MA2001

LIVE LECTURE 10

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Topics for week 10

5.3 Best Approximation

5.4 Orthogonal Matrices

6.1 Eigenvalues and Eigenvectors

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Least Squares Solution

$\boldsymbol{u}$ is the "best approximated" solution to an inconsistent $\boldsymbol{Ax} = \boldsymbol{b}$

$\boldsymbol{u}$ is the least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$

$\boldsymbol{u}$ is a solution of $\boxed{\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}}$ always consistent

$\boldsymbol{u}$ is a solution of $\boxed{\boldsymbol{Ax} = \boldsymbol{p}}$, the projection of $\boldsymbol{b}$ onto column space of $\boldsymbol{A}$

always consistent

We can always find least squares solution to any $\boldsymbol{Ax} = \boldsymbol{b}$

Least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ may be unique or infinite

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Find projection using least squares solution

Find projection of **w** onto a subspace V = span{$\boldsymbol{v}_1$, $\boldsymbol{v}_2$, …, $\boldsymbol{v}_n$}

i. Form matrix $\boldsymbol{A}$ using $\boxed{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n \text{ as columns}}$.

ii. Find the least squares solution of $\boxed{\boldsymbol{Ax} = \boldsymbol{w}}$.

iii. Find the solutions of $\boxed{\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{w}}$.

iv. Take any solution $\boldsymbol{u}$ in (iii).

v. $\boxed{\boldsymbol{Au}}$ gives the projection of $\boldsymbol{w}$ onto V.

Column space of $\boldsymbol{A}$ = V

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Orthogonal Matrices

$\boldsymbol{A}$ is n x n matrix

- $\boldsymbol{A}$ is an orthogonal matrix
- $\boldsymbol{A}^{-1} = \boldsymbol{A}^T$
- $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$

$\boldsymbol{A}^T$ is also orthogonal matrix

- The rows of $\boldsymbol{A}$ form an orthonormal basis for $\mathbf{R}^n$.
- The columns of $\boldsymbol{A}$ form an orthonormal basis for $\mathbf{R}^n$.

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} \qquad \boldsymbol{A}^T = (\boldsymbol{a}_1{}^T \quad \cdots \quad \boldsymbol{a}_n{}^T)$$

$$\boldsymbol{A}\boldsymbol{A}^T = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{a}_1{}^T & \boldsymbol{a}_1\boldsymbol{a}_2{}^T & \boldsymbol{a}_1\boldsymbol{a}_3{}^T \\ \boldsymbol{a}_2\boldsymbol{a}_1{}^T & \boldsymbol{a}_2\boldsymbol{a}_2{}^T & \boldsymbol{a}_2\boldsymbol{a}_3{}^T \\ \boldsymbol{a}_3\boldsymbol{a}_1{}^T & \boldsymbol{a}_3\boldsymbol{a}_2{}^T & \boldsymbol{a}_3\boldsymbol{a}_3{}^T \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{a}_1 & \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 & \boldsymbol{a}_1 \cdot \boldsymbol{a}_3 \\ \boldsymbol{a}_2 \cdot \boldsymbol{a}_1 & \boldsymbol{a}_2 \cdot \boldsymbol{a}_2 & \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 \\ \boldsymbol{a}_3 \cdot \boldsymbol{a}_1 & \boldsymbol{a}_3 \cdot \boldsymbol{a}_2 & \boldsymbol{a}_3 \cdot \boldsymbol{a}_3 \end{pmatrix}$$

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True or False

$\boldsymbol{A} = (\boldsymbol{c}_1\ \boldsymbol{c}_2\ \ldots\ \boldsymbol{c}_k)$ is an $n \times k$ matrix such that

the columns $\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_k\}$ of $\boldsymbol{A}$ form an orthonormal set.

Can we conclude that

(I) $\boldsymbol{A}^T\boldsymbol{A}$ = identity matrix and (II) $\boldsymbol{A}\boldsymbol{A}^T$ = identity matrix?

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} \boldsymbol{c}_1^T \\ \vdots \\ \boldsymbol{c}_k^T \end{pmatrix} (\boldsymbol{c}_1 \quad \cdots \quad \boldsymbol{c}_k) = \begin{pmatrix} \boldsymbol{c}_1^T\boldsymbol{c}_1 & \cdots & \boldsymbol{c}_1^T\boldsymbol{c}_k \\ \vdots & & \vdots \\ \boldsymbol{c}_k^T\boldsymbol{c}_1 & \cdots & \boldsymbol{c}_k^T\boldsymbol{c}_k \end{pmatrix}$$

True

$$\boldsymbol{A}\boldsymbol{A}^T = \begin{pmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_n \end{pmatrix} (\boldsymbol{r}_1^T \quad \cdots \quad \boldsymbol{r}_n^T) = \begin{pmatrix} \boldsymbol{r}_1\boldsymbol{r}_1^T & \cdots & \boldsymbol{r}_1\boldsymbol{r}_n^T \\ \vdots & & \vdots \\ \boldsymbol{r}_n\boldsymbol{r}_1^T & \cdots & \boldsymbol{r}_n\boldsymbol{r}_n^T \end{pmatrix}$$

False

Exercise 5 Q32 (Tutorial)

A be an orthogonal matrix:

a) $||\boldsymbol{u}|| = ||\boldsymbol{Au}||$;

b) $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{Au}, \boldsymbol{Av})$;

c) the angle between ***u*** and ***v*** is equal to the angle between ***Au*** and ***Av***.

Orthogonal matrix
preserves
norm, distance, angles

- Express each part using dot product
- Regard the dot product as matrix multiplication
- Use the fact that ***A*** is orthogonal

u and ***v*** regarded as column matrices

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

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Orthogonal matrix preserves
- Basis
- Orthogonal basis
- Orthonormal basis

Exercise 5 Q33 (Tutorial)

$\boldsymbol{A}$ be an orthogonal matrix and let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathbf{R}^n$.

(a) Show that $T = \{\boldsymbol{A}\boldsymbol{u}_1, \boldsymbol{A}\boldsymbol{u}_2, \ldots, \boldsymbol{A}\boldsymbol{u}_n\}$ is a basis for $\mathbf{R}^n$.

- Refer Ex 3.30(b)

(b) If S is orthogonal, show that T is orthogonal.

- Show $\boldsymbol{A}\boldsymbol{u}_i \cdot \boldsymbol{A}\boldsymbol{u}_j = 0$ for any $i \neq j$ using matrix multiplication

(c) If S is orthonormal, is T orthonormal?

- Refer to Q32

Transition matrix between orthonormal bases

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be two bases for $\mathbf{R}^n$

Transition matrix from S to T: $\boldsymbol{P} = (\,[\boldsymbol{u}_1]_T \ \ [\boldsymbol{u}_2]_T \cdots [\boldsymbol{u}_n]_T\,)$

S is an orthonormal basis for $\mathbf{R}^n$
T is the standard basis for $\mathbf{R}^n$

$$\boldsymbol{P} = (\,\boldsymbol{u}_1 \ \ \boldsymbol{u}_2 \cdots \boldsymbol{u}_n\,)$$

S is the standard basis for $\mathbf{R}^n$
T is an orthonormal basis for $\mathbf{R}^n$

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{pmatrix}$$

S is an orthonormal basis for $\mathbf{R}^n$
T is an orthonormal basis for $\mathbf{R}^n$

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{u}_1 \cdot \boldsymbol{v}_1 & \boldsymbol{u}_2 \cdot \boldsymbol{v}_1 & \cdots & \boldsymbol{u}_k \cdot \boldsymbol{v}_1 \\ \boldsymbol{u}_1 \cdot \boldsymbol{v}_2 & \boldsymbol{u}_2 \cdot \boldsymbol{v}_2 & \cdots & \boldsymbol{u}_k \cdot \boldsymbol{v}_2 \\ \vdots & \vdots & & \vdots \\ \boldsymbol{u}_1 \cdot \boldsymbol{v}_k & \boldsymbol{u}_2 \cdot \boldsymbol{v}_k & \cdots & \boldsymbol{u}_k \cdot \boldsymbol{v}_k \end{pmatrix}$$

$\boldsymbol{P}$ is an orthogonal matrix

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True or False

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$

two orthonormal bases for $\mathbf{R}^n$

$\boldsymbol{A} = (\boldsymbol{u_1}\ \boldsymbol{u_2} \ldots \boldsymbol{u_n})$ and $\boldsymbol{B} = (\boldsymbol{v_1}\ \boldsymbol{v_2} \ldots \boldsymbol{v_n})$ orthogonal matrices

$\boldsymbol{P}$ the transition matrix from S to T orthogonal matrices

True or false: $\boldsymbol{PA} = \boldsymbol{B}$

Recall: $\boldsymbol{P}[\boldsymbol{w}]_S = [\boldsymbol{w}]_T$

Correct relation: $\boldsymbol{A} = \boldsymbol{BP}$

$(\boldsymbol{B} \mid \boldsymbol{A}) \rightarrow (\boldsymbol{I} \mid \boldsymbol{P})$

Pre-multiply by $\boldsymbol{B}^{-1}$

$(\boldsymbol{B}^{-1}\boldsymbol{B} \mid \boldsymbol{B}^{-1}\boldsymbol{A})$

HW3 Q5(iv): $\boldsymbol{BC} = \boldsymbol{A}$

$\boldsymbol{A}$, $\boldsymbol{B}$ are not square matrices

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Orthonormal matrix ?

What is the difference between
orthogonal matrix and orthonormal matrix ?

The rows/columns of ***A*** form an orthonormal set

No such thing as orthonormal matrix !

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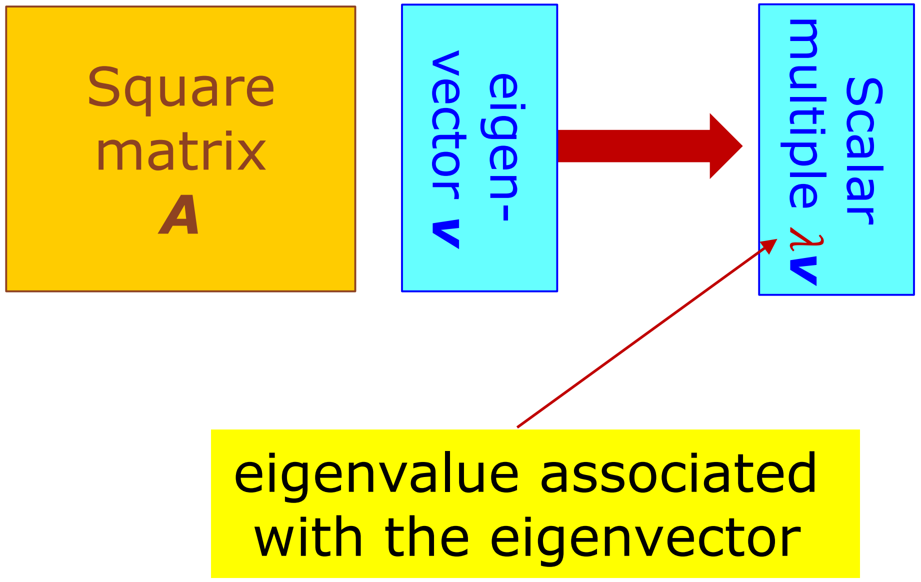
Eigenvalue and Eigenvector

Visualization



May not give the complete set of eigenvalues

Finding eigenvalue (given eigenvector)

$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ Are these eigenvectors of $\boldsymbol{A}$: $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$?

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector

with eigenvalue 1

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector

with eigenvalue 2

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is not an eigenvector

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector

with eigenvalue 0

n × n matrix has n eigenvalues, counting multiplicities

Finding eigenvalues (without eigenvector)

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

So the eigenvalues of ***A*** are 1 (repeated) and -1 (repeated).

Characteristic polynomial:

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{vmatrix}$$

$$= \lambda \times \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda & 0 \\ -1 & 0 & \lambda \end{vmatrix} - 1 \times \begin{vmatrix} 0 & \lambda & -1 \\ -1 & 0 & 0 \\ 0 & -1 & \lambda \end{vmatrix}$$

$$= \lambda(\lambda^3 - \lambda) - (1)(\lambda^2 - 1)$$

$$= \lambda^2(\lambda^2 - 1) - (1)(\lambda^2 - 1)$$

$$= (\lambda^2 - 1)(\lambda^2 - 1)$$

Multiplicities of the eigenvalues

$$= (\lambda - 1)(\lambda + 1)(\lambda - 1)(\lambda + 1) = (\lambda - 1)^2(\lambda + 1)^2$$

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Different perspectives of eigenvalues

- λ is an eigenvalue of $\boldsymbol{A}$
- $\boldsymbol{A}\mathbf{v} = \lambda\mathbf{v}$ for some nonzero column vector $\mathbf{v}$
- $(\lambda\boldsymbol{I} - \boldsymbol{A})\,\mathbf{v} = \mathbf{0}$ for some nonzero column vector $\mathbf{v}$
- $\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$ ← Solve this equation to find the eigenvalues of $\boldsymbol{A}$

$\mathbf{v}$ is a non-trivial solution of the homogeneous system $(\lambda\boldsymbol{I} - \boldsymbol{A})\,\mathbf{x} = \mathbf{0}$

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True or False

Let λ be an eigenvalue of an invertible matrix $\boldsymbol{A}$.
Then

1. 2λ is an eigenvalue of $2\boldsymbol{A}$.

2. λ^{-1} is an eigenvalue of $\boldsymbol{A}^{-1}$.

Bring in the corresponding eigenvector $\boldsymbol{v}$

Start with $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$

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Ways To Find Eigenvalues

- If an eigenvector **u** is given, multiply it by the matrix: $\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{u}$
- Solve characteristic equation $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$
- If the matrix is triangular, take the diagonal entries
- If λ is an eigenvalue of $\boldsymbol{A}$, then
  - $c\lambda$ is an eigenvalue of $c\boldsymbol{A}$
  - λ is an eigenvalue of $\boldsymbol{A}^T$
  - λ^n is an eigenvalue of $\boldsymbol{A}^n$
  - λ^{-1} is an eigenvalue of $\boldsymbol{A}^{-1}$ (when $\boldsymbol{A}$ is invertible)

Exercise 6 Q3

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Eigenspace

$\boldsymbol{A}$: n x n matrix and λ is (one of the) eigenvalue of $\boldsymbol{A}$

Solution space of $(\lambda \boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{x} = 0$ ⟷ eigenspace E_λ of $\boldsymbol{A}$ associated with eigenvalue λ

trivial solution

nontrivial solutions ⟷ eigenvectors of $\boldsymbol{A}$ associated with eigenvalue λ

Zero vector is not an eigenvector

Check: Eigenspace for λ = -1 : E_{-1} = span$\left\{\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}\right\}$

Finding eigenvector (given eigenvalue)

$$\boldsymbol{A} = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$$

Solve $(\lambda \boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{x} = \boldsymbol{0}$ for eigenvalues λ = 1 and -1

$$\begin{pmatrix}\lambda&0&-1&0\\0&\lambda&0&-1\\-1&0&\lambda&0\\0&-1&0&\lambda\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

λ = 1

$$\left(\begin{array}{cccc|c}1&0&-1&0&0\\0&1&0&-1&0\\-1&0&1&0&0\\0&-1&0&1&0\end{array}\right) \xrightarrow{G.E.} \left(\begin{array}{cccc|c}1&0&-1&0&0\\0&1&0&-1&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right) \Rightarrow$$

w = t, z = s, y = t, x = s

Gen. soln: $\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}s\\t\\s\\t\end{pmatrix} = s\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$

Eigenspace for λ = 1 : E_1 = span$\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\}$

Any non-trivial linear combination is an eigenvector associated to λ = 1

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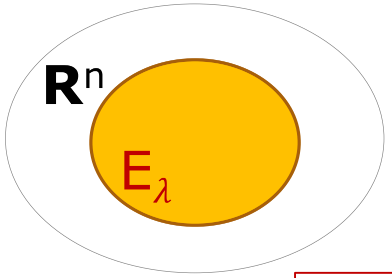
Complete process



n x n matrix $\boldsymbol{A}$ $\xrightarrow{\det(x\boldsymbol{I} - \boldsymbol{A})}$ characteristic polynomial CP

characteristic polynomial CP → solve CP for all the roots $(x - \lambda_1)(x - \lambda_2) \ldots (x - \lambda_n)$ → eigenvalues $\lambda_1\ \lambda_2 \ldots \lambda_n$

eigenvalues $\lambda_1\ \lambda_2 \ldots \lambda_n$ → Solve homogeneous system $(\lambda\boldsymbol{I} - \boldsymbol{A})\,\boldsymbol{x} = 0$ for the solution space → eigenspace E_λ

eigenspace E_λ → take any non-zero vectors in E_λ → eigenvectors $\boldsymbol{v}$

eigenspace E_λ → Separate parameters from the general solution → basis for E_λ

$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$

Exercise 6 Q7 (modified)

$\boldsymbol{A}$ is a 3 x 3 matrix with eigenvalue λ and
the associated eigenspace $E_{\lambda,\mathbf{A}}$ such that $\dim(E_{\lambda,\mathbf{A}}) = 2$.

$\boldsymbol{B}$ is another 3 x 3 matrix with eigenvalue μ and
the associated eigenspace $E_{\mu,\mathbf{B}}$ such that $\dim(E_{\mu,\mathbf{B}}) = 2$.

Show that $\lambda + \mu$ is an eigenvalue of the matrix $\boldsymbol{A} + \boldsymbol{B}$.

★★★ In general, $\lambda + \mu$ is not an eigenvalue of $\boldsymbol{A} + \boldsymbol{B}$

$E_{\lambda,\boldsymbol{A}}$ is a subspace of $\mathbf{R}^3$ with dimension 2

$E_{\mu,\boldsymbol{B}}$ is a subspace of $\mathbf{R}^3$ with dimension 2

Then $E_{\lambda,\boldsymbol{A}} \cap E_{\mu,\boldsymbol{B}}$ is a subspace of $\mathbf{R}^3$ with dimension 1 or 2

The two eigenspaces have non-trivial intersection

Take $\boldsymbol{v} \in E_{\lambda,\boldsymbol{A}} \cap E_{\mu,\boldsymbol{B}}$

$\boldsymbol{Av} = \lambda\boldsymbol{v}$ $\quad$ $\boldsymbol{Bv} = \mu\boldsymbol{v}$ $\quad$ $\Rightarrow (\boldsymbol{A}+\boldsymbol{B})\boldsymbol{v} = (\lambda + \mu)\boldsymbol{v}$

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Exercise 6 Q26

Let $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$.

Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are eigenvectors of $\boldsymbol{A}$ associating with two different eigenvalues.

Show that $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

Let $\lambda \neq \mu$ be eigenvalues for $\boldsymbol{u}$ and $\boldsymbol{v}$

$$\boldsymbol{Au} = \lambda \boldsymbol{u} \qquad \boldsymbol{Av} = \mu \boldsymbol{v}$$

$$\boldsymbol{v} \cdot \boldsymbol{Au} = \boldsymbol{Au} \cdot \boldsymbol{v} = (\boldsymbol{Au})^T \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{A}^T \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{A}^T \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{Av}$$

$$\Downarrow \qquad \Downarrow$$

$$\boldsymbol{v} \cdot (\lambda \boldsymbol{u}) \qquad \boldsymbol{u} \cdot (\mu \boldsymbol{v})$$

$$\Downarrow \qquad \Downarrow$$

$$\lambda(\boldsymbol{v} \cdot \boldsymbol{u}) \longleftrightarrow \mu(\boldsymbol{u} \cdot \boldsymbol{v})$$

$$\lambda(\boldsymbol{u} \cdot \boldsymbol{v}) = \mu(\boldsymbol{u} \cdot \boldsymbol{v}) \Leftrightarrow (\lambda - \mu)(\boldsymbol{u} \cdot \boldsymbol{v}) = 0 \Leftrightarrow \boldsymbol{u} \cdot \boldsymbol{v} = 0$$

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Exercise 6 Q8 (Tutorial)

$\{ \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n \}$ basis for $\mathbf{R}^n$ and $\boldsymbol{A}$ is an n x n matrix such that

$\boldsymbol{A}\boldsymbol{u}_i = \boldsymbol{u}_{i+1}$ for i = 1, 2, …, n-1 and $\boldsymbol{A}\boldsymbol{u}_n = \boldsymbol{0}$

Show that the only eigenvalue of $\boldsymbol{A}$ is 0 and find all the eigenvectors of $\boldsymbol{A}$.

Hint: Start with an eigenvalue λ and show $\lambda = 0$

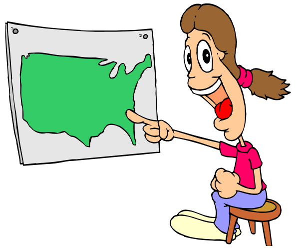
Suppose $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ with eigenvalue λ

$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$ ---(*)

$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots\ c_n\boldsymbol{u}_n$ ---(**)

Substitute (**) into (*) and use the given condition

Deduce $\lambda = 0$ by comparing the coefficients of the $\boldsymbol{u}_i$, and show $c_1 = c_2 = \ldots = c_{n-1} = 0$



to be continued

Map of LA

A is an $n \times n$ matrix

A is invertible	chapter 2	**A is not invertible**
det A $\neq$ 0	chapter 2	**det A = 0**
rref of A is identity matrix	chapter 1	**rref of A has a zero row**
AX= 0 has only the trivial solution	chapter 1	**AX= 0 has non-trivial solutions**
AX= B has a unique solution	chapter 1	**AX= B has no solution or infinitely many solutions**
rows (columns) of A are linearly independent	chapter 3	**rows (columns) of A are linearly dependent**
row (column) space of A = R^n	chapter 4	**row (column) space of A $\neq$ R^n**
rank(A) = n **nullity(A) = 0**	chapter 4	**rank(A) < n** **nullity(A) > 0**
0 is not an eigenvalue of A	chapter 6	**0 is an eigenvalue of A**