ST2334 (2021/22 Semester 2) Solution to Tutorial 9

Question 1

Let X be the time between two successive arrivals at the drive-up window of a fast-food restaurant. Then $X \sim Exp(1)$.

(a) $E(X) = 1/\lambda = 1$.

(b) $\sigma = 1/\lambda = 1$.

(c) $F_X(x) = \int_0^x e^{-t}\,dt = 1 - e^{-x}$, for $x > 0$.

$\Pr(X \le 4) = F_X(4) = 1 - e^{-1(4)} = 0.9817$.

$\Pr(2 \le X \le 5) = F_X(5) - F_X(2) = 1 - e^{-5} - (1 - e^{-2}) = 0.1286$.

(Excel: "=expon.dist(4,1,true)"; R: "pexp(4,1,lower=T)")

Question 2

Let X be the time until failure for the fan. Since $E(X) = 25000$, hence, $X \sim Exp(1/25000)$

(a) $\Pr(X > 20000) = e^{-20000/25000} = 0.4493$.

$\Pr(X \le 30000) = 1 - e^{-30000/25000} = 0.6988$.

$\Pr(20000 \le X \le 30000) = 0.6988 - (1 - 0.4493) = 0.1481$.

(b) $\sigma = 1/\lambda = 25000$. Therefore $\Pr(X > \mu + 2\sigma) = \Pr(X > 75000) = e^{-75000/25000} = 0.0498$.

Question 3

X = length of time to fail, in years

$X \sim Exp(1/2)$

(a) $V(X) = [E(X)]^2 = [2]^2 = 4$

(b) $\Pr(X < 1) = 1 - e^{-(1/2)(1)} = 0.39347$

Y = number of electrical switches out of 100 switches that fail during the first year.

$Y \sim Binomial(n = 100, p = 0.39347)$

$E(Y) = np = 39.35, \quad V(Y) = np(1-p) = 23.8651$

$$\Pr(Y \le 30) = \Pr(Y \le 30.5) \approx \Pr\left(Z \le \frac{30.5 - 39.35}{\sqrt{23.8651}}\right)$$
$$= \Pr(Z < -1.8116) = 0.03502$$

(Excel: "=1-norm.dist(30.5,39.35,sqrt(23.8651),true)";
R: "1-pnorm(30.5,39.35,sqrt(23.8651),lower=T)")

[Exact probability: $\Pr(Y \le 30) = 0.03347$.]

(Excel: "=binom.dist(30,100,0.39347,false)"; R: "dbinom(30,100,0.39347,lower=T)")

Question 4

$$\Pr(\mu - 3\sigma < X < \mu + 3\sigma) = \Pr(-3 < Z < 3) = \Pr(Z < 3) - \Pr(Z < -3)$$
$$= 0.99865 - 0.00135 = 0.9973$$

[Compare with $\Pr(\mu - 3\sigma < X < \mu + 3\sigma) \ge 8/9$ using Chebyshev's Inequality]

Question 5

X = amount of the soft drink

$X \sim Normal\ (\mu = 200;\ \sigma^2 = 15^2)$

(a) $\Pr(X > 224) = \Pr(Z > 1.60) = 0.05480$, where $Z \sim N(0,1)$

(Excel: "=1-norm.dist(224,200,15,true)"; R: "1-pnorm(224,200,15,lower=T)")

(b) $\Pr(191 < X < 209) = \Pr(-0.60 < Z < 0.60) = 0.4515$

(c) $\Pr(X > 230) = \Pr(Z > 2.00) = 0.02275 = p$

Y = number of cups out of 100 cups that overflow

$Y \sim Binomial\ (n = 1000, p = 0.02275)$
$E(Y) = np = 1000(0.02275) = 22.75 \approx 23$

(d) Let $z_{0.25}$ denote the 25th percentile of the standard normal distribution. That is, $\Pr(Z < z_{0.25}) = 0.25$, where $Z \sim N(0,1)$, Hence, $z_{0.25} = -0.6745$.
Note $Z = \frac{X-\mu}{\sigma}$ or $X = \mu + \sigma Z$. Therefore, $x_{0.25} = \mu + z_{0.25}\sigma = 200 + (-0.6745)(15) = 189.883$, where $x_{0.25}$ denotes the 25th percentile of the distribution for X (i.e., $\Pr(X < x_{0.25}) = 0.25$).
(Excel: “=norm.inv(0.25,200,15)”; R: “qnorm(0.25,200,15)”)

Question 6

$X =$ commute time from home to office
$X \sim Normal\ (\mu = 24;\ \sigma^2 = 3.8^2)$

(a) $\Pr(X > 30) = \Pr(Z > 1.57895) = 0.057174$

(b) $\Pr(X > 15) = \Pr(Z > -2.36842) = 1 - 0.0089321 = 0.991068 = 99.11\%$

(c) $Y=$ number of trips out of 3 trips that take at least half an hour
$Y \sim Binomial\ (n = 3, p = 0.057174)$

$$\Pr(Y = 2) = \binom{3}{2}(0.057174)^2(1 - 0.057174)^1 = 0.0092459$$

(Excel: “=binom.dist(2,3,0.057174,false)”; R: “dbinom(2,3,0.057174)”)

Question 7

$Y =$ number of head in 400 tosses of a coin
$Y \sim Binomial\ (n = 400,\ \ p = 0.5)$
$E(Y) = np = 400(0.5) = 200.\ \ V(Y) = np(1-p) = 400(0.5)(0.5) = 100$
$Y \dot\sim Normal\ (\mu = 200;\ \sigma^2 = 100)$

(a) $\Pr(185 \le Y \le 210) = \Pr(184.5 < Y < 210.5) = \Pr(-1.55 < Z < 1.05)$
$= \Pr(Z < 1.05) - \Pr(Z < -1.55) = 0.853141 - 0.060571 = 0.79257$

(b) $\Pr(Y = 205) = \Pr(204.5 < Y < 205.5) = \Pr(0.45 < Z < 0.55)$
$= \Pr(Z < 0.55) - \Pr(Z < 0.45) = 0.70884 - 0.67364 = 0.03520$

(c) $\Pr(Y < 176 \text{ or } Y > 227) = \Pr(Y < 175.5) + \Pr(Y > 227.5)$
$= \Pr(Z < -2.45) + \Pr(Z > 2.75) = 0.007143 + 0.002980 = 0.010123$

Question 8

$Y =$ number of drunk driver
$Y \sim Binomial\ (n = 400,\ \ p = 0.1)$
$E(Y) = np = 400(0.1) = 40.\ V(Y) = np(1-p) = 400(0.1)(0.9) = 36$
$Y \dot\sim Normal\ (\mu = 40;\ \sigma^2 = 6^2)$

(a) $\Pr(Y < 32) = \Pr(Y < 31.5) = \Pr(Z < -1.41667) = 0.07829$

(b) $\Pr(Y > 49) = \Pr(Y > 49.5) = \Pr(Z > 1.58333) = 0.056673$

(c) $\Pr(35 \le Y < 47) = \Pr(34.5 < Y < 46.5) = \Pr(-0.91667 < Z < 1.08333)$
$= \Pr(Z < 1.08333) - \Pr(Z < -0.91667) = 0.860669 - 0.179658 = 0.681011$

Question 9

$Y =$ number of defective parts
$Y \sim Binomial\ (n = 100,\ \ p = 0.05)$
$E(Y) = np = 100(0.05) = 5.\ \ V(Y) = np(1-p) = 100(0.05)(0.95) = 4.75$
$Y \dot\sim Normal\ (\mu = 5;\ \sigma^2 = 4.75)$

(a) $\Pr(Y > 2) = \Pr(Y > 2.5) \approx \Pr(Z > -1.14708) = 0.87433.$

(b) $\Pr(Y > 10) = \Pr(Y > 10.5) \approx \Pr(Z > 2.52357) = 0.0058085$

<u>Question 10</u>

(a) $\mu = \sum x f_X(x) = 4(0.2) + 5(0.4) + 6(0.3) + 7(0.1) = 5.3$

$$\sigma^2 = \sum (x-\mu)^2 f(x) = (4-5.3)^2(0.2) + (5-5.3)^2(0.4) + (6-5.3)^2(0.3) + (7-5.3)^2(0.1) = 0.81$$

(b) With $n = 36$, $\mu_{\bar{X}} = \mu = 5.3$; ${\sigma_{\bar{X}}}^2 = \frac{\sigma^2}{n} = \frac{0.81}{36} = 0.0225$

(c) Applying the Central Limit Theorem, $\bar{X}\ approx \sim N(5.3, 0.0255)$

$$\Pr(\bar{X} < 5.5) \approx \Pr\left(Z < \frac{5.5 - 5.3}{\sqrt{0.0225}}\right) = \Pr(Z < 1.33333) = 0.90879$$

<u>Question 11</u>

$X =$ amount of benzene. $E(X) = \mu$ and $V(X) = 100^2$

(a) $n = 25$. By the CLT, $\bar{X} \sim N\left(\mu, \frac{100^2}{25}\right)$. $\Pr(\bar{X} > 7950 | \mu = 7950) = \Pr(\bar{X} > \mu) = 0.5$

(b) $X \sim N(\mu, 100^2)$. Hence $\bar{X} \sim N\left(\mu, \frac{100^2}{25}\right)$.

$\Pr(\bar{X} \geq 7960 | \mu = 7950) = \Pr\left(Z > \frac{7960-7950}{100/\sqrt{25}}\right) = \Pr(Z > 0.5) = 0.30854$

No, there is no strong evidence that the population mean exceeds the government limit as it is likely to see a sample mean is equal to or larger than 7960 if the population mean equals to the government limit 7950.