MATHEMATICS 2090 Section 1
Exam II

Print name ______________________________

Last name First name

You must show your work in order to get full credit.

No.	Marks
1	
2	
3	
4	
5	
Total	

1. a) Use Gaussian elimination to determine the solution set to the system of linear equations: 9pt

$$\begin{array}{rrrrcl} 2x_1 & +2x_2 & +x_3 & & = & 3 \\ 2x_1 & +x_2 & & -x_4 & = & 2 \\ 3x_1 & +3x_2 & +x_3 & -x_4 & = & 5 \end{array}$$

b) Does the solution set above form a subspace of $\mathbb{R}^4$? If not, please explain why. 3pt

a) $$\left[\begin{array}{cccc|c} 2 & 2 & 1 & 0 & 3 \\ 2 & 1 & 0 & -1 & 2 \\ 3 & 3 & 1 & -1 & 5 \end{array}\right] \xrightarrow[A_{13}(-1)]{A_{12}(-1)} \left[\begin{array}{cccc|c} 2 & 2 & 1 & 0 & 3 \\ 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{A_{31}(-2)} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 2 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 2 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{A_{21}(1)} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{M_2(-1)} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{A_{32}(-1)} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{A_{31}(1)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

Let $x_4 = t$. $x_1 = 0$, $x_2 - t = 2$

So $x_2 = 2+t$, $x_3 + 2t = -1 \Rightarrow x_3 = -1-2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2+t \\ -1-2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

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b) The solution set $S = \left\{ \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} : t \in \mathbb{R} \right\}$ is

not a subspace of $\mathbb{R}^4$.

because it is not closed under scalar multiplication

$$2 \left(\begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 4 \\ -2 \\ 0 \end{bmatrix} + 2t \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

which does not belong to S.

2. a) Compute the determinant of $A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. 4pt

b) Is A invertible? If no, please explain why; if yes, please compute its inverse A^{-1}. 6pt

a) $\det A = 2 \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + 0 + \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} = 2(-1) + 1 = -1$

b) A is invertible because $\det A \neq 0$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 3 & -1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[A_{13}(-2)]{A_{12}(-3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & -3 & 0 & 1 & -3 \\ 0 & 0 & -1 & 1 & 0 & -2 \end{array}\right]$$

$$\xrightarrow{M_3(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & -3 & 0 & 1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right] \xrightarrow[A_{31}(-1)]{A_{32}(3)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -3 & 1 & 3 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{M_2(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 3 & -1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & -1 & -3 \\ -1 & 0 & 2 \end{bmatrix}$$

3. Compute the Wronskian of the functions $1, \cos x, \sin x$ defined over $\mathbb{R}$. Are they linearly dependent or independent? Please justify. 7pt

$$W(1, \cos x, \sin x) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - \cos x \begin{vmatrix} 0 & \cos x \\ 0 & -\sin x \end{vmatrix} + \sin x \begin{vmatrix} 0 & -\sin x \\ 0 & -\cos x \end{vmatrix}$$

$$= \sin^2 x + \cos^2 x - 0 + 0$$

$$= 1.$$

4. a) Show that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ span $\mathbb{R}^3$. 5pt

b) Express $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. 5pt

a) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \xrightarrow[A_{13}(1)]{A_{12}(-1)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix} \xrightarrow{A_{23}(1)} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[A_{32}(2)]{A_{31}(-2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has rank 3 then V_1, V_2, V_3 span $\mathbb{R}^3$.

b) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

5. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation 11 pt

$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 + 3x_3,\ x_1 + 2x_2 + 3x_3,\ x_1 + x_2 + x_3,\ x_1 - x_3).$$

a) Determine the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.
b) Find a basis for the kernel, $\mathrm{Ker}(T)$ of T, and determine its dimension.
c) Find a basis for the range, $\mathrm{Rng}(T)$ of T, and determine its dimension.

a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 \\ 3 & 3 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) Let $x_4 = t$, $x_2 = s$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 + x_4 \\ x_2 \\ -2x_4 \\ t \end{bmatrix} = \begin{bmatrix} -s + t \\ s \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

$\therefore \{(-1,1,0,0), (1,0,-2,1)\}$ is a basis of Ker T and dim Ker T = 2.

Range T is spanned by $(1,2,3)$ and $(1,1,1)$

$\therefore$ ~~dim Ker~~ dim Im T = 2.