MA 555 - Numerical Analysis

Assignment # 3
John Joseph

Problem 1

We can approximate the function $f(x) = \sqrt{x}$ with a linear Lagrange interpolant $\widetilde{y}(x)$ between some interval $0 \leq x \leq h$ like so:

$$\widetilde{y}(x) = L_1(x)f(x_1) + L_2(x)f(x_2) = \Big[\frac{x - x_2}{x_1 - x_2}\Big]\sqrt{x_1} + \Big[\frac{x - x_1}{x_2 - x_1}\Big]\sqrt{x_2}\Big|_{x_1=0, x_2=h}$$

$$\widetilde{y}(x) = 0 + \big(\frac{x}{h}\big)\sqrt{h} = \frac{x\sqrt{h}}{h}$$

Our error function $e(x) = |f(x) - \widetilde{y}(x)|$ has a maximum which can be found as follows:

$$\begin{aligned} e(x) &= \sqrt{x} - \frac{x\sqrt{h}}{h} \\ e'(x) &= \frac{1}{2}x^{-\frac{1}{2}} - \frac{\sqrt{h}}{h} = 0 \\ x &= \frac{h}{4} \\ e'(x < \frac{h}{4}) > 0; e'(x &> \frac{h}{4}) < 0 \\ e_{max}(x) = e(\frac{h}{4}) &= |\sqrt{\frac{h}{4}} - \frac{\sqrt{h}}{h}\frac{h}{4}| \\ e_{max}(x) &= \frac{\sqrt{h}}{4} \end{aligned}$$

Problem 2

We can find the parabolic Lagrange interpolant to the function $f(x) = x^3$ using the three points $x_1 = 1, x_2 = 2, x_3 = 3$ (solved with Mathematica).

$$\begin{aligned} \widetilde{y}(x) = &\Big[\frac{(x - x_2)(x - x_3}{(x_1 - x_2)(x_1 - x_3)}\Big]f(x_1) + \Big[\frac{(x - x_1)(x - x_3}{(x_2 - x_1)(x_2 - x_3)}\Big]f(x_2) \\ &+ \Big[\frac{(x - x_1)(x - x_2}{(x_3 - x_2)(x_3 - x_1)}\Big]f(x_3) = 6x^2 - 11x + 6 \end{aligned}$$

We can find the maximum error value by using the same method used above:

$$\begin{aligned}
e(x) &= |x^3 - (6x^2 - 11x + 6)| \\
e'(x) &= 3x^2 - 12x + 11 = 0 \\
x &= \frac{1}{3}(6 \pm \sqrt{3}) \\
e_{max}(x) &= e(\frac{1}{3}(6 \pm \sqrt{3})) = 0.3849
\end{aligned}$$

Problem 3

We can improve our approximations drastically by mapping the Chebyshev roots to be bound within our interval. To approximate the function $f(x) = \sqrt{1+x}$ linearly we need the two roots $\xi_{1,2} = \pm\frac{\sqrt{2}}{2}$ mapped to our interval $0 \leq x \leq \frac{1}{4}$, which we will call $c_{1,2}$.

$$\begin{aligned}
\xi \to x : x(\xi) &= \frac{x_1}{2}[1-\xi] + \frac{x_2}{2}[1+\xi]\Big|_{x_1=0, x_2=\frac{1}{4}} \\
x(\xi_{1,2}) = c_{1,2} &= \frac{1}{8}(1 \pm \frac{\sqrt{2}}{2}
\end{aligned}$$

Using these two Chebyshev roots, we can write our Lagrange Interpolant as follows:

$$\widetilde{y}(x) = \Big[\frac{x-c_2}{c_1-c_2}\Big]f(c_1) + \Big[\frac{x-x_1}{c_2-c_1}\Big]f(c_2) = 1.00087 + 0.471769x$$

Problem 4

This problem is similar to #3 except we now have three Chebyshev roots and are dealing with the function $f(x) = \frac{1}{1+x}$. We map the 3^{rd} degree Chebyshev roots, $\xi_{1,3} = \pm\frac{\sqrt{3}}{2}$, $\xi_2 = 0$, to our interval $0 \leq x \leq \frac{1}{2}$:

$$\begin{aligned}
\xi \to x : x(\xi) &= \frac{x_1}{2}[1-\xi] + \frac{x_3}{2}[1+\xi]\Big|_{x_1=0, x_2=\frac{1}{4}} \\
x(\xi_{1,3}) = c_{1,3} &= \frac{1}{4}(1 \pm \frac{\sqrt{3}}{2}) \\
x(\xi_2) = c_2 &= \frac{1}{4}
\end{aligned}$$

Having mapped the roots ont our interval, we use the Lagrange interpolants to find the parabola that passes through all three:

$$
\begin{aligned}
\widetilde{y}(x) = &\Big[\frac{(x-c_2)(x-c_3}{(c_1-c_2)(c_1-c_3)}\Big]f(c_1) + \Big[\frac{(x-c_1)(x-c_3}{(c_2-c_1)(c_2-c_3)}\Big]f(c_2) \\
&+ \Big[\frac{(x-c_1)(x-c_2}{(c_3-c_2)(c_3-c_1)}\Big]f(c_3) = \frac{4}{99}(64x^2+16x+25)
\end{aligned}
$$

Problem 5

We are asked to find the Bezier curve going through the points (0,0) and (2,0) with tangent vectors $< 0, 1 >$ and $< 0, -1 >$, respectively.

$$
\begin{aligned}
x(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\
x(0) = 0 = a_0, x(1) = 2 &= a_0 + a_1 + a_2 + a_3 \\
x'(t) &= a_1 + 2a_2 t + 3a_3 t^2 \\
x'(0) &= a_1 = 0 \\
x'(1) &= a_1 + 2a_2 + 3a_3 = 0
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= b_0 + b_1 t + b_2 t^2 + b_3 t^3 \\
y(0) = 0 = b_0, y(1) = 0 &= b_0 + b_1 + b_2 + b_3 \\
y'(t) &= b_1 + 2b_2 t + 3b_3 t^2 \\
y'(0) &= b_1 = 1 \\
y'(1) &= b_1 + 2b_2 + 3b_3 = -1
\end{aligned}
$$

From this we have 8 equations with which we can solve for our 8 unknowns. We write the system as a Matrix and solve it in Mathematica.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 3
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}
$$

From this we can solve for the coeffecient vector and see that

$$\begin{aligned} x(t) &= 6t^2 - 4t^3 \\ y(t) &= t - t^2 \end{aligned}$$

Problem 6

We can join the two splines by imposing certain continuity conditions, namely:

$$\begin{aligned} y_1 = s_1(-1) &= a_0 - a_1 + a_2 - a_3 \\ s_1''(-1) &= 0 = a_2 - 3a_3 \\ s_1(0) = s_2(0) &= y_2 = a_0 = b_0 \\ s_1'(0) &= s_2'(0) = a_1 = b_1 \\ s_1''(0) &= s_2''(0) = a_2 = b_2 \\ y_3 &= b_0 + b_1 + b_2 + b_3 \\ s_2''(1) &= 0 = b_2 + 3b_3 \end{aligned}$$

We again have 8 equations with 8 unknowns, which we can use to create a matrix representing our system of equations. We solve the matrix in Mathematica.

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ 0 \\ y_2 \\ 0 \\ y_2 \\ y_3 \\ 0 \\ 0 \end{bmatrix}$$

From which we get

$$
\begin{aligned}
a_0 = b_0 &= y2 \\
a_1 = b_1 &= \frac{1}{2}(y_3 - y_1) \\
a_2 = b_2 &= \frac{3}{4}(y_1 + y_3 - 2y_2) \\
a_3 &= \frac{1}{4}(y_1 + y_3 - 2y_2) \\
b_3 &= -\frac{1}{4}(y_1 + y_3 - 2y_2)
\end{aligned}
$$

Problem 7

This problem is like problem 6, except our curve is quadratic and we have real values for our variables.

$$
\begin{aligned}
s_1(-1) = 0 &= a_0 - a_1 + a_2 \\
s_2(2) = 0 &= b_0 + 2b_1 + 4b_2 \\
s_1(0) = s_2(0) = 1 = a_0 = b_0 s_1'(0) = s_2'(0) = \frac{1}{4} &= a_1 = b_1 \\
0 &= 1 - \frac{1}{4} + a_2 \\
a_2 &= -\frac{3}{4} \\
0 &= 1 + \frac{1}{2} + 4b_2 \\
b_2 &= -\frac{3}{8} \\
s_1(x) &= 1 + \frac{1}{4}x - \frac{3}{4}x^2 \\
s_2(x) &= 1 + \frac{1}{4}x - \frac{3}{8}x^2
\end{aligned}
$$