MA2001

LIVE LECTURE 7

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Topics for week 7

3.6 Dimensions

3.7 Transition Matrices

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Let’s revise – Linear dependency

1. If the vector equation $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0}$ has $\boxed{\text{only the trivial}}$ solution, then $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly independent
2. If the vector equation $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0}$ has $\boxed{\text{a non-trivial}}$ solution, then $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly dependent
3. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are $\boxed{\text{scalar multiples}}$ of each other, then $\{\boldsymbol{u}, \boldsymbol{v}\}$ is linearly dependent.
4. If S $\boxed{\text{contains } \mathbf{0},}$ then S is linearly dependent.
5. If one vector in S is $\boxed{\text{a linear combination of the other vectors}}$ in S, then S is linearly dependent

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Let's revise – Linear dependency & Span

6. If $\boldsymbol{u} \in \text{span}(S)$, then $S \cup \{\boldsymbol{u}\}$ is linearly $\boxed{\text{dependent}}$
7. If S is linearly independent and $\boldsymbol{u} \notin \text{span}(S)$, then $S \cup \{\boldsymbol{u}\}$ is linearly $\boxed{\text{independent.}}$
8. Let $\{\boldsymbol{u}, \boldsymbol{v}\} \in \mathbf{R}^2$.

   $\{\boldsymbol{u}, \boldsymbol{v}\}$ is linearly $\boxed{\text{independent}}$ iff $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \mathbf{R}^2$
9. Let $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \in \mathbf{R}^3$.

   $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly $\boxed{\text{independent}}$ iff $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} = \mathbf{R}^3$
10. If $S \in \mathbf{R}^n$ and S has more than n elements, then S is linearly $\boxed{\text{dependent.}}$

Vector Space

$\mathbf{R}^n$ (n-space) $\supseteq$ Subspace of $\mathbf{R}^n$

Let's revise – Bases

11. A subset S of a vector space V is called a basis for V if

    (i) span(S) = V and (ii) S is linearly independent

12. Every non-zero vector space has infinitely many different bases

13. The basis for the zero space is the empty set

14. All bases for the same vector space V has the same number of vectors

15. Every vector in a vector space can be expressed as linear combination of a given basis in a unique way

16. S is a basis for span(S) iff S is linearly independent

Dimension

dim V

dimension of V

Let V be a vector space which has a basis
$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ with k vectors.

1. Any subset of V with more than k vectors is always linearly dependent.

   > k : too many vectors to be a basis

2. Any subset of V with less than k vectors cannot span V.

   < k : too few vectors to be a basis

All bases for a vector space have the same number of vectors

Dimension of subspaces of $\mathbf{R}^3$

- $\{\mathbf{0}\}$ basis is empty set
  dimension 0
- lines through the origin span$\{\mathbf{u}\}$ with basis $\{\mathbf{u}\}$
  dimension 1
- planes containing the origin span$\{\mathbf{u}, \mathbf{v}\}$ with basis $\{\mathbf{u}, \mathbf{v}\}$
  dimension 2
- $\mathbf{R}^3$ span$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
  dimension 3

Dimension of linear span

- If $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ is linearly independent,

  then $\dim \operatorname{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\} = k$.
- If $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ is linearly dependent,

  then $\dim \operatorname{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\} < k$.

True or False:
$\dim \operatorname{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} < \dim \operatorname{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_5}\}$

True or False:
$\dim \operatorname{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} \leq \dim \operatorname{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}\}$

What's the dimension?

$V = \text{span}\{ (1,1,0,0), (0,0,2,2), (1,1,1,1), (4,4,3,3), (1,2,1,2)\}$

V is a subspace of $\mathbf{R}^4$

- dim V = 2
- dim V = 3
- dim V = 4
- dim V = 5

Dimension of solution space

homogeneous system $\longrightarrow$ row echelon form $\boldsymbol{R}$

number of non-pivot columns in $\boldsymbol{R}$

||

number of parameters in general solution

||

number of vectors in basis for solution space

||

the dimension of the solution space

$V_1 \subseteq V_2$: we say V_1 is a subspace of V_2

Exercise 3 Q39

Give an example of a family of subspaces $V_1, V_2, \ldots, V_n$ of $\mathbf{R}^n$ such that $\dim(V_i) = i$ and $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n$.

Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \ldots, \mathbf{u}_n\}$ be a basis for $\mathbf{R}^n$

- $V_1 = \text{span}\{\mathbf{u}_1\}$ dimension 1
- $V_2 = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ dimension 2
- $V_3 = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ dimension 3
- $\vdots$
- $V_{n-1} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \ldots, \mathbf{u}_{n-1}\}$ dimension n-1
- $V_n = \mathbf{R}^n$ dimension n

To show basis for vector space

To show a subset S of V is a basis for V :

S lin. indep S spans V	or	S lin. indep $\lvert S \rvert = \dim V$	or	S spans V $\lvert S \rvert = \dim V$

If $|S| = \dim V$, then

S is linearly independent $\Leftrightarrow$ S spans V

Identify bases for vector space

V = span{(1,0,0), (0,1,0), (1,1,0)}

I. {(1,0,0), (0,1,0)}
II. {(1,0,0), (1,-1,0)}
III. {(1,0,0), (0,0,1)}

I. {(1,0,0), (0,1,0)}

- V = span{(1,0,0), (0,1,0)} since (1,1,0) is redundant
- (1,0,0), (0,1,0) are linearly independent
- So {(1,0,0), (0,1,0)} is a basis for V

Identify bases for vector space

V = span{(1,0,0), (0,1,0), (1,1,0)}

I. {(1,0,0), (0,1,0)}
II. {(1,0,0), (1,-1,0)}
III. {(1,0,0), (0,0,1)}

II. {(1,0,0), (1,-1,0)}

- dim V = 2 (from I)
- (1,0,0), (1,-1,0) are linearly independent
- (1,0,0), (1,-1,0) belongs to V
- So {(1,0,0), (1,-1,0)} is a basis for V

Identify bases for vector space

V = span{(1,0,0), (0,1,0), (1,1,0)}

I. {(1,0,0), (0,1,0)}
II. {(1,0,0), (1,-1,0)}
III. {(1,0,0), (0,0,1)}

III. {(1,0,0), (0,0,1)}

- (1,0,0), (0,0,1) are linearly independent
- (0,0,1) does not belong to V
- So {(1,0,0), (0,0,1)} is not a basis for V

Deriving bases from a subset

V a vector space, and S, T are finite subsets of V.

- Suppose span(S) = V.

  We can find S′ ⊆ S such that S′ is a basis for V.

- Suppose T is a linearly independent subset of V.

  We can find T ⊆ T′ such that T′ is a basis for V.

Techniques in chapter 4

Dimensions give the “size” of subspaces

Let U and V be subspaces of $\mathbf{R}^n$

(i) If $U \subseteq V$, then $\dim(U) \leq \dim(V)$

(ii) If $U \subseteq V$ and $U \neq V$, then $\dim(U) < \dim(V)$

True or false

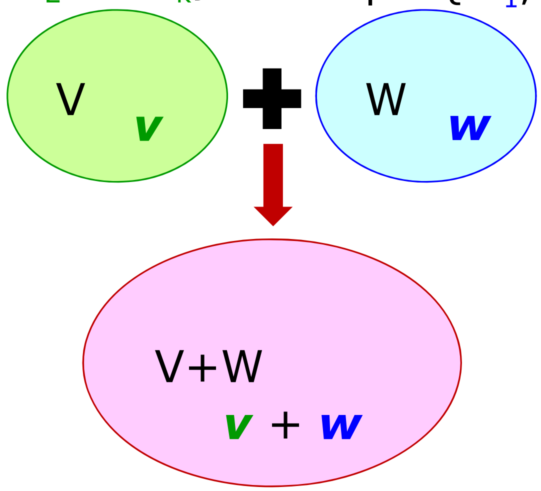
- If $\dim(U) = \dim(V)$, then $U = V$
- If $\dim(U) \leq \dim(V)$, then $U \subseteq V$
- If $U \subseteq V$ and $\dim(U) = \dim(V)$, then $U = V$

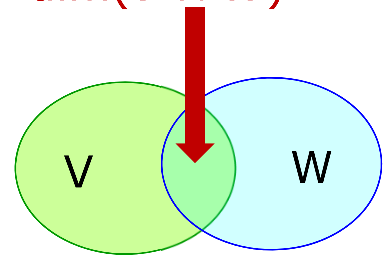
Exercise 3 Q43

V, W subspaces of $\mathbf{R}^n$. Show that:

$$\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$$

span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ span$\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_h\}$





span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k , \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_h\}$

Exercise 3 Q43

V, W subspaces of $\mathbf{R}^n$. Show that:

$$\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W)$$

Simple example in $\mathbf{R}^3$:

V, W: two lines through origin

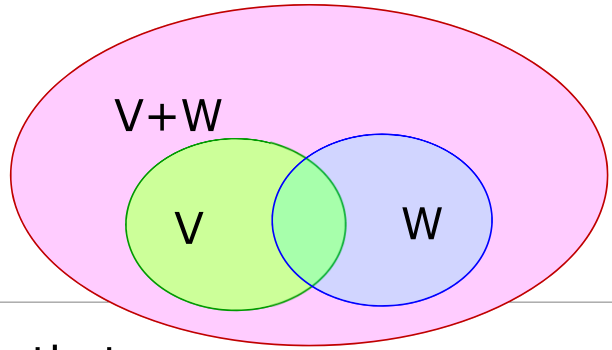
(i) If V, W represent the same line ℓ,

then $V \cap W = \ell$ and $V + W = \ell$

(ii) If V, W represent two different lines ℓ_1 and ℓ_2,

then $V \cap W = \{\mathbf{0}\}$ and $V + W =$ plane containing ℓ_1 and ℓ_2

Exercise 3 Q43



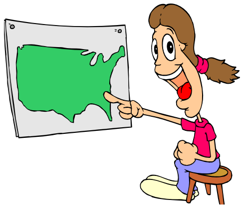
V, W subspaces of $\mathbf{R}^n$. Show that:

$$\dim(V + W) = \underbrace{\dim(V)}_{k} + \underbrace{\dim(W)}_{h} - \underbrace{\dim(V \cap W)}_{m}$$

Idea of proof:

- Start with a basis $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ for $V \cap W$
- Extend S to a basis for V: $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_k\}$
- Extend S to a basis for W: $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m, \mathbf{w}_{m+1}, \dots, \mathbf{w}_h\}$
- $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_k, \mathbf{w}_{m+1}, \dots, \mathbf{w}_h\} = V + W$
- Show $T = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_k, \mathbf{w}_{m+1}, \dots, \mathbf{w}_h\}$ is linearly independent (exercise)
- Then T is a basis for $V + W$, and $\dim(V+W) = k + h - m$

Map of LA



A **is an n×n matrix**

A is invertible	chapter 2	***A*** is not invertible
det ***A*** $\neq$ 0	chapter 2	det ***A*** = 0
rref of ***A*** is identity matrix	chapter 1	rref of ***A*** has a zero row
Ax = ***0*** has only the trivial solution	chapter 1	***Ax*** = ***0*** has non-trivial solutions
Ax = ***b*** has a unique solution	chapter 1	***Ax*** = ***b*** has no solution or infinitely many solutions
Columns (rows) of ***A*** are linearly independent — a basis for $\mathbf{R}^n$	chapter 3	Columns (rows) of ***A*** are linearly dependent — not a basis for $\mathbf{R}^n$

to be continued

Transition matrix

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$
two bases for a vector space V.

Given $\boldsymbol{w} \in V$

$$(\boldsymbol{w})_S \longrightarrow \boldsymbol{w} \longrightarrow (\boldsymbol{w})_T$$

Is there a direct method?

does not depend on $\boldsymbol{w}$

$[\boldsymbol{w}]_T = \boldsymbol{P}\, [\boldsymbol{w}]_S$ for some fixed $k \times k$ matrix $\boldsymbol{P}$

transition matrix

Finding transition matrix

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ two bases for a vector space V

1. Express each $\boldsymbol{u}_i$ as linear combination of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$
2. Form the (column) coordinate vectors w.r.t. T

$$[\boldsymbol{u}_1]_T = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{pmatrix} \quad [\boldsymbol{u}_2]_T = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k2} \end{pmatrix} \quad \cdots \quad [\boldsymbol{u}_k]_T = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{kk} \end{pmatrix}$$

3. Form the matrix $\boldsymbol{P} = ([\boldsymbol{u}_1]_T \ [\boldsymbol{u}_2]_T \ \ldots \ [\boldsymbol{u}_k]_T)$

$$\boldsymbol{P} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

transition matrix
from S to T

4. $\boldsymbol{P}[\boldsymbol{w}]_S = [\boldsymbol{w}]_T$ for any vector $\boldsymbol{w}$ in V.

Finding transition matrix

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ and $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ two bases for a vector space V

$$
\begin{array}{c}
\left(\begin{array}{ccc|c|c|c} 1 & 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 \end{array}\right) \\
\begin{array}{cccccc} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{array}
\end{array}
\xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}}
\begin{array}{c}
\qquad\qquad\qquad \boldsymbol{P} \\
\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right) \\
\qquad\qquad\qquad [\boldsymbol{u}_1]_T \, [\boldsymbol{u}_2]_T \, [\boldsymbol{u}_3]_T
\end{array}
$$

$\boldsymbol{P}$: the transition matrix from S to T

The transition matrix from T to S is given by $\boldsymbol{P}^{-1}$

Exercise 3 Q48

$V = \{(x, y, z) \mid 2x - y + z = 0\}$

$S = \{(0,1,1), (1,2,0)\}$ $\quad$ $T = \{(1,1,-1), (1,0,-2)\}$ Similar argument for T

a) Show that both S and T are bases for V.

b) Find the transition matrix from T to S and the transition matrix from S to T.

Check both $(0,1,1)$, $(1,2,0)$ satisfy the equation $2x - y + z = 0$

Also $\{(0,1,1), (1,2,0)\}$ is linearly independent

So $\text{span}\{(0,1,1), (1,2,0)\} = V$

So S is a basis for V.

a) Show that both S and T are bases for V.

b) Find the transition matrix from T to S and the transition matrix from S to T.

Exercise 3 Q48

V = {(x, y, z) | 2x – y + z = 0}

S = {(0,1,1), (1,2,0)}   T = {(1,1,-1), (1,0,-2)}

$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & -2 \end{array}\right) \xrightarrow{\text{G.J.E.}} \left(\begin{array}{cc|cc} 1 & 0 & \multicolumn{2}{c}{\overset{\boldsymbol{P}}{\boxed{\begin{matrix} -1 & -2 \\ 1 & 1 \end{matrix}}}} \\ 0 & 1 & & \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The transition matrix from *T* to *S* is $\boldsymbol{P} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$

The transition matrix from *S* to *T* is $\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$

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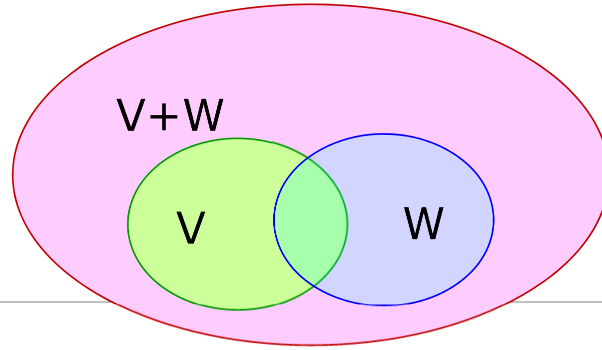
Announcement

- Homework 2
  - Deadline: 1 October (this Friday)
  - Submission folder will close at 11.59pm

Exercise 3 Q43



- basis $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ for $V \cap W$
- basis for V: $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{v}_{m+1}, \ldots, \mathbf{v}_k$
- basis for W: $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{w}_{m+1}, \ldots, \mathbf{w}_h$
- Show $T = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{v}_{m+1}, \ldots, \mathbf{v}_k, \mathbf{w}_{m+1}, \ldots, \mathbf{w}_h\}$ is linearly independent

$$c_1\mathbf{u}_1 + c_1\mathbf{u}_2 + \ldots + c_m\mathbf{u}_m + d_{m+1}\mathbf{v}_{m+1} + \ldots + d_k\mathbf{v}_k + e_{m+1}\mathbf{w}_{m+1} + \ldots + e_h\mathbf{w}_h = \mathbf{0}$$

$$\underbrace{c_1\mathbf{u}_1 + c_1\mathbf{u}_2 + \ldots + c_m\mathbf{u}_m + d_{m+1}\mathbf{v}_{m+1} + \ldots + d_k\mathbf{v}_k}_{\text{in } V} = \underbrace{-e_{m+1}\mathbf{w}_{m+1} - \ldots - e_h\mathbf{w}_h}_{\text{in } W} \quad \mathbf{(*)}$$

in $V \cap W$

$$-e_{m+1}\mathbf{w}_{m+1} - \ldots - e_h\mathbf{w}_h = f_1\mathbf{u}_1 + f_2\mathbf{u}_2 + \ldots + f_m\mathbf{u}_m$$

$$f_1\mathbf{u}_1 + f_2\mathbf{u}_2 + \ldots + f_m\mathbf{u}_m + e_{m+1}\mathbf{w}_{m+1} + \ldots + e_h\mathbf{w}_h = \mathbf{0} \quad \mathbf{(**)}$$

$$\mathbf{(**)} \Rightarrow f_1 = f_2 = \ldots = f_m = e_{m+1} = \ldots = e_h = 0$$

$$\mathbf{(*)} \Rightarrow c_1 = c_2 = \ldots = c_m = d_{m+1} = \ldots = d_k = 0$$