MATHEMATICS 2090 Section 3
Exam III

Print name ______________________
Last name First name

You must show your work in order to get full credit.

No.	Marks
1	
2	
3	
Total	

1. Let $L = (D^2 + 4)$ where $D = \frac{d}{dx}$.
a) Find the general solution to the homogeneous equation $Ly = 0$. 5pt
b) Find the general solution to the nonhomogenous equation $Ly = 16e^{2x}$. 5pt

a) $Ly = 0 \Rightarrow y'' + 4y = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$\therefore y_1 = \cos 2x$, $y_2 = \sin 2x$

The General solution is $y(x) = c_1 \cos 2x + c_2 \sin 2x$.

b) $D-2$ annihilate $16e^{2x}$

$(D+2i)(D-2i)(D-2) = 0$

$y(x) = c_1 \cos 2x + c_2 \sin 2x + A_0 e^{2x}$ Now we find A_0.

$(D^2+4)(A_0 e^{2x}) = 16e^{2x} \Rightarrow 4A_0 e^{2x} + 4A_0 e^{2x} = 16e^{2x}$

$\therefore A_0 = 2$

$\therefore y(x) = c_1 \cos 2x + c_2 \sin 2x + 2e^{2x}$

2. a) Find the general solution to the homogeneous differential equation $y'' - 4y' + 4y = 0$. 5pt
b) Find the general solution to the nonhomogenous differential equation 8pt

$$y'' - 4y' + 4y = \frac{e^{2x}}{x}$$

a) $r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$.

$y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$

$\therefore$ the general solution is $y(x) = c_1 e^{2x} + c_2 x e^{2x}$.

b) First, we find $y_p(x) = u_1 y_1 + u_2 y_2$

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}$$

$$u_1 = -\int \frac{xe^{2x} \cdot x^{-1}e^{2x}}{e^{4x}} = -\int dx = -x$$

$$u_2 = \int \frac{e^{2x} \cdot x^{-1} e^{2x}}{e^{4x}} = \int x^{-1} = \ln x$$

$\therefore y_p = -xe^{2x} + x \ln x e^{2x}$

$y(x) = c_1 e^{2x} + c_2 x e^{2x} - xe^{2x} + x \ln x e^{2x}$

5. Let $A = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$.

a) Compute all the eigenvalues of A, and determine a basis for each eigenspace of A. 8pt

$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 3 \\ 1 & 1-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda) - 3 = 0$

$\lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$

For $\lambda = 2$:

$A - 2I = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$.

$\therefore -x_1 + x_2 = 0 \Rightarrow x_1 = x_2$ so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ~~span~~ is a basis of the eigen space E_2 where $E_2 = \{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \}$

Similarly for $\lambda = -2$ we get $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is a basis for the eigen space E_{-2} when $E_{-2} = \{ t \begin{bmatrix} -3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \}$

b) Determine whether A is defective or nondefective. 1pt

nondefective

c) Determine two linearly independent solutions $\mathbf{x}_1$, $\mathbf{x}_2$ of the vector differential equation $\mathbf{x}' = A\mathbf{x}$ and find the general solution to this homogeneous equation. 5pt

$x_1(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_2(t) = e^{-2t} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

the general solution $x(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

b) Compute the Wronskian $W(\mathbf{x}_1, \mathbf{x}_2)$. 3pt

$W(x_1, x_2) = \begin{vmatrix} e^{2t} & -3e^{-2t} \\ e^{2t} & e^{-2t} \end{vmatrix} = e^{2t} \cdot e^{-2t} - (-3e^{-2t}) \cdot e^{2t}$

$= 1 + 3 = 4.$

c) Let $X = [\mathbf{x}_1, \mathbf{x}_2]$. Compute its inverse X^{-1}. 3pt

$X = \begin{bmatrix} e^{2t} & -3e^{-2t} \\ e^{2t} & e^{-2t} \end{bmatrix}$

$X^{-1} = \frac{1}{4} \begin{bmatrix} e^{-2t} & 3e^{-2t} \\ -e^{2t} & e^{2t} \end{bmatrix}$

d) Obtain the general solution to the vector equation $\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} 30e^{3t} \\ 24e^{t} \end{bmatrix}$ using variation of parameter, where A is given at the beginning of this question.

7pt

The fundamental matrix is $X(t) = \begin{bmatrix} e^{2t} & -3e^{-2t} \\ e^{2t} & e^{-2t} \end{bmatrix}$

$$u'(t) = X^{-1}(t) \cdot \begin{bmatrix} 30e^{3t} \\ 24e^{t} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} e^{-2t} & 3e^{-2t} \\ -e^{2t} & e^{2t} \end{bmatrix} \begin{bmatrix} 30e^{3t} \\ 24e^{t} \end{bmatrix}$$

$$\cancel{= \frac{1}{4} \begin{bmatrix} 30e^{t} - 72e^{-t} \\ 6e^{5t} \end{bmatrix}}$$

$$= \frac{1}{4} \begin{bmatrix} 30e^{t} + 72e^{-t} \\ -30e^{5t} + 24e^{3t} \end{bmatrix}$$

$$\therefore u(t) = \frac{1}{4} \begin{bmatrix} 30e^{t} - 72e^{-t} \\ -6e^{5t} + 8e^{3t} \end{bmatrix}$$

$$X_p(t) = X(t) \cdot u(t) = \frac{1}{4} \begin{bmatrix} e^{2t} & -3e^{-2t} \\ e^{2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 30e^{t} - 72e^{-t} \\ -6e^{5t} + 8e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} 12e^{3t} - 24e^{t} \\ 6e^{3t} - 16e^{t} \end{bmatrix}$$

The general solution is $c_1 X_1(t) + c_2 X_2(t) + X_p(t)$

$$= c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 12e^{3t} - 24e^{t} \\ 6e^{3t} - 16e^{t} \end{bmatrix}.$$