# 45 समाकलन ( INTEGRATION )

## महत्त्वपूर्ण सूत्र

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ जहाँ $n \neq -1$
2. $\int \frac{1}{x} dx = \log_e x + c$
3. $\int e^x dx = e^x + c$
4. $\int \text{Sin}x\, dx = -\text{Cos}x + c$
5. $\int \text{Cos}x\, dx = \text{Sin}x + c$
6. $\int \text{Sec}^2 x\, dx = \tan x + c$
7. $\int \text{Cosec}^2 x\, dx = -\text{Cot}x + c$
8. $\int \text{Sec}x.\tan x\, dx = \text{Sec}x + c$
9. $\int \text{Cosec}x.\text{Cot}x\, dx = -\text{Cosec}x + c$
10. $\int \frac{1}{\sqrt{1-x^2}} dx = \text{Sin}^{-1}x + c$
11. $\int \frac{-1}{\sqrt{1-x^2}} dx = \text{Cos}^{-1}x + c$
12. $\int \frac{1}{1+x^2} dx = \tan^{-1}x + c$
13. $\int \frac{-1}{1+x^2} dx = \text{Cot}^{-1}x + c$
14. $\int \frac{1}{x\sqrt{x^2-1}} dx = \text{Sec}^{-1}x + c$
15. $\int \frac{-1}{x\sqrt{x^2-1}} dx = \text{Cosec}^{-1}x + c$
16. $\int a^x dx = \frac{a^x}{\log_e a} + c$
17. $\int \tan x\, dx = \log(\text{Sec}x) + c$
18. $\int \text{Cot}x\, dx = \log \text{Sin}x + c$
19. $\int \text{Sec}x\, dx = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \log(\text{Sec}x + \tan x)$
20. $\int \text{Cosec}x.dx = \log\left|\tan\frac{x}{2}\right| = -\log(\text{Cosec}x + \text{Cot}x)$

## महत्त्वपूर्ण नियम

1. $\int dx = \int x^0.dx = \frac{x^{0+1}}{0+1} = x$
2. $\int k.f(x)dx = k.\int f(x)dx$ जहाँ k कोई constant है।

   जैसे - $\int 5x^4 dx = 5\int x^4 dx = 5.\frac{x^5}{5} = x^5 + c$
3. $\int (u \pm v \pm w \ldots)dx = \int u dx \pm \int v dx \pm \int w dx \pm \ldots$

   जहाँ u, v, w ..... इत्यादि सभी $x$ के फलन है।

   जैसे - $\int (4x^3 - 3e^x + 5)dx$

   $= 4\int x^3 dx - 3\int e^x dx + 5\int dx$

   $= 4.\frac{x^4}{4} - 3e^x + 5x + c = x^4 - 3e^x + 5x + c$
4. $\int uv dx = u\int v dx - \int\left[\int v dx.\frac{du}{dx}\right]dx$
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ जहाँ $a < c < b$

## समाकलन पर आधारित प्रश्न

1. $\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = ?$

***Speedy Solution*** :-

$$\int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = \int\left(x + 2 + \frac{1}{x}\right)dx = \frac{x^2}{2} + 2x + \log_e x + c$$

2. $\int \frac{1}{1+\text{Cos}x} dx = ?$

***Speedy Solution*** :-

$$\int \frac{1}{1+\text{Cos}x} dx = \int \frac{1}{2\text{Cos}^2\frac{x}{2}} dx = \frac{1}{2}\int \text{Sec}^2\frac{x}{2} dx$$

$$= \frac{1}{2} \times \frac{\tan\frac{x}{2}}{\frac{1}{2}} + c = \tan\frac{x}{2} + c$$

3. $\int \text{Cos}^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)dx = ?$

**Speedy Solution** :-

$$\int \text{Cos}^{-1}\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)dx$$

$$= \int \text{Cos}^{-1}(\text{Cos}\,2x)dx \quad \left[\because \frac{1-\tan^2 x}{1+\tan^2 x} = \text{Cos}\,2x\right]$$

$$= 2\int x\,dx = \frac{2x^2}{2} + c = x^2 + c$$

4. $\int \text{Cot}^{-1}\left(\frac{\sin 2x}{1-\cos 2x}\right)dx = ?$

**Speedy Solution** :-

$$\int \text{Cot}^{-1}\left(\frac{\sin 2x}{1-\cos 2x}\right)dx$$

$$= \int \text{Cot}^{-1}\left(\frac{2\,\text{Sin}\,x.\text{Cos}\,x}{2\,\text{Sin}^2 x}\right)dx$$

$$= \int \text{Cot}^{-1}(\text{Cot}\,x)dx = \int x\,dx = \frac{x^2}{2} + c$$

5. $\int \frac{1-\text{Cos}\,2x}{1+\text{Cos}\,2x}dx = ?$

**Speedy Solution** :-

$$\int \frac{1-\text{Cos}\,2x}{1+\text{Cos}\,2x}dx = \int \frac{2\,\text{Sin}^2 x}{2\,\text{Cos}^2 x}dx = \int \tan^2 x\,dx$$

$$= \int (\text{Sec}^2 x - 1)dx = \tan x - x + c$$

6. $\int \frac{dx}{1+\text{Sin}\,x} = ?$

**Speedy Solution** :-

$$\int \frac{dx}{1+\text{Sin}\,x} = \int \frac{1}{1+\text{Sin}\,x} \times \frac{1-\text{Sin}\,x}{1-\text{Sin}\,x}dx$$

$$= \int \frac{1-\text{Sin}\,x}{1-\text{Sin}^2 x}dx = \int \frac{1-\text{Sin}\,x}{\text{Cos}^2 x}dx = \int \tan x - \text{Sec}\,x + c$$

7. $\int \text{Cot}^2 x\,dx = ?$

**Speedy Solution** :-

$$\int \text{Cot}^2 x\,dx = \int (\text{Cosec}^2 x - 1)dx = -\text{Cot}\,x - x + c$$

8. $\int \text{Sin}^2 x\,dx = ?$

**Speedy Solution** :-

$$\int \text{Sin}^2 x\,dx = \int \frac{1-\text{Cos}\,2x}{2}dx = \frac{1}{2}\left(x - \frac{\text{Sin}\,2x}{2}\right) + c$$

9. $\int \frac{1}{1-\text{Cos}\,2x}dx = ?$

**Speedy Solution** :-

$$\int \frac{1}{1-\text{Cos}\,2x}dx = \int \frac{1}{2\,\text{Sin}^2 x}dx$$

$$= \frac{1}{2}\int \text{Cosec}^2 x\,dx = -\frac{1}{2}\text{Cot}\,x + c$$

10. $\int \sqrt{1-\text{Cos}\,2x}\,dx = ?$

**Speedy Solution** :-

$$\int \sqrt{1-\text{Cos}\,2x}\,dx = \int \sqrt{2\,\text{Sin}^2 x\,dx}$$

$$= \sqrt{2}\int \text{Sin}\,x\,dx = -\sqrt{2}\,\text{Cos}\,x + c$$

11. $\int \frac{x^2-1}{x^4-1}dx = ?$

**Speedy Solution** :-

$$\int \frac{x^2-1}{x^4-1}dx = \int \frac{x^2-1}{(x^2+1)(x^2-1)}dx = \int \frac{1}{1+x^2}dx = \tan^{-1}x + c$$

12. $\int \frac{x}{x+1}dx = ?$

**Speedy Solution** :-

$$\int \frac{x}{x+1}dx = \int \frac{x+1-1}{x+1}dx = \int \left(1 - \frac{1}{x+1}\right)dx = x - \log(x+1) + c$$

13. $\int x^{-7}dx = ?$

**Speedy Solution** :-

$$\int x^{-7}dx = \frac{x^{-7+1}}{-7+1} = \frac{-x^{-6}}{6} + c$$

14. $\int \frac{1}{1-\cos x}dx = ?$

**Speedy Solution** :-

$$\int \frac{1}{1-\cos x}dx = \int \frac{1}{2\sin^2\frac{x}{2}}dx = \frac{1}{2}\int \text{cosec}^2\frac{x}{2}dx$$

$$= \frac{1}{2} \times \frac{-\cot\frac{x}{2}}{\frac{1}{2}} = -\cot\frac{x}{2} + c$$

15. $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\}dx = ?$

**Speedy Solution** :-

$$\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\}dx = \int \tan^{-1}\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}dx = \int \tan^{-1}(\tan x)dx$$

$$= \int x.\,dx = \frac{x^2}{2} + c$$

## PREVIOUS YEAR'S RRB'S QUESTIONS

1. $\int_0^{2\pi}\sqrt{\left(1+\text{Sin}\frac{x}{2}\right)}\,dx$ का मान ज्ञात करें –

(A) 5 (B) 8 (C) 6 (D) 1

(RRB महेन्द्रूघाट Diesel Driver, 2001)

***Speedy Solution*** : (B)

$$\int_0^{2\pi}\sqrt{\left(1+\text{Sin}\frac{x}{2}\right)}\,dx$$

$$=\int_0^{2\pi}\sqrt{\left(\text{Sin}^2\frac{x}{4}+\text{Cos}^2\frac{x}{4}+2\text{Sin}\frac{x}{4}.\text{Cos}\frac{x}{4}\right)}dx$$

$$=\int_0^{2\pi}\sqrt{\left(\text{Sin}\frac{x}{4}+\text{Cos}\frac{x}{4}\right)^2}\,dx=\frac{1}{\left(\frac{1}{4}\right)}\left[-\text{Cos}\frac{x}{4}+\text{Sin}\frac{x}{4}\right]_0^{2\pi}$$

$$=4\left[\left(-\text{Cos}\frac{\pi}{2}+\text{Sin}\frac{\pi}{2}\right)-(-\text{Cos}\,0+\text{Sin}\,0)\right]$$

$$=4[(0+1)-1(-1+0)]=8$$

2. $\int\frac{dx}{1+\text{Sin}x}$ का मान निकालें –

(A) $\tan x-\text{Sec}\,x+c$ (B) $\text{Sin}x-\text{Sec}\,x+c$

(C) $\text{Sec}\,x+c$ (D) $-\tan x+c$

(RRB कोलकता Apprentice, Supervisor., 2001)

***Speedy Solution*** : (A)

$$\int\frac{1}{1+\text{Sin}x}dx$$

$$\int\frac{1.(1-\text{Sin}x)}{(1+\text{Sin}x).(1-\text{Sin}x)}dx$$

$$\int\frac{1-\text{Sin}x}{1-\text{Sin}^2x}dx=\int\frac{1-\text{Sin}x}{\text{Cos}^2x}dx$$

$$\int\frac{1}{\text{Cos}^2x}dx-\int\frac{\text{Sin}x}{\text{Cos}^2x}dx$$

$$\int\text{Sec}^2x\,dx-\int\tan x.\text{Sec}x\,dx=\tan x-\text{Sec}\,x+c$$

3. $\int_0^{\pi/2}\frac{\sqrt{\text{Sin}x}}{\sqrt{\text{Sin}x}+\sqrt{\text{Cos}x}}dx$ का मान क्या होगा ?

(A) $2\pi$ (B) $\pi$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

(RRB कोलकता Apprentice, Supervisor., 2001)

***Speedy Solution*** : (D)

माना कि $I=\int_0^{\pi/2}\frac{\sqrt{\text{Sin}x}}{\sqrt{\text{Sin}x}+\sqrt{\text{Cos}x}}dx$ ...(i)

तब, $I=\int_0^{\pi/2}\frac{\sqrt{\text{Sin}\left(\frac{x}{2}-x\right)}}{\sqrt{\text{Sin}\left(\frac{\pi}{2}-x\right)}+\sqrt{\text{Cos}\left(\frac{\pi}{2}-x\right)}}dx$

$$\left[\because\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

या, $I=\int_0^{\pi/2}\frac{\sqrt{\text{Cos}x}}{\sqrt{\text{Cos}x}+\sqrt{\text{Sin}x}}dx$ ...(ii)

समीकरण (i) और (ii) को जोड़ने पर

$$2I=\int_0^{\pi/2}\frac{\sqrt{\text{Sin}x}}{\sqrt{\text{Sin}x}+\sqrt{\text{Cos}x}}dx+\int_0^{\pi/2}\frac{\sqrt{\text{Cos}x}}{\sqrt{\text{Cos}x}+\sqrt{\text{Sin}x}}dx$$

$$=\int_0^{\pi/2}\frac{\sqrt{\text{Sin}x}+\sqrt{\text{Cos}x}}{\sqrt{\text{Sin}x}+\sqrt{\text{Cos}x}}dx=\int_0^{\pi/2}1dx=[x]_0^{\pi/2}=\frac{\pi}{2}$$

$$\therefore I=\frac{\pi}{2}$$

4. $\int\tan^2x\,dx$ का मान निकालें ?

(A) $\tan x-x+c$ (B) $\tan x-\text{Sec}\,x+c$

(C) $\text{Sec}\,x+c$ (D) $\text{Sec}\,x$

(RRB भुवनेश्वर Tech., 2001)

***Speedy Solution*** : (A)

$$\int\tan^2x\,dx=\int(\text{Sec}^2x-1)=\int\text{Sec}^2x\,dx-\int dx=\tan x-x+c$$

5. $\int\frac{dx}{1-\text{Sin}x}=?$

(A) $\tan x+\text{Sec}\,x+c$ (B) $\text{Sin}x+\text{Sec}\,x+c$

(C) $\text{Sin}x+\tan x$ (D) $\tan x+c$

(RRB मालदा A.S.M., 2004)

***Speedy Solution*** : (A)

$$\int\frac{dx}{1-\text{Sin}x}=\int\frac{1+\text{Sin}x}{(1-\text{Sin}x)(1+\text{Sin}x)}dx=\int\frac{1+\text{Sin}x}{1-\text{Sin}^2x}dx$$

$$= \int \frac{1+\sin x}{\cos^2 x}dx = \int\left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)dx$$

$$= \int \sec^2 x dx + \int \sec x.\tan x dx = \tan x + \sec x + c$$

6. $\int \frac{d\theta}{1+\cos\theta} = ?$

(A) $\sin\frac{\theta}{2}+c$ (B) $\tan\frac{\theta}{2}+c$ (C) $\sin\theta+c$ (D) $\tan\theta+c$

(RRB राँची A.S.M., 2002)

**Speedy Solution** : (B)

$$\int \frac{d\theta}{1+\cos\theta} = \int \frac{d\theta}{2\cos^2\frac{\theta}{2}} = \frac{1}{2}\int \sec^2\frac{\theta}{2}d\theta$$

$$= \frac{1}{2}.2\tan\frac{\theta}{2}+c = \tan\frac{\theta}{2}+c$$

7. $\int \frac{e^x}{2+e^x}dx = ?$

(A) $\log(2+e^x)+c$ (B) $\log e^x$

(C) $\log e^x + c$ (D) $\sin e^x + c$

(RRB गोरखपुर E.S.M. - II, 2003)

**Speedy Solution** : (A)

माना कि $2+e^x = y$

$\because e^x dx = dy$ $\therefore \int \frac{e^x}{2+e^x}dx$

$\therefore \int \frac{dy}{y} = \log y + c = \log(2+e^x)+c$

8. $\int \sqrt{1-\cos 2x}\,dx = ?$

(A) $\sin x + c$ (B) $\cos x + c$

(C) $-\sqrt{2}\sin x + c$ (D) $-\sqrt{2}\cos x + c$

(RRB चेन्नई J.E., 2002)

**Speedy Solution** : (D)

$$\int \sqrt{1-\cos 2x}\,dx = \int \sqrt{2\sin^2 x}dx = \sqrt{2}\int \sin x dx = -\sqrt{2}\cos x + c$$

9. $\int \frac{x^2-1}{x^4-1}dx = ?$

(A) $\sec^{-1}x + c$ (B) $\cos^{-1}x + c$

(C) $\tan^{-1}x + c$ (D) $\sin^{-1}x + c$

(RRB कोलकाता Asst. Driver, 2003)

**Speedy Solution** : (C)

$$\int \frac{x^2-1}{x^4-1}dx = \int \frac{x^2-1}{(x^2+1)(x^2-1)}dx = \int \frac{1}{1+x^2}dx = \tan^{-1}x + c$$

10. $\int \frac{1}{\sin^2 x.\cos^2 x}dx = ?$

(A) $-2\cot 2x + c$ (B) $\cot x + c$

(C) $\tan x + \cot x + c$ (D) $\tan x - \cot x + c$

(RRB गोरखपुर Goods Guard, 2003)

**Speedy Solution** : (A)

$$\int \frac{1}{\sin^2 x.\cos^2 x}dx = 4\int \frac{1}{4\sin^2 x.\cos^2 x}dx$$

$$= 4\int \frac{1}{(2\sin x.\cos x)^2}dx = 4\int \frac{1}{\sin^2 2x}dx$$

$$= 4\int \operatorname{cosec}^2 2x dx = 4\times -\frac{\cot 2x}{2}+c$$

$$= -2\cot 2x + c$$

11. $\int \tan^2 x dx = ?$

(A) $-\cot x - x + c$ (B) $\tan x - x + c$

(C) $\tan x - \frac{x^2}{2} + c$ (D) $\cot x + x + c$

(RRB मुम्बई Loco Pilot, 2004)

**Speedy Solution** : (B)

$$\int \tan^2 x dx = \int (\sec^2 x - 1)dx = \tan x - x + c$$

12. $\int \sin^2\frac{x}{2}dx = ?$

(A) $x - \sin x + c$ (B) $\frac{x}{2} - \sin x + c$

(C) $\frac{x}{2} - \frac{\sin x}{2} + c$ (D) $\frac{x}{2} - \cos x + c$

(RRB सिकन्दराबाद T.A., 2004)

**Speedy Solution** : (C)

$$\int \sin^2\frac{x}{2}dx = \int \frac{1-\cos x}{2}dx = \frac{1}{2}(x-\sin x)+c = \frac{x}{2}-\frac{\sin x}{2}+c$$

13. $\int (3x+5)^7 dx = ?$

(A) $\frac{(3x+5)^8}{8}+c$ (B) $\frac{(3x+5)^6}{6}+c$

(C) $\frac{(3x+5)^8}{24}+c$ (D) $18(3x+5)^6$

(RRB कोलकाता Diesel Driver, 2004)

**Speedy Solution** : (C)

$$\int (3x+5)^7 dx = \frac{(3x+5)^{7+1}}{(7+1)\times 3}+c = \frac{(3x+5)^8}{24}+c$$