COMPUTATIONAL PHYSICS: ASSIGNMENT #1

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1. Introduction

The Lane-Emden equation,

$$\frac{d^2w(z)}{dz^2} + \frac{2}{z}\frac{dw(z)}{dz} + w(z)^n = 0$$

describes the density profile of a star with polytropic index n. This equation can be written as a system of two coupled ODE's,

$$\begin{aligned} \frac{dw(z)}{dz} &= v(z) \\ \frac{dv(z)}{dz} &= -2\frac{v(z)}{z} - w(z)^n \end{aligned}$$

and then solved using various Runge-Kutta methods. We will describe our findings regarding numerical solutions of this equation using forward Euler, RK2, RK4 and RK5.

2. w profile

Figure 2 on the following page shows the results of solving Lane-Emden in the case where $n = 1$. This polytropic index is chosen because the analytic solution is known and thus we can compute the L1 error of the numerical solution. As such, we can get a sense of the convergence rates of the schemes since, with dt as high as $\frac{10}{9}$, RK4 and RK5 still provide quite accurate results. Interestingly, we see that RK5 does not greatly improve the results over RK4. This is a direct result of the theorem by Butcher which states that RK schemes greater than 4th order have lower order for systems of equations than for single scalar problems[1].

3. Convergence Rate

We can further analyze the properties of the various Runge-Kutta schemes by calculating the convergence rates. Figure 3 on page 3 shows how the L1 error scales with the chosen timestep. We can see that Euler is first order, midpoint is second order, RK4 is fourth order and finally that RK5 is effectively fourth order for this particular system. This graph outlines how much better fourth order schemes are: with every halving of the timestep, we get a quarter of the error!

4. Finding the Radius of Stars

Now that we have reliable numerical methods, we can solve for the w-profile of non-analytic polytropic indices. Figure 4 on page 4 shows the results of for the w-profile when $n \in 1, \frac{3}{2}, 3$ which we use to find the radius of the star.

For the previous solutions, we were not guaranteed to have integrated to large enough z in order to see the root of $w(z)$. In order to shed more light onto this issue, we have integrated all polyropes with polytropic indices $n \in \{0, 0.05, 0.10, \cdots, 4.95\}$ and founts the roots. Then, we solve using least squares to find a relation for the radius of the star given it's polytropic index. The solution is shown in Figure 4 on page 5. To further understand this, we first realize that the polytropic index defined a relationship between the pressure, P, and the density, ρ, of a system as $P \sim \rho^{1+\frac{1}{n}}$. Since $\frac{1+n}{n} = \frac{c_p}{c_v}$

[1] Given in class, Sept 16th, 2010

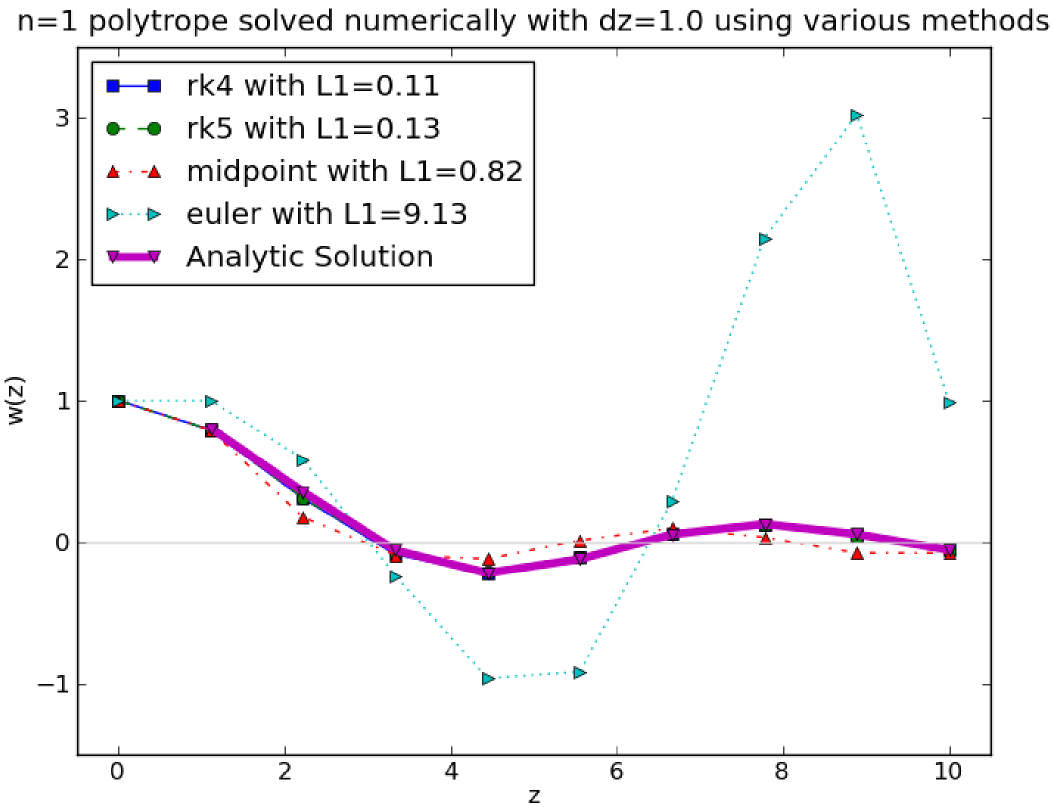

FIGURE 2.1. Solutions for Lane-Emden with n=1, $z_{max} = 10$ and $i_{max} = 10$ for forward Euler, RK2, RK4 and RK5.

where c_p is the specific heat at constant pressure and c_v is the specific heat at constant volume, this can be seen as a change of the compressibility of the gas for an adiabatic process.

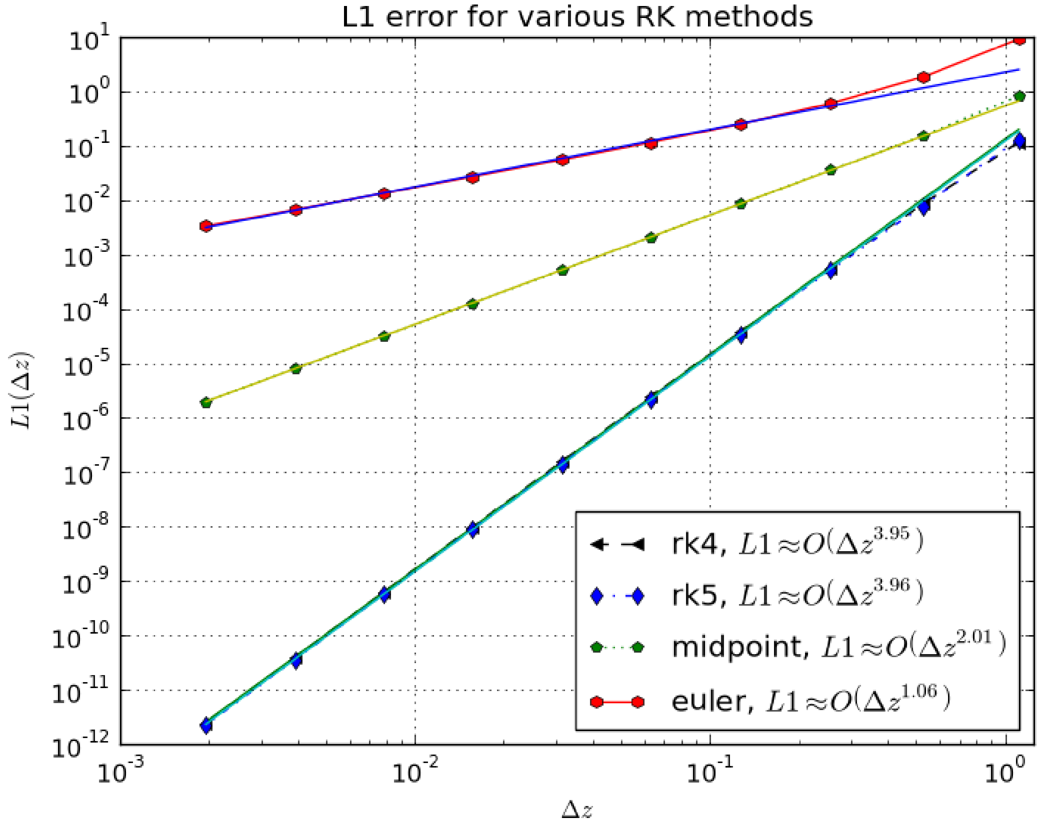

FIGURE 3.1. Convergence rates for forward Euler, RK2, RK4 and RK5

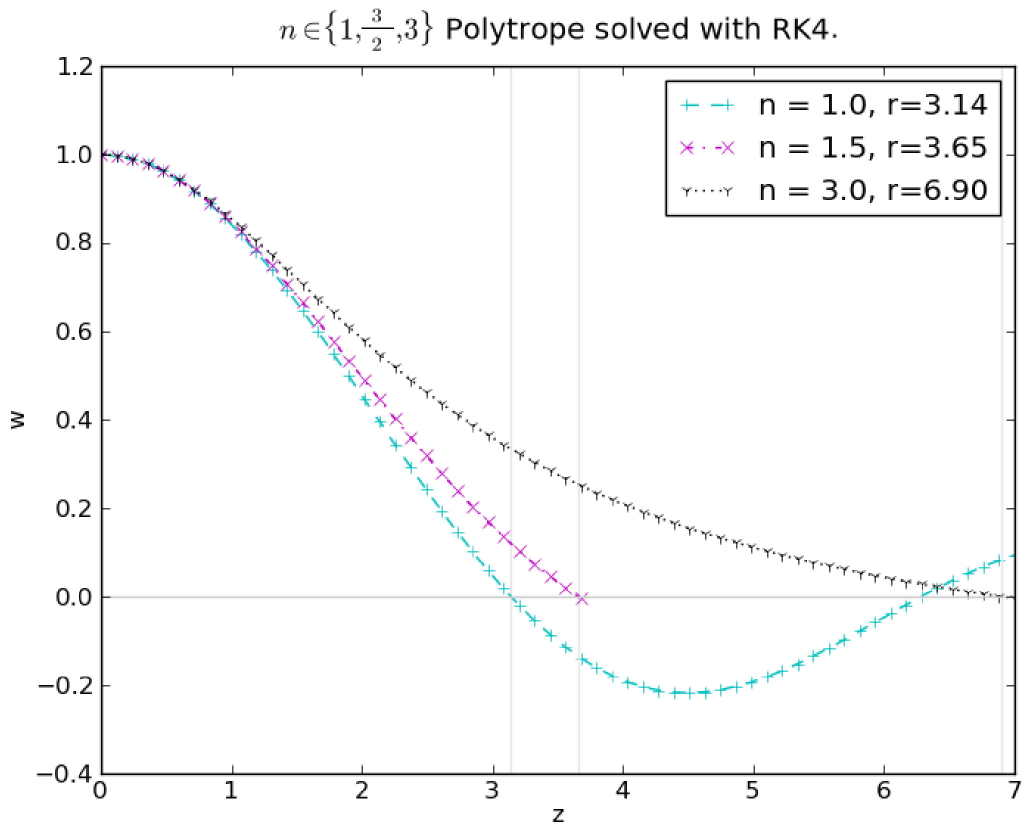

FIGURE 4.1. Solutions for Lane-Emden with $n \in 1, \frac{3}{2}, 3$ using RK4.

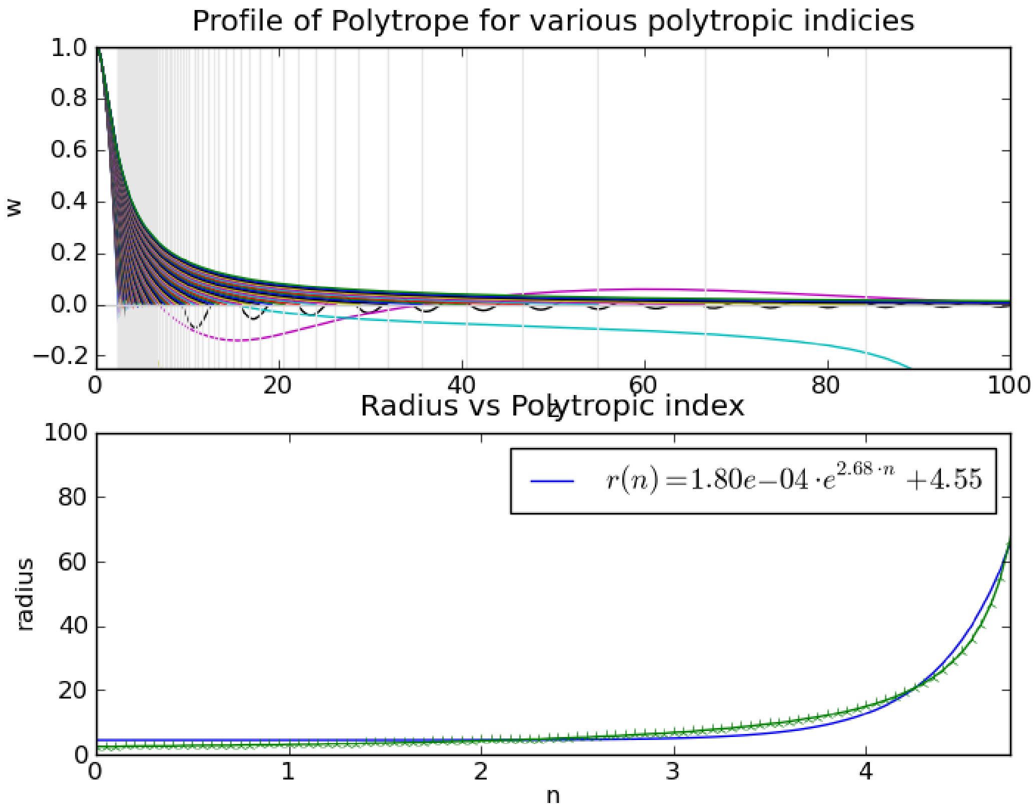

FIGURE 4.2. Using RK4 and Least Squares to find a relation for $r(n)$.