CS1231S Chapter 8

Induction and recursion

8.1 Mathematical Induction

Principle 8.1.1 (Mathematical Induction (MI))**.** Let $m \in \mathbb{Z}$. To prove that $\forall n \in \mathbb{Z}_{\geqslant m}\ \ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(m)$ is true; and

(induction step) show that $\forall k \in \mathbb{Z}_{\geqslant m}\ \ \big(P(k) \Rightarrow P(k+1)\big)$ is true.

Justification. The two steps ensure the following are true:

$$\begin{array}{ll}
P(m) & \text{by the base step;} \\
P(m) \Rightarrow P(m+1) & \text{by the induction step with } k = m; \\
P(m+1) \Rightarrow P(m+2) & \text{by the induction step with } k = m+1; \\
P(m+2) \Rightarrow P(m+3) & \text{by the induction step with } k = m+2; \\
\vdots &
\end{array}$$

We deduce that $P(m), P(m+1), P(m+2), \ldots$ are all true by a series of modus ponens. □

Terminology 8.1.2. In the induction step, we assume we have $k \in \mathbb{Z}_{\geqslant m}$ such that $P(k)$ is true, and then show $P(k+1)$ using this assumption. In this process, the assumption that $P(k)$ is true is called the *induction hypothesis*.

Example 8.1.3. $1 + 2 + \cdots + n = \frac{1}{2}\, n(n+1)$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

Proof. 1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition "$1 + 2 + \cdots + n = \frac{1}{2}\, n(n+1)$".
2. (Base step) $P(1)$ is true because $1 = \frac{1}{2} \times 1 \times (1+1)$.
3. (Induction step)
 3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., such that

$$1 + 2 + \cdots + k = \frac{1}{2}\, k(k+1).$$

 3.2. Then $1 + 2 + \cdots + k + (k+1)$
 3.3. $= \frac{1}{2}\, k(k+1) + (k+1)$ by the induction hypothesis $P(k)$;
 3.4. $= \Big(\frac{k}{2} + 1\Big)(k+1) = \frac{k+2}{2}\,(k+1)$
 3.5. $= \frac{1}{2}\,(k+1)((k+1)+1)$.
 3.6. So $P(k+1)$ is true.

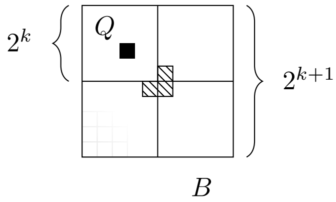


Figure 8.1: Covering a checkerboard with L-trominos

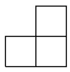
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}\ \ P(n)$ is true by MI. □

Terminology 8.1.4. We call the proof above an induction *on n* because n is the active variable in it.

Example 8.1.5. $n! > 2^n$ for all $n \in \mathbb{Z}_{\geqslant 4}$, where $n! = n \times (n-1) \times \cdots \times 1$.

Proof. 1. For each $n \in \mathbb{Z}_{\geqslant 4}$, let $P(n)$ be the proposition "$n! > 2^n$".
2. (Base step) $P(4)$ is true because $4! = 24 > 16 = 2^4$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 4}$ such that $P(k)$ is true, i.e., such that
$$k! > 2^k.$$
   3.2. Then $(k+1)! = (k+1) \times k!$ by the definition of !;
   3.3. $> (k+1) \times 2^k$ by the induction hypothesis $P(k)$;
   3.4. $> 2 \times 2^k$ as $k+1 \geqslant 4+1 > 2$;
   3.5. $= 2^{k+1}$.
   3.6. So $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 4}\ \ P(n)$ is true by MI. □

Example 8.1.6. An *L-tromino* is the following L-shape formed by three squares of the checkerboard:



For all $n \in \mathbb{Z}_{\geqslant 1}$, if one square is removed from a $2^n \times 2^n$ checkerboard, then the remaining squares can be covered by L-trominos.

Proof. 1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition

> if one square is removed from a $2^n \times 2^n$ checkerboard, then the remaining squares can be covered by L-trominos.

2. (Base step) $P(1)$ is true because such a board itself is an L-tromino.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true.
   3.2. 3.2.1. Let B be a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed.
      3.2.2. Divide B into four $2^k \times 2^k$ quadrants.
      3.2.3. Let Q be the quadrant containing the removed square.
      3.2.4. Remove one L-tromino from the centre of B in a way such that each quadrant other than Q has one square removed.
      3.2.5. We are left with four $2^k \times 2^k$ checkerboards, each with one square removed.
      3.2.6. By the induction hypothesis, each quadrant can be covered by L-trominos.
      3.2.7. Hence B can be covered by L-trominos.
   3.3. This shows $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}\ \ P(n)$ is true by MI. □

Example 8.1.7. All participants in this Zoom meeting have the same birthday. ✎ 8a

Proof. 1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition

if a Zoom meeting has exactly n participants, then all its participants have the same birthday.

2. (Base step) $P(1)$ is true because if a Zoom meeting has exactly 1 participant, then clearly all its participants have the same birthday.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true.
   - 3.2.
     - 3.2.1. Suppose a Zoom meeting has exactly $k+1$ participants.
     - 3.2.2. Pick two different participants a, b in the meeting.
     - 3.2.3. Ask a to leave the meeting.
     - 3.2.4. Since there are k people left in the meeting, by the induction hypothesis, all the remaining participants have the same birthday, including b.
     - 3.2.5. Tell a to join the meeting again, and then ask b to leave the meeting.
     - 3.2.6. Since there are k people left in the meeting, by the induction hypothesis, all the remaining participants have the same birthday, including a.
     - 3.2.7. The participants who stayed in the meeting throughout have the same birthday as both a and b.
     - 3.2.8. So a and b have the same birthday.
   - 3.3. This shows $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}\ \ P(n)$ is true by MI. ⊠

8.2 Strong Mathematical Induction

Principle 8.2.1 (Strong Mathematical Induction (Strong MI))**.** To prove that $\forall n \in \mathbb{Z}_{\geqslant m}\ \ P(n)$ is true, where each $P(n)$ is a proposition and $m \in \mathbb{Z}$, it suffices to choose some $\ell \in \mathbb{Z}_{\geqslant 0}$ and:

(base step) show that $P(m), P(m+1), \ldots, P(m+\ell-1)$ are true;

(induction step) show that

$$\forall k \in \mathbb{Z}_{\geqslant 0}\ \ \big(P(m) \wedge P(m+1) \wedge \cdots \wedge P(m+\ell-1+k) \Rightarrow P(m+\ell+k)\big)$$

is true.

Justification. The two steps ensure the following are true:

$$\begin{aligned}
&P(m) \wedge P(m+1) \wedge \cdots \wedge P(m+\ell-1) \\
&\qquad\qquad\qquad\qquad \text{by the base step;} \\
&P(m) \wedge P(m+1) \wedge \cdots \wedge P(m+\ell-1) \Rightarrow P(m+\ell) \\
&\qquad\qquad\qquad\qquad \text{by the induction step with } k=0; \\
&P(m) \wedge P(m+1) \wedge \cdots \wedge P(m+\ell-1) \wedge P(m+\ell) \Rightarrow P(m+\ell+1) \\
&\qquad\qquad\qquad\qquad \text{by the induction step with } k=1; \\
&P(m) \wedge P(m+1) \wedge \cdots \wedge P(m+\ell-1) \wedge P(m+\ell) \wedge P(m+\ell+1) \Rightarrow P(m+\ell+2) \\
&\qquad\qquad\qquad\qquad \text{by the induction step with } k=2; \\
&\qquad\qquad \vdots
\end{aligned}$$

We deduce that $P(m), P(m+1), P(m+2), P(m+3), \ldots$ are all true by a series of modus ponens. □

Definition 8.2.2. The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by setting

$$F_0 = 0 \quad \text{and} \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n$$

for each $n \in \mathbb{Z}_{\geqslant 0}$.

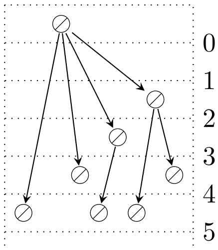


Figure 8.2: Rabbits

Example 8.2.3. $F_2 = 1+0 = 1, \quad F_3 = 1+1 = 2, \quad F_4 = 2+1 = 3, \quad F_5 = 3+2 = 5, \quad \ldots.$

Example 8.2.4.
- Initially, there is one pair of newly born matched rabbits.
- Each newly born rabbit takes one month to mature.
- Each mature pair of matched rabbits produces one pair of matched rabbits per month.

Let r_n denote the number of pairs of rabbits after n months. Then for every $n \in \mathbb{Z}_{\geqslant 0}$,

$$r_0 = 1 \quad \text{and} \quad r_1 = 1 \quad \text{and} \quad r_{n+2} = r_{n+1} + r_n,$$

where the r_{n+1} comes from the rabbits already present after $(n+1)$ months, and the r_n comes from the rabbits born after $(n+1)$ months.

Observation 8.2.5. $r_n = F_{n+1}$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

Example 8.2.6. $F_{n+1} \leqslant (7/4)^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. 1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition "$F_{n+1} \leqslant (7/4)^n$".
2. (Base step) $P(0)$ and $P(1)$ are true because

$$F_{0+1} = 1 \leqslant 1 = (7/4)^0 \quad \text{and} \quad F_{1+1} = 1 + 0 = 1 \leqslant 7/4 = (7/4)^1.$$

3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k+1)$ are true.
   3.2. Then $F_{(k+2)+1} = F_{k+3}$
   3.3. $= F_{k+2} + F_{k+1}$ by the definition of F_{k+3};
   3.4. $\leqslant (7/4)^{k+1} + (7/4)^k$ as $P(k)$ and $P(k+1)$ are true;
   3.5. $= (7/4)^k(7/4 + 1)$
   3.6. $< (7/4)^k(7/4)^2$ as $7/4 + 1 = 11/4 < 49/16 = (7/4)^2$;
   3.7. $= (7/4)^{k+2}$.
   3.8. So $P(k+2)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}\ \ P(n)$ is true by Strong MI. □

Remark 8.2.7. Given the same $P(n)$, Strong MI is more likely to succeed than usual MI, but the proof may be more cumbersome when written.

Remark 8.2.8. When $\ell = 0$ in Principle 8.2.1 (Strong MI), the base step is empty. Thus to prove that $\forall n \in \mathbb{Z}_{\geqslant m}\ \ P(n)$ is true, where each $P(n)$ is a proposition and $m \in \mathbb{Z}$, it suffices to show *only*

$$\forall k \in \mathbb{Z}_{\geqslant 0}\ \ \big(P(m) \wedge P(m+1) \wedge \cdots \wedge P(m+k-1) \Rightarrow P(m+k)\big).$$

(The conjunction of no formula is by convention always true.)

Example 8.2.9. (1) $S = \{x \in \mathbb{Z}_{\geqslant 0} : 0 < x < 5\}$ has smallest element 1.

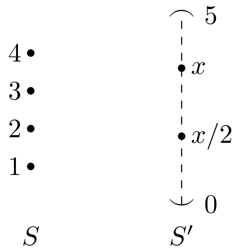


Figure 8.3: A difference between $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Q}_{\geqslant 0}$

(2) $S' = \{x \in \mathbb{Q}_{\geqslant 0} : 0 < x < 5\}$ has no smallest element because if $x \in S'$, then $x/2 \in S'$ and $x/2 < x$.

Theorem 8.2.10 (Well-Ordering Principle)**.** Every nonempty subset of $\mathbb{Z}_{\geqslant m}$, where $m \in \mathbb{Z}$, has a smallest element.

Proof. We prove this by Principle 8.2.1 (Strong MI) with $\ell = 0$.

1. Let $m \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}_{\geqslant m}$ with no smallest element.
2. For each $n \in \mathbb{Z}_{\geqslant m}$, let $P(n)$ be the proposition " $n \notin S$ ".
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(m), P(m+1), \ldots, P(m+k-1)$ are true, i.e., that $m, m+1, \ldots, m+k-1 \notin S$.
   3.2. 3.2.1. Suppose $m + k \in S$.
        3.2.2. Then $m + k$ is the smallest element of S by the induction hypothesis as $S \subseteq \mathbb{Z}_{\geqslant m}$.
        3.2.3. This contradicts our assumption that S has no smallest element on line 1.
   3.3. So $m + k \notin S$.
   3.4. Thus $P(m + k)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant m}\ \ P(n)$ is true by Strong MI.
5. This implies $S = \varnothing$ as $S \subseteq \mathbb{Z}_{\geqslant m}$. □

8.3 Recursively defined sequences

Terminology 8.3.1. A sequence $a_0, a_1, a_2, \ldots$ is said to be *recursively defined* if the definition of a_n involves $a_0, a_1, \ldots, a_{n-1}$ for all but finitely many $n \in \mathbb{Z}_{\geqslant 0}$.

Example 8.3.2. (1) Define $0!, 1!, 2!, \ldots$ by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$0! = 1 \quad \text{and} \quad (n+1)! = (n+1) \times n!.$$

Then $1! = 1 \times 1 = 1, \quad 2! = 2 \times 1 = 2, \quad 3! = 3 \times 2 = 6, \quad 4! = 4 \times 6 = 24, \quad \ldots.$

(2) The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ was defined in Definition 8.2.2 by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$F_0 = 0 \quad \text{and} \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n.$$

Then $F_2 = 1 + 0 = 1, \quad F_3 = 1 + 1 = 2, \quad F_4 = 2 + 1 = 3, \quad F_5 = 3 + 2 = 5, \quad \ldots.$

(3) Fix $r \in [0, 4]$ and $p_0 \in [0, 1]$. Define $p_1, p_2, \ldots$ by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$p_{n+1} = r(p_n - {p_n}^2).$$

If $r = 3$ and $p_0 = 1/2$, then

$$p_1 = 3\Big(\frac{1}{2} - \Big(\frac{1}{2}\Big)^2\Big) = \frac{3}{4}, \quad p_2 = 3\Big(\frac{3}{4} - \Big(\frac{3}{4}\Big)^2\Big) = \frac{9}{16}, \quad \ldots.$$

(4) Fix $a_0 \in \mathbb{Z}^+$. Define $a_1, a_2, a_3, \ldots$ by setting, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$a_{n+1} = \begin{cases} a_n/2, & \text{if } a_n \text{ is even;} \\ 3a_n + 1, & \text{if } a_n \text{ is odd.} \end{cases}$$

If $a_0 = 1$, then $a_1 = 3 \times 1 + 1 = 4, \quad a_2 = 4/2 = 2, \quad a_3 = 2/2 = 1, \quad \ldots.$

Exercise 8.3.3. Let $a_1 = 1$ and $a_{n+1} = a_n + (n+1)$ for all $n \in \mathbb{Z}_{\geqslant 1}$. Find a general formula for a_n in terms of n that does not involve $a_0, a_1, \ldots, a_{n-1}$. 8b

Proposition 8.3.4. There is a unique sequence $a_0, a_1, a_2, \ldots$ satisfying, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$a_0 = 0 \quad \text{and} \quad a_1 = 1 \quad \text{and} \quad a_{n+2} = a_{n+1} + a_n.$$

Proof (optional material). For the purpose of this proof, let us call a sequence $b_0, b_1, \ldots, b_{n-1}$ a *partial sequence* if for all $i \in \mathbb{Z}_{\geqslant 0}$ with $i < n$,

$$b_i = \begin{cases} 0, & \text{if } i = 0; \\ 1, & \text{if } i = 1; \\ b_{i-1} + b_{i-2}, & \text{if } i \geqslant 2. \end{cases}$$

1. First, we claim that there is a partial sequence of length n for every $n \in \mathbb{Z}_{\geqslant 0}$.
   1.1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition

   "there is a partial sequence of length n".

   1.2. (Base step) $P(0)$ is true because the empty sequence is trivially a partial sequence of length 0.
   1.3. (Induction step)
      1.3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(k)$ is true.
      1.3.2. This gives a partial sequence $b_0, b_1, \ldots, b_{k-1}$ of length k.
      1.3.3. Define
      $$b_k = \begin{cases} 0, & \text{if } k = 0; \\ 1, & \text{if } k = 1; \\ b_{k-1} + b_{k-2}, & \text{if } k \geqslant 2. \end{cases}$$
      1.3.4. Then $b_0, b_1, \ldots, b_k$ is a partial sequence of length $k + 1$ by the choice of b_k and because $b_0, b_1, \ldots, b_{k-1}$ is a partial sequence.
      1.3.5. So $P(k + 1)$ is true.
   1.4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0} \;\; P(n)$ is true by MI.
2. If $b_0, b_1, \ldots, b_{m-1}$ and $c_0, c_1, \ldots, c_{n-1}$ are partial sequences with $m \leqslant n$, then

$$\begin{aligned} b_0 &= 0 = c_0, \\ b_1 &= 1 = c_1, \\ b_2 &= b_1 + b_0 = c_1 + c_0 = c_2, \\ b_3 &= b_2 + b_1 = c_2 + c_1 = c_3, \\ &\vdots \\ b_{m-1} &= b_{m-2} + b_{m-3} = c_{m-2} + c_{m-3} = c_{m-1}. \end{aligned}$$

3. For each $n \in \mathbb{Z}_{\geqslant 0}$, define a_n to be the nth element of any partial sequence of length at least n.
4. Then the sequence $a_0, a_1, a_2, \ldots$ is well defined by lines 1 and 2.
5. This sequence $a_0, a_1, a_2, \ldots$ is what we want because it agrees with all the partial sequences, and the conditions in the definition of partial sequences match with the required conditions.

6. Let $b_0, b_1, b_2, \ldots$ be a sequence satisfying, for each $n \in \mathbb{Z}_{\geqslant 0}$,

$$b_0 = 0 \quad \text{and} \quad b_1 = 1 \quad \text{and} \quad b_{n+2} = b_{n+1} + b_n.$$

7. We show that $a_n = b_n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.
   7.1. Let $n \in \mathbb{Z}_{\geqslant 0}$.
   7.2. Note that $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ are partial sequences.
   7.3. So $a_n = b_n$ by line 2. □

8.4 Recursively defined sets

Theorem 8.4.1. $\mathbb{Z}_{\geqslant 0}$ is the unique set with the following properties.

(1) $0 \in \mathbb{Z}_{\geqslant 0}$. (base clause)

(2) If $x \in \mathbb{Z}_{\geqslant 0}$, then $x + 1 \in \mathbb{Z}_{\geqslant 0}$. (recursion clause)

(3) Membership for $\mathbb{Z}_{\geqslant 0}$ can always be demonstrated by (finitely many) successive applications of the clauses above. (minimality clause)

Example 8.4.2.

$$\begin{aligned} & 0 \in \mathbb{Z}_{\geqslant 0} && \text{by (1).} \\ \therefore\ & 1 \in \mathbb{Z}_{\geqslant 0} && \text{by (2) and the previous line.} \\ \therefore\ & 2 \in \mathbb{Z}_{\geqslant 0} && \text{by (2) and the previous line.} \end{aligned}$$

Remark 8.4.3. (1) and (2) are true when $\mathbb{Z}_{\geqslant 0}$ is changed to $\mathbb{Q}$, but (3) is not. So (1) and (2) are not enough to uniquely determine $\mathbb{Z}_{\geqslant 0}$.

Terminology 8.4.4. Theorem 8.4.1 gives a *recursive definition* of $\mathbb{Z}_{\geqslant 0}$.

Rough idea 8.4.5. A recursive definition of a set S consists of three types of clauses.

(base clause) Specify that certain elements, called *founders*, are in S: if c is a founder, then $c \in S$.

(recursion clause) Specify certain functions, called *constructors*, under which the set S is closed: if f is a constructor and $x \in S$, then $f(x) \in S$.

(minimality clause) Membership for S can always be demonstrated by (finitely many) successive applications of the clauses above.

In words, the members of S are precisely those objects that can be obtained from the founders by successively applying the constructors.

Rough idea 8.4.6 (structural induction)**.** Let S be a recursively defined set. To prove that $\forall x \in S \;\; P(x)$ is true, where each $P(x)$ is a proposition, it suffices to:

(base step) show that $P(c)$ is true for every founder c;

(induction step) show that $\forall x \in S \;\; \big(P(x) \Rightarrow P(f(x))\big)$ is true for every constructor f.

In words, if all the founders satisfy a property P, and P is preserved by all constructors, then all elements of S satisfy P.

Example 8.4.7. The set $2\mathbb{Z}$ of all even integers can be defined recursively as follows.

(1) $0 \in S$. (base clause)

(2) If $x \in S$, then $x - 2 \in 2\mathbb{Z}$ and $x + 2 \in 2\mathbb{Z}$. (recursion clause)

(3) Membership for $2\mathbb{Z}$ can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Theorem 8.4.8 (Structural induction over $2\mathbb{Z}$)**.** To prove that $\forall n \in 2\mathbb{Z}\ \ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(0)$ is true; and

(induction step) show that $\forall x \in 2\mathbb{Z}\ \ \big(P(x) \Rightarrow P(x-2) \land P(x+2)\big)$ is true.

Question 8.4.9. Define a set S recursively as follows. ✐ 8c

(1) $1 \in S$. (base clause)

(2) If $x \in S$, then $2x \in S$ and $3x \in S$. (recursion clause)

(3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the numbers $9, 10, 11, 12, 13$ are in S? Which are not?