ST2334 (2021/2022 Semester 2) Solutions to Questions in Tutorial 11

Question 1

X = lifetime. $X \sim N(\mu, 40^2)$

(a) Test $H_0: \mu = 800$ against $H_1: \mu \neq 800$

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{788 - 800}{40/\sqrt{30}} = -1.64$$

Since $|z_{obs}| = 1.64 < z_{0.025}$ $(= 1.96)$, therefore we do not reject H$_0$.
Alternatively, p-value $= 2\min\{\Pr(Z < -1.64), \Pr(Z > -1.64)\} = 2(0.0505) = 0.1010$. Since p-value $> \alpha\ (= 0.05)$, we do not reject H_0.

(b) 95% confidence interval for μ: $\bar{X} \pm z_{0.025}\frac{\sigma}{\sqrt{n}} = 788 \pm 1.96\frac{40}{\sqrt{30}} = (773.69, 802.31)$.
Yes, 800 is plausible.

(c) Under H$_0$, H$_0$ is not rejected if $-1.96 < Z < 1.96$ or $\mu - 1.96\frac{\sigma}{\sqrt{n}} < \bar{X} < \mu + 1.96\frac{\sigma}{\sqrt{n}}$ or $785.69 < \bar{X} < 814.31$.
When $\mu = 790$ (i.e. H$_0$ is false), $\bar{X} \sim N\left(790, \frac{40^2}{30}\right)$.
$\Pr(\text{Do not reject H}_0|\mu = 790) = \Pr(785.69 < \bar{X} < 814.31 \mid \mu = 790) = \Pr\left(\frac{785.69-790}{40/\sqrt{30}} < \frac{\bar{X}-790}{40/\sqrt{30}} < \frac{814.31-790}{40/\sqrt{30}}\right) = \Pr(-0.590 < Z < 3.329) = 0.9996 - 0.2774 = 0.7222$.

(d) When $\mu = 790$, Power $= 1 - \Pr(\text{Type II error } |\mu = 790) = 1 - 0.7222 = 0.2778$.

Question 2

X = content of lubricant. $X \sim N(\mu, \sigma^2)$

(a) Test H_0: $\mu = 10$ against H_1: $\mu \neq 10$
From the data, $\bar{x} = 10.06$, $s = 0.24585$. Hence,

$$t_{obs} = \frac{\bar{x} - 10}{s/\sqrt{10}} = \frac{10.06 - 10}{0.246/\sqrt{10}} = 0.772.$$

Since $|t_{obs}| = 0.772 < t_{9;0.005}(= 3.25)$, therefore we do not reject H$_0$
Alternatively, p-value $= 2\min\{\Pr(T < 0.772), \Pr(T > 0.772)\} = 0.4600$ (from statistical software). Since p-value > $\alpha(= 0.01)$, therefore we do not reject H$_0$.

(b) Test H_0: $\sigma^2 = 0.03$ against H_1: $\sigma^2 \neq 0.03$

$$\chi^2_{obs} = \frac{(n-1)s^2}{\sigma^2} = \frac{(9)(0.246)^2}{0.03} = 18.13$$

which falls between $\chi^2_{9;0.975}$ $(= 2.70)$ and $\chi^2_{9;0.025}(= 19.023)$. Hence, we do not reject H$_0$.
p-value $= 2\min\{Pr(\chi^2_9 > 18.13), Pr(\chi^2_9 < 18.13)\} = 0.0673$ from statistical software. Since the p-value > 0.05. We do not reject H$_0$.

(c) 99% confidence interval for σ^2 is given by

$$\left(\frac{(n-1)s^2}{{\chi^2}_{9,0.025}}, \frac{(n-1)s^2}{{\chi^2}_{9,0.975}}\right) = \left(\frac{9(0.246)^2}{19.023}, \frac{9(0.246)^2}{2.7}\right) = (0.0286, 0.2014)$$

Note: The constant, ${\chi^2}_{9,0.025}$, satisfies $\Pr\left(W > {\chi^2}_{9,0.025}\right) = 0.025$ with $W \sim \chi^2(9)$.

Question 3

X = amount of soft drink dispensed. $X \sim N(\mu, \sigma^2)$.
Test H_0: $\sigma^2 = 1.15$ against H_1: $\sigma^2 > 1.15$
From the data, we have $n = 25$, $s^2 = 2.03$. Hence

$$\chi^2_{obs} = \frac{(n-1)s^2}{\sigma^2} = \frac{(24)(2.03)}{1.15} = 42.37$$

Since the observed test statistic > $\chi^2_{24;0.05}$ $(= 36.415)$, we reject H$_0$ at 5% significance level.
Alternatively, p-value is between 0.01 and 0.025 as $\Pr(\chi^2_{24} > 39.364) = 0.025$ and $\Pr(\chi^2_{24} > 42.98) = 0.01$ [Exact p-value = 0.0117]

Question 4

X_A = tensile strength of thread A. $E(X_A) = \mu_A$ and $V(X_A) = 6.28^2$
X_B = tensile strength of thread B. $E(X_B) = \mu_B$ and $V(X_B) = 5.61^2$

(a) Test H$_0$: $\mu_A - \mu_B = 12$ against H$_1$: $\mu_A - \mu_B > 12$

Let $Z = \frac{(\bar{X}_A - \bar{X}_B) - (\mu_A - \mu_B)}{\sqrt{\sigma_A^2/50 + \sigma_B^2/50}}$. $Z\ approx \sim N(0,1)$ by CLT since both n_A and n_B are large.

From the data, we have $n_A = 50$, $\bar{x}_A = 86.7, n_B = 50$, $\bar{x}_B = 77.8$. Hence

$$z_{obs} = \frac{(86.7 - 77.8) - (12)}{\sqrt{\frac{6.28^2}{50} + \frac{5.61^2}{50}}} = -2.603$$

Since $z_{obs} < z_{0.05}$ $(= 1.645)$, we do not reject H$_0$.
Alternatively, p-value $= \Pr(Z > -2.603) = 1 - 0.0047 = 0.9954$.
Since p-value $> \alpha$ $(= 0.05)$. We do not reject H$_0$.

(b) We committed an error if our decision of not rejecting H$_0$ is wrong. Hence it is Type II error. (Type I error is committed if our decision of rejecting H$_0$ is wrong.)

Question 5

X_A = grades of students in the 3-semester-hour course $\sim N(\mu_A, \sigma^2)$
X_B = grades of students in the 4-semester-hour course $\sim N(\mu_B, \sigma^2)$
From the data, $n_A = 18$, $\bar{x}_A = 77$, $s_A = 6$; $n_B = 12$, $\bar{x}_B = 84$, $s_B = 4$. Hence,

$$s_p = \sqrt{\frac{(n_A - 1){s_A}^2 + (n_B - 1){s_B}^2}{n_A + n_B - 2}} = 5.3050$$

(a) 99% confidence interval for $\mu_B - \mu_A$ is given by

$(\bar{X}_B - \bar{X}_A) \pm t_{28,0.005} s_p \sqrt{\frac{1}{n_B} + \frac{1}{n_A}} = (84 - 77) \pm (2.763)(5.304)\sqrt{\frac{1}{12} + \frac{1}{18}} =$ $(1.537, 12.463)$.

Or 99% confidence interval for $\mu_B - \mu_A$ is given by

$(\bar{X}_A - \bar{X}_B) \pm t_{28,0.005} s_p \sqrt{\frac{1}{n_A} + \frac{1}{n_B}} = (77 - 84) \pm (2.763)(5.304)\sqrt{\frac{1}{18} + \frac{1}{12}} =$ $(-12.463, -1.537)$.

(b) Test H_0: $\mu_A - \mu_B = 0$ against H_1: $\mu_A - \mu_B > 0$

$$t_{obs} = \frac{\bar{x}_A - \bar{x}_B}{s_p\sqrt{\frac{1}{n_B} + \frac{1}{n_A}}} = \frac{77 - 84}{(5.304)\sqrt{\frac{1}{12} + \frac{1}{18}}} = -3.541$$

Since $t_{obs} = -3.541 < t_{28;0.05}(= 1.701)$, therefore, we do not reject H$_0$.
[Note: Exact p-value= $\Pr(T > -3.541) = 0.9993$ (from statistical software)]

Question 6

X_R = gasoline consumption by radial tires
X_B = gasoline consumption by belted tires

$d = X_R - X_B$. $d \sim N(\mu_d, \sigma_d^2)$
From the data, $n_d = 12$, $\bar{x}_d = 0.1417$, $s_d = 0.1975$

(a) 95% confidence interval for μ_d is given by

$$\bar{x}_d \pm t_{11,0.025}\frac{s_d}{\sqrt{n_d}} = 0.1417 \pm 2.201\frac{0.1975}{\sqrt{12}} = (0.0162, 0.2672)$$

(b) Test H_0: $\mu_d = 0$ against H_1: $\mu_d > 0$

$$t_{obs} = \frac{\bar{x}_d}{s_d/\sqrt{n}} = \frac{0.14167}{0.1975/\sqrt{12}} = 2.485$$

Since $t_{obs} > t_{11;\,0.05}$ $(= 1.796)$. Reject H0
Alternatively, p-value $= \Pr(T > 2.485) = 0.01515 < 0.05$. Reject H0.

Question 7

X_M= the length of time taken to assemble a product by men $\sim N(\mu_M, \sigma_M^2)$
X_W= the length of time taken to assemble a product by women $\sim N(\mu_W, \sigma_W^2)$
Test H_0: $\sigma_M^2 = \sigma_W^2$ against H_1: $\sigma_M^2 > \sigma_W^2$
From the data, $n_M = 11$, $s_M = 6.1$, $n_W = 14$, $s_W = 5.3$. Hence,

$$F_{obs} = \frac{{s_M}^2}{{s_W}^2} = \frac{6.1^2}{5.3^2} = 1.325.$$

Since $F_{obs} = 1.325 < F_{10,13;0.05} (= 2.67)$, therefore, we do not reject H0.
[Note: Exact p-value= $\Pr(F > 1.325) = 0.3117$ (from statistical software)]
At $\alpha = 0.05$, we do not have enough evidence to conclude that the variance of the times for women is less than that for men.

Question 8

X_1= the running times of film produced by company 1. Assume $X_1 \sim N(\mu_1, \sigma_1^2)$
X_2= the running times of film produced by company 2. Assume $X_2 \sim N(\mu_2, \sigma_2^2)$

(a) Test H_0: $\sigma_1^2 = \sigma_2^2$ against H_1: $\sigma_1^2 \neq \sigma_2^2$
From the data, $n_1 = 5$, ${s_1}^2 = 78.8$, $n_2 = 7$, ${s_2}^2 = 913.3333$. Hence,

$$F_{obs} = \frac{s_1^2}{s_2^2} = \frac{78.8}{913.3333} = 0.0863$$

Since $F_{obs} < F_{4,6;0.975}$ $(= 1/F_{6,4;0.025} = 1/9.20 = 0.1087)$. Reject H0.
Alternatively, p-value $= 2\min\{\Pr(F < 0.0863), \Pr(F > 0.0863)\} =$
$2\min\{0.01649,\ 0.9835\} = 2(0.01649) = 0.03298 < 0.05$. Reject H0.

(b) 95% confidence interval for $\frac{{\sigma_1}^2}{{\sigma_2}^2}$

$$\left(\frac{{s_1}^2}{{s_2}^2}\frac{1}{F_{4,6,0.025}}, \frac{{s_1}^2}{{s_2}^2}F_{6,4,0.025}\right) = \left(\frac{78.8}{913.33}\frac{1}{6.23}, \frac{78.8}{913.33}(9.20)\right) = (0.01385, 0.79375)$$

(c) 95% confidence interval for $\frac{\sigma_1}{\sigma_2}$ is given by

$$\left(\sqrt{0.01385}, \sqrt{0.79375}\right) = (0.1177, 0.8909)$$

Question 9

We have $E(W) = E(a_1X_1 + \cdots + a_nX_n) = a_1E(X_1) + \cdots + a_nE(X_n) = a_1\mu_1 + \cdots + a_n\,\mu_n$.
Also recall variance of sum of independent random variables is the sum of their variances. Therefore, $V(W) = V(a_1X_1 + \cdots + a_nX_n) = V(a_1X_1) + \cdots + V(a_nX_n) = a_1^2\sigma_1^2 + \cdots + a_n^2\sigma_n^2$.