Risk-Based Asset Allocation Using PCA

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1. Introduction

In regard to optimal portfolio construction, Markowitz's mean-variance framework, first introduced in 1952, still remains the standard way to allocate wealth across a portfolio of assets. Through setting up a problem where an investor wants to maximise expected returns for a given level of volatility, he showed the benefits of diversification that comes from holding a portfolio over a single asset.

However, there are known issues with implementing mean-variance efficient optimisation in practise. First, the derived mean-variance efficient portfolio is extremely sensitive to the input parameters, namely the asset expected returns and volatilities. Because one would need to forecast these inputs, there is the possibility of large estimation errors, particularly with returns. Since the estimation of such inputs can often have large errors, it is well-documented that these small changes in inputs can result in vastly different efficient portfolios obtained (Jorion, 1985), with claims being made that mean-variance optimisation is a 'estimation-error maximiser' (Michaud, 1989). The second issue is that the efficient portfolio tends to concentrate on a limited subset out of the full investment universe, which on the surface, seems to conflict with the intention of diversification. Hence, there have been attempts to deal with these robustness issues, but often due to additional computation and other problems, they do not result in better out-of-sample performance, as illustrated by Scherer (2007).

As a result, many investors have looked at computationally simpler and more robust asset allocation methods, such as minimum variance and equally weighted (or 1/N) portfolios. Minimum variance is easier to compute but suffers same portfolio concentration problem. Equally weighted portfolios are widely used and have done well out-of-sample (DeMiguel, Garlappi and Uppal, 2009).

From the 2008 financial crisis and the failure of mean-variance to perform in this environment, there has been an alarmingly growing amount of literature focused on risk management and diversification, particularly in risk parity (equal risk contribution) (Maillard et al., 2010) and diversified risk parity (Kind, 2013; Lohre et al., 2012, 2014). The general idea of these allocation methods is to make each component of the portfolio contribute an equal level of risk. The added benefit is that one avoids the estimation of returns that result in large estimation errors.

In this paper, we will particularly look at the application of principal components analysis (PCA) in the context of asset allocation. PCA is a statistical method that allows us to simplify and reduce the complexity of a problem – looking for a few uncorrelated principal components that can explain most of the variation given by those variances and covariances. In regards to applications in finance, some of the uses of PCA include the simplification of portfolio selection through simply choosing the assets based on a set of a few principal components rather than the full set of underlying assets, or reduce complexity of a portfolio through the transformation of a set of correlated assets into a new set of uncorrelated principal components that represent uncorrelated risk factors.

More specifically, we will have a particular interest in the diversified risk parity allocation method, which is unique because it is an allocation strategy based on principal components analysis (PCA). Because of the use of PCA, it uses a different definition of portfolio diversification and portfolio risk. In an attempt to find optimal portfolios that are well-diversified, Meucci (2009) builds on PCA of the portfolio assets to extract the main drivers of the asset's variability, which can be interpreted as uncorrelated risk sources. Hence, in this case, the well-diversified portfolio would be where the overall risk is evenly distributed across these principal components, which maximises the number of uncorrelated bets.

In this paper, we will compare the out-of-sample performance and properties of diversified risk parity against risk parity, minimum-variance, PCA 1/N and 1/N portfolios as well as the market cap index. Based on past literature such as Kind (2013), DRP has been compared with RP and 1/N futures contracts and found that DRP was unable to outperform. Using a sample of country indices, we will provide empirical illustrations and finally draw some conclusions.

2. Literature Review

2.1 Risk-Based Asset Allocation and Risk Parity

The general idea of mean-variance optimisation is to obtain the optimal risk and return trade-off through diversifying portfolio weights. Unfortunately, mentioned previously, the biggest issue is the difficulty of obtaining robust results, where estimating expected returns as inputs often has large estimation errors.

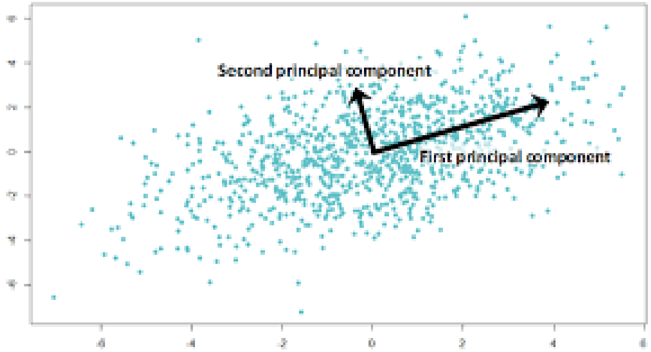
One way to avoid the estimation of expected returns is to use risk-based asset allocation, where the only inputs required is the variance-covariance matrix of asset returns. Under the Markowitz framework, this results in the well-known minimum variance portfolio, where a unique solution is obtained rather than the whole efficient frontier. However, minimum variance also suffers from the fact that it concentrates its portfolio on a few low-volatility assets.

One risk-based allocation method that has become popular as a result of the 2008 financial crisis is risk parity. The main idea of risk parity is to achieve an equal risk contribution of each asset in the portfolio.

2.2 Principal Components Analysis and Applications

PCA is a statistical method of dimension reduction, used to reduce the complexity of a data set while minimising the loss of information. It transforms a set of correlated variables into a new set of uncorrelated variables, labelled principal components., where they are linear combinations of the original variables. The idea is that the 1st principal component explains most of the variation in the data, the 2nd components explains the most variation given that it is orthogonal or statistically independent to the 1st component and so on. Hence, component's ability to explain the variation of the data set goes down as the order increases.

Let's consider a simple case with 2 variables: x_1 and x_2. Conducting PCA allows us to extract PC_1 and PC_2, where they are linear combinations $\alpha_1' x = a_{11}x_1 + \alpha_{12}x_2$ and $\alpha_2' x = a_{21}x_1 + \alpha_{22}x_2$. PC_1 looks for the linear combination that explains the most variation and PC_2 looks for a linear combination that explains as much variation possible given orthogonality. Visually, you have the following diagram:



Source: Analytics Vidhya

In terms of applications, PCA has been used in a wide range of areas for over 50 years (Jolliffe, 1986). Because financial markets are extremely complex, the ability of PCA to reduce the complexity and dimensions means PCA is quite attractive to use. In the past, PCA has been used to in the following ways:

- Market indices (Feeney and Hester, 1967)
- Identify common factors in bond returns (Driesson et al., 2003; Oerignon et al., 2007)
- Price swaps (Pelata et al., 2012)
- Analyse systemic risk (Billioand et al., 2012; Kritzman et al., 2011; Zheng et al., 2012)
- Determine how many stocks needed to have a 'diversified portfolio'

Partovi and Caputo (2003) first proposed the idea to use PCA to analyse the efficient portfolio problem, arguing that because there are no correlations, complexity in portfolio selection falls considerably.

2.3. Managing Diversification (Meucci, 2009) and Diversified Risk Parity

When discussing about the pros and cons about minimum variance and other asset allocation methods that aim to have diversification to improve risk return trade-offs, one must first establish what diversification actually means. In fact, in portfolio literature, there are actually several different views of what portfolio diversification means:

- Portfolio return variance (Frahm, Wiechers, 2013)
- Portfolio weights (Woerheide and Persson, 1993)
- Eliminate unsystematic risk
- Avoid exposure to single shocks or risk factors

In most cases, diversification brings benefits when combining assets that are not highly correlated with each other. Hence, Meucci (2009) takes this idea by constructing uncorrelated risk sources by applying PCA to the variance-covariance matrix of portfolio assets.

Consider a portfolio consisting of N assets with a return vector $\boldsymbol{R}$**.** Given weights $\boldsymbol{w}$**,** the resulting portfolio return is $R_w = \boldsymbol{w}'\boldsymbol{R}$. According to the spectral decomposition theorem, the covariance matrix $\boldsymbol{\Sigma}$ can be expressed as:

$$\boldsymbol{\Sigma} = \boldsymbol{E\Lambda E}'$$

Where $\boldsymbol{\Lambda} = diag(\lambda_1, \dots, \lambda_N)$ is a diagonal matrix, consisting of $\boldsymbol{\Sigma}'s$ eigenvalues that are assembled in descending order, $\lambda_1 \geq \cdots \geq \lambda_N$ i.e. PC1 will have the largest eigenvalue. The columns of matrix $\boldsymbol{E}$ represent the eigenvectors of $\boldsymbol{\Sigma}$. These eigenvectors define a set of N 'principal portfolios', coined by Partovi and Caputo (2004), whose returns given by $\widetilde{\boldsymbol{R}} = \boldsymbol{E}'\boldsymbol{R}$ are uncorrelated and their variances equal to $\lambda_1 \dots \lambda_N$. As a consequence, a given portfolio can be expressed in terms of its weights w in the original assets or in terms of its weights in the principal portfolios $\widetilde{\boldsymbol{w}} = \boldsymbol{E}'\boldsymbol{w}$.

Since principal portfolios are uncorrelated by design, the total portfolio variance emerges from simply computing a weighted average over the principal portfolio's variances:

$$Var(R_w) = \sum_{i=1}^{N} \widetilde{w}_i^2 \lambda_I$$

Normalising the principal portfolio's contributions by the portfolio variance then yields the diversification distribution:

$$p_i = \frac{\widetilde{w}_i^2 \lambda_I}{Var(R_w)}, i = 1, \dots, N$$

Building on this concept, Meucci (2009) conceives a portfolio to be well-diversified when the p_i are approximately equal, meaning the diversification distribution is close to uniform. This definition of a well-diversified portfolio coincides with allocating equal risk contributions to the principal portfolios, thus being dubbed the 'diversified risk parity' asset allocation strategy. Conversely, portfolios loading on a specific principal portfolio display a peaked diversified distribution. Meucci (2009) does use the properties of the diversification distribution, namely the exponential of its entropy, to obtain a single measure of diversification, which can intuitively be interpreted as the number of uncorrelated bets:

$$N_{Ent} = \exp\left(-\sum_{i=1}^{N} p_i \ln p_i\right)$$

For example, consider 2 extreme cases. For a completely concentrated portfolio, we have $p_1 = 1$ and $p_i = 0$ for the rest. This results in an entropy of 0, which implies a metric of 1, meaning the portfolio only makes 1 uncorrelated bet. Alternatively, a DRP portfolio with a metric equal to N, is completely homogenous in terms of uncorrelated risk sources and implies number of uncorrelated bets of N.

Hence, we can obtain maximum diversification / DRP weights by solving:

$$\boldsymbol{w}_{DRP} = \underset{\boldsymbol{w}\epsilon C}{\operatorname{argmax}}\, N_{Ent}(\boldsymbol{w})$$

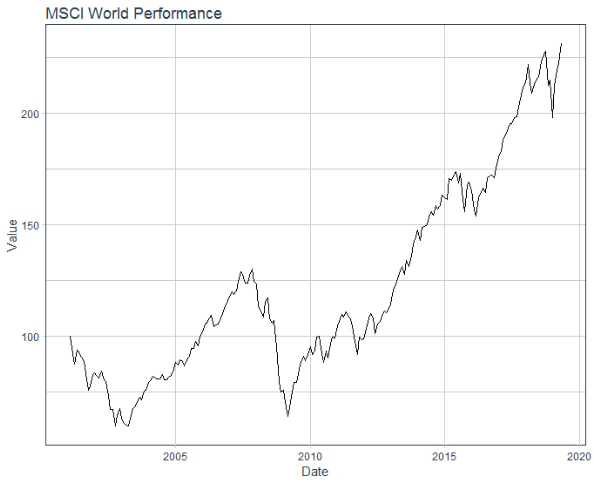
Investigating general risk budgeting strategies, Bruder and Roncalli (2012) and Roncalli and Weisang (2012) show that a unique diversified risk budgeting strategy can be obtained when imposing positively constraints with respect to the underlying risk factors. Hence, you obtain a unique DRP strategy if you impose sign constraints with respect to the principal portfolios.

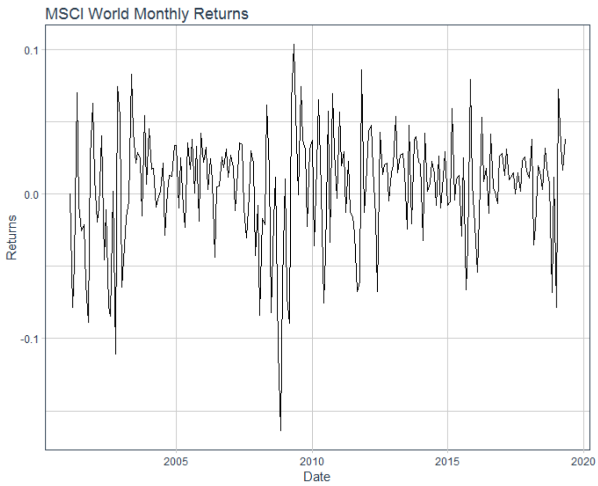
The idea is that extracted principal components (principal portfolios) represent the uncorrelated risk sources inherent in the set of asset returns. For a portfolio to be 'well-diversified', its overall risk should be therefore be evenly distributed across these principal portfolios.

In terms of the difference between diversified risk parity using PCA and the traditional risk parity allocation method, the maximum diversification portfolio budgets risk with respect to the extracted principal portfolios rather than the underlying portfolio assets. Hence, in certain situations where a set of assets are highly correlated with each other, traditional risk parity can be misleadingly concentrated and be subject to individual shocks. Consider an extreme case with all stocks perfectly positively correlated, then allocating equal risk budget to all stocks is actually the same as holding one stock, DRP doesn't have such a problem since it allocates risk based on uncorrelated risk factors, in the case of perfect correlations, all variation would be explained by the first principal component.

3. Data

In order to compare asset allocation methods, we obtained MSCI index monthly data from January 2001 to April 2019, obtaining net returns, thus including price performance and income from dividend payments and capital repayments. First, we obtained the MSCI World Index, which is a market-capitalisation index, covering 23 developed markets countries.





Within MSCI, the 23 developed market constituents are Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Hong Kong, Ireland, Israel, Italy, Japan, Netherlands, New Zealand, Norway, Portugal, Singapore, Spain, Sweden, Switzerland, the UK and the US, which we have also obtained.

Risk-free used is the 3-month US treasury rate. We also include the MSCI Emerging Markets Index to capture a different asset class and risk source.

Table provides descriptive statistics of the above classes over the whole sample (based on monthly local currency returns):

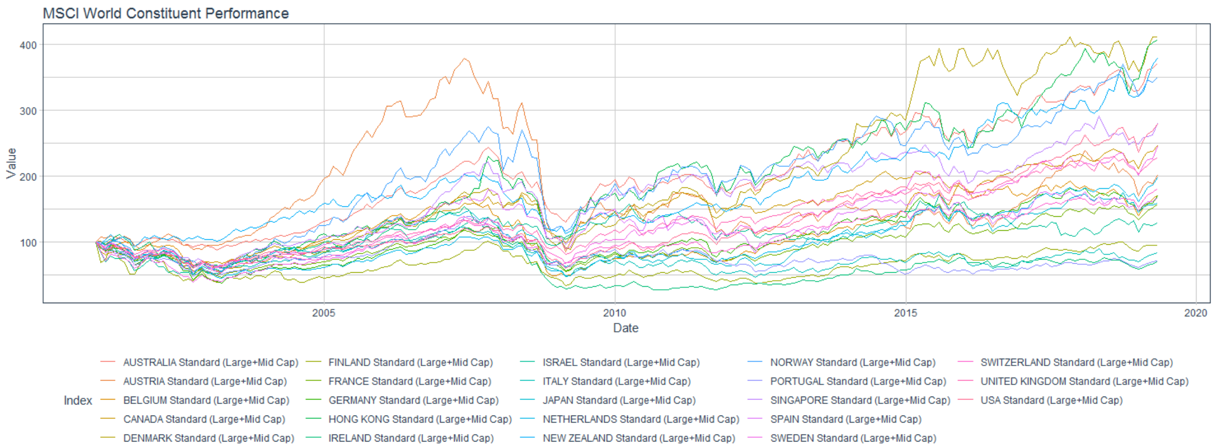
	Return p.a.	**Standard Deviation p.a.**	**Sharpe Ratio**
MSCI World	4.68%	13.75%	0.24

Over the whole sample period from January 2001 to April 2019, we observe an annualised return for the MSCI World Index of 4.68% and a standard deviation of 13.75%. Based on a 3-month US return over the same period of 1.4%, the Sharpe Ratio is 0.24.

Similarly, descriptive statistics for the constituents:

	Return p.a.	Standard Deviation p.a.	Sharpe Ratio
Australia	7.42%	12.66%	0.47
Austria	3.88%	22.38%	0.11
Belgium	2.96%	18.92%	0.08
Canada	5.05%	27.16%	0.13
Denmark	8.02%	17.47%	0.37
Finland	-0.31%	27.03%	-0.06
France	2.53%	16.94%	0.07
Germany	2.92%	20.32%	0.07
Hong Kong	7.95%	19.94%	0.32
Ireland	-1.92%	21.33%	-0.15
Israel	1.36%	19.12%	-0.00
Italy	-1.00%	19.52%	-0.01
Japan	2.56%	17.71%	0.06
Netherlands	3.78%	17.88%	0.01
New Zealand	7.55%	13.84%	0.44
Norway	7.08%	20.33%	0.28
Portugal	-1.89%	17.89%	-0.18
Singapore	5.78%	18.40%	0.23
Spain	2.86%	19.96%	0.07
Sweden	5.01%	20.04%	0.18
Switzerland	3.30%	13.45%	0.14
UK	4.60%	13.36%	0.24
US	5.76%	14.46%	0.30
MSCI EM	9.45%	16.74%	0.47

We see that country indices that performed better than the MSCI World were Australia, Denmark, Hong Kong, New Zealand, Norway and the US. Typically, the constituents has worse Sharpe Ratios, and typically had higher volatility.



4. Method

4.1. Extracting and Interpreting Principal Portfolios

Conducting PCA on the MSCI World constituent time series will provide us information about the uncorrelated risk sources inherent in the returns. We have the option to use either a correlation matrix or covariance matrix. However, in the literature, there are known problems when using the covariance matrix to conduct PCA. For example, if there are large differences between the variances of variables, then using a covariance matrix will result in low numbered PCs being dominated by variables that have large variance, which makes PCA less useful in extracting information in some cases (Jolliffe, 1986). Yang (2015) and Lohre (2012) also find that PCA using covariance matrix can be very misleading, and thus, we will only use the correlation matrix when extracting principal components.

Conducting principal components analysis gives us the following results:

	PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8	PC9
Std. Dev.	3.94	1.11	0.96	0.91	0.84	0.76	0.73	0.70	0.65
Variance	64.59%	5.09%	3.86%	3.48%	2.91%	2.41%	2.22%	2.02%	1.8%
Cumulative	64.59%	69.68%	73.55%	77.02%	79.92%	82.34%	84.56%	86.58%	88.4%
Eigenvalue	15.50	1.22	0.93	0.83	0.70	0.58	0.53	0.49	0.43

For 23 country indices, PCA will generate 23 PCs, a linear combination of the indices. However, generally we can explain most of the variance with just a few PCs, as the eigenvalues of the PCs typically decrease quickly, and the relevance of the PCs quickly drops off. From the decomposition of the correlation matrix by Kim and Jeong (2015), they break down the PCs into 3 kinds of fluctuations:

1. 1st PC represents market wide effect that influences all assets
2. Next PCs represent synchronised fluctuations that only happens to a group of assets
3. Remaining PCs represent randomness in fluctuations

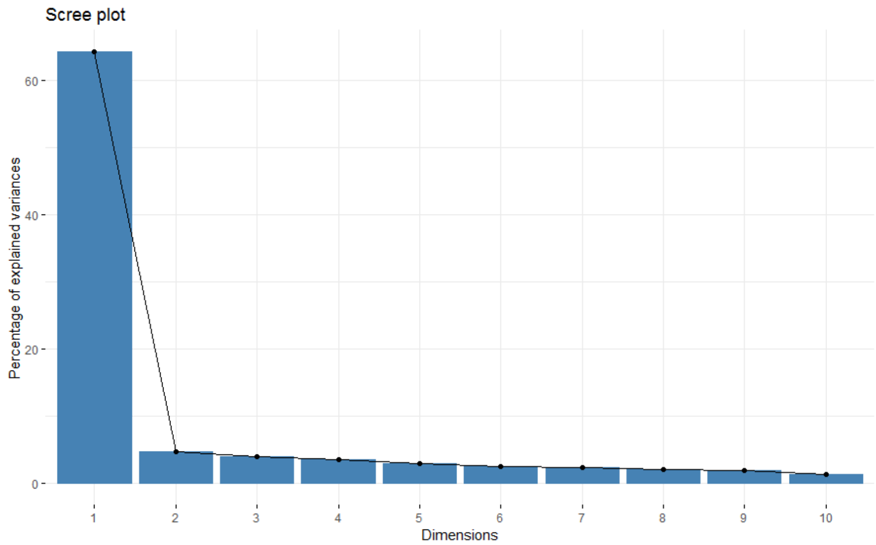
Hence, it is important to determine a threshold for cutting off the PCs that provide no information and only provide random noise, hence only preserving the main risk factors. In literature, there is not a single main way to determine the cut off point. In fact, there are 3 rules academics have used in the past:

1. Cumulative variance desired
2. Kaiser's rule (Kaiser, 1960)
3. Scree graph (Cattell, 1966)

In regard to cumulative variance desired, typically the literature aims between 70% to 90% for the cumulative proportion of variance. Based on the above results, this rule tell us to keep between 3 to more than 10 PCs.

Kaiser’s rule is a rule where we retain the principal components in which the eigenvalues are greater than 1. The idea is that only PCs with an eigenvalue less than 1 contain less information than one of the original variables and thus, is not worth retaining. However, Jolliffe (1986) advised a more conservative cut-off of 0.7. Hence based on this rule, it is advised to keep between 2 and 5 PCs.

Finally, the last rule is to use a Scree graph, which plots eigenvalues against the component number. The decision is to find the point in the graph where the slopes of the lines joining the plotted points are ‘steep’ to the left and have linear decay to the right. Tis is a more subjective method but tends to be used extensively in practice.



Based on the Scree plot, this suggests that between 2 and 3 PCs is recommended, as this where we see very little slop changes with higher numbered principal components.

Overall, the 3 methods give varying numbers of PCs to keep. The trade-off is the preservation of variance vs. avoiding keeping random, useless principal components. Since the ultimate aim is to provide greater out-of-sample performance for PCA asset allocation methods, using fewer principal components makes more sense here. Thus, we will use 3 PCs in the future.

In order to attempt to learn about the economic intuition of the principal portfolios, we look at the terms of the eigenvectors, which are the weights of the principal portfolios with respect to the original assets.

	PC1	**PC2**	**PC3**	**PC4**	**PC5**	**PC6**	**PC7**	**PC8**	**PC9**
Australia	0.21	-0.07	-0.02	0.25	0.04	-0.14	0.36	-0.07	0.24
Austria	0.20	-0.09	-0.38	0.09	-0.23	-0.17	0.01	0.19	0.17
Belgium	0.21	0.12	-0.13	0.11	-0.22	0.36	-0.17	0.00	-0.01
Canada	0.21	-0.23	0.03	-0.10	-0.20	-0.11	0.33	0.18	0.20
Denmark	0.20	0.12	0.14	0.13	-0.22	0.27	-0.14	0.42	-0.32
EM	0.22	-0.37	-0.05	-0.11	0.01	0.05	0.07	-0.01	-0.06

Finland	0.18	0.15	0.40	-0.09	0.29	-0.13	0.45	0.24	-0.31
France	0.24	0.18	0.00	-0.13	0.08	-0.01	0.03	-0.11	-0.04
Germany	0.23	0.13	0.08	-0.12	0.02	0.05	-0.05	-0.23	-0.07
Hong Kong	0.19	-0.46	-0.06	-0.09	0.12	0.13	-0.03	0.02	-0.15
Ireland	0.18	0.31	-0.01	0.29	-0.42	0.02	0.22	-0.08	0.09
Israel	0.15	-0.17	0.65	-0.07	-0.17	-0.18	-0.40	0.15	0.45
Italy	0.21	0.20	-0.16	-0.21	0.20	-0.12	-0.09	0.02	0.13
Japan	0.18	-0.05	-0.15	0.19	-0.07	-0.71	-0.29	0.05	-0.46
Netherlands	0.23	0.17	0.00	-0.04	-0.05	0.17	-0.11	-0.16	-0.11
New Zealand	0.14	-0.09	0.09	0.76	0.46	0.10	-0.07	-0.06	0.09
Norway	0.22	-0.14	-0.09	-0.06	-0.17	0.09	0.09	0.19	-0.08
Portugal	0.19	0.15	-0.20	-0.05	0.33	0.13	-0.15	0.57	0.21
Singapore	0.20	-0.41	-0.08	-0.03	0.05	0.21	-0.14	-0.25	-0.10
Spain	0.21	0.17	-0.19	-0.24	0.29	-0.11	-0.14	-0.10	0.19
Sweden	0.22	0.06	0.27	-0.09	0.03	0.10	-0.06	-0.14	-0.25
Switzerland	0.21	0.18	0.04	0.05	-0.06	-0.10	-0.21	-0.24	0.10
UK	0.22	0.05	-0.05	-0.09	0.03	0.06	0.20	-0.15	0.09
US	0.23	-0.04	0.09	-0.03	-0.04	-0.09	0.17	-0.20	0.10

The first principal component (PC1) very much looks like the market/equity risk factor, especially given the equal weights across all countries, and especially as it explains 64.31% of the total variation. This is consistent with the understanding that PC1 is normally understood to have roughly equal contribution of the underlying assets and all the coefficients being positive (Fenn et al., 2011; Kim and Jeong, 2005; Zheng et al., 2012).

Interpreting PC2, we have positive weights for Belgium, Denmark, Finland, France, Germany, Italy, Netherlands, Portugal, Spain, Sweden, Switzerland and the UK and negative weights for all other countries. Hence, PC2 appears to reflect some sort of European risk factor, and it is the 2nd largest risk factor, explaining 4.67% of variation.

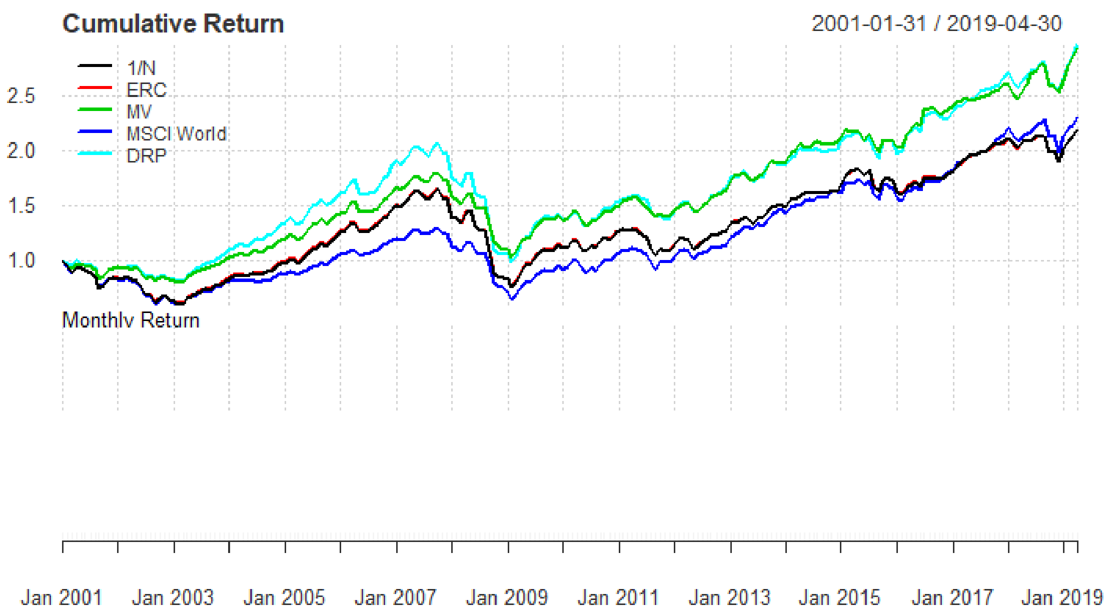
We see that 3 PCs explain 70% of the variation and after 4, the PCs start to explain very little, which tends to indicate across 23 developed market indices, there tends to be only 3 main risk factors that are inherent. Beyond a certain number of principal components, it is known that risk factors are no longer significant and simply generates random noise.

4.2. Constructing Principal Portfolios

We will move to construct our principal portfolios. Recall that PCA extracts uncorrelated linear combination of assets, which can reflect investible portfolios, that also represent uncorrelated risk sources in the market. Immediately, we see some purpose of doing this. For example, we see that to be purely exposure to the market risk factor and not be exposed to other risk factors, we simply invests equally across the underlying assets. Because we have our PPs, all asset allocations we have conduct on underlying assets we can now simply do on our PPs. Meucci (2009) says that the maximum diversification portfolio conducts equal risk contribution across these PPs. We will also consider a naïve PP strategy where we do a 1/N on the PPs.

5. Full-Sample Portfolio Performance Comparison

Based on the full-sample performance statistics, we see that DRP performs quite well relative to most of the other risk-based asset allocation strategies and outperforms against the market capitalisation index, but we do see that the minimum variance method outperforms against DRP. However, these results should not be used as a basis for deciding the optimal asset allocation method to use, particularly where it is expected that portfolio weights should change for most of the risk-based methods when the variances-covariances move, as expected during the 2008 financial crisis, etc. Hence, out-of-sample back-testing is also conducted in the next section.

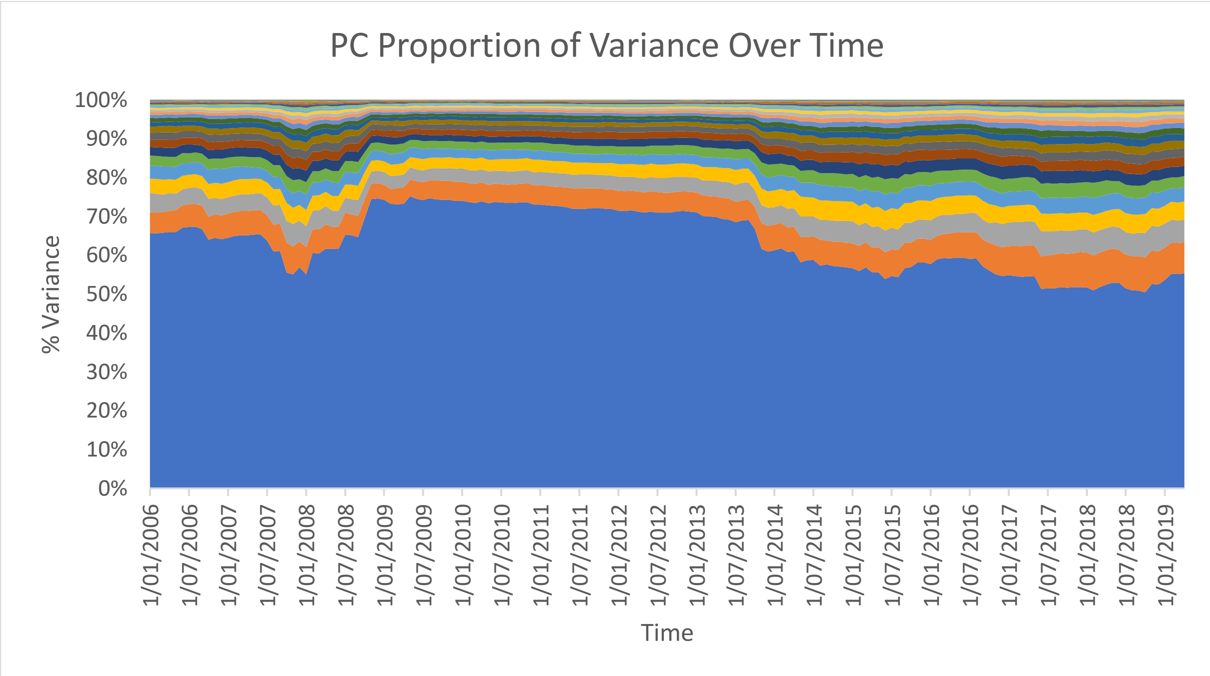


	Return p.a.	Standard Deviation p.a.	Sharpe Ratio
MSCI World	4.68%	13.75%	0.23
1/N	4.38%	14.12%	0.21
Risk Parity (ERC)	4.39%	13.94%	0.21
Minimum Variance	6.07%	10.89%	0.42
DRP	6.11%	12.98%	0.36
1/N PP	6.05%	13.05%	0.35

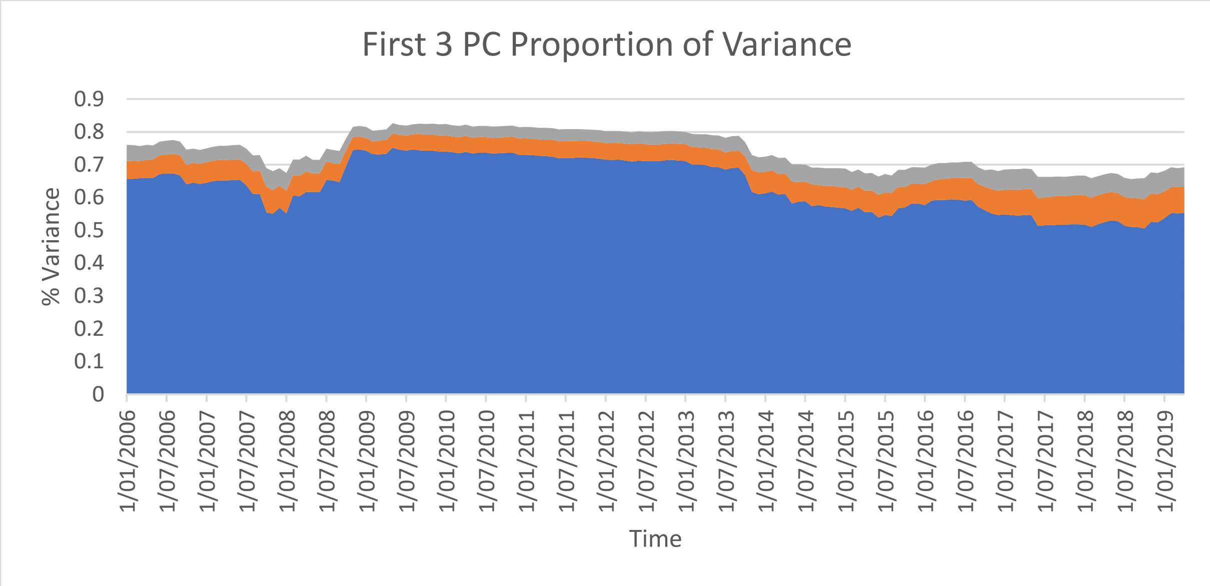
6. Out-Of-Sample Portfolio Performance Comparison

6.1. Rolling Window

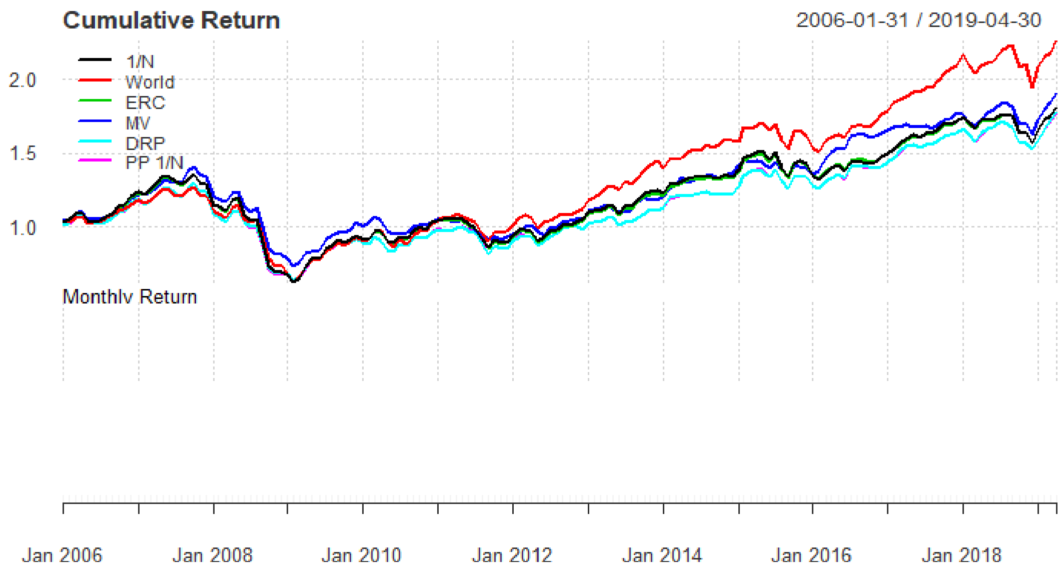
For estimating the principal portfolios over time, one has to make a choice with regard to the estimation window. Two of the most common approaches are considered: rolling window and an expanding window. In terms of how the two estimation methods differ, a rolling window may be more responsive to potential structural breaks, whereas expanding window is more appropriate when one wants to use all the information in order to obtain more robust results. In particular, we perform a PCA daily to extract the principal portfolios. For this section, a 60-month rolling window period is used, where rebalance monthly, having new weights and returns over 160 observations from January 2006 to April 2019. In doing so, we can obtain the variation of the principal portfolios' variances over time:



We observe that PP1 is fairly dominant by accounting for at least 55% of the underlying time series' variation at any point in time. We do see that it significantly increased upwards of 75% during the 2008 financial crisis period.



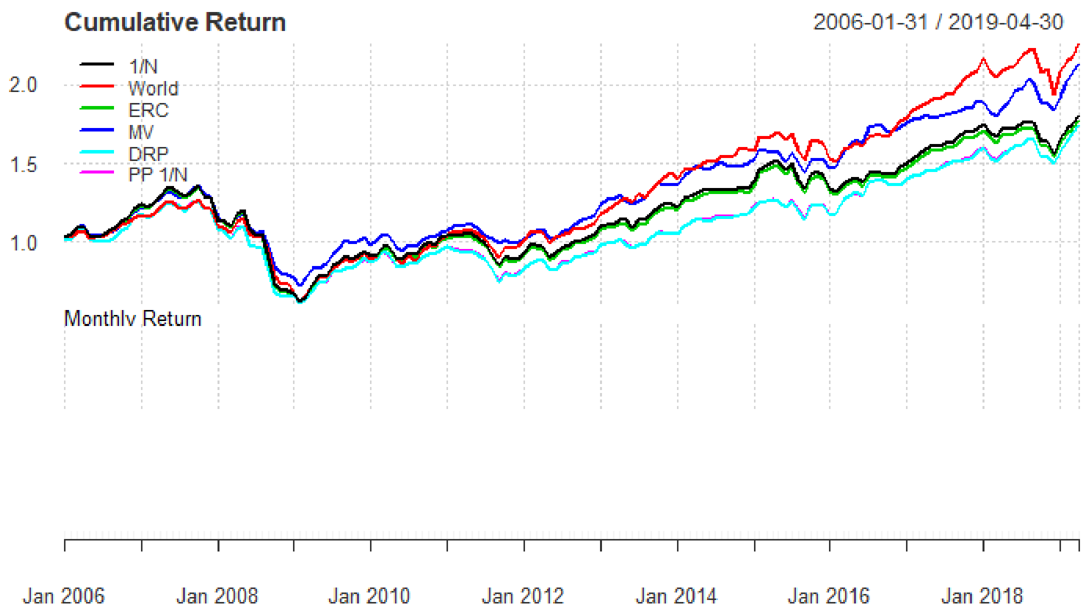
In terms of the second principal component, we do see that it did increase from 2016 onwards, which does seem to be consistent with negative news and uncertainty, for example Brexit was held on 23 June 2016.



	Return p.a.	Standard Deviation p.a.	Sharpe Ratio
MSCI World	6.32%	13.5%	0.36
1/N	4.55%	13.88%	0.22
Risk Parity (ERC)	4.56%	13.73%	0.23
Minimum Variance	4.97%	11.59%	0.30
DRP	4.65%	13.73%	0.23
PP 1/N	4.4%	13.34%	0.22

The table above provides performance and risk statistics of DRP, the naïve PCA strategy PP 1/N, risk-based allocations on underlying assets as well as market-capitalisation and minimum variance. We see that based on a 60-month rolling window, we see that DRP earns 4.65% p.a. with a standard deviation of 13.73%, which doesn't appear to be significantly different to traditional risk-based allocation methods as well as the naïve PCA method. Furthermore, we see that all of the risk-based methods have a significantly worse Sharpe ratio against the minimum variance and market-capitalisation methods. Furthermore, we tend to see a similar set of results using the expanding window method. Hence, this provides evidence for mean-variance frameworks relative to risk-based strategies, with or without PCA.

6.2 Expanding Window



	Return p.a.	Standard Deviation p.a.	Sharpe Ratio
MSCI World	6.32%	13.5%	0.36
1/N	4.55%	13.88%	0.22
Risk Parity (ERC)	4.41%	13.82%	0.21
Minimum Variance	5.89%	11.51%	0.38
DRP	4.29%	13.50%	0.21
PP 1/N	4.27%	13.52%	0.21

7. Conclusion

Motivated by the aim to find greater diversification against individual shocks such as the financial crisis in 2008 and by the ability of PCA to extract uncorrelated risk sources inherent in the asset returns, the aim of the paper looked at the properties of diversified risk parity relative to other asset allocation methods. Using PCA, 3 principal components were shown to represent the main risk factors inherent in the selection of country indices, which then became the principal portfolios used as part of the DRP portfolio construction process. Despite promising results from the in-sample performance comparisons, out-of-sample tests from both an expanding window and rolling window showed that the diversified risk parity strategy did not stand out compared to other risk-based asset allocation methods, especially against minimum-variance and the market-capitalisation index.

References

Bera, A. K., & Park, S. Y. (2008). Optimal portfolio diversification using the maximum entropy principle. Econometric Reviews, 27(4-6), 484-512.

Billioand, M., Getmansky, M., Lo, A.W., and Pelizzon, L. (2012). Econometric measures of connectedness and systemic risk in the finance and insurance sectors. Journal of Financial Economics, 104:535–559.

Bruder, B., & Roncalli, T. (2012). Managing risk exposures using the risk budgeting approach. Available at SSRN 2009778.

Chopra, V. K., & Ziemba, W. T. (1993). The Effect of Errors in Means. Variances and Covariances.

DeMiguel, V., Garlappi, L., & Uppal, R. (2007). Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy?. The review of Financial studies, 22(5), 1915-1953.

Driesson, J., Melenberg, B., and Nijman, T. (2003). Common factors in international bond returns. Journal of International Money and Finance, 22:629‑656.

Feeney, G. J. and Hester, D. D. (1967). Risk Aversion and Portfolio Choice. Wiley, New York.

Jolliffe, I. T. (1986). Principal component analysis. 1986. Spring-verlag, New York, 2, 29.

Jorion, P. (1985). International portfolio diversification with estimation risk. Journal of Business, 58(3):259–278.

Kind, C. (2013). Risk-based allocation of principal portfolios. Available at SSRN 2240842.

Kritzman, M., Li, Y., Page, S., and Rigobon, R. (2011). Principal Components as a measure of systemic risk. Journal of Portfolio Management, 37:112–126.

Lohre, H., Neugebauer, U., & Zimmer, C. (2012). Diversified risk parity strategies for equity portfolio selection. The Journal of Investing, 21(3), 111-128

Lohre, H., Opfer, H., & Orszag, G. (2014). Diversifying risk parity. Journal of Risk, 16(5), 53-79.

Maillard, S., Roncalli, T., & Teiletche, J. (2010). The properties of equally weighted risk contribution portfolios. The Journal of Portfolio Management, 36(4), 60-70.

Markowitz, H. (1952). Portfolio Selection. Journal of Finance, 7:77–91.

Meucci, A. (2010). Managing diversification.

Michaud, R. O. (1989). The Markowitz Optimisation Enigma: is 'optimised' optimal? Financial Analyst's Journal., 45(1), 31–42.

Partovi, M. H., & Caputo, M. (2004). Principal portfolios: Recasting the efficient frontier. Economics Bulletin, 7(3), 1-10.

Pelata, M., Giannopoulos, P. & Haworth, H. (2012). PCA Unleashed.

Qian, E. E. (2005). On the financial interpretation of risk contribution: Risk budgets do add up.

Qian, E. (2011). Risk parity and diversification. Journal of Investing, 20(1), 119.

Roncalli, T., & Weisang, G. (2012). Risk Parity Portfolios with Risk Factors, SSRN.

Yang, L. (2015). An Application of Principal Component Analysis to Stock Portfolio Management.

Zheng, Z., Podobnik, B., Feng, L., and Li, B. (2012). Changes in cross-correlations as an indicator for systemic risk. Scientific Reports, 2:888.

R Code

```r
setwd("~/R") #setwd("C:/Users/mruan/Downloads")
install.packages("tidyverse")
install.packages("tidyquant")
install.packages("readxl")
install.packages("factoextra")
install.packages("riskParityPortfolio")
install.packages("PortfolioAnalytics")
install.packages("fPortfolio")
install.packages("rgl")
install.packages("LICORS")
library(tidyquant)
library(tidyverse)
library(readxl)
library(factoextra)
library(riskParityPortfolio)
library(PortfolioAnalytics)
library(fPortfolio)
library(rgl)
library(LICORS)

### 2. Load MSCI World Index ###
msci_world <- read_excel("msci_world_index.xls", skip = 6, col_names = T)
msci_world <- msci_world %>% gather(Index, Value, -Date)
MSCI_World_returns <- msci_world %>% group_by(Index) %>%
  tq_transmute(select = Value, mutate_fun = periodReturn,
               period = 'monthly',type = 'arithmetic')

### 3. Load Risk-free Rate ###
risk_free <- read_excel("TB3MS.xls", skip = 10, col_names = T)
risk_free <- risk_free %>% mutate(Date = as.Date(Date))

### 4. MSCI World Descriptive Statistics ###
msci_world_ts <- xts(MSCI_World_returns[,3], order.by =
as.Date(MSCI_World_returns$Date))
risk_free_ts <- xts(risk_free[,2], order.by = as.Date(risk_free$Date))
SharpeRatio.annualized(msci_world_ts, Rf = risk_free_ts / 12,
                       scale = 12,geometric = TRUE)
MSCI_World_returns %>% tq_portfolio(assets_col = Index, weights = c(1),
                                    returns_col = monthly.returns) %>%
  tq_performance(Ra = portfolio.returns,
                 performance_fun = table.AnnualizedReturns) * 100

### 5. Load MSCI Country Indices / MSCI World Index Constituents ###
msci <- read_excel("historyIndex.xls", skip = 6, col_names = T)
```

```
msci <- msci %>% mutate(Date = as.Date(Date))
msci <- msci %>% gather(Index, Value, -Date)
msci <- msci %>% dplyr::filter(Date >= "2001-01-31")
returns <- msci %>% group_by(Index) %>%
  tq_transmute(select = Value, mutate_fun = periodReturn,
               period = 'monthly',type = 'arithmetic')
returns %>% tq_portfolio(assets_col = Index,
                         weights =
c(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1),
                         returns_col = monthly.returns) %>%
  tq_performance(Ra = portfolio.returns,
                 performance_fun = table.AnnualizedReturns, Rf = 0.014 / 12)
* 100

### 6. PCA Analysis ###
pca <- returns %>% spread(Index, monthly.returns) %>% select(-Date) %>%
  as.matrix() %>% prcomp(scale. = T, center = T)
eigenvectors <- pca$rotation
PCA_summary <- pca %>% summary()

### 7. Full-Sample Performance ###
#a. Equal Weight / 1/N
EW_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
rep(1/23,23),
                                       returns_col = monthly.returns)

#b. Risk parity / ERC
correlations <- returns %>% spread(Index, monthly.returns) %>% select(-Date)
%>% cor()
ERC <- riskParityPortfolio(correlations)
ERC_returns <- returns %>% tq_portfolio(assets_col = Index, weights = ERC$w,
                                        returns_col = monthly.returns)

#c. Minimum Variance
returns_ts <- as.timeSeries(returns %>% spread(Index, monthly.returns))
MV <- minvariancePortfolio(returns_ts,spec = portfolioSpec(), constraints =
"LongOnly")
MV_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
getWeights(MV),
                                       returns_col = monthly.returns)

#d. Diversified Risk Parity (DRP)
PP1_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,1]),
                                        returns_col = monthly.returns)
```

```r
PP2_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,2]),
                                        returns_col = monthly.returns)
PP3_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,3]),
                                        returns_col = monthly.returns)
PC_portfolios <- left_join(PP1_returns, PP2_returns, by = "Date")
PC_portfolios <- left_join(PC_portfolios, PP3_returns, by = "Date")
DRP <- riskParityPortfolio(PC_portfolios %>% select(-Date) %>% cor())
DRP_returns <- PC_portfolios %>% gather(PP, Returns, -Date) %>%
  tq_portfolio(assets_col = PP, weights = DRP$w, returns_col = Returns)

#e. PP 1/N
PP1_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,1]),
                                        returns_col = monthly.returns)
PP2_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,2]),
                                        returns_col = monthly.returns)
PP3_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,3]),
                                        returns_col = monthly.returns)
PC_portfolios <- left_join(PP1_returns, PP2_returns, by = "Date")
PC_portfolios <- left_join(PC_portfolios, PP3_returns, by = "Date")
PP_EW_returns <- PC_portfolios %>% gather(PP, Returns, -Date) %>%
tq_portfolio(assets_col = PP, weights = rep(1/3,3),
                                       returns_col = Returns)

#f. Trajectory
portfolios <- left_join(EW_returns, ERC_returns,by = "Date")
portfolios <- left_join(portfolios, MV_returns,by = "Date")
portfolios <- left_join(portfolios, MSCI_World_returns[,2:3]%>% mutate(Date
= as.Date(Date)),by = "Date")
portfolios <- left_join(portfolios, DRP_returns,by = "Date")
portfolios_xts<- as.xts(portfolios %>% remove_rownames %>%
column_to_rownames(var="Date"))
colnames(portfolios_xts) <- c("1/N", "ERC", "MV", "MSCI World", "DRP")
charts.PerformanceSummary(portfolios_xts, Rf = 0, main = NULL, geometric =
TRUE,methods = "none", width = 0,
                          event.labels = TRUE, ylog = FALSE,wealth.index =
TRUE, gap = 12, begin = c("first", "axis"),
                          legend.loc = "topleft", p = 0.95)

#g. Descriptive Statistics e.g. Sharpe Ratio
MSCI_World_returns %>% tq_portfolio(assets_col = Index, weights = c(1),
returns_col = monthly.returns) %>%
```

```
  tq_performance(Ra = portfolio.returns, performance_fun =
table.AnnualizedReturns, Rf = 0.014 / 12) * 100
EW_returns %>% tq_performance(Ra = portfolio.returns,
                              performance_fun = table.AnnualizedReturns, Rf
= 0.014 / 12) * 100
ERC_returns %>% tq_performance(Ra = portfolio.returns,
                               performance_fun = table.AnnualizedReturns, Rf
= 0.014 / 12) * 100
MV_returns %>% tq_performance(Ra = portfolio.returns,
                              performance_fun = table.AnnualizedReturns, Rf
= 0.014 / 12) * 100
DRP_returns %>% tq_performance(Ra = portfolio.returns,
                               performance_fun = table.AnnualizedReturns, Rf
= 0.014 / 12) * 100
PP_EW_returns %>% tq_performance(Ra = portfolio.returns,
                                 performance_fun = table.AnnualizedReturns,
Rf = 0.014 / 12) * 100

### 8. Out-Of-Sample Performance ###
#a. No. of PP
pca %>% fviz_eig()
pca %>% summary()
eigenvectors <- returns %>% spread(Index, monthly.returns) %>% select(-Date)
%>% as.matrix() %>%
  cor() %>% eigen()
eigenvectors$values

#b. Proportion of Variances over time
PCA_variance<- matrix(nrow = 160, ncol = 23)
for (btp in 1:160){
  i <- btp + 59
  backtestperiod <- (returns %>% spread(Index, monthly.returns))[btp:i,]
  pca <- backtestperiod %>% select(-Date) %>% as.matrix() %>% prcomp(scale.
= T, center = T)
  PCA_variance[btp,] <- pca$sdev^2 / sum(pca$sdev^2)
}

#c. Rolling Window Minimum Variance
bt_mv_returns <- tibble()
for (btp in 1:160){
  i <- btp + 59
  backtestperiod <- (returns %>% spread(Index, monthly.returns))[btp:i,]
  returns_ts <- as.timeSeries(backtestperiod)
  MV <- minvariancePortfolio(returns_ts,spec = portfolioSpec(), constraints
= "LongOnly")
```

```
  MV_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
getWeights(MV), returns_col = monthly.returns)
  bt_mv_returns <- bind_rows(bt_mv_returns, MV_returns[i + 1,])
}

#d. Rolling Window ERC
bt_ERC_returns <- tibble()
for (btp in 1:160){
  i <- btp + 59
  backtestperiod <- (returns %>% spread(Index, monthly.returns))[btp:i,]
  ERC <- riskParityPortfolio(backtestperiod %>% select(-Date) %>% cor())
  ERC_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
ERC$w, returns_col = monthly.returns)
  bt_ERC_returns <- bind_rows(bt_ERC_returns, ERC_returns[i + 1,])
}

#e. Rolling Window DRP & 1/N PP
bt_DRP_returns <- tibble()
bt_PP1N_returns <- tibble()
for (btp in 1:160){
  i <- btp + 59
  backtestperiod <- (returns %>% spread(Index, monthly.returns))[btp:i,]
  pca <- backtestperiod %>% select(-Date) %>% as.matrix() %>% prcomp(scale.
= T, center = T)
  PP1_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,1]), returns_col = monthly.returns)
  PP2_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,2]), returns_col = monthly.returns)
  PP3_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,3]), returns_col = monthly.returns)
  PC_portfolios <- left_join(PP1_returns, PP2_returns, by = "Date")
  PC_portfolios <- left_join(PC_portfolios, PP3_returns, by = "Date")
  DRP <- riskParityPortfolio(PC_portfolios[btp:i,] %>% select(-Date) %>%
cor())
  DRP_returns <- PC_portfolios %>% gather(PP, Returns, -Date) %>%
tq_portfolio(assets_col = PP, weights = DRP$w, returns_col = Returns)
  PP_EW_returns <- PC_portfolios %>% gather(PP, Returns, -Date) %>%
tq_portfolio(assets_col = PP, weights = rep(1/3,3),

returns_col = Returns)
  bt_DRP_returns <- bind_rows(bt_DRP_returns, DRP_returns[i + 1,])
  bt_PP1N_returns <- bind_rows(bt_PP1N_returns, PP_EW_returns[i + 1,])
}

#f. Rolling Window EW
bt_EW_returns <- tibble()
```

```
for (btp in 1:160){
  i <- btp + 59
  EW_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
rep(1/23,23), returns_col = monthly.returns)
  bt_EW_returns <- bind_rows(bt_EW_returns, EW_returns[i + 1,])
}

#g. Rolling Window MSCI World
bt_world_returns <- tibble()
for (btp in 1:160){
  i <- btp + 59
  bt_world_returns <- bind_rows(bt_world_returns, MSCI_World_returns[i +
1,])
}

#h. Descriptive Statistics e.g. Sharpe Ratio
bt_DRP_returns %>% tq_performance(Ra = portfolio.returns, performance_fun =
table.AnnualizedReturns, Rf = 0.014 / 12) * 100
bt_mv_returns %>% tq_performance(Ra = portfolio.returns, performance_fun =
table.AnnualizedReturns, Rf = 0.014 / 12) * 100
bt_ERC_returns %>% tq_performance(Ra = portfolio.returns, performance_fun =
table.AnnualizedReturns, Rf = 0.014 / 12) * 100
bt_EW_returns %>% tq_performance(Ra = portfolio.returns, performance_fun =
table.AnnualizedReturns, Rf = 0.014 / 12) * 100
bt_world_returns %>% tq_portfolio(assets_col = Index, weights = c(1),
returns_col = monthly.returns) %>%
  tq_performance(Ra = portfolio.returns, performance_fun =
table.AnnualizedReturns, Rf = 0.014 / 12) * 100
bt_PP1N_returns %>% tq_performance(Ra = portfolio.returns, performance_fun =
table.AnnualizedReturns, Rf = 0.014 / 12) * 100

#i. Trajectory
bt_portfolios <- left_join(bt_EW_returns, bt_world_returns[,2:3]%>%
mutate(Date = as.Date(Date)),by = "Date")
bt_portfolios <- left_join(bt_portfolios, bt_ERC_returns,by = "Date")
bt_portfolios <- left_join(bt_portfolios, bt_mv_returns,by = "Date")
bt_portfolios <- left_join(bt_portfolios, bt_DRP_returns,by = "Date")
bt_portfolios <- left_join(bt_portfolios, bt_PP1N_returns,by = "Date")
bt_portfolios_xts<- as.xts(bt_portfolios %>% remove_rownames %>%
column_to_rownames(var="Date"))
colnames(bt_portfolios_xts) <- c("1/N", "World", "ERC", "MV", "DRP", "PP
1/N")
charts.PerformanceSummary(bt_portfolios_xts, Rf = 0, main = NULL, geometric
= TRUE,methods = "none", width = 0,
                          event.labels = TRUE, ylog = FALSE,wealth.index =
TRUE, gap = 12, begin = c("first", "axis"),
```

```
                         legend.loc = "topleft", p = 0.95)

### 9. Expanding Window ###
#a. ERC
ex_ERC_returns <- tibble()
for (btp in 1:160){
  i <- btp + 59
  backtestperiod <- (returns %>% spread(Index, monthly.returns))[1:i,]
  ERC <- riskParityPortfolio(backtestperiod %>% select(-Date) %>% cor())
  ERC_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
ERC$w, returns_col = monthly.returns)
  ex_ERC_returns <- bind_rows(ex_ERC_returns, ERC_returns[i + 1,])
}
ex_ERC_returns %>% tq_performance(Ra = portfolio.returns, performance_fun =
table.AnnualizedReturns, Rf = 0.014 / 12) * 100

#b. MV
ex_mv_returns <- tibble()
for (btp in 1:160){
  i <- btp + 59
  backtestperiod <- (returns %>% spread(Index, monthly.returns))[1:i,]
  returns_ts <- as.timeSeries(backtestperiod)
  MV <- minvariancePortfolio(returns_ts,spec = portfolioSpec(), constraints
= "LongOnly")
  MV_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
getWeights(MV), returns_col = monthly.returns)
  ex_mv_returns <- bind_rows(ex_mv_returns, MV_returns[i + 1,])
}
ex_mv_returns %>% tq_performance(Ra = portfolio.returns, performance_fun =
table.AnnualizedReturns, Rf = 0.014 / 12) * 100

#c. DRP & PP 1/N
ex_DRP_returns <- tibble()
ex_PP1N_returns <- tibble()
for (btp in 1:160){
  i <- btp + 59
  backtestperiod <- (returns %>% spread(Index, monthly.returns))[1:i,]
  pca <- backtestperiod %>% select(-Date) %>% as.matrix() %>% prcomp(scale.
= T, center = T)
  PP1_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,1]), returns_col = monthly.returns)
  PP2_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,2]), returns_col = monthly.returns)
  PP3_returns <- returns %>% tq_portfolio(assets_col = Index, weights =
normalize(-pca$rotation[,3]), returns_col = monthly.returns)
  PC_portfolios <- left_join(PP1_returns, PP2_returns, by = "Date")
```

```
  PC_portfolios <- left_join(PC_portfolios, PP3_returns, by = "Date")
  DRP <- riskParityPortfolio(PC_portfolios[1:i,] %>% select(-Date) %>%
cor())
  DRP_returns <- PC_portfolios %>% gather(PP, Returns, -Date) %>%
tq_portfolio(assets_col = PP, weights = DRP$w, returns_col = Returns)
  PP_EW_returns <- PC_portfolios %>% gather(PP, Returns, -Date) %>%
tq_portfolio(assets_col = PP, weights = rep(1/3,3),

returns_col = Returns)
  ex_DRP_returns <- bind_rows(ex_DRP_returns, DRP_returns[i + 1,])
  ex_PP1N_returns <- bind_rows(ex_PP1N_returns, PP_EW_returns[i + 1,])
}
ex_PP1N_returns %>% tq_performance(Ra = portfolio.returns, performance_fun =
table.AnnualizedReturns, Rf = 0.014 / 12) * 100
ex_DRP_returns %>% tq_performance(Ra = portfolio.returns, performance_fun =
table.AnnualizedReturns, Rf = 0.014 / 12) * 100

#d. Trajectory
ex_portfolios <- left_join(bt_EW_returns, bt_world_returns[,2:3]%>%
mutate(Date = as.Date(Date)),by = "Date")
ex_portfolios <- left_join(ex_portfolios, ex_ERC_returns,by = "Date")
ex_portfolios <- left_join(ex_portfolios, ex_mv_returns,by = "Date")
ex_portfolios <- left_join(ex_portfolios, ex_DRP_returns,by = "Date")
ex_portfolios <- left_join(ex_portfolios, ex_PP1N_returns,by = "Date")
ex_portfolios_xts<- as.xts(ex_portfolios %>% remove_rownames %>%
column_to_rownames(var="Date"))
colnames(ex_portfolios_xts) <- c("1/N", "World", "ERC", "MV", "DRP", "PP
1/N")
charts.PerformanceSummary(ex_portfolios_xts, Rf = 0, main = NULL, geometric
= TRUE,methods = "none", width = 0,
                          event.labels = TRUE, ylog = FALSE,wealth.index =
TRUE, gap = 12, begin = c("first", "axis"),
                          legend.loc = "topleft", p = 0.95)
```