

## Two Dimensional Random Variables (Continued)

### Definition 3.3

1.  $(X, Y)$  is a two-dimensional **discrete** random variable if the possible values of  $(X(s), Y(s))$  are **finite or countable infinite**.  
 i.e. the possible values of  $(X(s), Y(s))$  may be represented as  

$$(x_i, y_j), i = 1, 2, 3, \dots; j = 1, 2, 3, \dots$$
2.  $(X, Y)$  is a two-dimensional **continuous** random variable if the possible values of  $(X(s), Y(s))$  can **assume all values in some region** of the Euclidean plane  $\mathbb{R}^2$ .

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To judge whether a two dimensional random vector  $(X, Y)$  is discrete or continuous, we can view  $X$  and  $Y$  separately.

- ✓ If both  $X$  and  $Y$  are discrete random variables, we say  $(X, Y)$  is a discrete random vector.
- ✓ Likewise, if both  $X$  and  $Y$  are continuous random variables, we say  $(X, Y)$  is a continuous random vector.
- ✓ Certainly, there are other cases. For example,  $X$  is discrete, but  $Y$  is continuous, or  $Y$  is neither a discrete nor a continuous random variable. But these are not the main focus of this module.

### 3.2.1 Joint Probability Function for Discrete RVs

#### Definition 3.4

- Let  $(X, Y)$  be a 2-dimensional **discrete** random variable defined on the sample space of an experiment. With each possible value  $(x_i, y_j)$ , we associate a number  $f_{X,Y}(x_i, y_j)$  representing  $\Pr(X = x_i, Y = y_j)$  and satisfying the following conditions:
  - $f_{X,Y}(x_i, y_j) \geq 0$  for all  $(x_i, y_j) \in R_{X,Y}$ .
  - $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{X,Y}(x_i, y_j) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \Pr(X = x_i, Y = y_j) = 1$  (3.1)

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Equation (3.1) on this page of the lecture slide essentially requires that the summation over all  $f(x_i, y_j) > 0$  equals 1. It can be equivalently written as

$$\sum_{(x_i, y_j): f_{X,Y}(x_i, y_j) > 0} f_{X,Y}(x_i, y_j) = 1.$$

Note that in this case,  $f_{X,Y}(x_i, y_j)$  may not be defined for some  $x_i$  and  $y_j$ ; see the distribution given on page 3-20. So, in this case, if you would like to add  $i = 0, 1, 2, 3$  and  $j = 0, 1, 2, 3$  freely, you need use 0 to replace those  $f_{X,Y}(x, y)$  who does not have a point mass on  $(x, y)$ .

## Solution to Example 3 (Continued)

The above p.f. are given explicitly in the following table.

$x$	$y$				Row Total
	0	1	2	3	
0	0	3/84	6/84	1/84	10/84
1	4/84	24/84	12/84	0	40/84
2	12/84	18/84	0	0	30/84
3	4/84	0	0	0	4/84
Column Total	20/84	45/84	18/84	1/84	1

## Joint pdf for Continuous RVs (Continued)

1.  $f_{X,Y}(x, y) \geq 0$  for all  $(x, y) \in R_{X,Y}$ .
- 2.

$$\iint_{(x,y) \in R_{X,Y}} f_{X,Y}(x, y) dx dy = 1$$

or

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1.$$

- ✓ In most cases of this module, when we do the bivariate integration, the integration region is a rectangular; therefore, the variables  $x$  and  $y$  can be integrated separately; and the order of which is integrated first does not matter. See examples 3-24, 3-28, and 3-29.
- ✓ However, we need to bear in mind that there are cases under which the integration region is NOT a rectangular, so that  $x$  and  $y$  can not move freely for a unified expression of  $f_{X,Y}(x, y)$ . See the example given on pages 3-25, 3-26, and 3-27 of the lecture slides: the region is defined by straight lines such as a triangle or a trapezium.

**Note:** when we integrate a two dimensional function in a region which is not a rectangular, we need to take care that  $x$  and  $y$  may not move freely! Based on mathematical theory, integrating which variable first won't change the outcome of the integration; however, a right choice of integration order may make the computation easier; read pages 3-25 to 3-26 carefully for such an example.

## Marginal Distributions (Continued)

- For **discrete** case,

$$f_X(x) = \sum_y f_{X,Y}(x, y) \quad \text{and} \quad f_Y(y) = \sum_x f_{X,Y}(x, y)$$

- For **continuous** case,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$$

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The practical interpretation of the marginal distribution for  $X$  is: focusing on viewing the distribution of  $X$  by ignoring the presence of  $Y$ . **Note that**

★  $f_X(x)$  should NOT involve  $y$ ; and

★ it is a pdf/pmf; so it must have all the properties of a pdf/pmf.

If  $(X, Y)$  is discrete, then the marginals are also discrete; likewise, if  $(X, Y)$  is continuous, the marginals are also continuous.

The meaning of the formulae for  $f_X(x)$  is that “for each given  $x$ , integrate (or sum) over all the value of  $y$  such that  $f_{X,Y}(x, y) > 0$ .” So, similar to the discussion of page 4 above, we need to take care of the region of  $y$  for each  $x$ .

## Conditional Distribution (Continued)

### Definition 3.7 (Continued)

- Then the conditional distribution of  $Y$  given that  $X = x$  is given by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}, \quad \text{if } f_X(x) > 0,$$

for each  $x$  within the range of  $X$ .

- ✓ The condition distribution is similar in meaning to the condition probability. It is the distribution of the random variable of  $Y$  when the random  $X$  is fixed at a certain value  $x$ .
- ✓ It is important to take note that it is a distribution for  $y$ , so it must satisfies all the properties of a pdf/pmf in terms of the argument  $y$  for every  $x$  that it is defined.
- ✓ It may or may not be a function of  $x$ . But it is defined only when  $x$  satisfies  $f_X(x) > 0$ . If it does not depend on  $x$ , then we have  $X$  and  $Y$  independent.
- ✓ It is not a pdf/pmf for  $x$ . So there is NO requirement that  $\int_{-\infty}^{\infty} f_{Y|X}(y|x)dx = 1$  when  $Y$  is continuous or  $\sum_x f_{Y|X}(y|x) = 1$ , when  $Y$  is discrete.
- ✓ Can you find  $f_{Y|X}(y|x)$  for the example given on page 5?

## Example 1 (Continued)

- $f_{X,Y}(x,y)$ ,  $f_X(x)$  and  $f_Y(y)$  are displayed in the following table

$y$	$x$						$f_Y(y)$
	0	1	2	3	4	5	
0	0	0.01	0.02	0.05	0.06	0.08	0.22
1	0.01	0.03	0.04	0.05	0.05	0.07	0.25
2	0.02	0.03	0.05	0.06	0.06	0.07	0.29
3	0.02	0.04	0.03	0.04	0.06	0.05	0.24
$f_X(x)$	0.05	0.11	0.14	0.20	0.23	0.27	1

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## Example 1 (Continued)

Outcome	HHH	THH	HTH	HHT	TTH	THT	HTT	TTT
$(x,y)$	(1,3)	(1,2)	(1,2)	(0,2)	(1,1)	(0,1)	(0,1)	(0,0)
$f_{X,Y}(x,y)$	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

- The joint probability distribution of  $(X, Y)$  is given in the following table:

$x$	$y$				$f_X(x)$
	0	1	2	3	
0	1/8	1/4	1/8	0	1/2
1	0	1/8	1/4	1/8	1/2
$f_Y(y)$	1/8	3/8	3/8	1/8	1

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For a discrete random vector  $(X, Y)$ . The two-dimensional tables as shown in these slides are particularly useful to help us understand the joint, marginal, and the conditional distributions.