ANTI-SYMMETRIC RELATIONS

Anti-symmetric Relation R.

- For each $x \neq y$, at most one of (x, y) and (y, x) is in R.

Notes.

- For a relation R on $X = \{a, b, c, d\}$ to be anti-symmetric, there are 6 conditions corresponding to $C(4, 2) = 6$ node pairs $\{x, y\}$,

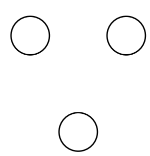
  At most one of (a, b) and (b, a) in R
  At most one of (a, c) and (c, a) in R
  ... ...
  At most one of (c, d) and (d, c) in R

- The presence or absence of a loop (i.e., a link of the form (x, x)) has no effect on the anti-symmetry property.

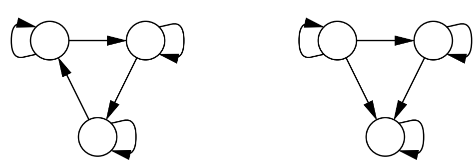
Example.

- Shown below are the structures of the extreme cases (i.e., having the minimum or maximum #(links)) of anti-symmetric relations on $X = \{a, b, c\}$.

  In (i), there are no links. In (ii), there is exactly one of the links (x, y) and (y, x) for $x \neq y$, in addition to all links (x, x).



(i) The structure of a smallest anti-symmetric relation.

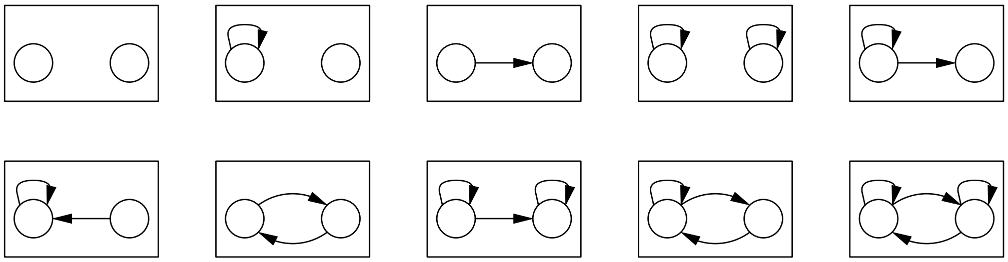


(ii) The two structures of a largest anti-symmetric relation.

- The two structures in (ii) are different because the one has a 3-cycle and the other does not.

Practice Questions.

1. Shown below are the structures of relations when $|X| = 2$. Mark the ones by "A" that correspond to anti-symmetric relations.

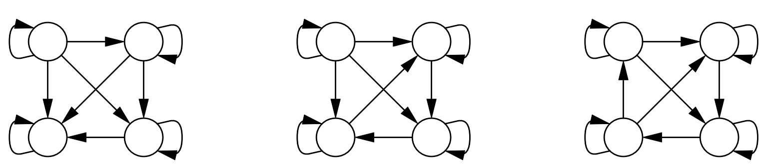


2. Show the structures of anti-symmetric relations without loops when $|X| = 3$ (loops are avoided to keep the number of structures small) other than the ones shown in the previous page. There will be at least 5 such structures; show the structures according to #links = 1, 2, and 3, in that order.

   Now focus on the one with #(links) = 1. Show that we get 3 additional structures if we allow 1 loop, 3 additional structures if we allow 2 loops, and 1 additional structure if we allow 3 loops. Draw the structures.

   Now repeat the process of allowing loops for all other structures you obtained above with #(links) $\geq$ 2 and no loops.

3. Give #(relations) on $X = \{a, b, c\}$ for each of the structures of anti-symmetric relations with minimum and maximum number of links (see the previous page) and then draw the associated digraphs for all those relations.
4. Argue that the maximum #(links) in an anti-symmetric relation on n items is $n(n+1)/2$.
5. Also, argue that there are are $2^{n(n-1)/2}$ anti-symmetric relations on n items with maximum #(links).
6. Shown below are three different structures of the largest anti-symmetric relations on 4 items; each of them has $n(n+1)/2 = 10$ links. The structure in (i) has no cycle, in (ii) the only cycle is a 3-cycle, and in (iii) we have one 4-cycle (and also two 3-cycles).



(i) (ii) (iii)

   Show another structure for the largest anti-symmetric relations on 4 items. Explain what makes the new structure different from the ones shown above. Also, argue that there is no other structure of the largest anti-symmetric relations on 4 items.

COUNTING ANTI-SYMMETRIC RELATIONS

Matrix of anti-symmetric relations on $X = \{a, b, c\}$.

- The diagonal items r_{ii}, $1 \le i \le 3$, can be chosen 0 or 1 arbitrarily.
- The upper diagonal items r_{ij}, $i < j$, can also be chosen 0 or 1 arbitrarily,
- Each lower diagonal item r_{ij}, $i > j$, is constrained by the corresponding upper diagonal item r_{ji} as shown below.

	a	b	c
a	r_{11}	r_{12}	r_{13}
b	$r_{21} + r_{12} \le 1$	r_{22}	r_{23}
c	$r_{13} + r_{31} \le 1$	$r_{23} + r_{32} \le 1$	r_{33}

#(Anti-symmetric relations for $|X| = n$).

- Two choices $r_{ii} = 0$ or 1 for each of n diagonal items.
- For each pair (r_{ij}, r_{ji}), $i \ne j$, three choices: (0, 0), (0, 1), or (1, 0).

  There are $(n-1) + (n-2) + \cdots + 1 = n(n-1)/2$ such pairs.

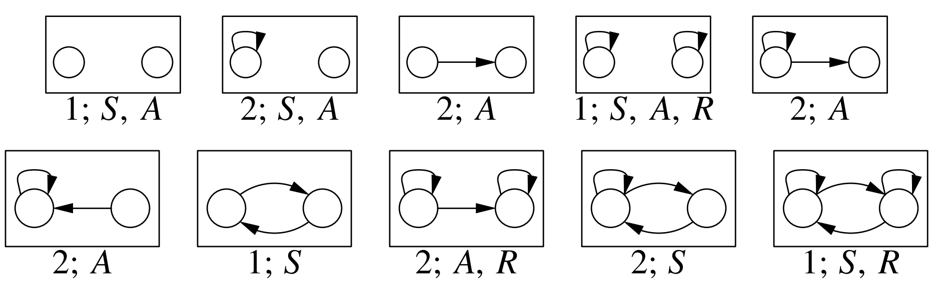
Thus, #(Anti-symmetric relations on n items) $= 2^n 3^{n(n-1)/2}$.

More Anti-symmetric Relations Than Symmetric Relations.

- Recall that #(Symmetric relations on n items) $= 2^n 2^{n(n-1)/2} = 2^{n(n+1)/2}$.
- Thus, there are many more anti-symmetric relations than symmetric relations.

Case of $X = \{a, b\}$ and $n = 2$.

- Shown below are the structures of these relations and #(relations) for each structure.
- Each structure is labeled with one or more of R (reflexive), S (Symmetric), and A (anti-symmetric) if the relation(s) with that structure have those properties.



- #(Anti-symmetric relations) = 12 > 8 = #(Symmetric relations).

Practice Questions.

1. Counting problems for combination of symmetric and anti-symmetric properties.
   (a) Give a detailed argument to show that #(Symmetric and anti-symmetric relations on n items) = 2^n. Verify the formula for $n = 2$ (by counting the appropriate number of relations in the notes).
   (b) Give a detailed argument to show that #(Symmetric but not anti-symmetric relations on n items) = $2^{n(n+1)/2} - 2^n$. Verify the formula for $n = 2$ (by counting the appropriate number of relations in the notes).
   (c) Give a detailed argument to show that #(Symmetric or anti-symmetric relations on n items) = $2^{(n(n+1)/2} + 2^n 3^{(n^2-n)/2} - 2^n$. Verify the formula for $n = 2$ (by counting the appropriate number of relations in the notes).
2. Give the structure (in digraph form) of a largest relation (having most number of links) on 3 items which is non-reflexive, non-symmetric, and non-anti-symmetric. Argue that there are 3×6 = 18 such relations on $X = \{a, b, c\}$. What would be the number of such relations be on a set X of size $n \geq 3$?