National University of Singapore
MA2001 Linear Algebra

MATLAB Worksheet 3
Working with Vector Spaces

Type `format rat`. Throughout the entire worksheet, we will use the rational format to read the entries of matrices.

A. Linear Combinations

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. A vector $\boldsymbol{u} \in \mathbb{R}^n$ is a **linear combination** of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ if there exist numbers $c_1, c_2, \ldots, c_k$ such that

$$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k.$$

If we view each $\boldsymbol{u}_i$ and $\boldsymbol{v}$ as column vectors, and write $\boldsymbol{A} = \begin{pmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_k\end{pmatrix}$. Then the above vector equation represents the linear system $\boldsymbol{Ax} = \boldsymbol{v}$ where the variable matrix $\boldsymbol{x} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$.

So to determine whether a vector $\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$, is equivalent to checking if the system $\boldsymbol{Ax} = \boldsymbol{v}$ is consistent.
Therefore, such linear combination problems can be solved using `rref` command.

For example, let $\boldsymbol{u}_1 = (1, 0, 1, 2, 3)$, $\boldsymbol{u}_2 = (2, 1, -1, 1, 0)$, $\boldsymbol{u}_3 = (1, 1, -2, -1, -3)$ and $\boldsymbol{u}_4 = (1, 2, 3, 1, 1)$. To see whether $\boldsymbol{u} = (2, 0, 0, 1, 0)$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$:

(i) Input $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ and $\boldsymbol{u}$ as column vectors in MATLAB. For example,

```
>> u1 = [1; 0; 1; 2; 3]
u1 =  1
      0
      1
      2
      3
```

(ii) Define the 5×4 matrix $\boldsymbol{A}$ whose columns are $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ and $\boldsymbol{u}_4$:

```
>> A = [u1 u2 u3 u4]
A =  1    2    1    1
     0    1    1    2
     1   -1   -2    3
     2    1   -1    1
     3    0   -3    1
```

(iii) Find the reduced row-echelon form of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{u})$ to check the consistency of $\boldsymbol{Ax} = \boldsymbol{u}$:

```
>>  rref([A u])
ans =  1    0    -1    0    0
       0    1    1     0    0
       0    0    0     1    0
       0    0    0     0    1
       0    0    0     0    0
```

Since the last column of the reduced row-echelon form of $(\boldsymbol{A} \mid \boldsymbol{u})$ is a pivot column, the system $\boldsymbol{Ax} = \boldsymbol{u}$ is inconsistent. Therefore, $\boldsymbol{u}$ is not a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$.

Repeat the same process by replacing $\boldsymbol{u}$ with $\boldsymbol{v} = (3, 5, 12, 6, 9)$.

```
>>  v = [3; 5; 12; 6; 9]
v =  3
     5
     12
     6
     9
>>  rref([A v])
ans =  1    0    -1    0    2
       0    1    1     0    -1
       0    0    0     1    3
       0    0    0     0    0
       0    0    0     0    0
```

Since the last column of the reduced row-echelon form of $(\boldsymbol{A} \mid \boldsymbol{v})$ is a non-pivot column, the system $\boldsymbol{Ax} = \boldsymbol{v}$ is consistent. Therefore, $\boldsymbol{v}$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$.
Moreover, from the pivot columns (1st, 2nd and 4th) together with the last column of the RREF, we can easily write down an explicit linear combination of $\boldsymbol{v}$ in terms of $\boldsymbol{u}_1, \boldsymbol{u}_2$ and $\boldsymbol{u}_4$:

$$\boldsymbol{v} = 2\boldsymbol{u}_1 + (-1)\boldsymbol{u}_2 + 3\boldsymbol{u}_4.$$

You may verify this directly by entering the linear combination:

```
>>  2*u1-u2+3*u4
ans =  3
       5
       12
       6
       9
```

which is exactly the vector $\boldsymbol{v}$.

To check whether a few vectors are linear combination of a fix set of vectors, $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$, we can perform the above process "concurrently". In our example above, we can consider the "double" augmented matrix $(\boldsymbol{A} \mid \boldsymbol{u} \mid \boldsymbol{v})$ and apply `rref`:

```
>>  rref([A u v])
ans =  1    0    -1    0    0    2
       0    1     1    0    0   -1
       0    0     0    1    0    3
       0    0     0    0    1    0
       0    0     0    0    0    0
```

Note that the above RREF is the "combination" of the two earlier RREF's for $\boldsymbol{u}$ and $\boldsymbol{v}$ separately.

B. Linear Spans

Let $S = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ and $T = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l\}$ be subsets of vectors in $\mathbb{R}^n$.
Let $U = \text{span}(S)$ and $V = \text{span}(T)$. Then

(i) $U \subseteq V$ if and only if every vector in S is a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l$.

(ii) $V \subseteq U$ if and only if every vector in T is a linear combination of $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$.

(Refer to Theorem 3.2.10 in the textbook.)

For example, let $U = \text{span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$ and $V = \text{span}\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4\}$, where

$$\boldsymbol{c}_1 = (1, 1, 2, 2, 3), \quad \boldsymbol{c}_2 = (1, 0, 2, 0, 3), \quad \boldsymbol{c}_3 = (1, 1, 1, 1, 1),$$

and

$$\boldsymbol{d}_1 = (3, 2, 5, 3, 7), \quad \boldsymbol{d}_2 = (0, 0, 1, 1, 2), \quad \boldsymbol{d}_3 = (2, 2, 1, 1, 0), \quad \boldsymbol{d}_4 = (1, -1, 3, -1, 5).$$

(i) Input $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ and $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ as column vectors in MATLAB. For example,

```
>>  c1 = [1; 1; 2; 2; 3]
c1 =  1
      1
      2
      2
      3
```

(ii) Form the matrices $\boldsymbol{C} = \begin{pmatrix}\boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3\end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix}\boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4\end{pmatrix}$. For example,

```
>>  C = [c1 c2 c3]
C =  1    1    1
     1    0    1
     2    2    1
     2    0    1
     3    3    1
```

(iii) In order to check whether $V \subseteq U$, we shall check if each $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ is a linear combination of $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$; i.e., if the linear systems $\boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}_i$, $i = 1, 2, 3, 4$, are consistent.

As mentioned previously, we may do this "concurrently" using $(\boldsymbol{C} \mid \boldsymbol{D}) = (\boldsymbol{C} \mid \boldsymbol{d}_1 \mid \boldsymbol{d}_2 \mid \boldsymbol{d}_3 \mid \boldsymbol{d}_4)$:

```
>>  rref([C D])

ans =  1    0    0    1    1    -1    0
       0    1    0    1    0     0    2
       0    0    1    1   -1     3   -1
       0    0    0    0    0     0    0
       0    0    0    0    0     0    0
```

Observe that the last four columns corresponding to $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ are all non-pivot columns. So $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$ are linear combinations of $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$. In fact, observing the entries of the last four columns, we can write down

$$\boldsymbol{d}_1 = \boldsymbol{c}_1 + \boldsymbol{c}_2 + \boldsymbol{c}_3, \quad \boldsymbol{d}_2 = \boldsymbol{c}_1 - \boldsymbol{c}_3, \quad \boldsymbol{d}_3 = -\boldsymbol{c}_1 + 3\boldsymbol{c}_3, \quad \boldsymbol{d}_4 = 2\boldsymbol{c}_2 - \boldsymbol{c}_3.$$

Hence, $V \subseteq U$.

(iv) In order to check whether $U \subseteq V$, we interchange $\boldsymbol{C}$ and $\boldsymbol{D}$. i.e. We shall check if each $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ is a linear combination of $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$. Similarly, consider $(\boldsymbol{D} \mid \boldsymbol{C}) = (\boldsymbol{D} \mid \boldsymbol{c}_1 \mid \boldsymbol{c}_2 \mid \boldsymbol{c}_3)$:

```
>>  rref([D C])

ans =  1    0    0    2       0     1     0
       0    1    0    -9/2    3/2   -2    1/2
       0    0    1    -5/2    1/2   -1    1/2
       0    0    0    0       0     0     0
       0    0    0    0       0     0     0
```

Again, we observe that the last three columns corresponding to $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are all non-pivot columns. So $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are linear combinations of $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{d}_4$. In fact,

$$\boldsymbol{c}_1 = \tfrac{3}{2}\boldsymbol{d}_2 + \tfrac{1}{2}\boldsymbol{d}_3, \quad \boldsymbol{c}_2 = \boldsymbol{d}_1 - 2\boldsymbol{d}_2 - \boldsymbol{d}_3, \quad \boldsymbol{c}_3 = \tfrac{1}{2}\boldsymbol{d}_2 + \tfrac{1}{2}\boldsymbol{d}_3.$$

Hence, $U \subseteq V$. We conclude that $U = V$.

Suppose we use the same U and V except $\boldsymbol{d}_4$ is replaced by $\boldsymbol{e}_4 = (1, -1, 3, -1, 0)$.

(i) Input $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ and $\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{e}_4$ as column vectors. In fact, we just need to define $\boldsymbol{e}_4$:

```
>>  e4 = [1; -1; 3; -1; 0]

e4 =  1
      -1
      3
```

```
    -1
    0
```

(ii) Form the matrices $\boldsymbol{C} = \begin{pmatrix}\boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3\end{pmatrix}$ and $\boldsymbol{E} = \begin{pmatrix}\boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{e}_4\end{pmatrix}$. In fact, we just need to input $\boldsymbol{E}$:

```
>>  E = [d1 d2 d3 e4]
E =  3    0    2    1
     2    0    2    -1
     5    1    1    3
     3    1    1    -1
     7    2    0    0
```

(iii) Check the consistency of $\boldsymbol{Cx} = \boldsymbol{d}_i$, $i = 1, 2, 3$, and $\boldsymbol{Cx} = \boldsymbol{e}_4$:

```
>>  rref([C E])
ans =  1    0    0    1    1    -1    0
       0    1    0    1    0     0    0
       0    0    1    1    -1    3    0
       0    0    0    0    0     0    1
       0    0    0    0    0     0    0
```

Since the column corresponding to $\boldsymbol{e}_4$ is a pivot column, the system $\boldsymbol{Cx} = \boldsymbol{e}_4$ is inconsistent; so $\boldsymbol{e}_4 \notin \operatorname{span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\} = U$. Consequently, $V \not\subseteq U$.

(iv) Check the consistency of $\boldsymbol{Ex} = \boldsymbol{c}_i$, $i = 1, 2, 3$.

```
>>  rref([E C])
ans =  1    0    0    0    0      1     0
       0    1    0    0    3/2    -2    1/2
       0    0    1    0    1/2    -1    1/2
       0    0    0    1    0      0     0
       0    0    0    0    0      0     0
```

Since the columns corresponding to $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are all non-pivot columns, the systems $\boldsymbol{Ex} = \boldsymbol{c}_i$, $i = 1, 2, 3$, are all consistent; so $\boldsymbol{c}_i \in \operatorname{span}\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \boldsymbol{e}_4\} = V$, $i = 1, 2, 3$. Consequently, $U \subseteq V$.

C. Linear Independence

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of vectors in $\mathbb{R}^n$. Then S is said to be **linearly independent** if the linear system $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$ has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$.

View each $\boldsymbol{v}_i$ and $\boldsymbol{0}$ as column vectors, and write $\boldsymbol{B} = \begin{pmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k\end{pmatrix}$. Then S is linearly independent if and only if the homogeneous linear system $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution.

For example, let $\boldsymbol{v}_1 = (1, 0, 2, 0, 3)$, $\boldsymbol{v}_2 = (1, 1, 0, 2, 2)$, $\boldsymbol{v}_3 = (1, -3, 8, -6, 6)$, $\boldsymbol{v}_4 = (1, 2, 3, 4, 1)$, $\boldsymbol{v}_5 = (0, -1, 1, -2, 1)$, $\boldsymbol{v}_6 = (1, 1, 1, 1, 1)$.

(i) Input $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_6$ and $\boldsymbol{w} = (0,0,0,0,0)$ as column vectors in MATLAB. For example,

```
>>  v1 = [1; 0; 2; 0; 3]

v1 =  1

      0

      2

      0

      3
```

(ii) Define the 5×6 matrix $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 & \boldsymbol{v}_5 & \boldsymbol{v}_6 \end{pmatrix}$.

```
>>  B = [v1 v2 v3 v4 v5 v6]

B =  1    1    1    1     0    1

     0    1   -3    2    -1    1

     2    0    8    3     1    1

     0    2   -6    4    -2    1

     3    2    6    1     1    1
```

(iii) Find the reduced row-echelon form of the augmented matrix $(\boldsymbol{B} \mid \boldsymbol{0})$ of the homogeneous linear system $\boldsymbol{Bx} = \boldsymbol{0}$. (Recall that $\boldsymbol{w} = \boldsymbol{0}$ is defined in Step (i).)

```
>>  rref([B w])

ans =  1    0    4     0    4/5    0    0

       0    1   -3     0   -3/5    0    0

       0    0    0     1   -1/5    0    0

       0    0    0     0     0     1    0

       0    0    0     0     0     0    0
```

Since the last column is a non-pivot column and the 3^{rd} and 5^{th} columns are non-pivot columns, the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has infinitely many non-trivial solutions (with 2 arbitrary parameters). As a conclusion, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_6\}$ is a linearly dependent set.

Note that in the above process, the last column of the reduced row-echelon form of $(\boldsymbol{B} \mid \boldsymbol{0})$ is the zero column. In fact, the RREF is $(\boldsymbol{R} \mid \boldsymbol{0})$ where $\boldsymbol{R}$ is the reduced row-echelon form of $\boldsymbol{B}$.
Therefore, to check for linear independence, in step (iii) above, we can drop the last zero column of the augmented matrix and simply use the command `rref(B)` to check whether the reduced row echelon form $\boldsymbol{R}$ of $\boldsymbol{B}$ has non-pivot columns.

D. Bases

Let T be a subset of vectors in $\mathbb{R}^n$. Then T is a **basis** for a vector space V if (i) $V = \text{span}(T)$ and (ii) T is linearly independent.

We shall illustrate using an example $V = \text{span}(S)$ where $S = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4\}$, and

$\boldsymbol{g}_1 = (1,1,1,1,1), \quad \boldsymbol{g}_2 = (1,-1,2,3,0), \quad \boldsymbol{g}_3 = (-1,-3,0,1,-2), \quad \boldsymbol{g}_4 = (0,1,1,-1,-1).$

Note that the spanning set S of V may not be a basis for V. Instead, we want to check whether another set of vectors T is a basis for V, where $T = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3\}$, and

$$\boldsymbol{h}_1 = (2,1,3,4,1), \quad \boldsymbol{h}_2 = (1,0,3,2,-1), \quad \boldsymbol{h}_3 = (1,2,2,0,0).$$

To do that, we shall verify:

(i) T is linearly independent;

(ii) $V \subseteq \text{span}(T)$, which is the same as $\text{span}(S) \subseteq \text{span}(T)$, i.e., every vector in S is a linear combination of vectors in T; and

(iii) $\text{span}(T) \subseteq V$, which is the same as $\text{span}(T) \subseteq \text{span}(S)$, i.e., every vector in T is a linear combination of vectors in S.

To input the relevant information into MATLAB, we create two matrices using the vectors of S and T respectively. For simplicity, we shall name the two corresponding matrices as S and T.

(i) Input $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4$ and $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ as column vectors in MATLAB.

(ii) Define the matrix $\boldsymbol{S} = \begin{pmatrix} \boldsymbol{g}_1 & \boldsymbol{g}_2 & \boldsymbol{g}_3 & \boldsymbol{g}_4 \end{pmatrix}$ and input to MATLAB:

```
>>  S = [g1 g2 g3 g4]
S =  1    1    -1    0
     1   -1    -3    1
     1    2     0    1
     1    3     1   -1
     1    0    -2   -1
```

(iii) Define the matrix $\boldsymbol{T} = \begin{pmatrix} \boldsymbol{h}_1 & \boldsymbol{h}_2 & \boldsymbol{h}_3 \end{pmatrix}$ and input to MATLAB:

```
>>  T = [h1 h2 h3]
T =  2    1    1
     1    0    2
     3    3    2
     4    2    0
     1   -1    0
```

(iv) Find the reduced row-echelon form of $\boldsymbol{T}$:

```
>>  rref(T)

ans =  1    0    0
       0    1    0
       0    0    1
       0    0    0
       0    0    0
```

Since the columns are all pivot columns, $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ are linearly independent.

(v) To check the consistency of $\boldsymbol{Tx} = \boldsymbol{g}_i$, $i = 1, \ldots, 4$, we find the reduced row-echelon form of $(\boldsymbol{T} \mid \boldsymbol{g}_1 \mid \boldsymbol{g}_2 \mid \boldsymbol{g}_3 \mid \boldsymbol{g}_4) = (\boldsymbol{T} \mid \boldsymbol{S})$:

```
>>  rref([T S])

ans =  1    0    0    1/2    1/2    -1/2    -1/2
       0    1    0   -1/2    1/2     3/2     1/2
       0    0    1    1/2   -1/2    -3/2     1/2
       0    0    0     0      0       0       0
       0    0    0     0      0       0       0
```

Since the columns corresponding to $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4$ are all non-pivot, each $\boldsymbol{g}_i$ is a linear combination of $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$. Hence, $V = \text{span}(S) \subseteq \text{span}(T)$.

(vi) To check the consistency of $\boldsymbol{Sx} = \boldsymbol{h}_i$, $i = 1, 2, 3$, we find the reduced row-echelon form of $(\boldsymbol{S} \mid \boldsymbol{h}_1 \mid \boldsymbol{h}_2 \mid \boldsymbol{h}_3) = (\boldsymbol{S} \mid \boldsymbol{T})$:

```
>>  rref([S T])

ans =  1    0   -2    0    1    0    1
       0    1    1    0    1    1    0
       0    0    0    1    0    1    1
       0    0    0    0    0    0    0
       0    0    0    0    0    0    0
```

Since the columns corresponding to $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ are all non-pivot, each $\boldsymbol{h}_i$ is a linear combination of $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4$. Hence, $\text{span}(T) \subseteq \text{span}(S)$.

Therefore, we conclude that T is a basis for V.

E. Practices

Use MATLAB to solve Questions 3.8, 3.9, 3.11, 3.12, 3.26(a), 3.32, 3.33 in the textbook Exercise 3.