Math2090, Section 1
Quiz 9, Total 10 points

Last name: First name:

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1. Use the variation-of-parameters method to find the general solution to the differential equation:

$$y'' - 4y' + 4 = x^{-2}e^{2x}, \quad x > 0.$$

$r^2 - 4r + 4 = 0 \Rightarrow (r-2)(r-2) = 0 \Rightarrow r = 2$

$\therefore\ Y_1(x) = e^{2x}, \quad Y_2(x) = xe^{2x}$

$Y_p = U_1 Y_1 + U_2 Y_2 = U_1 e^{2x} + U_2 x e^{2x}$

where U_1 and U_2 satisfy:

$Y_1 U_1' + Y_2 U_2' = 0$ and $Y_1' U_1' + Y_2' U_2' = x^{-2}e^{2x}$

$\therefore$ U_1 and U_2 satisfy:

$e^{2x} U_1' + x e^{2x} U_2' = 0 \quad \text{--} ①$

$2e^{2x} U_1' + (2xe^{2x} + e^{2x}) U_2' = x^{-2} e^{2x} \quad \text{--} ②$

Now, we solve (1) and (2) using Cramers rule

$$U_1' = \frac{\begin{vmatrix} 0 & xe^{2x} \\ x^{-2}e^{2x} & 2xe^{2x}+e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x}+e^{2x} \end{vmatrix}} = \frac{-xe^{4x}}{2xe^{4x} + e^{4x} - 2\ldots}$$

$\therefore\ U_1' = -x^{-1} \Rightarrow U_1 = \int -x^{-1} = -\ln x$

similarly, $U_2' = x^{-2} \Rightarrow U_2 = -x^{-1}$.

$\therefore\ Y_p = -\ln x\, e^{2x} - x^{-1}(xe^{2x}) = -\ln x e^{2x} - e^{2x}$

The general solution is

$Y = Y_p + C_1 Y_1 + C_2 Y_2$

$= -\ln x e^{2x} - e^{2x} + C_1 e^{2x} + C_2 x e^{2x}$.

2. Use substitution to solve the system of differential equations: $x_1' = 2x_2$, $x_2' = -2x_1$.

$X_1' = 2X_2 \Rightarrow X_1'' = 2X_2'$

but $X_2' = -2X_1 \Rightarrow X_1'' = 2(-2X_1) = -4X_1$

$\therefore\ X_1'' = -4X_1 \Rightarrow X_1'' + 4X_1 = 0$

$r^2 + 4 = 0 \Rightarrow r = \pm 2i$, so $X_1 = C_1 \sin 2x + C_2 \cos 2x$

$X_2 = \frac{1}{2} X_1' = \frac{1}{2}(C_1 \sin 2x + C_2 \cos 2x)'$

$= \frac{1}{2}(2C_1 \cos 2x - 2C_2 \sin 2x)$

$= C_1 \cos 2x - C_2 \sin 2x$.