MA2001

LIVE LECTURE 11

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Topics for week 11

6.2 Diagonalization

6.3 Orthogonal Diagonalization

6.4 for self-reading. Not in the exam scope

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Diagonalizable matrix

P diagonalizes **A**

$$\mathbf{P}^{-1}\boxed{\mathbf{A}}\,\mathbf{P} \longrightarrow \mathbf{D}$$

diagonalizable matrix

diagonal matrix

Diagonalization

Not possible to find **P** ↔ **A** non-diagonalizable

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Diagonalization

n x n matrix	**A**
characteristic polynomial	$(x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2}\cdots(x-\lambda_k)^{r_k}$ (group the repeated roots)
distinct eigenvalues	$\lambda_1, \lambda_2, \ldots, \lambda_k$
eigenspaces subspaces of $\mathbf{R}^n$	$E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_k}$
bases for eigenspaces	$S_{\lambda_1}, S_{\lambda_2}, \ldots S_{\lambda_k}$

<u>Case 1</u>

$S_{\lambda_1} \cup S_{\lambda_2} \cup \ldots \cup S_{\lambda_k}$ contains exactly n linearly independent eigenvectors

A diagonalizable

<u>Case 2</u>

$S_{\lambda_1} \cup S_{\lambda_2} \cup \ldots \cup S_{\lambda_k}$ contains less than n linearly independent eigenvectors

A not diagonalizable

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Case 1

$S_{\lambda_1} \cup S_{\lambda_2} \cup \dots \cup S_{\lambda_k}$ contains exactly n linearly independent eigenvectors

A diagonalizable

Finding P and D

n x n matrix	**A**
characteristic polynomial	$(x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2}\cdots(x-\lambda_k)^{r_k}$ (group the repeated roots)
distinct eigenvalues	$\lambda_1, \lambda_2, \dots, \lambda_k$
eigenspaces	$E_{\lambda_1}, E_{\lambda_2}, \dots, E_{\lambda_k}$
bases for eigenspaces	$S_{\lambda_1}, S_{\lambda_2}, \dots S_{\lambda_k}$

$$\mathbf{D} = \begin{pmatrix} \underbrace{\begin{matrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_1\end{matrix}}_{r_1} & & & \\ & \underbrace{\begin{matrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_2\end{matrix}}_{r_2} & & \\ & & \ddots & \\ & & & \underbrace{\begin{matrix}\lambda_k & & \\ & \ddots & \\ & & \lambda_k\end{matrix}}_{r_k} \end{pmatrix}$$

form matrix

$\mathbf{P} = (\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_n)$

$S_{\lambda_1} \cup S_{\lambda_2} \cup \dots \cup S_{\lambda_k} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$

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Many ways to diagonalize A

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 4 \end{pmatrix}$$

Characteristic polynomial: $(x - 3)^2(x - 6)$

eigenvalues: 3 (repeated), 6

A is diagonalizable $\left\{ \begin{array}{l} \text{eigenvectors associated to 3: } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \\ \text{eigenvectors associated to 6: } \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{array} \right.$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \quad \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \quad \mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\mathbf{Q} = \begin{pmatrix} 5 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & -2 & 2 \end{pmatrix} \quad \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & -3 \\ 2 & -2 & -3 \end{pmatrix} \quad \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

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Writing diagonalizable matrix as $\mathbf{PDP}^{-1}$

Given matrix **A** (diagonalizable)
Find eigenvalues and form **D**
Find eigenvectors and form **P**
Then $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$
So we can express **A** as $\mathbf{A} = \mathbf{PDP}^{-1}$

Suppose **A** unknown
Given all eigenvalues and eigenvectors of **A**
Form **D** (using eigenvalues)
Form **P** (using eigenvectors)
So we can find **A** using $\mathbf{A} = \mathbf{PDP}^{-1}$

Example

A is 2 x 2 (unknown), with eigenvalues 2, 0

$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

and respective eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

$$\mathbf{A} = \mathbf{PDP}^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 6 & -2 \\ 12 & -4 \end{pmatrix}$$

Multiplicity and dimension of eigenspace

Example

Characteristic polynomial of **B**: $(\lambda-1)^1(\lambda-2)^3(\lambda-4)^2$

Eigenvalues of **B** are 1, 2 and 4 with multiplicity 1, 3, 2 respectively

Eigenspaces of **B** are E_1 , E_2 and E_4:

$$\dim E_1 = 1 \qquad \dim E_2 \leq 3 \qquad \dim E_4 \leq 2$$

Characteristic polynomial : $(\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots(\lambda-\lambda_k)^{r_k}$

$$\boxed{\dim E_{\lambda_i} \leq r_i \text{ for all } i}$$

The number of basis vectors in each eigenspace cannot be more than the multiplicity of the eigenvalue in the characteristic polynomial.

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Dimension of eigenspace $(x-1)^a(x-2)^b$

Suppose 1 and 2 are the only eigenvalues of a 4x4 matrix $\boldsymbol{A}$.

- What are the possible characteristic polynomial of $\boldsymbol{A}$? degree 4
- What are the possible dimensions of the eigenspaces E_1 and E_2 ?

$\dim E_{\lambda_i} \leq r_i$

Three possible characteristic polynomials

- $(x-1)(x-2)^3$ dim $E_1 = 1$ and dim $E_2 = 1, 2$ or 3
- $(x-1)^3(x-2)$ dim $E_1 = 1, 2$ or 3 and dim $E_2 = 1$
- $(x-1)^2(x-2)^2$ dim $E_1 = 1$ or 2 and dim $E_2 = 1$ or 2

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Conditions on diagonalizable matrices

A is n x n matrix   char poly = $(x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2}\cdots(x-\lambda_k)^{r_k}$

- **A** is a diagonal matrix
- **A** is a symmetric matrix
- **A** has n ______ eigenvalues

Sufficient conditions
but not necessary conditions

- **A** has n ______ eigenvectors
- dim E_{λ_1} + dim E_{λ_2} + … + dim E_{λ_k} = n
- dim E_{λ_i} = r_i multiplicity of λ_i for every eigenvalue λ_i of **A**

Equivalent conditions

To show that a matrix is not diagonalizable:

Find one eigenvalue such that dim $E_{\lambda_i} < r_i$

For matrices with only 1 eigenvalue:
diagonalizable ⇔ scalar matrices

scalar matrix: $\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}$

Triangular matrices

Are these triangular matrices diagonalizable?

$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

3 distinct eigenvalues
diagonalizable

$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

only 1 eigenvalue
non-scalar matrix
non-diagonalizable

Find the dimension of E_1

$\mathbf{C} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

only 2 eigenvalues
diagonalizable
can't tell by inspection

dim $E_1 = 2$

$\mathbf{D} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

only 2 eigenvalues
non-diagonalizable
can't tell by inspection

dim $E_1 = 1$

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True or False

Every diagonalizable matrix is row equivalent to a diagonal matrix.

A. True
B. False

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

diagonalizable

not row equivalent to diagonal matrix

Diagonalization is NOT Gaussian elimination

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Powers of matrices

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \qquad \mathbf{D}^m = \begin{pmatrix} {\lambda_1}^m & 0 & \cdots & 0 \\ 0 & {\lambda_2}^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {\lambda_n}^m \end{pmatrix}$$

Diagonal matrix: $\mathbf{D}^k$ is easy to compute

Diagonalisable matrix: $\mathbf{A}^k$

- Find $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ is diagonal.
- Compute $\mathbf{D}^k$.
- $\mathbf{D}^k = (\mathbf{P}^{-1}\mathbf{AP})^k = \mathbf{P}^{-1}\mathbf{A}^k\mathbf{P}$.
- $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\,\mathbf{P}^{-1}$.

tends to infinity
(in the long run)

Fibonacci:
$a_0 = 0,\ a_1 = 1,\ a_n = a_{n-1} + a_{n-2}$ for $n \ge 2$

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

Solving linear recurrence relation

$a_0 = \mathrm{u}$ $\quad a_1 = \mathrm{v}$ $\quad a_n = \mathrm{p}a_{n-1} + \mathrm{q}a_{n-2}$ for $n \ge 2$

Form the recurrence matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ \mathrm{q} & \mathrm{p} \end{pmatrix}$$

Find the eigenvalues of $\boldsymbol{A}$

$$\lambda_1,\ \lambda_2$$

If $\boldsymbol{A}$ is diagonalizable, find the matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$

$$\boldsymbol{P} = (\ \boldsymbol{v}_1\ \ \boldsymbol{v}_2)$$

Set up $\boldsymbol{x}_\mathrm{n} = \boldsymbol{A}^\mathrm{n}\boldsymbol{x}_0$ and diagonalize $\boldsymbol{A}^\mathrm{n}$

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \boldsymbol{P} \begin{pmatrix} \lambda_1^{\,n} & 0 \\ 0 & \lambda_2^{\,n} \end{pmatrix} \boldsymbol{P}^{-1} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

Multiply out the RHS and equate the first component

$$a_n = s\left(\lambda_1\right)^n + t\left(\lambda_2\right)^n$$

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Forming recurrence matrix

$a_n = 3a_{n-1} + 5a_{n-2} + 7a_{n-3}$ with $a_0 = 0$, $a_1 = 1$ and $a_2 = 1$

Recurrence matrix (3 x 3)

$$\begin{pmatrix} a_n \\ a_{n+1} \\ a_{n+2} \end{pmatrix} = \begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \\ a_{n+1} \end{pmatrix}$$

$$\begin{aligned} a_n &= 0\,a_{n-1} + 1\,a_n + 0\,a_{n+1} \\ a_{n+1} &= 0\,a_{n-1} + 0\,a_n + 1\,a_{n+1} \\ a_{n+2} &= 7\,a_{n-1} + 5\,a_n + 3\,a_{n+1} \end{aligned}$$

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Orthogonally diagonalizable

A square matrix **A** is called diagonalizable
if there exists an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.
We say the matrix **P** diagonalizes **A**.

A square matrix **A** is called orthogonally diagonalizable ⟺ Symmetric matrix
if there exists an orthogonal matrix **P** such that $\mathbf{P}^{T}\mathbf{AP}$ is a diagonal matrix.
We say the matrix **P** orthogonally diagonalizes **A**.

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Orthogonal diagonalization

Symmetric matrix	$\mathbf{A}$
characteristic polynomial	$(x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2}\cdots(x-\lambda_k)^{r_k}$ (group the repeated roots)
distinct eigenvalues	$\lambda_1, \lambda_2, \ldots, \lambda_k$
eigenspaces	$E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_k}$
bases for eigenspaces	$S_{\lambda_1}, S_{\lambda_2}, \ldots S_{\lambda_k}$
	↓ Gram-Schmidt Process
orthonormal bases for eigenspaces	$T_{\lambda_1}, T_{\lambda_2}, \ldots T_{\lambda_k}$

$T_{\lambda_1} \cup T_{\lambda_2} \cup \ldots \cup T_{\lambda_k}$
$= \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

Form orthogonal matrix
$\mathbf{P} = (\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_n)$

$$\mathbf{D} = \begin{pmatrix} \overbrace{\lambda_1 \ \ddots \ \lambda_1}^{r_1} & & & \\ & \overbrace{\lambda_2 \ \ddots \ \lambda_2}^{r_2} & & \\ & & \ddots & \\ & & & \overbrace{\lambda_k \ \ddots \ \lambda_k}^{r_k} \end{pmatrix}$$

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True or False

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same size.

(I) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalizable, then $\boldsymbol{A} + \boldsymbol{B}$ is diagonalizable. False

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$ both diagonalizable

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$ not diagonalizable

(II) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonally diagonalizable, then $\boldsymbol{A} + \boldsymbol{B}$ is orthogonally diagonalizable. True

$\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric $\Rightarrow$ $\boldsymbol{A} + \boldsymbol{B}$ is symmetric

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$$\boldsymbol{A}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Exercise 6 Q16 (Tutorial)

A square matrix $(a_{ij})_{n\times n}$ is called a stochastic matrix if all $a_{ij} \geq 0$ and $a_{1i} + a_{2i} + \cdots + a_{ni} = 1$ for $i = 1, 2, \ldots, n$.

(b) Let $\boldsymbol{B} = \begin{pmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.95 & 0.05 \\ 0 & 0.05 & 0.95 \end{pmatrix}$

(i) Is $\boldsymbol{B}$ a stochastic matrix? Just use definition

(ii) Find a 3×3 invertible matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{B}$.

Go through the algorithm

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$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Exercise 6 Q16 (Tutorial)

A square matrix $(a_{ij})_{n\times n}$ is called a stochastic matrix if all $a_{ij} \geq 0$ and $a_{1i} + a_{2i} + \cdots + a_{ni} = 1$ for $i = 1, 2, \ldots, n$.

(a) Let $\boldsymbol{A}$ be a stochastic matrix.

(i) Show that 1 is an eigenvalue of $\boldsymbol{A}$.

(ii) If λ is an eigenvalue of $\boldsymbol{A}$, then $|\lambda| \leq 1$.

Hint: $\boldsymbol{A}$ and $\boldsymbol{A}^T$ have the same set of eigenvalues

(i) Find a special eigenvector of $\boldsymbol{A}^T$ with eigenvalue 1.

(Hint: Look at Q29)

(ii) Let $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be an eigenvector of $\boldsymbol{A}^T$ with eigenvalue λ.

Set up $\boldsymbol{A}^T\mathbf{v} = \lambda\mathbf{v}$.

Consider the component of $\mathbf{v}$ with the largest absolute value.

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$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise 6 Q29 (Tutorial)

(a) Show that $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$. Direct checking

(b) Let $\boldsymbol{v} = (a, b, c, d)^T$ be a nonzero vector.

Show that if $\boxed{\boldsymbol{v} \cdot \boldsymbol{u} = 0}$, then $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$.

use this

Show that $\boldsymbol{Av} = \boldsymbol{0}$

(c) Suppose $\boldsymbol{P} = \begin{pmatrix} 1/2 & a_1 & a_2 & a_3 \\ 1/2 & b_1 & b_2 & b_3 \\ 1/2 & c_1 & c_2 & c_3 \\ 1/2 & d_1 & d_2 & d_3 \end{pmatrix}$ is an orthogonal matrix.

Find $\boldsymbol{P}^T\boldsymbol{AP}$.

- Show that each column of $\boldsymbol{P}$ is an eigenvector of $\boldsymbol{A}$
  - Use part (a) for the first column
  - Use part (b) for the other columns
- Find the eigenvalues corresponding to each column of $\boldsymbol{P}$:

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Exercise 6 Q22

If $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}$ are eigenvectors of $\boldsymbol{A}$ with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$,

then $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is linearly independent

Proof by mathematical induction

Basis step: k = 1 Clearly $\{\boldsymbol{u}_1\}$ is linearly independent

Inductive step:

Assume $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\}$ is linearly independent

WTS: $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is linearly independent

Set up the vector equation: $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots c_k\boldsymbol{u}_k = \mathbf{0}$ ---(*)

Multiply (*) by $\boldsymbol{A}$: $c_1\boldsymbol{A}\boldsymbol{u}_1 + c_2\boldsymbol{A}\boldsymbol{u}_2 + \ldots c_k\boldsymbol{A}\boldsymbol{u}_k = \mathbf{0}$

which gives: $c_1\lambda_1\boldsymbol{u}_1 + c_2\lambda_2\boldsymbol{u}_2 + \ldots c_k\lambda_k\boldsymbol{u}_k = \mathbf{0}$ ---(**)

$\lambda_k \times$ (*): $c_1\lambda_k\boldsymbol{u}_1 + c_2\lambda_k\boldsymbol{u}_2 + \ldots c_k\lambda_k\boldsymbol{u}_k = \mathbf{0}$ ---(***)

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \dots c_k\boldsymbol{u}_k = \mathbf{0} \text{ ---}(*)$$

$$c_1\lambda_1\boldsymbol{u}_1 + c_2\lambda_2\boldsymbol{u}_2 + \dots c_k\lambda_k\boldsymbol{u}_k = \mathbf{0} \text{ ---}(**)$$

$$c_1\lambda_k\boldsymbol{u}_1 + c_2\lambda_k\boldsymbol{u}_2 + \dots c_k\lambda_k\boldsymbol{u}_k = \mathbf{0} \text{ ---}(***)$$

Exercise 6 Q22

If $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ are eigenvectors of $\boldsymbol{A}$ with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}$ is linearly independent

Assume $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_{k-1}\}$ is linearly independent

$(**) - (***)$: $\underbrace{c_1(\lambda_1-\lambda_k)}_{0}\boldsymbol{u}_1 + \underbrace{c_2(\lambda_2-\lambda_k)}_{0}\boldsymbol{u}_2 + \dots \underbrace{c_{k-1}(\lambda_{k-1}-\lambda_k)}_{0}\boldsymbol{u}_{k-1} = \mathbf{0}$

$c_i(\lambda_i-\lambda_k) = 0$

But $\lambda_i \neq \lambda_k$. This implies $c_i = 0$ for all $i = 1, 2, \dots, k-1$.

Substitute in $(*)$: $c_k\boldsymbol{u}_k = \mathbf{0}$ This implies $c_k = 0$ as well.

Hence $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ is linearly independent

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Announcement

- **Homework 4**
  - Deadline: 13 November
- **Zoom lecture next week (week 12)**
  - Live zoom ongoing
- **Examplify for Mock & final exam**
  - install in your PC
  - briefing and mock exam during week 13 zoom session
- **Past year papers**
  - Uploaded in LumiNUS > Files > Past Year Paper
- **Online quiz 11**
  - Due next Thursday