MA2001

LIVE LECTURE 9

Q&A: log in to PollEv.com/vtpoll

Q&A: log in to PollEv.com/vtpoll

Topics for week 9

5.1 Inner Products in $\mathbf{R}^n$

5.2 Orthogonal and Orthonormal Bases

5.3 Best Approximation (preview)

Q&A: log in to PollEv.com/vtpoll

Dot product

$\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ vectors in $\mathbf{R}^n$

The dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as

product of two vectors $\boxed{\boldsymbol{u} \cdot \boldsymbol{v}} = \boxed{u_1 v_1 + u_2 v_2 + \ldots + u_n v_n}$ scalar

$\boldsymbol{u}$ and $\boldsymbol{v}$ regarded as row matrices $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}^T$

$\boldsymbol{u}$ and $\boldsymbol{v}$ regarded as column matrices $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$

$$\boldsymbol{A}\boldsymbol{b} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} \boldsymbol{b} = \begin{pmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{b} \\ \boldsymbol{a}_2 \cdot \boldsymbol{b} \\ \vdots \\ \boldsymbol{a}_n \cdot \boldsymbol{b} \end{pmatrix}$$

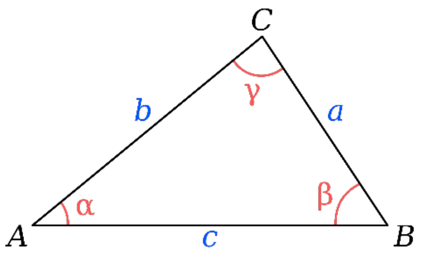
$\boldsymbol{u}$ regarded as row matrix, $\boldsymbol{v}$ regarded as column matrix ? $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}$

Q&A: log in to PollEv.com/vtpoll

Length, distance and angles in $\mathbf{R}^n$

$\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ vectors in $\mathbf{R}^n$

Dot product	$\boldsymbol{u} \cdot \boldsymbol{v}$	$u_1v_1 + u_2v_2 + \ldots u_nv_n$
Norm (length)	$\|\| \boldsymbol{u} \|\|$	$\sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$ $\sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$
Distance	$\|\| \boldsymbol{u} - \boldsymbol{v} \|\|$	$\sqrt{(\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v})}$ $\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$
Angle	between $\boldsymbol{u}$ and $\boldsymbol{v}$	$\cos^{-1}\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\|\boldsymbol{u}\|\| \, \|\|\boldsymbol{v}\|\|}\right)$ $\cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + \ldots + u_nv_n}{\|\|\boldsymbol{u}\|\| \, \|\|\boldsymbol{v}\|\|}\right)$



$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

Q&A: log in to PollEv.com/vtpoll

Comparing vectors

Among the three vectors $\mathbf{u}_1 = (1,1,0,0)$, $\mathbf{u}_2 = (0,1,1,0)$, $\mathbf{u}_3 = (1,0,0,1)$

which is the best approximation of $\mathbf{v} = (1,2,3,4)$?

Compare the distance between $\mathbf{v}$ and each of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$

$$\| \mathbf{v} - \mathbf{u}_1 \| = \sqrt{(1-1)^2+(2-1)^2+(3-0)^2+(4-0)^2} = \sqrt{26}$$

$$\| \mathbf{v} - \mathbf{u}_2 \| = \sqrt{(1-0)^2+(2-1)^2+(3-1)^2+(4-0)^2} = \sqrt{22}$$

$$\| \mathbf{v} - \mathbf{u}_3 \| = \sqrt{(1-1)^2+(2-0)^2+(3-0)^2+(4-1)^2} = \sqrt{22}$$

The smaller the distance between two vectors, the better is the approximation

The smaller the angle between two vectors, the better is the approximated ratio among the coordinates

Q&A: log in to PollEv.com/vtpoll

Properties of dot product

Let c be a scalar and $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ vectors in $\mathbf{R}^n$.

1. $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$ commutative law
2. $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$
   $\boldsymbol{w} \cdot (\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{w} \cdot \boldsymbol{u} + \boldsymbol{w} \cdot \boldsymbol{v}$ distributive law
3. $(c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v}) = c(\boldsymbol{u} \cdot \boldsymbol{v})$ scalar mult.
4. $||c\boldsymbol{u}|| = |c|\,||\boldsymbol{u}||$ norm (not $c\,||\boldsymbol{u}||$)
5. (i) $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$ $\boldsymbol{u} \cdot \boldsymbol{u} = ||\boldsymbol{u}||^2$
   (ii) $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ if and only if $\boldsymbol{u} = \mathbf{0}$.
6. $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\|$ Cauchy-Schwarz Inequality

Q&A: log in to PollEv.com/vtpoll

"Orthogonal" meaning

- Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

  (geometrically: $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular)

- A set $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is an orthogonal set if

  every pair of distinct vectors in S are orthogonal:

  $$\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 0, \ \ \boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = 0, \ \ \ldots \ \ \boldsymbol{u}_{k-1} \cdot \boldsymbol{u}_k = 0$$

- A set $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is an orthogonal basis of a

vector space V if S is an orthogonal set and a basis for V

- A vector $\boldsymbol{u}$ is orthogonal to a vector space V if

  $\boldsymbol{u}$ is orthogonal to every vector in V. i.e. $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ for all $\boldsymbol{v} \in V$

Q&A: log in to PollEv.com/vtpoll

Nullspace vs Row space

$\boldsymbol{Ab} = \mathbf{0}$

b belongs to the nullspace of ***A***

$$\begin{pmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{b} \\ \boldsymbol{a}_2 \cdot \boldsymbol{b} \\ \vdots \\ \boldsymbol{a}_n \cdot \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

b is orthogonal to every row of ***A***

Row space of ***A***

Nullspace of ***A***

$$\underbrace{(c_1\boldsymbol{a}_1 + c_2\,\boldsymbol{a}_2 + \ldots + c_n\boldsymbol{a}_n)}_{\text{Row space of } \boldsymbol{A}} \cdot \underbrace{\boldsymbol{b}}_{\text{Nullspace of } \boldsymbol{A}}$$

$$c_1\boldsymbol{a}_1 \cdot \boldsymbol{b} + c_2\,\boldsymbol{a}_2 \cdot \boldsymbol{b} + \ldots + c_n\boldsymbol{a}_n \cdot \boldsymbol{b} \ = 0$$

b is orthogonal to the row space of ***A***

Q&A: log in to PollEv.com/vtpoll

Orthogonal set VS orthonormal set

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ → S is an orthogonal set

$\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 0, \quad \boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = 0, \quad \ldots \quad \boldsymbol{u}_{k-1} \cdot \boldsymbol{u}_k = 0$

$||\boldsymbol{u}_1|| = ||\boldsymbol{u}_2|| = \ldots = ||\boldsymbol{u}_k|| = 1$

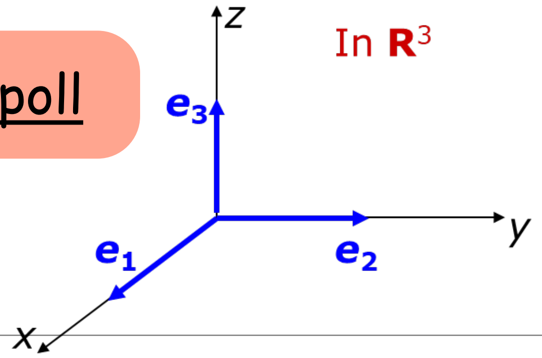
$T = \{\frac{1}{\|\boldsymbol{u}_1\|}\boldsymbol{u}_1, \frac{1}{\|\boldsymbol{u}_2\|}\boldsymbol{u}_2, \ldots, \frac{1}{\|\boldsymbol{u}_k\|}\boldsymbol{u}_k\}$ → S is an orthonormal set

$$\boldsymbol{u} \xrightarrow{\text{normalizing}} \frac{1}{||\boldsymbol{u}||}\boldsymbol{u}$$

any non-zero vector — unit vector

Q&A: log in to PollEv.com/vtpoll

Orthogonal basis



$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

- basis for $\mathbf{R}^3$
- orthogonal basis
- orthonormal basis

$\{(2,0,0), (0,1,1), (0,1,-1)\}$

- basis for $\mathbf{R}^3$
- orthogonal basis
- not orthonormal basis

$\{(1,0,0), (1,1,0), (1,1,1)\}$

- basis for $\mathbf{R}^3$
- not orthogonal basis
- not orthonormal basis

$\{(1,0,1), (0,1,0), (1,1,1)\}$

- not basis for $\mathbf{R}^3$
- not orthogonal basis
- not orthonormal basis

Q&A: log in to PollEv.com/vtpoll

Gram-Schmidt Process

Visualization tool

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$: a basis for a vector space V.

Define $\mathbf{v}_1 = \mathbf{u}_1$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

orthogonal to $\boldsymbol{v}_1$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

orthogonal to $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$

$$\vdots$$

$$\mathbf{v}_k = \mathbf{u}_k - \frac{\mathbf{u}_k \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{u}_k \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \cdots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k-1}}{\|\mathbf{v}_{k-1}\|^2}\mathbf{v}_{k-1}$$

orthogonal to $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}$

$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is an orthogonal basis for V.

Q&A: log in to PollEv.com/vtpoll

Coordinate vector w.r.t. orthogonal basis

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$: an orthogonal basis for V

For any vector $\boldsymbol{w}$ in V,

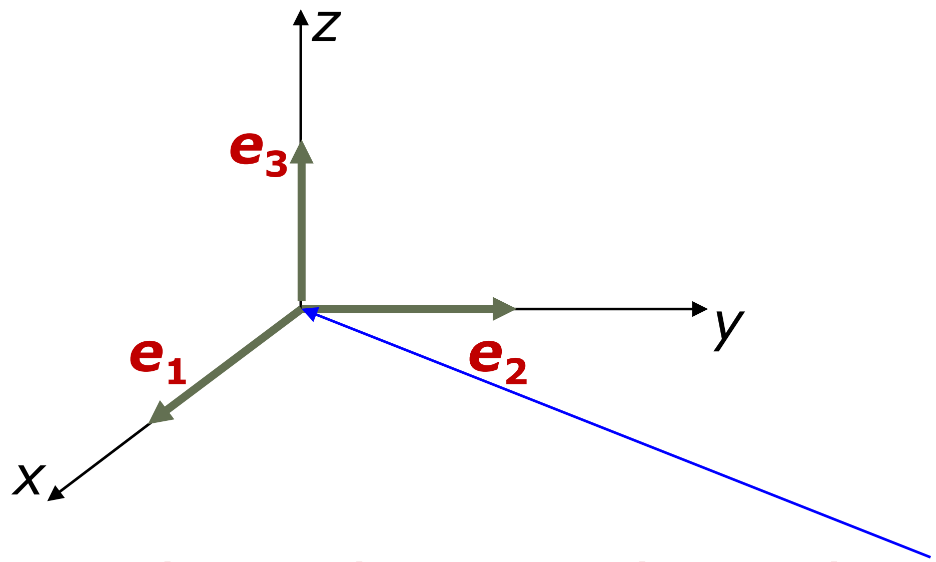
$$\boldsymbol{w} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \ldots + c_k\boldsymbol{u_k}$$

$$c_1 = \frac{\boldsymbol{w}\cdot\boldsymbol{u_1}}{||\boldsymbol{u_1}||^2} \qquad c_2 = \frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{||\boldsymbol{u}_2||^2} \qquad c_k = \frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{||\boldsymbol{u}_k||^2}$$

$$(\boldsymbol{w})_S = \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u_1}}{||\boldsymbol{u_1}||^2}, \frac{\boldsymbol{w}\cdot\boldsymbol{u_2}}{||\boldsymbol{u_2}||^2}, \ldots, \frac{\boldsymbol{w}\cdot\boldsymbol{u_k}}{||\boldsymbol{u_k}||^2}\right)$$

True or false

Can we find an orthogonal set of four vectors in $\mathbf{R}^3$?



$\boldsymbol{e_1} = (1, 0, 0)$ $\boldsymbol{e_2} = (0, 1, 0)$ $\boldsymbol{e_3} = (0, 0, 1)$ $\boldsymbol{0} = (0, 0, 0)$

Q&A: log in to PollEv.com/vtpoll

Orthogonal ⇒ Linearly independent

Let S be an orthogonal set of nonzero vectors in a vector space.

Then S is linearly independent.

To check whether S is an $\boxed{\text{orthogonal}}\,\boxed{\text{basis}}$ for V:

S is an orthogonal set ⇒ S spans V, S lin. indep

Only need to check:

(i) S is orthogonal and

(ii) span(S) = V

Q&A: log in to PollEv.com/vtpoll

A vector orthogonal to a subspace

To **show** a vector $\boldsymbol{v}$ is orthogonal to $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$

Show: $\boldsymbol{v} \cdot \boldsymbol{u_1} = 0, \boldsymbol{v} \cdot \boldsymbol{u_2} = 0, \ldots, \boldsymbol{v} \cdot \boldsymbol{u_k} = 0$

To **find** a vector $\boldsymbol{v}$ that is orthogonal to $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$

Let $\boldsymbol{v} = (x_1, x_2, \ldots, x_n)$ (unknowns)
Set up: $\boldsymbol{v} \cdot \boldsymbol{u_1} = 0, \boldsymbol{v} \cdot \boldsymbol{u_2} = 0, \ldots, \boldsymbol{v} \cdot \boldsymbol{u_k} = 0$
Convert into a homogeneous system.
Solve the system.

Q&A: log in to PollEv.com/vtpoll

$W^{\perp}$

Pronounced as W-perp

Let W be a subspace of $\mathbf{R}^n$.

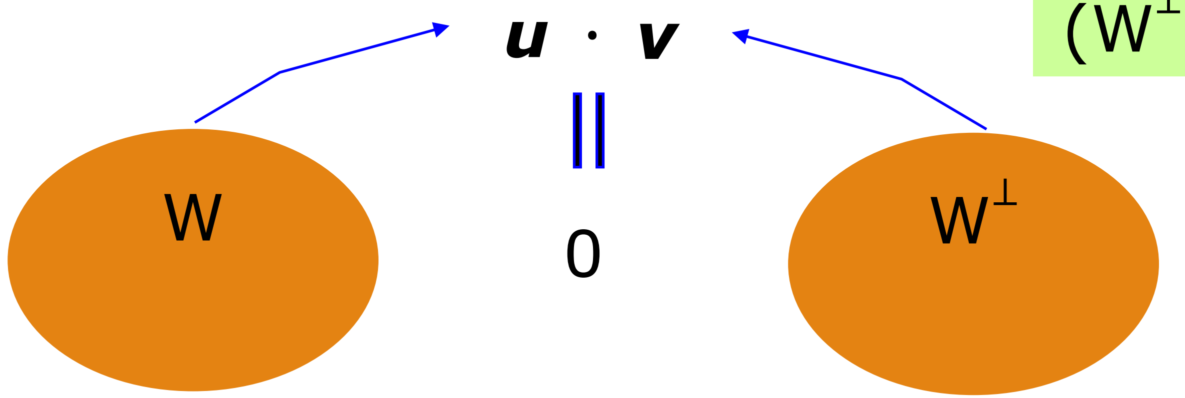
Define $W^{\perp} = \{ \boldsymbol{u} \in \mathbf{R}^n \mid \boldsymbol{u} \text{ is orthogonal to } W\}$

Exercise 5 Q7

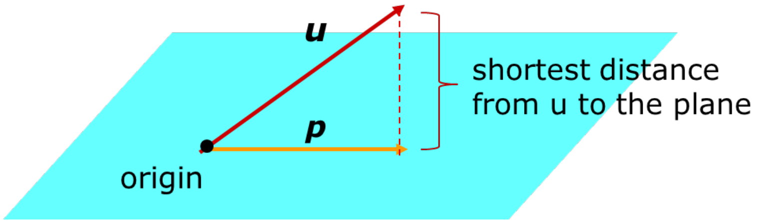
$W^{\perp}$ is also a subspace of $\mathbf{R}^n$

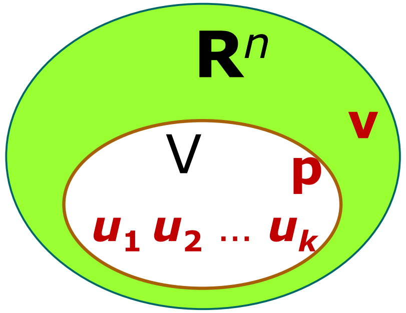
Every vector in $W^{\perp}$ is orthogonal to every vector in W.

$(W^{\perp})^{\perp} = W$



Projection





The projection ***p*** of ***v*** onto a subspace V:

p is the vector in V "nearest" to the given vector **v**

p is the best approximation of ***v*** in the subspace V

v – **p** is orthogonal to V

good for checking,
but not finding projection

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$: an orthogonal basis for V

The projection ***p*** of ***v*** onto V is given by:

$$\mathbf{p} = \frac{\boldsymbol{v} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\boldsymbol{v} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 + \cdots + \frac{\boldsymbol{v} \cdot \mathbf{u}_k}{\|\mathbf{u}_k\|^2}\mathbf{u}_k$$

Alternative method: use least squares solution

Q&A: log in to PollEv.com/vtpoll

Extending orthogonal basis

$\boldsymbol{u_1} = (1, 2, 2, -1)$, $\boldsymbol{u_2} = (1, 1, -1, 1)$, $\boldsymbol{u_3} = (-1, 1, -1, -1)$

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ orthogonal set

Extend S to an orthogonal basis for $\mathbf{R}^4$

Two methods:

$$\mathbf{p} = \frac{\boldsymbol{v} \cdot \mathbf{u}_1}{||\mathbf{u}_1||^2}\mathbf{u}_1 + \frac{\boldsymbol{v} \cdot \mathbf{u}_2}{||\mathbf{u}_2||^2}\mathbf{u}_2 + \cdots + \frac{\boldsymbol{v} \cdot \mathbf{u}_k}{||\mathbf{u}_k||^2}\mathbf{u}_k$$

1. Use row space method:
   i. find the "missing" row $\boldsymbol{r}$ in the row echelon form;
   ii. find the projection $\boldsymbol{p}$ of $\boldsymbol{r}$ onto span$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$
   iii. take the vector $\boldsymbol{r} - \boldsymbol{p}$
2. Find a non-zero vector $\boldsymbol{v}$ orthogonal to $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$

   Solve: $\boldsymbol{u_1} \cdot \boldsymbol{v} = 0$, $\boldsymbol{u_2} \cdot \boldsymbol{v} = 0$, $\boldsymbol{u_3} \cdot \boldsymbol{v} = 0$

$$\begin{pmatrix} \boldsymbol{u}_1 \cdot \mathbf{v} \\ \boldsymbol{u}_2 \cdot \mathbf{v} \\ \boldsymbol{u}_3 \cdot \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The projection $\boldsymbol{p}$ of $\boldsymbol{v}$ onto a subspace V: $\boldsymbol{v} - \boldsymbol{p}$ is orthogonal to V

Exercise 5 Q19 (Hint)

$\boldsymbol{A}^2 = \boldsymbol{A} = \boldsymbol{A}^T$

(a) show that $(\boldsymbol{Au}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (\boldsymbol{Av})$

- Regard dot product as matrix multiplication $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$

(b) show that $\boldsymbol{Aw}$ is the projection of $\boldsymbol{w}$ onto the subspace

$V = \{\boldsymbol{u} \in \mathbf{R}^n \mid \boldsymbol{Au} = \boldsymbol{u}\}$ of $\mathbf{R}^n$

Show: $\mathbf{w} - \mathbf{Aw}$ is orthogonal to V

Let $\boldsymbol{u} \in V$. Show $\boldsymbol{u} \cdot (\mathbf{w} - \mathbf{Aw}) = 0$

$\boldsymbol{Au} = \boldsymbol{u}$ (*)

Use part (a) and (*)

Q&A: log in to PollEv.com/vtpoll

Preview

Least squares solutions

When a linear system $\mathbf{Ax} = \mathbf{b}$ is inconsistent

A least squares solution $\mathbf{x}_0$ of $\mathbf{Ax} = \mathbf{b}$:

- $\mathbf{x}_0$ is the best approximation to a solution of $\mathbf{Ax} = \mathbf{b}$

$$\underset{\text{closest}}{\mathbf{Ax}_0 \approx \mathbf{b}} \qquad \underset{\text{smallest}}{\|\mathbf{b} - \mathbf{Ax}_0\|}$$

i.e. $\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_0\| \leq \|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{v}\|$ for all $\boldsymbol{v}$ in $\mathbf{R}^n$

To find the least squares solution $\mathbf{x}_0$ of $\mathbf{Ax} = \mathbf{b}$:

solve the new system $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$

Q&A: log in to PollEv.com/vtpoll

Example (Least Squares)

$$\begin{aligned} 2x + y &= 20 \\ 6x + y &= 18 \\ 20x + y &= 10 \\ 30x + y &= 6 \\ 40x + y &= 2 \end{aligned} \qquad \underbrace{\begin{pmatrix} 2 & 1 \\ 6 & 1 \\ 20 & 1 \\ 30 & 1 \\ 40 & 1 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 20 \\ 18 \\ 10 \\ 6 \\ 2 \end{pmatrix}}_{\mathbf{b}}$$

$\mathbf{Ax} = \mathbf{b}$ is inconsistent

Find least squares solution for $\mathbf{Ax} = \mathbf{b}$

$$\underbrace{\begin{pmatrix} 2940 & 98 \\ 98 & 5 \end{pmatrix}}_{\mathbf{A}^T\mathbf{A}} \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 608 \\ 56 \end{pmatrix}}_{\mathbf{A}^T\mathbf{b}}$$

Find the actual solutions of $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$

Q&A: log in to PollEv.com/vtpoll

$\mathbf{A^TAx = A^Tb}$

Always consistent !

$\mathbf{Ax = b}$

consistent

inconsistent

$\mathbf{A^TAx = A^Tb}$

gives the same

solution set as

$\mathbf{Ax = b}$

Ex 5 Q27

HW1 Q2(v)

$\mathbf{A^TAx = A^Tb}$

gives the least

squares solutions of

$\mathbf{Ax = b}$

There may be
more than one LSS

Q&A: log in to PollEv.com/vtpoll

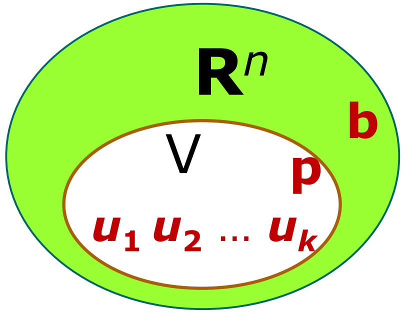
Finding projection using least squares

Let $\mathbf{x}_0$ be a least squares solution of $\mathbf{Ax} = \mathbf{b}$

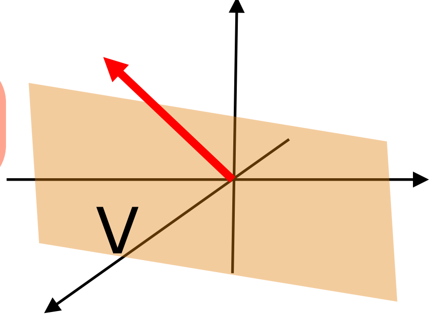
Then $\mathbf{Ax}_0$ = projection of $\mathbf{b}$ onto column space of $\mathbf{A}$

To find projection $\mathbf{p}$ of $\mathbf{b}$ onto a subspace V:

- Find any basis $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_k\}$ for V = column space of $\mathbf{A}$
- Form matrix $\mathbf{A}$ using $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ as column vectors
- Solve the system $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ to get $\mathbf{x}_0$
- $\mathbf{Ax}_0$ = projection $\mathbf{p}$



Q&A: log in to PollEv.com/vtpoll



Example (Projection)

Find the projection of (1,-1,1) onto the plane V: x + y + z = 0 in $\mathbf{R}^3$

V = span{(1,-1,0), (1,0,-1)}

Form matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$ and column vector $\boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Find the least squares solution of $\boldsymbol{Ax} = \boldsymbol{b}$

Solve $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$ $\quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4/3 \\ -2/3 \end{pmatrix}$

Projection of (1, -1, 1) on V: $\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 4/3 \\ -2/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -4/3 \\ 2/3 \end{pmatrix}$

Q&A: log in to PollEv.com/vtpoll

Announcement

- **Tutorial**
  - Tutorial set 8 (include Q23, 27)
  - Exercise set 5 solution (to be uploaded this weekend)
- **Homework**
  - HW2 results published
  - HW2 solutions available
  - HW3 due 22 October (next Friday)
- **MATLAB WS5**
  - OTOT starting this week
- **Online quiz 9**
  - Due next Thursday