MA2001

LIVE LECTURE 12

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Topics for week 12

7.1 Linear Transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$

7.2 Ranges and Kernel

Only section 7.1 and 7.2 are in the exam scope

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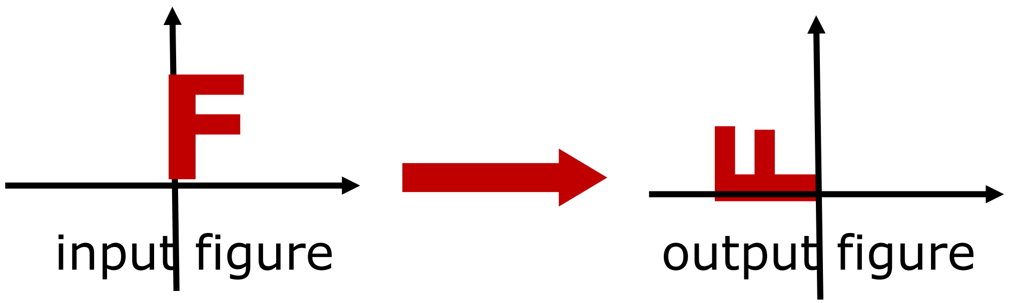
Geometrical Interpretation of Matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \boldsymbol{Au} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

input vector

output vector

A can be regarded as a transformation



$T : \mathbf{R}^2 \to \mathbf{R}^2$ (a mapping or function)

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -y \\ x \end{pmatrix} \qquad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$$

T : linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$

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Linear Transformation

A: m x n matrix

standard matrix of the linear transformation

$T : \mathbf{R}^n \to \mathbf{R}^m$ defined by $T(\mathbf{u}) = \mathbf{A}\mathbf{u}$ for all $\boldsymbol{u} \in \mathbf{R}^n$

T is called a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$

To show a mapping T is linear transformation, express T in terms of standard matrix

Linearity Conditions

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$
3. $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$
4. $T(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}) = c_1T(\boldsymbol{u_1}) + c_2T(\boldsymbol{u_2}) + \cdots + c_kT(\boldsymbol{u_k})$

To show a mapping T is NOT linear transformation, show T violates one of these conditions

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Example of Linear Transformations

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x - y + 5z \end{pmatrix} = x\begin{pmatrix} 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} 2 \\ -1 \end{pmatrix} + z\begin{pmatrix} 0 \\ 5 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 5 \end{pmatrix}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

standard matrix of T

So T(**u**) = **Au** for all $\boldsymbol{u} \in \mathbf{R}^3$

T is a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 0 \\ y - 2x \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -2 & 1 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix}$$

standard matrix of S

So S(**u**) = **Bu** for all $\boldsymbol{u} \in \mathbf{R}^2$

S is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$

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Example of Non-Linear Transformation

$$M\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\1\\y-2x\end{pmatrix}$$

$$M\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$M\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right] = \begin{pmatrix}2\\1\\-1\end{pmatrix}$$

Violate linearity

Violate linearity

$$M\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix} \qquad M\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow M\begin{pmatrix}1\\0\end{pmatrix} + M\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\2\\-1\end{pmatrix}$$

So M does not preserve vector addition

M is not a linear transformation

Linearity condition:

$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

$T(\mathbf{0}) = \mathbf{0}$

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Is this a linear transformation?

Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be given by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3$$

not the same as $\left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right]\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Try to find $\boldsymbol{A}$ such that $T(\boldsymbol{u}) = \boldsymbol{Au}$

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = (x + 2y + 3z)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 2x + 4y + 6z \\ 3x + 6y + 9z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So T is a linear transformation

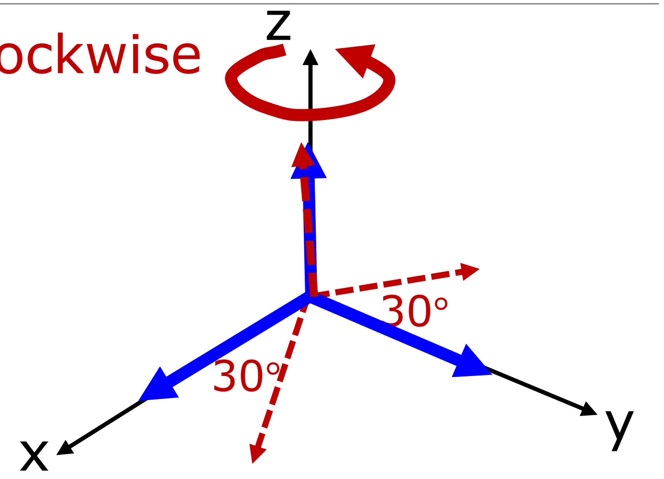
Standard matrix

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$: standard basis for $\mathbf{R}^n$

$$\mathbf{A} = \left(T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \ldots\ T(\mathbf{e}_n) \right)$$

A Geometrical Transformation

T : $\mathbf{R}^3 \to \mathbf{R}^3$ rotation about the z-axis with 30° anticlockwise



T is a linear transformation

Find the standard matrix for T.

Use the standard basis $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ for $\mathbf{R}^3$

We have $\mathbf{A}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos 30\\ \sin 30\\ 0\end{pmatrix}$, $\mathbf{A}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-\sin 30\\ \cos 30\\ 0\end{pmatrix}$ and $\mathbf{A}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

$$\mathbf{A}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0\\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0\\ 0 & 0 & 1\end{pmatrix} \Rightarrow \mathbf{A} = \begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0\\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0\\ 0 & 0 & 1\end{pmatrix}$$ the standard matrix of T

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Stacking Method

$T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation

Given $\mathbf{A}\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}2\\6\\6\end{pmatrix}$, $\mathbf{A}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}4\\8\\2\end{pmatrix}$ and $\mathbf{A}\begin{pmatrix}0\\2\\0\end{pmatrix} = \begin{pmatrix}6\\6\\6\end{pmatrix}$.

Find the standard matrix for T.

$$\mathbf{A}\begin{pmatrix}1 & 1 & 0\\2 & 1 & 2\\1 & 0 & 0\end{pmatrix} = \begin{pmatrix}2 & 4 & 6\\6 & 8 & 6\\6 & 2 & 6\end{pmatrix} \Longrightarrow \mathbf{A} = \begin{pmatrix}2 & 4 & 6\\6 & 8 & 6\\6 & 2 & 6\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\2 & 1 & 2\\1 & 0 & 0\end{pmatrix}^{-1}$$

Check:

$\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}$ form a basis for $\mathbf{R}^3$

$$\mathbf{A} = \begin{pmatrix}1 & 3 & -5\\5 & 3 & -5\\-1 & 3 & 1\end{pmatrix}$$

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Images of Bases Vectors

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$: a basis for $\mathbf{R}^n$

$T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation

If we know the images $T(\boldsymbol{u}_1)$, $T(\boldsymbol{u}_2)$,…, $T(\boldsymbol{u}_n)$, we can

- find the standard matrix of T
- determine the image $T(\boldsymbol{v})$ of any vector $\boldsymbol{v}$ in the domain $\mathbf{R}^n$.

Completely Determine a Linear Transformation

T: $\mathbf{R}^n \longrightarrow \mathbf{R}^m$

standard matrix of T

formula of T

images $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)$ of a basis for $\mathbf{R}^n$

In particular, can take the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$

Images of Incomplete Basis

$T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation

Linearity Condition

$$T(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}) = c_1T(\boldsymbol{u_1}) + c_2T(\boldsymbol{u_2}) + \cdots + c_kT(\boldsymbol{u_k})$$

Suppose we know the images of $T\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\1\\0\end{pmatrix}$.

Yes No

Can we find the images of $T\begin{pmatrix}2\\3\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\2\\3\end{pmatrix}$?

$$\begin{pmatrix}2\\3\\0\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\end{pmatrix} \Longrightarrow T\begin{pmatrix}2\\3\\0\end{pmatrix} = T\left[2\begin{pmatrix}1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\end{pmatrix}\right] = 2T\begin{pmatrix}1\\0\\0\end{pmatrix} + 3T\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$\begin{pmatrix}0\\2\\3\end{pmatrix}$ is not a linear combination of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$

Composition

Let $S : \mathbf{R}^n \to \mathbf{R}^m$ and $T : \mathbf{R}^m \to \mathbf{R}^k$ be linear transformations.

S, T have standard matrices $\boldsymbol{A}$, $\boldsymbol{B}$ respectively

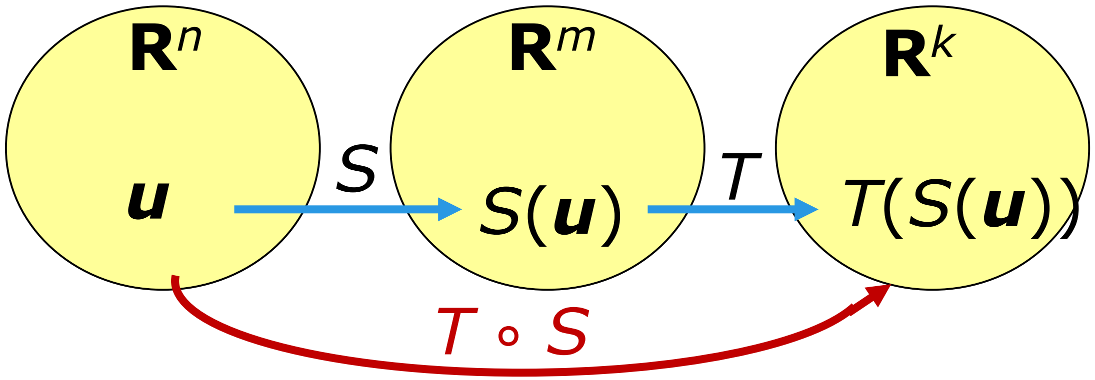
The composition of T with S, denoted by $T \circ S$

First S, then T

is a mapping from $\mathbf{R}^n$ to $\mathbf{R}^k$ such that

$T \circ S$ has standard matrix $\boldsymbol{BA}$

$(T \circ S)(\boldsymbol{u}) = T(S(\boldsymbol{u}))$ for all $\boldsymbol{u}$ in $\mathbf{R}^n$.



Notation:

$T \circ T$ by T^2

$T \circ T \circ T$ by T^3

etc

In a problem involving range, if you start with: $\mathbf{v} \in R(T)$,
you should follow by: $\mathbf{v} = T(\mathbf{u})$ for some $\mathbf{u} \in \mathbf{R}^n$.

Range of Linear Transformation

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation.

$\boldsymbol{A}$ the standard matrix for T

The range of T, $R(T)$ is the set of images of T.

$R(T) = \{T(\boldsymbol{u}) \mid \boldsymbol{u} \in \mathbf{R}^n\}$

$R(T)$ is a subspace of $\mathbf{R}^m$

= the column space of $\boldsymbol{A}$

$\text{rank}(T) = \text{rank}(\boldsymbol{A})$

$= \text{span}\{ T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), \ldots, T(\boldsymbol{e}_n) \}$

$= \text{span}\{ T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n) \}$ where $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is a basis for $\mathbf{R}^n$

In a problem involving kernel, if you start with: $\mathbf{v} \in \ker(T)$,
you should follow by: $T(\mathbf{v}) = \mathbf{0}$.

Kernel of Linear Transformation

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation.

$\boldsymbol{A}$ the standard matrix for T

The kernel of T, $\text{Ker}(T)$ is the set of vectors in $\mathbf{R}^n$ whose images under T are $\mathbf{0}$.

$$\text{Ker}(T) = \{\boldsymbol{u} \in \mathbf{R}^n \mid T(\boldsymbol{u}) = \mathbf{0}\}$$

$$= \text{the nullspace of } \boldsymbol{A}$$

$$= \text{solution space of } \boldsymbol{Ax} = \boldsymbol{0}$$

$\text{Ker}(T)$ is a subspace of $\mathbf{R}^n$

$\text{nullity}(T) = \text{nullity}(\boldsymbol{A})$

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Zero and identity transformation

Let $S_1 : \mathbf{R}^n \to \mathbf{R}^m$ be the zero transformation given by $S_1(\boldsymbol{u}) = \mathbf{0}$ for all $\boldsymbol{u}$ in $\mathbf{R}^n$

Standard matrix $\boldsymbol{A} = \mathbf{0}_{mxn}$

Let $S_2 : \mathbf{R}^n \to \mathbf{R}^n$ be the identity transformation given by $S_2(\boldsymbol{u}) = \boldsymbol{u}$ for all $\boldsymbol{u}$ in $\mathbf{R}^n$

Standard matrix $\boldsymbol{A} = \mathbf{I}_n$

What are $R(S_1)$ and $R(S_2)$?

$R(S_1) = \{\mathbf{0}\}$ $R(S_2) = \mathbf{R}^n$

What are $\ker(S_1)$ and $\ker(S_2)$?

$\ker(S_1) = \mathbf{R}^n$ $\ker(S_2) = \{\mathbf{0}\}$

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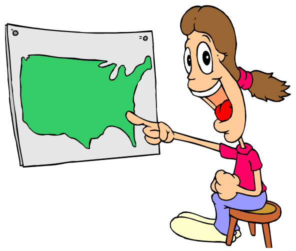
Linear transformation vs standard matrix

Linear Transformation	**Standard matrix**
$T: \mathbf{R}^n \to \mathbf{R}^m$	$\mathbf{A}$ is an $m \times n$ matrix
$T(\mathbf{u})$	$\mathbf{Au}$
$T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$	columns of $\mathbf{A}$
$R(T)$	column space of $\mathbf{A}$
$Ker(T)$	nullspace of $\mathbf{A}$
$rank(T)$	$rank(\mathbf{A})$
$nullity(T)$	$nullity(\mathbf{A})$
$S \circ T$	$\mathbf{BA}$

Map of LA



A is an n×n matrix

$T_{\mathbf{A}} : \mathbf{R}^n \to \mathbf{R}^n$

A is invertible	chapter 2	A is not invertible
det A $\neq$ 0	chapter 2	det A = 0
rref of A is identity matrix	chapter 1	rref of A has a zero row
AX= 0 has only the trivial solution	chapter 1	AX= 0 has non-trivial solutions
AX= b has a unique solution	chapter 1	AX= b has no solution or infinitely many solutions
rows (columns) of A are linearly independent	chapter 3	rows (columns) of A are linearly dependent
nullity(A) = 0 rank(A) = n	chapter 4	rank(A) < n nullity(A) > 0
0 is not an eigenvalue of A	chapter 6	0 is an eigenvalue of A
$\ker(T_A) = \{0\}$ $R(T_A) = R^n$	chapter 7	$R(T_A) \neq R^n$ $\ker(T_A) \neq \{0\}$

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Exercise 7 Q8

$T: \mathbf{R}^n \to \mathbf{R}^n$ linear transformation such that $T \circ T = T$

If T is not the zero transformation, show that
there exists a nonzero vector $\boldsymbol{u}$ in $\mathbf{R}^n$ such that $T(\boldsymbol{u}) = \boldsymbol{u}$.

There exist $\mathbf{v}$ in $\mathbf{R}^n$ s.t. $T(\mathbf{v}) \neq \mathbf{0}$.

Let $\mathbf{u} = T(\mathbf{v})$. Then $T(\mathbf{u}) = T \circ T(\mathbf{v}) = T(\mathbf{v}) = \mathbf{u}$

If T is not the identity transformation, show that
there exists a nonzero vector $\boldsymbol{v}$ in $\mathbf{R}^n$ such that $T(\boldsymbol{v}) = \boldsymbol{0}$.

There exist $\mathbf{w}$ in $\mathbf{R}^n$ s.t. $T(\mathbf{w}) \neq \mathbf{w}$.

Let $\mathbf{v} = T(\mathbf{w}) - \mathbf{w}$.

Then $T(\mathbf{v}) = T \circ T(\mathbf{w}) - T(\mathbf{w}) = T(\mathbf{w}) - T(\mathbf{w}) = \mathbf{0}$

Show the other direction:
If T is one-to-one, then ker(T) = {$\mathbf{0}$}.

Exercise 7 Q16

Cannot use invertibility of matrix

T: $\mathbf{R}^n \to \mathbf{R}^m$ linear transformation

Show that ker(T) = {$\mathbf{0}$} if and only if T is one-to-one.

If $\mathbf{u} \neq \mathbf{v}$, then $T(\mathbf{u}) \neq T(\mathbf{v})$

Suppose ker(T) = {$\mathbf{0}$}

If on the contrary, T is not one-to-one,
then there exists $\mathbf{u} \neq \mathbf{v}$ such that $T(\mathbf{u}) = T(\mathbf{v})$.
Let $\mathbf{w} = \mathbf{u} - \mathbf{v} \neq \mathbf{0}$.
Then $T(\mathbf{w}) = T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$
So $\mathbf{w} \in$ ker(T), a contradiction.
So T must be one-to-one.

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Exercise 7 Q9

F: $\mathbf{R}^n \to \mathbf{R}^n$ defined by $F(\mathbf{x}) = \mathbf{x} - 2(\mathbf{n} \cdot \mathbf{x})\,\mathbf{n}$

n is some fixed unit vector

a) Show that F is a linear transformation and find its standard matrix

$2(\mathbf{n} \cdot \mathbf{x})\,\mathbf{n} = 2\mathbf{n}\,(\mathbf{n} \cdot \mathbf{x}) = 2\mathbf{n}\,(\mathbf{n}^T\mathbf{x}) = 2(\mathbf{n}\mathbf{n}^T)\,\mathbf{x}$

$F(\mathbf{x}) = \mathbf{x} - 2(\mathbf{n}\mathbf{n}^T)\,\mathbf{x} = (\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)\,\mathbf{x}$

b) Prove that $F \circ F$ is the identity transformation

Same as showing: $(\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)^2 = \mathbf{I}$

Use the condition **n** is a unit vector

c) Show that the standard matrix for F is an orthogonal matrix

Show that: $(\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)^{-1} = (\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)^T$

Use part (b)

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Announcement

- Exam briefing ★★★★★
  - Week 13 zoom session
- Mock Exam ★★★★★
  - Week 13 zoom session
  - Install examplify to your PC before the mock exam
- Group Discussion 6
  - Week 13 tutorial
- Due next week
  - Homework 4: next Saturday
  - Online quiz 12: next Thursday
- Student Feedback exercise
  - Opened now till Nov 19