1.3 Laws of Limits

Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, where L and M are real numbers. Then the following apply:

1. $\lim_{x \to c} (f(x) + g(x)) = L + M$
2. $\lim_{x \to c} (f(x) - g(x)) = L - M$
3. $\lim_{x \to c} (kf(x)) = kL$
4. $\lim_{x \to c} (f(x) \bullet g(x)) = LM$
5. $\lim_{x \to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M},\ M \neq 0$
6. $\lim_{x \to c} (f(x))^n = L^n,\ n > 0$
7. $\lim_{x \to c} \left(\sqrt[n]{f(x)}\right) = L^{\frac{1}{n}},\ n > 0$

$O(1)$	Constant C
$O(\log n)$	Logarithmi
$O(n)$	Linear Con
$O(n \log n)$	log-linear C
$O(n^b)$	Polynomial
$O(b^n)$, where $b > 1$	Exponentia
$O(n!)$	Factorial C

reduce and conquer vs divide and conquer

in-place, stability, order-optimal
(secondary space) (keys are same) (best algorithm of type)

Begin with Let f and g be functions from $Z^+ \to R^+$

$O(n)$, $\Omega(n)$, $\Theta(n)$ Definition, Limit

- $f(n) \in O(g(n))$ iff $\exists\ c \in R^+$ and $\exists\ n_0 \in Z^+ \ni f(n) \le cg(n),\ \forall n \ge n_0$
- $f(n) \in \Omega(g(n))$ iff $\exists\ c \in R^+$ and $\exists\ n_0 \in Z^+ \ni f(n) \ge cg(n),\ \forall n \ge n_0$
- $f(n) \in \Theta(g(n))$ iff $\exists\ c \in R^+$ and $\exists\ n_0 \in Z^+ \ni c_1 g(n) \le f(n) \le c_2 g(n),\ \forall n \ge n_0$
- $f(n) \in O(g(n)) \leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = K,\ K \in [0, \infty)$
- $f(n) \in \Omega(g(n)) \leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = K,\ K \in (0, \infty]$
- $f(n) \in \Theta(g(n)) \leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = K,\ K \neq 0, \infty$

$o(n)$, $\omega(n)$ Definition, Limit

- $f(n) \in o(g(n))$ iff $\forall c \in R^+\ \exists\ n_0 \in Z^+ \ni f(n) < cg(n),\ \forall n \ge n_0$
- $f(n) \in \omega(g(n))$ iff $\forall c \in R^+\ \exists\ n_0 \in Z^+ \ni f(n) > cg(n),\ \forall n \ge n_0$
- $f(n) \in o(g(n)) \leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$
- $f(n) \in \omega(g(n)) \leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$

- For the growth rate function of an algorithm:
  - You can ignore lower terms: $O(n^3 + 4n^2 + 3n) = O(n^3)$
  - You can ignore coefficients: $\Theta(5n^3) = \Theta(n^3)$
  - They are additive: $\Omega(n^2) + \Omega(n) = \Omega(n^2 + n) = \Omega(n^2)$

Selection Sort Visual Presentation

$(n-1) + (n-2) + \dots + 2 + 1 = \frac{n(n-1)}{2} \in O(n^2)$

Insertion Sort

Binary Search

$$T(n) = 1 + T\left(\frac{n-1}{2}\right)$$
$$T(n) = 2 + T\left(\frac{n-3}{4}\right)$$
$$T(n) = 3 + T\left(\frac{n-7}{8}\right)$$
...
$$T(n) = k + T\left(\frac{n-2^k+1}{2^k}\right)$$

Algorithm	Worst case	Average ca
Selection sort	n^2	n^2
Bubble sort	n^2	n^2
Insertion sort	n^2	n^2
Merge sort	$n \lg n$	$n \lg n$
Quick sort	n^2	$n \lg n$
Radix sort	n	n
Treesort	n^2	$n \lg n$
Heapsort	$n \log n$	$n \lg n$

Figure 2: A Quick Sort of an Array of Eight Integers

3.1 Quick Sort

```
void sort(Item[] array, int from, int to)
{
   if (from >= to) {return; }
   p <- partition(array, from, to)
   sort(array, from,p);
   sort(array, p+1,to);
}
```

best: T(n)=2T(n/2)+alpha*n
worst: n+(n-1)+(n-2)+...+2
or T(n) = T(n-1) + alpha*n, T(1)=1

T(n)=2T(n/2)+alpha*n

Claim 1. $(n^3+4n-5)^2 \in \Theta(n^6)$

The claim is true.

A proof using the definition of Θ-notation.

Proof:

A. **Definition 1.** Let f and g be functions from $\mathbf{Z}^+ \to \mathbf{R}^+$ - that is, positive real-valued functions on the domain of positive integers. If $f(n) \in \Theta(g(n))$, then $g(n)$ is said to be an asymptotic tight bound for $f(n)$. Mathematically, there are constants $c_1 > 0$, $c_2 > 0$ and an integer constant $n_0 \geq 1$ such that $c_1 g(n) \leq f(n) \leq c_2 g(n)\ \forall n \geq n_o$.

B. Let $f(n) = (n^3+4n-5)^2$ and $g(n) = n^6$.

C. I want to find c_1, c_2 and n_o such that $c_1 n^6 < (n^3+4n-5)^2 < c_2 n^6$ where $n > n_o$.

D. Proving the first half of the inequality:

(a) $n^3 \leq n^3$, $n \geq 1$

(b) $0 < 4n-5$, $n > 2$

(c) Using the additive property of inequality and combining D.(a) and D.(b) and using the intersection of the half open intervals, we get $n^3 \leq n^3+4n-5$, $n \geq 2$.

(d) Squaring both sides of the inequality in D(c), we get $n^6 < (n^3+4n-5)^2$, $n \geq 2$

E. Proving the second half of the inequality:

(a) $n^3 < n^3$, $n > 1$

(b) $4n-5 \leq 4n^3$, $n \geq 1$

(c) Using the additive property of inequality and combining E.(a) and E.(b) and using the intersection of the half open intervals, we get $n^3+4n-5 \leq 5n^3$, $n \geq 1$.

(d) Squaring both sides of the inequality in E(c), we get $(n^3+4n-5)^2 < 25n^6$, $n > 1$

F. Combining D.(d) and E.(d) and using the intersection of the half open intervals, we get $n^6 < (n^3+4n-5)^2 < 25n^6$, $n > 2$. For $c_1 = 1$, $c_2 = 25$ and $n_o = 2$, we get $c_1 g(n) < f(n) < c_2 g(n), n > n_o$. Therefore $(n^3+4n-5)^2 \in \Theta(n^6)$.

Claim 2. $3n \lg n \in \Theta(n^2)$

The claim is false.

Proof:

A. Let f and g be functions from $\mathbf{Z}^+ \to \mathbf{R}^+$ - that is, positive real-valued functions on the domain of positive integers.
$f(n) \in \Theta(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c < \infty$.

B. Let $f(n) = 3n \lg n$ and $g(n) = n^2$.

C. I want to show that $\lim_{n\to\infty} \frac{n \lg n}{n^2} = c$, $0 < c < \infty$ is impossible.

D. $\lim_{n\to\infty} \frac{n \lg n}{n^2} = \lim_{n\to\infty} \frac{\lg n}{n} = \lim_{n\to\infty} \frac{(\lg n)'}{(n)'} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$
For $c = 0$, $0 < c < \infty$ is false. Therefore $3n \lg n \notin \Theta(n^2)$.

Heap: complete binary tree
- insert: put bottom, then trickle up if necessary
- remove: last node swapped with root, trickle down if necessary
- height: floor(lgn)
- insert: O(lgn) because height
- delete: 3(floor(lg(n-1))+1) is O(lgn)

Pre-order: +LR
In-order: L+R
Post-order: LR+

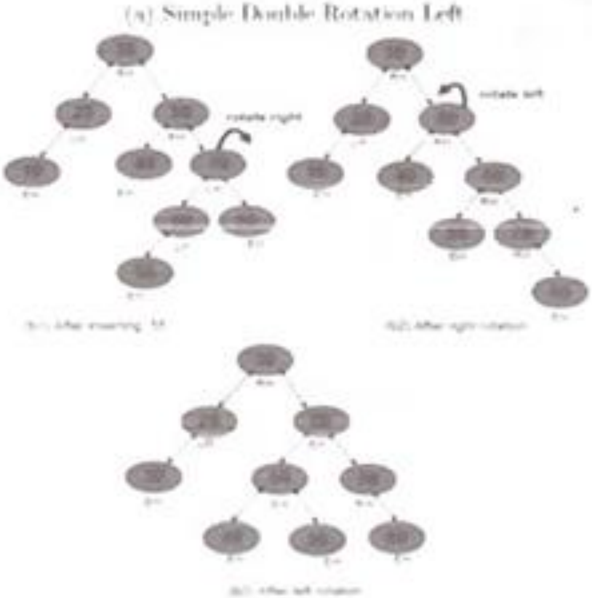
in-order predecessor: left, rightmost
in-order successor: right, leftmost

phi = (1 + sqrt(5))/2

fibbonaci(n) = (phi^n-(-phi)^(-n))/sqrt(5),

(n/2)lg(n/2) = (n/2)(lgn-lg2) = (n/2)(lgn-(1/2)lgn) n>=4
(n/4)lgn. c=1/4

(a) Simple Double Rotation Left



(b) Complex Double Rotation Left

Figure 7: Double Rotation Left

3. Right of Left: A subtree of a tree that is left high becomes right high.

leaf node: just remove it
only left/right node: replace w/ child
both: sucessor/predessor node

Determine x, y and z for the tri node restructure operation:

```
x = p                     ——> x = 44
y = taller child of x     ——> y = 77
z = taller child of y     ——> z = 50
```

Definition 15. The height of a tree is the number of edges on the longest path from the root to a leaf. An empty tree has a height of -1; a tree with only one node has a height of 0.

Definition 16. A perfect binary tree is a binary tree of height h with no missing nodes. All leaves are at level n and all other nodes each have two children.

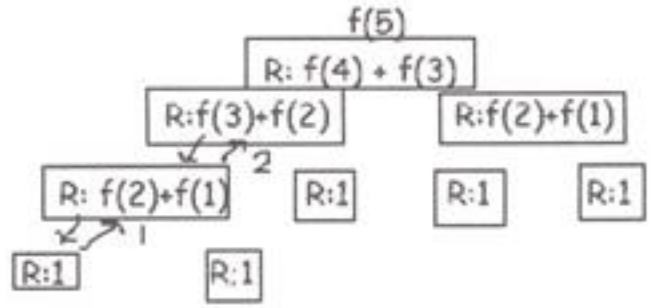
Definition 17. A full binary tree is a binary tree in which each node has 0 or two children.

Definition 18. A complete binary tree is a binary tree of height h that is perfect to level $h-1$ and has level h filled in from left to right.

Definition 19. A balanced binary tree is a binary tree in which the left and right subtrees of any node have heights that differ by at most 1.

tail-recursion: your last call is to the function ex: return n * f(n-1)
non-tail-recursion: not the top ex: return n * f'(n)
mutual recursion: ex: the even odd thing

box method trace of recursion



memoization: top-down
f(5) = 5*f(4) = 5*4*f(3)

tabulization: bottom-up
1*2*3*4*5....
1, 2, 6, 24, 120.....

fractional(continuous) knapsack: greedy
discrete (0/1) knapsack:

$$V[i,j] = \begin{cases} \max\left(V[i-1,j], V[i-1,j-w_i]+v_i\right), & w_i \le W, \\ V[i-1,j], & otherwise \end{cases}$$

0/1 Knapsack Dynamic Programming

Total wt = 7

val	wt	0	1	2	3	4	5	6	7
(1)	1	0	1	1	1	1	1	1	1
(4)	3	0	1	1	4	5	5	5	5
(5)	4	0	1	1	4	5	6	6	9
(7)	5	0	1	1	4	5	7	8	9 x

items

wt	val
1	1
3	4
4	5
5	7

claim: $3\ln n + 4 \in o(n)$

claim is true

A.) Definition: Let f and g be functions from $Z^+ \to R^+$

$f(n) \in o(g(n))$ iff $\forall c \in R^+ \; \exists n_0 \in Z^+ \ni f(n) < cg(n), \; \forall n \geq n_0$

B.) $f(n) \in o(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = K, \; K = 0$

C.) Let $f(n) = 3\ln n + 4$ and $g(n) = n$. I want to show that $\lim_{n\to\infty} \frac{3\ln n + 4}{n} = K, \; K = 0$

D.) $\lim_{n\to\infty} \frac{3\ln n + 4}{n} = \lim_{n\to\infty} \frac{(3\ln n + 4)'}{(n)'} = \lim_{n\to\infty} \frac{3/n}{1} = \lim_{n\to\infty} \frac{3}{n} = \frac{3}{\infty} = 0$

E.) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = K = 0$

$\therefore f(n) \in o(g(n)) \Rightarrow 3\ln n + 4 \in o(n)$

claim: Let f, g, h be functions from $Z^+ \to R^+$

If $f(n) \in o(g(n))$ and $g(n) \in o(h(n))$ then $f(n) \in o(g(n))$

A proof using limits

Proof:

A. Let f and g be functions from $\mathbb{Z}^+ \to \mathbb{R}^+$ – that is, positive real valued functions on the domain of positive integers.

$f(n) \in \Theta(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c < \infty$

B. Let $f(n) = (n^3 + 4n - 5)^2$ and $g(n) = n^6$

C. I want to show that $\lim_{n\to\infty} \frac{(n^3+4n-5)^2}{n^6} = c, \; 0 < c < \infty$

D. $\lim_{n\to\infty} \frac{(n^3+4n-5)^2}{n^6} = \lim_{n\to\infty} \left(\frac{n^3+4n-5}{n^3}\right)^2 = \lim_{n\to\infty} \left(\frac{n^3}{n^3} + \frac{4n}{n^3} - \frac{5}{n^3}\right)^2 = \left[\lim_{n\to\infty} \left(\frac{n^3}{n^3} + \frac{4}{n^2} - \frac{5}{n^3}\right)\right]^2 = (1 + 0 - 0)^2 = 1$

For $c = 1$, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c, 0 < c < \infty$. Therefore $(n^3 + 4n - 5)^2 \in \Theta(n^6)$

Prove the following theorem using the definition of the *Big-O* asymptotic notation

Theorem 3.

Suppose d, e, f and g are functions from $\mathbb{Z}^+ \to \mathbb{R}^+$. If $d(n) \in \mathbf{O}(e(n))$ and $f(n) \in \mathbf{O}(g(n))$, then $d(n) + f(n) \in \mathbf{O}(e(n) + g(n))$

Proof:

A. Given $d(n) \in \mathbf{O}(e(n)) \iff d(n) \leq c_1 e(n) \; \forall n \geq n_1$, where $c_1 \in \mathbb{R}^+$ and $n_1 \in \mathbb{Z}^+$

B. Given $f(n) \in \mathbf{O}(g(n)) \iff f(n) \leq c_2 g(n) \; \forall n \geq n_2$, where $c_2 \in \mathbb{R}^+$ and $n_2 \in \mathbb{Z}^+$

C. I want to find $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{Z}^+$ such that $d(n) + f(n) \leq c(e(n) + g(n))$ $\forall n \geq n_0$, where $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{Z}^+$.

D. Using the additive property of inequality on the inequalities in A and B and using the intersection of the half open intervals defined in A and B, we get: $d(n) + f(n) \leq c_1 e(n) + c_2 g(n) \; \forall n \geq max(n_1, n_2)$

E. Using the inequality in D, the fact that $c_1 e(n) + c_2 g(n) \leq max(c_1, c_2)(e(n) + g(n))$ $\forall n \geq max(n_1, n_2)$ and the transitive property of inequality, we get $d(n) + f(n) \leq max(c_1, c_2)(e(n) + g(n)) \; \forall n \geq max(n_1, n_2)$.

F. For $c = max(c_1, c_2)$ and $n_0 = max(n_1, n_2)$, we get $d(n) + f(n) \leq c(e(n) + g(n)) \; \forall n \geq n_0 \Rightarrow d(n) + f(n) \in \mathbf{O}(e(n) + g(n))$.

Therefore, if $d(n) \in \mathbf{O}(e(n))$ and $f(n) \in \mathbf{O}(g(n))$, then $d(n) + f(n) \in \mathbf{O}(e(n) + g(n))$.