TESTING REFLEXIVE PROPERTY OF A RELATION-MATRIX

Code for $n \times n$ Relation Matrix R.

```
for (i = 0; i < n; i++) //n = #(rows) = #(columns) in R
    if (R[i][i] == 0) return(false);
return(true);
```

Performance for `n = 4`.

- The diagonal items r_{11}, r_{22}, $\cdots$ are now $R_{0,0} = R[0][0]$, $R_{1,1} = R[1][1]$, $\cdots$. (Here, $R_{i,j}$ as a short form of $R[i][j]$.)
- Next to each '?' (which means "don't care") is shown in parentheses #(choices).

#(comparisons $R_{ii} = 0$)	Return value	$R_{0,0}$	$R_{1,1}$	$R_{2,2}$	$R_{3,3}$	Other 12 $R_{i,j}$'s	#(R)
1	false	0	? (2)	? (2)	? (2)	? (2^{12})	2^{15}
2	false	1	0	? (2)	? (2)	? (2^{12})	2^{14}
3	false	1	1	0	? (2)	? (2^{12})	2^{13}
4	false	1	1	1	0	? (2^{12})	2^{12}
4	true	1	1	1	1	? (2^{12})	2^{12}

- Aver. #(comparisons) = $2^{12}(1\times8 + 2\times4 + 3\times2 + 4\times1 + 4\times1)/2^{16} = 30/16 = 15/8 = 2(1 - 1/16)$.
- The general case:

$$
\begin{aligned}
\text{Aver. \#(comparisons)} &= (1.\,2^{n-1} + 2.\,2^{n-2} + 3.\,2^{n-3} + \cdots + n.\,1 + n.\,1)/2^n \\
&= [(1 + 2x + 3x^2 + \cdots + nx^{n-1}) + nx^{n-1}]/2, \text{ where } x = 1/2 \\
&= \frac{1}{2}\left[\frac{1 - x^n}{(1-x)^2} - \frac{nx^n}{1-x} + nx^{n-1}\right] \\
&= 2\left(1 - \frac{1}{2^n}\right).
\end{aligned}
$$

TESTING SYMMETRY PROPERTY OF A RELATION-MATRIX

Code for $n \times n$ Relation Matrix R.

```
for (i = 0; i < n; i++) //ith row
    for (j = i+1; j < n; j++) //i < j for upper-diagonal items
        if (R[i][j] != R[j][i]) return(false);
return(true);
```

- This is same as testing $H = W$, where H (W) corresponds to upper-diagonal items in R arranged row-by-row (and lower-diagonal items arranged column-by-column) as indicated below by successive underlined parts.

$$H = [\underline{R_{0,1},\ R_{0,2},\ \cdots,\ R_{0,n-1}},\ \underline{R_{1,2},\ R_{1,3},\ \cdots,\ R_{1,n-1}},\ \cdots,\ \underline{R_{n-1,n-1}}]$$
$$W = [\underline{R_{1,0},\ R_{2,0},\ \cdots,\ R_{n-1,0}},\ \underline{R_{2,1},\ R_{3,1},\ \cdots,\ R_{n-1,1}},\ \cdots,\ \underline{R_{n-1,n-1}}]$$

Example: $\begin{bmatrix} R_{0,0} & 1 & 1 \\ 1 & R_{1,1} & 0 \\ 0 & 1 & R_{2,2} \end{bmatrix}$ $\quad H = [\underline{1,\ 1},\ \underline{0}\ ]$, $W = [\underline{1,\ 0},\ \underline{1}\ ]$

Performance for `n = 3`.

- Each row below shows the values of different off-diagonal pairs (R_{ij}, R_{ji}) that together results in the appropriate return-value.
- Next to each '?' is shown in parentheses #(choices).

#(compares. $R_{ij} \neq R_{ji}$)	Return value	i=0, j=1 ($R_{0,1}$, $R_{0,1}$)	i=0, j=2 ($R_{0,2}$, $R_{0,2}$)	i=1, j=2 ($R_{1,2}$, $R_{2,1}$)	Diagonal 3 $R_{i,i}$'s	#(R)
1	false	(0,1), (1,0)	? (2^2)	? (2^2)	? (2^3)	$2.\ 2^7$
2	false	(0,0), (1,1)	(0,1), (1,0)	? (2^2)	? (2^3)	$2^2.\ 2^5$
3	false	(0,0), (1,1)	(0,0), (1,1)	(0,1), (1,0)	? (2^3)	$2^3 \cdot 2^3$
3	true	(0,0), (1,1)	(0,0), (1,1)	(0,0), (1,1)	? (2^3)	$2^3.\ 2^3$

- Aver. #(comparisons) = $2^6(1\times4 + 2\times2 + 3\times1 + 3\times1)/2^9 = 14/8 = 2(1 - 1/8)$.
- The general case is the same as testing $H = W$ for array-length $N = n(n-1)/2$:

$$\text{Aver. \#(comparisons)} = 2\left(1 - \frac{1}{2^N}\right) = 2\left(1 - \frac{1}{2^{n(n-1)/2}}\right).$$

Practice Questions.

1. Show the following code does not correctly test the symmetry property by giving an example 3×3 relation-matrix R that makes the code return wrong value in fewest #(comparisons). (Use '?' for items in R that are not looked at by the code before returning.)

```
for (i = 0; i < n; i++)
    for (j = i+1; j < n; j++)
        if (R[i][j] == R[j][i]) return(true);
return(false);
```

2. Clearly state when the code in Problem 1 returns the correct true/false values when R is an $n \times n$ relation-matrix. Also, give #(relations) for which it correctly returns true-value and for which it correctly returns false-value.
3. Clearly state the reason(s) for inefficiency in the following code that (correctly) tests the symmetry property of a relation-matrix R. In particular, give the extra #(comparisons $R[i][j] \neq R[i][j]$)" this code does in returning true-value compared to the original code given in the notes.

```
for (i = 0; i < n; i++)
    for (j = 0; j < n; j++)
        if (R[i][j] != R[j][i]) return(false);
return(true);
```

TESTING ANTI-SYMMETRY PROPERTY OF A RELATION-MATRIX

Code for $n \times n$ Realtion Matrix R.

```
for (i = 0; i < n; i++) //n = #(rows) = #(columns) in R
    for (j = i+1; j < n; j++) //i < j for upper-diagonal items
        if (R[i][j] + R[j][i] == 2) //same as R[i][j] + R[j][i] > 1
           return(false);
return(true);
```

- This is same as testing $H \cap W = \varnothing$, where H (W) corresponds to upper- (lower-) diagonal items in R, arranged row-by-row (column-by-column) as shown below.

$$H = [\underline{R_{0,1},\ R_{0,2},\ \cdots,\ R_{0,n-1}},\ \underline{R_{1,2},\ R_{1,3},\ \cdots,\ R_{1,n-1}},\ \cdots,\ \underline{R_{n-1,n-1}}]$$
$$W = [\underline{R_{1,0},\ C_{2,0},\ \cdots,\ R_{n-1,0}},\ \underline{R_{2,1},\ R_{3,1},\ \cdots,\ R_{n-1,1}},\ \cdots,\ \underline{R_{n-1,n-1}}]$$

Performance for `n = 3`.

- Each row below shows the values of different off-diagonal pairs (R_{ij}, R_{ji}) that together results in the appropriate return-value.
- Next to each '?' is shown in parentheses #(choices).

#(compares. $R_{ij}+R_{ji} = 2$	Return value	$(R_{0,1},\ R_{0,1})$	$(R_{0,2},\ R_{2,0})$	$(R_{1,2},\ R_{2,1})$	Diagonal 3 $R_{i,i}$'s	#(R)
1	false	(1,1)	? (2^2)	? (2^2)	? (2^3)	2^7
2	false	(0,0), (0,1), (1,0)	(1,1)	? (2^2)	? (2^3)	$3.\ 2^5$
3	false	(0,0), (0,1), (1,0)	(0,0), (0,1), (1,0)	(1,1)	? (2^3)	$3^{2.}2^3$
3	true	(0,0), (0,1), (1,0)	(0,0), (0,1), (1,0)	(0,0), (0,1), (1,0)	? (2^3)	$3^3.\ 2^3$

- Aver. #(comparisons) = $2^3(1\times2^4 + 2\times3\times2^2 + 3\times3^2\times1 + 3\times3^3\times1)/2^9 = 148/64 = 37/16$.
- The general case (testing $H \cap W = \varnothing$ for array-length $N = n(n-1)/2$):

$$\begin{aligned}
&\text{Aver. \#(comparisons)} \\
&= (2^n/2^{n^2})(1.\ 2^{2N-2} + 2.\ 3.\ 2^{2N-4} + 3.\ 3^2.\ 2^{2N-6} + \cdots + N.\ 3^{N-1}.\ 1 + N.\ 3^N) \\
&= (2^n/2^{n^2})(2^{2N-2}[1 + 2.\ (3/4) + 3.\ (3/4)^2 + \cdots + N.\ (3/4)^{N-1}] + N.\ 3^N) \\
&= \frac{2^n}{2^{n^2}}\left(2^{2N-2}\left[\frac{1-(3/4)^N}{(1-3/4)^2} - \frac{N(3/4)^N}{1-3/4}\right] + N3^N\right) = 4\left(1 - \frac{3^N}{4^N}\right).
\end{aligned}$$

Practice Questions.

1. Show the following code does not correctly test the anti-symmetry property by giving an example 3×3 relation-matrix R that makes the code return wrong value in fewest #(comparisons). (Use '?' for items in R that are not looked at by the code before returning.)

```
for (i = 0; i < n; i++)
    for (j = i+1; j < n; j++)
        if (R[i][j] + R[j][i] <= 1) return(true);
return(false);
```

2. Clearly state when the code in Problem 1 returns the correct true/false values when R is an n×n relation-matrix. Also, give #(relations) for which it correctly returns true-value and for which it correctly returns false-value.
3. Clearly state the reason(s) for inefficiency in the following code that (correctly) tests the anti-symmetry property of a relation-matrix R.

```
for (i = 0; i < n; i++)
    for (j = i+1; j < n; j++)
        if ((R[i][j] == 1) && (R[j][i] == 1)) return(false);
return(true);
```