CS1231S Chapter 10

Cardinality

10.1 Pigeonhole Principles

Theorem 10.1.1 (Pigeonhole Principle)**.** Let A and B be finite sets. If there is an injection $f\colon A \to B$, then $|A| \leqslant |B|$.

Proof. 1. Note that A is finite. Suppose $A = \{a_1, a_2, \ldots, a_m\}$, where $m = |A|$.
2. The injectivity of f tells us that, if $a_i \neq a_j$, then $f(a_i) \neq f(a_j)$.
3. So $f(a_1), f(a_2), \ldots, f(a_m)$ are m different elements of B.
4. This shows $|B| \geqslant m = |A|$. □

Theorem 10.1.2 (Dual Pigeonhole Principle)**.** Let A and B be finite sets. If there is a surjection $f\colon A \to B$, then $|A| \geqslant |B|$.

Proof. 1. Note that B is finite. Suppose $B = \{b_1, b_2, \ldots, b_n\}$, where $n = |B|$.
2. For each b_i, use the surjectivity of f to find $a_i \in A$ such that $f(a_i) = b_i$.
3. If $b_i \neq b_j$, then $f(a_i) \neq f(a_j)$, and so $a_i \neq a_j$ because f is a function.
4. So $a_1, a_2, \ldots, a_n$ are n different elements of A.
5. This shows $|A| \geqslant n = |B|$. □

Theorem 10.1.3. Let A and B be finite sets. Then there is a bijection $A \to B$ if and only if $|A| = |B|$.

Proof. 1. ("Only if") This follows directly from Theorem 10.1.1 and Theorem 10.1.2.
2. ("If")
   - 2.1. Suppose $|A| = |B| = n$.
   - 2.2. Let $a_1, a_2, \ldots, a_n$ be the n elements of A, and $b_1, b_2, \ldots, b_n$ be the n elements of B.
   - 2.3. Note that the list $a_1, a_2, \ldots, a_n$ cannot have repetition because $|A| = n$.
   - 2.4. Similarly, the list $b_1, b_2, \ldots, b_n$ has no repetition.
   - 2.5. Define functions $f\colon A \to B$ and $g\colon B \to A$ by setting $f(a_i) = b_i$ and $g(b_i) = a_i$ for all $i \in \{1, 2, \ldots, n\}$.
   - 2.6. As the lists $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ have no repetition, the functions f and g are well defined.
   - 2.7. Observe that $g = f^{-1}$ by the definition of inverses.
   - 2.8. So f is a bijection $A \to B$ by Theorem 9.3.19. □

Exercise 10.1.4. Prove the converse to Theorem 10.1.1. Prove also the converse to Theorem 10.1.2 when $B \neq \varnothing$. ✎ 10a

10.2 Same cardinality

Definition 10.2.1 (Cantor)**.** A set A is said to have the *same cardinality* as a set B if there is a bijection $A \to B$. In this case, we write $|A| = |B|$.

Note 10.2.2. We defined what $|A| = |B|$ means without defining what $|A|$ and $|B|$ mean.

Proposition 10.2.3. Let A, B, C be sets.

(1) $|A| = |A|$. (reflexivity)

(2) If $|A| = |B|$, then $|B| = |A|$. (symmetry)

(3) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. (transitivity)

Proof. 1. (Reflexivity.) It suffices to show that id_A is a bijection $A \to A$.
- 1.1. id_A is injective because if $x_1, x_2 \in A$ such that $\mathrm{id}_A(x_1) = \mathrm{id}_A(x_2)$, then $x_1 = x_2$.
- 1.2. id_A is surjective because given any $x \in A$, we have $\mathrm{id}_A(x) = x$.

2. (Symmetry.)
- 2.1. Suppose $|A| = |B|$.
- 2.2. Use the definition of same-cardinality to find a bijection $f : A \to B$.
- 2.3. Then Theorem 9.3.19 gives us an inverse of f; call it g.
- 2.4. By the definition of inverses, for all $x \in A$ and all $y \in B$,

$$y = f(x) \quad \Leftrightarrow \quad x = g(y).$$

- 2.5. This tells us that f is an inverse of g in view the definition of inverses.
- 2.6. Thus g is a bijection $B \to A$ by Theorem 9.3.19.
- 2.7. This shows $|B| = |A|$.

3. (Transitivity.)
- 3.1. Suppose $|A| = |B|$ and $|B| = |C|$.
- 3.2. Use the definition of same-cardinality to find a bijection $f : A \to B$ and a bijection $g : B \to C$.
- 3.3. Then Tutorial 7 Question 8 tells us $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- 3.4. In particular, this says $g \circ f$ has an inverse.
- 3.5. So $g \circ f$ is a bijection $A \to C$ by Theorem 9.3.19.
- 3.6. Hence $|A| = |C|$. □

10.3 Countability

Definition 10.3.1 (Cantor)**.** A set is *countable* if it is finite or it has the same cardinality as $\mathbb{Z}_{\geqslant 0}$.

Note 10.3.2. Some authors allow only infinite sets to be countable.

Example 10.3.3. (1) $|\mathbb{Z}_{\geqslant 0}| = |\mathbb{Z}_{\geqslant 0} \setminus \{0\}|$ because the function $f : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0} \setminus \{0\}$ satisfying $f(x) = x + 1$ for all $x \in \mathbb{Z}_{\geqslant 0}$ is a bijection. So $\mathbb{Z}_{\geqslant 0} \setminus \{0\} = \{1, 2, 3, \dots\}$ is countable.

(2) $|\mathbb{Z}_{\geqslant 0}| = |\mathbb{Z}_{\geqslant 0} \setminus \{1, 3, 5, \dots\}|$ because the function $g : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0} \setminus \{1, 3, 5, \dots\}$ satisfying $g(x) = 2x$ for all $x \in \mathbb{Z}_{\geqslant 0}$ is a bijection. So $\mathbb{Z}_{\geqslant 0} \setminus \{1, 3, 5, \dots\} = \{0, 2, 4, \dots\}$ is countable.

Note 10.3.4. An infinite set B is countable if and only if

there is a sequence $b_0, b_1, b_2, \dots \in B$ in which every element of B appears exactly once.

Proof. 1. ("If")
- 1.1. Let $b_0, b_1, b_2, \dots$ be a sequences of elements of B in which every element of B appears exactly once.
- 1.2. Define $f : \mathbb{Z}_{\geqslant 0} \to B$ by setting $f(i) = b_i$ for each $i \in \mathbb{Z}_{\geqslant 0}$.
- 1.3. Then f is well defined because $b_0, b_1, b_2, \dots \in B$.

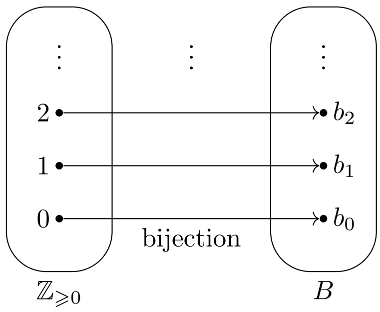


Figure 10.1: A countable infinite set B

- 1.4. (Surjectivity)
  - 1.4.1. Let $b \in B$.
  - 1.4.2. Then $b = b_i$ for some $i \in \mathbb{Z}_{\geqslant 0}$ because every element of B appears in $b_0, b_1, b_2, \ldots$.
  - 1.4.3. So $b = f(i)$ for some $i \in \mathbb{Z}_{\geqslant 0}$ by the definition of f.
- 1.5. (Injectivity)
  - 1.5.1. Let $i, j \in \mathbb{Z}_{\geqslant 0}$ such that $f(i) = f(j)$.
  - 1.5.2. Then $b_i = b_j$ by the definition of f.
  - 1.5.3. Thus $i = j$ because every element of B appears in $b_0, b_1, b_2, \ldots$ at most once.

2. ("Only if")
   - 2.1. Let f be a bijection $\mathbb{Z}_{\geqslant 0} \to B$.
   - 2.2. Define $b_0, b_1, b_2, \ldots$ to be $f(0), f(1), f(2), \ldots$ respectively.
   - 2.3. Then $b_0, b_1, b_2, \cdots \in B$ because the codomain of f is B.
   - 2.4. For every $b \in B$, there is $i \in \mathbb{Z}_{\geqslant 0}$ such that $b = f(i) = b_i$ by the surjectivity of f.
   - 2.5. So every element of B appears at least once in $b_0, b_1, b_2, \ldots$.
   - 2.6. Whenever $i, j \in \mathbb{Z}_{\geqslant 0}$ such that $b_i = b_j$, then $f(i) = f(j)$ and so $i = j$ by the injectivity of f.
   - 2.7. In particular, every element of B appears at most once in $b_0, b_1, b_2, \ldots$.
   - 2.8. Hence every element of B appears exactly once in $b_0, b_1, b_2, \ldots$. □

Lemma 10.3.5. An infinite set B is countable if and only if

there is a sequence $c_0, c_1, c_2, \ldots$ in which every element of B appears.

Proof. 1. ("Only if") This follows directly from Note 10.3.4.

2. ("If")
   - 2.1. Let $c_0, c_1, c_2, \ldots$ be a sequence in which every element of B appears.
   - 2.2. Remove those terms in the sequence that are not in B.
   - 2.3. If an element of B appears more than once, then remove all but the first appearance.
   - 2.4. The result is a sequence in which every element of B appears exactly once.
   - 2.5. So B is countable. □

Proposition 10.3.6. Any subset A of a countable set B is countable.

Proof. 1. If A is finite, then A is countable by definition.

2. So suppose A is infinite.
   - 2.1. Then B is infinite too as $A \subseteq B$.
   - 2.2. Use the countability of B to find a sequence $b_0, b_1, b_2, \ldots$ in which every element of B appears exactly once.
   - 2.3. This is a sequence in which every element of A appears.
   - 2.4. So A is countable by Lemma 10.3.5. □

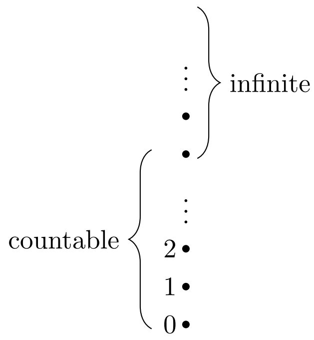


Figure 10.2: The smallest cardinalities

Proposition 10.3.7. Every infinite set B has a countable infinite subset.

Proof. 1. Keep choosing elements $b_0, b_1, b_2, \ldots$ from B. When we choose b_n, where $n \in \mathbb{Z}_{\geqslant 0}$, we can always make sure $b_n \neq b_i$ for any $i < n$, because otherwise B is equal to the finite set $\{b_0, b_1, \ldots, b_{n-1}\}$, which is a contradiction.
2. The result is a countable infinite set $\{b_0, b_1, b_2, \ldots\} \subseteq B$. $\square$

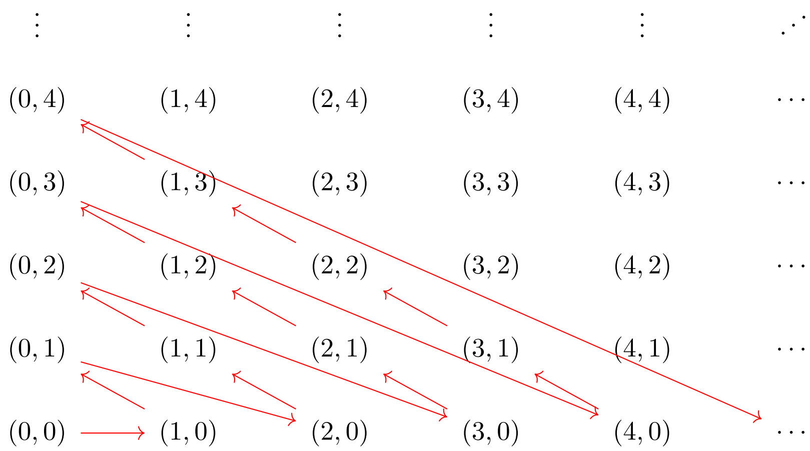
10.4 Set operations

Proposition 10.4.1. Let A, B be countable infinite sets. Then $A \cup B$ is countable.

Proof. 1. Apply Lemma 10.3.5 to find a sequence $a_0, a_1, a_2, \ldots$ in which every element of A appears.
2. Apply Lemma 10.3.5 to find a sequence $b_0, b_1, b_2, \ldots$ in which every element of B appears.
3. Then $a_0, b_0, a_1, b_1, a_2, b_2, \ldots$ is a sequence in which every element of $A \cup B$ appears.
4. So $A \cup B$ is countable by Lemma 10.3.5. $\square$

Theorem 10.4.2 (Cantor 1877)**.** $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is countable.

Proof sketch.



The figure above describes a sequence

$$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(3,0),(2,1),(1,2),(0,3),(4,0),(3,1),(2,2),(1,3),(0,4),\ldots$$

in which every element of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ appears. So $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ is countable by Lemma 10.3.5. $\square$

Theorem 10.4.3 (Cantor 1891)**.** Let A be a countable infinite set. Then $\mathcal{P}(A)$ is not countable.

Proof. Given any sequence of elements of $\mathcal{P}(A)$, we will produce an element of $\mathcal{P}(A)$ that does not appear in it. This will show that no sequence of elements of $\mathcal{P}(A)$ contains all the elements of $\mathcal{P}(A)$, and thus $\mathcal{P}(A)$ is uncountable by Note 10.3.4.

We organize all these into a proof by contradiction.

1. Suppose $\mathcal{P}(A)$ is countable.
2. We know $\mathcal{P}(A)$ is infinite because A is infinite and $\{a\} \in \mathcal{P}(A)$ for every $a \in A$.
3. Use the countability of $\mathcal{P}(A)$ to find a sequence $B_0, B_1, B_2, \ldots \in \mathcal{P}(A)$ in which every element of $\mathcal{P}(A)$ appears exactly once.
4. Use the countability of A to find a sequence $a_0, a_1, a_2, \ldots \in A$ in which every element of A appears exactly once.
5. Define $B = \{a_i : a_i \notin B_i\}$.
6. Note that $B \subseteq A$ since $a_0, a_1, a_2, \ldots \in A$.
7. 7.1. Let $i \in \mathbb{Z}_{\geqslant 0}$.
   7.2. If $a_i \notin B_i$, then $a_i \in B$ by the definition of B.
   7.3. if $a_i \in B_i$, then $a_i \notin B$ by the definition of B because no $j \neq i$ makes $a_j = a_i$ by the choice of $a_0, a_1, a_2, \ldots$.
   7.4. In either case, we know $B \neq B_i$.
8. This contradicts line 3 that every element of $\mathcal{P}(A)$ appears in $B_0, B_1, B_2, \ldots$. □

	a_0	a_1	a_2	a_3	a_4	$\dots$
B_0	$\boxed{\notin}$	$\in$	$\notin$	$\notin$	$\notin$	$\dots$
B_1	$\in$	$\boxed{\notin}$	$\in$	$\notin$	$\in$	$\dots$
B_2	$\notin$	$\in$	$\boxed{\in}$	$\notin$	$\in$	$\dots$
B_3	$\notin$	$\notin$	$\in$	$\boxed{\notin}$	$\notin$	$\dots$
B_4	$\in$	$\notin$	$\in$	$\in$	$\boxed{\notin}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
B	$\in$	$\in$	$\notin$	$\in$	$\in$	$\dots$

Figure 10.3: Illustration of Cantor's diagonal argument

Exercise 10.4.4. Which of the following is/are countable? Justify your answer. 10b

(1) $\mathbb{Z}$.

(2) $\mathbb{Q}$.

(3) $\mathbb{R}$.

(4) $\mathbb{C}$.

(5) The set of all finite sets of integers.

(6) The set of all strings over $\{\mathrm{s}, \mathrm{u}\}$.

(7) The set of all (infinite) sequences over $\{0, 1\}$.

(8) The set of all functions $A \to B$ where A, B are finite sets of integers.

(9) The set of all computer programs.