Practice Questions for Mar 07, 2019

1. Make sure you understand the steps below which show that the average size of subsets of an n-set is $n/2$. (We have shown in the class that this is true for $n = 4$ and 5.)

   (a) #(subsets of size m of an n-set) = $C(n, m)$ for $0 \le m \le n$.

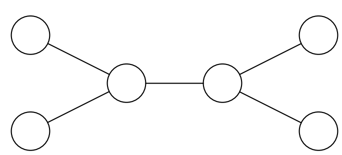
   (b) Total size of all subsets of an n-set

$$
\begin{aligned}
&= 0.\,C(n,0) + 1.\,C(n,1) + 2.\,C(n,2) + 3.\,C(n,3) + 4.\,C(n,4) + \cdots + n.\,C(n,n)\\
&= 1.\,C(n,1) + 2.\,C(n,2) + 3.\,C(n,3) + 4.\,C(n,4) + \cdots + n.\,C(n,n)\\
&= 1.\,n + 2.\,\frac{n(n-1)}{2.\,1} + 3.\,\frac{n(n-1)(n-2)}{3.\,2.\,1} + 4.\,\frac{n(n-1)(n-2)(n-3)}{4.\,3.\,2.\,1} + \cdots + n.\,\frac{n(n-1)(n-2)\cdots 3.\,2.\,1}{n(n-1)(n-2)\cdots 3.\,2.\,1}\\
&= n.\left[1 + \frac{(n-1)}{1} + \frac{(n-1)(n-2)}{2.\,1} + \frac{(n-1)(n-2)(n-3)}{3.\,2.\,1} + \cdots + \frac{(n-1)(n-2)\cdots 3.\,2.\,1}{(n-1)(n-2)\cdots 3.\,2.\,1}\right]\\
&= n[C(n-1,0) + C(n-1,1) + C(n-1,2) + C(n-1,3) + \cdots + C(n-1,n-1)]\\
&= n.\,2^{n-1}
\end{aligned}
$$

   (c) This gives the average(size of subsets of an n-set) = $\dfrac{n.\,2^{n-1}}{2^n} = n/2$.

   The equation $C(n, m) = \dfrac{n}{m}C(n-1, m-1)$ for $1 \le m \le n$, which is the same as $m.\,C(n, m) = nC(n-1, m-1)$, plays a critial role in showing that the total size of all susset of an n-set is $n.\,2^{n-1}$.

2. Consider the tree below and the labelings of its nodes using the labels $\{a, b, c, d, e, f\}$. As usual each node must have a distinct label. Show two labelings where the two nodes of degree 3 has the labels a and b. How many such labelings are there?



   Now give the total number of labelings of the tree.

   Also, give the total number of labelings of the tree using the labels $\{x_1, x_2, \cdots, x_n\}$, $n \ge 6$?

3. Express the number of items that belong to exactly 1 of A, B, and C in terms of the size of A, B, C, $A \cap B$, $\cdots$, etc. Do the same for the number of items that belong to at least 2 of A, B, and C.