Problem Set 1 Solutions: Set Theory, Calculus, and Limits

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For all problems, ensure that you show your work. Ensure you use the definitions, and present an argument, if needed, to prove your answer is correct. For example, if I ask you to find a relation that maps the even numbers in $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ to it's subsequent odd number in $B = \{1, 2, 3, 4, 5, 6, 7\}$, your answer would look like this:

We know the relation is defined with the first element being an even number in A and the second element being the next odd number in B. Let $E = \{x | x \in A \text{ and } x \text{ is even}\}$ and let $D = \{y | y \in B \text{ and } y \text{ is odd}\}$. Then $E = \{2, 4, 6\}$ and $D = \{1, 3, 5, 7\}$. So, the relation that maps the even to the next odd in these sets would be $R = \{(x, y) | x \in E \wedge y \in D \wedge y = x + 1\} = \{(2, 3), (4, 5), (6, 7)\}$.

1. **What set would represent all points, (x, y), on a standard 2-dimensional plane? Explain why.**
   Solution:
   We know that all coordinates in the standard 2-dimensional plane can be any real number. Therefore, $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Since each element of the set is an ordered pair (a, b), then the set is $\{(a, b) | a \in \mathbb{R}, b \in \mathbb{R}\}$. However, we should recognize that this is just the Cartesian product of the real numbers. Hence, the set that represents all points in the 2-dimensional plane would be $\mathbb{R} \times \mathbb{R}$ Sometimes, you will see this written as $\mathbb{R}^2 = R^2 = \mathbb{R} \times \mathbb{R}$. Generally, $R^n = \mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \ldots, x_n) | x_i \in \mathbb{R}\}$ is called the *n-Dimensional Euclidean Space*.

2. **Using Demorgan's Laws and the Laws of Sets (Theorems 1 - 5 in the notes), simplify the following expression**:

$$C \cap ((A \cup B)^c \cap (B \cup A))^c$$

   Solution:

$$\begin{aligned}
C \cap ((A \cup B)^c \cap (B \cup A))^c &= C \cap ((A \cup B)^c \cap (A \cup B))^c && \text{(Using the Commutative Property of Union)} \\
&= C \cap (\varnothing)^c && \text{(Using the rule } A \cap A^c = \varnothing\text{)} \\
&= C \cap U && \text{(Using the rule } \varnothing^c = U\text{)} \\
&= C && \text{(Using the rule } A \cap U = A\text{)}
\end{aligned}$$

3. **Let A be a countable set. Show that for subset $B \subset A$, B is also a countable set.**
   Solution:
   We will give a proof by contradiction. Assume that B is uncountable. We know that the

union of any set with an uncountable set is also uncountable. We know that if $B \subset A$, then $A = (A - B) \cup B$. This means that $(A - B) \cup B$ must also be uncountable since the union of a set with an uncountable set is also uncountable. However, this is a contradiction, because A is countable. Therefore, our original assumption of B being uncountable is incorrect, and we have thus shown that B must also be countable, otherwise we reach a contradiction.

4. **Show that the set $[5, 10]$ has the same cardinality as $\mathbb{R}$**

   Solution:

   We know the union of a countable and uncountable set is uncountable. We can write $[5, 10] = (5, 10) \cup \{5, 10\}$. We know that $[5, 10]$ is an uncountable set. Since $\{5, 10\}$ is a countable set, then it must be true that $(5, 10)$ is an uncountable set. Therefore, there exists a bijective function $f : [5, 10] \to (5, 10)$. If we can find a bijective function $g : (5, 10) \to \mathbb{R}$, then the composition function $g(f(x))$ for $x \in [5, 10]$ will be a bijective function from $[5, 10]$ to $\mathbb{R}$. This function can very easily be found by putting roots at 5 and 10, and defining a gap at $x = 7.5$. This is done very easily by definition the rational function in terms of the roots and gap: $\frac{(x-5)(x-10)}{x-7.5}$. We can see that this is 0 at $x = 5$ and $x = 10$. However, $5, 10$ are not in the set $(5, 10)$. If we define a function that that for all $x \in (5, 10) - \{7.5\}$, $g(x) = \frac{(x-5)(x-10)}{x-7.5}$, and for $x = 7.5$, $g(x) = 0$, then we can easily show that $g(x)$ is bijective on the domain $(5, 10)$. Let $y \in \mathbb{R} - \{0\}$. Then $y = g(x) = \frac{(x-5)(x-10)}{x-7.5}$

   First, let us show this is surjective. If $y = 0$, then by definition $x = 0$. Let $y \in \mathbb{R} - \{0\}$. Then $y = \frac{(x-5)(x-10)}{x-7.5}$. Rearranging:

$$\begin{aligned} y &= \frac{(x-5)(x-10)}{x-7.5} \\ y(x-7.5) &= (x-5)(x-10) \\ yx - 7.5y &= x^2 - 15x + 50 \\ x^2 - (15+y)x + (50 + 7.5y) &= 0 \end{aligned}$$

   Using the quadratic equation, we have:

$$\begin{aligned} x &= \frac{-b}{2} \pm \frac{\sqrt{b^2 - 4ac}}{2} \\ &= \frac{(15+y)}{2} \pm \frac{\sqrt{(-(15+y))^2 - 4(50+7.5y)}}{2} \\ &= \frac{(15+y)}{2} \pm \frac{\sqrt{y^2 + 30y + 225 - 200 - 30y}}{2} \\ &= \frac{(15+y)}{2} \pm \frac{\sqrt{y^2 + 25}}{2} \end{aligned}$$

   Plotting these two functions will show that $x \in (5, 10)$ for $y \in \mathbb{R} - \{0\}$. Hence, we have shown that $g(x)$ is surjective.

   Let $x_1, x_2 \in (5, 10)$. Let $x_2 = x_1 + \epsilon$. If we can show that $\epsilon = 0$, then we have shown $x_1 = x_2$.

Proceeding with the proof:

$$
\begin{aligned}
&g(x_1) = g(x_2) \\
&\frac{(x_1-5)(x_1-10)}{x_1-7.5} = \frac{(x_1+\epsilon-5)(x_1+\epsilon-10)}{x_2+\epsilon-7.5} \\
&(x_1-5)(x_1-10)(x_1+\epsilon-7.5) = (x_1-5+\epsilon)(x_1-10+\epsilon)(x_1-7.5) \\
&(x_1-5)(x_1-10)(x_1-7.5)+\epsilon(x_1-5)(x_1-10) = (x_1-7.5)[(x_1-5)(x_1-10)+\epsilon(x_1-5+x_1-10)+\epsilon^2] \\
&\epsilon(x_1-5)(x_1-10) = (x_1-7.5)[\epsilon(x_1-5+x_1-10)+\epsilon^2] \\
&(x_1-7.5)[\epsilon(x_1-5+x_1-10)+\epsilon^2]-\epsilon(x_1-5)(x_1-10) = 0 \\
&\epsilon[(x_1-7.5)(2x_1-15)+(x_1-7.5)\epsilon-(x_1-5)(x_1-10)] = 0
\end{aligned}
$$

So either $\epsilon = 0$ or $(x_1-7.5)(2x_1-15)+(x_1-7.5)\epsilon-(x_1-5)(x_1-10) = 0$. We will show that the second case is impossible for our values of $x_1, x_2 \in (5,10)$. Suppose $(x_1-7.5)(2x_1-15)+(x_1-7.5)\epsilon-(x_1-5)(x_1-10) = 0$, then:

$$
\begin{aligned}
(x_1-7.5)(2x_1-15)+(x_1-7.5)\epsilon-(x_1-5)(x_1-10) &= 0 \\
(x_1-7.5)\epsilon &= (x_1-5)(x_1-10)-(x_1-7.5)(2x_1-15) \\
\epsilon &= \frac{(x_1-5)(x_1-10)-(x_1-7.5)(2x_1-15)}{(x_1-7.5)}
\end{aligned}
$$

So, we must have:

$$
\begin{aligned}
x_2 &= x_1+\epsilon \\
&= x_1+\frac{(x_1-5)(x_1-10)-(x_1-7.5)(2x_1-15)}{(x_1-7.5)}
\end{aligned}
$$

It can be shown that $\epsilon \geq 5$ when $x_1 \in (5,10)$, which means $x_2 \geq 10$, which is out of the domain $(5,10)$. Hence, we must have $\epsilon = 0$, which means $x_1 = x_2$.

Therefore, the function $g(x)$ is bijective. Since $f : [0,5] \to (0,5)$ and $g : (0,5) \to \mathbb{R}$, and since these are both bijective, we must have the function $g(f) : [0,5] \to \mathbb{R}$ as bijective, and hence, $[0,5]$ and $\mathbb{R}$ therefore have the same cardinality.

5. **Suppose an investor would like to turn his initial investment of \$100 into \$10,000 in 365 days of trading on the stock market. If we assume that the investor earns the same percentage interest rate every single day on the market, what would this rate be? Make sure to show your entire derivation.**
Solution:
By compound interest, we know that the investment compounded t times at an interest rate of r percent (expressed in decimal) with a starting balance of \$100 is $P(t) = 100(1+r)^t$. In

our case, $t = 365$ and $P(365) = 10000$. We can hence set up an equation, leverage natural logs and exponentials, to solve for r:

$$\begin{aligned}
P(t) &= 100(1+r)^t \\
10000 &= 100(1+r)^{365} \\
\ln(10000) &= \ln(100(1+r)^{365}) \\
\ln(10000) &= \ln 100 + 365 \ln(1+r) \\
\ln(10000) - \ln 100 &= 365 \ln(1+r) \\
\ln(\frac{10000}{100}) &= 365 \ln(1+r) \\
\frac{\ln(100)}{365} &= \ln(1+r) \\
e^{\frac{\ln(100)}{365}} &= e^{\ln(1+r)} \\
e^{\frac{\ln(100)}{365}} &= 1+r \\
r &= e^{\frac{\ln(100)}{365}} - 1 \\
&= 0.012697
\end{aligned}$$

Therefore, the investor should aim to earn around 1.3% during the 365 days.

6. **If $f(x) = x^2 - 2x + 1$ and $g(x) = \frac{1}{x}$, and $h(x) = e^{-2x+5}$, then what is the function $h(g(f(x)))$? Again, make sure to show your entire derivation.**
Solution:
First we find $g(f(x))$. Recall that in order to find this function, we replace every occurrence of x in the equation for $g(x)$ with the entire equation for the function $f(x)$:

$$\begin{aligned}
g(f(x)) &= g(x^2 - 2x + 1) \\
&= \frac{1}{(x^2 - 2x + 1)}
\end{aligned}$$

Likewise, we now replace every occurrence of x in the equation for $h(x)$ with the equation for $g(f(x))$:

$$\begin{aligned}
h(g(f(x))) &= h\left(\frac{1}{(x^2 - 2x + 1)}\right) \\
&= e^{-2\left(\frac{1}{x^2-2x+1}\right)+5} \\
&= e^{5-\left(\frac{2}{x^2-2x+1}\right)}
\end{aligned}$$

7. Let $f(x) = -x^3 + x^2 + 5x$. Using the limit approach, select 10 values of x close to and to the left of $\frac{5}{3}$, compute an approximation to the derivative at $\frac{5}{3}$, and explain why the maximum of $f(x)$ for $x > 0$ is at the point $x = \frac{5}{3}$

Solution
The solution for this will vary depending on your choice of values. Recall that the equation of the derivative is $f'(x) = \lim\limits_{x \to a} \frac{f(a)-f(x)}{x-a}$. In this case, $a = \frac{5}{3}$. Let us pick 10 values that is less than $\frac{5}{3}$. Suppose we pick 1.1,1.2,1.3,1.4,1.5,1.6,1.66,1.666,1.6666,1.66666. Plugging each of these into the equation $\frac{f(\frac{5}{3})-f(x)}{\frac{5}{3}-x} = \frac{(-\frac{5}{3}^3+(\frac{5}{3})^2+5(\frac{5}{3}))-+x^3-x^2-5x}{\frac{5}{3}-x}$ gives us:

x	$\frac{f(a)-f(x)}{x-a}$
1.10000	1.94556
1.20000	1.64889
1.30000	1.33222
1.40000	0.99556
1.50000	0.63889
1.60000	0.26222
1.66000	0.02662
1.66600	0.00267
1.66660	0.00027
1.66666	0.00003

As we can see, the slope of the secant line is approaching 0. Recall that if $f'(x) = 0$ at a given x, then it is possible that the function is at it's maximum (or minimum) at this value of x. Since the slope of the secant lines are approaching 0 as we approach $\frac{5}{3}$, it is reasonable to assume that $f'(\frac{5}{3}) = 0$, which would indicate that there may be a maximum value at this point.

8. **Suppose a truck leaves a facility and within 60 minutes continuously picks up products and subsequently delivers them. Suppose for the first 30 minutes that the truck continuously picks up products, while the last 30 minutes the truck continuously drops off products. It costs the company \$0.20 per minute to hold a single product in the truck. The number of items in the truck (decimals are allowed) at time t is found using the equation $f(t) = -\frac{1}{10}t^2 + 6t$. Use the rectangle approach to approximate the total cost incurred by the company for holding the product over the time from $t \in [0, 60]$.**
Solution:
In order to find the total cost, we need to know how much inventory was held in the truck at each point in time. Since this is always changing, however, we will need to approximate this. Suppose we break the interval of time up into 10 intervals, each with width $\frac{60-0}{10} = 6$: $[0,6], [6,12], [12,18], [18,24], [24,30], [30,36], [36,42], [42,48], [48,54], [54,60]$. We can approximate how much inventory was held for 6 minutes by selecting a point in the interval, computing $f(t)$, and just assuming that this same level of inventory was held constant for the entire 6 minutes (even though it was not in reality, again, we are approximating).

We will pick a point within each of these intervals so that we can compute the inventory levels. You can use any to your choosing. In this solution, we will choose the right endpoint of each interval: 6,12,18,24,30,36,42,48,54,60. Passing each of these through the function $f(t)$, we have:

t	$f(t) = -\frac{1}{10}t^2 + 6t$
6.00000	32.40000
12.00000	57.60000
18.00000	75.60000
24.00000	86.40000
30.00000	90.00000
36.00000	86.40000
42.00000	75.60000
48.00000	57.60000
54.00000	32.40000
60.00000	0.00000

Each time interval is 6 minutes. We know that by approximation, for example in the first interval, we assume we held constant 32.4 units for the entire 6 minutes. Therefore, we were charged (0.2 /(minute unit))(6 minutes)(32.4 units) = .2 ((6)(32.4)) = .2 (interval width)(height of rectangle). Therefore, if we compute the area of each rectangle, and multiple each one by 0.2, we will be able to approximate the total cost:

$$\begin{aligned}
\text{Total Cost} &\approx 0.2 \sum_{i=1}^{10} f(x_i)\Delta x \\
&= 0.2\Delta x \sum_{i=1}^{10} f(x_i) \\
&= (0.2)(6)(32.4 + 57.6 + 75.6 + 86.4 + 90 + 86.4 + 75.6 + 57.6 + 32.4 + 0) \\
&= (0.2)(6)(594) \\
&= \$712.80
\end{aligned}$$

9. **Using the limit laws, compute the limit $\lim_{x \to 3} \frac{x^2+3x+2}{x^2-2x-3}$. If this limit does not exist, please explain why.**
Solution:
We can simplify this expression down first by factoring:

$$\frac{x^2+3x+2}{x^2-2x-3} = \frac{(x+1)(x+2)}{(x+1)(x-3)} = \frac{(x+2)}{(x-3)}$$

We notice that as $x \to 3$, the top number approaches $3 + 2 = 5$. However, the bottom number is problematic, since we will obtain 0 on the denominator. At first glance, we may be okay with this since it is possible for the limit to exist and just converge to ∞. The problem, however, is that if you take a left hand limit, then $x < 3$, and so this will switch the number to negative. Which means the function is approach $-\infty$ as we approach 3 from the left. When we approach it from the right, $x > 3$, and the function will be positive, which means we have the values of the function approaching $+\infty$. Since the left hand limit is $-\infty$ and the right hand limit is $+\infty$, and these do not equal each other, then by the limit laws, the limit does not exist at $x = 3$. Therefore:

$$\lim_{x \to 3} \frac{x^2 + 3x + 2}{x^2 - 2x - 3} = D.N.E$$

10. **Using the limit laws, compute the limit $\lim\limits_{x \to \infty} \frac{x^3+3x^2-x+3}{2x^3-5x+2}$. If the limit does not exist, please explain why.** *Solution:*
If we divide the top and bottom by x^3, we obtain:

$$\begin{aligned} \frac{x^3 + 3x^2 - x + 3}{2x^3 - 5x + 2} &= \frac{\frac{x^3}{x^3} + \frac{3x^2}{x^3} - \frac{x}{x^3} + \frac{3}{x^3}}{\frac{2x^3}{x^3} - \frac{5x}{x^3} + \frac{2}{x^3}} \\ &= \frac{1 + \frac{3}{x} - \frac{1}{x^2} + \frac{3}{x^3}}{2 - \frac{5}{x^2} + \frac{2}{x^3}} \end{aligned}$$

Using the limit rule $\lim_{x \to \infty} \frac{1}{x^n} = 0$, we have:

$$\begin{aligned} \lim_{x \to \infty} \frac{x^3 + 3x^2 - x + 3}{2x^3 - 5x + 2} &= \lim_{x \to \infty} \frac{1 + \frac{3}{x} - \frac{1}{x^2} + \frac{3}{x^3}}{2 - \frac{5}{x^2} + \frac{2}{x^3}} \\ &= \frac{\lim\limits_{x \to \infty} 1 + \lim\limits_{x \to \infty} \frac{3}{x} - \lim\limits_{x \to \infty} \frac{1}{x^2} + \lim\limits_{x \to \infty} \frac{3}{x^3}}{\lim\limits_{x \to \infty} 2 - \lim\limits_{x \to \infty} \frac{5}{x^2} + \lim\limits_{x \to \infty} \frac{2}{x^3}} \\ &= \frac{1 + 0 - 0 + 0}{2 - 0 + 0} \\ &= \frac{1}{2} \end{aligned}$$