National University of Singapore
MA2001 Linear Algebra

MATLAB Worksheet 6
Eigenvalues, Eigenvectors and Diagonalization

Let $\boldsymbol{A}$ be a square matrix of order n. If there exists a constant λ and a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ such that $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$, then λ is called an **eigenvalue** of $\boldsymbol{A}$, and $\boldsymbol{v}$ is an **eigenvector** of $\boldsymbol{A}$ associated to λ.

A. Characteristic Polynomial

Throughout the lesson, we illustrate using the matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ -5 & 0 & -5 & -5 & 0 & -3 \end{pmatrix}.$$

```
>> A = [2 0 0 0 0 0; 0 -3 0 0 0 0; 0 0 1 0 0 0; 0 0 1 2 0 0; 0 0 0
0 -3 0;-5 0 -5 -5 0 -3];
```

The **characteristic polynomial** of $\boldsymbol{A}$ is the polynomial given by

$$p_{\boldsymbol{A}}(\lambda) = \det(\lambda\boldsymbol{I}_n - \boldsymbol{A}).$$

Note that the degree of the characteristic polynomial is n and its leading coefficient is 1.

MATLAB provides several ways to find the characteristic polynomial of $\boldsymbol{A}$.

(a) `charpoly(A)` gives a **vector** with $n + 1$ components which are the coefficients of the characteristic polynomial in **descending** order. For example,

```
>> charpoly(A)
ans =  1     4     -10     -40     45     108     -108
```

The output means that the characteristic polynomial of $\boldsymbol{A}$ is (in variable λ)

$$p_{\boldsymbol{A}}(\lambda) = \lambda^6 + 4\lambda^5 - 10\lambda^4 - 40\lambda^3 + 45\lambda^2 + 108\lambda - 108.$$

(b) `charpoly(A, x)` gives the characteristic polynomial in variable `x`. In order to use x as the variable for the characteristic polynomial of $\boldsymbol{A}$, we shall use `syms` to declare it as a symbolic object:

```
>> syms x;
```

Then type

```
>> charpoly(A, x)
ans =  x^6 + 4*x^5 - 10*x^4 - 40 * x^3 + 45*x^2 + 108*x - 108
```

B. Eigenvalues

The eigenvalues of a square matrix $\boldsymbol{A}$ are precisely all the roots to the characteristic polynomial of $\boldsymbol{A}$.

(a) `solve` can be used to find the roots of an equation or a function.

```
>> solve(charpoly(A, x))
ans =  -3
       -3
       -3
       1
       2
       2
```

(b) MATLAB provides a simple commands `eig` to produce the eigenvalue of $\boldsymbol{A}$ as a column vector:

```
>> eig(A)
ans =  -3
       2
       -3
       2
       1
       -3
```

Using any of these methods, we see that the eigenvalues of $\boldsymbol{A}$ are -3, 2 and 1, with -3 and 2 being repeated eigenvalues.

C. Eigenvectors

Let λ be an eigenvalue of a matrix $\boldsymbol{A}$. Then the eigenvectors of $\boldsymbol{A}$ associated to λ are precisely all nonzero vectors in the nullspace of $\lambda\boldsymbol{I} - \boldsymbol{A}$. For this reason, the nullspace of $\lambda\boldsymbol{I} - \boldsymbol{A}$ is also called the **eigenspace** of $\boldsymbol{A}$ associated to λ. In our example above, there are three eigenspaces:

(i) $\lambda = -3$:

```
>> null(-3*eye(6) - A, 'r')
ans =  0    0    0
       1    0    0
       0    0    0
       0    0    0
       0    1    0
       0    0    1
```

The three columns above give a basis for the eigenspace E_{-3} of $\boldsymbol{A}$ associated to eigenvalue -3:

$$\{(0,1,0,0,0,0),(0,0,0,0,1,0),(0,0,0,0,0,1)\},$$

(ii) $\lambda = 2$:

```
>> null(2*eye(6) - A, 'r')
ans =  -1    -1
        0     0
        0     0
        1     0
        0     0
        0     1
```

The two columns above give a basis for the eigenspace E_2 of $\boldsymbol{A}$ associated to eigenvalue 2:

$$\{(-1,0,0,1,0,0),(-1,0,0,0,0,1)\},$$

(iii) $\lambda = 1$:

```
>> null(1*eye(6) - A, 'r')
ans =   0
        0
       -1
        1
        0
        0
```

The single column above gives a basis for the eigenspace E_1 of $\boldsymbol{A}$ associated to eigenvalue 1:

$$\{(0,0,-1,1,0,0)\}.$$

D. Diagonalization

A square matrix $\boldsymbol{A}$ is said to be **diagonalizable** if there exists an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$.

(i) The diagonal entries of $\boldsymbol{D}$ are the eigenvalues of $\boldsymbol{A}$.

(ii) The columns of $\boldsymbol{P}$ are the corresponding linearly independent eigenvectors of $\boldsymbol{A}$.

Recall that

> *A square matrix of order n is diagonalizable if and only if it has n linearly independent eigenvectors.*

We may use either of the following ways to determine whether a square matrix is diagonalizable and find the matrices $\boldsymbol{P}$ and $\boldsymbol{D}$.

(a) As mentioned above, the linearly independent eigenvectors form the columns of the matrix $\boldsymbol{P}$. We can simply put the basis vectors of all the eigenspaces together to get $\boldsymbol{P}$:

```
>>  V1 = null(-3*eye(6) - A, 'r');
>>  V2 = null(2*eye(6) - A, 'r');
>>  V3 = null(1*eye(6) - A, 'r');
```

Then

```
>>  P = [V1 V2 V3]
P =  0    0    0    -1    -1    0
     1    0    0    0     0     0
     0    0    0    0     0     -1
     0    0    0    1     0     1
     0    1    0    0     0     0
     0    0    1    0     1     0
```

In general, we can always form a matrix $\boldsymbol{P}$ whose columns are linearly independent eigenvectors of $\boldsymbol{A}$ regardless of whether $\boldsymbol{A}$ is diagonalizable. But $\boldsymbol{A}$ is diagonalizable if and only if there are enough linearly independent eigenvectors to form a square matrix $\boldsymbol{P}$.

In this case, $\boldsymbol{P}$ is a square matrix, so we can conclude that $\boldsymbol{A}$ is diagonalizable. We can verify this by checking $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ gives a diagonal matrix.

```
>>  inv(P)*A*P
ans =  -3    0    0    0    0    0
       0    -3    0    0    0    0
       0    0    -3    0    0    0
       0    0    0    2    0    0
       0    0    0    0    2    0
       0    0    0    0    0    1
```

Note that the diagonal entries are the eigenvalues of $\boldsymbol{A}$.

(b) There is a direct way to obtain $\boldsymbol{P}$ using `eig`.

  (i) Declare the matrix $\boldsymbol{A}$ as a symbolic object, say $\boldsymbol{A}_1$.

```
>>  A1 = sym(A);
```

  (ii) Input the matrices $\boldsymbol{P}_1$ and $\boldsymbol{D}_1$ as follow:

```
>>  [P1,D1] = eig(A1)
```

```
P1 =  [0,  -1,  -1,  0,  0,  0]
      [0,  0,  0,  1,  0,  0]
      [-1,  0,  0,  0,  0,  0]
      [1,  1,  0,  0,  0,  0]
      [0,  0,  0,  0,  1,  0]
      [0,  0,  1,  0,  0,  1]

D1 =  [1,  0,  0,  0,  0,  0]
      [0,  2,  0,  0,  0,  0]
      [0,  0,  2,  0,  0,  0]
      [0,  0,  0,  -3,  0,  0]
      [0,  0,  0,  0,  -3,  0]
      [0,  0,  0,  0,  0,  -3]
```

Note that $\boldsymbol{P}_1$ (respectively $\boldsymbol{D}_1$) are not the same as $\boldsymbol{P}$ (respectively $\boldsymbol{D}$). They differ by some permutation of the columns.

Remark. In general, the matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ are not unique:

(i) Eigenvectors may be replaced by nonzero constant multiples.

(ii) If an eigenspace has dimension ≥ 2, the eigenvectors can be recombined through linear combinations.

(iii) The columns of $\boldsymbol{P}$ may be permutated; then the diagonal entries of $\boldsymbol{D}$ shall be permutated accordingly.

E. Practices

Use MATLAB to solve Questions 6.1, 6.6, 6.7(a)(b), 6.10 (optional), 6.11, 6.16(b), 6.17, 6.18 in the textbook Exercise 6.