MA2001 Cheatsheet

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1 Important Identities & Techniques

1.1 Transpose

1. $(\boldsymbol{A}^T)^T = \boldsymbol{A}$
2. $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$
3. $(a\boldsymbol{A})^T = a\boldsymbol{A}^T$
4. $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$

1.2 Matrix Addition and Scalar Multiplication

1. Commutative Law: $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$
2. Associative Law: $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$

1.3 Matrix Multiplication

1. Associative Law: $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$
2. Distributive Law: $\boldsymbol{A}(\boldsymbol{B_1} + \boldsymbol{B_2}) = \boldsymbol{AB_1} + \boldsymbol{AB_2}$
   $(\boldsymbol{C_1} + \boldsymbol{C_2})\boldsymbol{A} = \boldsymbol{C_1A} + \boldsymbol{C_2A}$
3. Scalar Commutativity: $c(\boldsymbol{AB}) = (c\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(c\boldsymbol{B})$

1.4 Matrix Inverse

For an invertible matrix $\boldsymbol{A}$

1. $(a\boldsymbol{A})^{-1} = (1/a)\boldsymbol{A}^{-1}$
2. $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$
3. $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
4. $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
5. $(\boldsymbol{AB\ldots Z})^{-1} = \boldsymbol{Z}^{-1}\ldots\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

1.5 Determinants

For the elementary row operations,

1. $\boldsymbol{A} \xrightarrow{kR_i} \boldsymbol{B}$, $\det(\boldsymbol{B}) = k\det(\boldsymbol{A})$
2. $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}$, $\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$
3. $\boldsymbol{A} \xrightarrow{R_i + kR_j} \boldsymbol{B}$, $\det(\boldsymbol{B}) = \det(\boldsymbol{A})$

For the matrix operations on a square matrix of order n,

1. $\det(c\boldsymbol{A}) = c^n\det(\boldsymbol{A})$
2. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$
3. $\det(\boldsymbol{A}^T) = \det(\boldsymbol{A})$
4. $\det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})}$ if $\boldsymbol{A}$ is invertible

Using the determinant,

1. $\text{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T$ Where A_{ij} is the (i, j) cofactor of A
2. $\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\text{adj}(\boldsymbol{A})$
3. (Cramer's rule) For a system $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A}$ is invertible,

$$\boldsymbol{x} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} \det(A_1) \\ \det(A_2) \\ \vdots \\ \det(A_n) \end{pmatrix}$$

   Where A_i is the matrix obtained by replacing the ith column of $\boldsymbol{A}$ by $\boldsymbol{b}$

1.6 Dimension

1. For subspaces of $\mathbb{R}^n$ V and w, define $V + W = \{v + w | v \in V \wedge w \in W\}$. Then,
   - $\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$(Exercise 3 Q43)

1.7 Transition Matrices

For the bases $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ of a vector space V, we can obtain a transition matrix $\boldsymbol{P}$ from S to T:

$$\left(\begin{array}{cccc|c|c|c|c} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_k & \boldsymbol{u}_1 & \boldsymbol{u}_2 & \dots & \boldsymbol{u}_k \end{array} \right) \xrightarrow{GJE} \left(\begin{array}{c|c} \boldsymbol{I} & \boldsymbol{P} \end{array} \right)$$

Where the ith column of $\boldsymbol{P}$ is the coordinate vector of $\boldsymbol{u}_i$ in T, $[\boldsymbol{u}_i]_T$

1.8 Row Space, Column Space, Null Space, Rank, Nullity

For an $m \times n$ matrix A and $n \times p$ matrix B,

1. $\text{rank}(A) = \dim(\text{Row Space of A}) = \dim(\text{Column Space of A})$
2. $\text{nullity}(A) = \dim(\text{Null Space of A})$
3. $\text{rank}(A) \leqslant \min\{m, n\}$
4. $\text{rank}(A) = \min\{m, n\} \to A$ is full rank
5. For $v \in \mathbb{R}^n$, $Av \in$ column space of A
6. $\text{rank}(AB) \leqslant \min\{\text{rank}(A), \text{rank}(B)\}$ (Theorem 4.2.8)
7. $\text{rank}(A + B) \leqslant \text{rank}(A) + \text{rank}(B)$ (Exercise 4 Q23)
8. Number of columns of $A = \text{rank}(A) + \text{nullity}(A)$

1.9 Inner Products in $\mathbb{R}^n$

1. $u \cdot v = v \cdot u$
2. $(u + v) \cdot w = u \cdot w + v \cdot w$
3. $(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$
4. $||cu|| = |c|||u||$
5. $u \cdot u \geqslant 0$
6. $u \cdot u = 0 \iff u = 0$
7. $|u \cdot v| \leqslant ||u|| \cdot ||v||$ (Cauchy - Schwarz Inequality)

1.10 Orthogonal and Orthonormal Sets and Bases

1. To normalise a vector u_i, take $\frac{1}{||u_i||}u_i$

2. Gram-Schmidt Process: For a basis for vector space V, $\{u_1, u_2, \dots, u_k\}$

$$v_1 = u_1 \tag{1}$$
$$v_2 = u_2 - \frac{u_2 \cdot v_1}{||v_1||^2}v_1 \tag{2}$$
$$v_3 = u_3 - \frac{u_3 \cdot v_1}{||v_1||^2}v_1 - \frac{u_3 \cdot v_2}{||v_2||^2}v_2 \tag{3}$$
$$\vdots \tag{4}$$
$$v_k = u_k - \frac{u_k \cdot v_1}{||v_1||^2}v_1 - \frac{u_k \cdot v_2}{||v_2||^2}v_2 - \dots - \frac{u_k \cdot v_{k-1}}{||v_{k-1}||^2}v_{k-1} \tag{5}$$

   To obtain the orthogonal basis $\{v_1, v_2, \dots, v_k\}$

3. For a basis for vector space V, $S = \{u_1, u_2, \dots, u_k\}$ and any vector $w \in V$, $(w)_s = \{\frac{w \cdot u_1}{||u_1||^2}, \frac{w \cdot u_2}{||u_2||^2}, \dots, \frac{w \cdot u_k}{||u_k||^2}\}$

4. For a subspace of $\mathbb{R}^n, W$ $W^\perp$ is also a subspace of $\mathbb{R}^n$ (Exercise 5 Q7)

5. If p is a projection of v onto a subspace V where $S = \{u_1, u_2, \dots, u_k\}$ is a basis for V,

   - $v - p$ is orthogonal to V
   - $p = \frac{v \cdot u_1}{||u_1||^2} + \frac{v \cdot u_2}{||u_2||^2} + \dots + \frac{v \cdot u_k}{||u_k||^2}$

6. To extend an orthogonal set $S = \{u_1, u_2, u_3\}$to an orthogonal basis for $\mathbb{R}^4$:

   (a) Use row space method:

      i. Find "missing" row r in row echelon form

      ii. Find projection p of r onto span(S)

      iii. take vector $r - p$

   (b) Find non-zero vector v orthogonal to u_1, u_2, u_3

      i. Form matrix $A = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ using rows vector form of vectors in S

      ii. Solve $Av = 0$

1.11 Best Approximation, Orthogonal Matrices

1. For a matrix equation $Ax = b$, we can find the least squares solution by solving the system $A^TAx = A^Tb$

2. For a least square solution x_0, $Ax_0 = p$ where p is the projection of b onto column space of A

For an $n \times n$ orthogonal matrix A:

1. $A^{-1} = A^T$

2. $AA^T = I$

3. The rows and columns of A form an orthogonal basis for $\mathbb{R}^n$

(Exercise 5 Q32) For n-vectors u, v:

1. $||u|| = ||Au||$

2. $d(u, v) = d(Au, Av)$

3. The angle between u and v is equal to the angle between Au and Av

1.12 Eigenvalues & Eigenvectors

For a square matrix of order n:

1. $Ax = \lambda x$ where x is an eigenvector and λ is an eigenvalue of A
2. $\det(\lambda I - A)$ is the characteristic polynomial whose roots are the eigenvalues of A
3. The sum of the multiplicities of eigenvalues must always sum to n which is also the degree of the characteristic polynomial
4. For a given eigenspace E_λ associated to an eigenvalue λ, $\dim E_\lambda \leqslant$ multiplicity of λ

1.13 Diagonalization

1. For a square matrix A of order n, A is diagonalizable $\iff$ A has n linearly independent eigenvectors.
2. A is diagonalizable $\iff \forall \lambda_i, \dim(E_{\lambda_i}) = r_i$ where r_i is the multiplicity of λ_i
3. (Exercise 6 Q22) The union of the bases of eigenspaces is always linearly independent
4. (Theorem 6.2.7) A has n distinct eigenvalues $\Rightarrow$ A is diagonalizable
5. For a diagonalizable matrix A,
$$A = PDP^{-1}$$
Where P is an invertible matrix formed by the column vector form of the eigenvectors of A and D is a diagonal matrix with diagonal entries being the eigenvalues associated to the eigenvectors in P
6. For a diagonalizable matrix A, $P^{-1}AP = D$
7. $A^n = PD^nP^{-1}$
8. (Theorem 6.3.4) A square matrix is orthogonally diagonalizable $\iff$ it is symmetric
9. (Remark 6.3.6.1) The eigenvalues of a symmetric matrix are always real

1.14 Linear Transformation

The linearity conditions for a linear transformation T, vector u and constant c are:

1. $T(0) = 0$
2. $T(u + v) = T(u) + T(v)$
3. $T(cu) = cT(u)$
4. $T(c_1u_1 + c_2u_2 + \ldots + c_ku_k) = c_1T(u_1) + c_2T(u_2) + \ldots + c_kT(u_k)$

For a linear transformation T with standard matrix A:

1. range of T = column space of A
2. rank(T) = rank(A)
3. kernel of T = nullspace of A
4. nullity(T) = nullity(A)

2 Useful tricks

2.1 Linear Span

1. To show that $\text{span}(S) \subseteq \text{span}(T)$, show that every vector of S is a linear combination of T
2. To show that $\text{span}(S) = \text{span}(T)$, show that every vector of S is a linear combination of T and show that every vector of T is a linear combination of S. Thus $\text{span}(S) \subseteq \text{span}(T) \wedge \text{span}(T) \subseteq \text{span}(S)$
3. To show $\text{span}(S) \neq \text{span}(T)$, show $\text{span}(S) \not\subseteq \text{span}(T) \vee \text{span}(S) \not\supseteq \text{span}(T)$
4. To show $\text{span}(S) = \mathbb{R}^n$, sufficient to show that row echelon form of columns of S have no zero rows

2.2 Basis

1. To show S is a basis for $\mathbb{R}^n$,
   - Check S is linearly independent
   - Check S has n vectors $\rightarrow$ span$(S) = \mathbb{R}^n$
2. To show S is a basis for a subspace V of $\mathbb{R}^n$
   - Check S is linearly independent
   - Check span$(S) = V$
3. To check if S is an orthogonal basis for V:
   - Check S is orthogonal $\rightarrow$ S is linearly independent
   - Check span$(S) = V$

2.3 Eigenvalues & Eigenvectors

To find eigenvalues:

1. If an eigenvector u is given, multiply to matrix: $Au = \lambda u$
2. Solve the characteristic equation $\det(\lambda I - A) = 0$
3. If the matrix is triangular, take diagonal entries
4. (Exercise 6 Q3)If λ is an eigenvalue of A:
   (a) $c\lambda$ is an eigenvalue of cA
   (b) λ is an eigenvalue of A^T
   (c) λ^n is an eigenvalue of A^n
   (d) λ^{-1} is an eigenvalue of A^{-1} (when A is invertible)

2.4 Linear Transformation

Given $Au_1 = v_1, Au_2 = v_2, Au_3 = v_3$, to find A

$$\begin{pmatrix} Au_1 & Au_2 & Au_3 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \rightarrow A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}^{-1}$$

3 Definitions

1. A matrix is symmetric $\iff \boldsymbol{A} = \boldsymbol{A}^T$
2. If the vector equation
   $$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \boldsymbol{0}$$
   has only the trivial solution, $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is linearly independent
3. 2 vectors u and v are orthogonal if $u \cdot v = 0$
4. T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ if $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined by $\forall u \in \mathbb{R}^n \quad T(u) = Au$ where A is a $m \times n$ matrix.

4 Map of Linear Algebra

$\boldsymbol{A}$ is invertible	$\boldsymbol{A}$ is not invertible
$\det\boldsymbol{A} \neq 0$	$\det\boldsymbol{A} = 0$
RREF of $\boldsymbol{A}$ is identity matrix	RREF of $\boldsymbol{A}$ has a zero row
$\boldsymbol{A}x = \boldsymbol{0}$ has only the trivial solution	$\boldsymbol{A}x = \boldsymbol{0}$ has non-trivial solutions
$\boldsymbol{A}x = \boldsymbol{b}$ has a unique solution	$\boldsymbol{A}x = \boldsymbol{b}$ has no solutions or infinitely many solutions
rows (columns) of $\boldsymbol{A}$ are linearly independent	rows (columns) of $\boldsymbol{A}$ are linearly dependent
nullity$(\boldsymbol{A}) = 0$ and rank$(\boldsymbol{A}) = n$	nullity$(\boldsymbol{A}) > 0$ and rank$(\boldsymbol{A}) < n$
0 is not an eigenvalue of $\boldsymbol{A}$	0 is an eigenvalue of $\boldsymbol{A}$
$\ker(T_A) = \{0\}$ $\mathrm{R}(T_A) = \mathbb{R}^n$	$\ker(T_A) \neq \{0\}$ $\mathrm{R}(T_A) \neq \mathbb{R}^n$