

ST2234 (2021/22 Semester 2)  
Solution to Tutorial 6

Question 1

$f_{X,Y}(x, y)$		$x$			$f_Y(y)$
		1	2	3	
$y$	1	0.05	0.05	0.10	<b>0.20</b>
	2	0.05	0.10	0.35	<b>0.50</b>
	3	0	0.20	0.10	<b>0.30</b>
$f_X(x)$		<b>0.10</b>	<b>0.35</b>	<b>0.55</b>	1

(a)

$x$	1	2	3
$f_X(x)$	0.10	0.35	0.55

(b)

$y$	1	2	3
$f_Y(y)$	0.20	0.50	0.30

(c) Find  $\Pr(Y = 3 | X = 2)$ .

$$f_{Y|X}(y = 3 | x = 2) = \frac{f_{X,Y}(2, 3)}{f_X(2)} = \frac{0.20}{0.35} = \frac{4}{7} = 0.57143$$

(d)  $f_{Y|X}(y | x = 2) = f_{X,Y}(2, y) / f_X(2)$

$y$	1	2	3
$f_{Y X}(y   x = 2)$	$0.05/0.35 = 1/7$ $= 0.14286$	$0.10/0.35 = 2/7$ $= 0.28571$	$0.20/0.35 = 4/7$ $= 0.57143$

(e)  $X$  and  $Y$  are dependent if  $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$  for some values of  $x$  and  $y$ .

$$f_{X,Y}(1, 1) = 0.05$$

$$f_X(1)f_Y(1) = (0.10)(0.20) = 0.02$$

Since  $f_{X,Y}(1, 1) \neq f_X(1)f_Y(1)$ , therefore  $X$  and  $Y$  are dependent.

Question 2

(a) First, random variable  $X$  can only take values in 0, 1, 2, 3;  $Y$  in 0, 1, 2. As only 4 pieces of fruit is selected, therefore  $x + y \leq 4$ . Since there are only three bananas, one piece of the selected fruit must be either an orange or an apple, that is,  $x + y \geq 1$ .

$$f_{X,Y}(x, y) = \begin{cases} \frac{\binom{3}{x} \binom{2}{y} \binom{3}{4-x-y}}{\binom{8}{4}}, & x = 0, 1, 2, 3; y = 0, 1, 2; 1 \leq x + y \leq 4; \\ 0, & \text{otherwise.} \end{cases}$$

(b)  $\Pr(X = 1, Y = 1) = f_{X,Y}(1, 1) = \frac{\binom{3}{1} \binom{2}{1} \binom{3}{2}}{\binom{8}{4}} = \frac{9}{35} = 0.2571$

(c)  $\Pr(X + Y \leq 2) = f_{X,Y}(0, 1) + f_{X,Y}(0, 2) + f_{X,Y}(1, 0) + f_{X,Y}(1, 1) + f_{X,Y}(2, 0) = 0.5$

(d) Recall the possible values of  $X$  are 0, 1, 2, 3. Since 4 pieces of fruit are selected,  $(4 - X)$  pieces of fruit must be selected from 5 pieces of apples and bananas. That is,

$$f_X(x) = \begin{cases} \frac{\binom{3}{x} \binom{5}{4-x}}{\binom{8}{4}}, & x = 0, 1, 2, 3; \\ 0, & \text{otherwise.} \end{cases}$$

(e) For  $x = 2$ ,

$$f_{Y|X}(y|2) = \begin{cases} \frac{\binom{2}{y} \binom{3}{4-2-y}}{\binom{5}{4-2}}, & y = 0, 1, 2; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$f_{Y|X}(y|2) = \begin{cases} \frac{\binom{2}{y} \binom{3}{2-y}}{\binom{5}{2}} = \frac{1}{10} \binom{2}{y} \binom{3}{2-y}, & y = 0, 1, 2; \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{and } \Pr(Y = 0|X = 2) = \frac{1}{10} \binom{2}{0} \binom{3}{2} = \frac{1}{10} (1)(3) = 0.3.$$

### Question 3

Let  $D_1$  and  $D_2$  denote the number obtained by the first die and the second die respectively. The entries of the table below correspond to the values of  $(x, y)$  as defined in the question:

$d_2$	$d_1$					
	1	2	3	4	5	6
1	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)
2	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)
3	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)
4	(1, 0)	(1, 0)	(1, 0)	(2, 0)	(1, 1)	(1, 0)
5	(0, 1)	(0, 1)	(0, 1)	(1, 1)	(0, 2)	(0, 1)
6	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)

- (a) From the table above, we have

$f_{(X,Y)}(x, y)$		$x$			$f_Y(y)$
		0	1	2	
$y$	0	$16/36 = 4/9$	$8/36 = 2/9$	$1/36$	<b>25/36</b>
	1	$8/36 = 2/9$	$2/36 = 1/18$	0	<b>5/18</b>
	2	$1/36$	0	0	<b>1/36</b>
<b><math>f_X(x)</math></b>		<b>25/36</b>	<b>5/18</b>	<b>1/36</b>	1

(b)  $\Pr(2X + Y < 3) = f_{X,Y}(0, 0) + f_{X,Y}(0, 1) + f_{X,Y}(0, 2) + f_{X,Y}(1, 0)$   
 $= \frac{4}{9} + \frac{2}{9} + \frac{1}{36} + \frac{2}{9} = \frac{11}{12} = 0.91667.$

- (c)  $X$  and  $Y$  are dependent if  $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$  for some values of  $X$  and  $Y$ .  
 $f_X(2)f_Y(2) = (1/36)(1/36) = 1/1296 \neq f_{X,Y}(2, 2) (= 0)$   
 Since  $f_{X,Y}(2, 2) \neq f_X(2)f_Y(2)$ , therefore  $X$  and  $Y$  are dependent.

### Question 4

$$f_{X,Y}(x, y) = \begin{cases} k(x^2 + y^2), & 3 \leq x \leq 5; \quad 3 \leq y \leq 5; \\ 0, & \text{otherwise} \end{cases}$$

$$k \int_3^5 \int_3^5 (x^2 + y^2) dy dx = k \int_3^5 \left[ yx^2 + \frac{y^3}{3} \right]_3^5 dx$$

$$= \frac{2}{3}k \int_3^5 3x^2 + 49 dx = \frac{2}{3}k[x^3 + 49x]_3^5 = \frac{392}{3}k$$

Hence,  $k \int_3^5 \int_3^5 (x^2 + y^2) dy dx = 1$  implies  $\frac{392}{3}k = 1$  or  $k = \frac{3}{392}$

(b)  $\Pr(3 \leq X \leq 4 \text{ and } 4 \leq Y < 5)$

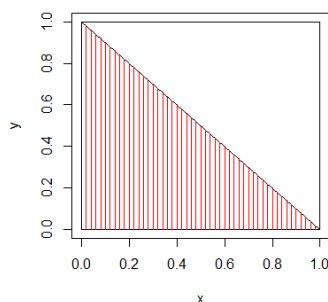
$$\begin{aligned} &= \frac{3}{392} \int_3^4 \int_4^5 (x^2 + y^2) dy dx = \frac{3}{392} \int_3^4 \left[ yx^2 + \frac{y^3}{3} \right]_4^5 dx \\ &= \frac{3}{392} \int_3^4 \left( x^2 + \frac{61}{3} \right) dx = \frac{1}{392} [x^3 + 61x]_3^4 = \frac{1}{392} (98) = \frac{1}{4} = 0.25 \end{aligned}$$

(c)  $f_X(x) = \frac{3}{392} \int_3^5 (x^2 + y^2) dy = \frac{3}{392} \left[ x^2y + \frac{y^3}{3} \right]_3^5 = \frac{3}{392} \left( 2x^2 + \frac{98}{3} \right) = \frac{1}{196} (3x^2 + 49)$ ,  
for  $3 \leq x \leq 5$

$$\begin{aligned} \Pr(3.5 < X < 4) &= \frac{1}{196} \int_{3.5}^4 (3x^2 + 49) dx \\ &= \frac{1}{196} [x^3 + 49x]_{3.5}^4 = \frac{1}{196} \frac{365}{8} = \frac{365}{1568} = 0.2328 \end{aligned}$$

### Question 5

$$f_{X,Y}(x,y) = \begin{cases} 24xy, & 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad x+y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$



(a)  $f_X(x) = \int_0^{1-x} (24xy) dy = [12xy^2]_0^{1-x} = 12x(1-x)^2$ , for  $0 \leq x \leq 1$

$f_Y(y) = \int_0^{1-y} (24xy) dx = [12x^2y]_0^{1-y} = 12y(1-y)^2$ , for  $0 \leq y \leq 1$

(b)  $f_{X,Y}(x,y) = 24xy \neq f_X(x)f_Y(y) (= 12x(1-x)^2 12y(1-y)^2)$ , for  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ . Hence  $X$  and  $Y$  are not independent.

Alternatively, we may consider a point, let say,  $(x,y) = (\frac{2}{3}, \frac{1}{2})$ . Then  $f_X(\frac{2}{3}) = \frac{8}{9}$  and

$f_Y(\frac{1}{2}) = \frac{3}{2}$ , while  $f_{X,Y}(\frac{2}{3}, \frac{1}{2}) = 0$

(c) For  $0 \leq x \leq 1$ ,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{24xy}{12x(1-x)^2} = \frac{2y}{(1-x)^2}, \text{ for } 0 \leq y \leq 1-x.$$

$$f_{Y|X}\left(y|x = \frac{3}{4}\right) = \frac{2y}{\left(1 - \frac{3}{4}\right)^2} = 32y, \text{ for } 0 \leq y \leq \frac{1}{4}$$

$$\begin{aligned} \Pr\left(Y < \frac{1}{8} \mid x = \frac{3}{4}\right) &= \int_0^{\frac{1}{8}} f_{Y|X}\left(y|x = \frac{3}{4}\right) dy = \int_0^{\frac{1}{8}} (32y) dy = [16y^2]_0^{\frac{1}{8}} = \frac{1}{4} \\ &= 0.25. \end{aligned}$$