MA2001

LIVE LECTURE 3

Q&A: log in to PollEv.com/vtpoll

Q&A: log in to PollEv.com/vtpoll

Topics for week 3

2.3 Inverses of Square Matrices

2.4 Elementary Matrices

2.5 Determinant

Invertible matrix

$\boldsymbol{A}$: square matrix of order n.

$\boldsymbol{A}$ is invertible

if there exists a square matrix $\boldsymbol{B}$ of order n such that

$$\boxed{\boldsymbol{A}^{-1}}\boldsymbol{A} = \boldsymbol{I} \text{ and } \boldsymbol{A}\boxed{\boldsymbol{A}^{-1}} = \boldsymbol{I}$$

only one

The matrix $\boldsymbol{B}$ here is called the inverse of $\boldsymbol{A}$.

We use $\boldsymbol{A}^{-1}$ to denote this unique inverse of $\boldsymbol{A}$.

A square matrix is called singular if it has no inverse.

Q&A: log in to PollEv.com/vtpoll

True or False

Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be square matrices of the same size

1. If $\boldsymbol{AB} = \boldsymbol{I}$, then $\boldsymbol{BA} = \boldsymbol{I}$ Always true. See Theorem 2.4.12.
2. $\boldsymbol{A} = \boldsymbol{B} \Rightarrow \boldsymbol{CA} = \boldsymbol{CB}$ Always true.
3. $\boxed{\boldsymbol{CA} = \boldsymbol{CB} \Rightarrow \boldsymbol{A} = \boldsymbol{B} \quad \text{True if } \boldsymbol{C} \text{ is invertible.}}$

Cancellation law Remark 2.3.4

$\boldsymbol{C}^{-1}\boldsymbol{CA} = \boldsymbol{C}^{-1}\boldsymbol{CB} \longrightarrow \boldsymbol{IA} = \boldsymbol{IB}$

Q&A: log in to PollEv.com/vtpoll

Show Inverse

Given $\boldsymbol{A}$ is a square matrix and $\boldsymbol{A}^2 + \boldsymbol{A} = \boldsymbol{I}$.

Show : $\boldsymbol{A}^{-1} = \boldsymbol{A} + \boldsymbol{I}$ ← to show $\boldsymbol{A} + \boldsymbol{I}$ is the inverse of $\boldsymbol{A}$

To show $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$, we just need to show $\boldsymbol{BA} = \boldsymbol{I}$ or $\boldsymbol{AB} = \boldsymbol{I}$

$$\boldsymbol{A}(\boldsymbol{A}+\boldsymbol{I}) = \boldsymbol{A}^2 + \boldsymbol{A} = \boldsymbol{I}$$

algebraic manipulation use given condition

This implies

- $\boldsymbol{A}$ is invertible
- $\boldsymbol{A}^{-1} = \boldsymbol{A} + \boldsymbol{I}$

Q&A: log in to PollEv.com/vtpoll

Invertibility and matrix operations

$\boldsymbol{A}, \boldsymbol{B}$: two invertible matrices (same size)

a : non-zero scalar

	Matrix	Invertible?	Inverse
Scalar multiplication	$a\boldsymbol{A}$	yes	$(a\boldsymbol{A})^{-1} = (1/a)\boldsymbol{A}^{-1}$
Transpose	$\boldsymbol{A}^T$	yes	$(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$
Inverse	$\boldsymbol{A}^{-1}$	yes	$(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
Matrix multiplication	$\boldsymbol{AB}$	yes	$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

$$(\boldsymbol{AB\ldots Z})^{-1} = \boldsymbol{Z}^{-1}\ldots\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

Note:

$\boldsymbol{A}$ and $\boldsymbol{B}$ invertible DOES NOT IMPLY $\boldsymbol{A} + \boldsymbol{B}$ is invertible

$(\boldsymbol{A}+\boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$ ← FALSE

Exercise 2 Q29

$\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible matrices of the same size. Suppose $\boldsymbol{A} + \boldsymbol{B}$ is invertible. Show that:

(i) $\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$ is invertible and

(ii) $(\boldsymbol{A}+\boldsymbol{B})^{-1} = \boldsymbol{A}^{-1}(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})^{-1}\boldsymbol{B}^{-1}$

For (i), not easy to find a matrix $\boldsymbol{M}$ such that $(\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1})\boldsymbol{M} = \boldsymbol{I}$

take the inverses of both sides

For (ii), we try to show: $(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{B}(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})\boldsymbol{A}$

$$\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{A} + \boldsymbol{B}\boldsymbol{B}^{-1}\boldsymbol{A}$$

$$\boldsymbol{B}\boldsymbol{I} + \boldsymbol{I}\boldsymbol{A}$$

$\mathbf{A}$ and $\mathbf{B}$ are invertible matrices of the same size. Suppose $\mathbf{A} + \mathbf{B}$ is invertible. Show that:

(i) $\mathbf{A}^{-1} + \mathbf{B}^{-1}$ is invertible and

(ii) $(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1}(\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}\mathbf{B}^{-1}$

Exercise 2 Q29

To show (i) $\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$ is invertible

From (ii), we have $(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{B}(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})\boldsymbol{A}$

$$\boldsymbol{B}^{-1}(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{A}^{-1} = (\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})$$

$\boldsymbol{B}^{-1}$, $\boldsymbol{A}+\boldsymbol{B}$ and $\boldsymbol{A}^{-1}$ are all invertible

so the product $\boldsymbol{B}^{-1}(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{A}^{-1}$ is invertible

This means $\boldsymbol{A}^{-1} + \boldsymbol{B}^{-1}$ is invertible

Q&A: log in to PollEv.com/vtpoll

Elementary matrices

A square matrix is called an elementary matrix
if it can be obtained from an identity matrix
by performing a single elementary row operation.

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2R_2} \boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{2R_2} \boldsymbol{B} = \boldsymbol{E}_1\boldsymbol{A}$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{R_2 \leftrightarrow R_3} \boldsymbol{C} = \boldsymbol{E}_2\boldsymbol{A}$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 + 2R_1} \boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{R_3 + 2R_1} \boldsymbol{D} = \boldsymbol{E}_3\boldsymbol{A}$$

Q&A: log in to PollEv.com/vtpoll

Which are elementary?

Which of the following are elementary matrices ?

$0R_2$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$R_1 + 2R_3$

$R_3 + R_1$

$R_1 \leftrightarrow R_3$

$R_2 \leftrightarrow R_1$

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$-R_1$

All elementary matrices are invertible

The inverse of an elementary matrix is also an elementary matrix

Inverse of elementary matrix

$$\boldsymbol{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{2R_2}\boldsymbol{E}_1=\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}\longrightarrow\boldsymbol{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\boldsymbol{E}_1^{-1}=\begin{pmatrix}1&0&0\\0&1/2&0\\0&0&1\end{pmatrix}$$

$$\boldsymbol{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{R_2\leftrightarrow R_3}\boldsymbol{E}_2=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\longrightarrow\boldsymbol{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\boldsymbol{E}_2^{-1}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$\boldsymbol{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{R_3+2R_1}\boldsymbol{E}_3=\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}\longrightarrow\boldsymbol{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\boldsymbol{E}_3^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\-2&0&1\end{pmatrix}$$

Q&A: log in to PollEv.com/vtpoll

Gaussian elimination

$$\boldsymbol{A} \xrightarrow{\text{ero}_1} \xrightarrow{\text{ero}_2} \xrightarrow{\text{ero}_3} \cdots \xrightarrow{\text{ero}_n} \boldsymbol{R}$$ REF

R can obtained from ***A*** by

pre-multiplying ***A*** with a series of elementary matrices

$$\boldsymbol{E}_n \ldots \boldsymbol{E}_3 \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{R}$$

$$\boldsymbol{A} \boldsymbol{E}_1 \boldsymbol{E}_2 \boldsymbol{E}_3 \ldots \boldsymbol{E}_n = \boldsymbol{R}$$

Incorrect

Q&A: log in to PollEv.com/vtpoll

True or false

Given $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent

I. There is an invertible matrix $\boldsymbol{C}$ such that $\boldsymbol{CA} = \boldsymbol{B}$

II. There is an invertible matrix $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{DB}$

$$\boldsymbol{A} \longrightarrow \quad \longrightarrow \quad \longrightarrow \quad \cdots \longrightarrow \boldsymbol{B}$$

$$\boldsymbol{E}_n \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{B} \quad \Rightarrow \boldsymbol{CA} = \boldsymbol{B} \quad \Rightarrow \boldsymbol{A} = \boldsymbol{C}^{-1}\boldsymbol{B}$$

Let $\boldsymbol{C} = \boldsymbol{E}_n \ldots \boldsymbol{E}_2 \boldsymbol{E}_1$

Let $\boldsymbol{D} = \boldsymbol{C}^{-1} = (\boldsymbol{E}_1)^{-1}(\boldsymbol{E}_2)^{-1} \ldots (\boldsymbol{E}_n)^{-1}$

Q&A: log in to PollEv.com/vtpoll

Finding inverse matrix

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{G.J.E.}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\boxed{(\,\boldsymbol{A} \mid \boldsymbol{I}\,) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} (\,\boldsymbol{I} \mid \boldsymbol{A}^{-1}\,)}$$

$$\boldsymbol{A} \xrightarrow{\text{GJE}} \boldsymbol{I} \qquad \boldsymbol{I} \xrightarrow{\text{GJE}} \boldsymbol{A}^{-1}$$

$$\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I} \Rightarrow \boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{I} = \boldsymbol{A}^{-1}$$

Q&A: log in to PollEv.com/vtpoll

Let's revise

- An elementary matrix can be obtained by performing exactly one e.r.o. on the identity matrix.
- The action of an e.r.o. on $\boldsymbol{A}$ is the same as pre-multiplying an elementary matrix on $\boldsymbol{A}$
- There are three types of elementary matrices
- All elementary matrices are invertible
- The inverse of an elementary matrix is an elementary matrix.

Q&A: log in to PollEv.com/vtpoll

Announcement

- Tutorial Session
  - Group Discussion 1 this week
  - No GD in week 4
  - Tutor will go through tutorial set 1 & 2
- MATLAB
  - Worksheet 2 this week (own pace)
  - Zoom recording on intro to MATLAB
- Textbook exercise
  - Exercise 1 solution in LumiNUS > Files
- Online quiz
  - Quiz 1 and 2 closed
  - Go back to LumiNUS Quiz to view your scores and correct answers
  - Quiz 3 due next Thursday

Q&A: log in to PollEv.com/vtpoll

A very important theorem

Let $\boldsymbol{A}$ be a square matrix.

The following statements are equivalent

1. $\boldsymbol{A}$ is invertible.
2. The linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
3. The reduced row-echelon form of $\boldsymbol{A}$ is an identity matrix.
4. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

Q&A: log in to PollEv.com/vtpoll

REF and invertible matrix

1. $\boldsymbol{A}$ is invertible.
2. The linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
3. The reduced row-echelon form of $\boldsymbol{A}$ is an identity matrix.
4. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

To check whether a square matrix is invertible:

$n \times n$

- Look at the RREF
  - RREF = $\boldsymbol{I}$ implies invertible — n non-zero rows
  - RREF ≠ $\boldsymbol{I}$ implies not invertible — < n non-zero rows
- Look at REF
  - REF has no zero row implies invertible — n non-zero rows
  - REF has zero rows implies not invertible — < n non-zero rows

1. $\boldsymbol{A}$ is invertible.
2. The linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
3. The reduced row-echelon form of $\boldsymbol{A}$ is an identity matrix.
4. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

Q&A: log in to PollEv.com/vtpoll

Exercise 2 Q44(b)

$\boldsymbol{A}$ is m x n and $\boldsymbol{B}$ is n x m.
If m > n, can $\boldsymbol{AB}$ be invertible?

So $\boldsymbol{AB}$ is NOT invertible

$\boldsymbol{Bx} = \boldsymbol{0}$

This homogeneous system has m variables and n equations

The system has infinitely many solutions

So it has a non-trivial solution, say $\boldsymbol{x} = \boldsymbol{u}$

$$\boldsymbol{Bu} = \boldsymbol{0}$$
$$\boldsymbol{ABu} = \boldsymbol{A0} = \boldsymbol{0}$$

So $\boldsymbol{ABx} = \boldsymbol{0}$ has a non-trivial solution $\boldsymbol{u}$

Every solution of $\boldsymbol{Bx} = \boldsymbol{0}$ is also a solution of $\boldsymbol{ABx} = \boldsymbol{0}$

Q&A: log in to PollEv.com/vtpoll

Product of elementary matrices

Express $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$ as a product of elementary matrices

1. $\boldsymbol{A}$ is invertible.
2. The linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
3. The reduced row-echelon form of $\boldsymbol{A}$ is an identity matrix.
4. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{-R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 - R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}$$

$$\boldsymbol{E_5 E_4 E_3 E_2 E_1 A} = \boldsymbol{I}$$

$$\boldsymbol{A} = \boldsymbol{E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} I}$$

Q&A: log in to PollEv.com/vtpoll

Elementary column operations

Perform e.c.o. C to a matrix $\boldsymbol{A}$ is the same as post-multiply a certain elementary matrix $\boldsymbol{E}$ to $\boldsymbol{A}$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2C_2} \boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{2C_2} \boldsymbol{B} = \boldsymbol{A}\boldsymbol{E}_1$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_2 \leftrightarrow C_3} \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{C_2 \leftrightarrow C_3} \boldsymbol{C} = \boldsymbol{A}\boldsymbol{E}_2$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_3 + 2C_1} \boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{A} \xrightarrow{C_3 + 2C_1} \boldsymbol{D} = \boldsymbol{A}\boldsymbol{E}_3$$

Q&A: log in to PollEv.com/vtpoll

True or false

We can reduce a matrix $\boldsymbol{A}$ to REF by performing a series of elementary column operations (e.c.o).

False

Counter-example $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

True or False

Exercise 2 Q45

$R_1, R_2, \ldots, R_n$ are e.r.o. corresponding to some elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_n$.
$C_1, C_2, \ldots, C_n$ are e.c.o. corresponding to the same elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_n$.

If

$$\boldsymbol{A} \xrightarrow{R_1} \xrightarrow{R_2} \cdots \xrightarrow{R_n} \boldsymbol{I}$$

$$\boldsymbol{E}_n \ldots \boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{I}$$

then

$$\boldsymbol{A} \xrightarrow{C_n} \xrightarrow{C_{n-1}} \cdots \xrightarrow{C_1} \boldsymbol{I}$$

$$\boldsymbol{A}\boldsymbol{E}_n \ldots \boldsymbol{E}_2\boldsymbol{E}_1 = \boldsymbol{I}$$

If $\boldsymbol{AB} = \boldsymbol{I}$, then $\boldsymbol{BA} = \boldsymbol{I}$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Determinant

For a 2 x 2 matrix, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

For a 3 x 3 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\boldsymbol{M}_{11}} - a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{\boldsymbol{M}_{12}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\boldsymbol{M}_{13}}$$

cofactor expansion along row 1

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Cofactor expansion

$$\det(\mathbf{A}) = a_{11}A_{11} + a_{12}A_{12} + \ \dots + a_{1n}A_{1n}$$

cofactor expansion along row 1

$A_{ij} = (-1)^{i+j}\det(\boldsymbol{M_{ij}})$ (i, j)-cofactor of $\boldsymbol{A}$

$$\det(\mathbf{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

cofactor expansion along row i

$$\det(\mathbf{A}) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

cofactor expansion along column j

Q&A: log in to PollEv.com/vtpoll

Finding determinant

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

cofactor expansion along column 3

$$\det(\boldsymbol{A}) = 2 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 2\times 1 \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2\times 1(2\times 2 - 2\times 1) = 4$$

cofactor expansion along column 1