MATHEMATICS 2090
Final Examination Practice Problems

Print name ______________________________
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Solution

1. Let $A = \begin{pmatrix} 1 & 4 & -1 & 3 \\ 2 & 9 & -1 & 7 \\ 2 & 8 & -2 & 6 \end{pmatrix}$.

a) Compute the rank and nullity of A.

b) Find a basis for the kernel of the linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ which is given by $T(\underline{v}) = A\underline{v}$.

Gauss elimination

$$\begin{pmatrix} 1 & 4 & -1 & 3 \\ 2 & 9 & -1 & 7 \\ 2 & 8 & -2 & 6 \end{pmatrix} \xrightarrow[A_{13}(-2)]{A_{12}(-2)} \begin{pmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank$(A) = 2$

nullity $= 4 - 2 = 2$

Kernel of $T\underline{v} = A\underline{v}$ $= \{ \underline{v} \in \mathbb{R}^4 \mid A\underline{v} = \underline{0} \}$

x_3, x_4 free variable let $x_3 = s$, $x_4 = t$

then $x_2 + x_3 + x_4 = 0$ implies $x_2 = -x_3 - x_4 = -s - t$

$x_1 + 4x_2 - x_3 + 3x_4 = 0$ implies

$x_1 = -4x_2 + x_3 - 3x_4 = -4(-s-t) + s - 3t$

$= 5s + 4t - 3t = 5s + t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} t$$

A basis for the kernel of T is given by

$$\left\{ \begin{pmatrix} 5 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2. Solve the linear system

$$\begin{aligned} -2x_1 + 4x_2 + x_3 &= -5 \\ 3x_1 - 2x_2 - x_3 &= 2 \\ 4x_1 - 3x_2 + 2x_3 &= 1 \end{aligned}$$

Augumented matrix

$$\left[\begin{array}{ccc|c} -2 & 4 & 1 & -5 \\ 3 & -2 & -1 & 2 \\ 4 & -3 & 2 & 1 \end{array}\right] \xrightarrow[A_{13}(2)]{A_{12}(1)} \left[\begin{array}{ccc|c} -2 & 4 & 1 & -5 \\ 1 & 2 & 0 & -3 \\ 0 & 5 & 4 & -9 \end{array}\right]$$

P_{12}

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ -2 & 4 & 1 & -5 \\ 0 & 5 & 4 & -9 \end{array}\right] \xrightarrow{A_{12}(2)} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 8 & 1 & -11 \\ 0 & 5 & 4 & -9 \end{array}\right] \xrightarrow{M_2(1/8)} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 1/8 & -11/8 \\ 0 & 5 & 4 & -9 \end{array}\right]$$

$$\xrightarrow{A_{23}(-5)} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 1/8 & -\frac{11}{8} \\ 0 & 0 & \frac{27}{32} & \frac{-72+55}{8} \end{array}\right] \xrightarrow{M_3(\frac{32}{27})} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 1/8 & -11/8 \\ 0 & 0 & 1 & \frac{-17}{27} \end{array}\right]$$

so $x_3 = -\frac{17}{27}$

$x_2 + \frac{1}{8}x_3 = -11/8$ $\quad x_2 = -11/8 + \frac{1}{8} \cdot \frac{17}{27} = \frac{-297+17}{8 \cdot 27} = \frac{-280}{8 \cdot 27} = -\frac{35}{27}$

$x_1 = -3 - 2x_2 = -3 + \frac{70}{27} = \frac{-11}{27}$

$x_1 = -11/27$, $x_2 = -\frac{35}{27}$, $x_3 = -17/27$

3. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$.

a) Compute its determinant. Is A invertible? If yes, please compute its inverse; if not, please justify.
b) Compute the characteristic polynomial, eigenvalues and eigenspaces of the matrix.
c) Is A defective or non-defective?

d) Determine the general solution of the differential equation system $\underline{x}' = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \underline{x}$.

a) $|A| \underset{A_{13}(1)}{\overset{A_{12}(-1)}{=}} \begin{vmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} \underset{\text{cofactor}}{=} 1\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -4 \neq 0$ Invertible

A^{-1} $\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{A_{13}(1)}{\xrightarrow{A_{12}(-1)}} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{P_{23}} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 & 1 & 0 \end{array}\right]$

$\underset{M_3(\frac{1}{2})}{\xrightarrow{M_2(\frac{1}{2})}} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \underset{A_{31}(1)}{\xrightarrow{A_{21}(-1)}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$ $A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

b) $|A - \lambda I_3| = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} \overset{A_{23}(1)}{=} \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ 0 & 2-\lambda & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 2-\lambda & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 2-\lambda & 2-\lambda \end{vmatrix}$

$= (1-\lambda)(2-\lambda)(-\lambda) - (2-\lambda)2$
$= (2-\lambda)(\lambda^2-\lambda-2) = -(\lambda-2)^2(\lambda+1)$

Eigenvalues $\lambda_1 = 2$ multiplicity $=2$ $\lambda_2 = -1$ multiplicity $=1$

Eigenspace

V_2 Augmented matrix

$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \underset{A_{13}(-1)}{\xrightarrow{A_{12}(1)}} \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 = s$, $x_3 = t$ free variables $x_1 = x_2 - x_3 = s-t$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t$ $\dim V_2 = 2$

V_{-1} $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] \underset{A_{23}(1)}{\xrightarrow{P_{12}}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \xrightarrow{A_{12}(-2)} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$

$\xrightarrow{A_{23}(1)} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{M_2(-1/3)} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_3 = s$
$x_2 = -x_3 = -s$
$x_1 = -2x_2 - x_3 = s$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} s$ $\dim V_{-1} = 1$

c) $\dim V_2 + \dim V_{-1} = 3$ Non-defective

d) $\underline{x} = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} c_1 e^{2t} - c_2 e^{2t} + c_3 e^{-t} \\ c_1 e^{2t} - c_3 e^{-t} \\ c_2 e^{2t} + c_3 e^{-t} \end{pmatrix}$

4. Solve the initial value problem $y'' + 6y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = 1$, using a method of your choice.

Solution Auxillany Poly $p(r) = r^2 + 6r + 13 = (r - r_1)(r - r_2)$

$$r_1, r_2 = \frac{-6 \pm \sqrt{36-52}}{2} = -3 \pm 2\sqrt{-1}$$

General solutions are

$$y = C_1 e^{-3x} \cos 2x + C_2 e^{-3x} \sin 2x$$

$$y(0) = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 = C_1 = 0 \qquad \text{so } C_1 = 0$$

$$y'(0) = (C_2 e^{-3x} \sin 2x)' \Big|_{x=0}$$

$$= (-3C_2 e^{-3x} \sin 2x + 2C_2 e^{-3x} \cos 2x) \Big|_{x=0}$$

$$= 2C_2 = 1$$

$$C_2 = 1/2 \qquad \text{so} \qquad y = \frac{1}{2} e^{-3x} \sin 2x$$

5. Use the variation-of-parameter technique to find the general solution to the order-1 differential equation system $\underline{x}' = \begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix} \underline{x} + \begin{pmatrix} 20e^{3t} \\ 12e^t \end{pmatrix}$.

1) $A = \begin{pmatrix} -1 & 1 \\ 3 & +1 \end{pmatrix} \quad |A - \lambda I| = \begin{vmatrix} -1-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-1) - 3 = \lambda^2 - 4 = (\lambda+2)(\lambda-2)$

V_2, $\lambda = 2$: $\left[A - 2I_2 \mid 0 \right] \to \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 3 & -1 & 0 \end{array} \right] \xrightarrow{A_{12}(1)} \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x_2 = s \\ x_1 = \frac{1}{3}s \end{array} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \frac{s}{3}$

$$x_1 = e^{+2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

V_{-2}: $\left[A + 2I_2 \mid 0 \right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 3 & 3 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x_2 = t \\ x_1 = -t \end{array} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} (t)$

$$x_2 = e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

A fundamental matrix $\Phi(t) = \begin{pmatrix} e^{2t} & e^{-2t} \\ 3e^{2t} & -e^{-2t} \end{pmatrix}$, $\Phi(t)^{-1} = -\frac{1}{4} \begin{bmatrix} -e^{-2t} & -e^{-2t} \\ -3e^{2t} & e^{2t} \end{bmatrix}$

So $y_p = \Phi(t) \int \Phi(s)^{-1} \begin{pmatrix} 20e^{3s} \\ 12e^s \end{pmatrix} ds = \Phi(t) \int (-1) \begin{bmatrix} -e^{-2s} & -e^{-2s} \\ -3e^{2s} & e^{2s} \end{bmatrix} \begin{bmatrix} 5e^{3s} \\ 3e^s \end{bmatrix} ds$

$$= \begin{bmatrix} e^{2t} & e^{-2t} \\ 3e^{2t} & -e^{-2t} \end{bmatrix} \cdot (-1) \int \begin{pmatrix} -5e^s - 3e^{-s} \\ -15e^{5s} + 3e^{3s} \end{pmatrix} ds$$

$$= \begin{bmatrix} e^{2t} & e^{-2t} \\ 3e^{2t} & -e^{-2t} \end{bmatrix} \begin{pmatrix} 5e^t - 3e^{-t} \\ 3e^{5t} - e^{3t} \end{pmatrix} = \begin{pmatrix} 5e^{3t} - 3e^t + 3e^{3t} - e^t \\ 15e^{3t} - 9e^t - 3e^{3t} + 1e^t \end{pmatrix}$$

$$= \begin{pmatrix} 8e^{3t} - 4e^t \\ 12e^{3t} - 8e^t \end{pmatrix}$$

$$X = C_1 e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 8e^{3t} - 4e^t \\ 12e^{3t} - 8e^t \end{pmatrix}$$

6. a) Are $\underline{x}_1 = \begin{pmatrix} t\sin(t) \\ \cos(t) \end{pmatrix}$, $\underline{x}_2 = \begin{pmatrix} -t\cos(t) \\ \sin(t) \end{pmatrix}$ linearly independent? Please justify your answer.

b) Are they both solutions of the linear system $\underline{x}' = \begin{pmatrix} 1/t & t \\ -1/t & 0 \end{pmatrix} \underline{x}$? Please justify your answer.

a) $\begin{vmatrix} t\sin t & -t\cos t \\ \cos t & \sin t \end{vmatrix} = t\sin^2 t + t\cos^2 t = t \neq 0$

So linearly independent.

b) $\frac{dx_1}{dt} = \begin{pmatrix} \sin t + t\cos t \\ -\sin t \end{pmatrix}$ equals $\begin{pmatrix} 1/t & t \\ -1/t & 0 \end{pmatrix}\begin{pmatrix} t\sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} \sin t + t\cos t \\ -\sin t \end{pmatrix}$

$\frac{dx_2}{dt} = \begin{pmatrix} -\cos t + t\sin t \\ \cos t \end{pmatrix}$ equals $\begin{pmatrix} 1/t & t \\ -1/t & 0 \end{pmatrix}\begin{pmatrix} -t\cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} -\cos t + t\sin t \\ \cos t \end{pmatrix}$

So they are both solutions of $\underline{x}' = \begin{pmatrix} 1/t & t \\ -1/t & 0 \end{pmatrix} \underline{x}$

7. Solve the initial value problem $y'' - 4y = 12e^{2t}$, $y(0) = 2, y'(0) = 3$, using a method of your choice.

#7 Solve $y'' - 4y = 12e^{2t}$ $y(0) = 2$ $y'(0) = 3$

Solution: $p(r) = r^2 - 4 = (r+2)(r-2)$

$y'' - 4y = 0$ general solution

$$y = c_1 e^{2t} + c_2 e^{-2t}$$

particular solution it is a solution of

$$y_p = c t e^{2t} \quad \text{as } (D-2)(D^2-4)$$

$y_p' = c e^{2t} + 2ct e^{2t}$

$y_p'' = 2c e^{2t} + 2c e^{2t} + 4ct e^{2t}$

$y_p'' - 4y_p = 4c e^{2t} + 4ct e^{2t} - 4ct e^{2t}$

$= 4c e^{2t}$

$= 12 e^{2t}$

$c = 3$

so $y = c_1 e^{2t} + c_2 e^{-2t} + 3t e^{2t}$

$y(0) = c_1 + c_2 = 2$

$y'(0) = 2c_1 - 2c_2 + 3 = 3$

so $c_1 = c_2 = 1$

$y = e^{2t} + e^{-2t} + 3t e^{2t}$

8. a) Compute the Laplace transform $L\left[e^{2t}\sin(3t) - t/2 + 4e^{t}\right](s)$

$$L[e^{2t}\sin 3t] - \tfrac{1}{2}L[t] + 4L[e^{t}]$$

$$= \frac{3}{(s-2)^2+3^2} - \frac{1}{s^2} + \frac{4}{s-1}$$

b) Compute the inverse Laplace transform $L^{-1}\left[\frac{2s}{s^2+2s+2} + \frac{12}{(s-2)^4}\right](t)$.

$$L^{-1}\left[\frac{2s}{s^2+2s+2}\right] + L\left[\frac{12}{(s-2)^4}\right]$$

$$= L^{-1}\left[\frac{2(s+1)-1}{(s+1)^2+1}\right] + 12L\left[\frac{1}{(s-2)^4}\right]$$

$$= 2e^{-t}\cos t - e^{-t}\sin t + e^{2t}\frac{t^3}{3!}$$

9. Use Laplace transform to solve the initial value problem $y' - 2y = u_2(t)e^{t-2}$, $y(0) = 2$ where $u_2(t)$ is the unit step function.

$$L[y'-2y] = L[y'] - 2L[y] = sL[y] - y(0) - 2L[y]$$

$$= (s-2)L[y] - 2$$

$$L[u_2(t)e^{t-2}] = e^{-2s}\frac{1}{s-1}$$

So $$(s-2)L[y] - 2 = e^{-2s}\frac{1}{s-1}$$

$$L[y] = \left(\frac{e^{-2s}}{s-1} + 2\right)\cdot\frac{1}{s-2}$$

$$= e^{-2s}\frac{1}{(s-1)(s-2)} + 2\frac{1}{s-2}$$

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} = \frac{As-2A+Bs-B}{(s-1)(s-2)} = \frac{(A+B)s-2A-B}{(s-1)(s-2)}$$

$$A+B=0 \qquad -2A-B=1 \qquad \text{so } A=-B$$

$$2B-B=1 \qquad B=1 \qquad A=-1$$

$$\frac{1}{(s-1)(s-2)} = \frac{-1}{s-1} + \frac{1}{s-2}$$

$$L^{-1}\left[\frac{1}{(s-1)(s-2)}\right] = L^{-1}\left[\frac{-1}{s-1} + \frac{1}{s-2}\right] = -e^{t} + e^{2t}$$

$$L^{-1}\left[e^{-2s}\frac{1}{(s-1)(s-2)}\right] = u_2(t)\left(-e^{t-2} + e^{2(t-2)}\right)$$

$$L^{-1}\left[2\frac{1}{s-2}\right] = 2e^{2t}$$

So $$y = u_2(t)\left(-e^{t-2} + e^{2(t-2)}\right) + 2e^{2t}$$