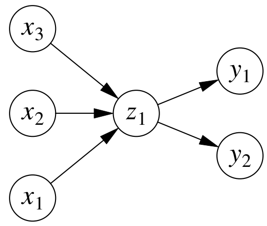
Solutions of PracticeQuestions on Digraphs.

We are going to use the short notation k-path for a path of length $k \geq 1$.

7. Consider the digraph below with 6 nodes and #(1-paths) = #(links) = 5. We have labeled the nodes here to talk about specific paths but recall that #(k-paths) is independent of node labels.



If we write $V_1 = \{x_1, x_2, x_3\}$ and $V_2 = \{y_1, y_2\}$, then the 2-paths are $\langle x_i, z_1, y_j \rangle$ and there are $|V_1| \times |V_2| = 3 \times 2 = 6$ such paths. Thus, #(2-paths) > #(1-paths).

You should be able to show now the following, by creating a suitable digraph for each case:

(a) (This requires a generalization of the digraph shown above.) There is a digraph with $n = p + q + 1$ nodes and $m = p + q$ links in which

$$\#(\text{2-paths}) = pq \text{ and } \#(\text{1-paths}) = p + q \text{ for any } p, q \geq 1.$$

Write down the 2-paths. (The digraph shown above corresponds to $p = 3$ and $q = 2$.)

(b) (This requires a modification of the digraph shown above.) There is a digraph with $n = 7$ nodes and $m = 6$ links in which

$$\#(\text{3-paths}) = 6 \text{ and } \#(\text{2-paths}) = 5.$$

Write down the 3-paths and 2-paths.

(c) (This requires a generalization of the digraph you obtained in (b).) Now assume that $k \geq 1$ (for the case (b), $k = 2$). There is a digraph with $n = 5+k$ and $m = 4+k$ links in which

$$\#((k+1)\text{-paths}) = 6 \text{ and } \#(\text{k-paths}) = 5.$$

Write down the $(k+1)$-paths and k-paths.

(d) (This requires a generalization of the digraph you obtained in (c).) Now assume that $k \geq 1$ as in (c). There is a digraph with $n = p + q + k$ nodes and $m = p + q + k - 1$ links in which

$$\#((k+1)\text{-paths}) = pq \text{ and } \#(k\text{-paths}) = p + q \text{ for any } p, q \geq 1.$$

In (d), you can choose p (≥ 3) and q (≥ 2) to show that #(($k+1$)-paths can be arbitrarily larger than #(k-paths).

In (d), you can choose $p = q = 1$ to show that #(($k+1$)-paths) = $pq = 1 < 2$ = #(k-paths).