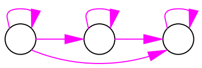
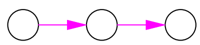
Solution Long Quiz #4.1 (23-Apr): CSC-2259: Discrete Structures, Sp 2020
Your answers must be to the point. Total = 50; marks for each question is shown in [ ].

LastName: FirstName

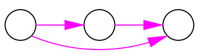
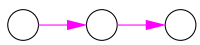
1. Answer the following questions on linear orders on a set X.

   (a) State the additional condition that a partial order R must satisfy in order to be a linear order. [2]
   For each $x \neq y$, exactly one of (x, y) and (y, x) is in R.

   (b) For $|X| = 3$, show the structure of a linear order, that of its Hasse-diagram, and #(linear orders on X). [3+3+3]

   
   
   #(linear orders on X) = 6.

   (c) Repeat (b) for a strict linear order. [1+1+1]

   
   
   #(strict linear orders on X) = 6.

   (d) Give a susbet $X = \{x_1, x_2, x_3, x_4\} \subseteq \{1, 2, \cdots, 10\}$ such that the "divide"-relation on X is a linear order. [3]
   $\{1, 2, 4, 8\}$

   (e) What is the maximum #(links, including loops) for a partial order on X, when $|X| = n$? Is it true that a partial order is a linear order if and only if #(links, including loops) is maximum? How can we use it to determine from the relation-matrix R of a partial-order whether it is a linear order or not? [3+2+3]
   Max. #(links) = $n(n+1)/2$. Yes. We count #(ones in R) and check whether this equals $n(n+1)/2$ or not.

   (f) **BONUS.** Let R be the relation-matrix of a linear order on X, $|X| = n$. What is wrong with the argument below. [3]
   For every $0 \leq i < j < n$, we have $R[i][j] + R[j][i] = 1$, i.e., $(R[i][j], R[j][i]) = (0, 1)$ or $(1, 0)$, i.e., 2 choices for each of $n(n-1)/2$ pairs $(R[i][j], R[j][i])$. Also, each $R[i][i]$ is 1. Thus, there are $2^{n(n-1)/2}$ many linear orders.
   The count $2^{n(n-1)/2}$ includes all linear orders and also some reflexive, anti-symmetric, and non-transitive relations when $|X| = n \geq 3$. (For $n = 3$, there are 2 such relations, both of which include a cycle of length 3).

2. Answer the following questions on probability.

   (a) Show the sample space S of the experiment "three tosses of a coin". [3]
   $\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

   (b) Show the subset of S in (a) for the event E_1 = "at least 2 heads". [3]
   $\{HHH, HHT, HTH, THH\}$.

   (c) If Prob(H) = 2/3 and the tosses in the experiment in (a) are independent, then what is Prob(HHT)? [2]
   4/27

   (d) Give Prob(E_1) in (b) based on Prob(H) = 2/3; show details. [3]
   8/27 + 4/27 + 4/27 + 4/27 = 20/27.

   (e) State the sum-rule for probabilities. [3]
   Propb($E_1 \cup E_2$) = Prob(E_1) + Prob(E_2) − Prob($E_1 \cap E_2$).

   (f) State the complement event of E_1 in (b) in Englilsh without using "not". Also, verify the complement-rule using the events E_1 and E_1^c based on Prob(H) = 2/3. (Show details of computing Prob(E_1^c).) [1+3]
   E_1^c = at most 1 head, Prob(E_1^c) = $3.(2/3).(1/3)^2 + (1/3)^3 = 7/27 = 1 - 20/27$.

   (g) Consider the spinning wheel discussed in the class with Prob(3) = 1/3 = Prob(8) and Prob(4) = 1/6 = Prob(5) when we turn the wheel once. Now consider the outcomes (s_1, s_2, s_3) in the experiment of three independent turns of the wheel.

   (g.1) Show the sample points (s_1, s_2, s_3) for the event $E_{1,2}$: "$s_1 = 3 = s_2$". [2]
   {(3,3,3), (3,3,4), (3,3,5), (3,3,8)}

   (g.2) Show the details in computing Prob($E_{1,2}$). [2] $2.(1/3)^3 + 2.(1/3)^2(1/6) = 6/54 = 1/9$

   (g.3) Consider the event $E_{1,2} \cup E_{2,3}$: "$s_1 = 3 = s_2$ or $s_2 = 3 = s_3$". Compute Prob($E_{1,2} \cup E_{2,3}$) using sum-rule. [3]
   Prob($E_{1,2}$) + Prob($E_{2,3}$) − Prob($E_{1,2} \cap E_{2,3}$) = 2/9 − 1/27 = 5/27