

ST2334 (2021/2022 Semester 2) Solutions to Questions in Tutorial 7

Question 1

$f_{(X,Y)}(x, y)$		x		$f_Y(y)$
		2	4	
y	1	0.10	0.15	0.25
	3	0.20	0.30	0.50
	5	0.10	0.15	0.25
$f_X(x)$		0.40	0.60	1

(a) It can be verified directly that

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) \text{ for } x = 2, 4 \text{ and } y = 1, 3, 5.$$

Hence, X and Y are independent

(b)

y	1	3	5
$f_{Y X}(y 2)$	$0.10/0.40 = 1/4$	$0.20/0.40 = 2/4$	$0.10/0.40 = 1/4$

$$E(Y|X = 2) = 1(1/4) + 3(2/4) + 5(1/4) = 3.$$

Alternatively, as X and Y are independent, then $E(Y|X = 2) = E(Y) = 1(0.25) + 3(0.5) + 5(0.25) = 3.$

(c)

x	2	4
$f_{X Y}(x 3)$	$0.20/0.50 = 2/5$	$0.30/0.50 = 3/5$

$$E(X|Y = 3) = 2(2/5) + 4(3/5) = 16/5 = 3.2$$

Alternatively, as X and Y are independent, then $E(X|Y = 3) = E(X) = 2(0.4) + 4(0.6) = 3.2.$

(d) $E(X) = 2(0.40) + 4(0.60) = 3.2$

$$E(Y) = 1(0.25) + 3(0.50) + 5(0.25) = 3$$

$E(2X - 3Y) = 2E(X) - 3E(Y) = 2(3.2) - 3(3) = -2.6.$ We used the results in parts (b) and (c) in the computation.

(e) $E(XY) = (2)(1)(0.10) + (2)(3)(0.10) + (2)(5)(0.10) + (4)(1)(0.10) + (4)(3)(0.10) + (4)(5)(0.10) = 9.6.$

Alternatively, as X and Y are independent, then $E(XY) = E(X)E(Y) = 3.2(3) = 9.6.$

(f) $E(X^2) = 2^2(0.4) + 4^2(0.6) = 11.2.$ Hence, $V(X) = E(X^2) - [E(X)]^2 = 11.2 - (3.2)^2 = 0.96$

Similarly, $E(Y^2) = 1^2(0.25) + 3^2(0.5) + 5^2(0.25) = 11.$ Hence, $V(Y) = E(Y^2) - [E(Y)]^2 = 11 - (3)^2 = 2$

(g) $\sigma_{X,Y} = 0$ as X and Y are independent.

$$\text{Alternatively, } \text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 9.6 - 3.2(3) = 0$$

$\rho_{X,Y} = 0$ as $\sigma_{X,Y} = 0.$ Alternatively, $\rho_{X,Y} = 0$ as X and Y are independent.

Question 2

By the definition of X and Y , Profit = $8X + 3Y - 10$; and the marginal distributions of X and Y are given in the following table.

$f_{(X,Y)}(x,y)$		x			$f_Y(y)$
		0	1	2	
y	0	0.01	0.01	0.03	0.05
	1	0.03	0.08	0.07	0.18
	2	0.03	0.06	0.06	0.15
	3	0.07	0.07	0.13	0.27
	4	0.12	0.04	0.03	0.19
	5	0.08	0.06	0.02	0.16
$f_X(x)$		0.34	0.32	0.34	1

$$E(X) = \sum x f_X(x) = 1. \quad E(X^2) = \sum x^2 f_X(x) = 1.68. \quad V(X) = E(X^2) - [E(X)]^2 = 0.68.$$

$$E(Y) = \sum y f_Y(y) = 2.85. \quad E(Y^2) = \sum y^2 f_Y(y) = 10.25. \quad V(Y) = E(Y^2) - [E(Y)]^2 = 2.1275.$$

$$\sigma_{X,Y} = E(XY) - E(X)E(Y) = (2.47) - (1)(2.85) = -0.38$$

$$\text{Profit} = 8X + 3Y - 10. \quad E(\text{Profit}) = 8E(X) + 3E(Y) - 10 = 6.55$$

$$V(\text{profit}) = V[8X + 3Y - 10] = 8^2 V(X) + 3^2 V(Y) + 2(8)(3) \text{Cov}(X, Y) = 44.4275.$$

Question 3

We first compute the marginal probability density functions of X and Y . They are also needed for parts (b) to (d).

$$f_X(x) = \int_0^1 \frac{2}{3}(x+2y) dy = \frac{2}{3} \left[xy + \frac{2y^2}{2} \right]_0^1 = \frac{2}{3}(x+1), \text{ for } 0 \leq x \leq 1; \text{ and } 0 \text{ otherwise}$$

$$f_Y(y) = \int_0^1 \frac{2}{3}(x+2y) dx = \frac{2}{3} \left[\frac{x^2}{2} + 2xy \right]_0^1 = \frac{2}{3} \left(\frac{1}{2} + 2y \right), \text{ for } 0 \leq y \leq 1; \text{ and } 0 \text{ otherwise,}$$

(a) Since $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, X and Y are dependent.

$$(b) E(X) = \frac{2}{3} \int_0^1 x(x+1) dx = \frac{2}{3} \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_0^1 = \frac{2}{3} \left(\frac{1}{3} + \frac{1}{2} \right) = \frac{5}{9} = 0.55556.$$

$$E(X^2) = \frac{2}{3} \int_0^1 x^2(x+1) dx = \frac{2}{3} \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_0^1 = \frac{2}{3} \left(\frac{1}{4} + \frac{1}{3} \right) = \frac{7}{18} = 0.38889.$$

$$V(X) = E(X^2) - [E(X)]^2 = 13/162 = 0.08024.$$

$$(c) E(Y) = \frac{2}{3} \int_0^1 y \left(\frac{1}{2} + 2y \right) dy = \frac{2}{3} \left[\frac{y^2}{4} + \frac{2y^3}{3} \right]_0^1 = \frac{2}{3} \left(\frac{1}{4} + \frac{2}{3} \right) = \frac{11}{18} = 0.61111.$$

$$E(Y^2) = \frac{2}{3} \int_0^1 y^2 \left(\frac{1}{2} + 2y \right) dy = \frac{2}{3} \left[\frac{y^3}{6} + \frac{y^4}{2} \right]_0^1 = \frac{2}{3} \left(\frac{1}{6} + \frac{1}{2} \right) = \frac{4}{9} = 0.44444.$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = 23/324 = 0.07099.$$

$$(d) E(XY) = \frac{2}{3} \int_0^1 \int_0^1 xy(x+2y) dx dy = \frac{2}{3} \int_0^1 \int_0^1 x^2 y + 2xy^2 dx dy$$

$$= \frac{2}{3} \int_0^1 \left[\frac{x^3 y}{3} + \frac{2x^2 y^2}{2} \right]_0^1 dy = \frac{2}{3} \int_0^1 \left(\frac{y}{3} + y^2 \right) dy = \frac{2}{3} \left[\frac{y^2}{6} + \frac{y^3}{3} \right]_0^1 = \frac{2}{3} \left(\frac{1}{6} + \frac{1}{3} \right) = \frac{1}{3} = 0.33333.$$

$$\text{Hence, } \sigma_{X,Y} = E(XY) - E(X)E(Y) = 1/3 - (5/9)(11/18) = -1/162 = -0.00617.$$

Question 4

$$f_X(x) = \int_0^1 \frac{3}{2}(x^2 + y^2) dy = \frac{3}{2} \left[x^2 y + \frac{y^3}{3} \right]_0^1 = \frac{3}{2} \left(x^2 + \frac{1}{3} \right), \text{ for } 0 \leq x \leq 1; \text{ and } 0 \text{ otherwise.}$$

$$f_Y(y) = \int_0^1 \frac{3}{2}(x^2 + y^2) dx = \frac{3}{2} \left[\frac{x^3}{3} + xy^2 \right]_0^1 = \frac{3}{2} \left(\frac{1}{3} + y^2 \right), \text{ for } 0 \leq y \leq 1; \text{ and } 0 \text{ otherwise.}$$

(a) Since $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, X and Y are dependent

$$(b) E(X) = \frac{3}{2} \int_0^1 x \left(x^2 + \frac{1}{3} \right) dx = \frac{3}{2} \left[\frac{x^4}{4} + \frac{x^2}{6} \right]_0^1 = \frac{3}{2} \left(\frac{1}{4} + \frac{1}{6} \right) = \frac{5}{8} = 0.625.$$

$$E(X^2) = \frac{3}{2} \int_0^1 x^2 \left(x^2 + \frac{1}{3} \right) dx = \frac{3}{2} \left[\frac{x^5}{5} + \frac{x^3}{9} \right]_0^1 = \frac{3}{2} \left(\frac{1}{5} + \frac{1}{9} \right) = \frac{7}{15} = 0.46667.$$

$$V(X) = E(X^2) - [E(X)]^2 = 73/960 = 0.07604.$$

(c) Method 1: Repeat the work in part (b)

$$E(Y) = \frac{3}{2} \int_0^1 y \left(y^2 + \frac{1}{3} \right) dy = \frac{3}{2} \left[\frac{y^4}{4} + \frac{y^2}{6} \right]_0^1 = \frac{3}{2} \left(\frac{1}{4} + \frac{1}{6} \right) = \frac{5}{8} = 0.625.$$

$$E(Y^2) = \frac{3}{2} \int_0^1 y^2 \left(y^2 + \frac{1}{3} \right) dy = \frac{3}{2} \left[\frac{y^5}{5} + \frac{y^3}{9} \right]_0^1 = \frac{3}{2} \left(\frac{1}{5} + \frac{1}{9} \right) = \frac{7}{15} = 0.46667.$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = 73/960 = 0.07604.$$

Method 2: Observe that the joint p.d.f. is symmetric in x and y (i.e. $f_{(X,Y)}(x,y) = f_{(X,Y)}(y,x)$), so $E(Y) = E(X) = 5/8 = 0.625$ and $V(Y) = V(X) = 73/960 = 0.07604$.

$$(d) E(XY) = \frac{3}{2} \int_0^1 \int_0^1 xy(x^2 + y^2) dx dy = \frac{3}{2} \int_0^1 \int_0^1 x^3 y + xy^3 dx dy$$

$$= \frac{3}{2} \int_0^1 \left[\frac{x^4 y}{4} + \frac{x^2 y^3}{2} \right]_0^1 dy = \frac{3}{2} \int_0^1 \left(\frac{y}{4} + \frac{y^3}{2} \right) dy = \frac{3}{2} \left[\frac{y^2}{8} + \frac{y^4}{8} \right]_0^1 = \frac{3}{2} \left(\frac{1}{8} + \frac{1}{8} \right) = \frac{3}{8} = 0.375.$$

$$\sigma_{X,Y} = E(XY) - E(X)E(Y) = (3/8) - (5/8)(5/8) = -1/64 = -0.01563.$$

$$(e) E(X + Y) = E(X) + E(Y) = 5/8 + 5/8 = 5/4 = 1.25.$$

$$V(X + Y) = V(X) + V(Y) + 2(\sigma_{X,Y}) = \frac{73}{960} + \frac{73}{960} + 2 \left(\frac{-1}{64} \right) = \frac{29}{240} = 0.12083.$$

Question 5

$$(a) f_X(x) = \int_0^1 (x + y) dy = \left[xy + \frac{y^2}{2} \right]_0^1 = x + \frac{1}{2}, \text{ for } 0 \leq x \leq 1; \text{ and } f_X(x) = 0 \text{ otherwise.}$$

$$E(X) = \int_0^1 x \left(x + \frac{1}{2} \right) dx = \left[\frac{x^3}{3} + \frac{x^2}{4} \right]_0^1 = \left(\frac{1}{3} + \frac{1}{4} \right) = \frac{7}{12} = 0.58333.$$

$$f_Y(y) = \int_0^1 (x + y) dx = \left[\frac{x^2}{2} + xy \right]_0^1 = y + \frac{1}{2}, \text{ for } 0 \leq y \leq 1; \text{ and } f_Y(y) = 0 \text{ otherwise.}$$

$$E(Y) = \int_0^1 y \left(y + \frac{1}{2} \right) dy = \left[\frac{y^3}{3} + \frac{y^2}{4} \right]_0^1 = \left(\frac{1}{3} + \frac{1}{4} \right) = \frac{7}{12} = 0.58333.$$

$$E(XY) = \int_0^1 \int_0^1 xy(x + y) dx dy = \int_0^1 \int_0^1 x^2 y + xy^2 dx dy = \int_0^1 \left[\frac{x^3 y}{3} + \frac{x^2 y^2}{2} \right]_0^1 dy$$

$$= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2} \right) dy = \left[\frac{y^2}{6} + \frac{y^3}{6} \right]_0^1 = \frac{1}{3} = 0.33333.$$

$$\sigma_{X,Y} = E(XY) - E(X)E(Y) = 1/3 - (7/12)(7/12) = -1/144 = -0.00694.$$

(b) To compute $E(Y|X = 0.2)$, we need to compute $f_{Y|X}(y|0.2)$. We shall restrict ourselves to $0 \leq y \leq 1$ because the conditional probability density function is 0 for $y \notin [0,1]$. We have

$$f_{Y|X}(y|x = 0.2) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x+y}{x+1/2} = \frac{0.2+y}{0.7} = \frac{2+10y}{7}, \text{ for } 0 \leq y \leq 1; \text{ and } f_{Y|X}(y|x = 0.2) = 0 \text{ otherwise.}$$

$$E(Y|X = 0.2) = \int_0^1 y \left(\frac{2+10y}{7} \right) dy = \frac{1}{7} \left[y^2 + \frac{10y^3}{3} \right]_0^1 = \frac{1}{7} \left(1 + \frac{10}{3} \right) = \frac{13}{21} = 0.61905.$$

(c) Similarly, $f_{X|Y}(x|y = 0.5) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{x+y}{y+1/2} = \frac{0.5+x}{1} = x + \frac{1}{2}$, for $0 \leq x \leq 1$; and $f_{X|Y}(x|y = 0.5) = 0$ otherwise.

$$E(X|Y = 0.5) = \int_0^1 x \left(x + \frac{1}{2} \right) dx = \left[\frac{x^3}{3} + \frac{x^2}{4} \right]_0^1 = \left(\frac{1}{3} + \frac{1}{4} \right) = \frac{7}{12} = 0.58333.$$

Question 6

$$V(Z) = V(-2X + 4Y) = (-2)^2 V(X) + 4^2 V(Y) + 2(-2)(4) \text{Cov}(X, Y)$$

(a) If X and Y are independent, then $\text{Cov}(X, Y) = 0$, hence,

$$V(Z) = V[-2X + 4Y - 3] = (-2)^2 V(X) + (4)^2 V(Y) = 4(5) + 16(3) = 68.$$

(b) If $\text{Cov}(X, Y) = 1$, then

$$\begin{aligned} V(Z) &= V[-2X + 4Y - 3] = (-2)^2 V(X) + (4)^2 V(Y) + 2(-2)(4) \text{Cov}(X, Y) \\ &= 4(5) + 16(3) + 2(-2)(4)(1) = 52. \end{aligned}$$

(c) From part (b), $\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{1}{\sqrt{5}\sqrt{3}} = 0.2582$.

Question 7

$X \sim \text{discrete uniform}$

(a) You can give your answer either in a table

x	1	2	3	4	5	6	7	8	9	10
$f_X(x)$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

or

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{10}, & x = 1, 2, \dots, 10; \\ 0, & \text{otherwise.} \end{cases}$$

(b) $\Pr(X < 4) = \sum_{x=1}^3 f(x) = \frac{3}{10} = 0.3$

(c) $\mu = \sum_{x=1}^{10} x \left(\frac{1}{10}\right) = 5.5. \sigma^2 = \sum_{x=1}^{10} (x - 5.5)^2 \frac{1}{10} = 8.25$

Alternatively, $E(X^2) = \sum_{x=1}^{10} x^2 \left(\frac{1}{10}\right) = 38.5. V(X) = 38.5 - 5.5^2 = 8.25.$