National University of Singapore

MA2101 Linear Algebra II Tutorial 2

1. Let S, T be subsets of a vector space V over a field F. Show that $\mathrm{Span}(S \cup T) = \mathrm{Span}(S) + \mathrm{Span}(T)$.

2a. Consider the line $L_1 = \{t(1,2,3) \mid t \in \mathbf{R}\}$ and the plane $P_1 = \{(x,y,z) \in \mathbf{R}^3 \mid x + 3y - z = 0\}$ in $\mathbf{R}^3$. Show that $\mathbf{R}^3 = L_1 \oplus P_1$.
2b. Consider the plane P_1 in **2a** and the plane $P_2 = \{(x,y,z) \in \mathbf{R}^3 \mid 2x - y - z = 0\}$ in $\mathbf{R}^3$. Show that $\mathbf{R}^3 = P_1 + P_2$. Is it a direct sum? Justify your answer.
2c. In general, suppose that L is line and P a plane in $\mathbf{R}^3$ such that both L and P pass through the origin $\mathbf{0} = (0,0,0)$, and L is not entirely inclued in P. Show that $\mathbf{R}^3 = L \oplus P$.
Is the conclusion still true when $L \subset P$?

3. Let $\mathrm{UT}_2 = \{\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \mid a_{ij} \in F\}$, $\mathrm{LT}_2 = \{\begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} \mid b_{ij} \in F\}$ be respectively the set of all upper triangular, and all lower triangular matrices in $M_2(F)$.
3a. Find the intersection $\mathrm{UT}_2 \cap \mathrm{LT}_2$. Justify your answer.
3b. Find the sum $\mathrm{UT}_2 + \mathrm{LT}_2$; is it a direct sum? Justify your answers.

4. Let $\mathrm{Sy}_n = \{S \in M_n(\mathbf{R}) \mid S^t = S\}$ and $\mathrm{Sk}_n = \{K \in M_n(\mathbf{R}) \mid K^t = -K\}$ be respectively the subset of all **symmetric**, and all **skew symmetric** matrices in $M_n(\mathbf{R})$. Show that both Sy_n and Sk_n are subspaces of $M_n(\mathbf{R})$ such that $M_n(\mathbf{R}) = \mathrm{Sy}_n \oplus \mathrm{Sk}_n$.
Hint. For any $A \in M_n(\mathbf{R})$, the matrix $(A + A^t)/2$ is symmetric while the matrix $(A - A^t)/2$ is skew-symmetric.

5. Let U_i, W_i, and U be subsets of a vector space V, which are not necessarily subspaces. Suppose that W is a vector subspace of V.
5a. Show that if $U_1 \subseteq U_2$ and $W_1 \subseteq W_2$ then $U_1 + W_1 \subseteq U_2 + W_2$. Is the converse true?
5b. Show that $W + \{\mathbf{0}\} = W$ and $W + W = W$.
5c. Show that $U + W = W \iff U \subseteq W$.

6. Let $W_1, \ldots, W_s; s \geq 2$ be subspaces of a vector space V, and $W := \sum_{i=1}^{s} W_i$. Show the following equivalence of the direct sum definition.
(i) $(\sum_{i=1}^{k-1} W_i) \cap W_k = \{\mathbf{0}\}$ $(\forall\ 2 \leq k \leq s)$.
(ii) $(\sum_{i \neq \ell} W_i) \cap W_\ell = \{\mathbf{0}\}$ $(\forall\ 1 \leq \ell \leq s)$.
(iii) Every vector $\mathbf{w} \in W$ can be expressed as $\mathbf{w} = \mathbf{w}_1 + \cdots + \mathbf{w}_s$ for some $\mathbf{w}_i \in W_i$ and such expression of $\mathbf{w}$ is unique: whenever $\mathbf{w} = \mathbf{w}'_1 + \cdots + \mathbf{w}'_s$ for some $\mathbf{w}'_i \in W_i$, we have $\mathbf{w}'_i = \mathbf{w}_i$.