Answers/Solutions of Exercise 5 (Q28-34)

28. (a) It is easy to check that U and V are orthogonal. Since $\dim(\mathbb{R}^3) = 3$, by Remark 5.2.6, U and V are bases for $\mathbb{R}^3$.

    (b) $U' = \{\frac{1}{\sqrt{5}}(2,1,0),\ (0,0,1),\ \frac{1}{\sqrt{5}}(-1,2,0)\}$

    $V' = \{\frac{1}{\sqrt{5}}(0,-1,2),\ \frac{1}{\sqrt{6}}(-1,2,1),\ \frac{1}{\sqrt{30}}(5,2,1)\}$

    (c) $\boldsymbol{P} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{\sqrt{5}} & -\frac{2}{5} \\ 0 & \frac{1}{\sqrt{6}} & \frac{5}{\sqrt{30}} \\ \frac{12}{5\sqrt{6}} & \frac{1}{\sqrt{30}} & -\frac{1}{5\sqrt{6}} \end{pmatrix}$.

    (d) Yes.

29. Let $\boldsymbol{R} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

    (a) $\boldsymbol{R}^{\mathrm{T}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{\sqrt{3}}{2} \\ \frac{1}{2} - \sqrt{3} \end{pmatrix}$

    (b) $\boldsymbol{R} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\sqrt{3}}{2} \\ \frac{1}{2} + \sqrt{3} \end{pmatrix}$

    (c) $\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \ \Leftrightarrow\ \begin{pmatrix} 1 & 1 \end{pmatrix} \boldsymbol{R} \begin{pmatrix} x' \\ y' \end{pmatrix} = 1 \ \Leftrightarrow\ (1+\sqrt{3})x' + (1-\sqrt{3})y' = 2$

30. $\boldsymbol{A} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

31. (a) $(\boldsymbol{u})_{S_1} = (1,4)$, $(\boldsymbol{v})_{S_1} = (-1,1)$, $(\boldsymbol{u})_{S_1} \cdot (\boldsymbol{v})_{S_1} = 3$.

    $(\boldsymbol{u})_{S_2} = (-\frac{7}{3}, \frac{5}{3})$, $(\boldsymbol{v})_{S_2} = (-1,0)$, $(\boldsymbol{u})_{S_2} \cdot (\boldsymbol{v})_{S_2} = \frac{7}{3}$.

    $(\boldsymbol{u})_{S_3} = (\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}})$, $(\boldsymbol{v})_{S_3} = (0, \sqrt{2})$, $(\boldsymbol{u})_{S_3} \cdot (\boldsymbol{v})_{S_3} = 3$.

    Note that $(\boldsymbol{u})_{S_1} \cdot (\boldsymbol{v})_{S_1} = (\boldsymbol{u})_{S_3} \cdot (\boldsymbol{v})_{S_3} \neq (\boldsymbol{u})_{S_2} \cdot (\boldsymbol{v})_{S_2}$. See (b) for an explanation.

    (b) Let $\boldsymbol{P}$ be the transition matrix from S to T. Since S and T are orthonormal bases, P is orthogonal, i.e. $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{P} = \boldsymbol{I}$. (To use the transition matrix, it is more convenient to write the coordinate vectors as column vectors, i.e. we use $[\boldsymbol{u}]_S$, $[\boldsymbol{v}]_S$, $[\boldsymbol{u}]_T$ and $[\boldsymbol{v}]_T$ in the following computation.)

    $$\begin{aligned} [\boldsymbol{u}]_T \cdot [\boldsymbol{v}]_T = ([\boldsymbol{u}]_T)^{\mathrm{T}}[\boldsymbol{v}]_T &= (\boldsymbol{P}[\boldsymbol{u}]_S)^{\mathrm{T}}(\boldsymbol{P}[\boldsymbol{v}]_S) \\ &= ([\boldsymbol{u}]_S)^{\mathrm{T}}\boldsymbol{P}^{\mathrm{T}}\boldsymbol{P}[\boldsymbol{v}]_S = ([\boldsymbol{u}]_S)^{\mathrm{T}}[\boldsymbol{v}]_S = [\boldsymbol{u}]_S \cdot [\boldsymbol{v}]_S. \end{aligned}$$

32. (a) $||\boldsymbol{Au}||^2 = (\boldsymbol{Au})^{\mathrm{T}}(\boldsymbol{Au}) = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Au} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = ||\boldsymbol{u}||^2$. Since both $||\boldsymbol{u}||$ and $||\boldsymbol{Au}||$ are nonnegative, we have $||\boldsymbol{Au}|| = ||\boldsymbol{u}||$.

(b) $d(\boldsymbol{Au}, \boldsymbol{Av}) = ||\boldsymbol{Au} - \boldsymbol{Av}|| = ||\boldsymbol{A}(\boldsymbol{u} - \boldsymbol{v})|| = ||\boldsymbol{u} - \boldsymbol{v}|| = d(\boldsymbol{u}, \boldsymbol{v})$

(c) $(\boldsymbol{Au}) \cdot (\boldsymbol{Av}) = (\boldsymbol{Au})^{\mathrm{T}}\boldsymbol{Av} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Av} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v}$. So

$$\begin{aligned}
\text{the angle between } \boldsymbol{u} \text{ and } \boldsymbol{v} &= \cos^{-1}\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{||\boldsymbol{u}||\,||\boldsymbol{v}||}\right) \\
&= \cos^{-1}\left(\frac{(\boldsymbol{Au}) \cdot (\boldsymbol{Av})}{||\boldsymbol{Au}||\,||\boldsymbol{Av}||}\right) \\
&= \text{the angle between } \boldsymbol{Au} \text{ and } \boldsymbol{Av}.
\end{aligned}$$

33. (a) Since $\boldsymbol{A}$ is invertible, by Question 3.30(b)(i), T is linearly independent. So T is a basis for $\mathbb{R}^n$ by Theorem 3.6.7.

(b) See Question 5.32.

(c) Yes.

34. (a) True. Note that $\boldsymbol{c_i} \cdot \boldsymbol{c_j} = 0$ if $i \neq j$ and $\boldsymbol{c_i} \cdot \boldsymbol{c_i} = 1$.

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \begin{pmatrix} \boldsymbol{c_1}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{c_k}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \boldsymbol{c_1} & \cdots & \boldsymbol{c_k} \end{pmatrix} = \begin{pmatrix} \boldsymbol{c_1} \cdot \boldsymbol{c_1} & \cdots & \boldsymbol{c_1} \cdot \boldsymbol{c_k} \\ \vdots & & \vdots \\ \boldsymbol{c_k} \cdot \boldsymbol{c_1} & \cdots & \boldsymbol{c_k} \cdot \boldsymbol{c_k} \end{pmatrix} = \boldsymbol{I}_k.$$

(b) False. For example, let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(c) False. For example, let $\boldsymbol{A} = \boldsymbol{I}_2$, $\boldsymbol{B} = -\boldsymbol{I}_2$.

(d) True. $(\boldsymbol{AB})^{\mathrm{T}}(\boldsymbol{AB}) = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{AB} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{B} = \boldsymbol{I}$.