Math2090, Section 1
Quiz 7, Total 10 points

Last name: First name:

Let $A = \begin{pmatrix} -1 & 3 & 0 \\ -3 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

a) Compute all eigenvalues of A and their multiplicities.
b) Compute a basis for each eigenspace and determine its dimension.
c) Determine whether A is defective or nondefective.

a) $\lambda I - A = \begin{pmatrix} \lambda+1 & -3 & 0 \\ 3 & \lambda-5 & 0 \\ 0 & 0 & \lambda-1 \end{pmatrix}$

$\det(\lambda I - A) = (\lambda+1)(\lambda-5)(\lambda-1) + 3(3(\lambda-1))$

$= (\lambda-1)(\lambda^2-4\lambda+4) = (\lambda-1)(\lambda-2)^2$

$\therefore$ eigenvalues $\lambda_1 = 1$ with multiplicity 1 and $\lambda_2 = 2$ with multiplicity 2.

b)
For $\lambda_1 = 1$, $A - I = \begin{bmatrix} -2 & 3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row-echelon}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (0,0,0) \Rightarrow x_1 = 0,\ x_2 = 0,\ x_3 = t$ free variable.

$\therefore$ The eigenspace for $\lambda_1 = 1$ is $\{ t(0,0,1) : t \in \mathbb{R} \} = \operatorname{span}\{(0,0,1)\}$

$\therefore (0,0,1)$ is a basis for the eigenspace $\Rightarrow$ the dimension is 1.

For $\lambda_2 = 2$, $A - 2I = \begin{bmatrix} -3 & 3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (0,0,0) \Rightarrow x_1 = x_2,\ x_2 = t$ and $x_3 = 0$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. So the eigenspace for $\lambda_2 = 2$ is spanned by $(1,1,0) \Rightarrow$ the eigenspace has dimension 1.

c) The matrix A has dimension 3×3, The total number of linearly independent eigenvectors is two which is less than 3
$\therefore$ The matrix is defective.