MA2001

LIVE LECTURE 8

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Topics for week 8

4.1 Row spaces and Column spaces

4.2 Ranks

4.3 Nullspaces and Nullities

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Row Space and Column Space

$$\boldsymbol{A} = \begin{array}{c} \\ \left(\begin{array}{cccc} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 9 & 9 & 5 \\ -2 & -1 & -1 & 1 \\ 5 & 8 & 9 & 4 \\ 4 & 2 & 7 & 3 \end{array}\right) \begin{array}{c} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \\ \mathbf{r}_5 \\ \mathbf{r}_6 \end{array} \\ \begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 \end{array} \end{array} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cccc} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \\ \\ \\ \end{array}$$

pivot columns

REF

row space of $\boldsymbol{A}$ = span$\{\mathbf{r}_1\ \mathbf{r}_2\ \mathbf{r}_3\ \mathbf{r}_4\ \mathbf{r}_5\ \mathbf{r}_6\}$

column space of $\boldsymbol{A}$ = span$\{\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3\ \mathbf{c}_4\}$

Basis for row space of $\boldsymbol{A}$ = $\{\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\}$

Basis for column space of $\boldsymbol{A}$ = $\{\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3\}$

Bases are not unique

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Row space, column space, nullspace

m x n matrix A	Subspace	Basis	Dimension
Row space	Subspace of $\mathbf{R}^n$	Non-zero rows in REF	Rank (# non-zero rows in REF)
Column space	Subspace of $\mathbf{R}^m$	Corresponding "pivot" columns in ***A***	Rank (# pivot columns in REF)
Nullspace	Subspace of $\mathbf{R}^n$ Same as solution space of ***Ax = 0***	From the spanning vectors in the general solution	Nullity (# parameters in general solution)

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Rank

If $\boldsymbol{R}$ is a row-echelon form of $\boldsymbol{A}$,

rank($\boldsymbol{A}$) = the number of nonzero rows of $\boldsymbol{R}$

= the number of leading entries in $\boldsymbol{R}$

= the number of pivot columns in $\boldsymbol{R}$

= largest number of linearly independent rows in $\boldsymbol{A}$ = dim (row space of $\boldsymbol{A}$)

= largest number of linearly independent columns in $\boldsymbol{A}$ = dim (column space of $\boldsymbol{A}$)

For an $m \times n$ matrix $\boldsymbol{A}$, rank($\boldsymbol{A}$) $\leq \min\{m, n\}$.

An $m \times n$ matrix $\boldsymbol{A}$ with rank($\boldsymbol{A}$) $= \min\{m, n\}$ is said to be of full rank.

Rank($\boldsymbol{A}$) = largest number of linearly independent rows in $\boldsymbol{A}$

= largest number of linearly independent columns in $\boldsymbol{A}$

Rank by inspection

What is the rank of each of the following matrices?

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$$

rank($\boldsymbol{A}$) = 1

All rows are scalar multiples of each other

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

rank($\boldsymbol{B}$) = 2

Two rows that are not scalar multiples of each other

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

rank($\boldsymbol{C}$) = 2

Third row is the sum of first two rows

Third column is the same as first column

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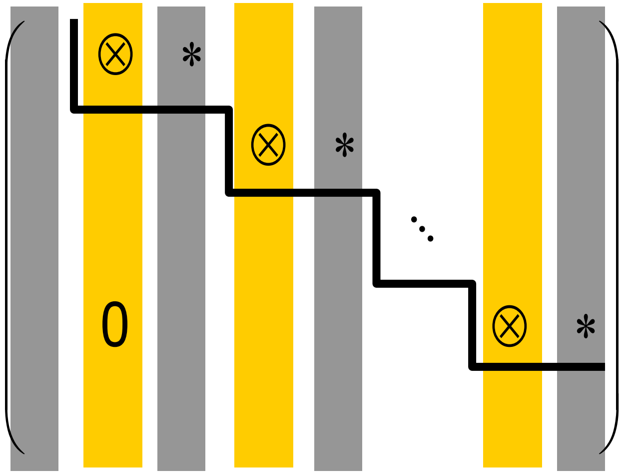
Dimension Theorem of Matrix

rank($\boldsymbol{A}$) + nullity($\boldsymbol{A}$) = # columns of $\boldsymbol{A}$

row-echelon form

$\boldsymbol{R}$ =



pivot columns

(correspond to basis for column space of $\boldsymbol{A}$) rank($\boldsymbol{A}$)

non-pivot columns

(correspond to parameters in general solutions) nullity($\boldsymbol{A}$)

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Row equivalence

A $\longrightarrow \longrightarrow \longrightarrow$ **B**

Preserve row space and rank

Preserve nullspace and nullity

Does not preserve column space, but preserve linearity relations among columns

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 9 & 9 & 5 \\ -2 & -1 & -1 & 1 \\ 5 & 8 & 9 & 4 \\ 4 & 2 & 7 & 3 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For both ***A*** and ***R***:

- column#1 + column#4 = column#3
- column#1, #3, #4 are linearly dependent
- column#2, #3, #4 are linearly independent

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True or False

If $\boldsymbol{A}$ is an n×n invertible matrix, then the row space of $\boldsymbol{A}$ is $\mathbf{R}^n$

The rows of $\boldsymbol{A}$ form a basis for $\mathbf{R}^n$

span{rows of $\boldsymbol{A}$} = $\mathbf{R}^n$

If $\boldsymbol{A}$ is an n×n invertible matrix, then the column space of $\boldsymbol{A}$ is $\mathbf{R}^n$

The columns of $\boldsymbol{A}$ form a basis for $\mathbf{R}^n$

span{columns of $\boldsymbol{A}$} = $\mathbf{R}^n$

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Basis for Linear Span

Let V = span{ $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ } Basis

We need to remove the redundant vectors among $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ to get a basis for V.

Column space method:

- Form a matrix **A** using $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ as its columns.
- Reduce **A** to row echelon form **R**
- Identify the pivot columns in **R**
- The corresponding columns in **A** form a basis for V.

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Example

span{(1,2,2,1), (3,6,6,3), (4,9,9,5), (-2,-1,-1,1), (5,8,9,4)}

$$\begin{pmatrix} \boxed{1} & 3 & \boxed{4} & -2 & \boxed{5} \\ \boxed{2} & 6 & \boxed{9} & -1 & \boxed{8} \\ \boxed{2} & 6 & \boxed{9} & -1 & \boxed{9} \\ \boxed{1} & 3 & \boxed{5} & 1 & \boxed{4} \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 3 & \boxed{4} & -2 & \boxed{5} \\ \boxed{0} & 0 & \boxed{1} & 3 & \boxed{-2} \\ \boxed{0} & 0 & \boxed{0} & 0 & \boxed{1} \\ \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} \end{pmatrix}$$

Basis for the linear span:
{(1,2,2,1), ~~(3,6,6,3)~~ , (4,9,9,5), ~~(-2,-1,-1,1)~~ , (5,8,9,4)}

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Deriving basis for R^n

$S = \{(2, -1, 0), (1, -1, 3), (-5, 1, 0), (1, 0, 1)\}$

How to get a basis from S for $\mathbf{R}^3$?

Throw out redundant vectors from S

Arrange the vectors as columns of a matrix

Look for pivot columns of the REF

$$T = \left\{ \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

How to extend T to a basis for $\mathbf{R}^4$?

Add on non-redundant vectors to T

Arrange the vectors as rows of a matrix

Look for 'missing' leading entries of the REF

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Extending basis

$S = \{(1, 4, -2, 5, 1), (2, 9, -1, 8, 2), (2, 9, -1, 9, 3)\}$

complete R to a 5x5 matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 2 & 9 & -1 & 8 & 2 \\ 2 & 9 & -1 & 9 & 3 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & x & * & * \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 & y \end{pmatrix}$$

form a basis for $\mathbf{R}^5$

are not redundant in row space of A $\Bigg\{$ E.g. $(0\ 0\ 1\ 0\ 0)$, E.g. $(0\ 0\ 0\ 0\ 1)$

$\{ (1, 4, -2, 5, 1), (2, 9, -1, 8, 2), (2, 9, -1, 9, 3),$
$(0, 0, 1, 0, 0), (0, 0, 0, 0, 1) \}$

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True or False

We can also use column space method to extend a set in $\mathbf{R}^n$ to a basis for $\mathbf{R}^n$

$$\boldsymbol{A} = (\boldsymbol{c_1}\ \boldsymbol{c_2}\ \dots\ \boldsymbol{c_k}) \xrightarrow{\text{row echelon form}} \boldsymbol{R} = \underbrace{(\boldsymbol{c_1}^*\ \boldsymbol{c_2}^*\ \dots\ \boldsymbol{c_k}^*)}_{\text{add the "missing" pivot columns}}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{G.E.}} \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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$Av \in$ Column space of A Visualization

$\boldsymbol{A}$: m × n

$\boldsymbol{A} = (\boldsymbol{a}_1 \ \boldsymbol{a}_2 \ \dots \ \boldsymbol{a}_n)$

$\boldsymbol{v}$: n × 1

$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$\boldsymbol{Av} = (\boldsymbol{a}_1 \ \boldsymbol{a}_2 \ \dots \ \boldsymbol{a}_n) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$= v_1\boldsymbol{a}_1 + v_2\boldsymbol{a}_2 + \cdots + v_n\boldsymbol{a}_n$

a linear combination of the columns of $\boldsymbol{A}$

belong to column space of $\boldsymbol{A}$

For any $\boldsymbol{v} \in \mathbf{R}^n$, $\boldsymbol{Av}$ belongs to the column space of $\boldsymbol{A}$

Every vector in the column space of $\boldsymbol{A}$ has the form $\boldsymbol{Av}$ for some $\boldsymbol{v} \in \mathbf{R}^n$

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Consistency of LS and column space

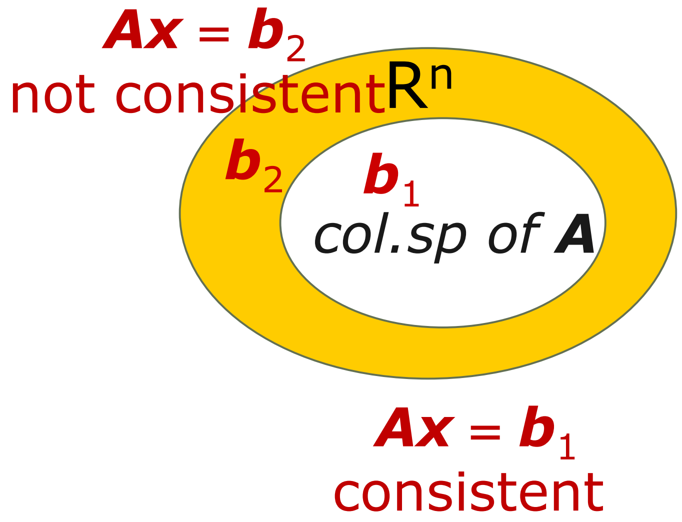
system $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution

$\Updownarrow$

$\boldsymbol{b}$ can be written as a linear combination of the columns of $\boldsymbol{A}$

$\Updownarrow$

$\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$



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Column space of *AB*

$\boldsymbol{A}$: m × n

$\boldsymbol{B}$: n × k $\quad\quad \boldsymbol{B} = (\boldsymbol{b}_1 \ \ \boldsymbol{b}_2 \ \ \ldots \ \ \boldsymbol{b}_k)$

$\boldsymbol{AB} = (\boldsymbol{Ab}_1 \ \ \boldsymbol{Ab}_2 \ \ \ldots \ \ \boldsymbol{Ab}_k)$

columns of $\boldsymbol{AB}$ in terms of columns of $\boldsymbol{B}$

each column of $\boldsymbol{AB} \in$ column space of $\boldsymbol{A}$

column space of $\boldsymbol{AB} \subseteq$ column space of $\boldsymbol{A}$

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Rank(AB) VS Rank(A)

column space of $\boldsymbol{AB} \subseteq$ column space of $\boldsymbol{A}$

$\Rightarrow$ rank($\boldsymbol{AB}$) $\leq$ rank($\boldsymbol{A}$) ✓

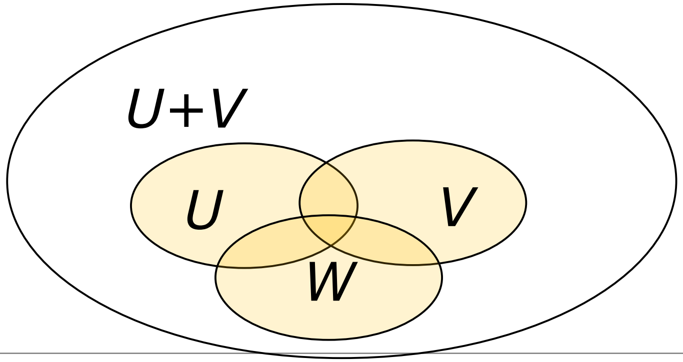
True or false:

column space of $\boldsymbol{AB} \subseteq$ column space of $\boldsymbol{B}$ ✗

rank($\boldsymbol{AB}$) $\leq$ rank($\boldsymbol{B}$) ✓

row space of $\boldsymbol{AB} \subseteq$ row space of $\boldsymbol{B}$ ✓

$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$



Exercise 4 Q23

$$\text{rank}(\boldsymbol{A} + \boldsymbol{B}) \leq \text{rank}(\boldsymbol{A}) + \text{rank}(\boldsymbol{B})$$

$\boldsymbol{A} = (\boldsymbol{a}_1\ \boldsymbol{a}_2\ \ \dots\ \ \boldsymbol{a}_n)$

U = Column space of $\boldsymbol{A}$ = span$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\}$

$\boldsymbol{B} = (\boldsymbol{b}_1\ \ \boldsymbol{b}_2\ \ \dots\ \ \boldsymbol{b}_n)$

V = Column space of $\boldsymbol{B}$ = span$\{\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_n\}$

$\boldsymbol{A} + \boldsymbol{B} = (\boldsymbol{a}_1+\boldsymbol{b}_1\ \boldsymbol{a}_2+\boldsymbol{b}_2\ \ \dots\ \boldsymbol{a}_n+\boldsymbol{b}_n)$

W = Column space of $\boldsymbol{A} + \boldsymbol{B}$ = span$\{\boldsymbol{a}_1+\boldsymbol{b}_1, \boldsymbol{a}_2+\boldsymbol{b}_2, \dots, \boldsymbol{a}_n+\boldsymbol{b}_n\}$

$U + V$ = span$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n, \boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_n\}$

$W \subseteq U + V \ \Rightarrow \dim W \leq \dim(U + V) \leq \dim(U) + \dim(V)$

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Exercise 4 Q20

Suppose $\boldsymbol{AB} = \mathbf{0}$.

Show that column space of $\boldsymbol{B} \subseteq$ nullspace of $\boldsymbol{A}$

$$\boldsymbol{B} = (\boldsymbol{b}_1 \;\; \boldsymbol{b}_2 \;\; \ldots \;\; \boldsymbol{b}_k) \qquad \boldsymbol{AB} = (\boldsymbol{Ab}_1 \;\; \boldsymbol{Ab}_2 \;\; \ldots \;\; \boldsymbol{Ab}_k)$$

$$\boldsymbol{AB} = \mathbf{0}$$

$$\Rightarrow (\boldsymbol{Ab}_1 \;\; \boldsymbol{Ab}_2 \;\; \ldots \;\; \boldsymbol{Ab}_k) = \mathbf{0}$$

$$\Rightarrow \boldsymbol{Ab}_1 = \mathbf{0}, \;\; \boldsymbol{Ab}_2 = \mathbf{0}, \ldots, \boldsymbol{Ab}_k = \mathbf{0}$$

$\Rightarrow \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k \in$ nullspace of $\boldsymbol{A}$

$\Rightarrow$ column space of $\boldsymbol{B} \subseteq$ nullspace of $\boldsymbol{A}$

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Solution set of non-homogeneous system

If we know the general solution of $\boldsymbol{Ax} = \mathbf{0}$
and one particular solution of $\boldsymbol{Ax} = \boldsymbol{b}$,
then we have the general solution of $\boldsymbol{Ax} = \boldsymbol{b}$.

general solution of $\boldsymbol{Ax} = \boldsymbol{b}$ = general solution of $\boldsymbol{Ax} = \boldsymbol{0}$ + $\boldsymbol{v}$ ← Fixed solution of $\boldsymbol{Ax} = \boldsymbol{b}$

nullspace of $\boldsymbol{A}$

The solution set of $\boldsymbol{Ax} = \boldsymbol{b}$ = { $\boldsymbol{u} + \boldsymbol{v}$ | $\boldsymbol{u}$ belongs to nullspace of $\boldsymbol{A}$}

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Solving Linear system with MATLAB

Ax = 0

```
>> rref( [A 0])
>> null(A)
```

(give basis for nullspace of A)

Ax = b (consistent)

```
>> rref( [A b])
>> A \ b
>> linsolve(A, b)
```

(give a particular solution)

general solution of ***Ax = b***

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Exercise 4 Q25(a) [Tutorial 7]

nullspace of $\boldsymbol{A}$ equals nullspace of $\boldsymbol{A}^T\boldsymbol{A}$

Strategy: Show $S \subseteq T$ and $T \subseteq S$

$\boldsymbol{v} \in$ nullspace of $\boldsymbol{A}$ $\Rightarrow \boldsymbol{Av} = \boldsymbol{0}$ $\Rightarrow \boldsymbol{A}^T\boldsymbol{Av} = \boldsymbol{0}$

$\Rightarrow \boldsymbol{v} \in$ nullspace of $\boldsymbol{A}^T\boldsymbol{A}$

So nullspace of $\boldsymbol{A} \subseteq$ nullspace of $\boldsymbol{A}^T\boldsymbol{A}$

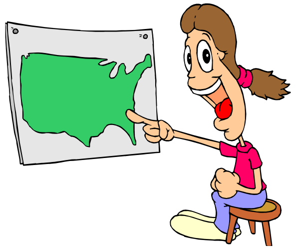
$\boldsymbol{v} \in$ nullspace of $\boldsymbol{A}^T\boldsymbol{A}$ $\Rightarrow \boldsymbol{A}^T\boldsymbol{Av} = \boldsymbol{0}$ $\Rightarrow \boldsymbol{v}^T\boldsymbol{A}^T\boldsymbol{Av} = 0$ $\Rightarrow (\boldsymbol{Av})^T\boldsymbol{Av} = 0$ $\Rightarrow \boldsymbol{Av} = \boldsymbol{0}$

hint: sum of squares

$\Rightarrow \boldsymbol{v} \in$ nullspace of $\boldsymbol{A}$

So nullspace of $\boldsymbol{A}^T\boldsymbol{A} \subseteq$ nullspace of $\boldsymbol{A}$

Map of LA



A is an $n\times n$ matrix

	A is invertible	chapter 2	A is not invertible
	$\det A \neq 0$	chapter 2	$\det A = 0$
	rref of A is identity matrix	chapter 1	rref of A has a zero row
	AX= 0 has only the trivial solution	chapter 1	AX= 0 has non-trivial solutions
	AX= B has a unique solution	chapter 1	AX= B has no solution or infinitely many solutions
	rows (columns) of A are linearly independent	chapter 3	rows (columns) of A are linearly dependent
	row (column) space of A = R^n	chapter 4	row (column) space of A $\neq R^n$
i.e. **A** is full rank	rank(A) = n	chapter 4	rank(A) < n
i.e. nullspace of **A** is the zero space	nullity(A) = 0	chapter 4	nullity(A) > 0

to be continued