MA 555 - Numerical Analysis

Assignment # 2

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Problem 1

Given the function $f(x) = \frac{1}{1+x}$, we can find an osculating parabola $y_p(x)$ that mimics it with some degree of error by solving the equation

$$e(x) = f(x) - y_p(x)$$

If we take $y_p(x)$ as a second order polynomial, we must solve the following equations at $x = 0$:

$$\begin{aligned}
y_p(x) &= a_0 + a_1 x + a_2 x^2 \\
e(0) = 0 = f(0) - y_p(0) &= 1 - a_0 \\
a_0 &= 1 \\
f'(x) &= \frac{-1}{(1+x)^2} \\
e'(0) = 0 = f'(0) - y_p'(0) &= -1 - a_1 \\
a_1 &= -1 \\
f''(x) &= \frac{2}{(1+x)^3} \\
e''(0) = 0 = f''(0) - y_p''(0) &= 2 - 2a_2 \\
a_2 &= 1 \\
y_p(x) &= 1 - x + x^2 \\
e(x) &= \frac{1}{1+x} - (1 - x + x^2)
\end{aligned}$$

$e(x)$ is our error function, and we can use Taylor's Theorem to solve for its lower and upper bounds. By Taylor's Theorem,

$$e(x) = x^3\left(\frac{1}{3!}e'''(\xi)\right), 0 < \xi < x, e'''(0) \neq 0$$

We can use this fact to find the asymptoticerror value as $x \to 0$ by dividing out the x^3 term as shown in class (note that any other x^n would cause the limit to either diverge or go to 0). Solving the following limit:

$$\lim_{x\to 0} \frac{x^3\left(\frac{1}{3!}e'''(\xi)\right)}{x^3} = \lim_{x\to 0} \frac{\frac{1}{1+x} - (1 - x + x^2)}{x^3} = -1$$

This limit was solved in Wolfram, and shows that the error asymptotically approaces $-x^3$, and that the error is $O(x^3)$.

Problem 2

Given

$$r(x) = (1+x)^{\frac{1}{2}} - (a_0 + a_1 x + a_2 x^2 + a_3 x^3)$$

Similarly to problem 1, we set $r(0) = r'(0) = r''(0) = r'''(0) = 0$ to get a third order polynomial that approximates our function.

$$\begin{aligned}
r(0) = 0 = f(0) - y_p(0) &= 1 - a_0 \\
a_0 &= 1 \\
f'(x) &= \frac{1}{2}(1+x)^{-\frac{1}{2}} \\
r'(0) = 0 = f'(0) - y_p'(0) &= \frac{1}{2} - a_1 \\
a_1 &= \frac{1}{2} \\
f''(x) &= -\frac{1}{4}(1+x)^{-\frac{3}{2}} \\
r''(0) = 0 = f''(0) - y_p''(0) &= -\frac{1}{4} - 2a_2 \\
a_2 &= -\frac{1}{8} \\
f'''(x) &= \frac{3}{8}(1+x)^{-\frac{5}{2}} \\
r'''(0) = 0 = f'''(0) - y_p'''(0) &= \frac{3}{8} - 6a_3 \\
a_3 &= \frac{1}{16} \\
r(x) = (1+x)^{\frac{1}{2}} - (1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3)
\end{aligned}$$

Again, by Taylor's Theorem:

$$r^{(4)}(x) \neq 0, r(x) = \frac{1}{4!}x^4 r^{(4)}(\xi), 0 < \xi < x$$

If we take the limit as $x \to 0$, we see that the error function $r(x)$ asymptotically approaches

$$\lim_{x\to 0} \frac{\frac{1}{4!}x^4 r^{(4)}(\xi)}{x^4} = \frac{1}{4!} r^{(4)}(0)$$

Problem 4

We want to fix the variables a and b so that the second order terms of the Taylor polynomial of the function $e(x)$ equals 0

$$e(x) = ln(1+x) - \frac{ax}{1+bx}$$

$$\begin{aligned} e(0) = 0 &= 0 - 0 \\ e'(0) = 0 = \frac{1}{(1+x)} &- \frac{a}{(bx+1)^2}\Big|_{x=0} \\ a &= 1 \\ e''(0) = 0 = -\frac{1}{(1+x)^2} &+ \frac{2b}{(bx+1)^3}\Big|_{x=0} \\ b &= \frac{1}{2} \\ e(x) = ln(1+x) - \frac{x}{x+\frac{1}{2}} &= ln(1+x) - \frac{2x}{2+x} \end{aligned}$$

Using Taylor's theorem, we also know that

$$e(x) = \frac{1}{3!} x^3 e'''(\xi), 0 < \xi < x$$

This is our error function,and by looking at it we can see that we must divide it by x^3 to get a non-zero / non-diverging value. We can evaluate the error asymptotically by taking the limit

$$\lim_{x\to 0} \frac{\frac{1}{3!}x^3 e'''(\xi)}{x^3} = \lim_{x\to 0} \frac{ln(1+x) - \frac{2x}{2+x}}{x^3} = \frac{1}{12}$$

Which was solved using Wolfram. This shows that our error goes asymptotically to $\frac{1}{12}x^3$ as $x \to 0$, so our error is $O(x^3)$.

Problem 3

We are solving the IVP

$$y'(x) = x^2 + y^2, y(1) = 1$$

And are asked to approximating with an osculating cubic Taylor polynomial centered on $x = 1$:

$$y_c(x) = y(1) + (x-1)y'(1) + \frac{1}{2!}(x-1)^2 y''(1) + \frac{1}{3!}(x-1)^3 y'''(1)$$

We do so as follows:

$$\begin{aligned}
y(1) &= 1 \\
y'(1) = x^2 + y^2\Big|_{x=1} &= 2 \\
y''(x) = 2x + 2y(x)y'(x)\Big|_{x=1} &= 6 \\
y'''(x) = 2 + 2[y'(x)^2 + y(x)y''(x)]\Big|_{x=1} &= 22 \\
y_c(x) = 1 + 2(x-1) + \frac{6}{2!}(x-1)^2 + \frac{22}{3!}&(x-1)^3
\end{aligned}$$

Problem 5

Using the following Python script, I found that $n = 31$.

```
import math

x=7.5
y=1.0
n=0
while (math.exp(x)-y) > 1e-7:
    n=n+1
    y=y+(1.0/math.factorial(n))*x**n
print n

...

31
```

The error of the polynomial when n=31 is 4.928e-08, which is less than 1e-07 (or 0.0000001). 30 iterations yields an error value of 2.121e-07, which is greater than 1e-07.