PARTIAL ORDERS

Partial Order.

- A relation R on a set X is partial order if it is reflexive, anti-symmetric, and transitive.
- We refer to these properties together, in short, as RAT-properties.

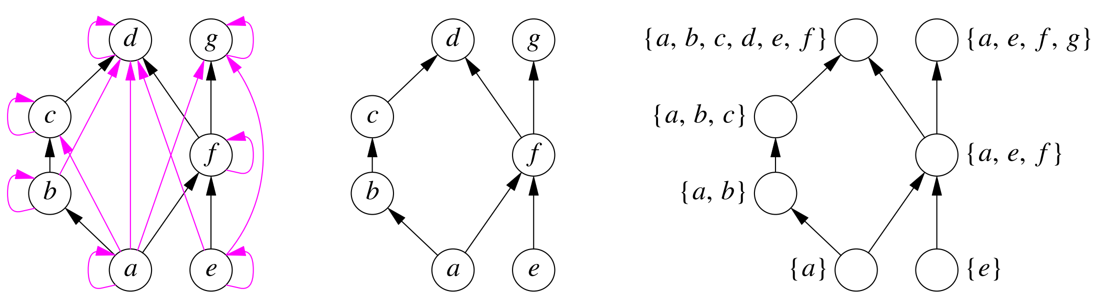
Anti-symmetry means (the digraph of) R has no cycle of length 2; this together with transitivity means R has no cycles. (Note: we do not consider loops to be cycles.)

Example.

- Shown in (i) below is the digraph of a partial order R on $X = \{a, b, c, d, e, f, g\}$.

  Here, the loops and the transitive-links, i.e., the links that can be obtained from the other links via transitivity are shown in color.
- Shown in (ii) is a simplified form it, without the loops and the transitive-links.

  Such a simplified form of the digraph of a partial order is called its Hasse-diagram.
- Shown in (iii) is the Hasse-diagram of the partial order given by $\subseteq$-relation on the sets $N^{-}(x) = \{y\colon (y, x) \in R\}$, where $x \in N^{-}(x)$ and $|N^{-}(x)| = \text{indegree}(x)$ in digraph of R.



(i) The digraph of a partial order R.

(ii) Its Hasse-diagram.

(iii) The same Hasse-diagram (partial order) for $\subseteq$-relation on the subsets $N^{-}(x)$, $x \in X$.

- The transitive closure of the Hasse-diagram in (iii) equals the original R minus the loops. It is the strict partial order corresponding to R and is anti-reflexive, anti-symmetric, and transitive.
- All partial orders can be viewed as $\subseteq$-relation on some set of subsets. See (iii).

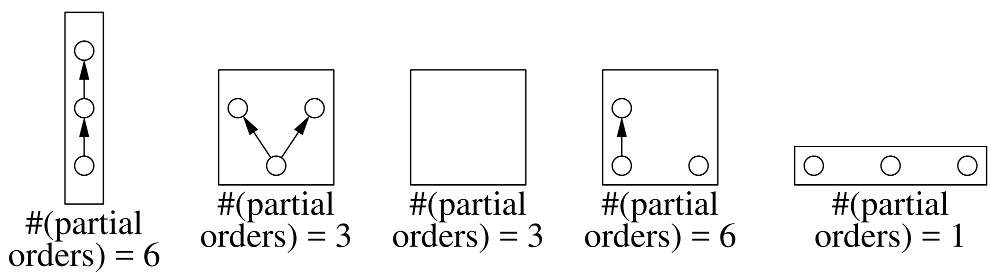
  All strict partial orders can be viewed as $\subset$-relation (proper subset) on some set of subsets if we define $N^{-}(x) = \{x\} \cup \{y\colon (y, x) \in R\}$ so that $x \in N^{-}(x)$ as before.

Maximal and Minimal Items.

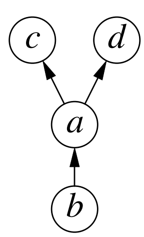
- An item x is maximal if there is no $y \neq x$ such that $(x, y) \in R$. A minimal item is defined similarly. (The partial order above has two maximal items d and g.)

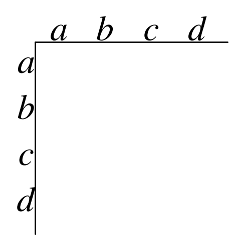
Practice Questions (Q5 modified, Apr 14, 2020)

1. Consider the "divide" relation on $X = \{1, 2, 3, 4, 5, 6\}$. Argue that it is a partial order and show its Hasse-diagram, the related sets $N^-(x)$ for each $x \in X$, and the minimal and maximal items. Show the effect on removing the item 1 from X on the divide-relation, its Hasse-diagram, the sets $N^-(x)$, and the minimal and maximal items. Do the same if we remove item 2 from X instead of 1. (Note: the "properly divide" (i.e., divide but not equal) relation is the strict partial order related to the "divide" relation.)
2. Give the digraph of a relation for each of the three cases where the relation has exactly two of the RAT-properties; keep #(nodes) ≥ 2 and as small as possible and also #(links) as small as possible. Indicate the missing property for each digraph.
3. Shown below are all but one structure of the Hasse-diagrams of partial orders on $n = 3$ items; they are grouped by #(links). Show the missing structure.



4. Consider the partial order R on $\{a, b, c, d\}$ whose Hasse-diagram is shown below.





   (i) Show $N^-(x)$ for the partial order R next to each node x.

   (ii) Show on the rightside above the relation-matrix of the partial order R.

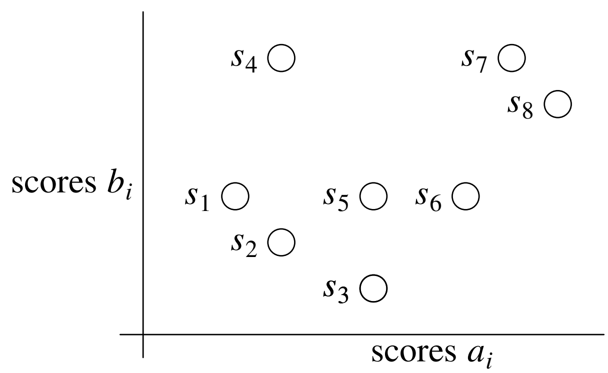
5. (Modified.) If we think of the Hasse-diagram (of a partial order R) itself as the digraph of a relation R', then what kind of relation is R'?
6. How the sets $N^-(x)$ for a partial order R are related to paths in the Hasse-diagram of R?
7. Complete the code below to compute $N^-(x_i)$ for a given item x_i and an $n \times n$ relation-matrix R. We represent $N^-(x_i)$ here by a binary array `neighborsTo` of 0/1.

```
int[] neighborsTo = new int[n]; //binary array for N⁻(xᵢ)
for (int j = 0; j < n; j++)
    if (1 == ...........) neighborsTo[j] = ... ;
```

8. Argue that for each partial order R (on a set X) the transitive closure of the Hasse-diagram of R gives back R except for the loops.

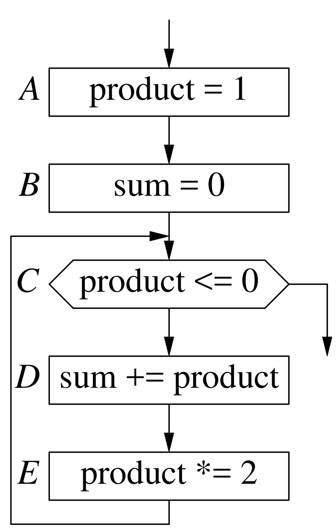
9. If G is the digraph of a partial order R, does the reverse digraph $r(G)$ correspond to a partial order? If so, how is the Hasse-diagram of the latter related to that of R? Verify your answer using the relation whose Hasse-diagram is shown in Problem 4.

10. Consider students s_i, $1 \le i \le 8$, and the pairs of scores (a_i, b_i) shown below for each student s_i, where a_i and b_i are the score of s_i in an Algebra-test and in a Biology-test. We say $s_i \le s_j$ if $a_i \le a_j$ and $b_i \le b_j$.
    (i) Which of the RAT-properties hold for the relation "≤" among s_i's?
    (ii) If "≤" is a partial order, then show the Hasse diagram. Who are the "best" performers (maximal items) and who are the "worst" performers (minimal items)?



11. Shown below is a code (that computes the sum of all powers of 2, starting with $2^0 = 1$, up to the largest power of $2 \le n$, for an input integer $n \ge 1$), and its flowchart. We define the "immediately precedes" relation (written, in short, as IP-relation) on the assignment and test operations O_i in the code/flowchart as follows: (O_i, O_j) is in IP-relation if there is directed link from O_i to O_j in the flowchart, i.e., O_i immediately precedes O_j. The matrix of IP-relation for the flowchart is shown on the right below.

```
int product = 1,
    sum = 0; //keep 0
while (product <= n)
{     sum += product;
      product *= 2;
}
```



	A	B	C	D	E
A	0	1	0	0	0
B	0	0	1	0	0
C	0	0	0	1	0
D	0	0	0	0	1
E	0	0	1	0	0

(a) What are the properties (reflexive, anti-reflexive, etc) of the IP-relation above?

(b) When can an assignment/test operation in a code immediately precede ≥ 2 items?

(c) What kind of code's IP-relation has a transitive-link (explain with example)?

(d) In what way the transitive-closure IP^+ of the IP-relation of a code/flowchart is important in understanding the code?

(e) For what kind of code, the transitive-closure IP^+ will be a strict partial order? When will the Hasse-diagram of that strict partial order equal the code's flowchart?

LINEAR ORDERS

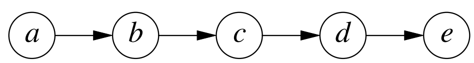
Linear Order.

- A partial-order on a set X is a linear order if for every $x \neq y$ in X, (exactly) one of (x, y) and (y, x) is in R.

  This is the same as saying that there are no incomparable items in the partial order R, where two items $x \neq y$ are called incomparable if neither of (x, y) and (y, x) is in R.
- A strict linear order is a strict partial order (anti-reflexive, anti-symmetric, and transitive) in which there are no incomparable items.

Example.

- Shown below is the Hasse-diagram of a linear order on $X = \{a, b, c, d, e\}$. It is a path connecting all the items. Here, the items may be numbers $a < b < \cdots < e$.



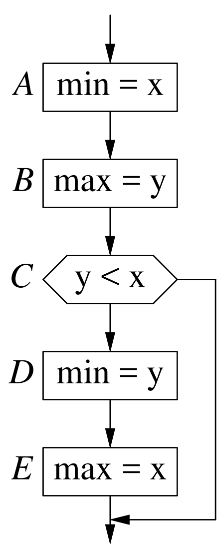
- There are $5! = 120$ possible linear orders on X.

Example.

- Shown below on the left below is a code ending in an if-statement. Also shown are its flowchart, the matrix of its IP-relation on the assignment and test operations, and the digraph of the IP-relation (which is anti-reflexive, anti-symmetric, and anti-transitive).

```
int min = x,
    max = y;
if (y < x)
{  min = y;
   max = x;
}
```

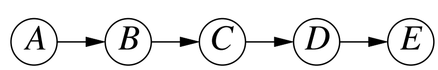
(i) A code.



(ii) Its flowchart.

	A	B	C	D	E
A	0	1	0	0	0
B	0	0	1	0	0
C	0	0	0	1	0
D	0	0	0	0	1
E	0	0	0	0	0

(iii) IP-relation of code/flowchart.



(iv) digraph of IP-relation.

- In this case, the transitive closure IP^+ of the IP-relation is a strict linear order.

Practice Questions.

1. Show the flowchart of the code below (which is a variation of the code shown in the previous page) and the matrix of its associated *IP*-relation. Is this *IP*-relation a partial order or linear order? How about the transitive closure IP^+? Is it a strict partial order or a strict linear order (recall that "strict" means replacing the reflexive-property by the anti-reflexive property)?

```
int min = x;
if (y < x)
   min = y;
int max = y;
if (y < x)
   max = x;
```

2. Argue that there is a unique maximal item in a linear order. (We call it the maximum item.) Likewise, argue that there is a unique minimal item in a linear order. (We call it the minimum item.)
3. Find out $|N^-(x)|$ for each item x in the transitive closure IP^+ of the *IP*-relation in Problem 1. (Note that a minimal item x in a partial-order has $|N^-(x)| = |\{x\}| = 1$. For a strict partial order R, we define $N^-(x) = \{x\} \cup \{y: (y, x) \in R\}$ so that $x \in N^-(x)$ as in the case of a partial order, and once again a minimal item x has $N^-(x) = \{x\}$ and hence $|N^-(x)| = 1$. In particular, for a linear order, its maximum item x has $N^-(x) = X$ and $|N^-(x)| = |X|$; the same is true for a strict linear order.)
4. What is a special property of the numbers $|N^-(x)|$ for a linear order, in addition to those indicated in Problem 3?
5. How can use the property in Problem 4 to test whether a partial order is a linear order or not?