Problem Set 4: Multi-Variable Optimization and Mathematical Programming

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For all problems, ensure that you show your work. Ensure you use the definitions, and present an argument, if needed, to prove your answer is correct. For example, if I ask you to find a relation that maps the even numbers in $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ to it's subsequent odd number in $B = \{1, 2, 3, 4, 5, 6, 7\}$, your answer would look like this:

We know the relation is defined with the first element being an even number in A and the second element being the next odd number in B. Let $E = \{x | x \in A \text{ and } x \text{ is even}\}$ *and let* $D = \{y | y \in B \text{ and } y \text{ is odd}\}$*. Then* $E = \{2, 4, 6\}$ *and* $D = \{1, 3, 5, 7\}$*. So, the relation that maps the even to the next odd in these sets would be* $R = \{(x, y) | x \in E \wedge y \in D \wedge y = x + 1\} = \{(2, 3), (4, 5), (6, 7)\}$.

1. The 2-Period newsvendor model involves a retailer trying to determine the optimal amount to order for time frame 1 (Q_1) and for time frame 2 (Q_2). In time frame 1, a typical newsvendor model can be constructed. However, one will need to append the total cost function for the 2nd time frame costs. There are two situations that can occur. If in the first period $D < Q$, then the newsvendor has $Q - D$ units left, and is incurred an inventory holding cost of h (the per-unit cost of the product, c, is not counted during this period if inventory is held, only the per-unit inventory holding cost of h). The newsvendor needs to take this information into consideration in order to determine how many units Q_2 they should order in the second period. Assume the demand in each period, D_1 and D_2, follows the same probability distribution (but the realizations of them may be different). Also assume that after time period one, knowledge of the value D_1 is known. With all of this in mind, construct the 2-Period newsvendor model, and find the first order conditions for it.

2. Find the maximum of the function $f(x, y) = xye^{-(x+x^2+y^2)}$. Ensure to show that it is a minimum or maximum via the Hessian Matrix.

3. Find the maximum or minimum of the function $f(x, y) = 3x - 6y$ subject to the constraint $g(x, y) = x^2 + y^2 = 5$. If there is no maximum or minimum, explain why. If there is, prove that it is a minimum or maximum.

4. Find the maximum or minimum of the function $f(x, y) = 2x^2 + 6y^2$ subject to the constraint $g(x, y) = y^3 - x^2 = 25$. If there is no maximum or minimum, explain why. If there is, prove that it is a minimum or maximum.

5. Suppose a firm would like to design an optimal production and inventory model for the next six months. Their production abilities are dependent on the number of employees

that are employed during those months. In the month 1, the firm has an initial number of employees, and no one is hired or laid off. In months 2 to 6, the firm can hire and layoff any employee, however, it comes at a per-employee hiring and firing cost. The firm has demand forecasts for every month that it must satisfy by either producing new products or using existing inventory. The firm is allowed to carry over inventory from month to month, however, for every unit of inventory held, they are incurred a unit cost per month. The firm also pays the employees an hourly wage, and the employees can only work a certain number of hours per day. In addition, we will assume that each month only has 20 working days in it, and that on any working day, all of the employees show up. We also know the number of hours that it takes to produce a single unit. The firm cannot outsource products, it MUST meet it's minimum demand through a combination of existing inventory and production activities. Design a mathematical program that will find the optimal number of employees, inventory, and production for each month in such a way that it minimizes it's total costs.