NATIONAL UNIVERSITY OF SINGAPORE

CS1231S DISCRETE STRUCTURES

(Semester 2: AY2019/2020)

Time Allowed: 2 Hours

INSTRUCTIONS TO STUDENTS

1. This assessment paper contains **FOUR** questions and comprises **FOUR** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated in brackets.
3. Write your answers on your own paper.

EXAMINER'S USE ONLY		
Question	Marks	Score
Q1	7	
Q2	8	
Q3	21	
Q4	14	
Total	50	

1. For a set X, the identity function is $i_X : X \to X$ such that $i_X(x) = x$ for all $x \in X$.

   (i) Give an example of a function $f : \{a, b\} \to \{a, b\}$ such that $f \neq i_{\{a,b\}}$ and f is bijective. [1 mark]

   Solution:

   $$f(a) = b,\ f(b) = a$$

   (ii) Give an example of a function $g : \{a, b, c\} \to \{a, b, c\}$ such that $g \neq i_{\{a,b,c\}}$ and $g \circ g$ is bijective. [2 marks]

   Solution:

   $g(a) = b,\ g(b) = a,\ g(c) = c$
   Then $g \circ g(a) = g(g(a)) = g(b) = a,\ g \circ g(b) = g(g(b)) = g(a) = b$
   and $g \circ g(c) = g(g(c)) = g(c) = c$
   so $g \circ g = i_{\{a,b,c\}}$, which is bijective.

   Note to grader:
   Many other possibilities.

   (iii) Suppose $h : X \to X$ is a function such that $h \circ h$ is 1-1 (injective). Prove that h is 1-1. [2 marks]

   Solution:

   $h(b) = h(c) \Rightarrow h(h(b)) = h(h(c)) \Rightarrow h \circ h(b) = h \circ h(c) \Rightarrow b = c$ since $h \circ h$ is 1-1.
   i.e. h is 1-1.
   Alternative:
   Tutorial 7, Problem 1(i): $f : X \to Y$, $g : Y \to Z$, $g \circ f$ is 1-1 $\Rightarrow$ f is 1-1.
   Let $X = Y = Z$ and $f = g = h$.

   (iv) Suppose $h : X \to X$ is a function such that $h \circ h$ is onto (surjective). Prove that h is onto. [2 marks]

   Solution:

   Consider any $b \in X$.
   $h \circ h$ is onto $\Rightarrow \exists a \in X$ such that $h \circ h(a) = b$,
   so $b = h(h(a)) = h(c)$ where $c = h(a) \in X$.
   Alternative:
   Tutorial 7, Problem 1(ii): $f : X \to Y$, $g : Y \to Z$, $g \circ f$ is onto $\Rightarrow$ g is onto.
   Let $X = Y = Z$ and $f = g = h$.

2. Recall from the Assignment and Quiz2 the equivalence relation $\approx$ on $\mathbb{R}$ defined by

$$\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ x \approx y \leftrightarrow \lfloor x \rfloor = \lfloor y \rfloor.$$

The equivalence classes are $I_n = \{x \in \mathbb{R} \mid n \leq x < n+1\}$, where $n \in \mathbb{Z}$.

(i) Let $k \in \mathbb{Z}$. Explain why, if $k \in I_n$, then $k = n$. [2 marks]

Solution:

The only integer in I_n is n,
so if k is an integer and $k \in I_n$, then $k = n$.

(ii) Let $\mathcal{I} = \{I_n \mid n \in \mathbb{Z}\}$. Prove that $\mathcal{I}$ is countable. [3 marks]

Solution:

Define $g : \mathbb{Z} \to \mathcal{I}$ by $g(n) = I_n$.
g is 1-1: Suppose $g(n) = g(k)$, where $n, k \in \mathbb{Z}$.
Then $k \in I_k = g(k) = g(n) = I_n$, so $k = n$ by (i).
g is onto: Consider any $I_n \in \mathcal{I}$. Then $I_n = g(n)$.
Thus g is bijective.
There is a bijection $f : \mathbb{N} \to \mathbb{Z}$,
so there is a bijection $g \circ f : \mathbb{N} \to \mathcal{I}$
Thus $\mathcal{I}$ is countable.

(iii) It is known that $\mathbb{R}$ is uncountable. Prove that I_n is uncountable for every $n \in \mathbb{Z}$. [3 marks]

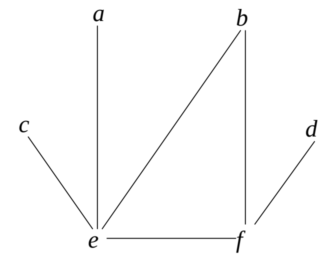
Solution:

For any n, there is a bijection $f : I_0 \to I_n$ defined by $f(x) = n + x$.
(f is well-defined: $x \in I_0 \Rightarrow 0 \leq x < 1 \Rightarrow n \leq n + x < n + 1 \Rightarrow f(x) \in I_n$.
f is 1-1: $f(b) = f(c) \Rightarrow n + b = n + c \Rightarrow b = c$.
f is onto: for any $y \in I_n$, $y - n \in I_0$ and $f(y - n) = n + (y - n) = y$.)
By Tutorial 7, Problem 7, $I = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has the same cardinality as $\mathbb{R}$,
so I is uncountable.
Since $I \subseteq I_0$, I_0 is uncountable (Tutorial 8, Problem 6(i)).
Since $f : I_0 \to I_n$ is bijective, I_n is also uncountable.

3. Let G be an undirected graph, and $G = (V, E)$. Recall that a subgraph of the form $(\{x_1, \ldots, x_p\}, \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{p-1}, x_p\}\})$ is called a **path** between x_1 and x_p in G, and this path has **length** $p - 1$.

Now, for integer $k \geq 1$, define
$E_k = \{\{x, y\} \mid x \neq y$ and there is a path of length k in G between x and $y\}$,
and let $G_k = (V, E_k)$. Note that $E_1 = E$ and $G_1 = G$.

Thus, for the undirected graph in Figure 1,
we have $\{e, b\} \in E_1$, $\{e, d\} \in E_2$, $\{e, d\} \in E_3$, etc.



(V, E)

Figure 1

(i) List the elements of V and E for Figure 1. [2 marks]

Solution:

$V = \{a, b, c, d, e, f\}$
$E = \{\{a, e\}, \{b, e\}, \{b, f\}, \{c, e\}, \{d, f\}, \{e, f\}\}$

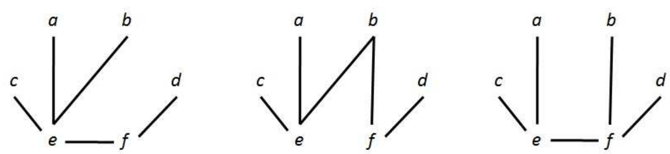
(ii) What is the length of the longest path in Figure 1? [1 mark]

Solution:

4 (e.g. c–e–b–f–d)

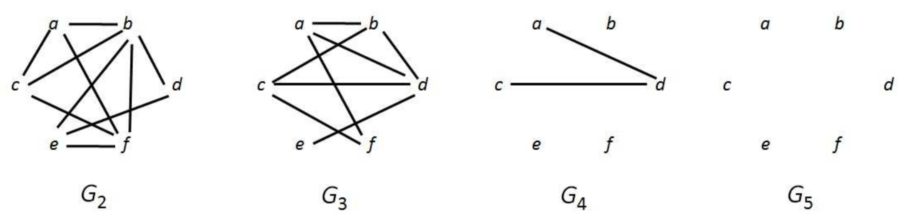
(iii) Draw all spanning trees for the graph in Figure 1. [3 marks]

Solution:



(iv) Draw G_2, G_3, G_4 and G_5 for the graph in Figure 1. [4 marks]

Solution:



(v) Identify all (if any) cyclic graphs in (iv). [1 mark]

Solution:

G_2 and G_3

Note to grader:

Grade (v) to (viii) according to student's answer to (iv), regardless of whether latter is correct.

(vi) Identify all (if any) connected graphs in (iv). [1 mark]

Solution:

G_2 and G_3

(vii) Among G_2, G_3, G_4 and G_5, which (if any) are trees? [1 mark]

Solution:

None

(viii) In (iv), how many connected components does G_4 have? [1 mark]

Solution:

4

(ix) For this part, consider any G (not just the one in Figure 1). Prove that G is connected if and only if $\{x, y\} \in \bigcup_{k=1}^{\infty} E_k$ for every $x, y \in V$ such that $x \neq y$. [2 marks]

Solution:

G is connected

$\Leftrightarrow \forall x \in U\ \forall y \in U\ x \neq y\ \rightarrow$ there is a path in G between x and y (by definition)

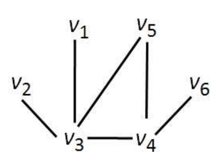
$\Leftrightarrow \forall x \in U\ \forall y \in U\ x \neq y\ \rightarrow\ \exists k \in \mathbb{Z}^+$ there is a path of length k in G between x and y

$\Leftrightarrow \forall x \in U\ \forall y \in U\ x \neq y\ \rightarrow\ \exists k \in \mathbb{Z}^+\ \{x, y\} \in E_k$ (by definition)

$\Leftrightarrow \forall x \in U\ \forall y \in U\ x \neq y\ \rightarrow\ \{x, y\} \in \bigcup_{k=1}^{\infty} E_k$

(x) Determine the number of graphs (with the same V) that are isomorphic to the graph in Figure 1. [5 marks]

Solution:



$\binom{6}{3}$ choices for triangle

$\binom{3}{1}$ choices for v_3; $\binom{3}{2}$ choices for v_1, v_2

$\binom{2}{1}$ choices for v_4

Multiplication Rule $\Rightarrow \binom{6}{3}\binom{3}{1}\binom{3}{2}\binom{2}{1} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} \cdot 3 \cdot 3 \cdot 2 = 360$ possibilites

Alternative:

6! permutations, v_1 and v_2 can be switched

$\Rightarrow \frac{6!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3 = 360$ possibilities

4. Let T be a rooted binary tree of height h. For $h \geq 1$, we call T a **strand** if and only if the following holds:

(I) there is exactly one leaf and one parent at every level ℓ, for $1 \leq \ell \leq h-1$ and

(II) there are exactly two leaves at level h.

Figure 2 below illustrates three strands T_1, T_2 and T_3.

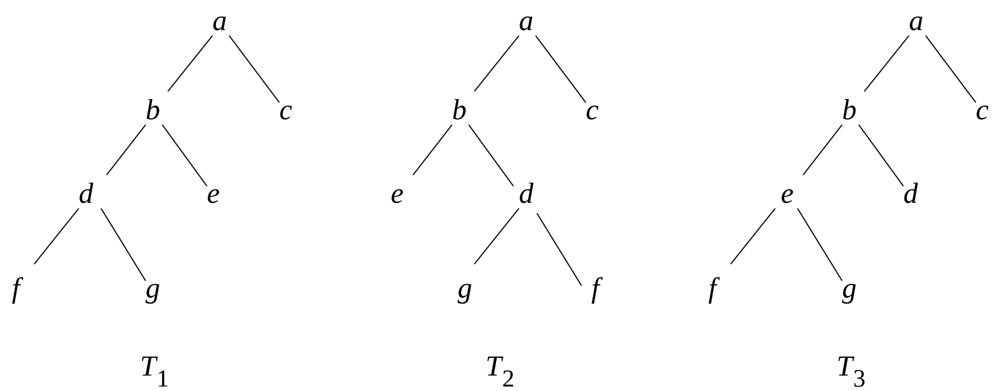


Figure 2

(i) Is $T_1 = T_2$? Is $T_2 = T_3$? Justify your answers. [2 marks]

Solution:

$$T_1 = (\{a,b,c,d,e,f,g\}, \{\{a,b\},\{a,c\},\{b,d\},\{b,e\},\{d,f\},\{d,g\})$$
$$= T_2$$

$T_2 \neq T_3$: d–g in T_2, d–g not in T_3.

(ii) Prove that, for any $h \geq 1$, a strand of height h has $2h+1$ nodes. [2 marks]

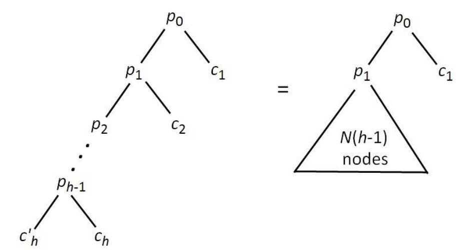
Solution:

Level 0 has 1 node.
(I) and (II) $\Rightarrow$ for $1 \leq \ell \leq h$, level ℓ has 2 nodes.
Total: $2h+1$ nodes.

Let $N(1) = 3$ and, for $h > 1$, let $N(h)$ be the number of different strands of height h, whose nodes are $\{v_1, v_2, \ldots, v_{2h+1}\}$.

(iii) Prove that $N(h) = 2(2h+1)hN(h-1)$ for integer $h > 1$. [5 marks]

Solution:



From $\{v_1, v_2, \ldots, v_{2h+1}\}$, there are

$\binom{2h+1}{1}$ choices for p_0

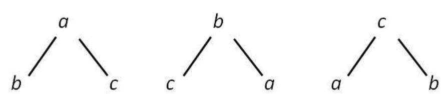
$\binom{2h+1-1}{1}$ choices for c_1

$N(h-1)$ strands rooted at p_1 if $h > 1$

total $= (2h+1)(2h)N(h-1)$

(iv) Use induction to prove that $N(h) = \frac{(2h+1)!}{2}$ for every positive integer h. [5 marks]

Solution:



Basis: $h = 1$

There are 3 strands, so $N(1) = 3 = \frac{3!}{2}$

so the claim is true for $h = 1$.

Induction Hypothesis: Suppose the claim is true if $h = k$, for some $k \geq 1$.

Induction Step: Consider a strand of height $k+1$.

$$\begin{aligned} N(k+1) &= 2(2(k+1)+1)(k+1)N(k) \text{ by (iii), since } k+1 > 1 \\ &= (2k+3)(2k+2)\tfrac{(2k+1)!}{2} \text{ by the Induction Hypothesis} \\ &= \tfrac{(2k+3)!}{2} \\ &= \tfrac{(2(k+1)+1)!}{2} \end{aligned}$$

so the claim is true for $h = k+1$.

By induction, the claim is true for all $h \geq 1$.

Note to grader: Partial credit for non-inductive proof.

Example: Permute all $2h+1$ nodes and divide by 2 since the lowest 2 leaves can be switched.