Answers/Solutions of Exercise 3

1. $\boldsymbol{u} = (1,\ \sqrt{3})$, $\boldsymbol{v} = (-\sqrt{3},\ -1)$, $\boldsymbol{u} + \boldsymbol{v} = (1 - \sqrt{3},\ -1 + \sqrt{3})$,
   $3\boldsymbol{u} - 2\boldsymbol{v} = (3 + 2\sqrt{3},\ 2 + 3\sqrt{3})$.

2. (a) Substituting $(x, y) = (1, 2)$ and $(2, -1)$ into the equation $ax + by = c$, we has a system of linear equations

$$\begin{cases} a + 2b - c = 0 \\ 2a - \ \ b - c = 0 \end{cases}$$

which implies $a = \frac{3}{5}c$ and $b = \frac{1}{5}c$. In set notation, the line is

$$\{(x, y) \mid 3x + y = 5\} \text{ (implicit)} \quad \text{and} \quad \{(\tfrac{5-t}{3}, t) \mid t \in \mathbb{R}\} \text{ (explicit)}.$$

(b) Substituting $(x, y, z) = (0, 1, -1)$, $(1, -1, 0)$ and $(0, 2, 0)$ into the equation $ax + by + cz = d$, we has a system of linear equations

$$\begin{cases} \quad\ \ b - c - d = 0 \\ a - \ b \qquad - d = 0 \\ \quad\ 2b \qquad - d = 0 \end{cases}$$

which implies $a = \frac{3}{2}d$, $b = \frac{1}{2}d$ and $c = -\frac{1}{2}d$. In set notation, the plane is

$$\{(x, y, z) \mid 3x + y - z = 2\} \text{ (implicit)} \quad \text{and} \quad \{(\tfrac{2-s+t}{3}, s, t) \mid s, t \in \mathbb{R}\} \text{ (explicit)}.$$

(c) In explicit form, the line is

$$\{(1, -1, 0) + t(-1, 2, -1) \mid t \in \mathbb{R}\} = \{(1 - t, -1 + 2t, -t) \mid t \in \mathbb{R}\}.$$

To find the implicit form, we need to find two non-parallel planes containing the two points $(0, 1, -1)$ and $(1, -1, 0)$. The intersection of the two planes will give us the required line. Substituting $(0, 1, -1)$ and $(1, -1, 0)$ into $ax + by + cz = d$ we has a system of linear equations

$$\begin{cases} \quad\ \ b - c - d = 0 \\ a - b \qquad - d = 0 \end{cases}$$

We obtain $a = c + 2d$ and $b = c + d$. There are infinitely many such planes. For example, we can write the line implicitly as

$$\{(x, y, z) \mid x + y + z = 0 \text{ and } 2x + y = 1\}.$$

3. $A = B = C = F$ and A, D, E are all different.

4. (a) U and V contains the origin but W does not.

   (b) $$\begin{cases} 2x - \ \ y + 3z = 0 \\ 3x + 2y - \ \ z = 0 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{5}{7}t \\ y = \frac{11}{7}t \quad \text{where } t \in \mathbb{R} \\ z = t \end{cases}$$

   So $U \cap V = \{\, (-\frac{5}{7}t,\ \frac{11}{7}t,\ t) \mid t \in \mathbb{R}\,\}$.

   $$\begin{cases} 3x + 2y - \ \ z = 0 \\ \ \ x - 3y - 2z = 1 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{11}(2 + 7t) \\ y = \frac{1}{11}(-3 - 5t) \quad \text{where } t \in \mathbb{R} \\ z = t \end{cases}$$

   So $V \cap W = \{\, (\frac{2+7t}{11},\ \frac{-3-5t}{11},\ t) \mid t \in \mathbb{R}\,\}$.

5. (a) A is a line joining the points $(1, 1, 1)$ and $(2, 3, 4)$.

   (b) Let $B = \{(x, y, z) \mid x + y - z = 1 \text{ and } x - 2y + z = 0\}$. Since $x + y - z = 1$ and $x - 2y + z = 0$ are two non-parallel planes, B is the line of intersection of the two planes. To show that $A = B$, it suffices to show that the line A lies on both planes. This is true because $(1 + t) + (1 + 2t) - (1 + 3t) = 1$ and $(1 + t) - 2(1 + 2t) + (1 + 3t) = 0$ for all $t \in \mathbb{R}$.

   (c) For example, $\boldsymbol{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

6. Since

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ b & c & d \end{vmatrix} - 0 + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & b & d \end{vmatrix} - 0 = a + b - d - c,$$

$V = \{(a, b, c, d) \mid a + b - d - c = 0\} = \{(x, y, z, w) \mid x + y - z - w = 0\} = T$.

On the other hand, $S \neq T$ because $(1, -1, 0, 0) \in T$ but $(1, -1, 0, 0) \notin S$.

7. (a) For example, $P = \{(1 + s - t,\ s, t) \mid s, t \in \mathbb{R}\}$.

   (b) A lies in P because $a - a + 1 = 1$. Since both B and C pass through $(0, 0, 0)$ and $(0, 0, 0) \notin P$, B and C does not lies in P.

   (c) B intersects P at one point, $(1, 0, 0)$.

   (d) The plane $x - y + z = 0$ contains C but not A and B.

(e) No. By Discussion 1.4.11, the solution set of a consistent nonzero linear system in three variables represents a point, a line or a plane in $\mathbb{R}^3$. Suppose we have a nonzero linear system whose solution set contains both B and C. Then the solution set must be a plane. However, the plane containing both B and C is the xz-plane which does not contain A. So the solution set cannot contain A.

8. $(2,3,-7,3)$, $(0,0,0,0)$ and $(-4,6,-13,4)$ are vectors in $\text{span}\{\boldsymbol{u_1},\boldsymbol{u_2},\boldsymbol{u_3}\}$ while $(1,1,1,1)$ is not.

9. S_4 and S_6 span $\mathbb{R}^3$ while S_1, S_2, S_3 and S_5 do not span $\mathbb{R}^3$.

10. (a) Since $(1,1,0)$ and $(5,2,3)$ satisfy the equation $x-y-z=0$, $(1,1,0),(5,2,3)\in V$ and hence $\text{span}(S)\subseteq V$.

Note that a general solution of $x-y-z=0$ is $x=s+t$, $y=s$, $z=t$ where $s,t\in\mathbb{R}$. Let $(s+t,s,t)$ be any vector in V. Consider the following equation:

$$a(1,1,0)+b(5,2,3)=(s+t,s,t) \quad\Leftrightarrow\quad \begin{cases} a+5b=s+t \\ a+2b=s \\ \quad\;\; 3b=t. \end{cases}$$

Since

$$\left(\begin{array}{cc|c} 1 & 5 & s+t \\ 1 & 2 & s \\ 0 & 3 & t \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c} 1 & 5 & s+t \\ 0 & 3 & t \\ 0 & 0 & 0 \end{array}\right),$$

the system is consistent for all $s,t\in\mathbb{R}$. So $V\subseteq\text{span}(S)$.

We have shown that $\text{span}\{(1,1,0),\,(5,2,3)\}=V$.

(b) Since

$$\begin{pmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 5 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

by Discussion 3.2.5, $\text{span}\{(1,1,0),\,(5,2,3),\,(0,0,1)\}=\mathbb{R}^3$.

11. (a) $$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & -2 \\ -5 & 1 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{array}\right)$$

Since $\boldsymbol{u_2}\notin\text{span}\{\boldsymbol{v_1},\boldsymbol{v_2}\}$, $\text{span}\{\boldsymbol{u_1},\boldsymbol{u_2},\boldsymbol{u_3}\}\neq\text{span}\{\boldsymbol{v_1},\boldsymbol{v_2}\}$.

(b) $\left(\begin{array}{ccc|c|c} 1 & 2 & -1 & 1 & 0 \\ 6 & 4 & 2 & -2 & 8 \\ 4 & -1 & 5 & -5 & 9 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussiann}} \left(\begin{array}{ccc|c|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -8 & 8 & -8 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

The systems are consistent and thus $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\} \subseteq \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$.

$\left(\begin{array}{cc|c|c|c} 1 & 0 & 1 & 2 & -1 \\ -2 & 8 & 6 & 4 & 2 \\ -5 & 9 & 4 & -1 & 5 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c|c|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 8 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

The systems are consistent and thus $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} \subseteq \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$.

So $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\} = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$.

12. (a) $\left(\begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 2 & 1 & 1 & 3 & 0 \\ 1 & 4 & 1 & -1 & 2 \\ 0 & 0 & 3 & 6 & 5 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 0 & 7 & 1 & -5 & 2 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

Since $\boldsymbol{u_2} \notin \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$, $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} \not\subseteq \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$.

(b) $\left(\begin{array}{cccc|c|c|c|c} 2 & 1 & 0 & 1 & -1 & 3 & 0 & -4 \\ 0 & 0 & 3 & 1 & 2 & 1 & 1 & 3 \\ 2 & 2 & 6 & 2 & 1 & 4 & 1 & -1 \\ 0 & 5 & 9 & -1 & 0 & 0 & 3 & 6 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cccc|c|c|c|c} 2 & 1 & 0 & 1 & -1 & 3 & 0 & -4 \\ 0 & 1 & 6 & 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 3 & 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 & 2 & 5 & 12 \end{array}\right)$

The systems are consistent and thus $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} \subseteq \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$.

(c) $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \mathbb{R}^4$.

(d) $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} \neq \mathbb{R}^4$.

13. By Example 3.2.8.2, S_1 does not span $\mathbb{R}^3$.

Since $\boldsymbol{w} - \boldsymbol{u} = -(\boldsymbol{u} - \boldsymbol{v}) - (\boldsymbol{v} - \boldsymbol{w})$, $\text{span}(S_2) = \text{span}\{\boldsymbol{u} - \boldsymbol{v},\ \boldsymbol{v} - \boldsymbol{w}\}$ and by Example 3.2.8.2, S_2 does not span $\mathbb{R}^3$.

S_3 spans $\mathbb{R}^3$: Since $\text{span}(S_3) \subseteq \mathbb{R}^3$, we only need to show $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \subseteq \text{span}(S_3)$. Note that

$$\begin{aligned} \boldsymbol{u} &= \tfrac{1}{2}[(\boldsymbol{u} - \boldsymbol{v}) + (\boldsymbol{v} - \boldsymbol{w}) + (\boldsymbol{u} + \boldsymbol{w})], \\ \boldsymbol{v} &= \tfrac{1}{2}[-(\boldsymbol{u} - \boldsymbol{v}) + (\boldsymbol{v} - \boldsymbol{w}) + (\boldsymbol{u} + \boldsymbol{w})], \\ \boldsymbol{w} &= \tfrac{1}{2}[-(\boldsymbol{u} - \boldsymbol{v}) - (\boldsymbol{v} - \boldsymbol{w}) + (\boldsymbol{u} + \boldsymbol{w})]. \end{aligned}$$

By Theorem 3.2.10, $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \subseteq \text{span}(S_3)$. Hence $\text{span}(S_3)$ spans $\mathbb{R}^3$.

Using the same argument as for S_3, we can show that both S_4 and S_5 also span $\mathbb{R}^3$.

14. (a) True. Let $\boldsymbol{u} = (u)$ for $u \neq 0$. Then for any $(c) \in \mathbb{R}^1$, $(c) = \frac{c}{u}\boldsymbol{u}$.
    (b) False. For example, let $\boldsymbol{u} = (1, 1)$, $\boldsymbol{v} = (2, 2)$.
    (c) False. For example, let $S_1 = \{(1, 0), (0, 1)\}$, $S_2 = \{(1, 0), (0, 2)\}$.
    (d) False. For example, let $S_1 = \{(1, 0)\}$, $S_2 = \{(0, 1)\}$.

15. (a) Yes. See Remark 3.3.3.1.
    (b) No. It does not contain the zero vector.
    (c) No. $(1, 1, 1)$ belongs to the set but $2(1, 1, 1)$ does not.
    (d) No. $(0, 0, 1)$ belongs to the set but $\frac{1}{2}(0, 0, 1)$ does not.
    (e) Yes. It is $\mathrm{span}\{(0, 0, 1)\}$.
    (f) No. It does not contain the zero vector.
    (g) No. $(1, 1, 0)$ and $(0, 0, 1)$ belong to the set but $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$ does not.
    (h) No. $(3, 2, 1)$ belongs to the set but $-(3, 2, 1)$ does not.
    (i) Yes. It is a solution set of a homogeneous linear system.
    (j) Yes. It is $\mathrm{span}\{(1, 0, 0),\ (0, 1, 1)\}$.
    (k) No. $(1, 1, 1)$ and $(2, 2, 4)$ belong to the set but $(1, 1, 1) + (2, 2, 4) = (3, 3, 5)$ does not.

16. (a) Yes. It is a solution set of a homogeneous linear system.
    (b) No. $(1, 0, 0, 1)$ and $(0, 2, 0, 1)$ belong to the set but $(1, 0, 0, 1) + (0, 2, 0, 1) = (1, 2, 0, 2)$ does not.
    (c) No. $(1, 1, -1, -1)$ and $(0, 4, 0, 2)$ belong to the set but $(1, 1, -1, -1) + (0, 4, 0, 2) = (1, 5, -1, 1)$ does not.
    (d) Yes. It is $\mathrm{span}\{(0, 1, 0, 0),\ (0, 0, 0, 1)\}$.
    (e) No. $(1, 0, 0, 0)$ and $(0, 0, 1, 0)$ belong to the set but $(1, 0, 0, 0) + (0, 0, 1, 0) = (1, 0, 1, 0)$ does not.
    (f) No. It does not contain the zero vector.
    (g) Yes. It is a solution set of a homogeneous linear system.
    (h) No. $(1, 0, 0, -1)$ and $(0, 0, 4, 1)$ belong to the set but $(1, 0, 0, -1) + (0, 0, 4, 1) = (1, 0, 4, 0)$ does not.

17. (a) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. (b) e.g. $\begin{pmatrix} 2 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. (c) e.g. $\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \end{pmatrix}$. (d) Not possible.

18. (a) $W + \boldsymbol{v}$ is the line $x + y = 2$ in $\mathbb{R}^2$.

(b) $W + \boldsymbol{v}$ is the line $\{(0,0,1) + c(1,1,1) \mid c \in \mathbb{R}\}$ in $\mathbb{R}^3$.

(c) $W + \boldsymbol{v}$ is the plane $x + y + z = 1$ in $\mathbb{R}^3$.

19. $U \cap V$ is a subspace of $\mathbb{R}^3$ because it is a line in $\mathbb{R}^3$ passing through the origin. $V \cap W$ is not a subspace since it does not contain the origin.

20. (a) Let $V = \text{span}\{\boldsymbol{v_1}, \dots, \boldsymbol{v_m}\}$ and $W = \text{span}\{\boldsymbol{w_1}, \dots, \boldsymbol{w_n}\}$. Then

$$\begin{aligned} V + W &= \{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{v} \in V \text{ and } \boldsymbol{w} \in W\} \\ &= \{a_1\boldsymbol{v_1} + \cdots + a_m\boldsymbol{v_m} + b_1\boldsymbol{w_1} + \cdots + b_n\boldsymbol{w_n} \mid a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{R}\} \\ &= \text{span}\{\boldsymbol{v_1}, \dots, \boldsymbol{v_m}, \boldsymbol{w_1}, \dots, \boldsymbol{w_n}\} \end{aligned}$$

Hence $V + W$ is a subspace of $\mathbb{R}^n$.

(b) (i) $V + W = \mathbb{R}^2$.

(ii) $V + W = \{s(1,1,1) + t(1,-1,0) \mid s, t \in \mathbb{R}\}$.

21. (a) Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{c_1} & \cdots & \boldsymbol{c_n} \end{pmatrix}$ where $\boldsymbol{c_1}, \dots, \boldsymbol{c_n}$ are columns of $\boldsymbol{A}$.

For any $\boldsymbol{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$, $\boldsymbol{A}\boldsymbol{u} = u_1\boldsymbol{c_1} + \cdots + u_n\boldsymbol{c_n}$.

Thus $V_{\boldsymbol{A}} = \text{span}\{\boldsymbol{c_1}, \dots, \boldsymbol{c_n}\}$ is a subspace of $\mathbb{R}^m$.

(b) (i) $V_{\boldsymbol{A}} = \mathbb{R}^2$.

(ii) $V_{\boldsymbol{A}} = \left\{ s\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}$.

22. (a) $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{u} \quad \Leftrightarrow \quad (\boldsymbol{A} - \boldsymbol{I})\boldsymbol{u} = \boldsymbol{0}$

$W_{\boldsymbol{A}}$ is the solution set of the homogeneous system $(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{u} = \boldsymbol{0}$. By Theorem 3.3.6, $W_{\boldsymbol{A}}$ is a subspace of $\mathbb{R}^n$.

(b) $\boldsymbol{A} - \boldsymbol{I} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

A general solution of $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is $x = s,\ y = t,\ z = 0$

where $s, t \in \mathbb{R}$. So $W_{\boldsymbol{A}} = \left\{ \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}$.

23. (a) False. $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$. (The xy-plane in $\mathbb{R}^3$ is written as $\{ (x, y, 0) \mid x, y \in \mathbb{R} \}$.)

    (b) True. The equation $x + 2y - z = 0$ forms a homogeneous system of linear equations (with one equation).

    (c) False. Note that $(0, 0, 0)$ is not a solution of $ax + by + cz = 1$. By Theorem 3.2.9.1, the solution set is not a subspace of $\mathbb{R}^3$.

    (d) True. See the proof of Question 3.20(a).

24. (a) We use Remark 3.3.8 to prove that $V \cap W$ is a subspace of $\mathbb{R}^n$:

    Since both V and W contain the zero vector, the zero vector is contained in $V \cap W$ and hence $V \cap W$ is nonempty.

    Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be any two vectors in $V \cap W$ and let a and b be any real numbers. Since $\boldsymbol{u}$ and $\boldsymbol{v}$ are contained in V, $a\boldsymbol{u} + b\boldsymbol{v}$ is also contained in V. Similarly, $a\boldsymbol{u} + b\boldsymbol{v}$ is also contained in W. Thus $a\boldsymbol{u} + b\boldsymbol{v}$ is contained in $V \cap W$.

    By Remark 3.3.8, $V \cap W$ is a subspace of $\mathbb{R}^n$.

    (b) Let $V = \{(x, 0) \mid x \in \mathbb{R}\}$ and $W = \{(0, y) \mid y \in \mathbb{R}\}$. Then both V and W are lines through the origin and hence are subspaces of $\mathbb{R}^n$. But $V \cup W$ is a union of two lines which is not a subspace of $\mathbb{R}^n$, see Discussion 3.2.14 and Remark 3.3.5.1.

    (c) ($\Leftarrow$) If $V \subseteq W$, then $V \cup W = W$ is a subspace of $\mathbb{R}^n$. If $W \subseteq V$, then $W \cup V = V$ is a subspace of $\mathbb{R}^n$.

    ($\Rightarrow$) Suppose $V \nsubseteq W$. We want to show that $W \subseteq V$.

    Take any vector $\boldsymbol{x} \in W$. Since $V \nsubseteq W$, there exists a vector $\boldsymbol{y} \in V$ but $\boldsymbol{y} \notin W$. As $V \cup W$ is a subspace of $\mathbb{R}^n$ and $\boldsymbol{x}, \boldsymbol{y} \in V \cup W$, we have $\boldsymbol{x} + \boldsymbol{y} \in V \cup W$, i.e. either $\boldsymbol{x} + \boldsymbol{y} \in V$ or $\boldsymbol{x} + \boldsymbol{y} \in W$.

    Assume $\boldsymbol{x} + \boldsymbol{y} \in W$. As W is a subspace of $\mathbb{R}^n$ and $-\boldsymbol{x} \in W$, we have $\boldsymbol{y} = (\boldsymbol{x} + \boldsymbol{y}) + (-\boldsymbol{x}) \in W$ which contradict that $\boldsymbol{y} \notin W$ as mentioned above.

    Hence we know that $\boldsymbol{x} + \boldsymbol{y} \in V$. As V is a subspace of $\mathbb{R}^n$ and $-\boldsymbol{y} \in V$, we have $\boldsymbol{x} = (\boldsymbol{x} + \boldsymbol{y}) + (-\boldsymbol{y}) \in V$.

    Since every vector in W is contained in V, $W \subseteq V$.

25. S_1 and S_4 are linearly independent while S_2, S_3, S_5 and S_6 are linearly dependent.

26. (a) $a(1,1,1,2,2)+b(0,0,1,1,1)+c(0,0,0,0,1)=(0,0,0,0,0) \quad \Rightarrow \quad \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$

Thus the nonzero rows of $\boldsymbol{R}$ are linearly independent.

(b) Yes.

We prove by mathematical induction on the number of nonzero rows of the matrix.

It is obvious that one nonzero row is linearly independent.

Assume that for any matrix in row-echelon form with less than k nonzero rows, the nonzero rows are linearly independent.

Let $\boldsymbol{R}$ be a matrix in row-echelon form with k nonzero rows. Let $\boldsymbol{r_i} = (r_{i1}, r_{i2}, \ldots, r_{in})$ be the ith row of $\boldsymbol{R}$ for $1 \le i \le k$. Since $\begin{pmatrix} \boldsymbol{r_2} \\ \vdots \\ \boldsymbol{r_k} \end{pmatrix}$ is a matrix with less than k nonzero rows and it is also in row-echelon form, by the inductive assumption, $\boldsymbol{r_2}, \ldots, \boldsymbol{r_k}$ are linearly independent.

Consider the vector equation:

$$c_1\boldsymbol{r_1} + c_2\boldsymbol{r_2} + \cdots + c_k\boldsymbol{r_k} = \boldsymbol{0} \;\Rightarrow\; \begin{cases} r_{11}c_1 + r_{21}c_2 + \cdots + r_{k1}c_k = 0 \\ r_{12}c_1 + r_{22}c_2 + \cdots + r_{k2}c_k = 0 \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ r_{1n}c_1 + r_{2n}c_2 + \cdots + r_{kn}c_k = 0 \end{cases}$$

Suppose r_{1s} is the leading entry of the first row of $\boldsymbol{R}$. By the definition of row-echelon form, $r_{is} = 0$ for all $i > 1$. Thus the sth equation of the linear system above is $r_{1s}c_1 + 0c_2 + \cdots + 0c_k = 0$ and hence $c_1 = 0$. Substituting $c_1 = 0$ into $c_1\boldsymbol{r_1} + c_2\boldsymbol{r_2} + \cdots + c_k\boldsymbol{r_k} = \boldsymbol{0}$, we get $c_2\boldsymbol{r_2} + \cdots + c_k\boldsymbol{r_k} = \boldsymbol{0}$. Since $\boldsymbol{r_2}, \ldots, \boldsymbol{r_k}$ are linearly independent, the equation above can only have trivial solution, i.e. $c_2 = c_3 = \cdots = 0$.

Thus the equation $c_1\boldsymbol{r_1}+c_2\boldsymbol{r_2}+\cdots+c_k\boldsymbol{r_k} = \boldsymbol{0}$ has only the trivial solution. The nonzero rows of $\boldsymbol{R}$ are linearly independent.

By mathematical induction, we have proven that the nonzero rows of any nonzero matrix in row-echelon form are linearly independent.

27. $a\boldsymbol{u} + b\boldsymbol{v} = \mathbf{0} \quad \Leftrightarrow \quad a\boldsymbol{u} + b\boldsymbol{v} + 0\boldsymbol{w} = \mathbf{0}$.
Since $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent, we have $a = 0$, $b = 0$. Thus S_1 is linearly independent.

Since $(\boldsymbol{u} - \boldsymbol{v}) + (\boldsymbol{v} - \boldsymbol{w}) + (\boldsymbol{w} - \boldsymbol{u}) = \mathbf{0}$, S_2 is linearly dependent.

$a(\boldsymbol{u}-\boldsymbol{v})+b(\boldsymbol{v}-\boldsymbol{w})+c(\boldsymbol{w}+\boldsymbol{u}) = \mathbf{0} \quad \Leftrightarrow \quad (a+c)\boldsymbol{u}+(-a+b)\boldsymbol{v}+(-b+c)\boldsymbol{w} = \mathbf{0}$.
Since $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent, we have

$$\begin{cases} a \qquad + c = 0 \\ -a + b \qquad = 0 \\ \qquad - b + c = 0. \end{cases}$$

The system has only the trivial solution $a = 0$, $b = 0$, $c = 0$. Thus S_3 is linearly independent.

Similarly, we can show that S_4 is linearly independent.

By Example 3.4.8.2, S_5 is linearly dependent.

28. $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} = \mathbf{0} \quad \Leftrightarrow \quad \begin{cases} ac_1 - \ \ c_2 + \ \ c_3 = 0 \\ c_1 + ac_2 - \ \ c_3 = 0 \\ -c_1 + \ \ c_2 + ac_3 = 0 \end{cases}$

Solving the system, we find that the system has exactly one solution if and only if $a \neq 0$. Thus $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ are linearly independent if and only if $a \neq 0$.

29. (a) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent, then the two planes V and W intersect at the line spanned by $\boldsymbol{u}$ and hence $V \cap W = \text{span}\{\boldsymbol{u}\}$.

(b) V and W are planes in $\mathbb{R}^3$. So $\boldsymbol{u}, \boldsymbol{v}$ are linearly independent and $\boldsymbol{u}, \boldsymbol{w}$ are linearly independent. If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly dependent, then $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ must lie on the same plane and hence $V = W = V \cap W$.

30. (a) Note that

$$\begin{aligned} & c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0} \\ \Rightarrow \quad & \boldsymbol{P}(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}) = \boldsymbol{P}\mathbf{0} \\ \Rightarrow \quad & c_1\boldsymbol{P}\boldsymbol{u_1} + c_2\boldsymbol{P}\boldsymbol{u_2} + \cdots + c_k\boldsymbol{P}\boldsymbol{u_k} = \mathbf{0}. \end{aligned}$$

Since $\boldsymbol{P}\boldsymbol{u_1}, \boldsymbol{P}\boldsymbol{u_2}, \ldots, \boldsymbol{P}\boldsymbol{u_k}$ are linearly independent, we conclude that $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_k = 0$. Thus $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly independent.

(b) (i) Note that

$$\begin{aligned} & c_1\boldsymbol{P}\boldsymbol{u_1} + c_2\boldsymbol{P}\boldsymbol{u_2} + \cdots + c_k\boldsymbol{P}\boldsymbol{u_k} = \mathbf{0} \\ \Rightarrow \quad & \boldsymbol{P}(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}) = \mathbf{0}. \\ \Rightarrow \quad & c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0} \qquad \text{(because } \boldsymbol{P} \text{ is invertible).} \end{aligned}$$

Since $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly independent, we conclude that $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_k = 0$. Thus $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \ldots, \boldsymbol{Pu_k}$ are linearly independent.

(ii) No conclusion.

For example, let $\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. It is obvious that $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are linearly independent.

If $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $\boldsymbol{Pu_1}$ and $\boldsymbol{Pu_2}$ are linearly independent.

If $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $\boldsymbol{Pu_1}$ and $\boldsymbol{Pu_2}$ are linearly dependent.

31. ($\Rightarrow$) If V is a subspace of $\mathbb{R}^n$, then by Theorem 3.2.9.2, for any $\boldsymbol{u}, \boldsymbol{v} \in V$ and $c, d \in \mathbb{R}$, $c\boldsymbol{u} + d\boldsymbol{v} \in V$.

($\Leftarrow$) Suppose for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $c, d \in \mathbb{R}$, $c\boldsymbol{u} + d\boldsymbol{v} \in V$. By applying this repeatedly, for any $\boldsymbol{u_1}, \ldots, \boldsymbol{u_k} \in V$, $\text{span}\{\boldsymbol{u_1}, \ldots, \boldsymbol{u_k}\} \subseteq V$.

If $V = \{\boldsymbol{0}\}$, then V is a subspace of $\mathbb{R}^n$, see Remark 3.3.3.1.

Suppose $V \neq \{\boldsymbol{0}\}$. Since V is a nonempty subset of $\mathbb{R}^n$, it has at least 1 and at most n linearly independent vectors, see Theorem 3.4.7. Let S be a largest set of linearly independent vectors in V. Then $\text{span}(S) = V$; if not, there exists $\boldsymbol{v} \in V$ but $\boldsymbol{v} \notin \text{span}(S)$ and by Theorem 3.4.10, $S \cup \{\boldsymbol{v}\}$ is linearly independent which violates our assumption on S. So V is a subspace of $\mathbb{R}^n$.

Remark on Question 3.32 to Question 3.49: Please note that bases for vector spaces are not unique. In the following, if a question asks for a basis, the answer given is only one of the possible answers.

32. (a) No. There are too few vectors.

(b) Yes.

(c) No. The vectors are linearly dependent: $3(1,0,-1)+2(-1,2,3)+2(0,3,3) = (0,0,0)$.

(d) No. There are too many vectors.

33. (a) A general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = r \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{where } r, s, t \in \mathbb{R}.$$

So $\{(-3, 1, 0, 0),\ (1, 0, 1, 0),\ (-2, 0, 0, 1)\}$ is a basis for the solution space.

(b) A general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 0 \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{where } s, t \in \mathbb{R}.$$

So $\{(0, \frac{1}{3}, 1, 0),\ (-2, 0, 0, 1)\}$ is a basis for the solution space.

(c) A general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 0 \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} \quad \text{where } t \in \mathbb{R}.$$

So $\{(0, \frac{1}{3}, 1, 0)\}$ is a basis for the solution space.

34. (a) $(1,\ -\frac{3}{2},\ \frac{8}{3})$. (b) $(-2,\ -1,\ 1)$.

35. (a) $V = \text{span}\{(1, 1, 0, 0),\ (1, 0, 0, 1),\ (0, 1, 1, 0),\ (0, 0, 1, 1)\}$ and hence is a subspace of $\mathbb{R}^4$.

Following the method discussed in Example 3.2.11, we can prove that

$$\begin{aligned} &\text{span}\{(1, 1, 0, 0),\ (1, 0, -1, 0),\ (0, -1, 0, 1)\} \\ &\qquad = \text{span}\{(1, 1, 0, 0),\ (1, 0, 0, 1),\ (0, 1, 1, 0),\ (0, 0, 1, 1)\}, \end{aligned}$$

i.e. $\text{span}(S) = V$. Also it is easy to check that S is linearly independent. So S is a basis for V.

(b) $(4, -3, 2)$.

(c) $(4, 2, -3, -1)$.

36. (a) The dimension is 2 and $\{(1, 1, 0),\ (-1, 0, 1)\}$ is a basis.

(b) The dimension is 2 and $\{(1, 1, 0),\ (0, 0, 1)\}$ is a basis.

(c) The dimension is 1 and $\{(1,-1,2)\}$ is a basis.

37. (a) The dimension is 2 and $\{\,(1,0,0,0),\ (0,0,1,0)\,\}$ is a basis.

(b) The dimension is 2 and $\{\,(1,0,0,1),\ (0,1,1,0)\,\}$ is a basis.

(c) The dimension is 2 and $\{\,(1,\frac{1}{2},\frac{1}{3},0),\ (0,0,0,1)\,\}$ is a basis.

(d) A general solution is

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{where } s, t \in \mathbb{R}.$$

So the dimension of the solution space is 2 and $\{(1,-1,1,0),\ (-2,1,0,1)\}$ is a basis for the solution space.

(e) $(w,x,y,z) = (w,\ x,\ w+x,\ w-x) = w(1,0,1,1) + x(0,1,1,-1)$

It is easy to check that $(1,0,1,1)$, $(0,1,1,-1)$ are linearly independent. So the dimension of the subspace is 2 and $\{\,(1,0,1,1),\ (0,1,1,-1)\,\}$ is a basis for the solution space.

38. (a) $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3} = \boldsymbol{0} \quad \Leftrightarrow \quad (c_1+c_2+c_3)\boldsymbol{u_1} + (c_2+c_3)\boldsymbol{u_2} + c_3\boldsymbol{u_3} = \boldsymbol{0}$

Since $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ are linearly independent, we have

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ \phantom{c_1 +{}} c_2 + c_3 = 0 \\ \phantom{c_1 + c_2 +{}} c_3 = 0. \end{cases}$$

The system has only the trivial solution. So $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ are linearly independent.

Since $\dim(V) = 3$, by Theorem 3.6.7, $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is a basis for V.

(b) No. The vectors are linearly dependent: $\boldsymbol{v_1} + \boldsymbol{v_2} + \boldsymbol{v_3} = \boldsymbol{0}$.

39. For example, for $i = 1, 2, \ldots, n$, let $V_i = \operatorname{span}\{\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_i}\}$, where $E = \{\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_n}\}$ is the standard basis for $\mathbb{R}^n$. It is obvious that $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n$.

As $\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_i}$ are linearly independent, $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_i}\}$ is a basis for V_i. Hence $\dim(V_i) = i$.

40. (a) For example, $a = -2$, $b = -1$, $c = 1$, $d = 0$.

(b) $\boldsymbol{u_3} = 2\boldsymbol{u_1} + \boldsymbol{u_2}$ and $\boldsymbol{u_4} = -2\boldsymbol{u_1} + \boldsymbol{u_2}$.

(c) $\{\boldsymbol{u_1},\ \boldsymbol{u_2}\}$ is a basis for V and $\dim(V) = 2$.

(d) For example, let $W = \text{span}\{\boldsymbol{u_1},\ \boldsymbol{u_2},\ (0, 0, 0, 1)\}$. Then $\dim(W) = 3$. Since $W \cap V = V$, $\dim(W \cap V) = \dim(V) = 2$.

41. Suppose $\dim(V) = n$.

(a) **(Throwing-Out Algorithm)** Since $\text{span}(S) = V$, by Theorem 3.6.1.2, $|S| \geq n$. If $|S| = n$, then by Theorem 3.6.7, S is a basis for V and we set $S' = S$.

Suppose $|S| > n$. Then S is linearly dependent. By Theorem 3.4.4, there exists a vector $\boldsymbol{v_1}$ such that $\boldsymbol{v_1}$ is a linear combination of other vectors in S. Let $S_1 = S - \{\boldsymbol{v_1}\}$. By Theorem 3.2.12, $\text{span}(S_1) = \text{span}(S) = V$.

If $|S_1| = n$, then S_1 is a basis for V and we set $S' = S_1$.

If not, we repeat the process above until we obtain a set $S_k = S - \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}\}$ such that $\text{span}(S_k) = V$ and $|S_k| = n$. Then S_k is a basis for V and we set $S' = S_k$.

(b) **(Adding-On Algorithm)** Since T is linearly independent, by Theorem 3.6.1.1, $|T| \leq n$. If $|T| = n$, then by Theorem 3.6.7, T is a basis for V and we set $T^* = T$.

Suppose $T < n$. Then $\text{span}(T) \neq V$. There exists a vector $\boldsymbol{v_1} \in V - \text{span}(T)$. Let $T_1 = T \cup \{\boldsymbol{v_1}\}$. By Theorem 3.4.10, T_1 is linearly independent. If $|T_1| = n$, then T_1 is a basis for V and we set $T^* = T_1$.

If not, we repeat the process above until we obtain a set $T_k = T \cup \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}\}$ such that T_k is linearly independent and $|T_k| = n$. Then T_k is a basis for V and we set $T^* = T_k$.

42. Take a basis $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ for V. Define $\boldsymbol{u_{n+1}} = -\boldsymbol{u_1} - \boldsymbol{u_2} - \cdots - \boldsymbol{u_n}$.

For any $\boldsymbol{v} \in V$, $\boldsymbol{v} = a_1\boldsymbol{u_1} + a_2\boldsymbol{u_2} + \cdots + a_n\boldsymbol{u_n}$ for some $a_1, a_2, \ldots, a_n \in \mathbb{R}$. Let $a = \min\{0, a_1, a_2, \ldots, a_n\}$. Then

$$\boldsymbol{v} = (a_1 - a)\boldsymbol{u_1} + (a_2 - a)\boldsymbol{u_2} + \cdots + (a_n - a)\boldsymbol{u_n} + (-a)\boldsymbol{u_{n+1}}$$

where $a_i - a \geq 0$, for $i = 1, 2, \ldots, n$, and $-a \geq 0$.

So every vector in V can be expressed as a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_n}$, $\boldsymbol{u_{n+1}}$ with non-negative coefficients.

43. Let $\{\boldsymbol{u_1}, \ldots, \boldsymbol{u_k}\}$ be a basis for $V \cap W$. By Question 3.41(b), there exists vectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_m} \in V$ such that $\{\boldsymbol{u_1}, \ldots, \boldsymbol{u_k}, \boldsymbol{v_1}, \ldots, \boldsymbol{v_m}\}$ is a basis for V and there exists vectors $\boldsymbol{w_1}, \ldots, \boldsymbol{w_n} \in W$ such that $\{\boldsymbol{u_1}, \ldots, \boldsymbol{u_k}, \boldsymbol{w_1}, \ldots, \boldsymbol{w_n}\}$ is a basis

for W.

It is easy to see that $V+W=\text{span}\{\boldsymbol{u_1},\dots,\boldsymbol{u_k},\boldsymbol{v_1},\dots,\boldsymbol{v_m},\boldsymbol{w_1},\dots,\boldsymbol{w_n}\}$.

Consider the vector equation

$$a_1\boldsymbol{u_1}+\cdots+a_k\boldsymbol{u_k}+b_1\boldsymbol{v_1}+\cdots+b_m\boldsymbol{v_m}+c_1\boldsymbol{w_1}+\cdots+c_n\boldsymbol{w_n}=\boldsymbol{0}. \qquad (*)$$

Since $c_1\boldsymbol{w_1}+\cdots+c_n\boldsymbol{w_n}=-(a_1\boldsymbol{u_1}+\cdots+a_k\boldsymbol{u_k}+b_1\boldsymbol{v_1}+\cdots+b_m\boldsymbol{v_m})\in V\cap W$, there exists $d_1,\dots,d_k\in\mathbb{R}$ such that $c_1\boldsymbol{w_1}+\cdots+c_n\boldsymbol{w_n}=d_1\boldsymbol{u_1}+\cdots+d_k\boldsymbol{u_k}$, i.e.

$$c_1\boldsymbol{w_1}+\cdots+c_n\boldsymbol{w_n}-d_1\boldsymbol{u_1}-\cdots-d_k\boldsymbol{u_k}=\boldsymbol{0}.$$

As $\{\boldsymbol{u_1},\dots,\boldsymbol{u_k},\boldsymbol{w_1},\dots,\boldsymbol{w_n}\}$ is linearly independent, $c_1=\cdots=c_n=d_1=\cdots=d_k=0$.

Substituting $c_1=\cdots=c_n=0$ into $(*)$, we have

$$a_1\boldsymbol{u_1}+\cdots+a_k\boldsymbol{u_k}+b_1\boldsymbol{v_1}+\cdots+b_m\boldsymbol{v_m}=\boldsymbol{0}.$$

As $\{\boldsymbol{u_1},\dots,\boldsymbol{u_k},\boldsymbol{v_1},\dots,\boldsymbol{v_m}\}$ is linearly independent, $a_1=\cdots=a_k=b_1=\cdots=b_m=0$.

So $(*)$ has only the trivial solution and hence $\{\boldsymbol{u_1},\dots,\boldsymbol{u_k},\boldsymbol{v_1},\dots,\boldsymbol{v_m},\boldsymbol{w_1},\dots,\boldsymbol{w_n}\}$ is linearly independent.

We have shown that $\{\boldsymbol{u_1},\dots,\boldsymbol{u_k},\boldsymbol{v_1},\dots,\boldsymbol{v_m},\boldsymbol{w_1},\dots,\boldsymbol{w_n}\}$ is a basis for $V+W$. Thus $\dim(V+W)=k+m+n=(k+m)+(k+n)-k=\dim(V)+\dim(W)-\dim(V\cap W)$.

44. As U and V are spanned by a set of three vectors, $\dim(U)\le 3$ and $\dim(V)\le 3$. On the other hand, since $\dim(U\cap V)=2$, $\dim(U)\ge 2$ and $\dim(V)\ge 2$.

    Suppose $\dim(U)=2$, then by Theorem 3.6.9, $U\cap V=U$. As the smallest subspace that contains both U and V, we have $W=V$ and hence $\dim(W)=\dim(V)=2$ or 3.

    Similarly, if $\dim(V)=2$, we have $W=U$ and hence $\dim(W)=\dim(U)=2$ or 3.

    Finally, if $\dim(U)=\dim(V)=3$, then by Question 3.43, $\dim(W)=3+3-2=4$.

    Therefore, the possible dimension of W are 2, 3 and 4.

45. (a) False. For example, let $S_1=\{(1,0),(0,1)\}$ and $S_2=\{(1,0),(0,2)\}$ where $V=W=\mathbb{R}^2$.

    (b) False. For example, let $S_1=\{(1,0)\}$ and $S_2=\{(1,1),(0,1)\}$ where $V=\text{span}(S_1)$ and $W=V+W=\mathbb{R}^2$. Note that $S_1\cup S_2$ is linearly dependent.

(c) True. See the proof of Question 3.43.

(d) True. See the proof of Question 3.43.

46. (a) $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 3 \end{vmatrix} = 7$. By Theorem 3.6.11, S is a basis for $\mathbb{R}^3$.

(b) $(\boldsymbol{w})_S = (1, -\frac{1}{7}, \frac{5}{7})$.

(c) $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

(d) $\dfrac{1}{8}\begin{pmatrix} 6 & 4 & -6 \\ -4 & 0 & 4 \\ -1 & -2 & 5 \end{pmatrix}$.

(e) $(\boldsymbol{w})_T = (\frac{1}{7}, -\frac{1}{7}, \frac{5}{14})$.

47. (a) $\begin{vmatrix} 3 & -2 & 5 \\ 1 & -4 & 4 \\ 0 & 3 & -2 \end{vmatrix} = -1$. By Theorem 3.6.11, S is a basis for $\mathbb{R}^3$.

(b)
$$\begin{aligned} & c_1(\boldsymbol{u_1} - \boldsymbol{u_2}) + c_2(\boldsymbol{u_1} + 2\boldsymbol{u_2} - \boldsymbol{u_3}) + c_3(\boldsymbol{u_2} + 2\boldsymbol{u_3}) = \boldsymbol{0} \\ \Leftrightarrow\ & (c_1 + c_2)\boldsymbol{u_1} + (-c_1 + 2c_2 + c_3)\boldsymbol{u_2} + (-c_2 + 2c_3)\boldsymbol{u_3} = \boldsymbol{0} \end{aligned}$$

By (a), S is linearly independent. Thus

$$\begin{cases} c_1 + c_2 = 0 \\ -c_1 + 2c_2 + c_3 = 0 \\ - c_2 + 2c_3 = 0. \end{cases}$$

The system has only the trivial solution. So T is linearly independent. Since $\dim(\mathbb{R}^3) = 3$, by Theorem 3.6.7, T is a basis for $\mathbb{R}^3$.

(c) $(1, -2, -2)$.

(d) $(3, 4, 1)$.

(e) $[\boldsymbol{u_1} - \boldsymbol{u_2}]_S = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $[\boldsymbol{u_1} + 2\boldsymbol{u_2} - \boldsymbol{u_3}]_S = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $[\boldsymbol{u_2} + 2\boldsymbol{u_3}]_S = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

The transition matrix from T to S is $P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}$ and the transition matrix from S to T is $P^{-1} = \dfrac{1}{7}\begin{pmatrix} 5 & -2 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 3 \end{pmatrix}$.

(f) $(2,3,3)$.

48. (a) Since $(0,1,1)$ and $(1,2,0)$ satisfy the equation $2x-y+z=0$, $S\subseteq V$. S is linearly independent because the two vectors are not scalar multiples of each other. As $\dim(V)=2$, by Theorem 3.6.7, S is a basis for V.

By the same argument, T is also a basis for V.

(b) $$\left(\begin{array}{cc|c|c} 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & -2 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cc|c|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $[(1,1,-1)]_S = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $[(1,0,-2)]_S = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

The transition matrix from T to S is $\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$.

The transition matrix from S to T is $\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$.

(c) $(\boldsymbol{w})_S = (-3,1)$ and $(\boldsymbol{w})_T = (-1,2)$.

49. (a) $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3} = \mathbf{0} \quad \Leftrightarrow \quad c_1\boldsymbol{u_1} + (c_1+c_2+c_3)\boldsymbol{u_2} + (c_1+c_2-c_3)\boldsymbol{u_3} = \mathbf{0}$

Since $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ are linearly independent, we have

$$\begin{cases} c_1 & = 0 \\ c_1 + c_2 + c_3 & = 0 \\ c_1 + c_2 - c_3 & = 0. \end{cases}$$

The system has only the trivial solution. So T is linearly independent.

Since $\dim(\mathbb{R}^3) = 3$, by Theorem 3.6.7, T is a basis for $\mathbb{R}^3$.

(b) $[\boldsymbol{v_1}]_S = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $[\boldsymbol{v_2}]_S = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $[\boldsymbol{v_3}]_S = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

The transition matrix from T to S is $P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ and the transition matrix from S to T is $P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.