NATIONAL UNIVERSITY OF SINGAPORE

CS1231 – Discrete Structures

(Semester 1: AY2017/18)

ANSWER SHEETS

Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES

1. These Answer Sheets consist of **SIX (6)** printed pages.
2. Fill in your **Student Number** clearly below with a pen.
3. You may write your answers in pencil.

STUDENT NUMBER:

A								

(Write your Student Number above legibly with a pen.)

SOLUTIONS

Answers for MCQs:

Q1. A	Q6. B	Q11. E
Q2. B	Q7. D	Q12. D
Q3. C	Q8. B	Q13. B
Q4. C	Q9. A	Q14. E
Q5. C	Q10. A	Q15. A

Section B (40 marks)

Q16. [14 marks]

(a) i. [2]

$x_1 + x_2 + x_3 = 10$

36

(a) ii. [2]

29! and **29!**

(a) iii. [3]

2/7

(b). [3]

Two consecutive positive integers are co-prime.

(Alternative solutions possible.)

1. Suppose not, that is, *x* and *x*+1 have a common divisor *d* > 1.
    1.1 From Theorem 4.1.1 (Linear Combination), *d* | (*x*+1) – *x* , or *d* | 1.
    1.2 From Theorem 4.3.1 (Epp), this implies *d* ≤ 1.
    1.3 This contradicts the claim that *d* > 1.
2. Therefore, *x* and *x*+1 must be co-prime.

(c). [4]

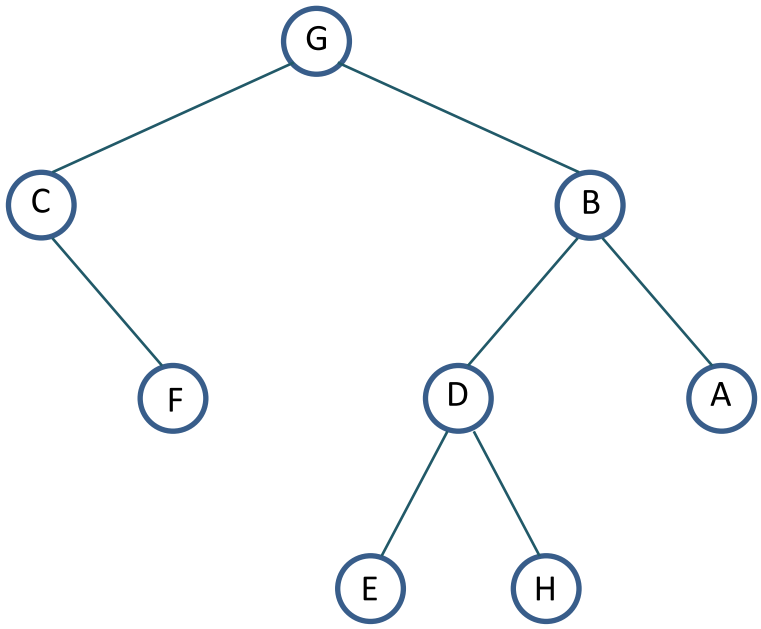
Pigeonhole Principle

1. We have shown in part (b) that consecutive positive integers are co-prime. Therefore, it suffices to show that we will get a pair of consecutive numbers among the $n + 1$ numbers.
2. Let the n pigeonholes be the following sets: $\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2n - 1, 2n\}$
3. Our pigeons are the $n + 1$ numbers we are choosing from the set $\{1, 2, 3, \ldots, 2n\}$.
4. By the PHP, two of the $n + 1$ numbers will be in the same pigeonhole.
5. Hence we will have a pair of consecutive numbers.

Q17. [14 marks]

(a)
[3]

Postorder: F C E H D A B G
Inorder: C F G E D H B A



(b)

Dijkstra's algorithm

[2] (i) Shortest distance from a to z = **21**

[4] (ii) Final $V(T)$ = { a, ***f*, *e*, *b*, *c*, *d*,** , z }

Q17.

(c) You should not write longer than the space provided in the box, or mark will be deducted.

[5]

A pile of n stones.

One can earn at most $n(n-1)/2$ dollars.

Represent each stone as a vertex in a graph. Two vertices representing two stones are connected if the stones are in the same pile. In the beginning, every stone is in the same pile as every other stone, so there are $n(n-1)/2$ edges in the graph. (It is a K_n complete graph.)

Separating a pile of k stones into two piles of k_1 and k_2 stones corresponds to removing some edges from the graph. The number of edges removed is exactly $k_1 \times k_2$.

The maximum amount of money one can earn is when the stones are separated into piles with single stone, that is, when all the edges are removed. Since the number of edges in K_n is $n(n-1)/2$, this is the maximum amount one can earn.

Q18. [12 marks]

(a) [2]

$f(1) =$ **1**
$f(2) =$ ***i***
$f(3) =$ **-*i***
$f(4) =$ **-1**

$G_2 \ncong G_3$

(b) [2]

In G_3, all elements, except identity e, have element order 2 (the diagonal entries in the Cayley table). But in G_2, i (which is not the identity) has element order 4. These two groups thus have different properties. Hence they are not isomorphic.

$\cong$ symmetry

(c) [4]

2.5 Then, $f\left(f^{-1}(x)f^{-1}(y)\right) = f\left(f^{-1}(x)\right) f\left(f^{-1}(y)\right)$, by the isomorphic property of f.

2.6 $= xy$, by Theorem 7.3.2 (Epp).

2.7 And thus, $f^{-1}(x)f^{-1}(y) = f^{-1}(xy)$, by definition of inverse relation f^{-1}.

$\boxed{x^n = e}$

(d)
[4]

Proof (Direct proof):

1. Let m be the size of G, i.e. G has m elements.
2. Take any $x \in G$.
3. Consider powers of x: $x^1, x^2, \cdots, x^{m+1}$, all of which are in G because of the Closure property.
4. There are $m + 1$ elements (pigeons) in the list above, but G has only m elements (pigeonholes). Thus, by the Pigeonhole Principle, two of the elements in the list, x^r, x^s must be equal.
5. Without loss of generality, assume $s > r$.
6. Then, $x^r = x^s = x^r x^{s-r}$, by the power laws.
7. By Axiom A4, $(x^r)^{-1}$ exists.
8. Pre-multiply by $(x^r)^{-1}$gives $(x^r)^{-1} x^r = (x^r)^{-1} x^r x^{s-r}$
9. Then, $e = ex^{s-r}$, by Associativity and Inverse Axioms.
10. Then, $e = x^n$, where $n = s - r$, by Identity Axiom.
11. Since $s > r$, then $n = s - r > 0$, by T19 of Appendix A (Epp).
12. Hence, $\forall x \in G, \exists n \in \mathbb{Z}^+$ such that $x^n = e$. ■

~~~ **END OF PAPER** ~~~
~~~