National University of Singapore
Department of Mathematics

MA2101 Linear Algebra II — Tutorial 3 $\supset$ **HW2**

In this Tutorial, you may use the **fact** *from linear algebra 1: n column vectors are L.I. in F_c^n iff the matrix A they form has $|A| \neq 0$.*

1a. Given vectors $\mathbf{v}_1 = (1,0,0)$ and $\mathbf{v}_2 = (0,1,0)$ in $\mathbf{R}^3$, find a vector $\mathbf{v}_3$ such that the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis of $\mathbf{R}^3$.

1b. In general, given Linearly Independent vectors $\mathbf{u}_1 = (a_1, b_1, c_1)$ and $\mathbf{u}_2 = (a_2, b_2, c_2)$, how to find *all* $\mathbf{u}_3 = (x, y, z)$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of $\mathbf{R}^3$?

2. Consider vectors $\mathbf{v}_1 = (2, -1, 0, 3)$, $\mathbf{v}_2 = (1, 2, 5, -1)$, $\mathbf{v}_3 = (7, -1, 5, 8)$ in F^4. Show that they are Linearly Dependent, and express one of them as a linear combination of the others.

[Suggested **Answers:** $\mathbf{v}_3 = 3\mathbf{v}_1 + \mathbf{v}_2$.]

3. Determine whether or not the subsets of $\mathbf{R}[x]$ below are Linearly Dependent:

3a. $\{\mathbf{u}_1 = 2 - x + 4x^2,\ \mathbf{u}_2 = 3 + 6x + 2x^2,\ \mathbf{u}_3 = 2 + 10x - 4x^2\}$.

3b. $\{\mathbf{v}_1 = 1 + 3x + 3x^2,\ \mathbf{v}_2 = x + 4x^2,\ \mathbf{v}_3 = 5 + 6x + 3x^2,$
$\mathbf{v}_4 = 7 + 2x - x^2\}$.

[Suggested **Answer:**. N, Y]

4. Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, $\mathbf{v}_3 = (3, 3, 4)$ form a basis of $\mathbf{R}^3$.

5. Find all $\lambda \in \mathbf{R}$ such that the vectors below are Linearly Dependent vectors in $\mathbf{R}^3$:

$$\mathbf{v}_1 = (\lambda, -\frac{1}{2}, -\frac{1}{2}), \mathbf{v}_2 = (-\frac{1}{2}, \lambda, -\frac{1}{2}), \mathbf{v}_3 = (-\frac{1}{2}, -\frac{1}{2}, \lambda).$$

[**Ans**. $\lambda = -\frac{1}{2}$ or 1.]

6. Let V be a (not necessarily finite-dimensional) vector space over a field F.

6a. Suppose that B is a basis of V. Decompose it as a disjoint union

$$B = B_1 \coprod \cdots \coprod B_s$$

of non-empty sets B_i. Show that B_i is a basis of $W_i := \mathrm{Span}(B_i)$ and

$$V = W_1 \oplus \cdots \oplus W_s$$

is a direct sum of nonzero vector subspaces W_i of V.

6b. Conversely, suppose that $V = W_1 \oplus \cdots \oplus W_s$ is a direct sum of nonzero vector subspaces W_i of V. Let B_i be a basis of W_i. Show that

$$B = B_1 \coprod \cdots \coprod B_s$$

is a basis of V and a disjoint union of non-empty sets B_i.

7. Suppose that

$$f_i \in C^{(n-1)}[x] \quad (i = 1, \ldots, n)$$

i.e., f_i can be differentiated $(n-1)$-times. The determinant below is called the **Wronskian** of f_i's:

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

7a. Show that if $W(x)$ is not identically zero on $(-\infty, +\infty)$ then $f_1, \ldots, f_n$ are Linearly Independent as vectors in the vector space $C^0[x]$ of all real-valued continuous functions.

7b. Hence show that $1, e^x, e^{2x}$ are Linearly Independent in $C^0[x]$.

Hint. If there is a relation $a_1 f_1 + \cdots + a_n f_n = 0$, defferentiate it $n-1$ times to get $n-1$ more linear equations and the coefficient matrix of these n equations is just the Wronskian of f.

8. (**Extra**) Suppose that V is a finite-dimensional vector space and W is a subspace of V. Find a subspace U of V such that $V = W \oplus U$.

Homework assignment 2(/5). Pls submit your solutions of Questions 1 and 6 to Canvas folders/Assignments/HW2 by Monday 11:59pm, 5th Sep. Late submission will not be accepted.