STRAIGHTLINE-INTERSECTION RELATION

Straightline-Intersection Relation I.

- For a set of (whole) straightlines $X = \{L_1, L_2, \cdots, L_n\}$, $n \geq 1$, we say $(L_i, L_j) \in I$, i.e., L_i is I-related to L_j if L_i and L_j intersects at one point (hence $i \neq j$). (In Practice Problems #4-#6, we consider the I-relation on a set of straightline-segments.)

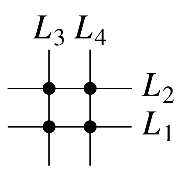
Example.

Shown below are:

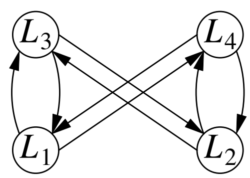
(i) A case of 4 straightlines $\{L_1, L_2, L_3, L_4\}$ with 4 intersection points among them. (Recall that the maximum #(intersection points among 4 lines) = $C(4, 2) = 6$.)

(ii) The digraph of I-relation among those lines, which is a symmetric relation.

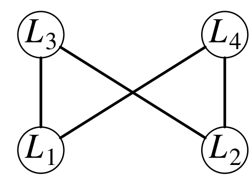
(iii) The simplified undirected graph form of the I-relation, with one undirected link for each pair of two-way (symmetric) directed links between two nodes.



(i)



(ii)



(iii)

The I-relation here depends only on the fact that the lines $\{L_1, L_2\}$ and $\{L_3, L_4\}$ are parallel, but not on the distance between the parallel-lines and the angle between other lines.

Two Properties of I-relation.

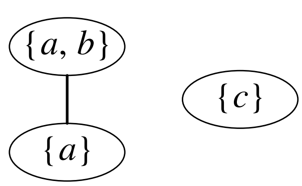
- It is a symmetric relation for every set of straightlines $X = \{L_1, L_2, \cdots, L_n\}$.
- Because $(L_i, L_i) \notin I$ for any L_i, we say that I is *anti-reflexive*. (Non-reflexive means at least one $(L_i, L_i) \notin I$; anti-reflexive means non-reflexive everywhere.)

Set-Intersection Relations $I_\cap$ and $I_\cap^-$.

- Let $X = \{S_1, S_2, \cdots, S_n\}$, where each S_i is a non-empty subset of some set. We say $(S_i, S_j) \in I_\cap$, i.e., S_i is $I_\cap$-related to S_j if $S_i \cap S_j$ is non-empty.
- $I_\cap$ is a symmetric relation; it is also reflexive because each $S_i \cap S_i = S_i \neq \emptyset$.
- Let $I_\cap^-$ be the reduced $I_\cap$-relation obtained by removing all (S_i, S_i) pairs from $I_\cap$. Then, $I_\cap^-$ is symmetric and anti-reflexive (similar to I-relation for straightlines).

  The relations $I_\cap^-$ and I have, however, different properties (see Practice Problems).

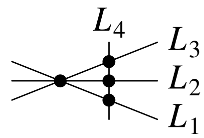
Example.



The undirected graph of the symmetric and anti-reflexive $I_\cap^-$-relation on $S_1 = \{a\}$, $S_2 = \{a, b\}$, and $S_3 = \{c\}$.

Practice Questions.

1. Shown below is another way that 4 straightlines $X = \{L_1, L_2, L_3, L_4$ can intersect in 4 points. Show the digraph and its undirected graph for the I-relation on X.



2. Repeat Problem 1 for all other cases of #(intersection points) = 0, 1, 3, 5, and 6 for 4 straightlines. Recall that #(intersection points) = 2 is not possible, and there are two ways you can get 3 intersection points.
3. Show a graph on 4 nodes which is different from each of the graphs of I-relations on 4 straightlines. (Use the results obtained in Problems 1 and 2.)
4. Show 4 straightline-segments $X = \{LS_1, LS_2, LS_3, LS_4\}$ (not whole straightlines) that has the I-relation corresponding to the graph you obtained in Problem 3.
5. Show all graphs on 4 unlabeled nodes having 0 to 6 links. (Hint: There are at least 11 such graphs; to avoid duplicate listing of such graphs, group the graphs by #(links).)
6. For each graph you found in Problem 5, show a set of 4 straightline-segments whose I-relation equals that graph. (Notes: (a) Two structurally different sets of line-segments, as in 4 line-segments having a common (intersection) point and having 6 intersection-points, can give the same I-relation. (b) The results in Problems 1-4 and 6 shows that the intersection of straightline-segments and of straightlines behave quite differently.)
7. Let G be the graph of I-relation on a set X of (whole) straightlines. Show that G has the following property, i.e., find the subsets X_i of X that satisfies conditions (a)-(b) below.

   There is a unique partition (decomposition) of X into disjoint subsets X_j such that:

   (a) $X = X_1 \cup X_2 \cup \cdots \cup X_k$ $(1 \le k \le n = |S|)$, and

   (b) there is a link connecting each node in X_i to each node in X_j for $1 \le i \ne j \le k$ and these are the only links in G. (G is called a complete k-partite graph.) If $k = n$ and hence each $|X_i| = 1$, then G has all $n(n-1)/2$ links connecting every pair of nodes and is called a complete graph. If $k = 1$, then G has no links.)

8. Use the property in Problem 7 (or otherwise) to argue that the unlabeled version of the set-intersection graph for sets $\{a\}$, $\{a, b\}$, and $\{c\}$ cannot correspond to an I-relation for 3 straightlines. On the other hand, show 3 straightline-segments whose I-relation has that graph. (Thus, the geometric property of straightlines makes I-relation of straightline-intersections a special case of, i.e., less general than $I^{-}_{\cap}$-relation of set-intersections.)
9. Argue that for each set of straightlines $X = \{L_i: 1 \le i \le n\}$, the unlabeled graph of I-relation on X equals to that of I-relation on some set of straightline-segments $XS = \{LS_i: 1 \le i \le n\}$ (find a way to create LS_i's from L_i's). Likewise, argue that for each $XS = \{LS_i: 1 \le i \le n\}$, the unlabeled graph of I-relation on XS equals to that of $I^{-}_{\cap}$-relation on some set of non-empty subsets $S = \{S_i: 1 \le i \le n\}$ (find a way to create S_i's from LS_i's).