Note on the Fourier transform of a real function.

Let us consider an array f_j with $2N$ real components. We extend the index j beyond the range $0 \leq j \leq 2N-1$ by assuming periodicity with period $2N$. Thus $f_j = f_{j+2mN}$ where m is any integer. We will use this extension in several equations that follow.
The Fourier transform of f is given by

$$F_k = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} f_j \, e^{-2\pi\imath \frac{jk}{2N}} \tag{1}$$

From this equation we see that

$$F^*_{-k} = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} \left[f_j \, e^{2\pi\imath \frac{jk}{2N}} \right]^* = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} f_j \, e^{-2\pi\imath \frac{jk}{2N}} = F_k \tag{2}$$

where we have used the fact that f_j is real. We will take the domain of definition of F_k to be $-N+1 \leq k \leq N$. In order to satisfy the constraint given by Eq. 2 we set

$$F_0 = 2G_0 \tag{3}$$

$$F_N = 2G_N \tag{4}$$

and

$$F_k = G_k - \imath H_k, \qquad F_{-k} = G_k + \imath H_k \tag{5}$$

for $1 \leq k \leq N-1$, with real G_k, H_k. Note that the number of the real variables G_k, H_k is $2N$, the same as the number of the (real) components of f.
We can now express the components of f in terms of the variables G_k and H_k. We have

$$f_j = \frac{1}{\sqrt{2N}} \sum_{k=-N+1}^{N} F_k \, e^{2\pi\imath \frac{jk}{2N}} =$$

$$\frac{1}{\sqrt{2N}} \left[F_0 + \sum_{k=1}^{N-1} F_k \, e^{2\pi\imath \frac{jk}{2N}} + \sum_{k=1}^{N-1} F_{-k} \, e^{2\pi\imath \frac{-jk}{2N}} + \frac{1}{\sqrt{2N}} F_N \, e^{-\pi\imath j} \right] =$$

$$\frac{1}{\sqrt{2N}} \left[2G_0 + \sum_{k=1}^{N-1} (G_k - \imath H_k) \, e^{2\pi\imath \frac{jk}{2N}} + \sum_{k=1}^{N-1} (G_k + \imath H_k) \, e^{2\pi\imath \frac{-jk}{2N}} + 2G_N (-1)^j \right] =$$

$$
\begin{aligned}
&\frac{1}{\sqrt{2N}}\Bigg[2G_0 + \sum_{k=1}^{N-1} G_k \left(e^{2\pi\imath\frac{jk}{2N}} + e^{2\pi\imath\frac{-jk}{2N}}\right) + 2G_N(-1)^j \\
&\qquad + \sum_{k=1}^{N-1}(-\imath H_k)\left(e^{2\pi\imath\frac{jk}{2N}} - e^{2\pi\imath\frac{-jk}{2N}}\right)\Bigg] = \\
&\frac{2}{\sqrt{2N}}\sum_{k=0}^{N} G_k \cos\left(\frac{\pi jk}{N}\right) + \frac{2}{\sqrt{2N}}\sum_{k=1}^{N-1} H_k \sin\left(\frac{\pi jk}{N}\right)\Bigg]
\end{aligned} \tag{6}
$$

The final line expresses f_j as a sum of two functions

$$
g_j = \sqrt{\frac{2}{N}}\sum_{k=0}^{N} G_k \cos\left(\frac{\pi jk}{N}\right) \tag{7}
$$

and

$$
h_j = \sqrt{\frac{2}{N}}\sum_{k=1}^{N-1} H_k \sin\left(\frac{\pi jk}{N}\right) \tag{8}
$$

where g_j is even under $j \to -j$ while f_j is odd for $j \to -j$ and vanishes for $j = 0$ and $j = N$. In the limit where $N \to \infty$ and the finite Fourier transform goes over a Fourier series, Eq. 7 gives the expansion into a cosine series of a function $g(x)$ which is even under $x \to -x$ and has vanishing derivatives at the boundaries of the domain of definition, while Eq. 8 gives the expansion into a sine series of a function $h(x)$ which is even under $x \to -x$ and vanishes at the boundaries of the domain of definition,