main result：

$$\beta_m \leq \left(\eta \alpha_\ell \alpha_\sigma v_\ell (\lambda_G^{max})^2 \sum_{t=1}^{T} (1 + \eta v_\ell v_\sigma (\lambda_G^{max})^2)^{t-1} \right) / m$$

$$\sup_{S,z} \left| \mathbf{E}_A[\ell(A_S, \mathbf{z})] - \mathbf{E}_A[\ell(A_{S^i}, \mathbf{z})] \right| \leq 2\beta_m$$

$$|\mathbf{E}_{\mathrm{SGD}}[\ell(A_S, y) - \ell(A_{S^i}, y)]| \leq \alpha_\ell \mathbf{E}_{\mathrm{SGD}}[|f(\mathbf{x}, \theta_S) - f(\mathbf{x}, \theta_{S^i})|]$$

$|\mathbf{E}[x]| \leq \mathbf{E}[|x|]$

Lipschitz continuous

$$\leq \alpha_\ell \mathbf{E}_{\mathrm{SGD}} \left[\left| \sigma \left(\sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_j \theta_S \right) - \sigma \left(\sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_j \theta_{S^i} \right) \right| \right]$$

σ−Lipschitz continuous

$$\leq \alpha_\ell \mathbf{E}_{\mathrm{SGD}} \left[\left| \sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_j \theta_S - \sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_j \theta_{S^i} \right| \right]$$

$$\leq \alpha_\ell \mathbf{E}_{\mathrm{SGD}} \left[\left| \sum_{\substack{j\in \\ \mathcal{N}(\mathbf{x})}} (e_{\cdot j}\mathbf{x}_j) \right| (|\theta_S - \theta_{S^i}|) \right]$$

$$\leq \alpha_\ell \left| \sum_{\substack{j\in \\ \mathcal{N}(\mathbf{x})}} (e_{\cdot j}\mathbf{x}_j) \right| (\mathbf{E}_{\mathrm{SGD}}[|\Delta\theta|])$$

$$\leq \alpha_\ell \mathbf{g}_\lambda \mathbf{E}_{\mathrm{SGD}}[|\Delta\theta|]$$

$$\boxed{\mathbf{g}_\lambda \coloneqq \sup_{\mathbf{x}} \left| \sum_{j\in\mathcal{N}(\mathbf{x})} e_{\cdot j}\mathbf{x}_{\mathbf{j}} \right|}$$

We will bound $\mathbf{g}_\lambda$ in terms of the largest absolute eigenvalue of the graph convolution filter $g(\mathbf{L})$ later.

Proof Part 2 (SGD Based Bounds For GCNN Weights):

$$\min_{\theta} \mathcal{L}(f(\mathbf{x}, \theta_S), y) = \frac{1}{m} \sum_{i=1}^{m} \ell(f(\mathbf{x}, \theta_S), y_i)$$

$$\boldsymbol{\theta}_{S,t+1} = \boldsymbol{\theta}_{S,t} - \eta \nabla \ell\left(f\left(\mathbf{x}_{\mathbf{i_t}}, \boldsymbol{\theta}_{S,t}\right), y_{i_t}\right)$$

$$\Delta\theta_t := \left|\theta_{S,t} - \theta_{S^i,t}\right|$$

$$Case1: |\Delta\boldsymbol{\theta}_{t+1}| \le |\Delta\boldsymbol{\theta}_t| + \eta\left|\nabla\ell\left(f\left(\mathbf{x}, \boldsymbol{\theta}_{S,t}\right), y\right) - \ell\left(f\left(\mathbf{x}, \theta_{S,t}\right), y\right)\right|$$

lemma1

$$Case2: \ |\Delta\theta_{t+1}| \le |\Delta\boldsymbol{\theta}_t| + \eta \mid \nabla\ell(f(\mathbf{x}_i, \theta_{S,t}), y_i) - \ell(f(\mathbf{x}'_i, \boldsymbol{\theta}_{S,t}), y'_i)|$$

lemma2

Lemma1(Case1):[GCNN Same Sample Loss Stability Bound]

$$\left|\nabla \ell(f(\mathbf{x}, \theta_{S,t}), y) - \nabla \ell(f(\mathbf{x}, \theta_{S^i,t}), y)\right| \leq v_\ell v_\sigma \mathbf{g}_\lambda^2 |\Delta\theta_t|.$$

$$\left|\nabla \ell(f(\mathbf{x}, \theta_{S,t}), y) - \nabla \ell(f(\mathbf{x}, \theta_{S^i,t}), y)\right|$$
$$\leq v_\ell \left|\nabla f(\mathbf{x}, \theta_{S,t}) - \nabla f(\mathbf{x}, \theta_{S^i,t})\right|$$

l-Lipschitz continuous and smooth

$$\leq v_\ell \mid \nabla\sigma\left(\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x})}} e_{\cdot j}\mathbf{x}_j\theta_{S,t}\right)\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x})}} e_{\cdot j}\mathbf{x}_j - \nabla\sigma\left(\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x})}} e_{\cdot j}\mathbf{x}_j\theta_{S^i,t}\right)\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x})}} e_{\cdot j}\mathbf{x}_j|$$

$$\leq v_\ell \left(\left|\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x})}} e_{\cdot j}\mathbf{x}_j\right|\right)\left|\nabla\sigma\left(\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x})}} e_{\cdot j}\mathbf{x}_j\theta_{S,t}\right) - \nabla\sigma\left(\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x})}} e_{\cdot j}\mathbf{x}_j\theta_{S^i,t}\right)\right|$$

$$\leq v_\ell v_\sigma \mathbf{g}_\lambda \left| \left(\sum_{\substack{j\in \\ \mathcal{N}(\mathbf{x})}} e_{\cdot j}\mathbf{x}_j \theta_{S,t} \right) - \left(\sum_{\substack{j\in \\ \mathcal{N}(\mathbf{x})}} e_{\cdot j}\mathbf{x}_j \theta_{S^i,t} \right) \right|$$

$$\leq v_\ell v_\sigma \mathbf{g}_\lambda \left(\left| \sum_{\substack{j\in \\ \mathcal{N}(\mathbf{x})}} e_{\cdot j}\mathbf{x}_j \right| \right) \left| \theta_{S,t} - \theta_{S^i,t} \right|$$

$$\leq v_\ell v_\sigma \mathbf{g}_\lambda^2 |\Delta\theta_t|$$

Lipschitz continuous and smooth

$$\boxed{\left| \sum_{\substack{j\in \\ \mathcal{N}(\mathbf{x})}} e_{\cdot j}\mathbf{x}_j \right| \leq \mathbf{g}_\lambda}$$

Lemma2(Case2):[GCNN Different Sample Loss Stability Bound]

$$\left|\nabla\ell(f(\mathbf{x}_i,\theta_{S,t}),y_i)-\nabla\ell\left(f\left(\mathbf{x}_i',\theta_{S^i,t}\right),y_i'\right)\right|\leq 2v_\ell\alpha_\sigma\mathbf{g}_\lambda.$$

$$\left|\nabla\ell(f(\mathbf{x},\boldsymbol{\theta}_{S,t}),y)-\nabla\ell\left(f\left(\mathbf{x}',\boldsymbol{\theta}_{S^i,t}\right),y'\right)\right|$$

$$\leq v_\ell\left|\nabla f(\mathbf{x},\boldsymbol{\theta}_{S,t})-\nabla f\left(\mathbf{x}',\theta_{S^i,t}\right)\right|$$

$\boxed{\text{l-Lipschitz continuous and smooth}}$

$$\leq v_\ell\left|\nabla\sigma\left(\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x})}}e_{\cdot j}\mathbf{x}_j\theta_{S,t}\right)\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x})}}e_{\cdot j}\mathbf{x}_j-\nabla\sigma\left(\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x}')}}e_{\cdot j}\mathbf{x}_j'\theta_{S^i,t}\right)\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x}')}}e_{\cdot j}\mathbf{x}_j'\right|$$

$$\boxed{|a-b|\leq|a|+|b|}$$

$$\leq v_\ell\left|\nabla\sigma\left(\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x})}}e_{\cdot j}\mathbf{x}_j\theta_{S,t}\right)\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x})}}e_{\cdot j}\mathbf{x}_j\right|+v_\ell\left|\nabla\sigma\left(\sum_{j\in,}e_{\cdot j}\mathbf{x}_j'\theta_{S^i,t}\right)\sum_{\substack{j\in\\ \mathcal{N}(\mathbf{x}')}}e_{\cdot j}\mathbf{x}_j'\right|\leq 2v_\ell\alpha_\sigma\mathbf{g}_\lambda$$

Lemma3: [GCNN SGD Stability Bound]

$$\mathbf{E}_{SGD}\left[\left|\boldsymbol{\theta}_{S,T} - \theta_{S^i,T}\right|\right] \leq \frac{2\eta v_\ell \alpha_\sigma \mathbf{g}_\lambda}{m} \sum_{t=1}^{T} \left(1 + \eta v_\ell v_\sigma \mathbf{g}_\lambda^2\right)^{t-1}$$

$$\mathbf{E}_{\mathrm{SGD}}[|\Delta\theta_{t+1}|]$$

$$\leq \left(1 - \frac{1}{m}\right)\mathbf{E}_{\mathrm{SGD}}[|\ (\theta_{S,t} - \eta\nabla\ell(f(\mathbf{x}, \theta_{S,t}), y)) - (\theta_{S^i,t} - \eta\nabla\ell\big(f\big(\mathbf{x}, \theta_{S^i,t}\big), y\big))\ |] +$$

$$\left(\frac{1}{m}\right)\mathbf{E}_{\mathrm{SGD}}[|\ (\theta_{S,t} - \eta\nabla\ell\big(f\big(\mathbf{x}', \theta_{S,t}\big), y'\big)) - (\theta_{S^i,t} - \eta\nabla\ell\big(f\big(\mathbf{x}'', \theta_{S^i,t}\big), y''\big))\ |]$$

$$\leq \left(1 - \frac{1}{m}\right)\mathbf{E}_{\mathrm{SGD}}[|\Delta\theta_t|] + \left(1 - \frac{1}{m}\right)\eta\mathbf{E}_{\mathrm{SGD}}[|\ \nabla\ell(f(\mathbf{x}, \theta_{S,t}), y) -$$

$$\nabla\ell\big(f\big(\mathbf{x}, \theta_{S^i,t}\big), y\big)\ |] + \left(\frac{1}{m}\right)\mathbf{E}_{\mathrm{SGD}}[|\Delta\boldsymbol{\theta}_t|] +$$

$$\left(\frac{1}{m}\right)\eta\mathbf{E}_{\mathrm{SGD}}\left[\left|\nabla\ell\big(f\big(\mathbf{x}', \theta_{S,t}\big), y'\big) - \nabla\ell\big(f\big(\mathbf{x}'', \boldsymbol{\theta}_{S^i,t}\big), y''\big)\right|\right]$$

$$
\begin{aligned}
=&\mathbf{E}_{\mathrm{SGD}}[|\Delta\theta_t|]+\\
&\left(1-\frac{1}{m}\right)\eta\mathbf{E}_{\mathrm{SGD}}\left[\left|\nabla\ell(f(\mathbf{x},\theta_{S,t}),y)-\nabla\ell\left(f\left(\mathbf{x},\theta_{S^i,t}\right),y\right)\right|\right]+\\
&\left(\frac{1}{m}\right)\eta\mathbf{E}_{\mathrm{SGD}}\left[\left|\left(\nabla\ell\left(f\left(\mathbf{x}',\theta_{S,t}\right),y'\right)\right)-\left(\nabla\ell\left(f\left(\mathbf{x}'',\theta_{S^i,t}\right),y''\right)\right)\right|\right].
\end{aligned}
$$

(Lemma1 && lemma2)

$$
\begin{aligned}
\mathbf{E}_{\mathrm{SGD}}[|\Delta\theta_{t+1}|] &\leq \mathbf{E}_{\mathrm{SGD}}[|\Delta\theta_t|]+\left(1-\frac{1}{m}\right)\eta v_\ell v_\sigma \mathbf{g}_\lambda^2\mathbf{E}_{\mathrm{SGD}}[|\Delta\theta_t|]\\
&\quad+\left(\frac{1}{m}\right)2\eta v_\ell\alpha_\sigma\mathbf{g}_\lambda\\
&=\left(1+\left(1-\frac{1}{m}\right)\eta v_\ell v_\sigma\mathbf{g}_\lambda^2\right)\mathbf{E}_{\mathrm{SGD}}[|\Delta\theta_t|]+\frac{2\eta v_\ell\alpha_\sigma\mathbf{g}_\lambda}{m}\\
&\leq\left(1+\eta v_\ell v_\sigma\mathbf{g}_\lambda^2\right)\mathbf{E}_{\mathrm{SGD}}[|\Delta\theta_t|]+\frac{2\eta v_\ell\alpha_\sigma\mathbf{g}_\lambda}{m}.
\end{aligned}
$$

Lastly, solving the $\mathbf{E}_{\mathrm{SGD}}[|\Delta\theta_t|]$ first order recursion yields,

$$\mathbf{E}_{\mathrm{SGD}}[|\Delta\boldsymbol{\theta}_T|] \leq \frac{2\eta v_\ell \alpha_\sigma \mathbf{g}_\lambda}{m} \sum_{t=1}^{T} \left(1 + \eta v_\ell v_\sigma \mathbf{g}_\lambda^2\right)^{t-1}$$

$$|[g_{\mathbf{x}}(\mathbf{L})\mathbf{h}_{\mathbf{x}}]_0| \leq \|g_{\mathbf{x}}(\mathbf{L})\mathbf{h}_{\mathbf{x}}\|_1 \leq \|\|g_{\mathbf{x}}(\mathbf{L})\|_2\|\mathbf{h}_{\mathbf{x}}\|_2 = \lambda_{G_{\mathbf{x}}}^{\max}$$

Cauchy–Schwarz Inequality

$$\lambda_{G_{\mathbf{x}}}^{\max} \leq \lambda_{G}^{\max} \qquad \mathbf{g}_{\lambda} \leq \lambda_{G}^{\max}$$

$$2\beta_m \leq \alpha_\ell \lambda_G^{\max} \mathbf{E}_{\mathrm{SGD}}[|\Delta\theta|]$$

$$\beta_m \leq \frac{\eta \alpha_\ell \alpha_\sigma v_\ell \left(\lambda_G^{\max}\right)^2 \sum_{t=1}^{T} \left(1 + \eta v_\ell v_\sigma \left(\lambda_G^{\max}\right)^2\right)^{t-1}}{m}$$

$$\beta_m \leq \frac{1}{m}\mathcal{O}\left((\lambda_G^{\max})^{2T}\right) \quad \forall T \geq 1$$