National University of Singapore
MA2001 Linear Algebra

MATLAB Worksheet 2
Working with Matrices

A. Input Matrices

Recall that to input a matrix to MATLAB, the entries of a matrix should be entered row by row, where the entries in each row are separated by spaces, and the rows are separated by semi-colon `;`. The entries should be enclosed by a pair of square brackets `[ ]`. For example,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

```
>> A = [1 2 3; 4 5 6]
A =  1  2  3
     4  5  6
```

The command `size` gives the number of rows and columns of a matrix. For example, for the above matrix $\boldsymbol{A}$,

```
>> size(A)
ans =  2  3
```

To extract a particular (i, j)-entry of matrix $\boldsymbol{A}$, we use the command `A(i,j)`. For example,

```
>> A(2,3)
ans =  6
```

To extract a particular i^{th} row of matrix $\boldsymbol{A}$, we use the command `A(i,:)`. For example,

```
>> A(2,:)
ans =  4  5  6
```

To extract a particular j^{th} column of matrix $\boldsymbol{A}$, we use the command `A(:,j)`. For example,

```
>> A(:,3)
ans =  3
       6
```

We can also extract a submatrix from a matrix with specific rows and columns. For example,

$$\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{pmatrix}.$$

```
>> B = [1 2 3 4 5; 2 3 4 5 6; 3 4 5 6 7; 4 5 6 7 8]
B =  1  2  3  4  5
     2  3  4  5  6
     3  4  5  6  7
     4  5  6  7  8
```

To extract the submatrix of $\boldsymbol{B}$ formed by the 2^{nd} and the 4^{th} rows of $\boldsymbol{B}$:

```
>> B([2 4], :)
ans =  2   3   4   5   6
       4   5   6   7   8
```

To extract the submatrix of $\boldsymbol{B}$ formed by the 1^{st}, 3^{rd} and the 5^{th} columns of $\boldsymbol{B}$:

```
>> B(:, [1 3 5])
ans =  1   3   5
       2   4   6
       3   5   7
       4   6   8
```

To extract the submatrix of $\boldsymbol{B}$ formed by the 2^{nd} and 3^{rd} rows, and the 3^{rd}, 4^{th} and 5^{th} columns of $\boldsymbol{B}$:

```
>> B([2 3], [3 4 5])
ans =  4   5   6
       5   6   7
```

B. Special Matrices

We can generate special matrices in MATLAB using the following commands:

(i) Zero matrix $\mathbf{0}_{m\times n}$ of size $m \times n$: `zeros(m,n)`.

```
>> zeros(2,3)
ans =  0   0   0
       0   0   0
```

(ii) Identity matrix $\boldsymbol{I}_n$ of order n: `eye(n)`.

```
>> eye(3)
ans =  1   0   0
       0   1   0
       0   0   1
```

(iii) Diagonal matrix with diagonal entries $a_1, \ldots, a_n$: `diag([a1 ... an])`.

```
>> diag([2 3 4 6])
ans =  2   0   0   0
       0   3   0   0
       0   0   4   0
       0   0   0   6
```

C. Matrix Operations

The matrix addition, subtraction and scalar multiplication can be evaluated using `+`, `-` and `*` respectively. For example,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}.$$

```
>> A = [1 2; 3 4];
>> B = [4 1; 2 5];
```

(i) Addition: $\boldsymbol{A} + \boldsymbol{B}$:

```
>>  A + B
ans =  5  3
       5  9
```

(ii) Subtraction: $\boldsymbol{A} - \boldsymbol{B}$:

```
>>  A - B
ans =  -3  1
       1  -1
```

(iii) Scalar multiplication: $c\boldsymbol{A}$:

```
>>  3 * A
ans =  3  6
       9  12
```

We illustrate more operations using the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ defined above.

(iv) Matrix product $\boldsymbol{AB}$, provided that the sizes are matched.

```
>>  A * B
ans =  8  11
       20  23
```

(v) Transpose $\boldsymbol{A}^{\mathrm{T}}$:

```
>>  A'
ans =  1  3
       2  4
```

(vi) Reduced row-echelon form of $\boldsymbol{A}$:

```
>>  rref(A)
ans =  1  0
       0  1
```

(vii) Powers $\boldsymbol{A}^n$, provided that $\boldsymbol{A}$ is a square matrix and n is an integer. If $n < 0$, $\boldsymbol{A}$ needs to be invertible (that is, non-singular).

```
>>  A ^ 10
ans =  4783807  6972050
       10458075  15241882
```

(viii) If $\boldsymbol{A}$ is invertible, its inverse can be evaluated using either `A^(-1)` or `inv(A)`.

```
>>  A ^ (-1)
ans =  -2.0000  1.0000
```

```
        1.5000  -0.5000
>>  inv(A)
ans =  -2.0000  1.0000
        1.5000  -0.5000
```

(ix) If $\boldsymbol{A}$ is a square matrix, its determinant can be evaluated using $\boxed{\texttt{det(A)}}$.

```
>>  det(A)
ans =  -2
>>  det(B)
ans =  18
```

D. Matrix Equations of Linear Systems

Recall that in MATLAB, we input a linear system as an augmented matrix and use the $\boxed{\texttt{rref}}$ command to solve the linear system.

In worksheet 1, we have the example:

$$\begin{cases} 2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 = -2 \\ x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 = -2 \\ 2x_1 \qquad\quad - 4x_3 + 2x_4 + x_5 = 3 \\ x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 = -7 \end{cases}$$

We input the coefficient matrix $\boldsymbol{A}$:

```
>>  A = [2 -3 -7 5 2; 1 -2 -4 3 1; 2 0 -4 2 1; 1 -5 -7 6 2]
A =  2  -3  -7  5  2
     1  -2  -4  3  1
     2  0  -4  2  1
     1  -5  -7  6  2
```

and the constant matrix $\boldsymbol{b}$:

```
>>  b = [-2; -2; 3; -7]
b =  -2
     -2
     3
     -7
```

Then we use the command

```
>>  rref([A b])
ans =  1  0  -2  1  0  1
       0  1  1  -1  0  2
       0  0  0  0  1  1
       0  0  0  0  0  0
```

to get the reduced row echelon form and use it to solve the system by hand to get the general solution $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2s - t + 1 \\ -s + t + 2 \\ s \\ t \\ 1 \end{pmatrix}$.

We learn that a linear system can be written in matrix equation form $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A}$ and $\boldsymbol{b}$ are the coefficient matrix and constant matrix as above, and $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is the variable matrix. When $\boldsymbol{x}$ is substituted with a solution of the linear system, the matrix equation will be satisfied. In other words, the matrix product $\boldsymbol{Ax}$ is equal to $\boldsymbol{b}$ if $\boldsymbol{x}$ represents a solution of the linear system.
To illustrate this in MATLAB using the above example, we need to declare the paramters s and t of the general solution as symbolic variables as follow.

```
>> syms s t
```

Then define the general solution

```
>> x = [2*s-t+1; -s+t+2; s; t; 1]
x =  2*s - t + 1
     t - s + 2
     s
     t
```

To check that $\boldsymbol{x}$ is a solution of the system, we evaluate $\boldsymbol{Ax}$ and compare it with $\boldsymbol{b}$:

```
>> A * x
ans =  -2
       -2
       3
       -7
```

which is indeed equal to $\boldsymbol{b}$.

Let's consider another linear system with the same number of equations and variables:

$$\begin{cases} x + y + 2z = 1 \\ 3x + 6y - 5z = -1 \\ 2x + 4y + 3z = 0 \end{cases}$$

Let's enter the coefficient matrix $\boldsymbol{A}$ and constant matrix $\boldsymbol{b}$:

```
>> A = [1 1 2; 3 6 -5; 2 4 3]
A =  1  1  2
     3  6  -5
     2  4  3
>> b = [1; -1; 0]
b =  1
     -1
     0
```

If the square matrix $\boldsymbol{A}$ is invertible, then the linear system has a unique solution which is given by $\boldsymbol{A}^{-1}\boldsymbol{b}$.
To check that the matrix $\boldsymbol{A}$ above is invertible, we can use either `rref` or `det`:

```
>> rref(A)
ans =  1  0  0
       0  1  0
       0  0  1
```

As the reduced row echelon form of $\boldsymbol{A}$ is the identity matrix, we can conclude that $\boldsymbol{A}$ is invertible. Alternatively,

```
>>  det(A)
ans =  19
```

As the determinant of $\boldsymbol{A}$ is non-zero, we can also conclude that $\boldsymbol{A}$ is invertible. Hence we can find the unique solution of the linear system $\boldsymbol{Ax} = \boldsymbol{b}$:

```
>>  x = inv(A)*b
x =  1.7368
     -0.9474
     0.1053
```

To get the solution in fraction form,

```
>>  format rat
>>  x
x =  33/19
     -18/19
     2/19
```

Again, to check that this is indeed a solution of the linear system, we just need to perform the matrix multiplication $\boldsymbol{Ax}$:

```
>>  A*x
ans =  1
       -1
       1/750599937895083
```

Note that instead of getting 0 for the third component, which is what we are expecting, we get a strange fraction with a very large denominator. This is the rounding error in MATLAB that may happen occasionally. If we change back the format to the default, the rounding error will be hidden.

```
>>  format
>>  A*x
ans =  1.0000
       -1.0000
       0.0000
```

If the coefficient matrix $\boldsymbol{A}$ of a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is not invertible, then we can't use this method to find the solution, but to fall back to the reduced row echelon form of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{b})$.

E. Practices

Use MATLAB to solve Questions 2.1, 2.25, 2.31, 2.37, 2.38, 2.47, 2.48 in the textbook Exercise 2.