Neutrino flavor conversions

Neutrinos are fundamental particles, they are fermions with spin 1/2; they have the smallest mass among fermions. Neutrinos interact weakly through Z or W boson exchange. The absence of electric charge and the weak cross section make neutrinos very useful probes in particle astrophysics.

Neutrino flavor conversions are determined by the mixing of the flavor eigenstates ν_α (with $\alpha = e, \mu, \tau$) with the mass eigenstates ν_i (with $i = 1, 2, 3$). The weak-interaction neutrino field ν_α can be expressed in terms of fields with definite mass ν_i through a unitary transformation:

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} \mathsf{U}^*_{\alpha i} |\nu_i\rangle \quad \text{and} \quad |\overline{\nu}_\alpha\rangle = \sum_{i=1}^{3} \mathsf{U}_{\alpha i} |\overline{\nu}_i\rangle \ . \tag{1}$$

The smallness of the absolute neutrino masses ($m_i \ll E$) allows for an important simplification of the neutrino propagation physics: since chirality flips ($\nu_L \to \nu_R$) have amplitude proportional to m_i/E in the Dirac equation, for typical neutrino energies ($\geq \mathcal{O}(\text{keV})$) one can neglect such rare transitions of chirality and forget about the spin structure of neutrinos. This means that we can treat neutrinos as wave functions (scalars) and use quantum-mechanics notation. We can identify $\nu = \nu_L$ and $\nu_R = \bar{\nu}$. The evolution equation then is

$$\imath \frac{\partial}{\partial t} \nu_\alpha = H_{\alpha\beta} \nu_\beta \ . \tag{2}$$

Let us define the mixing matrix $\mathbf{U}$. The usual parametrization is in terms of three Euler angles $(\theta_{12}, \theta_{13}, \theta_{23})$ and one CP-phase δ:

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

If no CP violation occurs, then $\mathbf{U}$ is real. If CP is violated, then $\mathbf{U}$ is complex. It can be shown that only one phase is relevant and can be placed in the inner rotation matrix. Note that CP violation requires $\theta_{13} \neq 0$ to be observable.

See also appendix B of arXiv:1910.11878.

Read this article for the latest developments on the CP-phase δ.

In the following, we will explore Hamiltonians of increasing complexity.

Example 1: Three massless neutrinos in vacuum

Let us write the evolution equation in the following way

$$\imath \frac{\partial}{\partial t} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = H \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \tag{4}$$

with $m(\nu_\alpha) = 0$. For a beam of momentum p, the Hamiltonian is

$$H = \begin{pmatrix} E_{\nu_e} & 0 & 0 \\ 0 & E_{\nu_\mu} & 0 \\ 0 & 0 & E_{\nu_\tau} \end{pmatrix} = p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

The solution to this equation is

$$|\nu_\alpha\rangle_t = e^{-\imath pt}|\nu_\alpha\rangle_0 \ . \tag{6}$$

This means that the flavor is conserved (i.e., no $\nu_\alpha \to \nu_\beta$). The flavor evolution introduces an overall phase for all three neutrinos

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \to e^{-\imath\phi} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \tag{7}$$

with $\phi = pt$. The phase ϕ is unobservable in squared amplitudes:

$$|\langle\nu_\alpha|\nu_\beta\rangle|^2 \to |\langle\nu_\beta|\nu_\alpha\rangle|^2 \ . \tag{8}$$

If $\mathbf{H} = \text{const.}\mathbf{1}+$something then the part proportional to the identity matrix is unobservable through propagation and only provides an overall phase.

Example 2: Three massive neutrinos in vacuum, no mixing

Let us assume that ν_e, ν_μ, ν_τ have masses m_1, m_2, m_3. For ultra-relativistic neutrinos ($m_i \ll E$), at first order one has a beam of neutrinos with momentum p

$$E_i = \sqrt{p^2 + m_i^2} \simeq p + \frac{m_i^2}{2p} = p + \frac{m_i^2}{2E} \ . \tag{9}$$

The Hamiltonian in the flavor basis is

$$H = \begin{pmatrix} E_{\nu_e} & 0 & 0 \\ 0 & E_{\nu_\mu} & 0 \\ 0 & 0 & E_{\nu_\tau} \end{pmatrix} = p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2E} \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} = p\mathbf{1} + \frac{\mathbf{M}^2}{2E} \ , \tag{10}$$

where $\mathbf{M}^2 = \text{diag}(m_1^2, m_2^2, m_3^2)$. The non-trivial part (i.e., the part non proportional to unity) is $\mathbf{M}^2/2E$. However, this part is diagonal in the flavor basis, and there are no flavor transitions. Each flavor state evolves independently, although with a different phase

$$|\nu_e\rangle_t = e^{-\imath pt}e^{-\imath m_1^2/2Et}|\nu_e\rangle_0 \tag{11}$$

$$|\nu_\mu\rangle_t = e^{-\imath pt}e^{-\imath m_2^2/2Et}|\nu_\mu\rangle_0 \tag{12}$$

$$|\nu_\tau\rangle_t = e^{-\imath pt}e^{-\imath m_3^2/2Et}|\nu_\tau\rangle_0 \tag{13}$$

and there is no $\nu_\alpha \to \nu_\beta$ with $\beta \neq \alpha$.

Example 3: Three massive neutrinos in vacuum with mixing

We assume that the flavor eigenstates are a linear combination of three mass eigenstates

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \mathbf{U}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3}\\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} \tag{14}$$

with $\mathbf{U}\mathbf{U}^\dagger = \mathbf{1}$.

The Hamiltonian is diagonal in the mass basis:

$$\mathbf{H}_{mb} = \frac{\mathbf{M}^2}{2E} + p\mathbf{1} = \frac{1}{2E}\begin{pmatrix}m_1^2 & 0 & 0\\ 0 & m_2^2 & 0\\ 0 & 0 & m_3^2\end{pmatrix} + \begin{pmatrix}p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{pmatrix} . \tag{15}$$

However, the Hamiltonian is not diagonal in the flavor basis $\nu_i = U^\dagger_{\alpha i}\nu_\alpha$

$$\imath\frac{\partial}{\partial t}\mathbf{U}^\dagger\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \mathbf{H}_{mb}\mathbf{U}^\dagger\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} \tag{16}$$

$$\imath\frac{\partial}{\partial t}\mathbf{U}\mathbf{U}^\dagger\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \mathbf{U}\mathbf{H}_{mb}\mathbf{U}^\dagger\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} \tag{17}$$

which implies

$$\imath\frac{\partial}{\partial t}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \mathbf{H}_{fb}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} \tag{18}$$

where $\mathbf{H}_{fb} = \mathbf{U}\mathbf{H}_{mb}\mathbf{U}^\dagger$. The Hamiltonian in the flavor basis is then

$$\mathbf{H}_{fb} = \frac{\mathbf{U}\mathbf{M}^2\mathbf{U}^\dagger}{2E} + p\mathbf{1} . \tag{19}$$

Neutrino flavor conversions with two neutrino species

Although three neutrino species exist and mix, let us consider a system with 2ν species only for simplicity

$$\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \begin{pmatrix}c_\theta & s_\theta\\ -s_\theta & c_\theta\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} . \tag{20}$$

with $s_\theta = \sin\theta$, $c_\theta = \cos\theta$, and

$$\mathbf{U} = \begin{pmatrix}c_\theta & s_\theta\\ -s_\theta & c_\theta\end{pmatrix} , \tag{21}$$

and $\mathbf{U}^\dagger = \mathbf{U}^T$, $\delta m^2 = m_2^2 - m_1^2$. The evolution equation in the flavor basis is

$$\imath \frac{\partial}{\partial t} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \mathbf{H}_{fb} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \tag{22}$$

where we are using $t \simeq x$ for ultrarelativistic neutrinos and

$$\mathbf{H}_{fb} = \frac{\mathbf{U}\mathbf{M}^2\mathbf{U}^\dagger}{2E} + p\mathbf{1} = \frac{1}{2E}\mathbf{U} \begin{pmatrix} -\frac{\delta m^2}{2} & 0 \\ 0 & \frac{\delta m^2}{2} \end{pmatrix} \mathbf{U}^\dagger + a\mathbf{1} \tag{23}$$

where a encloses all the multiplying factors proportional to the identiy matrix. Here we have used $m_1^2 = (m_1^2 + m_2^2)/2 - \delta m^2/2$ and $m_2^2 = (m_1^2 + m_2^2)/2 + \delta m^2/2$ where we shall drop the irrelevant part proportional to the identity matrix in the following. The evolution operator S which formally solves the evolution equation is

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}_x = S(x,0) \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}_0 \quad ; \tag{24}$$

S is obtained by exponentiation, since $\mathbf{H}_{fb}$ is x-independent:

$$S(x,0) = e^{-\imath H_{fb} x} = e^{\frac{-\imath}{4E}\mathbf{U}\mathrm{diag}(-\delta m^2, +\delta m^2)\mathbf{U}^\dagger x} = \begin{pmatrix} S_{ee} & S_{e\mu} \\ S_{\mu e} & S_{\mu\mu} \end{pmatrix} \tag{25}$$

If we start from a ν_e state

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{26}$$

after a pathlength x, we have

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}_x = \begin{pmatrix} S_{ee} & S_{e\mu} \\ S_{\mu e} & S_{\mu\mu} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} S_{ee} \\ S_{\mu e} \end{pmatrix} \tag{27}$$

and thus we have a non-zero amplitude for ν_μ

$$(\nu_\mu)_x = S_{\mu e} = -\imath \sin(2\theta) \sin\left(\frac{\delta m^2 x}{4E}\right) \tag{28}$$

and a finite probability that the initial flavor ν_e is transformed into ν_μ:

$$P(\nu_e \to \nu_\mu) = |S_{\mu e}|^2 = \sin^2(2\theta) \sin^2\left(\frac{\delta m^2 x}{4E}\right). \tag{29}$$

By unitarity, the survival probability of ν_e becomes

$$P(\nu_e \to \nu_e) = |S_{ee}|^2 = 1 - \sin^2(2\theta) \sin^2\left(\frac{\delta m^2 x}{4E}\right) \quad . \tag{30}$$

We have an oscillation pattern in $P(\nu_e \to \nu_\mu)$ with amplitude $\sin^2(2\theta)$ and phase $\delta m^2 x/(4E)$. Note that the oscillation physics in vacuum depends on $\sin^2(2\theta)$ instead of θ.

Three different regimes may be possible

- $\delta m^2 x/(4E) \gg 1$: fast oscillations, only average observed;

- $\delta m^2 x/(4E) \simeq 1$: oscillations visible;

- $\delta m^2 x/(4E) \ll 1$: vanishing oscillations.

Neutrino flavor conversions in matter

Let us work with three flavors and consider neutrino flavor conversions in matter.

Three massless neutrinos in matter

The evolution equation is:

$$\imath \frac{\partial}{\partial x} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = H \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \tag{31}$$

with $m(\nu) = 0$. The Hamiltonian is $\mathbf{H} = p\mathbf{1} + \mathbf{V}$ where $\mathbf{V}$ is the interaction energy in matter. The matter potential $\mathbf{V} = \mathrm{diag}(V_{NC}, V_{NC}, V_{NC}) + \mathrm{diag}(V_{CC}, 0, 0)$ where V_{NC} is the neutral current potential and $V_{CC} = \sqrt{2} G_F N_e$ is the charged current. V_{CC} is the relevant term to describe the interaction energy difference between ν_e and $\nu_{\mu,\tau}$. Frequently one defines $A = 2\sqrt{2} G_F N_e E$ which has the same units of δm^2.

Typical densities of electrons in astrophysical sources of interest are the electron density in the Sun $N_e(x) \simeq N_e(0) e^{-x/r_0}$ with $N_e(0) = 245$ mol/cm^3 and $r_0 = R_{sun}/10.54$. Electron density in a supernova $N_e \propto r^{-3}$.

Three massive neutrinos in matter with mixing

In the flavor basis, up to irrelevant terms proportional to the identity matrix

$$\mathbf{H}_{fb} = \frac{\mathbf{U}\mathbf{M^2}\mathbf{U}^\dagger}{2E} + \mathbf{V} = \frac{1}{2E}\mathbf{U}\mathrm{diag}(m_1^2, m_2^2, m_3^2)\mathbf{U}^\dagger + \sqrt{2} G_F N_e \mathrm{diag}(1, 0, 0) \ . \tag{32}$$

For antineutrinos $\mathbf{V} \to -\mathbf{V}$. Note that in general $N_e = N_e(x)$ and $\mathbf{H}$ is x-dependent. The evolution is non-trivial and it requires a case-by-case study, unless the electron density is constant $N_e =$ const. Let us focus on the simplified case of 2ν oscillations in constant electron density. This will first require the diagonalization of a 2×2 matrix.

Two flavor conversions in matter with a constant matter background

The Hamiltonian in the flavor basis is

$$\mathbf{H}_{fb} = \frac{\mathbf{UM^2U^\dagger}}{2E} + \mathbf{V} \tag{33}$$

$$= \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \frac{1}{2E} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} + \begin{pmatrix} \sqrt{2}G_F N_e & 0 \\ 0 & 0 \end{pmatrix} \tag{34}$$

by extracting a part proportional to **1**, we can always make this matrix traceless

$$\mathbf{H}_{fb} \to \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \frac{1}{2E} \begin{pmatrix} -\frac{\delta m^2}{2} & 0 \\ 0 & \frac{\delta m^2}{2} \end{pmatrix} \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} + \begin{pmatrix} +\frac{V}{2} & 0 \\ 0 & -\frac{V}{2} \end{pmatrix} \tag{35}$$

$$= \frac{1}{4E} \begin{pmatrix} A - \delta m^2 c_{2\theta} & \delta m^2 s_{2\theta} \\ \delta m^2 s_{2\theta} & -A + \delta m^2 c_{2\theta} \end{pmatrix} \tag{36}$$

with $A = 2\sqrt{2}G_F N_e E$ where we have a traceless matrix. The diagonalization angle is

$$\sin 2\tilde{\theta} = \frac{s_{2\theta}}{\sqrt{\left(c_{2\theta} - \frac{A}{\delta m^2}\right)^2 + s_{2\theta}^2}} \tag{37}$$

$$\cos 2\tilde{\theta} = \frac{c_{2\theta} - A/\delta m^2}{\sqrt{\left(c_{2\theta} - \frac{A}{\delta m^2}\right)^2 + s_{2\theta}^2}} \tag{38}$$

The eigenvalues if $\delta\tilde{m}^2$ is the squared mass difference in matter

$$\pm\frac{\delta\tilde{m}^2}{4E} = \pm\frac{\delta m^2}{4E}\frac{\sin 2\theta}{\sin 2\tilde{\theta}} \ . \tag{39}$$

and

$$\mathbf{H}_{fb} = \begin{pmatrix} c_{\tilde{\theta}} & s_{\tilde{\theta}} \\ -s_{\tilde{\theta}} & c_{\tilde{\theta}} \end{pmatrix} \begin{pmatrix} -\frac{\delta\tilde{m}^2}{4E} & 0 \\ 0 & +\frac{\delta\tilde{m}^2}{4E} \end{pmatrix} \begin{pmatrix} c_{\tilde{\theta}} & -s_{\tilde{\theta}} \\ s_{\tilde{\theta}} & c_{\tilde{\theta}} \end{pmatrix} \tag{40}$$

which is formally similar to the 2ν vacuum case with the replacements $\theta \to \tilde{\theta}$ and $\delta m^2 \to \delta\tilde{m}^2$.

The survival probability is

$$P(\nu_e \to \nu_\mu) = \sin^2 2\tilde{\theta} \sin^2\left(\frac{\delta\tilde{m}^2 L}{4E}\right) \tag{41}$$

The amplitude of oscillations in matter is governed by

$$\sin 2\tilde{\theta} = \frac{\sin 2\theta}{\sqrt{\left(\cos 2\theta - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta}} \tag{42}$$

This has the form of a Breit-Wigner resonance. This expression is such that even for small θ, one can have a resonant enhancement of the oscillation amplitude in vacuum at the resonance point. If $\cos 2\theta = A/\delta m^2$, then $\sin 2\tilde{\theta} \simeq 1$ (maximum oscillation amplitude). This is the so called MSW (Mikheyev-Smirnov-Wolfenstein) effect. Conversely if $A \gg \delta m^2$ (high density case), then $s_{2\tilde{\theta}} \to 0$ and oscillations are suppressed.

Matter can profoundly modify the oscillation behavior. Note that for $\nu \to \bar{\nu}$, one has $A \to -A$. Hence, the effects are different for neutrinos and antineutrinos.

See also appendix C of arXiv:1910.11878.