Fall 2019

MATHEMATICS 2090 Section 1
Exam I

Print name ______________________
Last name First name

You must show your work in order to get full credit.

No.	Marks
1	
2	
3	
4	
5	
Total	

1. Find a solution to the initial value problem $y' = y^3x - y^3$, $y(0) = 1$. (10pt)

$$\frac{dy}{dx} = y^3x - y^3$$

$$\frac{dy}{dx} = y^3(x-1) \Rightarrow \frac{1}{y^3}\,dy = (x-1)\,dx$$

$$\int y^{-3}dy = \int (x-1)\,dx \Rightarrow -\frac{1}{2}y^{-2} = \frac{1}{2}x^2 - x + c$$

$$y(0)=1 \Rightarrow -\frac{1}{2}(1)^{-2} = \frac{1}{2}\cdot(0)^2 - 0 + c$$

$$\Rightarrow -\frac{1}{2} = 0 + c \Rightarrow c = -\frac{1}{2}$$

$\therefore$ The solution is $-\frac{1}{2}y^{-2} = \frac{1}{2}x^2 - x - \frac{1}{2}$.

2. Find the general solution to the differential equation $x^2y' - 3xy = x^5e^x$. (10pt)

$x^2y' - 3xy = x^5e^x$, divide by x^2

$$y' - \frac{3}{x}y = x^3e^x$$

Find the integrating factor:

$$u(x) = e^{\int -\frac{3}{x}} = e^{-3\int\frac{1}{x}} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}.$$

The general solution is given by:

$$y = \frac{1}{u(x)}\int u(x)\cdot x^3e^x = \frac{1}{x^{-3}}\int x^{-3}\cdot x^3\cdot e^x = x^3\int e^x$$

$$= x^3\left[e^x + c\right]$$

$$= x^3e^x + cx^3.$$

3. (a) Determine whether the following differential equation is exact. Show your work. (4pt)

$$e^y dx + (xe^y + \cos(y))dy = 0$$

(b) Find the general solution to the differential equation. (10pt)

a) $\underbrace{(xe^y + \cos y)}_{M} dy + \underbrace{e^y}_{N} dx = 0$

$M_x = e^y$, $N_y = e^y \Rightarrow M_x = N_y$ so the differential equation is exact

$\phi = \int M\,dy = \int xe^y + \cos y \; dy = xe^y + \sin y + \psi(x)$

$\cancel{\phi'}$ $\phi_x = N \Rightarrow \frac{d}{dx}\left[xe^y + \sin y + \psi(x)\right] = e^y$

$\Rightarrow e^y + \psi'(x) = e^y \Rightarrow \psi'(x) = 0 \Rightarrow \psi(x) = C_1.$

$\therefore \phi(x,y) = xe^y + \sin y + C_1$

$\therefore$ The general solution is $xe^y + \sin y + C_1 = C_2$ when C_1, C_2 are constants.

4. If $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix}$. Please compute the following products if they are well-defined:

a). AB b). BA c). BA^T (6pt)

a) $AB = \begin{pmatrix} 4 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$

b) BA is not defined because the dimension of B is 3×3 and the dimension of A is 2×3 $\quad \underbrace{3\times3 , 2\times3}$

they should be the same, otherwise we cannot multiply the matrices !!.

c) $A^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}$

$BA^T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -3 & -2 \\ 2 & 3 \end{pmatrix}$

5. Use elementary row operations to reduce the following matrix to a REDUCED row-echelon matrix. Then find the rank of this matrix. (10pt)

$$\begin{pmatrix} 2 & -1 & 3 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 3 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 4 \end{pmatrix} \xrightarrow[\text{and row}_2]{\text{interchange row}_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 3 & -1 \\ 1 & 1 & 3 & 4 \end{pmatrix} \xrightarrow[\text{and add to row}_3]{\text{multiply row}_1 \text{ by } (-1)} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 3 & -1 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

~~multiply row (-1)~~

multiply row$_1$ by (-2) and add to row$_2$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{pmatrix} \xrightarrow{\text{add row}_2 \text{ to row}_3} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

multiply row$_2$ by (-1)

$$\longrightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and we are done ^^

The rank is $\underline{\underline{2}}$, because we have two linearly independent rows.