CSc 3102: (Dis)Proving Asymptotic Bounds

Worked Examples

Consider the claims below. If the claim is true, prove the claim using (a.) the definition of the Θ-notation and (b.) limits. If the claim is false, disprove it using a proof-by-contradiction involving limits.

Claim 1. $(n^3 + 4n - 5)^2 \in \Theta(n^6)$

The claim is true.

A proof using the definition of Θ-notation.

Proof:

A. **Definition 1.** Let f and g be functions from $\mathbb{Z}^+ \to \mathbb{R}^+$ - that is, positive real-valued functions on the domain of positive integers. If $f(n) \in \Theta(g(n))$, then $g(n)$ is said to be an asymptotic tight bound for $f(n)$. Mathematically, there are constants $c_1 > 0$, $c_2 > 0$ and an integer constant $n_0 \geq 1$ such that $c_1 g(n) \leq f(n) \leq c_2 g(n) \; \forall \, n \geq n_o$.

B. Let $f(n) = (n^3 + 4n - 5)^2$ and $g(n) = n^6$.

C. I want to find c_1, c_2 and n_o such that $c_1 n^6 \leq (n^3 + 4n - 5)^2 \leq c_2 n^6$ where $n \geq n_o$.

D. Proving the first half of the inequality:

   (a) $n^3 \leq n^3$, $n \geq 1$

   (b) $0 \leq 4n - 5$, $n \geq 2$

   (c) Using the additive property of inequality and comibining D.(a) and D.(b) and using the intersection of the half open intervals, we get $n^3 \leq n^3 + 4n - 5$, $n \geq 2$.

(d) Squaring both sides of the inequality in D(c), we get $n^6 \le (n^3 + 4n - 5)^2$, $n \ge 2$

E. Proving the second half of the inequality:

(a) $n^3 \le n^3$, $n \ge 1$

(b) $4n - 5 \le 4n^3$, $n \ge 1$

(c) Using the additive property of inequality and comibining E.(a) and E.(b) and using the intersection of the half open intervals, we get $n^3 + 4n - 5 \le 5n^3$, $n \ge 1$.

(d) Squaring both sides of the inequality in E(c), we get $(n^3 + 4n - 5)^2 \le 25n^6$, $n \ge 1$

F. Combining D.(d) and E.(d) and using the intersection of the half open intervals, we get $n^6 \le (n^3 + 4n - 5)^2 \le 25n^6$, $n \ge 2$. For $c_1 = 1$, $c_2 = 25$ and $n_o = 2$, we get $c_1 g(n) \le f(n) \le c_2 g(n), n \ge n_o$. Therefore $(n^3 + 4n - 5)^2 \in \Theta(n^6)$.

□

A proof using limits.

Proof:

A. Let f and g be functions from $\mathbb{Z}^+ \to \mathbb{R}^+$ - that is, positive real-valued functions on the domain of positive integers.
$f(n) \in \Theta(g(n)) \Leftrightarrow \lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c < \infty$.

B. Let $f(n) = (n^3 + 4n - 5)^2$ and $g(n) = n^6$.

C. I want to show that $\lim\limits_{n\to\infty} \frac{(n^3+4n-5)^2}{n^6} = c,\ 0 < c < \infty$.

D. $\lim\limits_{n\to\infty} \frac{(n^3+4n-5)^2}{n^6} = \lim\limits_{n\to\infty} \left(\frac{n^3+4n-5}{n^3}\right)^2 = \lim\limits_{n\to\infty} \left(\frac{n^3}{n^3} + \frac{4n}{n^3} - \frac{5}{n^3}\right)^2 =$
$\left[\lim\limits_{n\to\infty} \left(\frac{n^3}{n^3} + \frac{4}{n^3} - \frac{5}{n^3}\right)\right]^2 = (1 + 0 - 0)^2 = 1$
For $c = 1$, $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = c,\ 0 < c < \infty$. Therefore $(n^3 + 4n - 5)^2 \in \Theta(n^6)$.

□

Claim 2. $3n \lg n \in \Theta(n^2)$

The claim is false.

Proof:

A. Let f and g be functions from $\mathbb{Z}^+ \to \mathbb{R}^+$ - that is, positive real-valued functions on the domain of positive integers.
$f(n) \in \Theta(g(n)) \Leftrightarrow \lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c < \infty.$

B. Let $f(n) = 3n \lg n$ and $g(n) = n^2$.

C. I want to show that $\lim\limits_{n\to\infty} \frac{n \lg n}{n^2} = c,\ 0 < c < \infty$ is impossible.

D. $\lim\limits_{n\to\infty} \frac{n \lg n}{n^2} = \lim\limits_{n\to\infty} \frac{\lg n}{n} = \lim\limits_{n\to\infty} \frac{(\lg n)'}{(n)'} = \lim\limits_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim\limits_{n\to\infty} \frac{1}{n} = 0$
For $c = 0$, $0 < c < \infty$ is false. Therefore $3n \lg n \notin \Theta(n^2)$.

$\square$

Prove the following theorem using the defintion of the *Big*-O asymptotic notation.

Theorem 3.
Suppose d, e, f and g are functions from $\mathbb{Z}^+ \to \mathbb{R}^+$. If $d(n) \in \mathrm{O}(\mathrm{e(n)})$ and $f(n) \in \mathrm{O}(\mathrm{g(n)})$, then $d(n) + f(n) \in \mathrm{O}(\mathrm{e(n) + g(n)})$.

Proof:

A. Given $d(n) \in \mathrm{O}(\mathrm{e(n)}) \iff \mathrm{d(n) \leq c_1 e(n)\ \forall n \geq n_1}$, where $c_1 \in \mathbb{R}^+$ and $n_1 \in \mathbb{Z}^+$.

B. Given $f(n) \in \mathrm{O}(\mathrm{g(n)}) \iff \mathrm{f(n) \leq c_2 g(n)\ \forall n \geq n_2}$, where $c_2 \in \mathbb{R}^+$ and $n_2 \in \mathbb{Z}^+$.

C. I want to find $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{Z}^+$ such that $d(n) + f(n) \leq c(e(n) + g(n))$ $\forall n \geq n_0$, where $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{Z}^+$.

D. Using the additive property of inequality on the inequalities in A and B and using the intersection of the half open intervals defined in A and B, we get: $d(n) + f(n) \leq c_1 e(n) + c_2 g(n)\ \ \forall n \geq max(n_1, n_2)$.

E. Using the inequality in D, the fact that $c_1 e(n) + c_2 g(n) \leq max(c_1, c_2)\,(e(n) + g(n))$ $\forall n \geq max(n_1, n_2)$ and the transitive property of inequality, we get $d(n) + f(n) \leq max(c_1, c_2)\,(e(n) + g(n))\ \ \forall n \geq max(n_1, n_2)$.

F. For $c = max(c_1, c_2)$ and $n_o = max(n_1, n_2)$, we get $d(n) + f(n) \leq c\,(e(n) + g(n))\ \ \forall n \geq n_0 \Rightarrow d(n) + f(n) \in \mathrm{O}\,(\mathrm{e(n) + g(n)})$. Therefore, if $d(n) \in \mathrm{O}\,(\mathrm{e(n)})$ and $f(n) \in \mathrm{O}\,(\mathrm{g(n)})$, then $d(n) + f(n) \in \mathrm{O}\,(\mathrm{e(n) + g(n)})$.

$\square$