CS1231S Chapter 9

Functions

9.1 Basics

Definition 7.2.1 (again)**.** Let A, B be sets. A *function* or a *map* from A to B is an assignment to each element of A exactly one element of B. We write $f\colon A \to B$ for "f is a function from A to B". Suppose $f\colon A \to B$.

(1) Let $x \in A$. Then $f(x)$ denotes the element of B that f assigns x to. We call $f(x)$ the *image* of x under f. If $y = f(x)$, then we say that f *maps* x *to* y, and we may write $f\colon x \mapsto y$.

(2) Here A is called the *domain* of f, and B is called the *codomain* of f.

Remark 9.1.1. (1) A sequence $a_0, a_1, a_2, \ldots$ can be represented by the function a whose domain is $\mathbb{Z}_{\geqslant 0}$ that satisfies $a(n) = a_n$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

(2) In this sense, any function whose domain is $\mathbb{Z}_{\geqslant m}$ for some $m \in \mathbb{Z}$ represents a sequence.

Example 9.1.2. One can represent the Fibonacci sequence $F_0, F_1, F_2, \ldots$ by the unique function $F\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ that satisfies, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$F(0) = 0 \quad \text{and} \quad F(1) = 1 \quad \text{and} \quad F(n+2) = F(n+1) + F(n).$$

Such an F exists and is unique, essentially by Proposition 8.3.4.

Definition 9.1.3. Let A be a set. A *string* or a *word* over A is an expression of the form

$$a_0 a_1 \ldots a_{\ell-1}$$

where $\ell \in \mathbb{Z}_{\geqslant 0}$ and $a_0, a_1, \ldots, a_{\ell-1} \in A$. Here ℓ is called the *length* of the string. Let A^* denote the set of all strings over A. The *empty string*, denoted ε, is the string of length 0.

Example 9.1.4. Let $A = \{\mathrm{s}, \mathrm{u}\}$. The following are strings over A:

$$\mathrm{s}, \quad \mathrm{ssuu}, \quad \mathrm{susususu}, \quad \mathrm{uuuuuuu}, \quad \ldots.$$

Their lengths are respectively 1, 4, 8, and 7.

Remark 9.1.5. Let A be a set.

(1) One can represent a string $a_0 a_1 \ldots a_{\ell-1}$ over A by the function $a\colon \{0, 1, \ldots, \ell-1\} \to A$ satisfying $a(n) = a_n$ for all $n \in \{0, 1, \ldots, \ell-1\}$.

(2) Every function $a\colon \{m, m+1, \ldots, m+\ell-1\} \to A$, where $m \in \mathbb{Z}$ and $\ell \in \mathbb{Z}_{\geqslant 0}$, represents a string of length ℓ over A, namely $a(m)\, a(m+1)\, \ldots\, a(m+\ell-1)$.

Definition 9.1.6. Two functions $f\colon A \to B$ and $g\colon C \to D$ are *equal* if

(1) $A = C$ and $B = D$; and

(2) $f(x) = g(x)$ for all $x \in A$.

In this case, we write $f = g$.

Example 9.1.7. Let $f\colon \{0,2\} \to \mathbb{Z}$ and $g\colon \{0,2\} \to \mathbb{Z}$ defined by setting, for all $x \in \{0,2\}$,

$$f(x) = 2x \quad \text{and} \quad g(x) = x^2.$$

Then $f = g$ because their domains are the same, their codomains are the same, and $f(x) = g(x)$ for every $x \in \{0,2\}$.

Example 9.1.8. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Q}$ defined by setting, for all $x \in \mathbb{Z}$,

$$f(x) = x^3 = g(x).$$

Then $f \neq g$ because they have different codomains.

9.2 Composition

Definition 9.2.1. Let $f\colon A \to B$ and $g\colon B \to C$. Then $g \circ f\colon A \to C$ such that for every $x \in A$,

$$(g \circ f)(x) = g(f(x)).$$

We read $g \circ f$ as "g composed with f", or "g circle f".

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, dashed, "g \circ f"'] & & C \end{tikzcd}$$

Figure 9.1: Function composition

Note 9.2.2. For $g \circ f$ to be defined, the codomain of f must equal the domain of g.

Example 9.2.3. Let $f\colon A \to B$.

(1) $f \circ \mathrm{id}_A = f$ because

- the domain of $f \circ \mathrm{id}_A$ and the domain of f are both A;
- the codomain of $f \circ \mathrm{id}_A$ and the codomain of f are both B; and
- $(f \circ \mathrm{id}_A)(x) = f(\mathrm{id}_A(x)) = f(x)$ for all $x \in A$.

(2) $\mathrm{id}_B \circ f = f$ because

- the domain of $\mathrm{id}_B \circ f$ and the domain of f are both A;
- the codomain of $\mathrm{id}_B \circ f$ and the codomain of f are both B;
- $(\mathrm{id}_B \circ f)(x) = \mathrm{id}_B(f(x)) = f(x)$ for all $x \in A$.

Question 9.2.4. Which of the following define a function $f\colon \mathbb{Z} \to \mathbb{Z}$ that satisfies $f \circ f = f$? 9a

(1) $f(x) = 1231$ for all $x \in \mathbb{Z}$.

(2) $f(x) = x$ for all $x \in \mathbb{Z}$.

(3) $f(x) = -x$ for all $x \in \mathbb{Z}$.

(4) $f(x) = 3x + 1$ for all $x \in \mathbb{Z}$.

(5) $f(x) = x^2$ for all $x \in \mathbb{Z}$.

Example 9.2.5. Let $f, g\colon \mathbb{Z} \to \mathbb{Z}$ such that for every $x \in \mathbb{Z}$,

$$f(x) = 3x \quad \text{and} \quad g(x) = x + 1.$$

Then for every $x \in \mathbb{Z}$,

$$(g \circ f)(x) = g(f(x)) = g(3x) = 3x + 1 \quad \text{and} \quad (f \circ g)(x) = f(g(x)) = f(x + 1) = 3(x + 1).$$

Note $(g \circ f)(0) = 1 \neq 3 = (f \circ g)(0)$.

$$\begin{tikzcd}
 & x \arrow[rr, mapsto, "f"] & & 3x \arrow[dd, mapsto, "g"] \\
x \arrow[dd, mapsto, "g"'] & \mathbb{Z} \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{Z} \arrow[d, "g"] & \\
 & \mathbb{Z} \arrow[r, "f"'] & \mathbb{Z} & 3x+1 \\
x+1 \arrow[rr, mapsto, "f"'] & & 3(x+1) &
\end{tikzcd}$$

Figure 9.2: The two paths from the top left to the bottom right are not the same

Theorem 9.2.6 (associativity of function composition)**.** Let $f\colon A \to B$ and $g\colon B \to C$ and $h\colon C \to D$. Then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, dashed, "g \circ f"'] & B \arrow[d, "g"] \arrow[dr, dashed, "h \circ g"] & \\
 & C \arrow[r, "h"'] & D
\end{tikzcd}$$

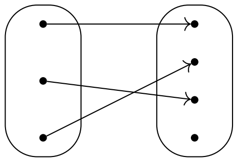
Figure 9.3: All paths from A to D are the same

Proof. 1. The domains of $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are both A.
2. The codomains of $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are both D.
3. For every $x \in A$,

$$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x))) = h((g \circ f)(x)) = (h \circ (g \circ f))(x). \qquad \square$$

9.3 Inverse

Arrow diagrams.



The figure above represents a function in the following sense.

- The dots on the left denote the elements of the domain.
- The dots on the right denote the elements of the codomain.
- An arrow from a left dot to a right dot indicates that the left dot is assigned the right dot.

Since every dot on the left is joined to exactly one dot on the right in the figure above, this function is well defined.

Definition 9.3.1. Let $f\colon A \to B$.

(1) If $X \subseteq A$, then let $f(X) = \{f(x) : x \in X\}$.

(2) If $Y \subseteq B$, then let $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$.

We call $f(X)$ the *(setwise) image* of X, and $f^{-1}(Y)$ the *(setwise) preimage* of Y under f.

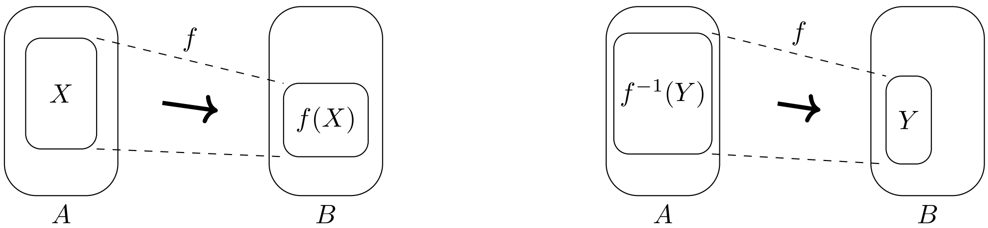


Figure 9.4: Setwise image and setwise preimage

Example 9.3.2. Define $g\colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$.

(1) If $X = \{-1, 0, 1\}$, then $g(X) = \{g(-1), g(0), g(1)\} = \{1, 0, 1\} = \{0, 1\}$.

(2) If $Y = \{0, 1, 2\}$, then $g^{-1}(Y) = \{0, -1, 1\}$.

Note 9.3.3. Let $f\colon A \to B$.

(1) If $X \subseteq A$, then $f(X) = \{f(x) : x \in X\}$, which is a set. If $x \in A$, then $f(x) \in B$.

(2) If $Y \subseteq B$, then $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$, which exists even when the inverse function f^{-1} does not. If $y \in B$ and f^{-1} exists, then $f^{-1}(y) \in A$.

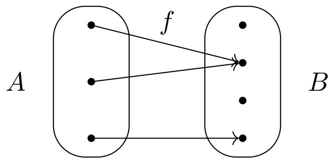
The inverse of a function will be defined in Definition 9.3.14.

Question 9.3.4. As in Example 9.3.2, define $g\colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. ✎ 9b
Which of the following are true statements?

- $g(0) = 0$.
- $g(0) = \{0\}$.

- $g(\{0\}) = 0$.
- $g(\{0\}) = \{0\}$.

Note 9.3.5. In general, we cannot make f^{-1} operate on elements instead of subsets.



Definition 9.3.6. Let $f\colon A \to B$.

(1) f is *surjective* or *onto* if

$$\forall y \in B \ \ \exists x \in A \ \ (y = f(x)).$$

A *surjection* is a surjective function.

(2) f is *injective* or *one-to-one* if

$$\forall x_1, x_2 \in A \ \ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2).$$

An *injection* is an injective function.

(3) f is *bijective* if it is both surjective and injective, i.e.,

$$\forall y \in B \ \ \exists! x \in A \ \ (y = f(x)).$$

A *bijection* is a bijective function.

Example 9.3.7. The function $f\colon \mathbb{Q} \to \mathbb{Q}$, defined by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$, is surjective.

Proof. 1. Take any $y \in \mathbb{Q}$.
2. Let $x = (y - 1)/3$.
3. Then $x \in \mathbb{Q}$ and $f(x) = 3x + 1 = y$. □

Remark 9.3.8. A function $f\colon A \to B$ is *not* surjective if and only if

$$\exists y \in B \ \ \forall x \in A \ \ (y \neq f(x)).$$

Example 9.3.9. As in Example 9.3.2, define $g\colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not surjective.

Proof. 1. Note $g(x) = x^2 \geqslant 0 > -1$ for all $x \in \mathbb{Z}$.
2. So $g(x) \neq -1$ for all $x \in \mathbb{Z}$, although $-1 \in \mathbb{Z}$. □

Example 9.3.10. As in Example 9.3.7, define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$. Then f is injective.

Proof. 1. Let $x_1, x_2 \in \mathbb{Q}$ such that $f(x_1) = f(x_2)$.
2. Then $3x_1 + 1 = 3x_2 + 1$.
3. So $x_1 = x_2$. □

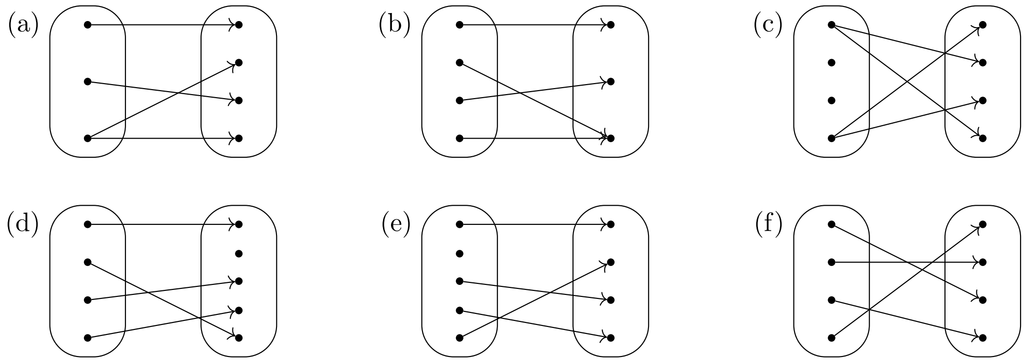
Remark 9.3.11. A function $f\colon A \to B$ is *not* injective if and only if

$$\exists x_1, x_2 \in A \ \ (f(x_1) = f(x_2) \wedge x_1 \neq x_2).$$

Example 9.3.12. As in Example 9.3.2, define $g\colon \mathbb{Z} \to \mathbb{Z}$ by setting $g(x) = x^2$ for every $x \in \mathbb{Z}$. Then g is not injective.

Proof. Note $g(1) = 1^2 = 1 = (-1)^2 = g(-1)$, although $1 \neq -1$. $\square$

Question 9.3.13. Which of the arrow diagrams below represent a function from the LHS set to the RHS set? Amongst those that represent a function, which ones represent injections, which ones represent surjections, and which ones represent bijections? ✐ 9c



Definition 9.3.14. Let $f\colon A \to B$. Then $g\colon B \to A$ is an *inverse* of f if

$$\forall x \in A \;\; \forall y \in B \;\; \big(y = f(x) \quad \Leftrightarrow \quad x = g(y)\big).$$

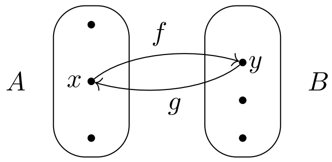


Figure 9.5: An inverse of a function

Example 9.3.15. As in Example 9.3.10, define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(x) = 3x + 1$ for all $x \in \mathbb{Q}$. Note that for all $x, y \in \mathbb{Q}$,

$$y = 3x + 1 \quad \Leftrightarrow \quad x = (y - 1)/3.$$

Let $g\colon \mathbb{Q} \to \mathbb{Q}$ such that $g(y) = (y - 1)/3$ for all $y \in \mathbb{Q}$. Then the equivalence above tells us

$$\forall x, y \in \mathbb{Q} \;\; (y = f(x) \Leftrightarrow x = g(y)).$$

So g is an inverse of f.

Note 9.3.16. We have no guarantee of a description of an inverse of a general function that is much different from what is given by the definitions.

Proposition 9.3.17 (uniqueness of inverses)**.** If g_1, g_2 are inverses of $f\colon A \to B$, then $g_1 = g_2$.

Proof. 1. Note $g_1, g_2\colon B \to A$.
2. Since g_1, g_2 are inverses of f, for all $x \in A$ and all $y \in B$,

$$x = g_1(y) \quad \Leftrightarrow \quad y = f(x) \quad \Leftrightarrow \quad x = g_2(y).$$

3. So $g_1 = g_2$. $\square$

Definition 9.3.18. The inverse of a function f is denoted f^{-1}.

Theorem 9.3.19. A function $f\colon A \to B$ is bijective if and only if it has an inverse.

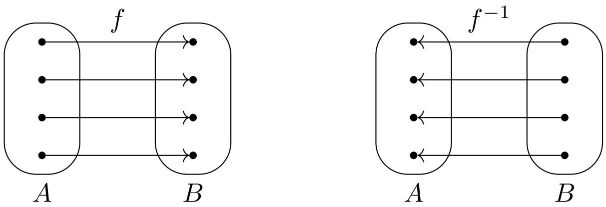


Figure 9.6: A bijective function and its inverse

Proof. 1. ("If")

1.1. Suppose f has an inverse, say $g\colon B \to A$.

1.2. We first show injectivity.

1.2.1. Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$.

1.2.2. Define $y = f(x_1) = f(x_2)$.

1.2.3. Then $x_1 = g(y)$ and $x_2 = g(y)$ as g is an inverse of f.

1.2.4. Thus $x_1 = x_2$.

1.3. Next we show surjectivity.

1.3.1. Let $y \in B$.

1.3.2. Define $x = g(y)$.

1.3.3. Then $y = f(x)$ as g is an inverse of f.

2. ("Only if")

2.1. Suppose f is bijective.

2.2. Then $\forall y \in B \;\; \exists! x \in A \;\; (y = f(x))$.

2.3. Define the function $g\colon B \to A$ by setting $g(y)$ to be the unique $x \in A$ such that $y = f(x)$ for all $y \in B$.

2.4. This g is well defined and is an inverse of f by the definition of inverse functions. $\square$