CS1231/CS1231S (AY2020/21 Semester 1) Exam Answer Keys

(Workings/explanations on page 5 onwards.)

Part A:

1. **D**	2. **B**	3. **D**	4. **C**	5. **D**	6. **C**
7. **A**	8. **C**	9. **B**	10. **D**	11. **C**	12. **D**

Part B:

13. **A, B, C**	14. **A, B**	15. **B, D**	16. **B, D**	17. **A, B ,C**

18. [Total: 5 marks]

Answer:

(a) 1. (Reflexivity) For all $(a, b) \in A$, we have $ab = ab$ and so $(a, b)\ R\ (a, b)$.

2. (Symmetry)
   - 2.1. Let $(a, b), (c, d) \in A$ such that $(a, b)\ R\ (c, d)$.
   - 2.2. Then $ab = cd$ by the definition of R.
   - 2.3. Thus $cd = ab$ by the symmetry of $=$.
   - 2.4. So $(c, d)\ R\ (a, b)$ by the definition of R.

3. (Transitivity)
   - 3.1. Let $(a, b), (c, d), (e, f) \in A$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$.
   - 3.2. Then $ab = cd$ and $cd = ef$ by the definition of R.
   - 3.3. Thus $ab = ef$ by the transitivity of $=$.
   - 3.4. So $(a, b)\ R\ (e, f)$ by the definition of R.

(b) $[(1,1)] = \{(a, b) \in A : (1,1)\ R\ (a, b)\} = \{(a, b) \in A : ab = 1 \times 1 = 1\} = \{(1,1)\}$.

$[(4,3)] = \{(a, b) \in A : (4,3)\ R\ (a, b)\} = \{(a, b) \in A : ab = 4 \times 3 = 12\}$
$= \{(1,12), (2,6), (3,4), (4,3), (6,2), (12,1)\}$.

19. [Total: 4 marks]

Answer:

(a) $\{\{1231,2040,3230\}, \{1101,2030,2103\}, \{2100,2106\}\}$,

$\{\{1231,2040,3230\}, \{2030,2103\}, \{1101,2100,2106\}\}$.

(b) $\{\{1101,1231\}, \{2030,2040,2100\}, \{2103,2106,3230\}\}$,

$\{\{1101,1231\}, \{2030,2040,2106\}, \{2100,2103,3230\}\}$,

$\{\{1101,2040\}, \{1231,2030,2100\}, \{2103,2106,3230\}\}$,

$\{\{1101,2040\}, \{1231,2030,2106\}, \{2100,2103,3230\}\}$.

20. Counting and Probability [Total: 20 marks]

(a) $\mathbf{3240}$. [3 marks]

(b) (i) **15** [1 mark]; (ii) **55** or **35** [2 marks]

(c) $\mathbf{\frac{1}{8}}$ [3 marks]

(d) (i) **512** [1 mark]; (ii) **448** [2 marks]

(e) $\mathbf{\frac{16}{23}}$ or $\mathbf{0.696}$. (See page 8 for working.) [4 marks]

(f)

Suppose all the scores are different. We may then arrange them in strictly increasing order: $s_1 < s_2 < \cdots < s_{21}$. The smallest possible scores are $s_1 = 0, s_2 = 1, \cdots, s_{21} = 20$. Summing the scores we have $0 + 1 + 2 + \cdots + 20 = 210 > 200$, hence a contradiction. Therefore, the scores cannot be all different. [4 marks]

21. Graphs and Trees [Total: 18 marks]

(a) Weight of MST = **251** [1 mark]

List of edges: $\{e, g\}, \{g, h\}, \{b, d\}, \{d, g\}, \{f, g\}, \{a, d\}, \{c, e\}$. [2 marks]

(b)

$\boldsymbol{\frac{n(n-1)}{4}}$ [2 marks]

By the definition of complement graph, the union graph of G and $\bar{G}$ is a complete graph with n vertices, which has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. Since G has half of this number of edges, therefore it has $\boldsymbol{\frac{n(n-1)}{4}}$ edges.

(It's okay if students give the correct answer without working/explanation.)

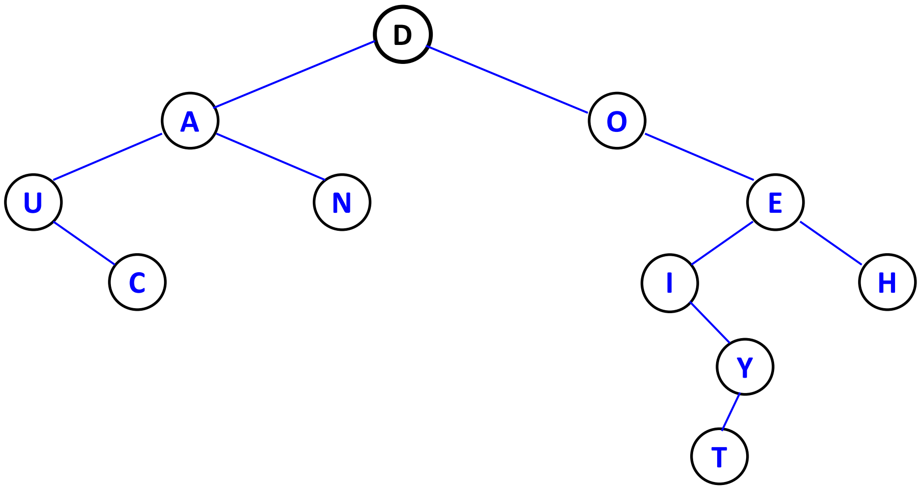
(c)

From part (b), a self-complementary graph with n vertices has $\frac{n(n-1)}{4}$ edges.

If $n = 4k + 2$, then there are $\frac{(4k+2)(4k+1)}{4} = \frac{(2k+1)(4k+1)}{2}$ edges. Since both $(2k + 1)$ and $(4k + 1)$ are odd, their product is odd (Tutorial #1 question 9 or Lemma 1). As odd number is not divisible by 2, the number of edges would not be an integer. Therefore, there are no self-complementary simple undirected graph with $4k + 2$ vertices. [4 marks]

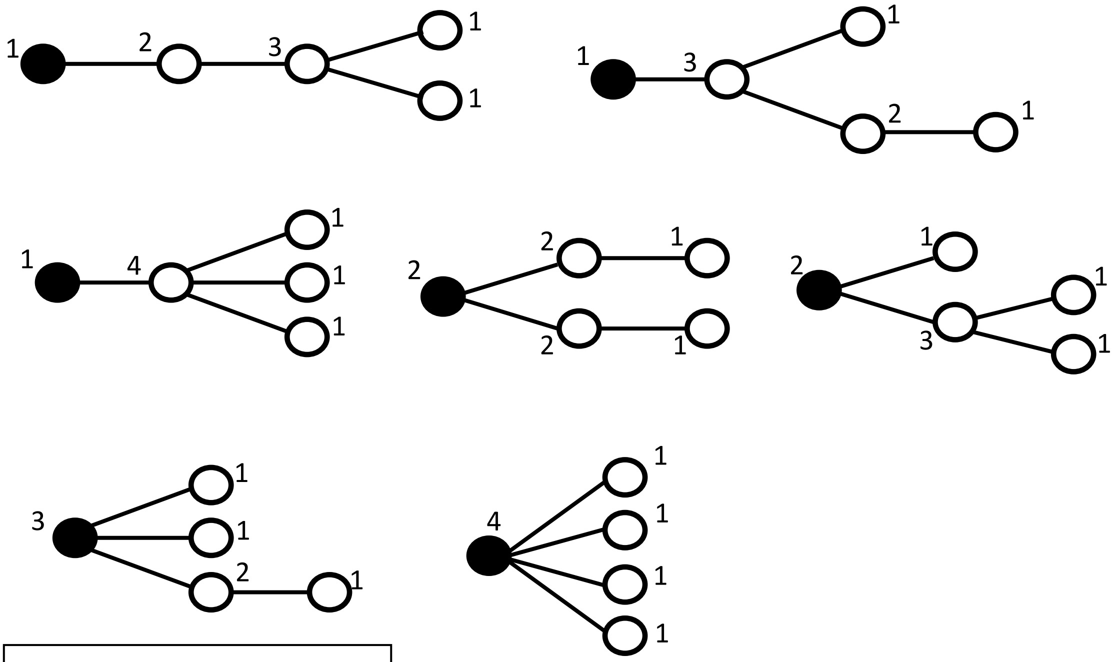
(It is okay if students didn't quote Tutorial #1 question 9, as that is a basic result.)

(d) [4 marks]



(e) [5 marks]

There are 9 non-isomorphic rooted trees with 5 vertices. Students need to draw the remaining 7 of them.



Numbers beside the vertices are the degrees, for reference only.

22. $n = \mathbf{212}$, $x = \mathbf{2}$, $y = \mathbf{5}$, $z = \mathbf{5}$. [3 marks] (See explanation on page 9.)

23. [Total: 8 marks]

(a) No. It can be readily verified that there is exactly one equivalence class with respect to R_6. (More generally, the counterexamples are precisely those R_n's where n is a product of at least two distinct primes; see the theorem on page 9.)

(b) Yes.

1. Let n be a prime number.
2. We prove by contradiction that $a^k \not\equiv 0 \pmod{n}$ for all $a \in A_n$ and all $k \in \mathbb{Z}^+$.
   - 2.1. Let $a \in A_n$ and $k \in \mathbb{Z}^+$ such that $a^k \equiv 0 \pmod{n}$.
   - 2.2. Then $\gcd(a, n) = 1$ by Proposition P.
   - 2.3. Use Theorem 8.6.19 to find a multiplicative inverse b of a modulo n.
   - 2.4. Then we can deduce from line 2.1 that $a^k b^{k-1} \equiv 0 \cdot b^{k-1} = 0 \pmod{n}$.
   - 2.5. Note $a^k b^{k-1} = \underbrace{a \cdot a \cdot \cdots \cdot a}_{k\ a\text{'s}} \cdot \underbrace{b \cdot b \cdot \cdots \cdot b}_{(k-1)\ b\text{'s}} = a\,\underbrace{(ab)(ab)\ldots(ab)}_{(k-1)\ ab\text{'s}}$

     $\equiv a \cdot 1 \cdot 1 \cdot \cdots \cdot 1 \pmod{n}$ as b is a multiplicative inverse of a modulo n;

     $= a$.
   - 2.6. Combining lines 2.4 and 2.5, we have $a \equiv 0 \pmod{n}$.
   - 2.7. So Proposition P implies $\gcd(a, n) \neq 1$, which contradicts line 2.2.
3. It follows from line 2 that all elements of A_n are R_n-related.
4. So there is exactly one equivalence class with respect to R_n.

Alternative argument for block 2.

2. We claim that $a^k \not\equiv 0 \pmod{n}$ for all $a \in A_n$ and all $k \in \mathbb{Z}^+$.
   - 2.1. Let $a \in A_n$ and $k \in \mathbb{Z}^+$.
   - 2.2. Then $a \not\equiv 0 \pmod{n}$ by Proposition P.
   - 2.3. $\therefore$ $n \nmid a$ by the alternative definitions of congruence.
   - 2.4. $\therefore$ $n \nmid a^k$ by Euclid's Lemma, as n is prime.
   - 2.5. $\therefore$ $a^k \not\equiv 0 \pmod{n}$ by the alternative definitions of congruence.

Explanations/workings

Part A

Q1. Answer: D.

$1 \mid 0$ as $0 = 1 \times 0$. Remember from our definition of remainders that $0 \leqslant -1 \underline{\text{ mod }} 12 < 12$.

Q2. Answer: B.

- Take any integer $n \geqslant 2$. Then $(n+2)! + 2, (n+2)! + 3, \dots, (n+2)! + (n+2)$ are all composite.
- Let $n = 3$. Then whenever a is a positive integer, there must be an even number in $a, a+1, a+2, a+3$ that is at least 4. This even number is not prime.

Q3. Answer: D.

Let $a = 10$ and $b = 2$. Then $\gcd(a, a+2) = 2$ and $\gcd(a, b) = 2 \neq 4 = \gcd(a+b, a-b)$

Q4. Answer: C.

(i) If $x, y, z \in \mathbb{Z}$, then $10x + 15y - 35z \equiv 5 \pmod 5$ but $2 \not\equiv 5 \pmod 5$.

(ii) Note $\gcd(10,15) = 5$. Apply Bézout's Lemma to find $r, s \in \mathbb{Z}$ such that $5 = 10r + 15s$. Next, note that $\gcd(5,42) = 1$. Apply Bézout's Lemma to find $w, z \in \mathbb{Z}$ such that $1 = 5w + 42z$. Then $1 = (10r + 15s)w + 42z = 10(rw) + 15(sw) + 42z$.

Q5. Answer: D.

Recall that there are infinitely many primes. Let n be a prime number and $a \in \mathbb{Z}$ satisfying $a \not\equiv 0 \pmod n$. Then $\gcd(a, n) = 1$ and so a has a multiplicative inverse modulo n.

Q6. Answer: C.

(i) For example, $2 \equiv 4 \pmod 2$ but $2 \not\equiv 4 \pmod 4$.

(ii) If $mn \mid (a-b)$, then $n \mid (a-b)$.

Q7. Answer: A.

Let $R_1 = \{(0,1), (1,2)\}$ and $R_2 = \{(2,3), (3,4)\}$. Then $\bigcap_{i=1}^{n} R_i = \emptyset$ is both symmetric and transitive, but R_1 and R_2 are neither symmetric nor transitive. If $\bigcap_{i=1}^{n} R_i$ is reflexive, then (x, x) is in each R_i for all $x \in \mathbb{Z}$, and thus each R_i is reflexive.

Q8. Answer: C.

- $(1,2)\ R\ (2,1)$ and $(2,1)\ R\ (1,2)$, but $(1,2) \neq (2,1)$.
- If $a, b, c, d, e, f \in \mathbb{Z}^+$ such that $ab \leqslant cd$ and $cd \leqslant ef$, then $ab \leqslant ef$.

Q9. Answer: B.

With respect to the partial order "divides" on $\mathbb{Z}_{\geqslant 2}$, the number 2 is minimal, but it is not smallest and it is not the unique minimal element.

Q10. Answer: D.

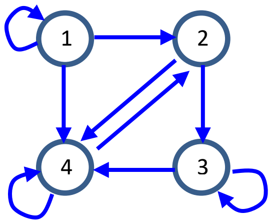
Expected value is $\frac{1}{6} \times (1 + 2 + 3 + 4 + 5 + 6) = 3.5$ for rolling a fair die.

By linearity of expectation, expected sum for rolling three dice is $3.5 \times 3 = 10.5$.

Q11. Answer: C.

The given statement can be simplified to $c \vee {\sim}d \vee {\sim}e$. Since the only assignment to make this statement false is $c = false, d = true, e = true$, the probably is $1/2^3$.

Q12. Answer: D.



The graph is shown above.

There are **7** walks of length 3: 1 → 1 → 1 → 4; 1 → 1 → 2 → 4; 1 → 4 → 2 → 4; 1 → 2 → 3 → 4; 1 → 1 → 4 → 4; 1 → 2 → 4 → 4; 1 → 4 → 4 → 4.

Alternatively, compute A^3 and obtain its A^3_{14} value.

$$A^2 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 2 & 1 & 3 \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} ? & ? & ? & \mathbf{7} \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix}$$

Part B.

Q13. Answer: A,B,C.

Q14. Answer: A,B.

C. $2^3 = 8$ and $2^6 = 64$. Note $3 \equiv 6 \pmod 3$ but $2^3 \not\equiv 2^6 \pmod 3$.

D. $\gcd(2,3) = 1$ and $\gcd(2,6) = 2$. Note $3 \equiv 6 \pmod 3$ but $\gcd(2,3) \not\equiv \gcd(2,6) \pmod 3$.

Q15. Answer: B,D.

A. Following the left path up, we see that n is a product of 3 primes. Following the right path up, we see that n is a product of 2 primes. This is impossible by the Fundamental Theorem of Arithmetic.

B. Let $n = pq^2$, where p, q are distinct primes.

C. There are exactly 3 vertices adjacent to the minimum element. So n has exactly 3 distinct prime factors. However, following the paths up, we see that n is a product of 2 primes. This is impossible by the Fundamental Theorem of Arithmetic.

D. Let $n = pqr$, where p, q, r are distinct primes.

Q16. Answer: B,D.

A. 180 is not largest because $21 \nmid 180$.

B. 180 is maximal because no $a \in A$ that is different from 180 satisfies $180 \mid a$.

C. 42 is not largest because $180 \nmid 42$.

D. 42 is maximal because no $a \in A$ that is different from 42 satisfies $42 \mid a$.

Q17. Answer: A,B,C.

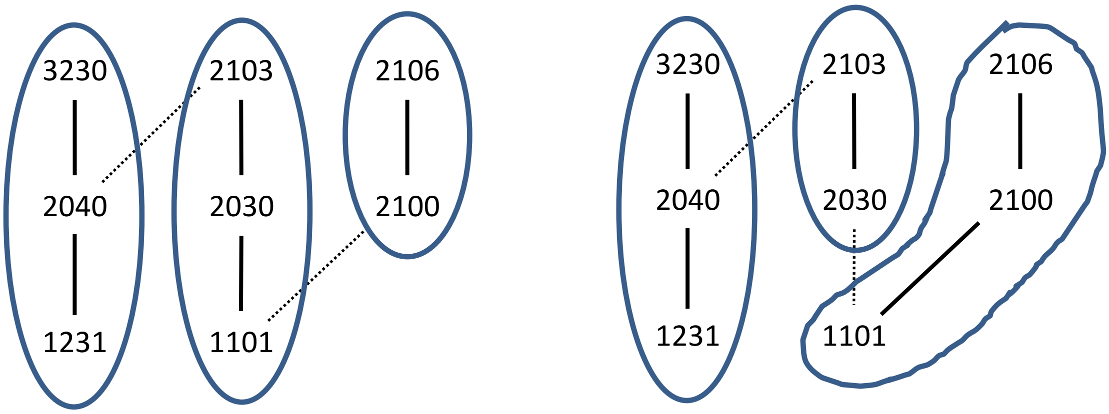
With respect to this partial order, the numbers 4 and 18 are incomparable. So one can linearize to make either bigger than the other.

Note that $6 \mid 36$. So 6 must be smaller than 36 in any linearization.

Part C

Q19.

(a)


3230
2103
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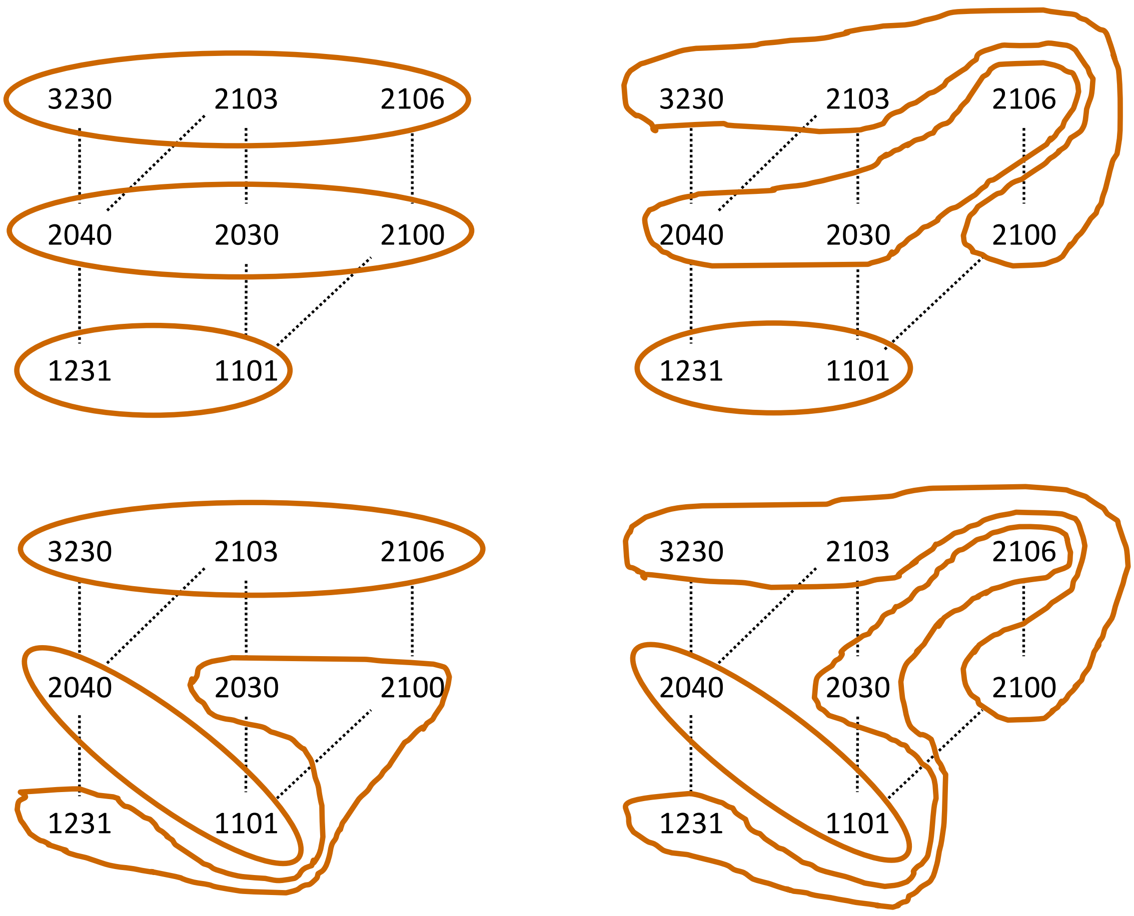
(b)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Q20.

(a)

The problem is equivalent to finding the number of non-negative solutions for the following equation: $x' + y' + z' = 79$.

Using the multiset formula: $n = 3, r = 79$; $\binom{r+n-1}{r} = \binom{79+3-1}{79} = \binom{81}{79} = \binom{81}{2} = \mathbf{3240}$.

(If students solve the given equation assuming x, y, z are non-negative integers, the answer would be $\binom{90}{88} = 4005$.)

(b)

(i) The two 'I's are fixed. Therefore, there are $\binom{6}{2} = \mathbf{15}$ ways. (1 mark)

(ii) Interpretation 1: The I's are distinguishable. There are $\binom{8}{4} = 70$ ways to choose 4 tiles, out of which 15 are with duplicates. Therefore, there are $70 - 15 = \mathbf{55}$ ways to choose 4 tiles without duplicates.

Interpretation 2: The I's are indistinguishable. There are $\binom{6}{4} = 15$ ways for no I and $\binom{6}{3} = 20$ ways for one I. So there are $15 + 20 = \mathbf{35}$ ways.

We accept both interpretations. (2 marks)

(c)

$$\binom{10}{4}\left(\frac{1}{2\sqrt{x}}\right)^6\left(-\frac{1}{2}\right)^4 = 210\left(\frac{1}{64x^3}\right)\left(\frac{1}{16}\right) = 105$$

$$x^3 = \frac{210}{(64)(16)(105)} = \frac{1}{512} = \frac{1}{2^9}$$

$$\therefore x = \frac{1}{2^3} = \mathbf{\frac{1}{8}}$$

(d)

(i) There are $2^{(n^2)}$ directed graphs on n vertices. For $n = 3$, $2^{(n^2)} = 2^{(3^2)} = 2^9 = \mathbf{512}$. (1 mark)

(ii) There are $2^6 = 64$ directed graphs on 3 vertices a, b, c without any loops. Therefore there are $512 - 64 = \mathbf{448}$ directed graphs on three vertices with a least a loop. (2 marks)

(e)

Let A: Bag 1 from 1994 and bag 2 from 1996, and B: Bag 1 from 1996 and bag 2 from 1994.

$P(A) = P(B) = ½$.

Let E: yellow M&M from bag 1, green M&M from bag 2.

Then $P(E|A) = 0.2 \times 0.16 = 0.032$ (yellow from 1994 and green from 1996)

and $P(E|B) = 0.14 \times 0.1 = 0.014$ (yellow from 1996 and green from 1994)

By Bayes' Theorem,

$$P(A|E) = \frac{P(E|A) \cdot P(A)}{P(E|A) \cdot P(A) + P(E|B) \cdot P(B)} = \frac{0.032 \times \frac{1}{2}}{0.032 \times \frac{1}{2} + 0.014 \times \frac{1}{2}} = \frac{0.016}{0.023} = \mathbf{\frac{16}{23}} \text{ or } \mathbf{0.696}.$$

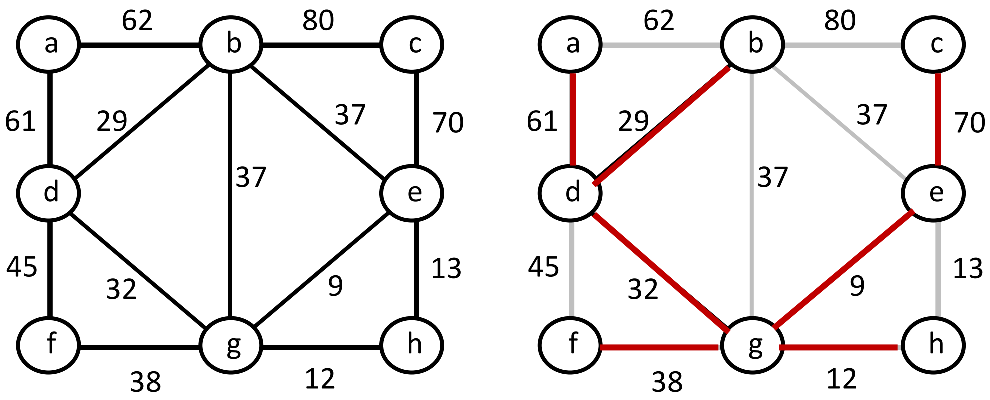
Alternatively:

$$P(Y1994 \mid Y\&G) = \frac{P(Y1994 \ \wedge G1996)}{P(Y\&G)} = \frac{P(Y1994 \wedge G1996)}{P(Y1994 \wedge G1996) + P(G1994 \wedge Y1996)}$$

$$= \frac{0.2 \times 0.16}{(0.2 \times 0.16) + (0.1 \times 0.14)} = \mathbf{\frac{16}{23}}.$$

Q21.

(a)



Q22.

Explanation:

Suppose $n = (xyz)_9 = (zyx)_6$. Then $9^2x + 9y + z = n = 6^2z + 6y + x$. From this, we deduce that $80x + 3y - 35z = 0$.

Since $5 \mid 35z$ and $5 \mid 80x$, we know $5 \mid 35z - 80x$ by the Closure Lemma. So $5 \mid 3y$. Hence Euclid's Lemma tells us $5 \mid y$. This implies $y = 0$ or $y = 5$ as $y \in \{0,1,2,3,4,5\}$, being a digit in the base-6 representation of a number.

Suppose $y = 0$. Then $80x - 35z = 0$. This simplifies to $16x - 7z = 0$ or $16x = 7z$. By successively applying Euclid's Lemma 4 times, we deduce that $16 \mid z$. This implies $z = 0$ as $z \in \{0,1,2,3,4,5\}$, being a digit in the base-6 representation of a number. Substituting back gives $x = 0$. All these tell us that $n = 0$, which is not a case we are interested in because the n we want is positive.

So it must be the case that $y = 5$. Then $80x + 15 - 35z = 0$, and thus $16x + 3 - 7z = 0$. If z is even, then $3 = 7z - 16x$ is also even, which is not true. So z is odd. This implies $z \in \{1,3,5\}$, as z is a digit in the base-6 representation of a number. Note that $7 \times 1 - 3 = 4$ and $7 \times 3 - 3 = 18$, both of which are not multiples of 16. So $z = 5$ and $x = (7 \times 5 - 3)/16 = 2$. Hence $n = 5 \times 6^2 + 5 \times 6 + 2 = 212$. It can be directly verified that $212 = (255)_9$.

Q23.

Additional information:

Theorem. Let $n \in \mathbb{Z}_{\geqslant 2}$. Then R_n has exactly one equivalence class if and only if n is a product of distinct primes.

Proof. Consider first the "only if" direction. Suppose n is not a product of distinct primes, say,

$$n = p^2 m,$$

where p is a prime number and $m \in \mathbb{Z}^+$. Let $a = pm$. It can readily be verified that $a \in A_n$. On the one hand, we have

$$a^2 = (pm)^2 = p^2 m \cdot m = nm \equiv 0 \pmod n.$$

On the other hand, Proposition P implies $1^2 = 1 \not\equiv 0 \pmod n$. So $\sim(a\ R_n\ 1)$. It follows that $[1]$ and $[a]$ are different equivalence classes.

Next, consider the "if" direction. Suppose n is a product of distinct primes, say,

$$n = p_1 p_2 \dots p_\ell,$$

where $p_1, p_2, \dots, p_\ell$ are distinct prime numbers and $\ell \in \mathbb{Z}^+$. It suffices to show that $a^k \not\equiv 0 \pmod n$ for all $a \in A_n$ and all $k \in \mathbb{Z}^+$.

Let $a \in A_n$ and $k \in \mathbb{Z}^+$. If $\gcd(a, n) = 1$, then we can follow the argument in (b). So suppose not. Then there must be $I \subsetneq \{1,2,\dots,\ell\}$ such that $a = \prod_{i \in I} p_i$. Fix such an I, and pick any $j \in \{1,2,\dots,\ell\} \setminus I$. Now

	$p_j \nmid (\prod_{i \in I} p_i)^k$	as the p_i's are distinct and $j \notin I$;
$\therefore$	$p_j \nmid a^k$	by the choice of I;
$\therefore$	$n \nmid a^k$	by the transitivity of divisibility, as $p_j \mid n$;
$\therefore$	$a^k \not\equiv 0 \pmod n$	by the alternative definitions of congruence. $\square$