Problem Set 2: Derivatives, Sequences, Series, and Integrals

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For all problems, ensure that you show your work. Ensure you use the definitions, and present an argument, if needed, to prove your answer is correct. For example, if I ask you to find a relation that maps the even numbers in $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ to it's subsequent odd number in $B = \{1, 2, 3, 4, 5, 6, 7\}$, your answer would look like this:

We know the relation is defined with the first element being an even number in A and the second element being the next odd number in B. Let $E = \{x | x \in A \text{ and } x \text{ is even}\}$ *and let* $D = \{y | y \in B \text{ and } y \text{ is odd}\}$*. Then* $E = \{2, 4, 6\}$ *and* $D = \{1, 3, 5, 7\}$*. So, the relation that maps the even to the next odd in these sets would be* $R = \{(x, y) | x \in E \wedge y \in D \wedge y = x + 1\} = \{(2, 3), (4, 5), (6, 7)\}$*.*

1. Explain why the derivative of the function $f(x) = |x|$ does not exist at $x = 0$.

2. Let $f(x) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, and assume that σ, μ and π are numbers (i.e. constants). Find, and prove, the maximum of this function using first and second order conditions.

3. Find an equation for the nth derivative of the function $f(x) = \frac{1}{x}$.

4. The Cobb-Douglas Production function is defined as $P = bL^{\alpha}K^{1-\alpha}$, where P represent production levels, L represents the amount of labor in person-hours, and K represents the amount of capital invested, α and b are parameters in the model. Use implicit differentiation to find an expression for the rate of change of capital with respect to labor (Hint: it is okay to have a rate of change in your expression, but you should be able to find an explicit expression for the rate of change of capital with respect to labor).

5. Firms often try to maximize their profits as one of their primary objectives. Generally, profit is defined as total revenue minus total cost, or $\pi(Q) = TR(Q) - TC(Q)$, where Q is the total number of units produced. Use the first and second order conditions to characterize the firm's optimal production policy. That is, why conditions need to be true so that the firm is running at an optimal level of profit?

6. Let $f(x) = \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, and assume that σ, μ and π are numbers (i.e. constants). Find the Taylor series for this function.

7. Using the Taylor series for the function, find

$$\int_{\mu}^{2\mu} \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

8. In probability theory, the *exponential distribution* is often used to model the probability of having to wait a given amount of time for the next customer to arrive at a store. The probability density function for this is $f(x) = \lambda e^{-\lambda x}$, where λ represents the average number of customers that arrive per time unit. To use this, we often leverage the cumulative distribution function to find the probability that the amount of time we need to wait, X, for the next arrival is less than or equal to x. That is, we can use the PDF to find $P(X < x) = F(x) = \int_0^x f(t)dt$, where $f(t)$ is the probability density function. If $\lambda = 10$, then find $P(X < 5)$

9. Find
$$\int_2^{16} xe^{-x}\ dx$$

10. Find
$$\int_1^2 k^5 \ln k\ dk$$