MATH 2090 Problem Collection

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Louisiana State University

Section 1.1

For Problems 1 and 3, determine the order of the differential equation:

1. $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = e^x$

3. $y'' + xy' + e^x y = y'''$

5. Verify that, for $t > 0$, $y(t) = \ln t$ is a solution to the differential equation:

$$2\left(\frac{dy}{dt}\right)^3 = \frac{d^3y}{dt^3}$$

7. Verify that $y(x) = e^x \sin x$ is a solution to the differential equation:

$$2y\cot x - \frac{d^2y}{dx^2} = 0$$

Section 1.2

For Problems 1, 3, and 5, determine whether the differential equation is linear or nonlinear:

1. $\dfrac{d^2y}{dx^2} + e^x\dfrac{dy}{dx} = x^2$

3. $yy'' + x(y') - y = 4x\ln x$

5. $\dfrac{d^4y}{dx^4} + 3\dfrac{d^2y}{dx^2} = x$

For Problems 7, 9, and 11, verify that the given function is a solution to the given differential equation (c_1 and c_2 are abitrary constants), and state the maximum interval over which the solution is valid:

7. $y(x) = c_1e^{-5x} + c_2e^{5x}, \qquad y'' - 25y = 0$

9. $y(x) = c_1e^{x} + c_2e^{-2x}, \qquad y'' + y' - 2y = 0$

11. $y(x) = c_1x^{1/2}, \qquad y' = \dfrac{y}{2x}$

For Problems 23 and 25, determine all values of the constant r such that the given function solves the given differential equation.

23. $y(x) = e^{rx}, \qquad y'' + 6y' + 9y = 0$

25. $y(x) = x^r, \qquad x^2y'' + 5xy' + 4y = 0$

27. Determine a solution to the differential equation

$$(1 - x^2)y'' - xy' + 4y = 0$$

of the form $y(x) = a_0 + a_1x + a_2x^2$ satisfying the normalization condition $y(1) = 1$.

Section 1.4

For Problems 1-9, solve the given differential equation:

1. $\dfrac{dy}{dx} = 2xy$.

3. $e^{x+y}dy - dx = 0$.

5. $ydx - (x - 2)dy = 0$.

7. $y - x\dfrac{dy}{dx} = 3 - 2x^2\dfrac{dy}{dx}$.

9. $\dfrac{dy}{dx} = \dfrac{x(y^2 - 1)}{2(x - 2)(x - 1)}$.

15. Solve the given initial-value problem:

$$y' = y^3 \sin x, \qquad y(0) = 0$$

17. An object of mass m falls from rest, starting at a point near the earth's surface. Assuming that air resistance varies as the square of the velocity of the object, a simple application of Newton's second law yields the initial-value problem for the velocity, $v(t)$, of the object at time t:

$$m\frac{dv}{dt} = mg - kv^2, \qquad v(0) = 0,$$

where k, m, g are positive constants.
(a) Solve the foregoing initial-value problem for v in terms of t.
(b) Does the velocity of the object increase indefinitely? Justify.
(c) Determine the position of the object at time t.

Section 1.6

For Problems 1-5, solve the given differential equation:

1. $\dfrac{dy}{dx} + y = 4e^x$.

3. $x^2y' - 4xy = x^7 \sin x, \qquad x > 0$.

5. $\dfrac{dy}{dx} + \dfrac{2x}{1-x^2}y = 4x, \qquad -1 < x < 1.$

19. Solve the given initial-value problem:

$$(y - e^x)dx + dy = 0, \qquad y(0) = 1.$$

Section 1.9

For Problems 1-3, determine whether the given differential equation is exact:

1. $ye^{xy}dx + (2y - xe^{xy})dy = 0.$

3. $(y + 3x^2)dx + xdy = 0.$

For Problems 5-9, solve the given differential equation:

5. $2xydx + (x^2 + 1)dy = 0.$

7. $(4e^{2x} + 2xy - y^2)dx + (x - y)^2dy = 0$

9. $[y\cos(xy) - \sin x]dx + x\cos(xy)dy = 0$

17. Solve the given initial-value problem:

$$(3x^2 \ln x + x^2 - y)dx - xdy = 0, \qquad y(1) = 5$$

21. Determine whether the given function is an integrating factor for the given differential equation:

$$I(x, y) = y^{-2}e^{-x/y}, \qquad y(x^2 - 2xy)dx - x^3dy = 0$$

For Problems 23-25, determine an integrating factor for the given differential equation, and hence find the general solution:

23. $(y - x^2)dx + 2xdy = 0, \qquad x > 0.$

25. $x^2ydx + y(x^3 + e^{-3y}\sin y)dy = 0$

Section 2.1

1. If $A = \begin{bmatrix} 1 & -2 & 3 & 2 \\ 7 & -6 & 5 & -1 \\ 0 & 2 & -3 & 4 \end{bmatrix}$, determine:

(a) a_{31}, a_{24}, a_{14}, a_{32}, a_{21}, and a_{34},

(b) all pairs (i, j) such that $a_{ij} = 2$.

For problems 3-9, write the matrix with the given elements. In each case, specify the dimensions of the matrix.

3. $a_{11} = 1$, $a_{21} = -1$, $a_{12} = 5$, $a_{22} = 3$.

7. $a_{12} = -1$, $a_{13} = 2$, $a_{23} = 3$, $a_{ji} = -a_{ij}$, $1 \leq i \leq 3$, $1 \leq j \leq 3$.

9. $a_{ij} = i + j$, $1 \leq i \leq 4$, $1 \leq j \leq 4$.

11. Determine $\text{tr}(A)$ for the given matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & -2 \\ 7 & 5 & -3 \end{bmatrix}$$

For Problems 13-15, write the column vectors and row vectors of the given matrix:

13. $A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$

15. $A = \begin{bmatrix} 2 & 10 & 6 \\ 5 & -1 & 3 \end{bmatrix}$

17. If $a_1 = \begin{bmatrix} -2 & 0 & 4 & -1 & -1 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 9 & -4 & -4 & 0 & 8 \end{bmatrix}$, write the matrix

$$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

and determine the column vectors of A.

For Problems 21-23, give an example of a matrix of the specified form. (In some cases, many examples may be possible.)

21. 3×3 diagonal matrix.

23. 4×4 skew-symmetric matrix.

Section 2.2

For Problem 1, let:

$$A = \begin{bmatrix} -2 & 6 & 1 \\ -1 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 4 & -4 \end{bmatrix}, \quad C = \begin{bmatrix} 1+i & 2+i \\ 3+i & 4+i \\ 5+i & 6+i \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 1 \\ 1 & 2 & 5 \\ 3 & 1 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 2 & -5 & -2 \\ 1 & 1 & 3 \\ 4 & -2 & -3 \end{bmatrix}, \quad F = \begin{bmatrix} 6 & 2-3i & i \\ 1+i & -2i & 0 \\ -1 & 5+2i & 3 \end{bmatrix}$$

1. Compute each of the following:

(a) $5A$

(b) $-3B$

(c) iC

(d) $2A - B$

(e) $A + 3C^T$

(f) $3D - 2E$

(g) $D + E + F$

(h) the matrix G such that $2A + 3B - 2G = 5(A + B)$

(i) the matrix H such that $D + 2F + H = 4E$

(j) the matrix K such that $K^T + 3A - 2B = 0_{2\times 3}$

For Problem 3, let:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix},\ B = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 1 & -2 \\ 4 & 6 & -2 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & -2 & 3 \end{bmatrix},\ E = \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix},\ F = \begin{bmatrix} i & 1-3i \\ 0 & 4+i \end{bmatrix}$$

3. For each item, decide whether or not the given expression is defined. For each item that is defined, compute the result.

(a) AB

(b) BC

(c) CA

(d) A^TE

(e) CD

(f) C^TA^T

(g) F^2

(h) BD^T

(i) $A^T A$

(j) FE

5. Let $A = \begin{bmatrix} -3 & 2 & 7 & -1 \\ 6 & 0 & -3 & -5 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 8 \\ 8 & -3 \\ -1 & -9 \\ 0 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} -6 & 1 \\ 1 & 5 \end{bmatrix}$. Compute ABC and CAB.

7. Determine Ac by computing an appropriate linear combination of the column vectors of A:

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 5 \\ 7 & -6 & 3 \end{bmatrix}, \; c = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

13. If $A = \begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix}$, calculate A^2 and verify that A satisfies $A^2 + 4A + 18I_2 = 0_2$.

Section 2.4

For Problems 1-5, determine whether the given matrices are in reduced row-echelon form, row-echelon form but not reduced row-echelon form, or neither.

1. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3. $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

5. $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

For Problems 9-13, use elementary row operations to reduce the given matrix to row-echelon form, and hence determine the rank of each matrix.

9. $\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$.

11. $\begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 3 & 5 \end{bmatrix}$.

13. $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

For Problems 19-21, reduce the given matrix to reduced row-echelon form and hence determine the rank of each matrix.

19. $\begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix}$.

21. $\begin{bmatrix} 3 & 7 & 10 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$.

Section 2.5

1. $\begin{aligned} x_1 - 5x_2 &= 3, \\ 3x_1 - 9x_2 &= 15. \end{aligned}$

3. $\begin{aligned} 7x_1 - 3x_2 &= 5, \\ 14x_1 - 6x_2 &= 10. \end{aligned}$

5. $\begin{aligned} 3x_1 - x_2 &= 1, \\ 2x_1 + x_2 + 5x_3 &= 4, \\ 7x_1 - 5x_2 - 8x_3 &= -3. \end{aligned}$

13. Use Gauss-Jordan elimination to determine the solution set to the given system:

$$\begin{aligned} 2x_1 - x_2 - x_3 &= 2, \\ 4x_1 + 3x_2 - 2x_3 &= -1, \\ x_1 + 4x_2 + x_3 &= 4. \end{aligned}$$

19. Determine the solution set to the system Ax=b for the given coefficient matrix A and right-hand side vector b:

$$A = \begin{bmatrix} 1 & -3 & 1 \\ 5 & -4 & 1 \\ 2 & 4 & -3 \end{bmatrix}, \ \mathrm{b} = \begin{bmatrix} 8 \\ 15 \\ -4 \end{bmatrix}$$

25. Determine all values of the constant k for which the following system has (a) no solution, (b) an infinite number of solutions, and (c) a unique solution:

$$\begin{aligned} 2x_1 + x_2 - x_3 + x_4 &= 0, \\ x_1 + x_2 + x_3 - x_4 &= 0, \\ 4x_1 + 2x_2 - x_3 + x_4 &= 0, \\ 2x_1 - x_2 + x_3 + kx_4 &= 0. \end{aligned}$$

Section 2.6

For Problems 1-3, verify by direct multiplication that the given matrices are inverses of one another:

1. $A = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 7 & -9 \\ -3 & 4 \end{bmatrix}$

3. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, provided $ad - bc \neq 0$.

For Problems 5-9, determine A^{-1}, if possible, using the Gauss-Jordan method. If A^{-1} exists, check your answer by verifying that $AA^{-1} = I_n$.

5. $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

7. $A = \begin{bmatrix} 1 & -i \\ -1+i & 2 \end{bmatrix}$.

9. $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 11 \\ 4 & -3 & 10 \end{bmatrix}$.

21. Use A^{-1} to find the solution to the given system.

$$\begin{aligned} 6x_1 + 20x_2 &= -8, \\ 2x_1 + 7x_2 &= 2. \end{aligned}$$

Section 3.1

For Problems 17-45, evaluate the determinant of the given matrix / matrix function:

17. $A = \begin{bmatrix} 6 & -3 \\ -5 & -1 \end{bmatrix}$

19. $A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$

25. $A = \begin{bmatrix} 6 & -1 & 2 \\ -4 & 7 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

35. $A = \begin{bmatrix} 0 & 0 & 0 & -3 \\ 0 & 0 & -7 & -1 \\ 0 & 2 & 6 & 9 \\ 1 & 8 & -8 & -9 \end{bmatrix}$

45. $A(t) = \begin{bmatrix} e^{2t} & e^{3t} & e^{-4t} \\ 2e^{2t} & 3e^{3t} & -4e^{-4t} \\ 4e^{2t} & 9e^{3t} & 16e^{-4t} \end{bmatrix}$

Section 3.2

For Problems 3-11, evaluate the determinant of the given matrix by first using elementary row operations to reduce it to upper triangular form:

3. $\begin{vmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ -2 & 1 & 4 \end{vmatrix}$

7. $\begin{vmatrix} 1 & -1 & 2 & 4 \\ 3 & 1 & 2 & 4 \\ -1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 2 \end{vmatrix}$

11. $\begin{vmatrix} 2 & -1 & 3 & 4 \\ 7 & 1 & 2 & 3 \\ -2 & 4 & 8 & 6 \\ 6 & -6 & 18 & -24 \end{vmatrix}$

17. Use Theorem 3.2.5 to determine whether the given matrix is invertible or not:

$$\begin{bmatrix} -1 & 2 & 3 \\ 5 & -2 & 1 \\ 8 & -2 & 5 \end{bmatrix}$$

For Problems 37-43, let A and B be 4×4 matrices such that $\det(A) = 5$ and $\det(B) = 3$. Compute the determinant of the given matrix:

37. AB^T

39. $(A^{-1}B^2)^3$

41. $(5A)(2B)$

43. $B^{-1}(2A)B^T$

47. Without expanding the determinant, determine all values of x for which $\det(A) = 0$ if

$$A = \begin{bmatrix} 1 & -1 & x \\ 2 & 1 & x^2 \\ 4 & -1 & x^3 \end{bmatrix}$$

63. Determine all values for a for which

$$\begin{bmatrix} 1 & 2 & 3 & 4 & a \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ a & 4 & 3 & 2 & 1 \end{bmatrix}$$

is invertible.

Section 3.3

5. If $A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 3 & 4 & 1 & 2 \\ 7 & 1 & 4 & 6 \\ 5 & 0 & 1 & 2 \end{bmatrix}$,

determine the minors M_{12}, M_{31}, M_{23}, M_{42} and the corresponding cofactors.

For Problems 25-27, use elementary row operations together with the Cofactor Expansion Theorem to evaluate the given determinant:

25. $\begin{vmatrix} 2 & -7 & 4 & 3 \\ 5 & 5 & -3 & 7 \\ 6 & 2 & 6 & 3 \\ 4 & 2 & -4 & 5 \end{vmatrix}$

27. $\begin{vmatrix} -3 & 2 & 0 & -1 & -4 \\ 1 & 3 & 1 & 3 & 5 \\ 0 & 2 & 1 & -3 & -1 \\ 2 & 0 & 4 & -2 & -3 \\ -1 & 2 & 6 & 0 & -1 \end{vmatrix}$

For Problems 35-37, determine the eigenvalues of the given matrix A. That is, determine the scalars λ such that $\det(A - \lambda I) = 0$.

35. $\begin{bmatrix} 2 & 0 & 0 \\ 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$

37. $\begin{bmatrix} -5 & -5 & 0 \\ -8 & 1 & 0 \\ -5 & 3 & 7 \end{bmatrix}$

Section 4.2

For Problems 3-13, determine whether the given set S of vectors is closed under addition and closed under scalar multiplication. In each case, take the set of scalars to be set of all real numbers.

3. The set $S := \{A \in M_n(\mathbb{R}) : A$ is upper or lower triangular$\}$.

5. The set $S := \{a_0 + a_1x + a_2x^2 : a_0 + a_1 + a_2 = 1\}$

7. The set S of all solutions to the differential equation $y' + 3y = 0$. (Do not solve the differential equation).

9. The set $S := \{A \in M_2(\mathbb{R}) : \det(A) = 0\}$.

13. The set S of all polynomials of degree *exactly* 2.

Section 4.3

1. Let $S = \{\mathrm{x} \in \mathbb{R}^3 : \mathrm{x} = (r - 2s, 3r + s, s), r, s \in \mathbb{R}\}$.

(a) Show that S is a subspace of $\mathbb{R}^3$.

(b) Show that the vectors in S lie on the plane with equation $3x - y + 7z = 0$.

For Problems 3-21, express S in set notation and determine whether it is a subspace of the given vector space V.

3. $V = \mathbb{R}^3$, and S is the set of all vectors (x, y, z) in V such that $z = 3x$ and $y = 2x$.

7. $V = \mathbb{R}^n$, and S is the set of all solutions to the nonhomogeneous linear system $A\mathrm{x}=\mathrm{b}$, where A is a fixed $m \times n$ matrix and $\mathrm{b}(\neq 0)$ is a fixed vector.

19. $V = P_2(\mathbb{R})$, and S is the subset of $P_2(\mathbb{R})$ consisting of all polynomials of the form $p(x) = ax^2 + b$.

21. $V = C^2(I)$, and S is the subset of V consisting of those functions satisfying the differential equation

$$y'' + 2y' - y = 0$$

on I.

For Problems 23-29, determine the null space of the given matrix A:

25. $A = \begin{bmatrix} 2 & -4 \\ 1 & 2 \\ -3 & -5 \end{bmatrix}$

27. $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & -2 \\ -1 & 3 & 4 \end{bmatrix}$

Section 4.4

For Problems 1-3, determine whether the given set of vectors spans $\mathbb{R}^2$:

1. $\{(5, -1)\}$

3. $\{(2, 5), (0, 0)\}$

5. Recall that three vectors v_1, v_2, v_3 in $\mathbb{R}^3$ are coplanar if and only if

$$\det([\mathrm{v}_1,\ \mathrm{v}_2,\ \mathrm{v}_3]) = 0.$$

Use this result to determine whether the given set of vectors spans $\mathbb{R}^3$:

$$\{(1, -1, 1), (2, 5, 3), (4, -2, 1)\}.$$

9. Show that the set of vectors

$$\{(-4, 1, 3), (5, 1, 6), (6, 0, 2)\}$$

does not span $\mathbb{R}^3$, but that it does span the subspace of $\mathbb{R}^3$ consisting of all vectors lying in the plane with equation $x + 13y - 3z = 0$.

11. Show that $\mathrm{v}_1 = (2, -1)$, $\mathrm{v}_2 = (3, 2)$ span $\mathbb{R}^2$, and express the vector $\mathrm{v} = (5, -7)$ as a linear combination of v_1, v_2.

13. Show that $\mathrm{v}_1 = (1, -3, 2)$, $\mathrm{v}_2 = (1, 0, -1)$, $\mathrm{v}_3 = (1, 2, -4)$ span $\mathbb{R}^3$, and express $\mathrm{v} = (9, 8, 7)$ as a linear combination of v_1, v_2, v_3.

Section 4.5

For Problems 1-7, determine whether the given set of vectors is linearly independent or linearly dependent in $\mathbb{R}^n$. In the case of linear dependence, find a dependency relationship.

1. $\{(3, 6, 9)\}$

3. $\{(2, -1), (3, 2), (0, 1)\}$

5. $\{(1, 2, 3), (1, -1, 2), (1, -4, 1)\}$

7. $\{(1, -1, 2), (2, 1, 0)\}$

11. Let $\mathrm{v}_1 = (1, 2, 3)$, $\mathrm{v}_2 = (4, 5, 6)$, $\mathrm{v}_3 = (7, 8, 9)$. Determine whether $\{\mathrm{v}_1,\ \mathrm{v}_2,\ \mathrm{v}_3\}$ is linearly independent in $\mathbb{R}^3$. Describe

$$\mathrm{span}\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$$

geometrically.

13. Determine all values of the constant k for which the vectors $(1, 1, k), (0, 2, k)$ and $(1, k, 6)$ are linearly dependent in $\mathbb{R}^3$.

17. Determine whether the given set of vectors is linearly independent in $M_2(\mathbb{R})$:

$$A_1 = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$$

19. Determine whether the given set of vectors is linearly independent in $P_2(\mathbb{R})$:

$$p_1(x) = 1 - x, \qquad p_2(x) = 1 + x$$

33. Use the Wronskian to show that the given functions are linearly independent on the given interval I:

$$f_1(x) = \sin x, \; f_2(x) = \cos x, \; f_3(x) = \tan x, \; I = (-\pi/2, \pi/2)$$

37. Show that the Wronskian of the given function is identically zero on $(-\infty, \infty)$. Determine whether the functions are linearly independent or linearly dependent on that interval:

$$f_1(x) = 1, \; f_2(x) = x, \; f_3(x) = 2x - 1$$

Section 4.6

For Problems 1-7, determine whether the given set S of vectors is a basis for $\mathbb{R}^n$:

1. $S = \{(-6, -1)\}$

3. $S = \{(1, 2, 1), (3, -1, 2), (1, 1, -1)\}$

5. $S = \{(1, 1, -1, 2), (1, 0, 1, -1), (2, -1, 1, -1)\}$

7. $S = \{(7, 1, -3), (6, 1, 0), (-5, -1, -2), (0, -3, 8)\}$

9. Determine whether the given set S of vector is a basis for $P_n(\mathbb{R})$:

$$n = 1 : S = 2 - 5x, 3x, 7 + x$$

15. Determine whether the given set S of vectors is a basis for $M_{m \times n}(\mathbb{R})$:

$$m = n = 2 : S = \left\{ \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & -5 \\ 6 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & -4 \end{bmatrix}, \begin{bmatrix} 6 & -2 \\ -3 & -4 \end{bmatrix} \right\}$$

For Problems 19-23, find the dimension of the null space of the given matrix A:

19. $A = \begin{bmatrix} 8 & -9 & 3 & 3 & -5 \end{bmatrix}$

21. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Section 6.1

For Problems 1-7, verify directly from Definition 6.1.3 that the given mapping is a linear transformation:

1. $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x_1, x_2, x_3) = (x_1 + 3x_2 + x_3, x_1 - x_2)$$

3. $T : C^2(I) \to C^0(I)$ defined by

$$T(y) = y'' + a_1 y' + a_2 y,$$

where a_1 and a_2 are functions defined on I.

5. $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by

$$T(A) = AB - BA,$$

where B is a fixed $n \times n$ matrix.

7. $T : M_n(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \mathrm{tr}(A)$, where $\mathrm{tr}(A)$ denotes the trace of A.

For Problems 9-13, show that the given mapping is a nonlinear transformation:

9. $T : P_2(\mathbb{R}) \to \mathbb{R}$ defined by $T(a + bx + cx^2) = a + b + c + 1$.

13. $T : M_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$T(A) = \det(A)$$

For Problems 15-17, determine the matrix of the given transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m :$$

15. $T(x_1, x_2) = (x_1 + 3x_2, 2x_1 - 7x_2, x_1)$

17. $T(x_1, x_2, x_3) = x_1 + 5x_2 - 3x_3$

For Problems 19-21, determine the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ that has the given matrix:

19. $A = \begin{bmatrix} 1 & 3 \\ -4 & 7 \end{bmatrix}$

21. $A = \begin{bmatrix} 2 & 2 & -3 \\ 4 & -1 & 2 \\ 5 & 7 & -8 \end{bmatrix}$

Section 6.3

1. Consider $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $T(\mathrm{x}) = A\mathrm{x}$, where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \\ 8 & 16 \end{bmatrix}$. For each x below, find $T(\mathrm{x})$ and thereby determine whether x is in $\mathrm{Ker}(T)$.

(a) $\mathrm{x} = (-10, 5)$

(b) $\mathrm{x} = (1, -1)$

(c) $\mathrm{x} = (2, -1)$

For Problems 3-7, find $\mathrm{Ker}(T)$ and $\mathrm{Rng}(T)$, and give a geometrical description of each. Also, find $\dim[\mathrm{Ker}(T)]$ and $\dim[\mathrm{Rng}(T)]$, and verify Theorem 6.3.8.

3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathrm{x}) = A\mathrm{x}$, where

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

5. $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathrm{x}) = A\mathrm{x}$, where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ 5 & -8 & -1 \end{bmatrix}$$

7. $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\mathrm{x}) = A\mathrm{x}$, where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

15. Consider the linear transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T(ax^2 + bx + c) = ax^2 + (a + 2b + c)x + (3a - 2b - c),$$

where a, b, and c are arbitrary constants.

17. Consider the linear transformation $T : P_1(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T(ax + b) = (b - a) + (2b - 3a)x + bx^2.$$

Determine $\mathrm{Ker}(T)$, $\mathrm{Rng}(T)$, and their dimensions.

Section 7.1

For Problems 1-3, use Equation (7.1.1) to verify that λ and $\mathbf{v}$ are an eigenvalue/eigenvector pair for the given matrix A.

1. $\lambda = 2$, $\mathbf{v} = (2, 9)$, $A = \begin{bmatrix} -7 & 2 \\ -9 & 4 \end{bmatrix}$.

3. $\lambda = 3$, $\mathbf{v} = (2, 1, -1)$, $A = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix}$.

For Problems 13-23, determine all eigenvalues and corresponding eigenvectors of the given matrix:

13. $\begin{bmatrix} 1 & 6 \\ 2 & -3 \end{bmatrix}$

17. $\begin{bmatrix} -2 & -6 \\ 3 & 4 \end{bmatrix}$

21. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

23. $\begin{bmatrix} 6 & 3 & -4 \\ -5 & -2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

Section 7.2

For Problems 1-9, determine the multiplicity of each eigenvalue and a basis for each eigenspace of the given matrix A. Hence, determine the dimension of each eigenspace and state whether the matrix is defective or nondefective.

1. $A = \begin{bmatrix} -7 & 0 \\ -3 & -7 \end{bmatrix}$

3. $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

7. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -2 & 1 \end{bmatrix}$

9. $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 0 & -4 \\ 1 & 4 & 0 \end{bmatrix}$

Section 8.1

For Problems 1-3, find Ly for the given differential operator if **(a)** $y(x) = 2e^{3x}$, **(b)** $y(x) = 3\ln x$, **(c)** $y(x) = 2e^{3x} + 3\ln x$:

1. $L = D - x$

3. $L = D^3 - 2xD^2$

5. Verify that the given function is in the kernal of L:

$$y(x) = xe^{2x}, \quad L = D^2 - 4D + 4$$

11. Compute $\mathrm{Ker}(L)$:

$$L = D^2 + 1$$

15. Find L_1L_2 and L_2L_1 for the given differential operators, and determine whether $L_1L_2 = L_2L_1$:

$$L_1 = D + x, \quad L_2 = D + (2x - 1)$$

17. Write the given nonhomogeneous differential equation as an operator equation, and give the associated homogenous differential equation:

$$y''' + x^2y'' - (\sin x)y' + e^x y = x^3$$

21. Determine which of the following sets of vectors is a basis for the solution space to the differential equation $y'' - 16y = 0$:

$$S_1 = \{e^{4x}\},\ S_2 = \{e^{2x}, e^{4x}, e^{-4x}\},\ S_3 = \{e^{4x}, e^{2x}\},$$

$$S_4 = \{e^{4x}, e^{-4x}\},\ S_5 = \{e^{4x}, 7e^{4x}\},\ S_6 = \{\cosh 4x, \sinh 4x\},$$

23. Determine two linearly independent solutions to the given differential equation of the form $y(x) = e^{rx}$, and thereby determine the general solution to the differential equation:

$$y'' - 2y' - 3y = 0$$

27. Determine three linearly independent solutions to the given differential equation of the form $y(x) = e^{rx}$, and thereby determine the general solution to the differential equation:

$$y''' - 3y'' - y' + 3y = 0$$

Section 8.2

For Problems 1-3, determine a basis for the solution space of the given differential equation:

1. $y'' + 2y' - 3y = 0$

3. $y'' - 6y' + 25y = 0$

For Problems 9-13, determine the general solution to the given differential equation:

9. $y'' - 6y' + 9y = 0$

11. $(D+1)(D-5)y = 0$

13. $y'' - 6y' + 34y = 0$

Section 8.3

For Problems 1-16, determine the annihilator of the given function:

1. $F(x) = 5e^{-3x}$

3. $F(x) = \sin x + 3xe^{2x}$

5. $F(x) = 4e^{-2x}\sin x$

For Problems 17-21, determine the general solution to the given differential equation. Derive your trial solution using the annihilator technique.

17. $(D-1)(D+2)y = 5e^{3x}$

21. $(D-2)(D+1)y = 4x(x-2)$

33. Solve the given initial-value problem:

$$y'' + 9y = 5\cos 2x, \quad y(0) = 2, \quad y'(0) = 3$$

Section 8.7

For Problems 1-5, use the variation-of-parameters method to find the general solution to the given differential equation:

1. $y'' - 6y' + 9y = 4e^{3x}\ln x, \qquad x > 0$

3. $y'' + 9y = 18\sec^3(3x), \qquad |x| < \pi/6$

5. $y'' - 4y = \dfrac{8}{e^{2x}+1}$

Section 9.1

For Problems 1-3, solve the given system of differential equations:

1. $x_1' = 2x_1 + x_2, \quad x_2' = 2x_1 + 3x_2$

3. $x_1' = 4x_1 + 2x_2, \quad x_2' = -x_1 + x_2$

For Problems 9-11, solve the given initial-value problem:

9. $x_1' = 2x_2, \quad x_2' = x_1 + x_2, \quad x_1(0) = 3, \quad x_2(0) = 0$

11. $x_1' = 2x_1 + x_2, \quad x_2' = -x_1 + 4x_2, \quad x_1(0) = 1, \quad x_2(0) = 3$

13. Solve the given nonhomogeneous system:

$$x_1' = -2x_1 + x_2 + t, \quad x_2' = -2x_1 + x_2 + 1$$

15. Convert the given system of differential equations to a first-order linear system:

$$\frac{dx}{dt} - ty = \cos t, \quad \frac{d^2y}{dt^2} - \frac{dx}{dt} + x = e^t$$

For Problems 17-19, convert the given linear differential equations to a first-order linear system:

17. $y'' + 2ty' + y = \cos t$

19. $y''' + t^2y' - e^ty = t$

Section 9.2

For Problems 1-5, show that the given vector functions are linearly independent on $(-\infty, \infty)$:

1. $\mathbf{x}_1(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}, \mathbf{x}_2(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix},$

3. $\mathbf{x}_1(t) = \begin{bmatrix} t+1 \\ t-1 \\ 2t \end{bmatrix}, \mathbf{x}_2(t) = \begin{bmatrix} e^t \\ e^{2t} \\ e^{3t} \end{bmatrix}, \mathbf{x}_3(t) = \begin{bmatrix} 1 \\ \sin t \\ \cos t \end{bmatrix},$

5. Is there an interval on which $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ in this exercise are not linearly dependent?

$$\mathbf{x}_1(t) = \begin{bmatrix} \sin t \\ \cos t \\ 1 \end{bmatrix}, \mathbf{x}_2(t) = \begin{bmatrix} t \\ 1-t \\ 1 \end{bmatrix}, \mathbf{x}_3(t) = \begin{bmatrix} \sinh t \\ \cosh t \\ 1 \end{bmatrix}.$$

7. Show that the given vector functions are linearly dependent on $(-\infty, \infty)$:

$$\mathbf{x}_1(t) = \begin{bmatrix} t^2 \\ 6 - t + t^3 \end{bmatrix}, \mathbf{x}_2(t) = \begin{bmatrix} -3t^2 \\ -18t + 3t^2 - 3t^3 \end{bmatrix}.$$

Section 9.3

For Problems 1-5, show that the given functions are solutions of the system $\mathbf{x}'(t) = A(x)\mathbf{x}(t)$ for the given matrix A, and hence, find the general solution to the system (remember to check linear independence). If auxiliary conditions are given, find the particular solution that satisfies these conditions:

1. $\mathbf{x}_1(t) = \begin{bmatrix} \sin 3t \\ \cos 3t \end{bmatrix}$, $\mathbf{x}_2(t) = \begin{bmatrix} -\cos 3t \\ \sin 3t \end{bmatrix}$, $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

3. $\mathbf{x}_1(t) = \begin{bmatrix} e^{-t}\cos 2t \\ e^{-t}\sin 2t \end{bmatrix}$, $\mathbf{x}_2(t) = \begin{bmatrix} -e^{-t}\sin 2t \\ e^{-t}\cos 2t \end{bmatrix}$, $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

5. $\mathbf{x}_1(t) = \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix}$, $\mathbf{x}_2(t) = \begin{bmatrix} e^{2t} \\ 3e^{2t} \\ e^{2t} \end{bmatrix}$, $\mathbf{x}_3(t) = \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix}$, $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$

Section 9.4

For Problems 1-7, determine the general solution to the system $\mathbf{x}' = A\mathbf{x}$ for the given matrix A:

1. $\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 0 & 0 \\ 1 & 5 & -1 \\ 1 & 6 & -2 \end{bmatrix}$

Section 9.6

For Problems 1-5, use the variation-of-parameters technique to find a particular solution $\mathbf{x}_p$ to $\mathbf{x}' = A\mathbf{x}+\mathbf{b}$, for the given A and $\mathbf{b}$. Also obtain the general solution to the system of differential equations:

1. $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$, $\quad b = \begin{bmatrix} e^{2t} \\ e^{t} \end{bmatrix}$

5. $A = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}$, $\quad b = \begin{bmatrix} 54te^{3t} \\ 9e^{3t} \end{bmatrix}$

11. Consider the nonhomogenous system

$$\begin{aligned} x_1' &= 2x_1 - 3x_2 + 34\sin t, \\ x_2' &= -4x_1 - 2x_2 + 17\cos t, \end{aligned}$$

Find the general solution to this system by first solving the associated homogeneous system, and then using the *method of undetermined coefficients* to obtain a particular solution.

[**Hint:** The form of the nonhomogeneous term suggests a trial solution of the form

$$\mathbf{x}_p(t) = \begin{bmatrix} A_1\cos t + B_1\sin t \\ A_2\cos t + B_2\sin t \end{bmatrix},$$

where the constants A_1, A_2, B_1, and B_2 can be determined by substituting into the given system.]

Section 10.1

For Problems 1-9, use (10.1.1) to determine $L[f]$:

1. $f(t) = t - 1$

3. $f(t) = te^t$

5. $f(t) = \sinh bt$, where b is constant.

7. $f(t) = 3e^{2t}$

9. $f(t) = \begin{cases} t^2, & 0 \le t \le 1, \\ 1, & t > 1. \end{cases}$

Section 10.2

For Problems 7-13, determine the inverse Laplace transform of the given function:

7. $F(s) = \dfrac{3}{s-2}$

9. $F(s) = \dfrac{1}{s^2+4}$

11. $F(s) = \dfrac{4}{s^3}$

13. $F(s) = \dfrac{2s+1}{s^2+16}$

Section 10.4

For Problems 1-13, use the Laplace transform to solve the given initial-value problem:

1. $y' - 2y = 6e^{5t}, \quad y(0) = 3$

5. $y' - y = 6\cos t, \quad y(0) = 2$

9. $y'' + 4y = 0, \quad y(0) = 5, \quad y'(0) = 1$

13. $y'' - 3y' + 2y = 4e^{3t}, \quad y(0) = 0, \quad y'(0) = 0$

Section 10.5

For Problems 1-5, determine $f(t-a)$ for the given function f and the given constant a:

1. $f(t) = 2t, \quad a = 1$

5. $f(t) = e^{3t}, \quad a = 2$

13. Determine $f(t)$:

$$f(t-1) = (t-2)^2$$

For Problems 19-21, determine the Laplace transform of f:

19. $f(t) = e^{-4t} \sin 5t$

21. $f(t) = 3te^{-t}$

For Problems 29-31, determine $L^{-1}[F]$:

29. $F(s) = \dfrac{4}{(s+2)^3}$

31. $F(s) = \dfrac{2}{(s-1)^2+4}$

Section 10.6

For Problems 1-3, make a sketch of the given function on the interval $[0, \infty)$:

1. $f(t) = 3(u_2(t) - u_4(t))$

3. $f(t) = 1 + (t-1)u_1(t)$

For Problems 8-15, make a sketch of the given function on $[0, \infty)$ and express it in terms of the unit step function:

9. $f(t) = \begin{cases} t^2, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$

11. $f(t) = \begin{cases} 2, & 0 \le t < 1, \\ 2e^{(t-1)}, & t > 1. \end{cases}$

Section 10.7

For Problems 1-5, determine the Laplace transform of the given function f:

1. $f(t) = u_2(t) - u_3(t)$

3. $f(t) = e^{3(t-2)}u_2(t)$

5. $f(t) = \cos t \; u_\pi(t)$

For Problems 13-15, determine the inverse Laplace transform of F:

13. $F(s) = \dfrac{e^{-s}}{s+1}$

15. $F(s) = \dfrac{se^{-s}}{s^2+4}$

For Problems 27-41, solve the given initial-value problem:

27. $y' + 2y = 2u_1(t), \quad y(0) = 1$

29. $y' - y = 4u_{\pi/4}(t)\cos(t - \pi/4), \quad y(0) = 1$

31. $y' + 3y = f(t), \quad y(0) = 1$, where

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & t \ge 1. \end{cases}$$