

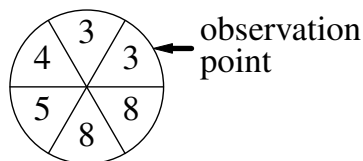
# PROBABILITY

## Why Use Probability.

- We use it to describe behavior of uncertain things.  
Unlike  $2^3$  always gives 8, we can't say whether a throw of a dice will give 2 or not. (We know it won't give 7 or  $-2$ .)
- However, one thing is certain: If we collect data from a very large number of throws of a **perfect dice** (a perfect cube, with its center of gravity at the cube's center), each of the numbers 1, 2, ..., 6 will appear **approx.  $1/6$**  of the times.
- Probability Theory finds the things that are **certain** even in presence of **uncertainties**.

## Sample Space of an Experiment.

- Sample space  $S$ : set of all possible outcomes of the experiment.
  - (a) Experiment: A toss of a coin.  
 $S: \{H, T\}$ ; here,  $H$  = head and  $T$  = tail.
  - (b) Experiment: Two tosses of a coin.  
 $S: \{HH, HT, TH, TT\}$ ; here  $HT$  means 1st toss is  $H$  and 2nd is  $T$ .
  - (c) Experiment: The number in the sector pointed to (or on its right if it falls on the line between two sectors) by the fixed arrow when we turn the spinning wheel shown below, with 6 equal size sectors labeled 3, 3, 8, 8, 4, and 5 (in some order).  
 $S: \{3, 4, 5, 8\}$ .



## Basic Probabilities of A Sample Point $x \in S$ .

- It is the idealized ratio of frequency of  $x$  and the number repetitions of the experiment, when the later is very very large (tends to infinity) and the experiments are "independent" in that one experiment's result doesn't affect the other experiment's result.
  - (a) If the coin is ideal ("unbiased"), then  $\text{Prob}(H) = 1/2 = \text{Prob}(T)$ .
  - (b) If addition, if the two throws in the experiment are "independent", then  $\text{Prob}(HH) = \text{Prob}(HT) = \text{Prob}(TH) = \text{Prob}(TT) = 1/4$ .
  - (c) No matter how the numbers in the wheel are arranged,  $\text{Prob}(3) = 2/6 = 1/3 = \text{Prob}(8)$  and  $\text{Prob}(4) = 1/6 = \text{Prob}(5)$ .

**Practice Questions.**

1. Create a spinning wheel with its equal size sectors marked  $H$  and  $T$  so that  $\text{Prob}(H) = 1/5$  and  $\text{Prob}(T) = 4/5$ .
2. Suppose our experiment consists of three turns of a spinning wheel as in Problem 1, and we write the outcome of an experiment as a triplet  $(s_1, s_2, s_3)$  or simply as  $s_1s_2s_3$  (in short), where  $s_i$  is the outcome ( $H$  or  $T$ ) of  $i$ th turn of the wheel. Show the sample space  $S$  and the basic probabilities  $\text{Prob}(x)$  for each  $x \in S$ . (Hint: the three turns can produce  $5 \times 5 \times 5 = 125$  possible combinations of sectors and the outcome  $(H, T, T)$  corresponds to  $1 \times 4 \times 4 = 16$  of them, giving its probability  $16/125 = (1/5)(4/5)(4/5)$ .)
3. Verify the following two properties of  $\text{Prob}(x)$ 's for  $x \in S$  in Problem 2. (They hold for the sample space of any experiment; for example, they hold for each of the three sample spaces in the previous page.)
  - (a) For each  $x \in S$ ,  $0 \leq \text{Prob}(x) \leq 1$ .
  - (b)  $\sum_{x \in S} \text{Prob}(x) = 1$ .

## COMPLEX EVENTS AND THEIR PROBABILITIES

### Events.

- An **event**  $E$  is simply a subset of the sample space  $S$ ; thus, there are  $2^{|S|}$  many events, including  $E = \emptyset$  and  $E = S$ .
- $\text{Prob}(E) = \sum_{x \in E} \text{Prob}(x)$ ; in particular, we always have  $\text{Prob}(\emptyset) = 0$  and  $\text{Prob}(S) = 1$ .

### Examples.

Consider the experiment "turning of the wheel", whose sample space  $S = \{3, 4, 5, 8\}$ , with  $\text{Prob}(3) = 1/3 = \text{Prob}(8)$  and  $\text{Prob}(4) = 1/6 = \text{Prob}(5)$ .

- (a) For  $E_1 = \{4, 5, 8\} \subseteq S$ , i.e., the outcome is at least 4 (which is the same as "outcome is not 3"),  $\text{Prob}(E_1) = \text{Prob}(4) + \text{Prob}(5) + \text{Prob}(8) = 1/6 + 1/6 + 1/3 = 4/6 = 2/3$ .
- (b) For  $E_2 = \{3\} \subseteq S$ , i.e., the outcome is less than 4,  $\text{Prob}(E_2) = \text{Prob}(3) = 1/3$ .
- (c) For  $E_3 = \{4, 5\} \subseteq S$ ,  $\text{Prob}(E_3) = \text{Prob}(4) + \text{Prob}(5) = 1/6 + 1/6 = 1/3$ .
- (d) For  $E_4 = \{3, 4, 5\} \subseteq S$ ,  $\text{Prob}(E_4) = \text{Prob}(3) + \text{Prob}(4) + \text{Prob}(5) = 1/3 + 1/6 + 1/6 = 2/3$ .

### Two Basic Rules of Probability of Events.

- Sum-rule:  $\text{Prob}(E \cup E') = \text{Prob}(E) + \text{Prob}(E') - \text{Prob}(E \cap E')$ .  
A special case: for disjoint events  $E$  and  $E'$ ,  $\text{Prob}(E \cup E') = \text{Prob}(E) + \text{Prob}(E')$ .
- Complement-rule: For an event  $E$  and its complement  $E^c = \text{not } E$ ,  $\text{Prob}(E) + \text{Prob}(E^c) = 1$ , i.e.,  $\text{Prob}(E^c) = 1 - \text{Prob}(E)$ .

It is a special case of sum-rule;  $\text{Prob}(E) + \text{Prob}(E^c) = \text{Prob}(E \cup E^c) = \text{Prob}(S) = 1$ .

### Notes.

- The sum-rule is a generalization of  $|A \cup B| = |A| + |B| - |A \cap B|$  for set-cardinalities.
- The sum-rule is one of those **certain** things in presence of **uncertainties**.

### Examples.

- (i) In Examples (a), (c), and (d) above, we have  $\text{Prob}(S) = \text{Prob}(E_1 \cup E_4) = 1 = 2/3 + 2/3 - 1/3 = \text{Prob}(E_1) + \text{Prob}(E_4) - \text{Prob}(E_1 \cap E_4)$  because  $E_1 \cap E_4 = \{4\}$ .
- (ii) In Examples (a)-(b) above, we have  $E_2 = E_1^c$  and  $\text{Prob}(E_1) + \text{Prob}(E_1^c) = 2/3 + \text{Prob}(E_2) = 2/3 + 1/3 = 1$ .

**Practice Problems.**

1. Consider three turns of a spinning wheel with 5 equal size sectors labeled  $H, T, T, T$ , and  $T$  in some order, giving  $\text{Prob}(H) = 1/5$  and  $\text{Prob}(T) = 4/5$  in each turn of the wheel. Determine the set of sample points  $s_1 s_2 s_3$ , where each  $s_i = H$  or  $T$  (outcome of  $i$ th turn of the wheel), for each of the following events and compute probabilities of those events.
  - (a)  $E_1 = \{s_1 s_2 s_3 : \#(H) \leq 2\}$ .
  - (b)  $E_2 = \{s_1 s_2 s_3 : \#(H) \geq 2\}$ .
  - (c)  $E_3 = \{s_1 s_2 s_3 : \#(H) \text{ equals } 2\} = E_1 \cap E_2$ .
  - (d)  $E_4 = \{s_1 s_2 s_3 : \#(H) \leq 1\} = E_2^c$ .
2. Verify the sum-rule and the complement-rule involving the events in Problem 1.
3. Consider three turns of a spinning wheel with 6 equal size sectors, where two sectors are labeled 3, two sectors are labeled 8, and the other two sectors are labeled 4 and 5, one each. Determine the sample points  $(s_1, s_2, s_3)$  corresponding to the following events and determine probabilities of the events.
  - (a) There is an equilateral triangle with the sides  $s_1, s_2$ , and  $s_3$ .
  - (b) There is a right-angled triangle with the sides  $s_1, s_2$ , and  $s_3$ .
  - (c) There is an equilateral or right-angled triangle with the sides  $s_1, s_2$ , and  $s_3$ .
  - (d) There is an isosceles triangle with the sides  $s_1, s_2$ , and  $s_3$ .
  - (e) There is no triangle with the sides  $s_1, s_2$ , and  $s_3$ .
4. Verify the sum-rule involving the events (a)-(c) in Problem 3.

## BINOMIAL PROBABILITY DISTRIBUTION

### Probability of $X = \#(H)$ in $n = 2$ Tosses of A Coin.

- Assume  $p = \text{Prob}(H)$  and  $q = 1 - p = \text{Prob}(T)$ .
- Shown below are values of  $X$ , the associated sample points, and the probabilities.

$X$	Sample points	Probability
0	$TT$	$\text{Prob}(X = 0) = q^2$
1	$HT, TH$	$\text{Prob}(X = 1) = pq + qp = 2pq$
2	$HH$	$\text{Prob}(X = 2) = p^2$

### Binomial Probability Distributions.

- The case  $n = 1$ :  
 Sample space of values of  $X$  is  $\{0, 1\}$ ;  
 Probabilities  $\text{Prob}(0) = q$ ,  $\text{Prob}(1) = p$ .  
 The sum of these probabilities =  $p + q = 1$ .
- The case  $n = 2$ :  
 Sample space of values of  $X$  is  $\{0, 1, 2\}$ ;  
 Probabilities  $\text{Prob}(0) = q^2$ ,  $\text{Prob}(1) = 2pq$ , and  $\text{Prob}(2) = p^2$ .  
 The sum of these probabilities =  $p^2 + 2pq + q^2 = (p + q)^2 = 1$ .
- The case  $n = 3$ :  
 Sample space of values of  $X$  is  $\{0, 1, 2, 3\}$ ;  
 Probabilities  $\text{Prob}(0) = q^3$ ,  $\text{Prob}(1) = 3pq^2$ ,  $\text{Prob}(2) = 3p^2q$ , and  $\text{Prob}(3) = p^3$ .  
 The sum of these probabilities =  $p^3 + 3p^2q + 3pq^2 + q^3 = (p + q)^3 = 1$ .
- The general case  $n \geq 1$ :  
 Sample space of values of  $X$  is  $\{0, 1, \dots, n\}$ ;  
 Probabilities  $\text{Prob}(i) = C(n, i)p^i q^{n-i}$  for  $0 \leq i \leq n$ .  
 The sum of these probabilities =  $\sum_{0 \leq i \leq n} C(n, i)p^i q^{n-i} = (p + q)^n = 1$ .

## EXPECTED VALUES

### A Finite Discrete Random Variable $X$ .

- A random variable  $X$  has a **probability** associated with each of its possible value and those values form a finite sample space  $S_X$  of **numbers**.

For example,  $X = \#(H \text{ in } n \text{ tosses of a coin})$ ,  $S_X = \{0, 1, 2, \dots, n\}$ , and  $\text{Prob}(i) = C(n, i)p^i q^{n-i}$ . (There should be no confusion between  $S_X$  and the sample space  $S$  of the underlying experiment.)

- We allow both values 0 and 1 for  $\text{Prob}(x)$ ,  $x \in S_X$ . (If  $S_X = \{x\}$ ,  $\text{Prob}(x) = 1$ ; allowing  $\text{Prob}(x) = 0$  for some  $x \in S_X$  makes comparison of different random variables easier.)

Note: An ordinary variable  $x$  has no notion of associated probability  $\text{Prob}(x)$ .

### Expected Value $E(X)$ of A Random Variable $X$ .

- $$E(X) = \sum_{x \in S_X} x \cdot \text{Prob}(x).$$

### Example.

Let us consider Binomial probability distributions.

- For  $n = 1$ ,  $E(X) = 0 \cdot q + 1 \cdot p = p$ .
- For  $n = 2$ ,  $E(X) = 0 \cdot q^2 + 1 \cdot 2pq + 2 \cdot p^2 = 2p(q + p) = 2p$ .
- For  $n = 3$ ,  $E(X) = 0 \cdot q^3 + 1 \cdot 3pq^2 + 2 \cdot 3p^2q + 3 \cdot p^3 = 3p(q^2 + 2pq + q^2) = 3p$ .
- For the general case  $n \geq 1$ ,

$$\begin{aligned}
 E(X) &= \sum_{0 \leq i \leq n} i \cdot C(n, i) q^i p^{n-i} \\
 &= \sum_{1 \leq i \leq n} i \cdot C(n, i) p^i q^{n-i} \\
 &= np \cdot \sum_{1 \leq i \leq n} C(n-1, i-1) p^{i-1} q^{n-i} \text{ because } i \cdot C(n, i) = n \cdot C(n-1, i-1) \\
 &= np \cdot \sum_{0 \leq j \leq n-1} C(n-1, j) p^j q^{(n-1)-j} \\
 &= np \cdot (p + q)^{n-1} = np.
 \end{aligned}$$

**Practice Questions.**

1. Compute expected value of  $X$ , where  $S_X = \{3, 4, 5, 8\}$  with  $\text{Prob}(3) = 2/3 = \text{Prob}(8)$  and  $\text{Prob}(4) = 1/3 = \text{Prob}(5)$ .
2. Let  $c \neq 0$  be a constant and let  $Y = cX$ . Then, the possible values of  $Y$  are  $S_Y = \{3c, 4c, 5c, 8c\}$  and  $\text{Prob}(Y = 3c) = \text{Prob}(X = 3) = \text{Prob}(3) = 2/3$ , etc. Compute  $E(Y)$  and show that it equals  $cE(X)$ .

Note that if  $c = 0$ , then  $S_Y = \{0\}$  and thus  $E(Y) = 0$ .  $\text{Prob}(Y = 0) = 0 \times 1 = 0 = 0 \cdot E(X)$ . Thus,  $E(cX) = c \cdot E(X)$  for all  $c$ .

The equality  $E(cX) = c \cdot E(X)$  holds for all random variable  $X$ . (This is one of those **cer-tain** things in presence of **uncertainties** that Probability Theory finds out.)

3. If we consider  $Y = X^2$ , where  $X$  is as in Problem 1, then what are the possible values of  $Y$  and what are the probabilities associated with those values?
4. Compute  $E(Y)$  based on your solution of Problem 3.