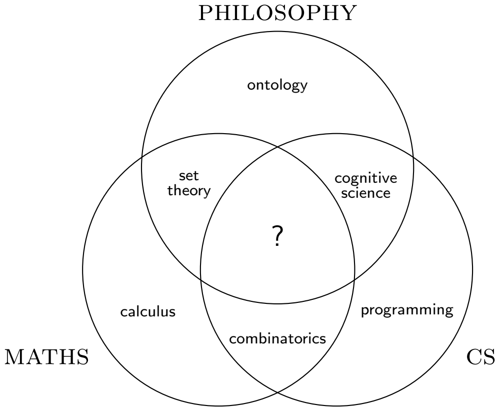
Chapter 5: Sets

CS1231S Discrete Structures

Wong Tin Lok

National University of Singapore

2021/21 Semester 1



What can one put in the centre?

Answer at `https://pollev.com/wtl`.

- There is no need to log in to pollev.
- Use hyphens (-) for spaces in multi-word answers.

About me

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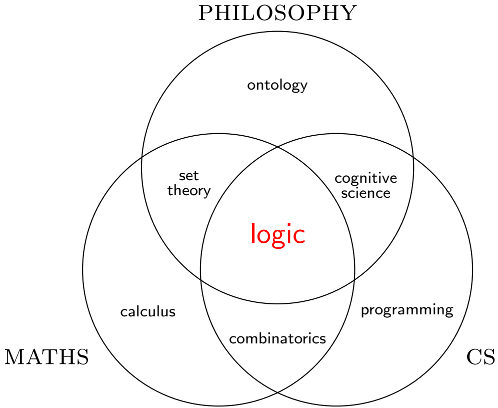
- `https://blog.nus.edu.sg/matwong/`

- definitions $\to$ undefinables
- proofs $\to$ (true) unprovables
- necessary truth $\to$ possible truth



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Practicalities

- Lectures: Zoom (Mute yourself when you are not speaking.)
  - Thursday 12:00 – ~~2:00pm~~ 1:35pm, with a 5-minute "break" in the middle
  - Friday 3:00 – ~~4:00pm~~ 3:45pm
- Slides and notes are posted on LumiNUS (`https://luminus.nus.edu.sg`) and on the module website (`https://www.comp.nus.edu.sg/~cs1231s/`).
- Try out the questions marked with ✎ in the notes. Answers will be provided.
- There are a pre-lecture and a post-lecture version of slides. Pages that are different are marked by a red line near the top left-hand corner.
- If you have any questions/comments for me during lectures, then you can unmute yourself and speak, or ask at the CS1231S Telegram group (or at the Zoom chat?).
- Consultation: online
  - preferably immediately after the lectures (or by individual/group appointment)
  - LumiNUS Forum
- Additional resources: search for "discrete mathematics" on the Internet or in the library (catalogue).
- Weeks 4–9: sets, relations, induction/recursion, functions, cardinality — *proofs*

Sets

Why sets?

- The language of sets is an important part of modern mathematical discourse.
- Sets are interesting mathematical objects.
- For this module, they provide a topic on which we practise writing and understanding proofs.

Young man, in mathematics you don't understand things. You just get used to them. (reportedly) John von Neumann

Definition 5.1.1

(1) A *set* is an unordered collection of objects.

(2) These objects are called the *members* or *elements* of the set.

(3) Write

$$\begin{array}{rll} x \in A & \text{for} & x \text{ is an element of } A; \\ x \notin A & \text{for} & x \text{ is not an element of } A; \\ x, y \in A & \text{for} & x, y \text{ are elements of } A; \\ x, y \notin A & \text{for} & x, y \text{ are not elements of } A; \end{array}$$

etc.

(4) We may read $x \in A$ also as "x is in A" or "A *contains* x (as an element)".

Warning 5.1.2. Some use "contains" for the subset relation, but we do *not*.

Common sets (Table 5.1)

Note 5.1.3. Some define $0 \notin \mathbb{N}$, but we do *not*.

Symbol	Meaning	Examples	Non-examples
$\mathbb{N}$	the set of all natural numbers	$0, 1, 2, 3, 31 \in \mathbb{N}$	$-1, \frac{1}{2} \notin \mathbb{N}$
$\mathbb{Z}$	the set of all integers	$0, 1, -1, 2, -10 \in \mathbb{Z}$	$\frac{1}{2}, \sqrt{2} \notin \mathbb{Z}$
$\mathbb{Q}$	the set of all rational numbers	$-1, 10, \frac{1}{2}, -\frac{7}{5} \in \mathbb{Q}$	$\sqrt{2}, \pi, \sqrt{-1} \notin \mathbb{Q}$
$\mathbb{R}$	the set of all real numbers	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi \in \mathbb{R}$	$\sqrt{-1}, \sqrt{-10} \notin \mathbb{R}$
$\mathbb{C}$	the set of all complex numbers	$-1, 10, -\frac{3}{2}, \sqrt{2}, \pi, \sqrt{-1}, \sqrt{-10} \in \mathbb{C}$	
$\mathbb{Z}^+$	the set of all positive integers	$1, 2, 3, 31 \in \mathbb{Z}^+$	$0, -1, -12 \notin \mathbb{Z}^+$
$\mathbb{Z}^-$	the set of all negative integers	$-1, -2, -3, -31 \in \mathbb{Z}^-$	$0, 1, 12 \notin \mathbb{Z}^-$
$\mathbb{Z}_{\geqslant 0}$	the set of all non-negative integers	$0, 1, 2, 3, 31 \in \mathbb{Z}_{\geqslant 0}$	$-1, -12 \notin \mathbb{Z}_{\geqslant 0}$

$\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{Q}_{\geqslant m}$, $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}_{\geqslant m}$, etc. are defined similarly.

$\mathbb{Z}$ is for *Zahlen*.

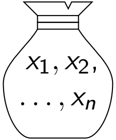
$\mathbb{Q}$ is for quotients.

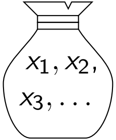
"Positive" means > 0.

"Negative" means < 0.

"Non-negative" means $\geqslant 0$.

Specifying a set by listing out all its elements





Definition 5.1.4 (roster notation)

(1) The set whose only elements are $x_1, x_2, \dots, x_n$ is denoted $\{x_1, x_2, \dots, x_n\}$.

(2) The set whose only elements are $x_1, x_2, x_3, \dots$ is denoted $\{x_1, x_2, x_3, \dots\}$.

Example 5.1.5

(1) The only elements of $A = \{1, 5, 6, 3, 3, 3\}$ are 1, 5, 6 and 3.
So $6 \in A$ but $7 \notin A$.

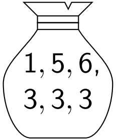
(2) The only elements of $B = \{0, 2, 4, 6, 8, \dots\}$ are the non-negative even integers.
So $4 \in B$ but $5 \notin B$.

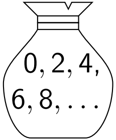
To check whether an object z is an element of a set $S = \{x_1, x_2, \dots, x_n\}$

If z is in the list $x_1, x_2, \dots, x_n$, then $z \in S$, else $z \notin S$.

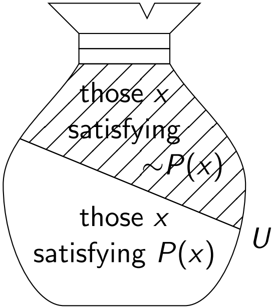
Question

What are the elements of $\{2, 3, \dots\}$? All integers $x \geqslant 2$?





Specifying a set by describing its elements



Definition 5.1.6 (set-builder notation)

Let U be a set and $P(x)$ be a predicate over U. Then the set of all elements $x \in U$ such that $P(x)$ is true is denoted

$$\{x \in U : P(x)\} \quad \text{or} \quad \{x \in U \mid P(x)\}.$$

This is read as "the set of all x in U such that $P(x)$".

Example 5.1.7

(1) The elements of $C = \{x \in \mathbb{Z}_{\geqslant 0} : x \text{ is even}\}$ are precisely the elements of $\mathbb{Z}_{\geqslant 0}$ that are even, i.e., the non-negative even integers. So $6 \in C$ but $7 \notin C$.

(2) The elements of $D = \{x \in \mathbb{Z} : x \text{ is a prime number}\}$ are precisely the elements of $\mathbb{Z}$ that are prime numbers, i.e., the prime integers. So $7 \in D$ but $9 \notin D$.

To check whether an object z is an element of $S = \{x \in U : P(x)\}$

If $z \in U$ and $P(z)$ is true, then $z \in S$, else $z \notin S$. Hence $z \notin U$ implies $z \notin S$, and $P(z)$ is false implies $z \notin S$.

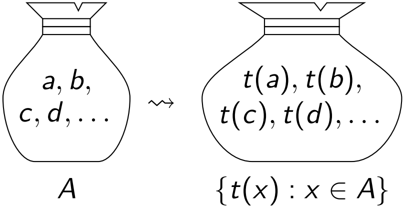
Specifying a set by replacement

Definition 5.1.8 (replacement notation)

Let A be a set and $t(x)$ be a term in a variable x. Then the set of all objects of the form $t(x)$ where x ranges over the elements of A is denoted

$$\{t(x) : x \in A\} \quad \text{or} \quad \{t(x) \mid x \in A\}.$$

This is read as "the set of all $t(x)$ where $x \in A$".



Example 5.1.9

(1) The elements of $E = \{x + 1 : x \in \mathbb{Z}_{\geqslant 0}\}$ are precisely those $x + 1$ where $x \in \mathbb{Z}_{\geqslant 0}$, i.e., the positive integers. So $1 = 0 + 1 \in E$ but $0 \notin E$.

(2) The elements of $F = \{x - y : x, y \in \mathbb{Z}_{\geqslant 0}\}$ are precisely those $x - y$ where $x, y \in \mathbb{Z}_{\geqslant 0}$, i.e., the integers. So $-1 = 1 - 2 \in F$ but $\sqrt{2} \notin F$.

To check whether an object z is an element of $S = \{t(x) : x \in A\}$

If there is $x \in A$ such that $t(x) = z$, then $z \in S$, else $z \notin S$.

Equality of sets

Convention 5.1.11. This is the *only* situation in mathematics when "if" should be understood as a (special) "if and only if".

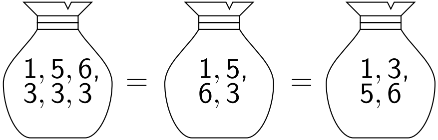
Definition 5.1.10

Two sets are *equal* if they have the same elements, i.e., for all sets A, B,

$$A = B \quad \Leftrightarrow \quad \forall z \ (z \in A \Leftrightarrow z \in B).$$

Example 5.1.12

$\{1,5,6,3,3,3\} = \{1,5,6,3\} = \{1,3,5,6\}$.



Slogan 5.1.13. Order and repetition do not matter.

Example 5.1.14

$$\{y^2 : y \text{ is an odd integer}\} = \{x \in \mathbb{Z} : x = y^2 \text{ for some odd integer } y\}$$
$$= \{1^2, 3^2, 5^2, \ldots\}.$$

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Example 5.1.15

$\{x \in \mathbb{Z} : x^2 = 1\} = \{1, -1\}$.

Slogan 5.1.13. Order and repetition do not matter.

Proof

1. $(\Rightarrow)$
   1.1. Take any $z \in \{x \in \mathbb{Z} : x^2 = 1\}$.
   1.2. Then $z \in \mathbb{Z}$ and $z^2 = 1$.
   1.3. So $z^2 - 1 = (z-1)(z+1) = 0$.
   1.4. $\therefore$ $z - 1 = 0$ or $z + 1 = 0$.
   1.5. $\therefore$ $z = 1$ or $z = -1$.
   1.6. This means $z \in \{1, -1\}$.
2. $(\Leftarrow)$ . . .

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Example 5.1.15

$\{x \in \mathbb{Z} : x^2 = 1\} = \{1, -1\}$.

Slogan 5.1.13. Order and repetition do not matter.

Proof

1. $(\Rightarrow)$ ...
2. $(\Leftarrow)$
   2.1. Take any $z \in \{1, -1\}$.
   2.2. Then $z = 1$ or $z = -1$.
   2.3. In either case, we have $z \in \mathbb{Z}$ and $z^2 = 1$.
   2.4. So $z \in \{x \in \mathbb{Z} : x^2 = 1\}$. $\square$

The empty set

Definition 5.1.10. For all sets A, B,

$$A = B \quad \Leftrightarrow \quad \forall z\ (z \in A \Leftrightarrow z \in B).$$

Theorem 5.1.17

There exists a unique set with no element, i.e.,

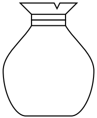
- there is a set with no element; and (existence part)
- for all sets A, B, if both A and B have no element, then $A = B$. (uniqueness part)

Proof

1. (existence part) The set $\{\}$ has no element.
2. (uniqueness part)
   - 2.1. Let A, B be sets with no element.
   - 2.2. Then vacuously,

     $$\forall z\ (z \in A \Rightarrow z \in B) \quad \text{and} \quad \forall z\ (z \in B \Rightarrow z \in A)$$

     because the hypotheses in the implications are never true.
   - 2.3. So $A = B$. $\square$



Definition 5.1.18

The set with no element is called the *empty set*. It is denoted by $\varnothing$.

Inclusion of sets

Let A, B be sets.

Definition 5.1.10. For all sets A, B,
$$A = B \quad \Leftrightarrow \quad \forall z\, (z \in A \Leftrightarrow z \in B).$$

Definition 5.1.19

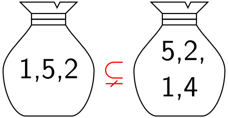
Call A a *subset* of B, and write $A \subseteq B$, if
$$\forall z\, (z \in A \Rightarrow z \in B).$$

Alternatively, we may say that B *includes* A, and write $B \supseteq A$ in this case.

Example 5.1.21 and Example 5.1.24

(1) $\{1,5,2\} \subsetneq \{5,2,1,4\}$ but $\{1,5,2\} \not\subseteq \{2,1,4\}$.

(2) $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$. All these inclusions are proper.



Remark 5.1.22

(1) $$A \not\subseteq B \quad \Leftrightarrow \quad \exists z\, (z \in A \text{ and } z \notin B).$$

(2) $$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

(3) $$\varnothing \subseteq A \quad \text{and} \quad A \subseteq A.$$

Note 5.1.20. We avoid using the symbol $\subset$ because it may have different meanings to different people.

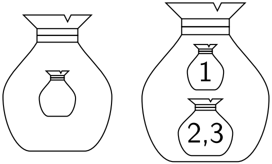
Definition 5.1.23

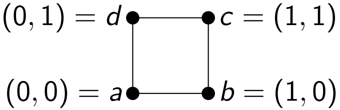
Call A a *proper subset* of B, write $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$. In this case, we may say that the inclusion of A in B is *proper* or *strict*.

Sets of sets

Note 5.1.25

Sets can be elements of sets.





Example 5.1.26

(1) The set $A = \{\varnothing\}$ has exactly 1 element, namely the empty set. So A is not empty.

(2) The set $B = \{\{1\}, \{2,3\}\}$ has exactly 2 elements, namely $\{1\}$, $\{2,3\}$. So $\{1\} \in B$, but $1 \notin B$.

How can one use a set to represent the square above?

- If one only wants to represent the connectivity between the points, then use

$$\big\{\{a,b\},\{b,c\},\{c,d\},\{d,a\}\big\}.$$

- If one also wants to represent the positions of the lines, then use

$$\{(x,y) : (x = 0 \text{ and } y \in [0,1]) \text{ or } (x = 1 \text{ and } y \in [0,1])$$
$$\text{or } (y = 0 \text{ and } x \in [0,1]) \text{ or } (y = 1 \text{ and } x \in [0,1])\}.$$

Checkpoint

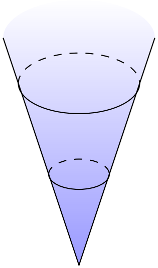
Question 5.1.28

Let $C = \{\{1\}, 2, \{3\}, 3, \{\{4\}\}\}$. Which of the following are true?

- $\{1\} \in C$. $\checkmark$
- $\{2\} \in C$.
- $\{3\} \in C$. $\checkmark$
- $\{4\} \in C$.
- $\{1\} \subseteq C$.
- $\{2\} \subseteq C$. $\checkmark$
- $\{3\} \subseteq C$. $\checkmark$
- $\{4\} \subseteq C$.

So far

- membership
- equality of sets
- inclusion



Next

new sets from old ones

Zoom poll

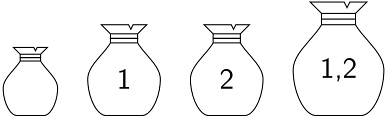
Power set

Let A be a set.

Definition 5.2.1

The set of all subsets of A, denoted $\mathcal{P}(A)$, is called the *power set* of A.

Example 5.2.2



(1) $\mathcal{P}(\varnothing) = \{\varnothing\}$.

(2) $\mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}$.

(3) $\mathcal{P}(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$.

(4) The following are subsets of $\mathbb{Z}_{\geqslant 0}$ and thus are elements of $\mathcal{P}(\mathbb{Z}_{\geqslant 0})$.

$$\varnothing, \{0\}, \{1\}, \{2\}, \ldots \{0,1\}, \{0,2\}, \{0,3\} \ldots \{1,2\}, \{1,3\}, \{1,4\} \ldots$$
$$\{2,3\}, \{2,4\}, \{2,5\} \ldots \{0,1,2\}, \{0,1,3\}, \{0,1,4\}, \ldots$$
$$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \ldots \{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \ldots \; \ldots$$
$$\mathbb{Z}_{\geqslant 0}, \mathbb{Z}_{\geqslant 1}, \mathbb{Z}_{\geqslant 2}, \ldots \{0,2,4,\ldots\}, \{1,3,5,\ldots\}, \{2,4,6,\ldots\}, \{3,5,7,\ldots\}, \ldots$$
$$\{x \in \mathbb{Z}_{\geqslant 0} : (x-1)(x-2) < 0\}, \{x \in \mathbb{Z}_{\geqslant 0} : (x-2)(x-3) < 0\}, \ldots$$
$$\{3x+2 : x \in \mathbb{Z}_{\geqslant 0}\}, \{4x+3 : x \in \mathbb{Z}_{\geqslant 0}\}, \{5x+4 : x \in \mathbb{Z}_{\geqslant 0}\}, \ldots \; \ldots$$

Cardinality of the power set

Let A be a set.

Definition 5.2.1

The set of all subsets of A, denoted $\mathcal{P}(A)$, is called the *power set* of A.

Example 5.2.2 and Example 5.2.5

(1) $\mathcal{P}(\varnothing) = \{\varnothing\}$. $\quad |\varnothing| = 0$ and $|\mathcal{P}(\varnothing)| = 1 = 2^0$.

(2) $\mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}$. $\quad |\{1\}| = 1$ and $|\mathcal{P}(\{1\})| = 2 = 2^1$.

(3) $\mathcal{P}(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$. $\quad |\{1,2\}| = 2$ and $|\mathcal{P}(\{1,2\})| = 4 = 2^2$.

Definition 5.2.3

(1) A set is *finite* if it has finitely many (distinct) elements. It is *infinite* if it is not finite.

(2) Suppose A is a finite set. The *cardinality* of A, or the *size* of A, is the number of (distinct) elements in A. It is denoted by $|A|$.

(3) Sets of size 1 are called *singletons*.

Theorem 5.2.4

Suppose A is a finite set. Then $|\mathcal{P}(A)| = 2^{|A|}$.

Ordered pairs and Cartesian products

Definition 5.2.6

An *ordered pair* is an expression of the form (x, y). Let (x_1, y_1) and (x_2, y_2) be ordered pairs. Then $(x_1, y_1) = (x_2, y_2)$ if

$$x_1 = x_2 \quad \text{and} \quad y_1 = y_2.$$

Example 5.2.7

(1) $(1, 2) \neq (2, 1)$, although $\{1, 2\} = \{2, 1\}$.

(2) $(3, 0.5) = (\sqrt{9}, \frac{1}{2})$.

read as "A cross B"

Definition 5.2.8

Let A, B be sets. The *Cartesian product* of A and B, denoted $A \times B$, is defined to be

$$\{(x, y) : x \in A \text{ and } y \in B\}.$$

Example 5.2.9

$$\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}.$$

$$\begin{tikzcd}
(a,3) \arrow[r, dash] \arrow[d, dash] & (b,3) \arrow[d, dash] \\
(a,2) \arrow[r, dash] \arrow[d, dash] & (b,2) \arrow[d, dash] \\
(a,1) \arrow[r, dash] & (b,1)
\end{tikzcd}$$

Note 5.2.10

$$|\{a, b\} \times \{1, 2, 3\}| = 6 = 2 \times 3 = |\{a, b\}| \times |\{1, 2, 3\}|.$$

Ordered n-tuples and Cartesian products

Let $n \in \{x \in \mathbb{Z} : x \geqslant 2\}$.

Definition 5.2.11

An *ordered n-tuple* is an expression of the form $(x_1, x_2, \ldots, x_n)$. Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be ordered n-tuples. Then $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ if

$$x_1 = y_1 \quad \text{and} \quad x_2 = y_2 \quad \text{and} \quad \ldots \quad \text{and} \quad x_n = y_n.$$

Example 5.2.12

(1) $(1, 2, 5) \neq (2, 1, 5)$, although $\{1, 2, 5\} = \{2, 1, 5\}$.

(2) $(3, (-2)^2, 0.5, 0) = (\sqrt{9}, 4, \frac{1}{2}, 0)$

Definition 5.2.13

Let $A_1, A_2, \ldots, A_n$ be sets. The *Cartesian product* of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is defined to be

$$\{(x_1, x_2, \ldots, x_n) : x_1 \in A_1 \text{ and } x_2 \in A_2 \text{ and } \ldots \text{ and } x_n \in A_n\}.$$

If A is a set, then $A^n = \underbrace{A \times A \times \ldots \times A}_{n\text{-many } A\text{'s}}$.

Example 5.2.14

$\{0, 1\} \times \{0, 1\} \times \{x, y\} = \{(0, 0, x), (0, 0, y), (0, 1, x), (0, 1, y), (1, 0, x), (1, 0, y), (1, 1, x), (1, 1$

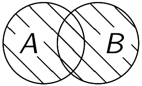
Boolean operations

Let A, B be sets.

Definition 5.3.1

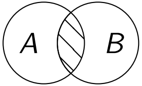
(1) The *union* of A and B, denoted $A \cup B$, is defined by

read as ‘A union B’ $\longrightarrow$ $A \cup B = \{x : x \in A \text{ or } x \in B\}.$



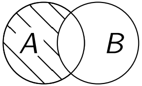
(2) The *intersection* of A and B, denoted $A \cap B$, is defined by

read as ‘A intersect B’ $\longrightarrow$ $A \cap B = \{x : x \in A \text{ and } x \in B\}.$



(3) The *complement* of B in A, denoted $A - B$ or $A \setminus B$, is defined by

read as ‘A minus B’ $\longrightarrow$ $A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$

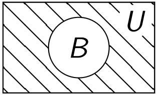


Convention and terminology 5.3.2

When working in a particular context, one usually works within a fixed set U which includes all the sets one may talk about, so that one only needs to consider the elements of U when proving set equality and inclusion. This U is called a *universal set*.

Definition 5.3.3

In a context where U is the universal set (so that implicitly $U \supseteq B$), the *complement* of B, denoted $\overline{B}$ or B^{c}, is defined by $\overline{B} = U \setminus B$.



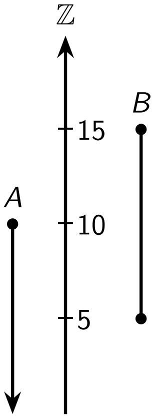
Example 5.3.4 on Boolean operations

For all sets A, B,

$$A \cup B = \{x : (x \in A) \vee (x \in B)\},$$
$$A \cap B = \{x : (x \in A) \wedge (x \in B)\},$$
$$A \setminus B = \{x : (x \in A) \wedge (x \notin B)\},$$
$$\overline{B} = \{x \in U : x \notin B\}, \quad \text{in a context where } U \text{ is the universal set.}$$

Let $A = \{x \in \mathbb{Z} : x \leqslant 10\}$ and $B = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 15\}$. Then

$$A \cup B = \{x \in \mathbb{Z} : (x \leqslant 10) \vee (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x \leqslant 15\};$$
$$A \cap B = \{x \in \mathbb{Z} : (x \leqslant 10) \wedge (5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 10\};$$
$$A \setminus B = \{x \in \mathbb{Z} : (x \leqslant 10) \wedge \sim(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : x < 5\};$$
$$\overline{B} = \{x \in \mathbb{Z} : \sim(5 \leqslant x \leqslant 15)\} = \{x \in \mathbb{Z} : (x < 5) \vee (x > 15)\},$$



in a context where $\mathbb{Z}$ is the universal set. To show the first equation, one shows

$$\forall x \in \mathbb{Z}\ \big((x \leqslant 10) \vee (5 \leqslant x \leqslant 15) \Leftrightarrow (x \leqslant 15)\big), \qquad \text{etc.}$$

Set identities (Theorem 5.3.5)

For all set A, B, C in a context where U is the universal set, the following hold.

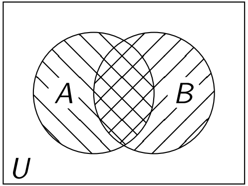
Identity Laws	$A \cup \varnothing = A$	$A \cap U = A$
Universal Bound Laws	$A \cup U = U$	$A \cap \varnothing = \varnothing$
Idempotent Laws	$A \cup A = A$	$A \cap A = A$
Double Complement Law	$\overline{(\overline{A})} = A$	
Commutative Laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative Laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive Laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption Laws	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Complement Laws	$A \cup \overline{A} = U$	$A \cap \overline{A} = \varnothing$
Set Difference Law	$A \setminus B = A \cap \overline{B}$	
Top and Bottom Laws	$\overline{\varnothing} = U$	$\overline{U} = \varnothing$

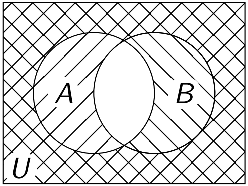
Venn diagrams

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

In the left diagram, hatch the regions representing A and B with and respectively.
In the right diagram, hatch the regions representing $\overline{A}$ and $\overline{B}$ with and respectively.





Then the region represents $\overline{A \cup B}$ on the left diagram, and the region represents $\overline{A} \cap \overline{B}$ on the right diagram. Since these regions occupy the same region in the respective diagrams, we infer that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Note 5.3.6. This argument depends on the fact that each possibility for membership in A and B is represented by a region in the diagram.

Proving set identities using truth tables

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Proof #1

The rows in the following table list all the possibilities for an element $x \in U$:

$x \in A$	$x \in B$	$x \in A \cup B$	$x \in \overline{A \cup B}$	$x \in \overline{A}$	$x \in \overline{B}$	$x \in \overline{A} \cap \overline{B}$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Since the columns under "$x \in \overline{A \cup B}$" and "$x \in \overline{A} \cap \overline{B}$" are the same, for any $x \in U$,

$$x \in \overline{A \cup B} \quad \Leftrightarrow \quad x \in \overline{A} \cap \overline{B}$$

no matter in which case we are. So $\overline{A \cup B} = \overline{A} \cap \overline{B}$. $\square$

Proving set identities directly

One of De Morgan's Laws. Work in the universal set U. For all sets A, B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Proof #2

1. Let $z \in U$.
2.
   2.1. Then $z \in \overline{A \cup B}$

   2.2. $\Leftrightarrow$ $z \notin A \cup B$ by the definition of $\overline{\cdot}$;

   2.3. $\Leftrightarrow$ $\sim\big((z \in A) \vee (z \in B)\big)$ by the definition of $\cup$;

   2.4. $\Leftrightarrow$ $(z \notin A) \wedge (z \notin B)$ by De Morgan's Laws for propositions;

   2.5. $\Leftrightarrow$ $(z \in \overline{A}) \wedge (z \in \overline{B})$ by the definition of $\overline{\cdot}$;

   2.6. $\Leftrightarrow$ $z \in \overline{A} \cap \overline{B}$ by the definition of $\cap$. $\square$

Applications of the set identities

Fix a universal set U. The following are true for all sets A, B, C.

Identity Law	$A \cap U = A.$
Distributive Law	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$
Complement Law	$A \cup \overline{A} = U.$
Set Difference Law	$A \setminus B = A \cap \overline{B}.$

Example 5.3.7

Under the universal set U, show that $(A \cap B) \cup (A \setminus B) = A$ for all sets A, B.

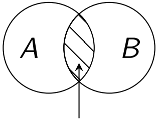
Proof

1.	$(A \cap B) \cup (A \setminus B)$	$= (A \cap B) \cup (A \cap \overline{B})$	by the Set Difference Law;
2.		$= A \cap (B \cup \overline{B})$	by the Distributive Law;
3.		$= A \cap U$	by the Complement Law;
4.		$= A$	by the Identity Law. $\square$

Boolean operations and inclusion

Example 5.3.8

Show that $A \cap B \subseteq A$ for all sets A, B.



$A \cap B$

Let A, B be sets.

Definition 5.1.19. $A \subseteq B \quad \Leftrightarrow \quad \forall z\ (z \in A \Rightarrow z \in B)$.

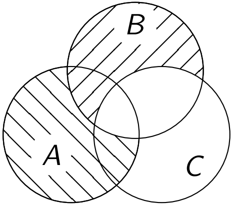
Definition 5.3.1(2). $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Proof

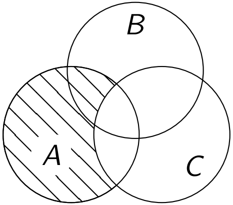
1. Let $z \in A \cap B$.
2. Then $z \in A$ and $z \in B$ by the definition of $\cap$.
3. In particular, we know $z \in A$. $\square$

Example 5.3.9: Is the following true?

$$\boxed{(A \setminus B) \cup (B \setminus C) = A \setminus C \quad \text{for all sets } A, B, C.}$$



$(A \setminus B) \cup (B \setminus C)$



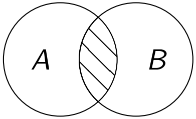
$A \setminus C$

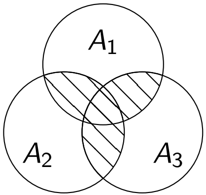
No. For a counterexample, let $A = C = \varnothing$ and $B = \{1\}$. Then

$$(A \setminus B) \cup (B \setminus C) = \varnothing \cup \{1\} = \{1\} \neq \varnothing = A \setminus C.$$

Zoom poll

Cardinality of a union





Definition 5.3.10

(1) Two sets A, B are *disjoint* if $A \cap B = \varnothing$.

(2) Sets $A_1, A_2, \ldots, A_n$ are *pairwise disjoint* or *mutually disjoint* if $A_i \cap A_j = \varnothing$ for all distinct $i, j \in \{1, 2, \ldots, n\}$.

Example 5.3.11

The sets $A = \{1, 3, 5\}$ and $B = \{2, 4\}$ are (pairwise) disjoint. Note

$$|A \cup B| = |\{1, 2, 3, 4, 5\}| = 5 = 3 + 2 = |A| + |B|.$$

Theorem 5.3.12

(1) Let A, B be disjoint finite sets. Then

$$|A \cup B| = |A| + |B|.$$

(2) Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Proof

Count the elements set by set. Every element in the union is counted exactly once because the sets are (pairwise) disjoint. □

Theorem 5.3.13 (Inclusion–Exclusion Principle).

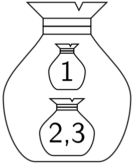
For all finite sets A, B,

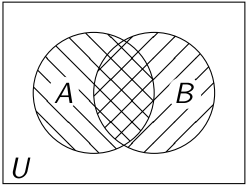
$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Checkpoint

What we saw

- membership, inclusion, and equality of sets
- power sets and Cartesian products
- union, intersections, complements
- set identities and their proofs
- Venn diagrams
- cardinalities of finite sets





Questions

- Is there any other set operation?
- Are sets simply predicates in disguise?
- Why do we work with a universal set?

Search for "Russell's Paradox".

Next

how sets can represent mathematical objects