Solution to Example 7

(a) $\Pr(B|A) = \Pr(A \cap B) \,/\, \Pr(A) = 0.1/0.3 = 1/3.$

(b) $\Pr(B'|A) = 1 - \Pr(B \mid A) = 1 - 1/3 = 2/3.$

(c) $\Pr(B|A') = \Pr(A' \cap B) \,/\, \Pr(A')$
$= [\Pr(B) - \Pr(B \cap A)]/(1 - \Pr(A))$
$= (0.2 - 0.1)/(1 - 0.3) = 1/7.$

(d) $\Pr(B'|A') = 1 - \Pr(B|A') = 1 - 1/7 = 6/7.$

(e) $\Pr(A \mid B) = \Pr(A \cap B)/\Pr(B) = 0.1/0.2 = 1/2.$

Note that $\Pr(A' \cap B) = \Pr(B) - \Pr(B \cap A)$ is based on the fact

$$B = S \cap B = (A' \cup A) \cap B = (A' \cap B) \cup (A \cap B),$$

and we can check that $A' \cap B$ and $A \cap B$ are mutually exclusive. Therefore

$$\Pr(B) = \Pr(A' \cap B) + \Pr(A \cap B).$$

1.6.3 Multiplication Rule of Probability

$$\Pr(A \cap B) = \Pr(A)\Pr(B|A) \quad \text{or}$$
$$\Pr(A \cap B) = \Pr(B)\Pr(A|B),$$

providing $\Pr(A) > 0, \Pr(B) > 0$.

- This rule enables us to calculate the probability that two events will both occur.
- The probability that both events occur is the product of the probability of one event occurs and the conditional probability that the other event occurs given that the first event has occurred.

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These formulae are derived very easily from the definition of conditional probability:

$$\Pr(B|A) = \frac{\Pr(B \cap A)}{\Pr(A)},$$

by multiplying $\Pr(A)$ on both sides.

"$\Pr(A) > 0$" is to ensure that $\Pr(B|A)$ is correctly defined; and likewise $\Pr(B) > 0$ is to guarantee that $\Pr(A|B)$ is correctly defined.

Take notes of these formulae, they are very useful.

Multiplication Rule of Probability (Continued)

- It can be extended to more than 2 events:
  $$\Pr(A \cap B \cap C) = \Pr(A)\Pr(B|A)\Pr(C|A \cap B),$$
  providing that $\Pr(A \cap B) > 0$.

- In general
  $$\Pr(A_1 \cap \cdots \cap A_n) = \Pr(A_1)\Pr(A_2 \mid A_1) \Pr(A_3 \mid A_1 \cap A_2) \cdots \Pr(A_n \mid A_1 \cap \cdots \cap A_{n-1}),$$
  providing that $\Pr(A_1 \cap \cdots \cap A_{n-1}) > 0$.

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$\Pr(A \cap B) > 0$ implies $\Pr(A) > 0$. This is because $(A \cap B) \subset A$. Then applying the conclusion given on page 1-145 that "$A \subset B$ *implies* $\Pr(A) \leq \Pr(B)$", we have $\Pr(A) \geq \Pr(A \cap B) > 0$.

Similarly, we can derive that $\Pr(A_1 \cap A_2 \cap \cdots \cap A_n) > 0$ is sufficient to ensure that all the conditional probabilities in the second formula in the slide is appropriately defined.

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Example 1

- Suppose that among 12 shirts, 3 are white.
- Two shirts are chosen randomly one by one **without replacement**.
  (a) What is the probability that **both shirts** that being picked are white?
  (b) What is the probability that there is **only one** white shirt being picked?
  (c) If 3 shirts are chosen at random, what is the probability that they are **all** white?

In the lecture video, Prof. Chan used the technics of conditional probabilities to sovle these problems. These are very important methods that we must learn.

On the other hand, one interesting thinking is: since the shirts are chosen randomly, can we use the old method to solve this problem? Think about the formula:

$$\Pr(A) = \frac{Number\ of\ Sample\ Points\ in\ A}{Number\ of\ Sample\ points\ in\ S}$$

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Solution to Example 1 (Continued)

(b) We have $\Pr(A_1) = 3/12$ and $\Pr(A_1') = 1 - 3/12 = 9/12$.

$\Pr(A_2|A_1) = 2/11$ and $\Pr(A_2'|A_1) = 1 - 2/11 = 9/11$.

If the first shirt is not white, then there are 3 white shirts among the remaining 11 shirts. Hence

$$\Pr(A_2|A_1') = 3/11.$$

$$\begin{aligned}\Pr((A_1 \cap A_2') \cup (A_1' \cap A_2)) &= \Pr(A_1 \cap A_2') + \Pr(A_1' \cap A_2) \\ &= \Pr(A_1)\Pr(A_2'|A_1) + \Pr(A_1')\Pr(A_2|A_1') \\ &= \left(\frac{3}{12}\right)\left(\frac{9}{11}\right) + \left(\frac{9}{12}\right)\left(\frac{3}{11}\right) = \frac{9}{22}.\end{aligned}$$

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In the lecture video, Prof. Chan mentioned with the help of the venn diagram that $\Pr\big((A_1 \cap A_2') \cup (A_1' \cap A_2)\big) = \Pr(A_1 \cup A_2) - P(A_1 \cap A_2)$.

We can also observe this from the discussion in page 6 of the complimentary notes for week 1. There, we have concluded that $A_1 \cup A_2$ can be partitioned to be the union of three disajoint subsets: $A_1 \cap A_2'$, $A_1 \cap A_2$, and $A_1' \cap A_2$, therefore,

$$\Pr(A_1 \cup A_2) = \Pr((A_1 \cap A_2') \cup (A_1' \cap A_2)) + \Pr(A_1 \cap A_2)$$

Take note that this formula is correct; however, to use this formula to compute may not be easy, as we still find the value for $\Pr(A_1 \cup A_2)$. So, think about how to compute this.

If the question is to evaluate A_2, we can do it as follows.

Recall that we have this partition of an event multiple times: $A_2 = (A_2 \cap A_1) \cup (A_2 \cap A_1')$ and $\emptyset = (A_2 \cap A_1) \cap (A_2 \cap A_1')$. Therefore

$$\begin{aligned}\Pr(A_2) &= \Pr(A_2 \cap A_1) + \Pr(A_2 \cap A_1') \\ &= \Pr(A_1)\Pr(A_2|A_1) + \Pr(A_1')\Pr(A_2|A_1') \\ &= \left(\frac{3}{12}\right)\left(\frac{2}{11}\right) + \left(\frac{9}{12}\right)\left(\frac{3}{11}\right)\end{aligned}$$

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Example 3

- Traffic police plan to enforce speed limits by using speed cameras at 4 different locations of the expressway.
- The speed cameras at each of the locations L_1, L_2, L_3 and L_4 are operated 40%, 30%, 20% and 30% of the time respectively.
- If a driver who is speeding on his way to work has probabilities 0.2, 0.1, 0.5 and 0.2 respectively, of passing through these locations, what is the probability that
  - (a) he will receive a speeding ticket?
  - (b) he passed through the radar trap at location L_2 if he received a speeding ticket?

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Solution to Example 3

Let B = {The driver receives a speeding ticket}

A_i = {He past through the location L_i} for $i = 1, \cdots, 4$

(a) $\Pr(B) = \Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2) + \Pr(A_3)\Pr(B|A_3) + \Pr(A_4)\Pr(B|A_4)$

$= 0.2(0.4) + 0.1(0.3) + 0.5(0.2) + 0.2(0.3) = 0.27.$

(b) $\Pr(A_2|\,B) = \Pr(A_2 \cap B)\,/\Pr(B)$

$= \Pr(A_2)\Pr(B|A_2)/\Pr(B)$

$= [0.1(0.3)]/0.27 = 1/9.$

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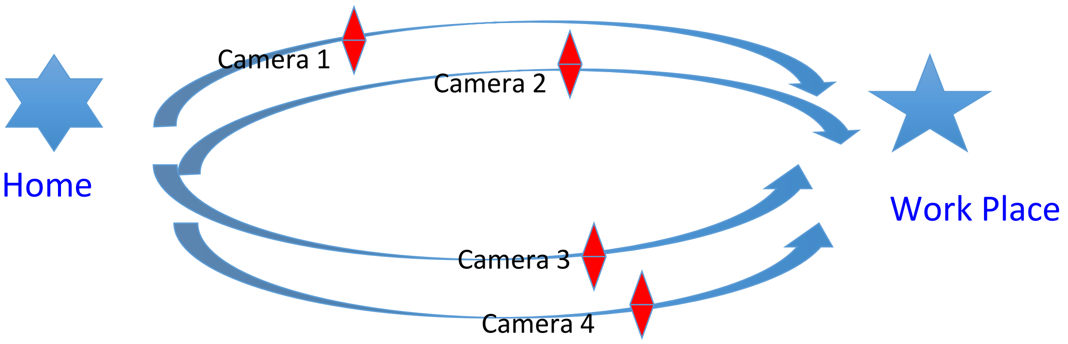
The practical senario of this question can be illustrated by the following figure:



In other words, it has potentially assumed that the drive has and only has four possible and separate routes to go for work, on each route, there is a carmera, since the probabilities of A_i's are summing to 1, i.e., 0.2+0.1+0.5+0.2 = 1. Therefore, $A_1 \cup A_2 \cup A_3 \cup A_4$ is equal to the sample space, and the Law of Total Probability formula given on page 1-230 applies.

1.6.5 Bayes' Theorem

Bayes' Theorem

- Let $A_1, A_2, \cdots, A_n$ be **a partition** of the sample space S. Then
$$\Pr(A_k \mid B) = \frac{\Pr(A_k)\Pr(B \mid A_k)}{\sum_{i=1}^{n} \Pr(A_i)\Pr(B \mid A_i)}$$
for $k = 1, \cdots, n$.

Note : The denominator is just $\Pr(B)$. That is,
$$\Pr(A_k \mid B) = \frac{\Pr(A_k)\Pr(B \mid A_k)}{\Pr(B)}$$

In Bayesian statistics, $\Pr(A_k|B)$ is called the posterior probability. $\Pr(A_k)$ is called the prior probability. B in some sense can be treated as practical observation. Therefore, the direct application of Bayes Theorem is to update the prior probability knowledge of A_k (i.e., $\Pr(A_k)$) by observing B occured, namely posterior probability $\Pr(A_k|B)$. Page 1-244 gives a good example: we update the probability that the design for a house is faulty when observing that it collapsed.

Example 1

- Suppose that there is a chance for a newly constructed house to collapse whether the design is faulty or not.
- The chance that the design is faulty is 1%.
- The chance that the house collapses if the design is faulty is 75% and otherwise it is 0.01%.
- It is seen that the house collapsed.
- What is the probability that it is due to faulty design?

Example 7

(The Monty Hall Problem)

- Suppose you are on a game show, and you are given the choice of three doors: behind one door is a car; behind the others, goats.
- You pick a door, say No. 1, and the host Monty, who knows what is behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?"
- Is it to your advantage to switch your choice?

Consider events: $W = \{win\}$; $A = \{choose\ the\ car\ door\ in\ the\ first\ pick\}$. Then this problem is to compute $\Pr(W)$ under different strategies: Stick and Switch.

No matte which strategy you use, you can always formulate:

$$\Pr(W) = \Pr(A)\Pr(W|A) + \Pr(A')\Pr(W|A')$$

and always, $\Pr(A) = \frac{1}{3}$ and $\Pr(A') = \frac{2}{3}$, so

$$\Pr(W) = \frac{1}{3}\Pr(W|A) + \frac{2}{3}\Pr(W|A')$$

- If you use the "Stick" strategy, $\Pr(W|A)$ means the probability of win given your first pick is the car; as you stick to your original choice, this probability is 1; likewise $\Pr(W|A') = 0$. So $\Pr(W) = \frac{1}{3}$.
- If you use the "Switch" strategy, $\Pr(W|A)$ still means the probability of win given your first pick is the car; but now you switch to another door, so for sure you will lose, therefore, this probability is 0; similarly, we can see $\Pr(W|A') = 1$ however. As a consequence, $\Pr(W) = \frac{2}{3}$.

1.7.2 Independent Events

Definition:

- Two events A and B are said to be ***independent*** if and only if

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

- Two events A and B that are not independent are said to be **dependent**.

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Independence is a very important terminology in statitics. The practical implication is that wehhter A occur is not related to whether B occurs, and vice versa.

The red formula in the slide is the mathematically definition of indepence of two events. Here are two tips:

- If $\Pr(A) = 0$, then based on this definition, it is independent with any event. The sample space S is also independent with any event.
- If $\Pr(A) \neq 0$, this definition is equivalent to $\Pr(B|A) = \Pr(B)$; similarly if $\Pr(B) \neq 0$, this definition can also be written as $\Pr(A|B) = \Pr(A)$.

Take note that independence and mutually exclusive are totally different terminologies. Furthermore, If $\Pr(A) > 0, \Pr(B) > 0$, then

- If $A\ and\ B$ are independent, we are able to assert $A\ and\ B$ are not mutually exclusive.
- If $A\ and\ B$ are mutually exclusive, we can assert that $A\ and\ B\ are\ not\ independent.$

Do not try to use Venn diagram to illustrate independence!