# MA 555 - Numerical Analysis

## Assignment # 6


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### Problem 1

We aim to numerically solve the IVP $yy' = 1$, $y(0) = 1$ using three methods.

Euler Explicit

$$
\begin{aligned}
yy' &= 1 \\
y' &= \frac{1}{y} \\
y_1 = y_0 + (\frac{1}{y_0})\tau &= 1 + \tau
\end{aligned}
$$

Euler Implicit

$$
y_1 = y_0 + (\frac{1}{y_1})\tau = \frac{\tau}{1+\tau}
$$

Predictor-Corrector

$$
\begin{aligned}
k_1 &= \frac{1}{y_0} \\
k_2 &= \frac{1}{y_0 + k_1\tau} \\
y_1 = y_0 + \frac{tau}{2}(1 + \frac{1}{1+\tau}) = 1 + &\frac{2\tau + \tau^2}{2(1+\tau)}
\end{aligned}
$$

### Problem 2

Applying one iteration of Euler Explicit to the given system, $\tau = 0.1$, gives

$$
\begin{aligned}
y_1 &= y_0 + y'\tau \\
y_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \tau &\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
y_1 &= \begin{bmatrix} 0.8 \\ 0.1 \end{bmatrix}
\end{aligned}
$$

## Problem 3

Solving the second order system using Euler Explicit. $y'' = -y$, $y(0) = 1$, $y'(0) = 0$ goes as follows.

$$y_1 = y_0 + y'\tau$$
$$y_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \tau \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We can analyze its stability by looking at the eigenvalues of the matrix representing the differential equation.

$$\left| \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \right| = 0$$

From which we see $\lambda = \pm i$. The solution to this system goes as $|1 - \lambda\tau|$.

$$|1 - \lambda\tau| = \sqrt{1 + \tau^2} > 1$$

Which implies instability.

## Problem 4

I'm not entirely sure how to do this problem, but here is my best guess. For the IVP $y' = -y$, $y(0) = 1$ we are given $2\alpha_0 + \alpha_1 = 1$. We can write this system as a matrix

$$y_1' = \tau \begin{bmatrix} \alpha_0 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

The eigenvalues of this matrix are $\lambda_1 = \alpha_1$, $\lambda_{2,3} = \alpha_0$. If it's still true that $|1 - \lambda\tau| < 1$, then We must restrict $\alpha$ (since no restriction on $\tau$ is imposed). From this we see that $\alpha$ must be less than 1 but greater than 0.

## Problem 5

We apply the classical RK method to the IVP $y' = -\lambda y$, the solution of which is actually $y(t) = e^{-\lambda t}$. After much algebra we can see that using the RK4 method gets us

$$y_1 = y_0 - \lambda\tau + \frac{1}{2}\tau^2\lambda^2 - \frac{1}{6}\tau^3\lambda^3 + \frac{1}{24}\tau^4\lambda^4$$

Which is the Taylor series of the actual solution up to the fourth term. In terms of stability the linear term dominates, meaning

$$|y_0 - \lambda\tau| < 1$$

Since $y_0 = 1$, $\lambda\tau < 2$ for stability (assuming both are $> 0$). In other words, $\tau < \frac{2}{\lambda}$.

**Problem 6**

We apply the explicit predictor-corrector method to the IVP $y' = -\lambda y$, the solution of which is $y(t) = e^{-\lambda t}$.

$$y_1 = y_0 + \tau(\alpha y_0' + \beta(\tau y_0'' + \frac{1}{2}\tau^2 y_0'''))$$
$$y_1 = y_0 + \tau\alpha y_0' + \tau\beta(\tau y_0'' + \frac{1}{2}\tau^2 y_0''')$$
$$y_1 = y_0 + \tau\alpha y_0' + \tau^2\beta y_0'' + \frac{1}{2}\tau^3 y_0'''$$

From matching the Taylor series we can see that $\alpha = -1$, $\beta = \frac{1}{2}$, although we can never eliminate the third term (unless I've made an error.) Again, the linear term governs stability.

$$|y_0 - \lambda\tau| < 1$$

Since $y_0 = 1$, $\lambda\tau < 2$ for stability (assuming both are $> 0$). In other words, $\tau < \frac{2}{\lambda}$, making the stability conditional.

**Problem 7**

We attempt to approximate the IVP $y' = -y$ with a multistep method, the characteristic equation for which is

$$z^2 + 2\tau z - y_0 = 0$$

Solving for $z$, we see that

$$z = \frac{-2\tau \pm \sqrt{4(\tau^2+1)}}{2}$$
$$z = -\tau \pm \sqrt{\tau^2+1}$$

In order for this to be stable, $|z| < 1$. It is clear that this is impossible because $|-\tau - \sqrt{\tau^2+1}| > 1$ for any $\tau$. Therefore this approximation is not stable.

**Problem 8**

In approximating $y'' = -y$, we will use a solution whose characteristic equation is

$$z^2 + (\tau - 2)z + 1 = 0$$

Solving for $z$, we see that

$$z = \frac{2 - \tau^2 \pm \sqrt{(\tau^2-2)^2 - 4y_0}}{2}$$

In order for this to be stable, the discriminant $(\tau^2-2)^2 - 4y_0$ must be negative (make $z$ complex). In order for that to happen, $0 < \tau < 2$.