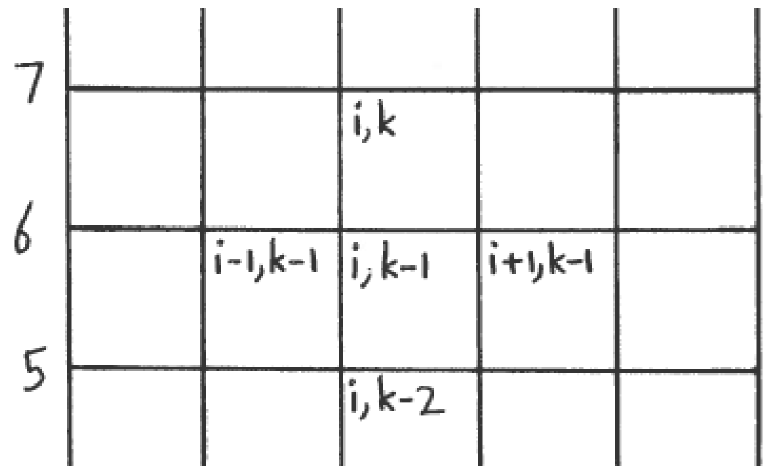
Derivation of Equation for $u_{i,k}$ in Sample 12

One-Dimensional Wave Equation for u(x,t)

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2} + f(x,t)$$

Before writing the function wave1, obtain the equation to be used in the explicit scheme by doing the following:

1) Approximate the second order partial derivatives in the One-Dimensional Wave Equation by the 3-point second order central difference formula, using the point with indices i,k-1 as the central point:



$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2} + f(x,t)$$

$$\frac{u_{i,k-2} - 2u_{i,k-1} + u_{i,k}}{h_t^2} = a\frac{u_{i-1,k-1} - 2u_{i,k-1} + u_{i+1,k-1}}{h_x^2} + f_{i,k-1}$$

where $u_{i,k} = u(i,k)$, $f_{i,k-1} = f(x_i, t_{k-1})$, and h_x and h_t are the stepsizes in the x and t intervals.

2) Solve the equation for $u_{i,k}$:

$$u_{i,k} = \frac{a h_t^2}{h_x^2}\left(u_{i-1,k-1} - 2u_{i,k-1} + u_{i+1,k-1}\right) + h_t^2 f_{i,k-1} + 2u_{i,k-1} - u_{i,k-2}$$

NOTE: This equation can be used only for $k \geq 3$. <u>It cannot be used for $k = 2$</u> because when $k = 2$, the term $u_{i,k-2}$ becomes $u_{i,0}$ which is undefined because the second index cannot be less than 1 (1 corresponds to t=0; if the second index were less than 1, it would correspond to a negative time).

3) In order to obtain an equation that can be used when k = 2, do the following:

3a) The velocity v is the first derivative of u with respect to time:

$$v(x,t) = \frac{\partial u}{\partial t}$$

3b) Approximate the first order partial derivative in the above equation by the 2-point backward difference formula, using the point with indices i,k-1 as the central point:

$$v_{i,k-1} = \frac{u_{i,k-1} - u_{i,k-2}}{h_t}$$

where $v_{i,k-1} = v(x_i, t_{k-1})$.

3c) Solve this equation for $u_{i,k-2}$:

$$u_{i,k-2} = u_{i,k-1} - h_t v_{i,k-1}$$

3d) Substitute this equation for $u_{i,k-2}$ into the equation in step 2 above:

$$u_{i,k} = \frac{a h_t^2}{h_x^2}\left(u_{i-1,k-1} - 2u_{i,k-1} + u_{i+1,k-1}\right) + h_t^2 f_{i,k-1} + u_{i,k-1} + h_t v_{i,k-1}$$

<u>Use the above equation for k = 2</u> in which case $v_{i,k-1}$ becomes $v_{i,1}$ which is given by the initial velocity v0 since $t_1 = 0$:

$v_{i,1} = v(x_i, t_1) = v(x_i, 0) = v0(x_i)$.