1) Let $V = \mathbb{R}^3$, and S be the set of all vectors (x, y, z) in V such that $x + 2y - 3z = 1$.

We will show that S is not a subspace of V.

Let (x_1, y_1, z_1) be element of S then $x_1 + 2y_1 - 3z_1 = 1$

But $2(x_1, y_1, z_1)$ is not in S because $2x_1 + 2(2y_1) - 3(2z_1) = 2(x_1 + 2y_1 - 3z_1) = 2 \cdot 1$
$= 2 \neq 1$

So S is not closed under scalar multiplication

2) $V = \mathbb{R}^3$ and S is the set of all vectors (x, y, z) in V such that $z = 2x$, $y = 5x$.

We will show that S is a subspace of V.

Why S is not empty?

First, we will show that S is closed under addition

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be elements in S.

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$
$$= (x_1 + x_2, 5x_1 + 5x_2, 2x_1 + 2x_2)$$
$$= (x_1 + x_2, 5(x_1 + x_2), 2(x_1 + x_2))$$
$$= (a, b, c) \text{ where } a = x_1 + x_2,\ b = 5a,\ c = 2a$$

Therefore $(x_1, y_1, z_1) + (x_2, y_2, z_2)$ is in S.

Let $K \in S$, $K(x_1, y_1, z_1) = (Kx_1, Ky_1, Kz_1) = (Kx_1, 5Kx_1, 2Kx_1)$
$= (a, b, c)$ where $a = Kx_1$, $b = 5a$ and $c = 2a$

Therefore $K(x_1, y_1, z_1)$ is in S.

So S is closed under addition and scalar multiplication. Hence, S is a subspace.

3)
$V = P_2(\mathbb{R})$ and S is the set of real polynomials $a_0 + a_1 x + a_2 x^2$
such that $a_0 + a_1 + a_2 = 1$.
We will show that S is not a subspace of V.
Let $f(x) = \frac{1}{2} + \frac{1}{2} x^2$ and $g(x) = \frac{1}{3} + \frac{1}{3} x + \frac{1}{3} x^2$
$f(x)$ is in S because $\frac{1}{2} + \frac{1}{2} = 1$ and $g(x)$ is in S because $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$
But $f(x) + g(x) = (\frac{1}{2} + \frac{1}{3}) + \frac{1}{3} x + (\frac{1}{2} + \frac{1}{3}) x^2$ which is not in S
because $\frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = 2 \neq 1$.

4) V be the vector space of all real valued functions
S the set of all solutions to $y' - x^2 y = 0$.
We will show that S is a subspace of V.

Why S is not empty?

a) S is closed under addition:
Let y_1 and y_2 be in S, we will show $y_1 + y_2$ in S.

$$(y_1 + y_2)' - x^2 (y_1 + y_2) \overset{?}{=} 0$$

$$\begin{aligned}(y_1 + y_2)' - x^2 (y_1 + y_2) &= y_1' + y_2' - x^2 y_1 - x^2 y_2 \\ &= \underbrace{(y_1' - x^2 y_1)}_{\substack{\text{zero because} \\ y_1 \text{ is in } S}} + \underbrace{(y_2' - x^2 y_2)}_{\substack{\text{zero because} \\ y_2 \text{ is in } S}} = 0 + 0 = 0\end{aligned}$$

Therefore $y_1 + y_2$ is in S.
b) S is closed under scalar multiplication:
Let y_1 be in S and k be in $\mathbb{R}$.

$$(k y_1)' - x^2 (k y_1) \overset{?}{=} 0$$

$$k y_1' - x^2 k y_1 = k (y_1' - x^2 y_1) = k \cdot 0 = 0 \Rightarrow k y_1 \text{ is in } S$$

Q2: Determine whether the vectors $(1,2,-1)$, $(0,1,2)$ and $(1,0,-3)$ span $\mathbb{R}^3$.

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & -3 \end{bmatrix}$

$$\text{Det } A = 1 \cdot \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} - 2 \begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -3 - 2(-2) - (-1) = -3 + 4 + 1 = 2 \neq 0$$

Since the determinant is not zero then the vectors are linearly independent.

Therefore the vectors span $\mathbb{R}^3$.