main result:

$$\beta_m \leq \left(\eta \alpha_\ell \alpha_\sigma v_\ell (\lambda_G^{max})^2 \sum_{t=1}^T (1 + \eta v_\ell v_\sigma (\lambda_G^{max})^2)^{t-1} \right) / m$$

$$\sup_{S,z} \lvert \mathbf{E}_A[\ell(A_S,\mathbf{z})] - \mathbf{E}_A[\ell(A_{S^i},\mathbf{z})]
vert \leq 2eta_m$$

 $|\mathbf{E}_{ ext{SGD}}[\ell(A_S,y) - \ell(A_{S^i},y)]| \leq lpha_\ell \mathbf{E}_{ ext{SGD}}[|f(\mathbf{x}, heta_S) - f(\mathbf{x}, heta_{S^i})|]$

$$|\mathrm{E}[x]| \leq \mathbf{E}[|x|]$$

Lipschitz continuous

$$\leq lpha_{\ell} \mathbf{E}_{ ext{SGD}} \left[\left| \sigma \left(\sum_{\substack{j \in \ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_{j} heta_{S}
ight) - \sigma \left(\sum_{\substack{j \in \ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_{j} heta_{S^{i}}
ight)
ight|
ight]$$

 σ –Lipschitz continuous

$$\leq lpha_{\ell} \mathbf{E}_{ ext{SGD}} \left[\left| \sum_{\substack{j \in \ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_{j} heta_{S} - \sum_{\substack{j \in \ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_{j} heta_{S^{i}}
ight|
ight]$$

$$\leq lpha_{\ell} \mathbf{E}_{ ext{SGD}} \left[\left| \sum_{\substack{j \in \ \mathcal{N}(\mathbf{x})}} (e_{\cdot j} \mathbf{x}_{j}) \right| (| heta_{S} - heta_{S^{i}}|)
ight] \ \leq lpha_{\ell} \left| \sum_{\substack{j \in \ \mathcal{N}(\mathbf{x})}} (e_{\cdot j} \mathbf{x}_{j}) \left| (\mathbf{E}_{ ext{SGD}}[|\Delta heta|])
ight.$$

$$\leq lpha_\ell \mathbf{g}_\lambda \mathbf{E}_{\mathrm{SGD}}[|\Delta heta|]$$

$$\mathbf{g}_{\lambda} \coloneqq \sup_{\mathbf{x}} \left| \sum_{j \in \mathcal{N}(\mathbf{x})} e_{\cdot j} \mathbf{x}_{\mathbf{j}} \right|$$

We will bound \mathbf{g}_{λ} in terms of the largest absolute eigenvalue of the graph convolution filter $g(\mathbf{L})$ later.

Proof Part 2 (SGD Based Bounds For GCNN Weights):

$$\min_{ heta} \mathcal{L}(f(\mathbf{x}, heta_S), y) = rac{1}{m} \sum_{i=1}^m \ell(f(\mathbf{x}, heta_S), y_i)$$

$$\boldsymbol{\theta}_{S,t+1} = \boldsymbol{\theta}_{S,t} - \eta \nabla \ell(f(\mathbf{x}_{i_t}, \boldsymbol{\theta}_{S,t}), y_{i_t})$$

$$\Delta\theta_t \coloneqq \left|\theta_{S,t} - \theta_{S^i,t}\right|$$

Case1:
$$|\Delta \boldsymbol{\theta}_{t+1}| \le |\Delta \boldsymbol{\theta}_{t}| + \eta |\nabla \ell(f(\mathbf{x}, \boldsymbol{\theta}_{S,t}), y) - \ell(f(\mathbf{x}, \theta_{S,t}), y)|$$

lemma1

Case2:
$$|\Delta \theta_{t+1}| \leq |\Delta \theta_t| + \eta |\nabla \ell(f(\mathbf{x}_i, \theta_{S,t}), y_i) - \ell(f(\mathbf{x}_i', \theta_{S,t}), y_i')|$$

lemma2

Lemma1(Case1):[GCNN Same Sample Loss Stability Bound]

$$\left|
abla \ell(f(\mathbf{x}, heta_{S,t}), y) -
abla \ellig(fig(\mathbf{x}, heta_{S^i,t}ig), yig)
ight| \leq v_\ell v_\sigma \mathbf{g}_\lambda^2 |\Delta heta_t|.$$

$$\begin{aligned} & \left| \nabla \ell(f(\mathbf{x}, \theta_{S,t}), y) - \nabla \ell(f(\mathbf{x}, \theta_{S^{i},t}), y) \right| \\ & \leq v_{\ell} \left| \nabla f(\mathbf{x}, \theta_{S,t}) - \nabla f(\mathbf{x}, \theta_{S^{i},t}) \right| \end{aligned}$$

1-Lipschitz continuous and smooth

$$\leq v_{\ell} \mid \nabla \sigma \left(\sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{.j} \mathbf{x}_{j} \theta_{S,t} \right) \sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{.j} \mathbf{x}_{j} - \nabla \sigma \left(\sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{.j} \mathbf{x}_{j} \theta_{S^{i},t} \right) \sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{.j} \mathbf{x}_{j}$$

$$\leq v_{\ell} \left(\left| \sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{.j} \mathbf{x}_{j} \right| \right) \left| \nabla \sigma \left(\sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{.j} \mathbf{x}_{j} \theta_{S,t} \right) - \nabla \sigma \left(\sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{.j} \mathbf{x}_{j} \theta_{S^{i},t} \right) \right|$$

$$egin{aligned} & \leq v_\ell v_\sigma \mathbf{g}_\lambda \left| \left(\sum_{\substack{j \in \ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_j heta_{S,t}
ight) - \left(\sum_{\substack{j \in \ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_j heta_{S^i,t}
ight)
ight| \ & \leq v_\ell v_\sigma \mathbf{g}_\lambda^2 \left| \Delta heta_t
ight| \ & \leq v_\ell v_\sigma \mathbf{g}_\lambda^2 \left| \Delta heta_t
ight| \end{aligned}$$

Lipschitz continuous and smooth

$$\left| \sum_{\substack{j \in \ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_j
ight| \leq \mathbf{g}_{\lambda}$$

Lemma2(Case2):[GCNN Different Sample Loss Stability Bound]

$$\left|
abla \ell(f(\mathbf{x}_i, heta_{S,t}), y_i) -
abla \ellig(fig(\mathbf{x}_i', heta_{S^i,t}ig), y_i'ig)
ight| \leq 2v_\ell lpha_\sigma \mathbf{g}_\lambda.$$

$$egin{aligned} igg|
abla \ell(f(\mathbf{x},oldsymbol{ heta}_{S,t}),y) -
abla \ell(f(\mathbf{x}',oldsymbol{ heta}_{S^i,t}),y') igg| \ \le & v_\ell igg|
abla f(\mathbf{x},oldsymbol{ heta}_{S,t}) -
abla f(\mathbf{x}',oldsymbol{ heta}_{S^i,t}) igg| \end{aligned}$$

1-Lipschitz continuous and smooth

$$\leq v_{\ell} \left| \nabla \sigma \left(\sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_{j} \theta_{S,t} \right) \sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_{j} - \nabla \sigma \left(\sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x}')}} e_{\cdot j} \mathbf{x}_{j}' \theta_{S^{i},t} \right) \sum_{\substack{j \in \\ \mathcal{N}(\mathbf{x}')}} e_{\cdot j} \mathbf{x}_{j}' \right| \qquad |a - b| \leq |a| + |b|$$

$$|a - b| \le |a| + |b|$$

$$\leq v_{\ell} \left|
abla \sigma \left(\sum_{\substack{j \in \ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_{j} heta_{S,t}
ight) \sum_{\substack{j \in \ \mathcal{N}(\mathbf{x})}} e_{\cdot j} \mathbf{x}_{j} \right| + v_{\ell} \left|
abla \sigma \left(\sum_{j \in ,} e_{\cdot j} \mathbf{x}_{j}' heta_{S^{i},t}
ight) \sum_{\substack{j \in \ \mathcal{N}(\mathbf{x}')}} e_{\cdot j} \mathbf{x}_{j}'
ight| \ \leq 2 v_{\ell} lpha_{\sigma} \mathbf{g}_{\lambda}$$

Lemma3: [GCNN SGD Stability Bound]

$$\begin{split} \mathbf{E}_{SGD}\big[\big|\boldsymbol{\theta}_{S,T} - \boldsymbol{\theta}_{S^{i},T}\big|\big] &\leq \frac{2\eta v_{\ell}\alpha_{\sigma}\mathbf{g}_{\lambda}}{m} \sum_{t=1}^{T} \left(1 + \eta v_{\ell}v_{\sigma}\mathbf{g}_{\lambda}^{2}\right)^{t-1} \\ \mathbf{E}_{SGD}\big[\big|\Delta\boldsymbol{\theta}_{t+1}\big|\big] \\ &\leq \left(1 - \frac{1}{m}\right) \mathbf{E}_{SGD}\big[\big|\left(\theta_{S,t} - \eta \nabla \ell(f(\mathbf{x},\theta_{S,t}),y)) - \left(\theta_{S^{i},t} - \eta \nabla \ell(f(\mathbf{x},\theta_{S^{i},t}),y)\right)\big|\big] + \\ &\left(\frac{1}{m}\right) \mathbf{E}_{SGD}\big[\big|\left(\theta_{S,t^{-}} - \eta \nabla \ell(f(\mathbf{x}',\theta_{S,t}),y')) - \left(\theta_{S^{i},t} - \eta \nabla \ell(f(\mathbf{x}'',\theta_{S^{i},t}),y'')\right)\big|\big] \\ &\leq \left(1 - \frac{1}{m}\right) \mathbf{E}_{SGD}\big[\big|\Delta\boldsymbol{\theta}_{t}\big|\big] + \left(1 - \frac{1}{m}\right) \eta \mathbf{E}_{SGD}\big[\big|\nabla \ell(f(\mathbf{x},\theta_{S,t}),y) - \nabla \ell(f(\mathbf{x},\theta_{S^{i},t}),y')\big|\big] \\ &\nabla \ell(f(\mathbf{x},\theta_{S^{i},t}),y) \mid\big] + \left(\frac{1}{m}\right) \mathbf{E}_{SGD}\big[\big|\Delta\boldsymbol{\theta}_{t}\big|\big] + \\ &\left(\frac{1}{m}\right) \eta \mathbf{E}_{SGD}\big[\big|\nabla \ell(f(\mathbf{x}',\theta_{S,t}),y') - \nabla \ell(f(\mathbf{x}'',\boldsymbol{\theta}_{S^{i},t}),y'')\big|\big] \end{split}$$

$$\begin{split} &= &\mathbf{E}_{\mathrm{SGD}}[|\Delta\theta_t|] + \\ &\left(1 - \frac{1}{m}\right) \eta \mathbf{E}_{\mathrm{SGD}}\big[\big| \nabla \ell(f(\mathbf{x}, \theta_{S,t}), y) - \nabla \ell(f(\mathbf{x}, \theta_{S^i,t}), y) \big| \big] + \\ &\left(\frac{1}{m}\right) \eta \mathbf{E}_{\mathrm{SGD}}\big[\big| \big(\nabla \ell(f(\mathbf{x}', \theta_{S,t}), y')) - \big(\nabla \ell(f(\mathbf{x}'', \theta_{S^i,t}), y'')) \big| \big]. \end{split}$$

(Lemma1 && lemma2)

$$egin{align*} \mathbf{E}_{ ext{SGD}}[|\Delta heta_{t+1}|] &\leq \mathbf{E}_{ ext{SGD}}[|\Delta heta_t|] + \left(1 - rac{1}{m}
ight)\eta v_\ell v_\sigma \mathbf{g}_\lambda^2 \mathbf{E}_{ ext{SGD}}[|\Delta heta_t|] \ &+ \left(rac{1}{m}
ight)2\eta v_\ell lpha_\sigma \mathbf{g}_\lambda \ &= \left(1 + \left(1 - rac{1}{m}
ight)\eta v_\ell v_\sigma \mathbf{g}_\lambda^2
ight)\mathbf{E}_{ ext{SGD}}[|\Delta heta_t|] + rac{2\eta v_\ell lpha_\sigma \mathbf{g}_\lambda}{m} \ &\leq \left(1 + \eta v_\ell v_\sigma \mathbf{g}_\lambda^2
ight)\mathbf{E}_{ ext{SGD}}[|\Delta heta_t|] + rac{2\eta v_\ell lpha_\sigma \mathbf{g}_\lambda}{m}. \end{split}$$

Lastly, solving the $\mathbf{E}_{\mathrm{SGD}}[|\Delta heta_t|]$ first order recursion yields,

$$\mathbf{E}_{ ext{SGD}}[|\Deltaoldsymbol{ heta}_T|] \leq rac{2\eta v_\ell lpha_\sigma \mathbf{g}_\lambda}{m} \sum_{t=1}^T \left(1 + \eta v_\ell v_\sigma \mathbf{g}_\lambda^2
ight)^{t-1}$$

$$|[g_{\mathbf{x}}(\mathbf{L})\mathbf{h}_{\mathbf{x}}]_0| \leq \|g_{\mathbf{x}}(\mathbf{L})\mathbf{h}_{\mathbf{x}}\|_1 \leq \|\|g_{\mathbf{x}}(\mathbf{L})\|_2\|\mathbf{h}_{\mathbf{x}}\|_2 = \lambda_{G_{\mathbf{x}}}^{\max}$$

Cauchy–Schwarz Inequality

$$\lambda_{G_{\mathrm{x}}}^{\mathrm{max}} \leq \lambda_{G}^{\mathrm{max}} \qquad \mathbf{g}_{\lambda} \leq \lambda_{G}^{\mathrm{max}}$$

$$2eta_m \leq lpha_\ell \lambda_G^{ ext{max}} \mathbf{E}_{ ext{SGD}}[|\Delta heta|]$$

$$eta_m \leq rac{\eta lpha_\ell lpha_\sigma v_\ell ig(\lambda_G^{ ext{max}}ig)^2 \sum_{t=1}^T ig(1 + \eta v_\ell v_\sigma ig(\lambda_G^{ ext{max}}ig)^2ig)^{t-1}}{m} \ eta_m \leq rac{1}{m} \mathcal{O}ig((\lambda_G^{ ext{max}})^{2T}ig) \quad orall T \geq 1$$