]y^3g^5 + ([\sum_{m=1}^{2} m \sum_{i=1}^{m+1} i \sum_{j=1}^{i+1} j)g^4y
\end{aligned}
$$

We start to notice a pattern for certain terms. To help simplify some of the work, we will define a new function of two variables to help reduce some of the notation down. Let $h(m,n) = \begin{cases} 1 & \text{if } m < 1 \\ \sum_{i_1=1}^{n} i_1 \sum_{i_2=1}^{i_1+1} i_2 \sum_{i_3=1}^{i_2+1} i_3 \ldots i_{m-1} \sum_{i_m=1}^{i_{m-1}+1} i_m & \text{if } m \geq 1 \end{cases}$. Then we can simplify our 7th derivative as:

$$\begin{aligned} f^{(7)}(y) &= k(-y^7g^7 + (\sum_{i=1}^{6} i)y^5g^6 - [(\sum_{i=1}^{4} i \sum_{j=1}^{i+1} j)]y^3g^5 + ([\sum_{m=1}^{2} m \sum_{i=1}^{m+1} i \sum_{j=1}^{i+1} j)g^4y \\ &= k(-y^7g^7 + h(1,6)y^5g^6 - h(2,4)y^3g^5 + h(3,2)g^4y) \end{aligned}$$

Continuing on with the pattern:

$$\begin{aligned} f^{(8)}(y) &= \frac{d}{dy}[k](-y^7g^7 + h(1,6)y^5g^6 - h(2,4)y^3g^5 + h(3,2)g^4y) \\ &+ k\frac{d}{dy}[(-y^7g^7 + h(1,6)y^5g^6 - h(2,4)y^3g^5 + h(3,2)g^4y)] \\ &= (-kgy)(-y^7g^7 + h(1,6)y^5g^6 - h(2,4)y^3g^5 + h(3,2)g^4y) \\ &+ k[(-7y^6g^7 + 5h(1,6)y^4g^6 - 3h(2,4)y^2g^5 + h(3,2)g^4)] \\ &= k(y^8g^8 - h(1,6)y^6g^7 + h(2,4)y^4g^6 - h(3,2)g^5y^2) \\ &+ k[(-7y^6g^7 + 5h(1,6)y^4g^6 - 3h(2,4)y^2g^5 + h(3,2)g^4)] \\ &= k(y^8g^8 - h(1,7)y^6g^7 + h(2,5)y^4g^6 - h(3,3)g^5y^2 + h(3,2)g^4) \\ f^{(9)}(y) &= \frac{d}{dy}[k(y^8g^8 - h(1,7)y^6g^7 + h(2,5)y^4g^6 - h(3,3)g^5y^2 + h(3,2)g^4)] \\ &= \frac{d}{dy}[k](y^8g^8 - h(1,7)y^6g^7 + h(2,5)y^4g^6 - h(3,3)g^5y^2 + h(3,2)g^4) \\ &+ k\frac{d}{dy}[(y^8g^8 - h(1,7)y^6g^7 + h(2,5)y^4g^6 - h(3,3)g^5y^2 + h(3,2)g^4)] \\ &= (-kgy)(y^8g^8 - h(1,7)y^6g^7 + h(2,5)y^4g^6 - h(3,3)g^5y^2 + h(3,2)g^4) \\ &+ k(8y^7g^8 - 6h(1,7)y^5g^7 + 4h(2,5)y^3g^6 - 2h(3,3)g^5y) \\ &= k(-y^9g^9 + h(1,7)y^7g^8 - h(2,5)y^5g^7 + h(3,3)g^6y^3 - h(3,2)g^5y + 8y^7g^8 - 6h(1,7)y^5g^7 \\ &+ 4h(2,5)y^3g^6 - 2h(3,3)g^5y) \\ &= k(-y^9g^9 + h(1,8)y^7g^8 - h(2,6)y^5g^7 + h(3,4)g^6y^3 - h(4,2)g^5y) \\ f^{(10)}(y) &= \frac{d}{dy}[k(-y^9g^9 + h(1,8)y^7g^8 - h(2,6)y^5g^7 + h(3,4)g^6y^3 - h(4,2)g^5y)] \\ &= k(y^{10}g^{10} - h(1,8)y^8g^9 + h(2,6)y^6g^8 - h(3,4)g^7y^4 + h(4,2)g^6y^2) \\ &+ k(-9y^8g^9 + 7h(1,8)y^6g^8 - 5h(2,6)y^4g^7 + 3h(3,4)g^6y^2 - h(4,2)g^5) \\ &= k(y^{10}g^{10} - h(1,9)y^8g^9 + h(2,7)y^6g^8 - h(3,5)g^7y^4 + h(4,3)g^6y^2 - h(4,2)g^5) \end{aligned}$$

At this point, we can see a clear pattern emerge. When we center the taylor series at $x = \mu$, it is centered at $y = 0$. This means that $a = 0$ and $k = be^{-a} = b$. Plugging this into the terms of the derivatives yields:

$$\begin{aligned}
f^{(0)}(0) &= k = b \\
f^{(1)}(0) &= 0 \\
f^{(2)}(0) &= -gk = -gb \\
f^{(3)}(0) &= 0 \\
f^{(4)}(0) &= kh(1,2)g^2 = bh(1,2)g^2 \\
f^{(5)}(0) &= 0 \\
f^{(6)}(0) &= -kh(2,2)g^3 = -bh(2,2)g^3 \\
f^{(7)}(0) &= 0 \\
f^{(8)}(0) &= kh(3,2)g^4 = bh(3,2)g^4 \\
f^{(9)}(0) &= 0 \\
f^{(10)}(0) &= -kh(4,2)g^5
\end{aligned}$$

since we have 0 terms on the odd terms in the Taylor Series, we see that our series will only apply for even numbers. Hence, our series will be:

$$\begin{aligned}
f(y) &= b - \frac{gb}{2!}y^2 + \frac{bh(1,2)g^2}{4!}y^4 - \frac{bh(2,2)g^3}{6!}y^6 + \frac{bh(3,2)g^4}{8!}y^8 - \frac{bh(4,2)g^5}{10!}y^{10} + \dots \\
&= b(1 - \frac{g}{2!}y^2 + \frac{h(1,2)g^2}{4!}y^4 - \frac{h(2,2)g^3}{6!}y^6 + \frac{h(3,2)g^4}{8!}y^8 - \frac{h(4,2)g^5}{10!}y^{10} + \dots) \\
&= b\sum_{i=0}^{\infty}(-1)^i \frac{h(i-1,2)g^i}{(2i)!}y^{2i}
\end{aligned}$$

7. **Using the Taylor series for the function, find**

$$\int_{\mu}^{2\mu} \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

NOTE: As it turns out, I made a very bad mistake in my initial solution. It turns out there is no closed form for this problem. If you attempted it, you will receive FULL credit. However, we can still find an expression for the integral in terms of the Taylor Series, and use the Taylor Series to roughly approximate the solution:
First, we convert the limits: $y_1 = x_1 - \mu = 2\mu - \mu = \mu$ and $y_0 = x_0 - \mu = \mu - \mu = 0$. So we have:

$$
\begin{aligned}
\int_0^{\mu} f(y)dy &= \int_0^{\mu} b\sum_{i=0}^{\infty}(-1)^i\frac{h(i-1,2)g^i}{(2i)!}y^{2i}dy \\
&= b\sum_{i=0}^{\infty}(-1)^i\frac{h(i-1,2)g^i}{(2i)!}\int_0^{\mu} y^{2i}dy \\
&= b\sum_{i=0}^{\infty}(-1)^i\frac{h(i-1,2)g^i}{(2i)!(2i+1)}(\mu^{2i+1}-(0)^{2i+1}) \\
&= b\sum_{i=0}^{\infty}(-1)^i\frac{h(i-1,2)g^i}{(2i+1)!}\mu^{2i+1} \\
&= b\mu - b\frac{g}{3!}\mu^3 + b\frac{h(1,2)g^2}{5!}\mu^5 - \ldots
\end{aligned}
$$

8. **In probability theory, the *exponential distribution* is often used to model the probability of having to wait a given amount of time for the next customer to arrive at a store. The probability density function for this is $f(x) = \lambda e^{-\lambda x}$, where λ represents the average number of customers that arrive per time unit. To use this, we often leverage the cumulative distribution function to find the probability that the amount of time we need to wait, X, for the next arrival is less than or equal to x. That is, we can use the PDF to find $P(X < x) = F(x) = \int_0^x f(t)dt$, where $f(t)$ is the probability density function. If $\lambda = 10$, then find $P(X < 5)$**

   We are looking for the integral of the function $10e^{-10x}$. from 0 to 5:

$$\int_0^5 10e^{-10x}dx$$

Let $u = -10x,$ so we have $-du = 10dx.$

Making these substitutions, we have:

$$
\begin{aligned}
\int_0^5 10e^{-10x}dx &= -\int_0^{-50} e^u du \\
&= -(e^{-50} - e^0) = 1 - \frac{1}{e^{50}}du \approx 1
\end{aligned}
$$

9. **Find**

$$\int_2^{16} xe^{-x}\,dx$$

   We can start by doing integration by parts:

Let $u = x$ and $dv = e^{-x}$. Then $du = dx$ and $v = -e^{-x}$. Using the equation, we have

$$\begin{aligned}
\int_2^{16} xe^{-x}dx &= uv - \int vdu \\
&= -xe^{-x} + \int e^{-x}du \\
&= -xe^{-x} - e^{-x} \\
&= -e^{-x}(x+1) \\
&= -e^{16}(16+1) + e^2(2+1) = 3e^2 - 17e^{16}
\end{aligned}$$

10. **Find**

$$\int_1^2 k^5 \ln k \, dk$$

Let $u = \ln k$ and $dv = k^5 dk$. Then $du = \frac{1}{k}dk$ and $v = \frac{1}{6}k^6$. Using our integration by parts formula, we have:

$$\begin{aligned}
\int xe^{-x}dx &= uv - \int vdu \\
&= \ln k \frac{1}{6}k^6 - \int \frac{1}{6}k^6 \frac{1}{k}dk \\
&= \frac{k^6 \ln k}{6} - \frac{1}{6}\frac{1}{6}k^6 \\
&= \frac{k^6}{6}(\ln k - \frac{1}{6})
\end{aligned}$$

Now we can find the definite integral:

$$\begin{aligned}
\int xe^{-x}dx &= \frac{(2)^6}{6}(\ln 2 - \frac{1}{6}) - \frac{1^6}{6}(\ln 1 - \frac{1}{6}) \\
&= \frac{2^5}{3}\ln 2 - \frac{2^6}{36} + \frac{1}{36} \\
&= \frac{2^5}{3}\ln 2 - \frac{63}{36} \\
&= \frac{32}{3}\ln 2 - \frac{7}{4}
\end{aligned}$$