L2: Substitution Model, Recursion

CS1101S: Programming Methodology

Martin Henz

August 18, 2021

Before we start...

Where are we in CS1101S?

What is CS1101S? SICP JS, Source Academy, Studios, Missions, Piazza, Paths

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- Use ModReg appeals to secure a Reflection or Studio session for **Week 3 onwards**

Earning Experience Points

- Missions (basic homework): your primary source of XP

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- Avengers can "unsubmit" your submission, within reasonable limits
- If you submit less than 2 days after release of Mission/Quest, Avengers will give feedback less than 4 days after release.
- Path: XP for submission
- Early path submission: 25 XP within 1 day; bonus decays after that

Earning Experience Points

- Missions (basic homework): your primary source of XP
- Quests (optional homework): supplement your XP
- Early Mission/Quest submission bonus: 75 XP within 3 days; bonus decays after that
- Avengers can "unsubmit" your submission, within reasonable limits
- If you submit less than 2 days after release of Mission/Quest, Avengers will give feedback less than 4 days after release.
- Path: XP for submission
- Early path submission: 25 XP within 1 day; bonus decays after that
- No "unsubmit" for paths

1. Substitution model, 1.1.3 and 1.1.5
2. Recursion and iteration, 1.2.1 and 1.2.2

1. Substitution model, 1.1.3 and 1.1.5
   - Evaluating combinations, 1.1.3
   - Evaluating function application, 1.1.5
   - Applicative vs normal order reduction, 1.1.5
2. Recursion and iteration, 1.2.1 and 1.2.2

Review: Evaluating arithmetic expressions, see 1.1.3

```
1 + 2 * 3 + 4
```

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Replace "eligible" subexpression with result

```
(1 + (2 * 3)) + 4
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```
(1 + (2 * 3)) + 4

(1 + 6) + 4
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Review: Evaluating arithmetic expressions, see 1.1.3

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1 + 2 * 3 + 4
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Replace “eligible” subexpression with result

```
(1 + (2 * 3)) + 4

(1 + 6) + 4

7 + 4
```

Review: Evaluating arithmetic expressions, see 1.1.3

```
1 + 2 * 3 + 4
```

Replace "eligible" subexpression with result

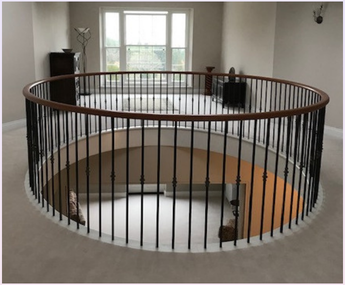
```
(1 + (2 * 3)) + 4

(1 + 6) + 4

7 + 4

11
```

Example for substitution model 1.1.5



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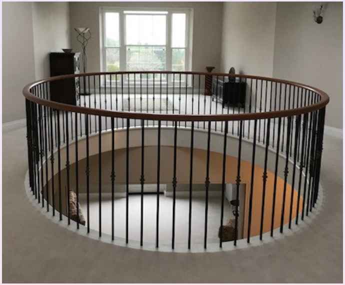
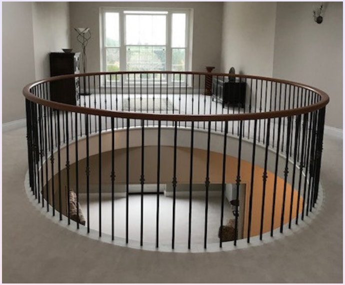


Problem

You need to find the cost of a circular handrail, knowing that the cost of the rail per meter is 199.50 dollars, and the radius of the circle is 2.1 meters.

Example for substitution model 1.1.5



Problem

You need to find the cost of a circular handrail, knowing that the cost of the rail per meter is 199.50 dollars, and the radius of the circle is 2.1 meters.

Solution

Evaluation of function application (1.1.5)

```
const cost_per_meter = 199.95;
function circumference(radius) {
    return 2 * math_PI * radius;}
function cost_of_circular_handrail(r) {
    return cost_per_meter * circumference(r);}
```

Evaluation of function application (1.1.5)

```
const cost_per_meter = 199.95;
function circumference(radius) {
    return 2 * math_PI * radius;}
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cost_of_circular_handrail(2.1)
```

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```
cost_of_circular_handrail(2.1)
```

```
-> 199.5 * circumference(2.1)
```

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```

```
cost_of_circular_handrail(2.1)
```

```
-> 199.5 * circumference(2.1)
```

```
-> 199.5 * ((2 * 3.141592) * 2.1)
```

Evaluation of function application (1.1.5)

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const cost_per_meter = 199.95;
function circumference(radius) {
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cost_of_circular_handrail(2.1)

-> 199.5 * circumference(2.1)

-> 199.5 * ((2 * 3.141592) * 2.1)

-> 199.5 * (6.283185 * 2.1)
```

Evaluation of function application (1.1.5)

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const cost_per_meter = 199.95;
function circumference(radius) {
    return 2 * math_PI * radius;}
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```

```
cost_of_circular_handrail(2.1)

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-> 199.5 * 13.19468
```

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-> 199.5 * (6.283185 * 2.1)

-> 199.5 * 13.19468

-> 2632.340
```

Applicative order reduction, 1.1.5

```
function sq(x) { return x * x; }
function sum_of_sqs(x,y) { return sq(x)+sq(y);}
function f(a) { return sum_of_sqs(a+1, a*2);}
```

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f(5)
```

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```
->  sum_of_sqs(5 + 1, 5 * 2)
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f(5)

->   sum_of_sqs(5 + 1, 5 * 2)

->   sum_of_sqs(6, 10)
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->  sum_of_sqs(5 + 1, 5 * 2)

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->  sq(6) + sq(10)
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f(5)

->  sum_of_sqs(5 + 1, 5 * 2)

->  sum_of_sqs(6, 10)

->  sq(6) + sq(10)

->  (6 * 6) + (10 * 10)
```

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->  sq(6) + sq(10)

->  (6 * 6) + (10 * 10)

⋮

->  136
```

Normal order reduction, 1.1.5

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function sq(x) { return x * x; }
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function sq(x) { return x * x; }
function sum_of_sqs(x,y) { return sq(x)+sq(y);}
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->  sum_of_sqs(5 + 1, 5 * 2)

->  sq(5 + 1) + sq(5 * 2)
```

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f(5)

->  sum_of_sqs(5 + 1, 5 * 2)

->  sq(5 + 1) + sq(5 * 2)

->  ((5 + 1) * (5 + 1)) +
    ((5 * 2) * (5 * 2))
```

Normal order reduction, 1.1.5

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function sq(x) { return x * x; }
function sum_of_sqs(x,y) { return sq(x)+sq(y);}
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->  sum_of_sqs(5 + 1, 5 * 2)

->  sq(5 + 1) + sq(5 * 2)

->  ((5 + 1) * (5 + 1)) +
    ((5 * 2) * (5 * 2))

⋮

->  136
```

Substitution model for applicative-order languages

Primitive expressions : take the value

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Operator combinations : evaluate operands, apply operator

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- if function is primitive: apply the primitive function

Substitution model for applicative-order languages

Primitive expressions : take the value

Operator combinations : evaluate operands, apply operator

Constant declaration : evaluate the value expression and replace the name by value

Function application : evaluate component expressions

- if function is primitive: apply the primitive function
- if function is compound: substitute argument values for parameters in body of declaration

Substitution model for runes

```
function turn_upside_down(rune) {
  return quarter_turn_right(
           quarter_turn_right(rune));
}
function quarter_turn_left(rune) {
  return quarter_turn_right(
           turn_upside_down(rune));
}
```

Substitution model for runes

```
function turn_upside_down(rune) {
  return quarter_turn_right(
           quarter_turn_right(rune));
}
function quarter_turn_left(rune) {
  return quarter_turn_right(
           turn_upside_down(rune));
}

quarter_turn_left(heart)
```

1. Substitution model, 1.1.3 and 1.1.5
2. Recursion and iteration, 1.2.1 and 1.2.2
   - `stackn` and `repeat_pattern`
   - Simple factorial and its recursive process, 1.2.1
   - Fibonacci function, 1.2.2

A new predeclared combination: `stack_frac`

`stack_frac(r, heart, sail)`

splits available bounded box such that
heart occupies top fraction `r` of box
and sail occupies remaining `1 - r` of box

Examples

`stack_frac(87 / 100, heart, sail)`

splits available bounded box such that heart occupies the top 87% of box and sail occupies remaining 13% of box

Examples

stack_frac(87 / 100, heart, sail)

splits available bounded box such that heart occupies the top 87% of box and sail occupies remaining 13% of box

Trisection of the heart

```
stack_frac(
    1 / 3,
    heart,
    stack_frac(
        1 / 2,
        heart,
        heart))
```

Can we define stackn in Source?

Trisection of the heart

```
stack_frac(1 / 3, heart,
  stack_frac(1 / 2, heart, heart))
```

Can we define stackn in Source?

Trisection of the heart

```
stack_frac(1 / 3, heart,
  stack_frac(1 / 2, heart, heart))
```

Quadrisection of the heart

```
stack_frac(1 / 4, heart,
  stack_frac(1 / 3, heart,
    stack_frac(1 / 2, heart,
                heart)))
```

Can we define stackn in Source?

Trisection of the heart

```
stack_frac(1 / 3, heart,
  stack_frac(1 / 2, heart, heart))
```

Quadrisection of the heart

```
stack_frac(1 / 4, heart,
  stack_frac(1 / 3, heart,
    stack_frac(1 / 2, heart,
               heart)))
```

Can we generalise this idea?

A *Recursive* Function, first try

```
function stackn(n, rune) {
    return stack_frac(1 / n,
                      rune,
                      stackn(n - 1, rune));
}

stackn(3, heart)
```

A *Recursive* Function, first try

```
function stackn(n, rune) {
    return stack_frac(1 / n,
                      rune,
                      stackn(n - 1, rune));
}

stackn(3, heart)
```

Computers will follow our orders

We need to *precisely* describe
how a computational process
should be executed.

The correct version

```
function stackn(n, rune) {
    return n === 1
           ? rune
           : stack_frac(1 / n,
                        rune,
                        stackn(n - 1, rune));
}
```

The correct version

```
function stackn(n, rune) {
    return n === 1
           ? rune
           : stack_frac(1 / n,
                        rune,
                        stackn(n - 1, rune));
}
```

Observation

Solution for n computed using solution $n-1$,

solution for $n-1$ is computed using solution $n-2$, ...

until we reach trivial case.

A Recipe

```
function stackn(n, rune) {
    return n === 1
           ? rune
           : stack_frac(1 / n,
                        rune,
                        stackn(n - 1, rune));
}
```

A Recipe

```
function stackn(n, rune) {
    return n === 1
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```

Recipe for recursion

A Recipe

```
function stackn(n, rune) {
    return n === 1
           ? rune
           : stack_frac(1 / n,
                        rune,
                        stackn(n - 1, rune));
}
```

Recipe for recursion

- Figure out trivial *base case*

A Recipe

```
function stackn(n, rune) {
    return n === 1
           ? rune
           : stack_frac(1 / n,
                        rune,
                        stackn(n - 1, rune));
}
```

Recipe for recursion

- Figure out trivial *base case*
- Assume you know how to solve problem for $n - 1$.

A Recipe

```
function stackn(n, rune) {
    return n === 1
           ? rune
           : stack_frac(1 / n,
                        rune,
                        stackn(n - 1, rune));
}
```

Recipe for recursion

- Figure out trivial *base case*
- Assume you know how to solve problem for $n - 1$.
  How can we solve problem for n?

Can we define repeat_pattern in Source?

Remember pre-defined repeat_pattern in module rune

```
repeat_pattern(3, make_cross, sail)
// should lead to
make_cross(make_cross(make_cross(sail)))
```

Can we define repeat_pattern in Source?

Remember pre-defined repeat_pattern in module rune

```
repeat_pattern(3, make_cross, sail)
// should lead to
make_cross(make_cross(make_cross(sail)))
```

Another example

```
function square(x) { return x * x; }
repeat_pattern(3, square, 2)
// should lead to
square(square(square(2)))
```

repeat_pattern, our first version

```
function repeat_pattern(n, pat, init) {
  return n === 0
         ? init
         : pat(repeat_pattern(n - 1, pat, init));
}
```

repeat_pattern, our first version

```
function repeat_pattern(n, pat, init) {
  return n === 0
         ? init
         : pat(repeat_pattern(n - 1, pat, init));
}
repeat_pattern(
    3,
    square,
    2);
```

repeat_pattern, our first version

```
function repeat_pattern(n, pat, init) {
  return n === 0
         ? init
         : pat(repeat_pattern(n - 1, pat, init));
}
repeat_pattern(
    3,
    square,
    2);
```

Recursive process

The applications of pat
accumulate as result of recursive calls.
They are *deferred* operations.

repeat_pattern, second version

```
function repeat_pattern(n, pat, rune) {
  return n === 0
         ? rune
         : repeat_pattern(n - 1, pat, pat(rune));
}
```

repeat_pattern, second version

```
function repeat_pattern(n, pat, rune) {
  return n === 0
         ? rune
         : repeat_pattern(n - 1, pat, pat(rune));
}
repeat_pattern(
    3,
    square,
    2);
```

repeat_pattern, second version

```
function repeat_pattern(n, pat, rune) {
  return n === 0
         ? rune
         : repeat_pattern(n - 1, pat, pat(rune));
}
repeat_pattern(
    3,
    square,
    2);
```

Iterative process

With applicative order reduction,
pat function is applied *before*
the recursive call.
There is no deferred operation.

Another example: Factorial 1.2.1

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Factorial

$n\,! = n(n-1)(n-2)\cdots 1$

Another example: Factorial 1.2.1

Factorial

$n\,! = n(n-1)(n-2)\cdots 1$

Grouping

$n\,! = n((n-1)(n-2)\cdots 1)$

Another example: Factorial 1.2.1

Factorial

$n\,! = n(n-1)(n-2)\cdots 1$

Grouping

$n\,! = n((n-1)(n-2)\cdots 1)$

Replacement

$n\,! = n(n-1)\,!$

Another example: Factorial 1.2.1

Factorial

$n\,! = n(n-1)(n-2)\cdots 1$

Grouping

$n\,! = n((n-1)(n-2)\cdots 1)$

Replacement

$n\,! = n(n-1)\,!$

Remember the base case

$$
\begin{array}{ll}
n\,! = 1 & \text{if } n = 1 \\
n\,! = (n-1)\,! & \text{if } n > 1
\end{array}
$$

Translation into Source

Remember the base case

$$
\begin{array}{lll}
n\,! & = & 1 \qquad\qquad\quad \text{if } n = 1 \\
n\,! & = & n(n-1)\,! \quad \text{if } n > 1
\end{array}
$$

Factorial in Source

```
function factorial(n) {
  return n === 1
         ? 1
         : n *
           factorial(n - 1);
}
```

Example execution using Substitution Model

```
function factorial(n) {
    return n === 1 ? 1 : n * factorial(n - 1);
}

factorial(4)
4 * factorial(3)
4 * (3 * factorial(2))
4 * (3 * (2 * factorial(1)))
4 * (3 * (2 * 1))
4 * (3 * 2)
4 * 6
24
```

Example execution using Substitution Model

```
function factorial(n) {
    return n === 1 ? 1 : n * factorial(n - 1);
}
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Recursive process.

A Closer look at performance

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- Time:
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  to run the program

Time for calculating n!

Number of operations

grows linearly proportional to n.

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Space for calculating n!

Deferred operations: Number of "things to remember"

grows linearly proportional to n.

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```

Can we write an iterative factorial?

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```
function factorial(n) {
  return iter(1, 1, n);
}
function iter(product, counter, n) {
  return counter > n
         ? product
         : iter(counter * product,
                counter + 1,
                n);
}
```

Fibonacci: Computing $F(n)$, first attempt

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```
function fib(n) {
    return n <= 1
           ? n
           : fib(n - 1) + fib(n - 2);
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$$F(n) = \left\lfloor \frac{\phi^n}{\sqrt{5}} + \frac{1}{2} \right\rfloor, \text{where } \phi = \frac{1+\sqrt{5}}{2}$$

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Can we write an *efficient iterative* version?

Fibonacci: Computing $F(n)$, iterative solution

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```
function fib(n) {
    return fib_iter(1, 0, n);
}
function fib_iter(a, b, count) {
    return count === 0
           ? b
           : fib_iter(a + b, a, count - 1);
}
```

Fibonacci: Computing $F(n)$, iterative solution

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    return fib_iter(1, 0, n);
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function fib_iter(a, b, count) {
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  a bad one and a good one!