MA2001

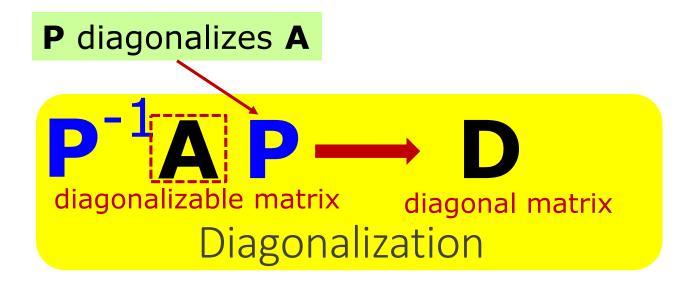
LIVE LECTURE 11

Q&A: log in to PollEv.com/vtpoll

Topics for week 11

- **6.2** Diagonalization
- 6.3 Orthogonal Diagonalization
- 6.4 for self-reading. Not in the exam scope

Diagonalizable matrix



Not possible to find **P** ↔ **A** non-diagonalizable

Diagonalization

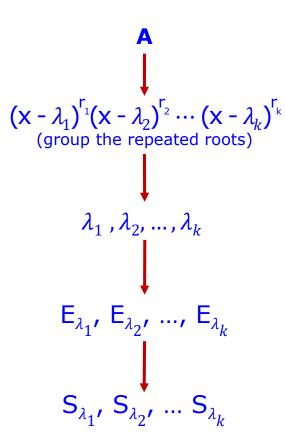
n x n matrix

characteristic polynomial

distinct eigenvalues

eigenspaces subspaces of **R**ⁿ

bases for eigenspaces



Case 1

 $S_{\lambda_1} \cup S_{\lambda_2} \cup ... \cup S_{\lambda_k}$ contains exactly n linearly independent eigenvectors

A diagonalizable

Case 2

 $S_{\lambda_1} \cup S_{\lambda_2} \cup ... \cup S_{\lambda_k}$ contains less than n linearly independent eigenvectors

A not diagonalizable

Finding P and D

Case 1

 $S_{\lambda_1} \cup S_{\lambda_2} \cup ... \cup S_{\lambda_k}$ contains exactly n linearly independent eigenvectors

A diagonalizable

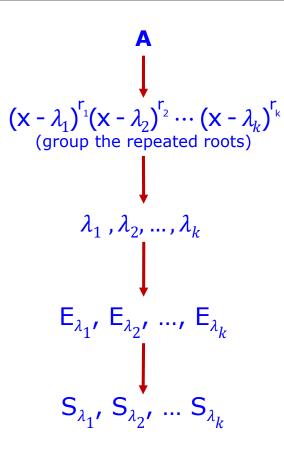
n x n matrix

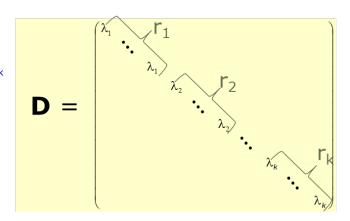
characteristic polynomial

distinct eigenvalues

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form matrix
$$P = (u_1 u_2 \cdots u_n)$$

$$S_{\lambda_1} \cup S_{\lambda_2} \cup ... \cup S_{\lambda_k}$$

= $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$

Many ways to diagonalize A

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 4 \end{pmatrix}$$
 Characteristic polynomial: $(\mathbf{x} - 3)^2(\mathbf{x} - 6)$ eigenvalues: 3 (repeated), 6
$$\mathbf{A} \text{ is diagonalizable}$$
 eigenvectors associated to $3: \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ eigenvectors associated to $6: \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \quad \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad \mathbf{R} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad \mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\mathbf{Q} = \begin{pmatrix} 5 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & -2 & 2 \end{pmatrix} \quad \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & -3 \\ 2 & -2 & -3 \end{pmatrix} \quad \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Writing diagonalizable matrix as PDP⁻¹

Given matrix **A** (diagonalizable)

Find eigenvalues and form **D**

Find eigenvectors and form **P**

Then $P^{-1}AP = D$

So we can express \mathbf{A} as $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$

Suppose **A** unknown

Given all eigenvalues and eigenvectors of A

Form **D** (using eigenvalues)

Form **P** (using eigenvectors)

So we can find **A** using $A = PDP^{-1}$

Example

A is 2 x 2 (unknown), with eigenvalues 2, 0

$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

and respective eigenvectors $\binom{1}{2}$ and $\binom{1}{3}$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

A = PDP⁻¹ =
$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 6 & -2 \\ 12 & -4 \end{pmatrix}$$

Multiplicity and dimension of eigenspace

Example

```
Characteristic polynomial of B: (\lambda - 1)^{1}(\lambda - 2)^{3}(\lambda - 4)^{2}
```

Eigenvalues of **B** are 1, 2 and 4 with multiplicity 1, 3, 2 respectively Eigenspaces of **B** are E_1 , E_2 and E_4 :

$$\dim E_1 = 1 \qquad \dim E_2 \leq 3 \qquad \dim E_4 \leq 2$$

Characteristic polynomial
$$(\lambda - \lambda_1)^{r_1} (\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_k)^{r_k}$$

$$\dim E_{\lambda_i} \leq r_i \text{ for all } i$$

The number of basis vectors in each eigenspace cannot be more than the multiplicity of the eigenvalue in the characteristic polynomial.

Dimension of eigenspace (x - 1)a(x - 2)b

Suppose 1 and 2 are the only eigenvalues of a 4x4 matrix A.

- What are the possible characteristic polynomial of **A** ? degree 4
- What are the possible dimensions of the eigenspaces E_1 and E_2 ?

Three possible characteristic polynomials

 $\dim E_{\lambda_i} \leq r_i$

•
$$(x-1)(x-2)^3$$
 dim $E_1 = 1$ and dim $E_2 = 1$, 2 or 3

•
$$(x-1)^3(x-2)$$
 dim $E_1 = 1$, 2 or 3 and dim $E_2 = 1$

•
$$(x-1)^2(x-2)^2$$
 dim $E_1 = 1$ or 2 and dim $E_2 = 1$ or 2

Conditions on diagonalizable matrices

A is n x n matrix char poly =
$$(x - \lambda_1)^{r_1}(x - \lambda_2)^{r_2} \cdots (x - \lambda_k)^{r_k}$$

- A is a diagonal matrix
- A is a symmetric matrix
- A has n eigenvalues
- A has n eigenvectors
- dim E_{λ_1} + dim E_{λ_2} + ... + dim E_{λ_k} = n Equivalent conditions
- dim $E_{\lambda_i} = r_i$ multiplicity of λ_i for every eigenvalue λ_i of **A**

To show that a matrix is not diagonalizable:

Find one eigenvalue such that dim $E_{\lambda_i} < r_i$

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Sufficient conditions

but not necessary conditions

For matrices with only 1 eigenvalue: diagonalizable ⇔ scalar matrices

scalar matrix:
$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$$

Triangular matrices

Are these triangular matrices diagonalizable?

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

3 distinct eigenvalues diagonalizable

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

only 2 eigenvalues diagonalizable can't tell by inspection

$$\dim E_1 = 2$$

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

only 1 eigenvalue non-scalar matrix non-diagonalizable

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{Find the dimension of } \mathbf{E_1} \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

only 2 eigenvalues non-diagonalizable can't tell by inspection

$$\dim E_1 = 1$$

True or False

Every diagonalizable matrix is row equivalent to a diagonal matrix.

A. True

B. False

Diagonalization is **NOT** Gaussian elimination

Powers of matrices
$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \quad \mathbf{D}^m = \begin{pmatrix} \lambda_1^m & 0 & \cdots & 0 \\ 0 & \lambda_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^m \end{pmatrix}$$

Diagonal matrix: **D**^k is easy to compute

Diagonalisable matrix: Ak

- Find **P** such that $P^{-1}AP = D$ is diagonal.
- Compute \mathbf{D}^k .
- $\mathbf{D}^{k} = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^{k} = \mathbf{P}^{-1}\mathbf{A}^{k}\mathbf{P}$.

tends to infinity (in the long run)

Fibonacci:

$$a_0 = 0$$
, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n \ge 2$

$$\binom{a_n}{a_{n+1}} = \mathbf{A} \binom{a_{n-1}}{a_n}$$

Solving linear recurrence relation

$$a_0 = U$$
 $a_1 = V$ $a_n = pa_{n-1} + qa_{n-2}$ for $n \ge 2$

Form the recurrence matrix **A**

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$$

Find the eigenvalues of **A**

$$\lambda_1, \lambda_2$$

If **A** is diagonalizable, find the matrix **P** that diagonalizes **A**

$$P = (v_1 v_2)$$

Set up $\mathbf{x}_n = \mathbf{A}^n \mathbf{x}_0$ and diagonalize \mathbf{A}^n

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \mathbf{P} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \mathbf{P}^{-1} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

Multiply out the RHS and equate the first component

$$a_n = s(\lambda_1)^n + t(\lambda_2)^n$$

Forming recurrence matrix

$$a_n = 3a_{n-1} + 5a_{n-2} + 7a_{n-3}$$
 with $a_0 = 0$, $a_1 = 1$ and $a_2 = 1$

Recurrence matrix (3×3)

$$\begin{pmatrix} a_{n} \\ a_{n+1} \\ a_{n+2} \end{pmatrix} = \begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_{n} \\ a_{n+1} \end{pmatrix}$$

$$a_{n} = 0 a_{n-1} + 1 a_{n} + 0 a_{n+1}$$
 $a_{n+1} = 0 a_{n-1} + 0 a_{n} + 1 a_{n+1}$
 $a_{n+2} = 7 a_{n-1} + 5 a_{n} + 3 a_{n+1}$

Orthogonally diagonalizable

A square matrix $\bf A$ is called diagonalizable if there exists an invertible matrix $\bf P$ such that $\bf P^{-1}AP$ is a diagonal matrix.

We say the matrix **P** diagonalizes **A**.

A square matrix **A** is called orthogonally diagonalizable Symmetric matrix if there exists an orthogonal matrix **P** such that **P**^T**AP** is a diagonal matrix. We say the matrix **P** orthogonally diagonalizes **A**.

Orthogonal diagonalization

Symmetric matrix

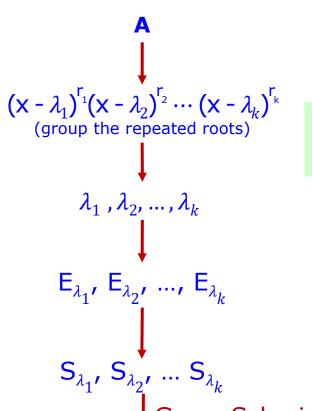
characteristic polynomial

distinct eigenvalues

eigenspaces

bases for eigenspaces

orthonormal bases for eigenspaces

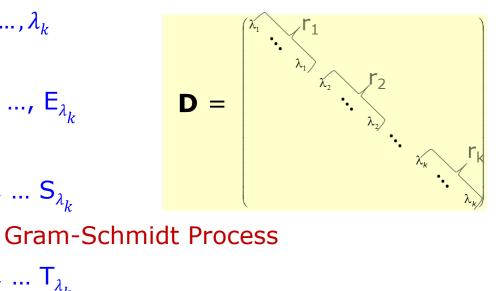


$$T_{\lambda_1} \cup T_{\lambda_2} \cup ... \cup T_{\lambda_k}$$

$$= \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$$

Form orthogonal matrix

$$\mathbf{P} = (\mathbf{v_1} \ \mathbf{v_2} \ \cdots \ \mathbf{v_n})$$



True or False

Let **A** and **B** be square matrices of the same size.

(I) If \mathbf{A} and \mathbf{B} are diagonalizable, then $\mathbf{A} + \mathbf{B}$ is diagonalizable. False

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$$
 , $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ both diagonalizable

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$
 not diagonalizable

(II) If **A** and **B** are orthogonally diagonalizable, True then **A** + **B** is orthogonally diagonalizable.

A and **B** are symmetric \Rightarrow **A** + **B** is symmetric

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Exercise 6 Q16 (Tutorial)

A square matrix $(a_{ij})_{n\times n}$ is called a stochastic matrix if all $a_{ij} \ge 0$ and $a_{1i} + a_{2i} + \cdots + a_{ni} = 1$ for $i = 1, 2, \ldots, n$.

(b) Let
$$\mathbf{B} = \begin{pmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.95 & 0.05 \\ 0 & 0.05 & 0.95 \end{pmatrix}$$

- (i) Is **B** a stochastic matrix? Just use definition
- (ii) Find a 3×3 invertible matrix **P** that diagonalizes **B**.

Go through the algorithm

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Exercise 6 Q16 (Tutorial)

A square matrix $(a_{ij})_{n\times n}$ is called a stochastic matrix if all $a_{ij} \ge 0$ and $a_{1i} + a_{2i} + \cdots + a_{ni} = 1$ for $i = 1, 2, \ldots, n$.

- (a) Let **A** be a stochastic matrix.
 - (i) Show that 1 is an eigenvalue of A.
 - (ii) If λ is an eigenvalue of \boldsymbol{A} , then $|\lambda| \leq 1$.

Hint: \mathbf{A} and \mathbf{A}^{T} have the same set of eigenvalues

(i) Find a special eigenvector of \mathbf{A}^{T} with eigenvalue 1.

(Hint: Look at Q29)

(ii) Let $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be an eigenvector of \mathbf{A}^T with eigenvalue λ .

Set up $\mathbf{A}^{\mathsf{T}}\mathbf{v} = \lambda \mathbf{v}$.

Consider the component of \mathbf{v} with the largest absolute value.

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

- (a) Show that \boldsymbol{u} is an eigenvector of \boldsymbol{A} . Direct checking
- (b) Let $\mathbf{v} = (a, b, c, d)^T$ be a nonzero vector.

Show that if $\mathbf{v} \cdot \mathbf{u} = 0$, then \mathbf{v} is an eigenvector of \mathbf{A} .

use this

Show that Av = 0

(c) Suppose
$$\mathbf{P} = \begin{pmatrix} 1/2 & a_1 & a_2 & a_3 \\ 1/2 & b_1 & b_2 & b_3 \\ 1/2 & c_1 & c_2 & c_3 \\ 1/2 & d_1 & d_2 & d_3 \end{pmatrix}$$
 is an orthogonal matrix.

Find P^TAP .

- Show that each column of P is an eigenvector of A
 - Use part (a) for the first column
 - Use part (b) for the other columns
- Find the eigenvalues corresponding to each column of **P**:

Exercise 6 Q22

```
If u_1, u_2, ..., u_n are eigenvectors of A with distinct eigenvalues \lambda_1, \lambda_2, ..., \lambda_n,
then \{\boldsymbol{u}_1, \boldsymbol{u}_2, ..., \boldsymbol{u}_n\} is linearly independent
Proof by mathematical induction
Basis step: k = 1 Clearly \{u_1\} is linearly independent
Inductive step:
        Assume \{u_1, u_2, ..., u_{k-1}\} is linearly independent
       WTS: \{u_1, u_2, ..., u_k\} is linearly independent
Set up the vector equation: c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2 + \dots c_k \boldsymbol{u}_k = \boldsymbol{0} - - - (*)
        Multiply (*) by A: c_1Au_1 + c_2Au_2 + ... c_kAu_k = 0
               which gives: c_1\lambda_1\boldsymbol{u}_1 + c_2\lambda_2\boldsymbol{u}_2 + \dots c_k\lambda_k\boldsymbol{u}_k = \boldsymbol{0} ---(**)
                      \lambda_k \times (*): C_1 \lambda_k \mathbf{u}_1 + C_2 \lambda_k \mathbf{u}_2 + \dots C_k \lambda_k \mathbf{u}_k = \mathbf{0} - - - (***)
```

$$c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \dots c_k \mathbf{u}_k = \mathbf{0} - --(*)$$

 $c_1 \lambda_1 \mathbf{u}_1 + c_2 \lambda_2 \mathbf{u}_2 + \dots c_k \lambda_k \mathbf{u}_k = \mathbf{0} - --(**)$
 $c_1 \lambda_k \mathbf{u}_1 + c_2 \lambda_k \mathbf{u}_2 + \dots c_k \lambda_k \mathbf{u}_k = \mathbf{0} - --(***)$

Exercise 6 Q22

If u_1 , u_2 , ..., u_n are eigenvectors of A with distinct eigenvalues λ_1 , λ_2 , ..., λ_n , then $\{u_1, u_2, ..., u_n\}$ is linearly independent

Assume $\{u_1, u_2, ..., u_{k-1}\}$ is linearly independent

(**) - (***):
$$c_1(\lambda_1 - \lambda_k) \mathbf{u}_1 + c_2(\lambda_2 - \lambda_k) \mathbf{u}_2 + \dots c_{k-1}(\lambda_{k-1} - \lambda_k) \mathbf{u}_{k-1} = \mathbf{0}$$

$$c_i(\lambda_i - \lambda_k) = 0$$

But $\lambda_i \neq \lambda_k$. This implies $c_i = 0$ for all i = 1, 2, ..., k-1.

Substitute in (*): $c_k u_k = 0$ This implies $c_k = 0$ as well.

Hence $\{u_1, u_2, ..., u_k\}$ is linearly independent

Announcement

- Homework 4
 - Deadline: 13 November
- Zoom lecture next week (week 12)
 - Live zoom ongoing
- Examplify for Mock & final exam
 - install in your PC
 - briefing and mock exam during week 13 zoom session
- Past year papers
 - Uploaded in LumiNUS > Files > Past Year Paper
- Online quiz 11
 - Due next Thursday