Class 4 Notes: Sequences, Series, and Single Variable Integration

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1 Introduction

In this lecture, we will review through the fundamental concepts of Sequences, Series, and Integration. All of these concepts have many applications in business problems. Sequences have direct applications in stochastic modeling of customer arrival problems. Series, finite and infinite, have applications in vehicle routing problems, inventory problems, and maintenance policies. Integration, where does one even begin all of it's applications! One can leverage integration for determine consumer surplus as well as total inventory held over periods of time. This is not even including all of the physical applications in the fields of physics and engineering.

We will begin our discussion with sequences, discuss the definition of them, some examples, as well as some common convergence properties. We will then move to discuss series. Partial sums will be the first topic of discussion, and we will then move to the definition of infinite series, as well as some convergence properties of these. We will then review through some common types of series, and last hold a discussion on three common, and very practical, types of series called power, Taylor, and Maclaurin Series. We will then move to the basics of integration. First we will offer a precise definition of integration, and distinguish between indefinite and definite integration. We will then connect the concept of antiderivatives and integrals, as these two ideas are amazingly related to each other with the Fundamental Theorem of Calculus. We will finish our lecture by discussing some methods by which we can integrate, and wrap up our lecture by discussing improper integrals, or otherwise known as limits of integrals.

2 Sequences

2.1 Definition and Limits of Sequences

In some sense, we have already seen the fundamental and intuitive idea of sequences, both in this class, and in our every day lives. For example, draw a triangle, then draw a square, then a pentagon, and so on and so forth. Keep adding sides, and you will see that the "sequence" of these shapes converges, that is, gets "really close to" the shape of a circle.

We also had seen the concept of sequences appear in some business applications. For example, suppose we put \$100 under our bed, and left it there for a year. Then it would be worth 2% less than the following year, leading us to only have \$98. If we continue to deduct 2% every as a result of inflation, then we can see from Table 1 that the \$100 devalues to practically 0 over time. In other words, sequences comprise of a discrete collection of numbers, with the idea of a number that comes before another number, and a number that comes after another number. We also notice that sequences "converge". That is, as we continue to move further out into the sequence, we begin to notice that the numbers in the sequence look "the same". They approach, but sometimes never reach, a specific number, called a *limit*, which we have already discussed in the previous chapter. This is an important and useful concept that we will continue to leverage in our study of Calculus.

Let us now formalize the concept of a sequence. Sequences can be written down explicitly as follows: $a_1, a_2, a_3, \ldots, a_n, \ldots$. So we know that sequences have a concept of enumeration, which means that they are countable. Furthermore, we know that the specific number in any sequence, as we saw in the prior example, can commonly be any real number $\mathbb{R}$. In addition, there is the concept of ordering of these numbers. This would imply that a sequence is nothing more than a member of an infinite Cartesian product of the real numbers. On the other hand, since we can list each number in a specific order, they are countable, which means there must exist a function between the natural numbers and the real numbers. Hence, we can define sequences in multiple ways:

Number of Years Passed	Present Value of \$100
10.00	81.71
20.00	66.76
30.00	54.55
40.00	44.57
50.00	36.42
60.00	29.76
70.00	24.31
80.00	19.86
90.00	16.23
100.00	13.26
110.00	10.84
120.00	8.85
130.00	7.23
140.00	5.91
150.00	4.83
160.00	3.95
170.00	3.22
180.00	2.63
190.00	2.15
200.00	1.76

Table 1: The present value of \$100 over the next 200 years

Definition 1 *A sequence, $\{a_n\} = \{a_1, a_2, a_3, \dots\}$ can be defined as a function $f : \mathbb{N} \to \mathbb{R}$. Likewise, a sequence can be defined as an element of the set $R^\infty = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \{(a_1, a_2, \dots)|a_i \in \mathbb{R}\}$.*

Example 1 *The sequence $\{1, 3, 6, 10, 15, 21, 28, \dots\}$ can be represented in short hand [1] by using the notation $a_n = \frac{n(n+1)}{2}$.*

Example 2 *Another sequence is the well-known Fibonacci Sequence [2], which is defined as $\{1, 1, 2, 3, 5, 8, 13, \dots\}$. Interestingly, this sequence has many applications in nature, including predicting stock market returns for assets.*

Example 3 *The sequence $\{2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, \dots\}$ can be represented by the equation[3] $a_n = \frac{n^2+n+2}{2}$.*

Example 4 *The sequence $\{2, 3, 5, 7, 11, 13, 17, \dots\}$ is the sequence of prime numbers. As of the time of this writing, there is no known formula that will give the nth prime number!*

[1] The nth term in this sequence actually represents the sum of integers from 1 to n. It was first discovered by Gauss at the age of 9, so the story goes. His teacher asked the class to add the numbers from 1 to 100. Gauss responded with the correct answer in a very short time. The teacher thought he was cheating, and asked him to explain. He mentioned that if you rearrange the sequence of numbers $(1, 2, 3, 4, \dots, 100)$ to the sequence $(1, 100, 2, 99, 3, 98, \dots, 50, 51)$, and added every two terms together, one would obtain the sequence $(1 + 100, 2 + 99, \dots, 50 + 51)$. Since there are 50 terms with the sum of 101, the answer is easy: $101(50) = 5050$.

[2] This sequence was discovered by Fibonacci (Leonardo of Pisa), who was trying to model the breeding behavior of rabbits.

[3] This is called the "Lazy Caterers Sequence", and it represents the maximum number of slices that can be made from a pizza with n cuts.

We can clearly see that these sequences were described in difference ways. The first sequence was defined through a formula, while the second sequence was defined by performing an action on the previous two numbers in the sequence. The third sequence is another sequence that was defined by a formula, while the last sequence has no formula or rule that combines the prior values, but rather rests on a different type of rule. This motivates us to distinguish between two common types of sequences:

Definition 2 *A sequence is said to be defined in closed form if there exists an equation for the function $f(n)$ that will give the nth number in the sequence.*

Definition 3 *A sequence is said to be defined in recursive form if there exists an equation that combines the previous numbers in the sequence before the nth term to provide the value for the nth term.*

Example 5 *We can see that the Fibonacci Sequence can be defined recursively by the following formula: $F_n = F_{n-1} + F_{n-2}$, where $F_1 = F_2 = 1$.*

Some sequences exhibit the behavior of *converging*. We discussed the general idea of this with functions in previous lectures. However, we do have precise definitions of convergence for sequences. First, let us define this idea informally:

Definition 4 *A sequence $\{a_n\}$ is said to converge to the limit L if we can the number a_n as we bring the value of n very close to ∞. In such an instance, we write:*

$$\lim_{n\to\infty} a_n = L$$

On the other hand, if a_n does not exhibit this behavior as we bring n very large, then we say that the sequence is divergent, or, that it diverges.

Just like in the case of functions, the limit of sequences can also be defined in a more mathematically formal manner by using an approach that is analogous to the $\delta - \epsilon$ approach:

Definition 5 *A sequence $\{a_n\}$ is said to converge to the limit L, where $L \in \mathbb{R}$, if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. whenever $n > N$, then it is true that $|a_n - L| < \epsilon$.*

Example 6 *We can show that our earlier example of the present value of \$100 will approach 0 as we take the number of years out to infinity. We need to show that $\lim_{n\to\infty} 100(0.98^n) = 0$. Just like we did with function, we can work backwards to find an N that satisfies this condition:*

$$\begin{aligned}
|a_n - L| = |100(0.98^n) - 0| &< \epsilon \\
100(0.98^n) &< \epsilon \\
0.98^n &< \frac{\epsilon}{100} \\
\log(0.98^n) &< \log\left(\frac{\epsilon}{100}\right) \\
n\log(0.98) &< \log\left(\frac{\epsilon}{100}\right) \\
n &> \frac{\log\left(\frac{\epsilon}{100}\right)}{\log(0.98)}
\end{aligned}$$

Notice that we had to switch the direction of the inequality since $\log(0.98) < 0$. *Notice that the right hand side is nothing more than a change of base formula, and we can simplify it as* $\frac{\log\left(\frac{\epsilon}{100}\right)}{\log(0.98)} = \log_{0.98}\left(\frac{\epsilon}{100}\right)$. *Therefore, we can set* $N = \log_{0.98}\left(\frac{\epsilon}{100}\right)$. *Now comes the proof. Recall that when* $k < 1$ *and* $n > N$ *that* $k^n < k^N$. *This can easily be seen in the case where* $k = \frac{1}{2}$ *and* $n = 2, N = 1$. *Notice that* $k^n = (\frac{1}{2}^2) = \frac{1}{4}$ *and* $k^N = (\frac{1}{2})^1 = \frac{1}{2}$, *and clearly,* $\frac{1}{4} < \frac{1}{2}$. *Also recall from Algebra that* $a^{\log_a x} = x$. *Using this fact, if* $n > N$, *then we have:*

$$\begin{aligned}
|a_n - L| &= |100(0.98^n) - 0| \\
&= 100(0.98^n) \\
&< 100(0.98^N) \\
&= 100(0.98^{\log_{0.98}\left(\frac{\epsilon}{100}\right)}) \\
&= 100(\frac{\epsilon}{100}) \\
&= \epsilon
\end{aligned}$$

Limits can also go towards "infinity". That is, if the number in the sequence continues to get bigger, and does not converge to any real number, then we can write $\lim_{n\to\infty} a_n = \infty$. We can formally define this case as well:

Definition 6 *The sequence* $\{a_n\}$ *is said to tend to infinity if* $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ *s.t. whenever* $n > N$, $a_n > M$.

Sequences also follow the same limit laws that functions follow, since, well, sequences also happen to be functions themselves, only with a different domain. With that said, finding limits using the limit laws for sequences is the same as when we find them for functions.

Example 7 *Find* $\lim_{n\to\infty} \frac{3+2n}{n+1}$. *If we divide the top of bottom of the equation by n, we obtain:* $\frac{\frac{3}{n}+\frac{2n}{n}}{\frac{n}{n}+\frac{1}{n}}$. *Hence, we have:*

$$\begin{aligned}
\lim_{n\to\infty} \frac{3+2n}{n+1} &= \lim_{n\to\infty} \frac{\frac{3}{n}+\frac{2n}{n}}{\frac{n}{n}+\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\lim_{n\to\infty}\frac{3}{n}+\lim_{n\to\infty}\frac{2n}{n}}{\lim_{n\to\infty}\frac{n}{n}+\lim_{n\to\infty}\frac{1}{n}} \\
&= \frac{0+2}{1+0} \\
&= 2
\end{aligned}$$

2.2 Convergence Properties of Sequences

Certain types of sequences have interesting properties regarding their convergence, which is referred to here as their convergence properties. In this section, we will explore some of these properties. We first have:

Theorem 1 *If* $\lim_{n\to\infty} |a_n| = 0$ *then* $\lim_{n\to\infty} a_n = 0$

Another common sequence we come across is the sequence $\{a^n\}$. This is quite common in many applications. As such, we state the conditions under which this sequence is convergent:

Theorem 2 *The sequence $\{a^n\}$ converges to 0 if $a \in (-1, 1)$. It converges to 1 if $a = 1$. If $a \notin (-1, 1]$, then it diverges.*

Some properties of sequences are of immense importance to understanding whether or not if they converge. Such as:

Definition 7 *A sequence $\{a_n\}$ is said to be an increasing sequence if $a_n < a_{n+1} \forall n \geq 1$. It is said to be a decreasing sequence if $a_n > a_{n+1} n \geq 1$.*

Definition 8 *If the sequence $\{a_n\}$ is either increasing or decreasing, it is said to be monotonic.*

Definition 9 *A sequence is said to be bounded above if $\exists M \in \mathbb{R}$ such that $a_n \leq M \ \forall n \geq 1$.*
Likewise, a sequence is said to be bounded below if $\exists m \in \mathbb{R}$ such that $a_n \geq m \ \forall n \geq 1$.

These definitions of properties of sequences, if they have them, can be combined into a very useful theorem that allows us to speculate as to whether or not a sequence converges to a number or not:

Theorem 3 *Let $\{a_n\}$ be a sequence. If $\{a_n\}$ is bounded above and bounded below, and the sequence is monotonic, then the sequence will converge to a number $L \in \mathbb{R}$*

We can think of sequences from a different perspective as well. Rather than characterizing sequences as "convergent to a number", we can think about the distance between the subsequent numbers in the sequence:

Definition 10 *A Cauchy Sequence is a sequence $\{a_n\}$ such that the distance $|a_{n+1} - a_n|$ approaches 0 as $n \to \infty$. In other words, it is sequence where the numbers become very close to each other as we progress further in the sequence.*

Theorem 4 *Every sequence that is convergent is a Cauchy Sequence. Likewise, all Cauchy Sequences are also Convergent Sequences.*

3 Series

3.1 Definition and Limits of Series

We saw in the previous section that sequences can be defined by closed or recursive formulas, and at times, they may converge to a specific number or to $\pm\infty$. Related to sequences is the concept of a series. Let us take a practical example to motivate the idea. Suppose that a company would like to expand it's capacity by building a new facility every 5 years. The firm will pay a fixed cost to build and operate the facility every time it builds one. Suppose the demand is linear over time. That is, the demand is equal to tD at year t, where $D = 1000$ is the increase in demand per year. Suppose that $f(tD) = tD$ is the cost to build a facility with capacity tD. Then if the firm builds a new facility, the demand would have increased a total of $5D$ over the 5 years, which means the new capacity of the building would need to be $5D$ to satisfy this additional demand of $5D$. We assume that the cost to build a facility does not change over time.

If the firm continues to do this out to infinity time, what is the total cost to the firm for building every 5 years with a capacity of $5D$? Intuitively we may initially think that it is ∞. Shockingly, however, it is not. If the value of money is devalued by 2% every year, then the value of a single dollar would be 0.98^t in t years. This means that the first year the firm builds the facility, it will cost them:

$$f(5D)(\text{Year } 0)$$

However, in 5 years, when it decides to build a new facility, it will cost them only $0.98^5 f(5D)$ (since we assumed the cost to build the facility over time does not change). Therefore, the total cost at year 5 would be:

$$\begin{aligned}
s_1 &= f(5D) = 5(1000) = 5000 && \text{(Year 0)} \\
s_2 &= f(5D) + (0.98)^5 f(5D) = 5(1000) + (0.98)^5(5)(1000) = 9519.60 && \text{(Year 5)}
\end{aligned}$$

On year 10, the firm will build year another facility, and notice that at this point in time, one dollar today is worth only $(0.98)^{10}$ on year 10. So the total cost by year 10 would be:

$$\begin{aligned}
s_1 &= f(5D) = 5(1000) = 5000 && \text{(Year 0)} \\
s_2 &= f(5D) + (0.98)^5 f(5D) = 5(1000) + (0.98)^5(5)(1000) = 9519.60 && \text{(Year 5)} \\
s_3 &= f(5D) + (0.98)^5 f(5D) + (0.98)^{10} f(5D) = 5(1000) + (0.98)^5(5)(1000) + (0.98)^{10}(5)(1000) = 13694.97 && \text{(Year 10)}
\end{aligned}$$

Notice that as every 5 years goes by, we are adding less and less to the total sum. We can start to think about what the implication of that means. If in 5 years, we are adding less cost to the total cost than 5 years before, than perhaps the total cost over infinite time will converge to a real number rather than to infinity, as a result of devaluation of the dollar. We notice that if we continue this pattern, we have:

$$\begin{aligned}
s_1 &= f(5D) = 5(1000) = 5000 && \text{(Year 0)} \\
s_2 &= f(5D) + (0.98)^5 f(5D) = 5(1000) + (0.98)^5(5)(1000) = 9519.60 && \text{(Year 5)} \\
s_3 &= f(5D) + (0.98)^5 f(5D) + (0.98)^{10} f(5D) = 5(1000) + (0.98)^5(5)(1000) + (0.98)^{10}(5)(1000) = 13694.97 && \text{(Year 10)} \\
s_4 &= f(5D) + (0.98)^5 f(5D) + (0.98)^{10} f(5D) = 5(1000) + (0.98)^5(5)(1000) + (0.98)^{10}(5)(1000) + (0.98)^{15}(5)(1000) = 17297.81 && \text{(Year 15)} \\
s_5 &= f(5D) + (0.98)^5 f(5D) + (0.98)^{10} f(5D) = 5(1000) + (0.98)^5(5)(1000) + (0.98)^{10}(5)(1000) + (0.98)^{15}(5)(1000) + (0.98)^{20}(5)(1000) = 20635.85 && \text{(Year 20)}
\end{aligned}$$

Notice how when we add the following 5 year's cost onto the prior summation, we are in essence forming a new number in a sequence. That is, in our case, the sequence $\{s_i\}$ is being formed by adding the next term to the prior total summation. If we were to continue to do this out to infinity, then we are in essence adding an "infinite" amount of terms. We hence have the following:

Definition 11 *Let $\{a_n\}$ be a sequence. Then the summation $s_n = \sum_{i=1}^{n} a_n$ is called a partial sum.*

Definition 12 *Let $\{a_n\}$ be a sequence. Then the limit of the sequence s_n, that is, the limit of the partial summations of $\{a_n\}$, is called an infinite series, and we write:*

$$\sum_{i=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_n = \lim_{n\to\infty} s_n$$

Returning back to our example, we can see a general pattern in the partial summations begin to emerge:

$$\begin{aligned} s_n &= \sum_{i=0}^{n} (0.98)^{5i}(5000) \\ &= (5000)\sum_{i=0}^{n} (0.98)^{5i} \\ &= (5000)\sum_{i=0}^{n} (0.98^5)^i \end{aligned}$$

Let $r = 0.98^5$. Then if we can find an expression for $\sum_{i=0}^{n} r^i$, then we can apply out theory of limits to determine what the value of the infinite series is. We can do this by first setting $S_n = r^0 + r^1 + \cdots + r^n$. If we multiply both sides of this equation by r, the exponents on r all increment by 1, and we hence obtain: $rS_n = r^1 + r^2 + \cdots + r^{n+1}$. Now, if we subtract these two equations, we obtain:

$$\begin{aligned} S_n - rS_n &= (r^0 + r^1 + \cdots + r^n) - (r^1 + r^2 + \cdots + r^{n+1}) = r^0 - r^{n+1} \\ S_n(1-r) &= 1 - r^{n+1} \\ S_n &= \frac{1 - r^{n+1}}{(1-r)} \end{aligned}$$

So, we can now finish our derivation of the cost by replacing the summation with our closed-form expression for the partial summation of that series:

$$\begin{aligned} s_n &= (5000)\sum_{i=0}^{n} r^i \\ &= (5000)\frac{1 - r^{n+1}}{(1-r)} \end{aligned}$$

Now, we know that if $r < 1$ (which in our case it is), that $\lim_{n\to\infty} r^{n+1} = 0$. So, we finally have:

$$\begin{aligned}
\sum_{i=0}^{\infty}(5000)r^i &= \lim_{n\to\infty}(5000)\sum_{i=0}^{n} r^i \\
&= \lim_{n\to\infty}(5000)\frac{1-r^{n+1}}{(1-r)} \\
&= (5000)\frac{1}{(1-r)} \\
&= (5000)\frac{1}{(1-(0.98^5))} \\
&= 52040.40
\end{aligned}$$

Note that our assumption behind the reason why the sequence s_n converged to 52040.40 was due to (among other reasons) that fact that $r < 1$. If $r \geq 1$, then the limit of the series would have *diverged* to infinity. This leads us to a fundamental definition for our series:

Definition 13 *Suppose that* $\sum\limits_{i=0}^{\infty} a_n = \lim\limits_{n\to\infty} s_n$ *is an infinite series. Then we say that the series converges to the number s if* $s = \lim\limits_{n\to\infty} s_n$. *Otherwise, we say the series diverges.*

Since a series is itself a sequence of partial sums of a different sequence, we also can express some series in closed form or recursive form. We saw this for the *geometric series*, which was the series $a + ar + ar^2 + \dots$, and we found that the expression for the nth term in the sequence was $\frac{1-r^{n+1}}{1-r}$. We also could have expressed the series in recursive form as $s_n = rs_{n-1}$, and set $s_0 = 1$. We will study an approach to finding closed form of an entire series when given the recursive form when we discuss our Linear Algebra lecture.

3.2 Convergence of Series

In the previous section, we defined what it means for a series to converge. In this section, we will explore the typical approach to determine if a series is convergent or divergent, and if it is convergent, how we can find the number to which it converges. We begin with a very simple theorem:

Theorem 5 *Consider the series* $\sum\limits_{i=0}^{\infty} a_n$. *If* $\lim\limits_{n\to\infty} a_n \neq 0$, *then the series* $\sum\limits_{i=0}^{\infty} a_n$ *is divergent.*

This should make sense intuitively. Think about what would happen if the limit of the sequence $\{a_n\}$ either diverged, went to infinity, or converged to a number. The earlier two cases is obvious: we would always have a bigger number, or another number that is not decreasing, that we are adding onto the end of the series in the next partial sum, which means the total sum would just diverge to $\pm\infty$. If the sequence converged to a number k other than 0, then after a certain integer N (that is, if $n > N$), then we would have $\sum\limits_{i=n}^{\infty} a_n \approx \sum\limits_{i=n}^{\infty} k$. This would mean the end of the sum would be represented by nk, and if $n \to \infty$, then the end of the sum would also go to ∞.

Be careful with this theorem, however. Many students introduced to calculus quite often mix this statement up. The converge of this statement is not necessarily true. That is, if the sequence

$\{a_n\}$ converges to a number other than 0, then we do know that the series is divergent. But the theorem does **not** say that if $\{a_n\} \to 0$, then the series is convergent. In fact, the *harmonic series*, defined as $\sum_{i=1}^{\infty} \frac{1}{i}$ does not converge, despite the fact that $\frac{1}{i} \to 0$. This is very important to keep in mind. That is, this is a basic test of convergence where the only information we can ascertain is about its **divergence**, not its **convergence**, and that is a very important consideration to keep in mind!

Example 8 *Determine if the series* $\sum\limits_{i=0}^{\infty} \frac{1}{5+\frac{1}{i}}$ *is divergent. We can see that*

$$\begin{aligned} \lim_{i\to\infty} \frac{1}{5+\frac{1}{i}} &= \frac{\lim_{i\to\infty} 1}{\lim_{i\to\infty} 5 + \lim_{i\to\infty} \frac{1}{i}} \\ &= \frac{1}{5+0} = \frac{1}{5} \end{aligned}$$

Since $\lim\limits_{i\to\infty} a_n \neq 0$*, we can conclude that this series will diverge.*

Example 9 *Determine if the series* $\sum\limits_{i=0}^{\infty} \frac{1}{i!}$ *is divergent or convergent. Taking the limit as* $i \to \infty$*, we can clearly see that this is a sub-sequence of* $\frac{1}{n}$*. Since* $\frac{1}{n} \to 0$*, we must have* $\frac{1}{i!} \to 0$*. However, since* $a_n \to 0$*, we cannot conclude from this test alone if the series is divergent or convergent, and we would need a different test to make this final determination.*

In addition to the convergence test listed above, we also can "break apart" various series into other series, contingent on if the other series themselves are convergent:

Theorem 6 *Suppose that the series* $\sum\limits_{i=0}^{\infty} a_n$ *and* $\sum\limits_{i=0}^{\infty} b_n$ *are both convergent and let* $k \in \mathbb{R}$*. Then the following is true:*

$$\begin{aligned} \sum_{i=0}^{\infty} [a_n + b_n] &= \sum_{i=0}^{\infty} a_n + \sum_{i=0}^{\infty} b_n \\ \sum_{i=0}^{\infty} [a_n - b_n] &= \sum_{i=0}^{\infty} a_n - \sum_{i=0}^{\infty} b_n \\ \sum_{i=0}^{\infty} k a_n &= k \sum_{i=0}^{\infty} a_n \end{aligned}$$

Example 10 *Find* $\sum\limits_{i=0}^{\infty} (\frac{1}{3^n} + \frac{1}{2^n})$*. Applying the series rules and the formula from our prior observations* *that* $\sum\limits_{i=0}^{\infty} r^i = \frac{1}{1-r}$ *for* $|r| < 1$*, we see that:*

$$\begin{aligned}\sum_{i=0}^{\infty}(\frac{1}{3^n} + \frac{1}{2^n}) &= \sum_{i=0}^{\infty}\frac{1}{3^n} + \sum_{i=0}^{\infty}\frac{1}{2^n} \\ &= \sum_{i=0}^{\infty}(\frac{1}{3})^n + \sum_{i=0}^{\infty}(\frac{1}{2})^n \\ &= \frac{1}{1-\frac{1}{3}} + \frac{1}{1-\frac{1}{2}} \\ &= \frac{3}{2} + \frac{4}{2} \\ &= \frac{7}{2}\end{aligned}$$

3.3 Ratio Test for Convergence

In some cases, the convergence test that we discussed will not provide us enough information as to whether or not a series will converge. If we find that $a_n \to 0$, then we cannot conclude anything definitive about the convergence of a series, and hence, we typically need to find a different test (see the Stewart textbook for a larger body of tests to run to determine convergence, as we only discuss two such tests). One of these different tests is the ratio test:

Definition 14 *Let* a_n *be a sequences and let* $\sum\limits_{i=0}^{\infty}$ *be an infinite series. Then we can conclude the following about the series if:*

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} < 1 & \textit{Convergent} \\ > 1 & \textit{Divergent} \\ = 1 & \textit{Inconclusive}\end{cases}$$

Example 11 *We found in the previous section that* $\sum\limits_{i=0}^{\infty}\frac{1}{i!}$ *has a sequence* $a_n \to 0$*, and hence, we were unable to conclude if the series itself converges. If we apply the ratio test, we can see:*

$$\begin{aligned}\lim_{i\to\infty}\left|\frac{\frac{1}{(i+1)!}}{\frac{1}{i!}}\right| &= \lim_{i\to\infty}\left|\frac{i!}{(i+1)!}\right| \\ &= \lim_{i\to\infty}\left|\frac{i!}{(i+1)(i!)}\right| \\ &= \lim_{i\to\infty}\left|\frac{1}{i+1}\right| = 0\end{aligned}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$*, we can conclude that the series converges.*

Example 12 *We mentioned that the series* $\sum\limits_{i=0}^{\infty}\frac{1}{n}$ *was divergent but did not show why. Our first test was* *inconclusive, since $a_n \to 0$. If we apply the ratio test, we can see:*

$$\lim_{i\to\infty}\left|\frac{\frac{1}{(i+1)}}{\frac{1}{i}}\right| = \lim_{i\to\infty}\left|\frac{i}{(i+1)}\right|$$
$$= \lim_{i\to\infty}\left|\frac{\frac{i}{i}}{(\frac{i+1}{i})}\right|$$
$$= \lim_{i\to\infty}\left|\frac{1}{1+\frac{1}{i}}\right| = 1$$

Therefore, we would also conclude that this test inconclusive.

Example 13 *A neat feature about the ratio test is that it provides us a mechanism to understand where a series is convergent if the series is expressed in terms of another variable. For example, the geometric series, $\sum\limits_{i=0}^{\infty} r^i$ is convergent when the limit of the absolute value of the ratio is less than 1, and from this expression, we can determine the interval of convergence:*

$$\lim_{i\to\infty}\left|\frac{r^{n+1}}{r^n}\right| = \lim_{i\to\infty}\left|\frac{r\cdot r^n}{r^n}\right| = |r|$$

Using the ratio test conditions, we know then that the series is convergent if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$. Therefore, since $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |r|$, the series is convergent if $|r| < 1$. By the same reasoning, if $|r| > 1$, then by the ratio test, we know the series will be divergent.

3.4 Power, Taylor, and Maclaurin Series

There is a special class of series that typically that has astonishing properties which in some situations makes our Calculus a lot easier to handle. This type of series is known as a *Power Series*, which we define as:

Definition 15 *The series $\sum\limits_{i=0}^{\infty} c_i(x-a)^i$ is called a Power Series. This series is a function of x, and hence can be expressed as: $f(x) = \sum\limits_{i=0}^{\infty} c_i(x-a)^i$. There are three situations as to when this series is convergent:*

1. *The series is convergent only when $x = a$.*
2. *The series is convergent $\forall x \in \mathbb{R}$*
3. *The series is convergent $\forall x \in (a-R, a+R)$, where R is called the radius of convergence.*

A very interesting propery of these Power Series is that we can express many different functions $f(x)$ as a Power Series, a special type of series that is called a Taylor Series:

Definition 16 *Let $f(x)$ be a continuous and differential function. Then the function can be expressed as a Power Series, centered at $a \in \mathbb{R}$:*

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

where $f^{(i)}(a)$ is the ith derivative of $f(x)$ evaluated at a, $f^{(0)}(x) = f(x)$ (the 0 derivative is assumed to just equal the function itself), and a is any real number chosen by the individual who wants to find the Taylor Series representation of $f(x)$. When $a = 0$, the Taylor Series is commonly referred to as a MacLaurin Series.

Example 14 *We can express the function $f(x) = e^x$ as a power series by leveraging the Taylor Series. The first step is to pick a value for a. We try to do this in such a way where the derivatives all exist at a, but are easy to evaluate and offer a general closed-form expression for the sequence $\{f^{(n)}(a)\}$. We know that $e^0 = 1$, so perhaps this is the best starting point. The next step is to find a closed form equation for the ith term for the sequence $\{f^{(n)}(a)\}$. We can do that by listing the first few in the sequence:*

$$\begin{aligned}
f^{(0)}(x) &= e^x \\
f^{(1)}(x) &= e^x \\
f^{(2)}(x) &= e^x \\
\vdots &= \vdots \\
f^{(n)}(x) &= e^x
\end{aligned}$$

Therefore, our general sequence at $x = a = 0$ would be $1, 1, 1, \ldots$ since $e^0 = 1$. Given this, we can now plug all of our information into the formula to obtain the power series for the function:

$$\begin{aligned}
e^x &= \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i \\
&= \sum_{i=0}^{\infty} \frac{1}{i!}(x-0)^i \\
&= \sum_{i=0}^{\infty} \frac{1}{i!}x^i
\end{aligned}$$

We can now find the values for x for which the series converges by using the ratio test:

$$\begin{aligned}
\lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| = \left|\frac{\frac{1}{(i+1)!}x^{(i+1)}}{\frac{1}{i!}x^i}\right| \\
= \lim_{i\to\infty} \left|\frac{i!}{(i+1)!}x\right| \\
= \lim_{i\to\infty} \left|\frac{1}{(i+1)}x\right| = 0
\end{aligned}$$

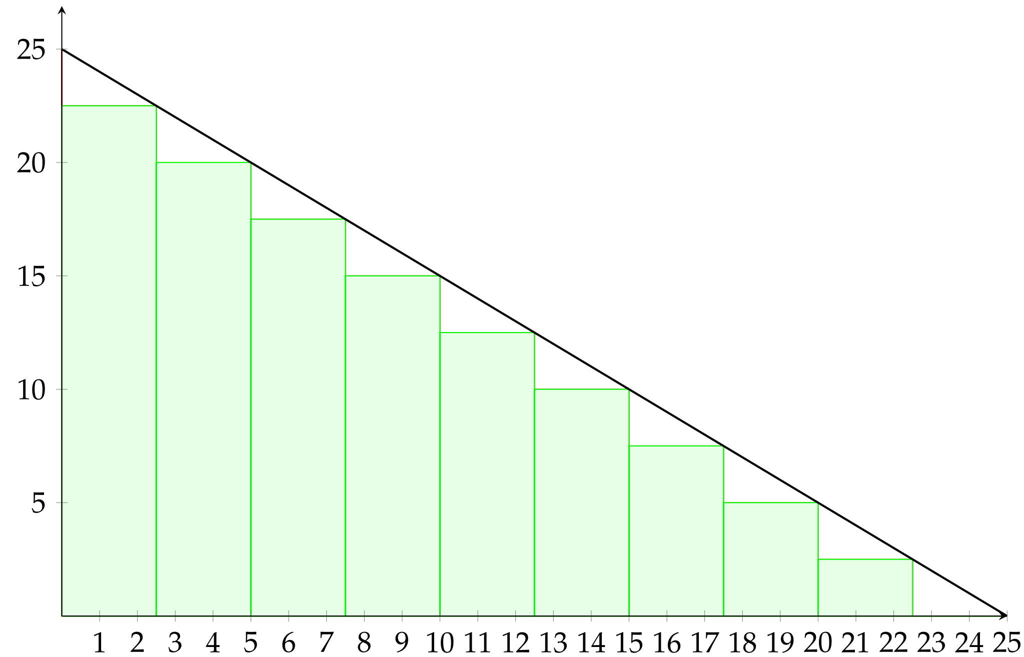

Figure 1: The inventory level over 25 days if the demand per day is $\lambda = 1$ unit and our ordering quantity is Q = 25

.

We can see that for any value of x, this limit is always equal to 0, and hence, by the ratio test, is convergent. Therefore, the series is convergent $\forall x \in \mathbb{R}$, and it's interval of convergence is $(-\infty, \infty)$. We can see that this is a very useful representation, since we can use this to approximate the value of the number e: $e = e^{(1)} = \sum_{i=0}^{\infty} \frac{1}{i!}(1)^i = \sum_{i=0}^{\infty} \frac{1}{i!}$. Therefore, e can be approximated by taking a finite number of partial sums of this series.

4 Fundamentals of Integration

4.1 Definition of Integration

We will derive the intuitive and mechanical definitions of the integral by exploring an application to inventory theory known as the Economic Ordering Quantity (EOQ). Suppose that an inventory manager at a retail store would like to determine the optimal ordering quantity Q to order every T weeks. We assume that the demand rate, which is the number of units that will leave the inventory per day, is λ. Furthermore, we assume that units leave the inventory continuously. In practice, inventory leaves only discretely. However, when we model environments like those of inventory, the mathematics is much easier if we assume, as strange as it may sound, that our inventory reduces by continuous quantities. This means that at any time t, the inventory level could be different than at any other time.

If we further suppose that the manager does not place an order until the inventory reaches 0, then the inventory level at time t looks like the function in Figure 1. If we assume *instantaneous replenishment*, then when the inventory manager places an order, the inventory level is immediately replenished to the level Q. Therefore, to study the cost of the decision Q, the manager only needs to study a single *inventory cycle*, which is the time interval from when the level is equal to Q ($t = 0$) to when the inventory level is equal to 0 ($t = T$).

There are three costs that the inventory manager must face throughout the time interval $[0, T]$: fixed ordering cost, variable ordering cost, and inventory holding cost. If we assume that K is the fixed ordering cost (the fixed number of dollars incurred anytime an order is placed, and not dependent on the order quantity) and if c is the per unit variable ordering cost (the cost per unit when ordered), then the total ordering cost would be $K + cQ$. The problem lies in computing the total inventory holding cost. If our inventory quantities were discrete for fixed intervals of time, then this is easy to compute. For example, if I told you that it costs $2 per unit per day to hold inventory, and for 2 days you held 3 units (and this number did not change in the 2 days), then your total cost would be $\frac{\text{Cost}}{\text{Day}\cdot\text{Unit}}(Days)(Units) = (2)(2)(3) = 12$. However, the $(Units)$ in this equation for our situation is difficult to plug into this equation, since it changes with every change in time. As such, we need to approximate the total cost.

If we split the time interval into n intervals of time, and we make the simplifying assumption that we hold the same level of inventory for that entire interval of time, then we can easily compute the total cost for that specific interval. In Figure 1, we notice that the interval $[0, 25]$ (if $T = 25$ and $Q = 25$) is split into 10 intervals of equal length:
$[0, 2.5], [2.5, 5.0], [5.0, 7.5], [7.5, 10], [10, 12.5], [12.5, 15.0], [15.0, 17.5], [17.5, 20]$
Now, despite the fact that the inventory level in each of these intervals change over the interval, we need to pick an inventory level for that specific time interval to use as an approximation to the inventory level for that interval. For example, in the first interval, we notice that $IL(0) = 25$ while $IL(2.5) = 22.5$. We need to fix a level between these two points. This can be done by picking any point t^* in the interval $[0, 2.5]$ and subsequently finding $f(t^*)$. We notice then that the total cost over that interval would be, using our equation before if h is the cost per day per unit, $hf(t^*)2.5$, since we assume that we hold the level $f(t^*)$ of inventory for the quantity of time of 2.5 days, despite it actually changing from 25 to 22.5 over that time (but this is why we call this approximation).

Given this for the first interval, we can do this for all the intervals to get an approximated total cost $\sum_{i=1}^{10} hf(t_i^*)2.5$. In this example, we would have $f(t) = Q - \lambda t$. Hence, our approximated summation would be $\sum_{i=1}^{10} h(Q - \lambda t_i^*)2.5$. Notice that the length of time 2.5 was found very easily by dividing the interval length of $[0, T]$ by the number of rectangles used. If we generalize this, we can denote the length of every interval as $\Delta t = \frac{T-0}{n} = \frac{T}{n}$. To pick a value for t_i, this can be any number in any of the intervals. We will choose the right end point of the intervals. These would be $\Delta t, 2\Delta t, 3\Delta t, \ldots, n\Delta t$. With all of this in mind, we can generalize our approximation cost with n rectangles to:

$$
\begin{aligned}
\text{Approx Total Cost} &= \sum_{i=1}^{n} hf(t_i^*)\Delta t \\
&= \sum_{i=1}^{n} h(Q - \lambda t_i^*)\frac{T}{n} \\
&= \sum_{i=1}^{n} h(Q - \lambda(i\Delta t))\frac{T}{n} \\
&= \sum_{i=1}^{n} h(Q - \lambda(i\frac{T}{n}))\frac{T}{n} \\
&= \sum_{i=1}^{n} hQ\frac{T}{n} - h\sum_{i=1}^{n} \lambda i\frac{T^2}{n^2} \\
&= nhQ\frac{T}{n} - h\lambda\frac{T^2}{n^2}\sum_{i=1}^{n} i \\
&= hQT - h\lambda\frac{T^2}{n^2}\frac{n(n+1)}{2} \\
&= hQT - h\lambda\frac{T^2}{n^2}\frac{n^2+n}{2} \\
&= hQT - h\frac{1}{2}\lambda T^2(1 + \frac{1}{n})
\end{aligned}
$$

Now if we would like that actual total cycle cost, we can take the number of intervals to infinity, and this would then give us the total inventory cost under the curve, since we would have no error:

$$
\begin{aligned}
\text{Actual Total Cost} &= \lim_{n\to\infty}\sum_{i=1}^{n} hf(t_i^*)\Delta t \\
&= \lim_{n\to\infty}[hQT - h\frac{1}{2}\lambda T^2(1 + \frac{1}{n})] \\
&= \lim_{n\to\infty} hQT - h\lim_{n\to\infty}\frac{1}{2}\lambda T^2(1 + \frac{1}{n}) \\
&= hQT - h\frac{1}{2}\lambda T^2(\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n}) \\
&= hQT - h\frac{1}{2}\lambda T^2(1 + (0)) \\
&= hQT - h\frac{1}{2}\lambda T^2
\end{aligned}
$$

Therefore, the total cycle cost would be $K + cQ + hQT - \frac{1}{2}h\lambda T^2$. If we divide by the interval length T, then we obtain the average daily cost: $\frac{K}{T} + \frac{cQ}{T} + hQ - \frac{1}{2}h\lambda T$. Notice that when $t = T$, $0 = f(T) = Q - \lambda T$, and so $T = \frac{Q}{\lambda}$. Making this substitution, we have the average daily cost at $\frac{K\lambda}{Q} + c\lambda + hQ - \frac{1}{2}Qh = \frac{K\lambda}{Q} + c\lambda + \frac{1}{2}Qh$, which is the same equation we discussed in an earlier lecture when we found the optimal ordering quantity Q.

Notice in our derivation for the inventory holding cost that when we found the holding cost in the interval, we needed to multiply the length of the interval by the inventory level. Geometrically, we notice that this is the area of each rectangle in Figure 1. This area represents the approximate total inventory held over that time interval. We can generalize our approach for finding "total quantities" over other quantities by find the area of functions $f(x)$. The areas can be found by this approximation method of splitting the interval into equal sized intervals, picking a value of x in each interval, using this as the height of the rectangle, finding the area, adding them all together, and taking the limit of the approximate area as the number of rectangles goes to infinity. This limit, which represents the exact area under the curve in the interval, is referred to as an *integral*:

Definition 17 *Let $f(x)$ be a function. Then the area under $f(x)$ in the interval $[a, b]$ is called the integral of the function between $[a, b]$, and is defined mathematically by:*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Where $\Delta x = \frac{b-a}{n}$, and $x_i^* \in [a + (i-1)\Delta x, a + i\Delta x]$

More precisely, we have:

Definition 18 *Let $\int\limits_a^b f(x)dx$ represent the area under the function $f(x)$ between a and b. Then $\forall \epsilon > 0, \exists N > 1$ such that whenever $n > N$, then*

$$\left| \int\limits_a^b f(x)dx - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \epsilon$$

4.2 Rules of Integration

There are some fundamental rules of integration that help us simplify functions down more easily, which allow us an easier path the solving for integrals. We states these without proof:

Theorem 7 *Let $c \in \mathbb{R}$, and let $f(x)$ and $g(x)$ be functions. Then the following is true:*

$$\int_a^b c dx = c(b-a) \tag{1}$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \tag{2}$$

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx \tag{3}$$

$$\int_a^b f(x)dx = \int_a^k f(x)dx + \int_k^b f(x)dx \text{ if } k \in [a,b] \tag{4}$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \tag{5}$$

$$\text{If } f(x) > 0 \forall x \in [a,b], \text{ then } \int_a^b f(x)dx > 0 \tag{6}$$

$$\text{If } f(x) \geq g(x) \forall x \in [a,b], \text{ then } \int_a^b f(x)dx \geq \int_a^b g(x)dx \tag{7}$$

Example 15 *Find $\int\limits_1^2 x^2 + 3dx$. Using our integration rules, we have:*

$$
\begin{aligned}
\int_1^2 x^2 + 3dx &= \int_1^2 x^2 dx + \int_1^2 3dx \\
&= \int_1^2 x^2 dx + 3(2-1) \\
&= \int_1^2 x^2 dx + 3 \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} x_i^2 \Delta x + 3 \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} x_i^2 \frac{2-1}{n} + 3 \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} (1 + i\Delta x)^2 \frac{1}{n} + 3 \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} (1 + 2i\Delta x + i^2\Delta x^2) \frac{1}{n} + 3 \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{n} + \frac{2\Delta x}{n} \sum_{i=0}^{n} i + \sum_{i=0}^{n} i^2 \Delta x^2 \frac{1}{n} + 3 \\
&= \lim_{n\to\infty} n\frac{1}{n} + \frac{2}{n^2} \frac{n(n+1)}{2} + \sum_{i=0}^{n} i^2 \frac{1}{n^3} + 3 \\
&= \lim_{n\to\infty} 1 + (1 + \frac{1}{n}) + \sum_{i=0}^{n} i^2 \frac{1}{n^3} + 3 \\
&= \lim_{n\to\infty} 1 + \lim_{n\to\infty} (1 + \frac{1}{n}) + \lim_{n\to\infty} \sum_{i=0}^{n} i^2 \frac{1}{n^3} + 3 \\
&= 1 + 1 + \lim_{n\to\infty} \sum_{i=0}^{n} i^2 \frac{1}{n^3} + 3 \\
&= 5 + \lim_{n\to\infty} \sum_{i=0}^{n} i^2 \frac{1}{n^3}
\end{aligned}
$$

It can be shown that $\sum\limits_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$*, and so, using this, we have:*

$$
\begin{aligned}
\int\limits_1^2 x^2 + 3dx &= \int\limits_1^2 x^2 dx + \int\limits_1^2 3dx \\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=0}^{n} i^2 + 5 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} + 5 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \frac{(n^2+n)(2n+1)}{6} + 5 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \frac{(2n^3+n^2+2n^2+n)}{6} + 5 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \frac{(2n^3+3n^2+n)}{6} + 5 \\
&= \lim_{n\to\infty} \frac{(2+3\frac{1}{n}+\frac{1}{n^2})}{6} + 5 \\
&= \frac{(2+0+0)}{6} + 5 \\
&= \frac{1}{3} + 5 \\
&= \frac{16}{3}
\end{aligned}
$$

4.3 The Fundamental Theorem of Calculus

We had seen that the integral rules help in simplifying our work. However, it still is laborious to compute integrals by using the limiting rectangle approach. As it turns out, one of the most astonishing and beautiful theorems in mathematics can help us here. There is not much intuition here, however, it can be shown, amazingly, that integrals (areas under curves) and antiderivatives (a function $F(x)$ whose derivative equals a given function $f(x)$) are directly related to each other. This implies that computing integrals, in most cases, can be simplified down even more, since we know that there are nice and easy tools to be able to compute anti-derivatives for many different functions:

Theorem 8 *(The Fundamental Theorem of Calculus) Let f(x) be a function and let $F(x)$ be an antiderivative of $f(x)$. Assume this antiderivative exists (that is, $F(x)$ is a function such that $F'(x) = f(x)$. Then:*

1. *If $g(x) = \int_a^x f(t)dt$, then g'(x) = f(x). This would imply that $F(x) = g(x) = \int_a^x f(t)dt$*
2. $\int\limits_a^b f(x)dx = F(b) - F(a)$

In other words, if we seek to find the integral of $f(x)$ over an interval $[a, b]$, there is no need for us to take a limit approach to finding the integral, despite this definition of the integral being intuitive. If we can find a formula for an antiderivative of $f(x)$, then we can very easily compute the area, by taking the difference of the antiderivative evaluated at the limits of the integral. This astonishing theorem has led to many great advancements in mathematics, not to mention, a very quick and easy way to compute integrals!

Example 16 *Find* $\int_1^2 x^2 + 3dx$. *We know that this integral can be split into two integrals:* $\int_1^2 x^2 + \int_1^2 3dx$. *The second integral we already have a rule for, and that reduces down to 3. Given our new theorem, if we can find an antiderivative for the function* x^2*, then our computations are very easy. We know from an earlier lecture that the antiderivative of* $f(x) = x^n$ *is* $F(x) = \frac{1}{n+1}x^{n+1} + C$*, where* $C \in \mathbb{R}$*. Using the Fundamental Theorem of Calculus, we then have:*

$$\begin{aligned}
\int_1^2 x^2 + 3dx &= \int_1^2 x^2dx + \int_1^2 3dx \\
&= F(2) - F(1) + 3(2-1) \\
&= \frac{1}{2+1}(2)^{2+1} + C - \frac{1}{2+1}(1)^{2+1} - C + 3 \\
&= \frac{8}{3} - \frac{1}{3} + 3 \\
&= \frac{7}{3} + 3 \\
&= \frac{16}{3}
\end{aligned}$$

Essentially, the basic strategy for integrating a function first involves using the integration rules to break the integral into smaller pieces if the function is defined as a sum, difference, or scalar multiple of a function. For each integral, we then try to find an antiderivative for the function. If we can do this using the basic antiderivative rules, then we simply compute the difference in antiderivatives of the limit points on the integral. If we cannot find an antiderivative that matches the function we have, we need to pursue a more complicated approach, which we will discuss in later sections.

4.4 Indefinite vs. Definite Integration

One point of note that should be made is that when we defined the integral, we defined it as an area under a function $f(x)$ for a given interval $[a, b]$. In other words, integrals were defined as *numbers*. Given our new theorem, however, an argument can be made that we can define integrals as functions as well, analogous to how we defined derivatives of functions as functions themselves. This is quite easy to do. Since the integral of a function between two points is just the difference in the anti-derivative, we can argue that the antiderivative is itself a special type of integral, which is a function that will tell us what the area under the curve is when given two limits. This means that we in fact have two types of integrals:

Definition 19

1. *A definite integral is a number and represents the area under the function* $f(x)$ *in a given interval* $[a, b]$*, and is denoted as* $\int_a^b f(x)dx$.
2. *An indefinite integral is a function that will give us the area under a curve* $f(x)$ *if provided two points, and is denoted as* $\int f(x)dx$.

Recall from our discussion on antidifferentiation that when we take an antiderivative, the antiderivative of 0 is a number C. Hence, when we find indefinite integrals in a general sense, we are actually finding *families* of functions, rather than a specific function. For example, we have $\int x^2 dx = \frac{1}{3}x^3 + C$. The derivative and indefinite integral are inverse operations of each other (much like addition/subtraction, multiplication/division, etc). If we take the derivative of an indefinite integral, we obtain the original function we started with. We must include the C however since any number will yield the same derivative for the function. However, if we are given a *boundary condition*, or an *initial condition*, such as $F(0) = 10$, then our indefinite integral does not represent a family of functions, but rather, a specific function:

$$\begin{aligned} F(x) = \int x^2 dx = \frac{1}{3}x^3 + C \\ F(0) = 10 \\ \frac{1}{3}(0)^3 + C = 10 \\ C = 10 \\ \therefore \text{ if } f(x) = x^2 \text{ and } F(0) = 10, \text{ then } \int x^2 dx = \frac{1}{3}x^3 + 10 \end{aligned}$$

Definition 20 *If we are given a function $f(x)$ and a boundary condition $F(a) = b$, then the indefinite integral of a function $f(x)$ is a single function, and we can solve for C. Otherwise, when we take an indefinite integral of $f(x)$ and we are not given a boundary condition, the indefinite integral is a family of possible functions.*

With this stated, we can find the families of some common types of functions if they are presented to us in a particular form:

Theorem 9 *Let $C \in \mathbb{R}$. Then*

Example 17 *Find $\int 3x^2 + 2e^x - \frac{3}{x}dx$. We can first use the addition/subtraction rule, and the constant rule, to simplify this integral down:*

$$\int 3x^2 + 2e^x - \frac{3}{x}dx = 3\int x^2 dx + 2\int e^x dx - 3\int \frac{1}{x}dx$$

Now we can apply our rules listed above, since all of the functions exactly match a function listed in the rules:

$$\begin{aligned} \int 3x^2 + 2e^x - \frac{3}{x}dx &= 3\int x^2 dx + 2\int e^x dx - 3\int \frac{1}{x}dx \\ &= 3(\frac{1}{3}x^3 + C_1) + 2(e^x + C_2) - 3(\ln|x| + C_3) \\ &= 3\frac{1}{3}x^3 + 2e^x - 3\ln|x| + (3C_1 + 2C_2 - 3C_3) \end{aligned}$$

Now since C_1, C_2, C_3 are all constants, any arithmetic combination of them will also be a constant, and hence, we can just write $C = 3C_1 + 2C_2 - 3C_3$, without having to worry about solving for the individual constants. If provided a boundary condition, we need not solve for the three constants, but only for C. Hence, we have:

$$\int 3x^2 + 2e^x - \frac{3}{x}dx = x^3 + 2e^x - 3\ln|x| + C$$

It must be made explicitly known here to take caution. One may be tempted, for example, to say that $\int e^{-2x} = e^{-2x} + C$ since we have the rule $\int e^x dx = e^x + C$. However, this is not correct! In order to use these rules, we need our function to look EXACTLY like the formulas we have. Since $x \neq -2x$, we cannot use this rule directly. Instead, we need to force the exponent to look like the e^x, typically by performing some type of substitution. If we do $u = -2x$, then we have $\int e^{-2x}dx = \int e^u dx$. This is close to looking like our rule, except for the fact that we have a dx and not a du. Later, we will demonstrate how to convert the dx into a du. However, the student of calculus should be very very careful applying these rules. It can ONLY be applied under two conditions: (1) The formula looks exactly the same as in the rules and (2) the variable letter in the d portion of the integral is the same. If either of these conditions fail, you cannot use the rules, it is as simple as that!

5 Integration Methods

5.1 Integration with Taylor Series

We posed the question earlier, how can we solve $\int e^{-2x}$ when we do not have a rule for it? One could use the limiting sums approach, but even this is too complicated, and we may not be able to find a closed form of the partial sum, which makes our derivations even more complicated. However, when all else fails, if the function can be represented by a Taylor series, then we should be able to solve for the integral by finding it's Taylor series and then integrating the entire series. Because the series is itself nothing more than powers of $(x - a)$, we can very easily compute the integral by leveraging the power rule (if $a = 0$), or some form of the power rule with substitution (to be explored later). Let us illustrate this.

Example 18 *Find $\int e^{-2x}dx$. First, let us find it's MacLaurin Series. Set $a = 0$. Then the first few derivatives of $f(x)$ are:*

$$\begin{aligned}
f^{(0)}(x) &= e^{-2x} \\
f^{(1)}(x) &= -2e^{-2x} \\
f^{(2)}(x) &= (-2)(-2)e^{-2x} \\
&\vdots \\
f^{(n)}(x) &= (-2)^n e^{-2x}
\end{aligned}$$

Now, if we set $a = 0$, then the specific derivatives would be:

$$\begin{aligned}
&f^{(0)}(0) = e^{-2(0)} = 1 \\
&f^{(1)}(0) = -2e^{-2(0)} = -2 \\
&f^{(2)}(0) = (-2)(-2)e^{-2(0)} = (-2)(-2) \\
&\vdots \\
&f^{(n)}(0) = (-2)^n e^{-2(0)} = (-2)^n
\end{aligned}$$

So we have the MacLaurin series as:

$$\begin{aligned}
f(x) = e^{-2x} &= \sum_{i=0}^{\infty} \frac{f^{(i)(0)}}{i!}(x)^i \\
&= \sum_{i=0}^{\infty} \frac{(-2)^i}{i!}(x)^i
\end{aligned}$$

Now let us take the integral:

$$\begin{aligned}
\int f(x)dx = \int e^{-2x}dx &= \int \sum_{i=0}^{\infty} \frac{(-2)^i}{i!}(x)^i dx \\
&= \sum_{i=0}^{\infty} \int \frac{(-2)^i}{i!}(x)^i dx \\
&= \sum_{i=0}^{\infty} \frac{(-2)^i}{i!(i+1)}(x)^{(i+1)} + C
\end{aligned}$$

Now we need to find a function that has this specific series as it's Taylor Series representation. This is the hard part, but not impossible. We are basically reversing the process by manipulating the series to "force it" to "look like" a series we already know. We notice that this series looks vary similar to e^{-2x}, so let us try to make some small changes. First, we want all the i's to be consistent. That is, we dont want a mix of i and $i+1$, we want them to all be either i or $i+1$, but not both. Let us try to force this to be $i+1$. We can force the exponent of -2 to be $i+1$ by multiplying the term by another -2. To keep things balanced, we need to also divide by -2. Hence, we have:

$$\begin{aligned}
&\sum_{i=0}^{\infty} \frac{(-2)^i}{i!(i+1)}(x)^{(i+1)} + C \\
&= (1) \sum_{i=0}^{\infty} \frac{(-2)^i}{i!(i+1)}(x)^{(i+1)} + C \\
&= (\frac{-2}{-2}) \sum_{i=0}^{\infty} \frac{(-2)^i}{i!(i+1)}(x)^{(i+1)} + C \\
&= \frac{1}{-2} \sum_{i=0}^{\infty} \frac{(-2)^{i+1}}{i!(i+1)}(x)^{(i+1)} + C
\end{aligned}$$

We also notice that $i!(i+1) = (i+1)!$ by definition, so we have:

$$\begin{aligned}
&\frac{1}{-2} \sum_{i=0}^{\infty} \frac{(-2)^{i+1}}{i!(i+1)}(x)^{(i+1)} + C \\
&= \frac{1}{-2} \sum_{i=0}^{\infty} \frac{(-2)^{i+1}}{(i+1)!}(x)^{(i+1)} + C
\end{aligned}$$

Now let $j = i+1$. Then when $i = 0$, we have $j = 1$. Making this substitution, we have:

$$\begin{aligned}
&\frac{1}{-2} \sum_{i=0}^{\infty} \frac{(-2)^{i+1}}{(i+1)!}(x)^{(i+1)} + C \\
&= \frac{1}{-2} \sum_{j=1}^{\infty} \frac{(-2)^j}{j!} x^j + C
\end{aligned}$$

Now notice that if $j = 0$, then we are left with the term $-\frac{1}{2}\frac{(-2)^0}{0!}x^0 = -\frac{1}{2}(1) = -\frac{1}{2}$. So if we were to *add and subtract $\frac{1}{2}$, we have:*

$$\begin{aligned}
&\frac{1}{-2}\sum_{j=1}^{\infty}\frac{(-2)^j}{j!}x^j + C \\
&= \frac{1}{-2}\sum_{j=1}^{\infty}\frac{(-2)^j}{j!}x^j + 0 + C \\
&= \frac{1}{-2}\sum_{j=1}^{\infty}\frac{(-2)^j}{j!}x^j + (\frac{1}{2} - \frac{1}{2}) + C \\
&= \frac{1}{-2}\sum_{j=1}^{\infty}\frac{(-2)^j}{j!}x^j + \frac{1}{-2}(1) + \frac{1}{2} + C \\
&= \frac{1}{-2}(\sum_{j=1}^{\infty}\frac{(-2)^j}{j!}x^j + 1) + \frac{1}{2} + C \\
&= \frac{1}{-2}(\sum_{j=1}^{\infty}\frac{(-2)^j}{j!}x^j + \frac{(-2)^{(0)}}{(0)!}x^{(0)}) + \frac{1}{2} + C \\
&= \frac{1}{-2}(\sum_{j=0}^{\infty}\frac{(-2)^j}{j!}x^j) + \frac{1}{2} + C
\end{aligned}$$

Since $\frac{1}{2}$ is a constant, we can just fold that into the constant C, leaving us with our final result:

$$= \frac{1}{-2}(\sum_{j=0}^{\infty}\frac{(-2)^j}{j!}x^j) + C$$

Notice, however, that from before, we found that $e^{-2x} = (\sum_{i=0}^{\infty}\frac{(-2)^i}{i!}x^i)$. Other than the fact that we used i instead of j, we notice that these two equations are exactly the same. So, we must therefore have:

$$\begin{aligned}
&= \frac{1}{-2}(\sum_{j=0}^{\infty}\frac{(-2)^j}{j!}x^j) + C \\
&= \frac{1}{-2}e^{-2x} + C
\end{aligned}$$

Hence, we have shown that the Taylor series can be used to find integrals that are difficult to compute. The hardest part about this: finding a function that has the same series you found. If you cannot find such a function, then unfortunately, you are out of luck. However, the Taylor series can still be used, even in that event, to find definite integrals of functions that may not have a nice equation for it's antiderivative. With all of this said, we have finally concluded that:

$$\int e^{-2x}dx = -\frac{1}{2}e^{-2x} + C$$

5.2 Substitution

Earlier, we encountered a possible way to solve for $\int e^{-2x}$ by noticing that if we set $u = -2x$, then would have $\int e^u dx$. This is close to our rule of $\int e^x dx = e^x + C$, but we're not there just yet. The problem lies in the fact that dx is still there, and we need this to be du. If we take the derivative of u, then we have $\frac{du}{dx} = -2$. If we multiply both sides by dx. then they cancel on the left side, leaving us with: $du = -2dx$. Hence, if we solve for dx, then we can turn the entire integral into nothing by u's: $dx = -\frac{1}{2}du$. Making this substitution, we have:

$$\begin{aligned}
\int e^{-2x} dx &= \int e^u(-\frac{1}{2}du) \\
&= -\frac{1}{2}\int e^u du \\
&= -\frac{1}{2}e^u + C \\
&= -\frac{1}{2}e^{-2x} + C
\end{aligned}$$

This leads us to be able to generalize the methodology of substitution:

Theorem 10 *Suppose $f(x) = f(g(x))$. Let $u = g(x)$. Then $g'(x) = \frac{du}{dx}$. This means that $du = g'(x)dx$. If we have an integral that "looks like" $\int f(g(x))g'(x)dx$, then $\int f(g(x))g'(x)dx = \int f(u)du$.*

Example 19 *Find $\int \frac{1}{2x-3}dx$. Let $u = 2x - 3$. Then $du = 2dx$, or $dx = \frac{1}{2}du$. Then making the substitutions, we have $\int \frac{1}{2x-3}dx = \int \frac{1}{u}\frac{1}{2}du = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}ln|u| + C = \frac{1}{2}ln|2x-3| + C$.*

So when can't we use substitution? There are a few occasions where this happens. If we cannot get the function in the form of all u's, and we are still left with x's, then we cannot apply the substitution rules:

Example 20 *Find $\int e^{x^2}dx$. Let $u = x^2$. Then $du = 2xdx$. Then $dx = \frac{1}{2x}du$. Making the substitution, we have $\int e^{x^2}dx = \int e^u(\frac{1}{2x}du) = \frac{1}{2}\int \frac{e^u}{x}du$. Notice that we still have an x in our integral. This is problematic. We could try to make another substitution, but this will result in an even more complicated integral. As such, we cannot further simplify this integral down using substitution.*

5.3 Integration by Parts

If we notice the product rule from our derivative lecture, we can observe the following:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Taking the integral of both sides with respect to x leaves us with:

$$\begin{aligned}
\int \frac{d}{dx}dx[f(x)g(x)] &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \\
f(x)g(x) &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \\
\int f(x)g'(x)dx &= f(x)g(x) - \int f'(x)g(x)dx
\end{aligned}$$

Now, if we let $u = f(x)$ and $v = g(x)$, then we have $\frac{du}{dx} = f'(x)$ and $\frac{dv}{dx} = g'(x)$, or $du = f'(x)dx$ and $dv = g'(x)dx$. Making these substitutions, we see that we have:

$$\int udv = uv - \int vdu$$

We can use this rule then if we can do something simliar as we did in substitution. That is, if we can find a proper selective of u and v so that our integrals look like the equation above, then we can simplify the integral. The result is what is called integration by parts.

Example 21 *Find $\int x^2e^xdx$. If we let $u = x^2$ and $dv = e^xdx$, then we have $du = 2xdx$ and $v = e^x$ (take the integral of both sides of dv to find v). Now we can use the integration by parts to solve:*

$$\begin{aligned}
\int x^2e^xdx &= \int (x^2)(e^xdx) \\
&= \int udv \\
&= uv - \int vdu \\
&= (x^2)(e^x) - \int e^x(2xdx) \\
&= x^2e^x - 2\int xe^xdx
\end{aligned}$$

Now let us apply this again to this new integral. Let $u = x$ and let $dv = e^xdx$. Then $du = dx$ and $v = e^x$. Making the substitutions, we have:

$$\begin{aligned}
\int xe^xdx &= \int (x)(e^xdx) \\
&= \int udv \\
&= uv - \int vdu \\
&= (x)(e^x) - \int e^xdx \\
&= xe^x - e^x + C
\end{aligned}$$

Plugging this into our original problem, we have:

$$\begin{aligned}
\int x^2e^xdx &= \int (x^2)(e^xdx) \\
&= x^2e^x - 2\int xe^xdx \\
&= x^2e^x - 2(xe^x - e^x + C) \\
&= x^2e^x - 2xe^x + 2e^x + C
\end{aligned}$$

5.4 Improper Integrals

As it turns out, we can also define integrals that have infinite limits. Such integrals are referred to as *improper integrals*.

Definition 21 *Let $f(x)$ be a function. Then we define $\int\limits_a^\infty f(x)dx$ as:*

$$\int\limits_a^\infty f(x)dx = \lim_{t\to\infty} \int\limits_a^t f(x)dx$$

.

Likewise, we define $\int\limits_{-\infty}^a f(x)dx$ as

$$\int\limits_{-\infty}^a f(x)dx = \lim_{t\to-\infty} \int\limits_t^a f(x)dx$$

.

Last, we define:

$$\int_{-\infty}^\infty f(x)dx = \lim_{t\to-\infty} \int\limits_t^a f(x)dx + \lim_{t\to\infty} \int\limits_a^t f(x)dx$$

Example 22 *Find $\int_1^\infty \frac{1}{x^2}$. Apply the integration rules, we have:*

$$\begin{aligned}\int_1^\infty \frac{1}{x^2} &= \lim_{t\to\infty} \int_1^t x^{-2} \\ &= \lim_{t\to\infty} 1 - \frac{1}{t} \\ &= 1\end{aligned}$$