Answers/Solutions of Exercise 6 (Q1-8)

1. (a) The characteristic equation is $(\lambda+1)(\lambda-3)=0$; eigenvalues are -1 and 3; $\{(0,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and $\{(1,2)^{\mathrm{T}}\}$ is a basis for E_3.

   (b) The characteristic equation is $(\lambda-2)^2=0$; the eigenvalue is 2; $\{(1,1)^{\mathrm{T}}\}$ is a basis for E_2.

   (c) The characteristic equation is $\lambda^2-4=0$; eigenvalues are -2 and 2; $\{(-2,1)^{\mathrm{T}}\}$ is a basis for E_{-2} and $\{(2,1)^{\mathrm{T}}\}$ is a basis for E_2 .

   (d) The characteristic equation is $\lambda^2=0$; the eigenvalue is 0; $\{(1,0),(0,1)^{\mathrm{T}}\}$ is a basis for E_0.

   (e) The characteristic equation is $\lambda(\lambda-2)^2=0$; eigenvalues are 0 and 2; $\{(-1,1,0)^{\mathrm{T}}\}$ is a basis for E_0 and $\{(1,1,0)^{\mathrm{T}}\}$ is a basis for E_2.

   (f) The characteristic equation is $(\lambda-2)(\lambda^2-9)=0$; eigenvalues are 2, -3 and 3; $\{(0,0,1)^{\mathrm{T}}\}$ is a basis for E_2, $\{(-1,3,0)^{\mathrm{T}}\}$ is a basis for E_{-3} and $\{(1,3,0)^{\mathrm{T}}\}$ is a basis for E_3.

   (g) The characteristic equation is $(\lambda-1)^3=0$; the eigenvalue is 1; $\{(0,0,1)^{\mathrm{T}}\}$ is a basis for E_1.

   (h) The characteristic equation is $(\lambda+1)(\lambda-1)^2=0$; eigenvalues are -1 and 1; $\{(-1,-1,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and $\{(1,2,0)^{\mathrm{T}},(1,0,2)^{\mathrm{T}}\}$ is a basis for E_1.

   (i) The characteristic equation is $(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)=0$; eigenvalues are 1,2,3 and 4; $\{(0,0,0,1)^{\mathrm{T}}\}$is a bais for E_1, $\{(0,0,1,1)^{\mathrm{T}}\}$ is a basis for E_2, $\{(0,2,4,3)^{\mathrm{T}}\}$ is a basis for E_3 and $\{(3,9,12,8)^{\mathrm{T}}\}$ is a basis for E_4.

   (j) The characteristic equation is $\lambda^4-2\lambda^2+1=0$; eigenvalues are -1 and 1; $\{(-1,0,1,0)^{\mathrm{T}},(0,-1,0,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and $\{(1,0,1,0)^{\mathrm{T}},(0,1,0,1)^{\mathrm{T}}\}$ is a basis for E_1.

2. (a) $\det(\lambda\boldsymbol{I}-\boldsymbol{A})=\begin{vmatrix}\lambda-a & -b\\ -c & \lambda-d\end{vmatrix}=\lambda^2+(-a-d)\lambda+(ad-bc)$

   Hence $m=-a-d=-\mathrm{tr}(\boldsymbol{A})$ and $n=\det(\boldsymbol{A})$.

   (b) Direct verification shows that $\boldsymbol{A}^2+m\boldsymbol{A}+n\boldsymbol{I}=\boldsymbol{0}$.

3. (a) Let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{A}$ associated with λ, i.e. $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$. We prove that $\boldsymbol{A}^n\boldsymbol{x}=\lambda^n\boldsymbol{x}$ by induction on n.

   It is given that $\boldsymbol{A}^1\boldsymbol{x}=\lambda^1\boldsymbol{x}$. Assume that $\boldsymbol{A}^k\boldsymbol{x}=\lambda^k\boldsymbol{x}$. Then

$$\boldsymbol{A}^{k+1}\boldsymbol{x}=\boldsymbol{A}(\boldsymbol{A}^k\boldsymbol{x})=\boldsymbol{A}(\lambda^k\boldsymbol{x})=\lambda^k\boldsymbol{A}\boldsymbol{x}=\lambda^k\lambda\boldsymbol{x}=\lambda^{k+1}\boldsymbol{x}.$$

By mathematical induction, $\boldsymbol{A}^n\boldsymbol{x} = \lambda^n\boldsymbol{x}$ and hence λ^n is an eigenvalue of $\boldsymbol{A}$ for all positive integer n.

(b) Let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{A}$ associated with λ. Then

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \quad\Rightarrow\quad \boldsymbol{x} = \boldsymbol{A}^{-1}(\lambda\boldsymbol{x}) = \lambda\boldsymbol{A}^{-1}\boldsymbol{x} \quad\Rightarrow\quad \tfrac{1}{\lambda}\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{x}.$$

Thus $\frac{1}{\lambda}$ is an eigenvalue of $\boldsymbol{A}^{-1}$.

(c) λ is an eigenvalue of $\boldsymbol{A}$
$$\begin{aligned} &\Rightarrow \det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0 \\ &\Rightarrow \det\left((\lambda\boldsymbol{I} - \boldsymbol{A})^{\mathrm{T}}\right) = 0 \\ &\Rightarrow \det(\lambda\boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}}) = 0 \\ &\Rightarrow \lambda \text{ is an eigenvalue of } \boldsymbol{A}^{\mathrm{T}}. \end{aligned}$$

4. (a) Let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{A}$ associated with λ, i.e. $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ and $\boldsymbol{x}$ is a nonzero vector. Then

$$\boldsymbol{A}^2 = \boldsymbol{A} \quad\Rightarrow\quad \boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \quad\Rightarrow\quad \lambda^2\boldsymbol{x} = \lambda\boldsymbol{x} \quad\Rightarrow\quad \lambda(\lambda-1)\boldsymbol{x} = \boldsymbol{0}$$

Since $\boldsymbol{x}$ is nonzero, $\lambda = 0$ or 1.

(b) Since $\boldsymbol{A}$ has 2 distinct eigenvalues, it is diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible matrix such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} ad & -ab \\ cd & -cb \end{pmatrix} \quad \text{where } ad - bc \neq 0.$$

We can simplify the expression to $\boldsymbol{A} = \begin{pmatrix} r & s \\ t & 1-r \end{pmatrix}$ where $st = r(1-r)$.

5. (a) Let $\boldsymbol{x}$ be a nonzero eigenvector of $\boldsymbol{A}$ associated with λ, i.e. $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$.

$$\boldsymbol{A}^2 = \boldsymbol{0} \quad\Rightarrow\quad \boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{0}\boldsymbol{x} \quad\Rightarrow\quad \boldsymbol{A}(\lambda\boldsymbol{x}) = \boldsymbol{0} \quad\Rightarrow\quad \lambda^2\boldsymbol{x} = \boldsymbol{0}$$

Since $\boldsymbol{x}$ is nonzero, $\lambda = 0$.

(b) No. Suppose $\boldsymbol{A}$ is diagonalizable. Then there exists invertible $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{0}$. Then $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{0}\boldsymbol{P}^{-1} = \boldsymbol{0}$, a contradiction.

(c) Consider the vector equation

$$a\boldsymbol{u} + b\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}. \qquad (*)$$

Pre-multiplying $\boldsymbol{A}$ to both side of $(*)$, we have

$$\boldsymbol{A}(a\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{0} \quad\Rightarrow\quad a\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}. \quad (\because \boldsymbol{A}^2 = \boldsymbol{0}.)$$

As $\boldsymbol{Au} \neq \mathbf{0}$, $a = 0$. Substituting $a = 0$ into $(*)$, we have $b\boldsymbol{Au} = \mathbf{0}$ and hence $b = 0$. Since $(*)$ has only the trivial solution, $\boldsymbol{u}$ and $\boldsymbol{Au}$ are linearly independent.

(d) Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{u} & \boldsymbol{Au} \end{pmatrix}$. By (c), $\boldsymbol{P}$ is invertible. Since

$$\boldsymbol{AP} = \begin{pmatrix} \boldsymbol{Au} & \boldsymbol{A}^2\boldsymbol{u} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Au} & \mathbf{0} \end{pmatrix}$$

and

$$\boldsymbol{P}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0\boldsymbol{u} + \boldsymbol{Au} & 0\boldsymbol{u} + 0\boldsymbol{Au} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Au} & \mathbf{0} \end{pmatrix},$$

$\boldsymbol{AP} = \boldsymbol{P}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ which implies $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

6. (a) Since $\det(-\boldsymbol{I} - \boldsymbol{A}) = 0$, -1 is an eigenvalue of $\boldsymbol{A}$.

(b) $\{(1,1,0)^{\mathrm{T}}, (0,0,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and hence $\dim(E_{-1}) = 2$.

(c) For example, $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

7. (a) Since $\det(2\boldsymbol{I} - \boldsymbol{A}) = 0$, 2 is an eigenvalue of $\boldsymbol{A}$.

(b) $\{(1,2,0)^{\mathrm{T}}, (-3,0,1)^{\mathrm{T}}\}$ is a basis for the eigenspace associated with 2.

(c) Let E_2 be the eigenspace of $\boldsymbol{A}$ associated with 2 and let E'_λ be the eigenspace of $\boldsymbol{B}$ associated with λ.

Since E_2 and E'_λ are subspaces of $\mathbb{R}^3$ and have dimension 2, they are two planes in $\mathbb{R}^3$ that contain the origin. So $E_2 \cap E'_\lambda$ is either a line through the origin or a plane containing the origin. In both cases, we can find a nonzero vector $\boldsymbol{u} \in E_2 \cap E'_\lambda$, i.e. $\boldsymbol{Au} = 2\boldsymbol{u}$ and $\boldsymbol{Bu} = \lambda\boldsymbol{u}$, such that

$$(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{u} = \boldsymbol{Au} + \boldsymbol{Bu} = 2\boldsymbol{u} + \lambda\boldsymbol{u} = (2 + \lambda)\boldsymbol{u}.$$

So $2 + \lambda$ is an eigenvalue of $\boldsymbol{A} + \boldsymbol{B}$.

8. Note that for $i = 1, 2, \ldots, n$, $\boldsymbol{A}^n\boldsymbol{u_i} = \boldsymbol{A}^{n-1}\boldsymbol{u_{i+1}} = \cdots = \boldsymbol{A}^i\boldsymbol{u_n} = \mathbf{0}$.

Let $\boldsymbol{v} \in \mathbb{R}^n$ be an eigenvector of $\boldsymbol{A}$ associated with eigenvalue λ, i.e. $\boldsymbol{Av} = \lambda\boldsymbol{v}$. Since $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ is a basis for $\mathbb{R}^n$,

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_n\boldsymbol{u_n}$$

for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Then

$$\boldsymbol{A}^n\boldsymbol{v} = c_1\boldsymbol{A}^n\boldsymbol{u_1} + c_2\boldsymbol{A}^n\boldsymbol{u_2} + \cdots + c_n\boldsymbol{A}^n\boldsymbol{u_n} = \mathbf{0}.$$

From the proof of Question 6.3(a), $\boldsymbol{A}^n \boldsymbol{v} = \lambda^n \boldsymbol{v}$. Since $\boldsymbol{v} \neq \mathbf{0}$, $\lambda = 0$. Hence we have shown that $\boldsymbol{A}$ has only one eigenvalue 0.

As $\lambda = 0$, we get $\boldsymbol{A}\boldsymbol{v} = \mathbf{0}$. Then

$$\mathbf{0} = \boldsymbol{A}\boldsymbol{v} = c_1 \boldsymbol{A}\boldsymbol{u_1} + c_2 \boldsymbol{A}\boldsymbol{u_2} + \cdots + c_n \boldsymbol{A}\boldsymbol{u_n} = c_1 \boldsymbol{u_2} + c_2 \boldsymbol{u_3} + \cdots + c_{n-1} \boldsymbol{u_n}.$$

Since $\boldsymbol{u_2}, \boldsymbol{u_3}, \ldots, \boldsymbol{u_n}$ are linearly independent, $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_{n-1} = 0$, i.e. $\boldsymbol{v} = c_n \boldsymbol{u_n}$. Hence all eigenvectors of $\boldsymbol{A}$ are scalar multiples of $\boldsymbol{u_n}$.