①

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + 5x_2 + x_3 &= 3 \\ 2x_1 + 6x_2 + 7x_3 &= 1 \end{aligned} \quad \leftrightsquigarrow \quad \begin{pmatrix} 1 & 2 & 1 \\ 3 & 5 & 1 \\ 2 & 6 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{array}\right) \xrightarrow[A_{13}(-2)]{A_{12}(-3)} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 5 & -1 \end{array}\right) \xrightarrow[A_{23}(2)]{A_{21}(1)} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow[A_{32}(2)]{A_{31}(3)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{M_2(-1)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \Rightarrow \boxed{x_1 = -2,\ x_2 = 2,\ x_3 = -1}$$

$\square$

② a) Use the cofactor expansion along the second column:

$$\det A = (-1)^{2+1} \cdot (-2) \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{vmatrix} = 4$$

b) Because $\det A = 4 \neq 0$, A is invertible.

c)
$$\left(\begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{ero} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & 1/2 & 3/2 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 1/2 & 1/2 & \frac{1}{2} \end{array}\right).$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 0 & \frac{3}{2} & \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & 1/2 & 3/2 & \frac{1}{2} \\ 0 & 1/2 & 1/2 & \frac{1}{2} \end{pmatrix}$$

and $AA^{-1} = I_4$.

$\square$

(3)

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(2-\lambda) + (1-\lambda) = (1-\lambda)\left[(1-\lambda)(2-\lambda)+1\right]$$

$$= (1-\lambda)(\lambda^2-3\lambda+3).$$

$$p(\lambda)=0 \Rightarrow (1-\lambda)(\lambda^2-3\lambda+3)=0 \Rightarrow \boxed{\lambda = 1,\ \frac{3\pm\sqrt{3}\,i}{2}.}$$

□

(5) The Wronskian of $\{\overset{f_1}{\overset{\shortparallel}{x}}, \overset{f_2}{\overset{\shortparallel}{\sin x}}, \overset{f_3}{\overset{\shortparallel}{\cos x}}\}$ is

$$W[f_1,f_2,f_3](x) = \begin{vmatrix} x & \sin x & \cos x \\ 1 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix} = -x\cos^2 x - \cancel{\sin x\cos x} + \cancel{\sin x\cos x} - x\sin^2 x$$

$$= -x(\sin^2 x+\cos^2 x) = -x.$$

Since $W[f_1,f_2,f_3](x) = -x \neq 0$ (except $x=0$), the functions $\{x, \sin x, \cos x\}$ are linearly independent.

(6) Use the Corollary 4.5.17

$$\det\begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 1 & 1 & 3 \end{pmatrix} = 3-2-1=0 \Rightarrow \left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}\right\} \text{ is linearly dependent.}$$

$$c_1\underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{\vec{v}_1} + c_2\underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{\vec{v}_2} + c_3\underbrace{\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}}_{\vec{v}_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives solutions: $c_1=-t$, $c_2=-2t$, $c_3=t$, $t\in\mathbb{R}$. Take $t=1$, we have $\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$. Also, $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent. Thus, the dimension of $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is 2.

□

(7) First, check the linear independence : Set

$$c_1(1+x) + c_2(2+x+x^2) + c_3(x-x^2) = 0 + 0x + 0x^2$$

$$\Rightarrow (c_1+2c_2) + (c_1+c_2+c_3)x + (c_2-c_3)x^2 = 0 + 0x + 0x^2$$

$$\Rightarrow \begin{cases} c_1+2c_2=0 \\ c_1+c_2+c_3=0 \\ c_2-c_3=0 \end{cases} \Rightarrow \begin{cases} c_1+2c_2=0 \\ c_2=c_3. \end{cases}$$

Take $c_1=-2$, $c_2=1$, $c_3=1$, then

$$-2(1+x) + (2+x+x^2) + (x-x^2) = 0+0x+0x^2 = 0.$$

That is, $\{1+x, 2+x+x^2, x-x^2\}$ is NOT linearly independent. Thus, it is NOT a basis for $\mathbb{P}_2(\mathbb{R})$.

(4)
$$\left[\begin{array}{ccc:c} -1 & 1 & 2 & 0 \\ 3 & -2 & 1 & 1 \\ 2 & 1 & 1 & -1 \end{array}\right] \xrightarrow{ero} \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & -3/4 \\ 0 & 0 & 1 & 1/4 \end{array}\right]$$

$$\therefore \text{ rank of } \begin{bmatrix} -1 & 1 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = 3$$

$$\Rightarrow \vec{v_1}, \vec{v_2}, \vec{v_3} \text{ span } \mathbb{R}^3$$

and $(0,1,-1) = -\frac{1}{4}\vec{v_1} - \frac{3}{4}\vec{v_2} + \frac{1}{4}\vec{v_3}$