Chapter 7: Modular arithmetic and partial orders

CS1231S Discrete Structures

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2021/22 Semester 1

> We have concluded that the trivial mathematics is, on the whole, useful, and that the real mathematics, on the whole, is not.
>
> G.H. Hardy 1940
>
> G. H. Hardy would have been surprised and probably displeased with the increasing interest in number theory for application to "ordinary human activities" such as information transmission (error-correcting codes) and cryptography (secret codes).
>
> N. Koblitz 1987

Mass consultation

Indicate your availability on LumiNUS Polls before 23:59 on Saturday 18 September if you are interested.

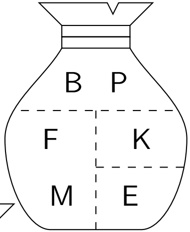
What we saw

Let A be a set.

Definition 6.1.1. A *partition* of A is a set $\mathscr{C}$ of *nonempty* subsets of A such that

$$\forall x \in A \ \exists! S \in \mathscr{C} \ \ (x \in S).$$

Definitions 6.1.5 and 6.2.1. A *relation on A* is a subset of A^2.

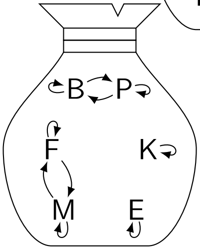


Definition 6.1.5. If R is a relation on A, then we write $x \mathrel{R} y$ for $(x, y) \in R$.

Definitions 6.2.4 and 6.2.13. A relation R on A is an *equivalence relation* if

- (reflexivity) $\forall x \in A \quad (x \mathrel{R} x)$;
- (symmetry) $\forall x, y \in A \quad (x \mathrel{R} y \Rightarrow y \mathrel{R} x)$; and
- (transitivity) $\forall x, y, z \in A \quad (x \mathrel{R} y \wedge y \mathrel{R} z \Rightarrow x \mathrel{R} z)$.

Proposition 6.2.16. The same-component relation with respect to a partition is an equivalence relation.



Definition 6.4.1. Let $\sim$ be an equivalence relation on A. Then

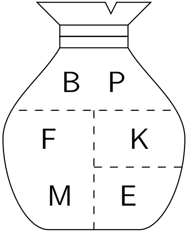
$$A/{\sim} = \{[x]_\sim : x \in A\},$$

where $[x]_\sim = \{y \in A : x \sim y\}$.

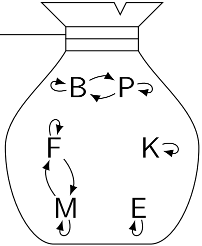
Theorem 6.4.9. If $\sim$ is an equivalence relation on A, then $A/{\sim}$ is a partition of A.

Informal descriptions of the terms

1.	underlying set	A	the set to be “partitioned”
2.	components	S	subsets of A, mutually disjoint, together union to A
3.	partition	$\mathscr{C}$	the set of all components
4.	same-component relation	$\sim$	equivalence relation



1.	underlying set	A	the set of all vertices
2.	relation	R	the set of all arrows
3.	equivalence relation	$\sim$	If ignoring directions of arrows one can walk from x to y, then there is an arrow from x to y.
4.	equivalence classes	$[x]$, where $x \in A$	connected components
5.	quotient	$A/{\sim}$	the set of all connected components



Why partitions and equivalence relations

There are numerous situations when we want to treat different individuals as the same.

Example (take with a pinch of salt)

When should two programs be considered the same?

https://tex.stackexchange.com/a/413506

- To the file system: they are the same copy of a file.
- To the programmer: they have the same code.
- To the theoretical computer scientist: the underlying algorithms are the same.
- To the complexity theorist: they require the same (time or memory) resources.
- To the recursion theorist: on the same input, they give the same output.

Each view is represented by an equivalence relation on the set of all programs.

> The formal mathematical language, together with the *univalence axiom*, fulfills the mathematicians' dream: a language for mathematics invariant under equivalence and thus freed from irrelevant details and able to merge the results of mathematicians taking different but equivalent approaches. D.R. Grayson 2018

Congruence

Fix $n \in \mathbb{Z}^+$.

Definition 6.3.1

Let $a, b \in \mathbb{Z}$. Then *a is congruent to b modulo n* if $a - b = nk$ for some $k \in \mathbb{Z}$. In this case, we write $a \equiv b \pmod{n}$.

Proposition 6.3.4

Congruence-mod-n is an equivalence relation on $\mathbb{Z}$.

Examples 6.4.3 and 6.4.8

The equivalence classes with respect to the congruence-mod-n relation $\sim_n$ on $\mathbb{Z}$ are of the form

$$[x] = \{y \in \mathbb{Z} : x \equiv y \pmod{n}\} = \{nk + x : k \in \mathbb{Z}\},$$

where $x \in \mathbb{Z}$. So

$$\mathbb{Z}/{\sim_n} = \{[x] : x \in \mathbb{Z}\}$$
$$= \big\{\{nk : k \in \mathbb{Z}\}, \{nk+1 : k \in \mathbb{Z}\}, \dots, \{nk + (n-1) : k \in \mathbb{Z}\}\big\}.$$

Hence $\mathbb{Z}/{\sim_2} = \big\{\{2k : k \in \mathbb{Z}\}, \{2k+1 : k \in \mathbb{Z}\}\big\}$.

$\mathbb{Z}$

$\vdots$	$\vdots$	$\vdots$	$\vdots$
-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\mathbb{Z}$

$\vdots$	$\vdots$
-4	-3
-2	-1
0	1
2	3
4	5
$\vdots$	$\vdots$

Generalizing the arithmetic on even/odd from mod-2 to mod-n

We can do arithmetic on the equivalence classes with respect to the congruence-mod-2 relation.

$+$	even	odd
even	even	odd
odd	odd	even

$\times$	even	odd
even	even	even
odd	even	odd

How to come up with these tables

$[x] + [y] = [x + y]$

Denote by $\sim_2$ the congruence-mod-2 relation on $\mathbb{Z}$, so that

$$\mathbb{Z}/{\sim_2} = \big\{\{m \in \mathbb{Z} : m \text{ is even}\}, \{m \in \mathbb{Z} : m \text{ is odd}\}\big\}.$$

1. Let $S_1, S_2 \in \mathbb{Z}/{\sim_2}$, say $S_1 = \{m \in \mathbb{Z} : m \text{ is odd}\}$ and $S_2 = \{m \in \mathbb{Z} : m \text{ is even}\}$.
2. We want to define $S_1 + S_2$ and $S_1 \times S_2$, which will again be elements of $\mathbb{Z}/{\sim_2}$.
3. Pick a representative $x \in S_1$ and a representative $y \in S_2$, say $x = 3$ and $y = -2$.
4. Sum the representatives; here $x + y = 3 + (-2) = 1$. ($[x] = S_1$ and $[y] = S_2$.)
5. There is a unique equivalence class S containing $x + y$ as $\mathbb{Z}/{\sim_2}$ is a partition of $\mathbb{Z}$.
6. Define $S_1 + S_2$ to be this S; since $x + y = 1 \in \{m \in \mathbb{Z} : m \text{ is odd}\}$, we define $S_1 + S_2 = \{m \in \mathbb{Z} : m \text{ is odd}\}$ here. ($[x + y] = S$.)

Exercise 7.1.2. An element $x \in \mathbb{Z}$ is in an equivalence class S if and only if $[x] = S$.

Defining arithmetic modulo n in terms of representatives

Let $n \in \mathbb{Z}^+$.

Definition 7.1.1

A *representative* of an equivalence class is an element of the equivalence class.

Definition 7.1.4

The quotient $\mathbb{Z}/{\sim_n}$, where $\sim_n$ is the congruence-mod-n relation on $\mathbb{Z}$, is denoted $\mathbb{Z}_n$ or $\mathbb{Z}/n\mathbb{Z}$. Define addition and multiplication on $\mathbb{Z}_n$ as follows: whenever $[x], [y] \in \mathbb{Z}_n$,

$$[x] + [y] = [x + y] \quad \text{and} \quad [x] \cdot [y] = [x \cdot y].$$

https://tex.stackexchange.com/a/413506

What can go wrong?

Say $n = 2$.

- We have $[0] + [1] = [1]$ as *some* even number plus *some* odd number is odd.
- What if *some* even number plus *some* odd number is even, so that $[0] + [1] = [0]$ too? Then what is $[0] + [1]$?
- ... Fortunately, *any* even number plus *any* odd number is odd.

Exercise 7.1.2. An element $x \in \mathbb{Z}$ is in an equivalence class S if and only if $[x] = S$.

Example 7.1.3: same distance from 0

Define an equivalence relation $\sim$ on $\mathbb{Z}$ by setting, for all $x, y \in \mathbb{Z}$,

$$x \sim y \quad \Leftrightarrow \quad x = y \text{ or } x = -y.$$

Hence $x \sim y$ means $|x| = |y|$. Note

$$[0] = \{0\}, \quad [1] = \{1, -1\} = [-1], \quad [2] = \{2, -2\} = [-2], \quad \ldots$$

So $\mathbb{Z}/\sim = \big\{\{0\}, \{1, -1\}, \{2, -2\}, \ldots\big\}$. Define addition and multiplication on $\mathbb{Z}/\sim$ as follows: whenever $[x], [y] \in \mathbb{Z}/\sim$,

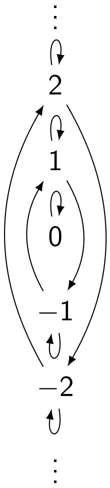
$$[x] + [y] = [x + y] \quad \text{and} \quad [x] \cdot [y] = [x \cdot y].$$

Then $+$ is not well defined because $[1] = [1]$ and $[2] = [-2]$, but

$$[1] + [2] = [1 + 2] = [3] \neq [-1] = [1 + (-2)] = [1] + [-2].$$

Note $\cdot$ is well defined because whenever $[x_1], [y_1], [x_2], [y_2] \in \mathbb{Z}/\sim$,

$$[x_1] = [x_2] \text{ and } [y_1] = [y_2] \quad \Rightarrow \quad [x_1] \cdot [y_1] = [x_2] \cdot [y_2].$$



Definitions 6.4.1 and 6.4.6. Let $\sim$ be an equivalence relation on a set A. Then $A/\sim = \{[x] : x \in A\}$, where $[x] = \{y \in A : x \sim y\}$.

Addition on $\mathbb{Z}_n$ is well defined

Definitions 6.3.1 and 7.1.4. $\mathbb{Z}_n = \mathbb{Z}/\sim_n$, where
$$x \sim_n y \quad \Leftrightarrow \quad \exists k \in \mathbb{Z} \ (x - y = kn).$$

Proposition 7.1.5

For all $n \in \mathbb{Z}^+$ and all $[x_1], [y_1], [x_2], [y_2] \in \mathbb{Z}_n$,
$$[x_1] = [x_2] \text{ and } [y_1] = [y_2] \quad \Rightarrow \quad [x_1] + [y_1] = [x_2] + [y_2] \text{ and } [x_1] \cdot [y_1] = [x_2] \cdot [y_2].$$

Proof

$[x] + [y]$
$= [x + y]$.
$[x] \cdot [y]$
$= [x \cdot y]$.

1. Let $[x_1], [y_1], [x_2], [y_2] \in \mathbb{Z}_n$ such that $[x_1] = [x_2]$ and $[y_1] = [y_2]$.
2. Then Lemma 6.4.4 implies $x_1 \equiv x_2 \pmod{n}$ and $y_1 \equiv y_2 \pmod{n}$.
3. Use the definition of congruence to find $k, \ell \in \mathbb{Z}$ such that
$$x_1 - x_2 = nk \quad \text{and} \quad y_1 - y_2 = n\ell.$$
4. 4.1. Note $(x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2) = nk + n\ell = n(k + \ell)$, where $k + \ell \in \mathbb{Z}$.
   4.2. So the definition of congruence tells us $x_1 + y_1 \equiv x_2 + y_2 \pmod{n}$.
   4.3. Hence $[x_1] + [y_1] = [x_1 + y_1] = [x_2 + y_2] = [x_2] + [y_2]$ by Lemma 6.4.4.
5. We show that $[x_1] \cdot [y_1] = [x_2] \cdot [y_2]$. . . .

Lemma 6.4.4. TFAE: (i) $x \sim y$; (ii) $[x] = [y]$; (iii) $[x] \cap [y] \neq \varnothing$.

Multiplication on $\mathbb{Z}_n$ is well defined

Definitions 6.3.1 and 7.1.4. $\mathbb{Z}_n = \mathbb{Z}/{\sim_n}$, where
$$x \sim_n y \quad \Leftrightarrow \quad \exists k \in \mathbb{Z}\ (x - y = kn).$$

Proposition 7.1.5

For all $n \in \mathbb{Z}^+$ and all $[x_1], [y_1], [x_2], [y_2] \in \mathbb{Z}_n$,
$$[x_1] = [x_2] \text{ and } [y_1] = [y_2] \quad \Rightarrow \quad [x_1] + [y_1] = [x_2] + [y_2] \text{ and } [x_1] \cdot [y_1] = [x_2] \cdot [y_2].$$

Proof

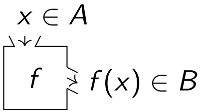
$[x] + [y] = [x + y]$.

$[x] \cdot [y] = [x \cdot y]$.

1. Let $[x_1], [y_1], [x_2], [y_2] \in \mathbb{Z}_n$ such that $[x_1] = [x_2]$ and $[y_1] = [y_2]$.
2. Then Lemma 6.4.4 implies $x_1 \equiv x_2 \pmod{n}$ and $y_1 \equiv y_2 \pmod{n}$.
3. Use the definition of congruence to find $k, \ell \in \mathbb{Z}$ such that
$$x_1 - x_2 = nk \quad \text{and} \quad y_1 - y_2 = n\ell.$$
4. (It is shown that $[x_1] + [y_1] = [x_2] + [y_2]$.)
5. 5.1. Note $(x_1 \cdot y_1) - (x_2 \cdot y_2) = (nk + x_2)(n\ell + y_2) - x_2 y_2 = n^2 k\ell + nky_2 + n\ell x_2 = n(nk\ell + ky_2 + \ell x_2)$, where $nk\ell + ky_2 + \ell x_2 \in \mathbb{Z}$.
   5.2. So the definition of congruence tells us $x_1 \cdot y_1 \equiv x_2 \cdot y_2 \pmod{n}$.
   5.3. Hence $[x_1] \cdot [y_1] = [x_1 \cdot y_1] = [x_2 \cdot y_2] = [x_2] \cdot [y_2]$ by Lemma 6.4.4. $\square$

Lemma 6.4.4. TFAE: (i) $x \sim y$; (ii) $[x] = [y]$; (iii) $[x] \cap [y] \neq \varnothing$.

"Well-defined function"

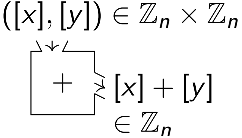


Definition 7.2.1

Let A, B be sets. A *function* or a *map* from A to B is an assignment to each element of A exactly one element of B. We write $f\colon A \to B$ for "f is a function from A to B". Suppose $f\colon A \to B$.

(1) Let $x \in A$. Then $f(x)$ denotes the element of B that f assigns x to. We call $f(x)$ the *image* of x under f. If $y = f(x)$, then we say that *f maps x to y*, and we may write $f\colon x \mapsto y$.

(2) Here A is called the *domain* of f, and B is called the *codomain* of f.



Convention 7.2.2

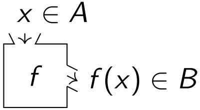
Instead of $+(x, y)$ and $\cdot(x, y)$, people usually write $x + y$ and $x \cdot y$ respectively.

Convention 7.2.3

In mathematics, one can read "Define $f\colon A \to B$ by .... Then f is well defined." as "The condition '...' defines a function $f\colon A \to B$. We use '...' to define f."

$$\boxed{[x] + [y] = [x + y] \qquad \text{and} \qquad [x_1] = [x_2] \text{ and } [y_1] = [y_2] \;\Rightarrow\; [x_1] + [y_1] = [x_2] + [y_2]}$$

A polynomial function



Definition 7.2.1

Let A, B be sets. A *function* or a *map* from A to B is an assignment to each element of A exactly one element of B. We write $f\colon A \to B$ for "f is a function from A to B". Suppose $f\colon A \to B$.

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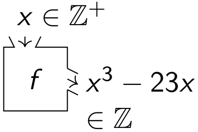


Example 7.2.4

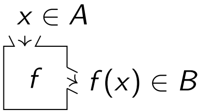
Define $f\colon \mathbb{Z}^+ \to \mathbb{Z}$ by setting, for each $x \in \mathbb{Z}$,

$$f(x) = x^3 - 23x.$$

Then the domain of f is $\mathbb{Z}^+$ and codomain of f is $\mathbb{Z}$. We know

$$f(1) = 1^3 - 23 \times 1 = -22 \quad \text{and} \quad f(2) = 2^3 - 23 \times 2 = -38.$$

Identity functions



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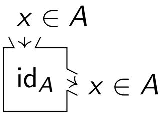


(2) Here A is called the *domain* of f, and B is called the *codomain* of f.

Definition 7.2.5 and Remark 7.2.6

Let A be a set. Then the *identity function* on A, denoted id_A, is the function $A \to A$ which satisfies, for all $x \in A$,

$$\mathrm{id}_A(x) = x.$$

The domain and the codomain of id_A are both A.

Bad definitions

Definition 7.2.1

A *function* $A \to B$ is an assignment to each element of A exactly one element of B.

Question 7.2.7

Define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(x) = 2^x$ for all $x \in \mathbb{Q}$. Why is f not well defined?

Answer

It does not make f assign any value in the codomain $\mathbb{Q}$ to the element $1/2$ of the domain $\mathbb{Q}$. (Recall $2^{1/2} = \sqrt{2} \notin \mathbb{Q}$.)

Define $g\colon \mathbb{Q} \to \mathbb{Q}$ by setting

$$g(x) = \frac{x^2+1}{x^2+2x+1}$$

for all $x \in \mathbb{Q}$.

$g(-1) = ??$

Define $h\colon \mathbb{Q} \to \mathbb{Z}$ by setting $h(m/n) = m$ for all $m, n \in \mathbb{Z}$ where $n \neq 0$.

$h(1/2) = 1 \neq 2 = h(2/4)$, although $1/2 = 2/4$.

Zoom poll

Checkpoint

$+$	even	odd
even	even	odd
odd	odd	even

$\times$	even	odd
even	even	even
odd	even	odd

We have $+$ and $\cdot$ on $\mathbb{Z}_n$ for any $n \in \mathbb{Z}^+$.

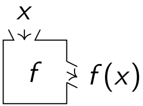
We can then...

- try to do subtraction and division;
- solve equations and systems of equations;
- develop the RSA cryptosystem;
- build combinatorial designs;
- etc.

4	9	2
3	5	7
8	1	6

We also saw a precise definition of functions.

We will return to the study of functions later.



Next

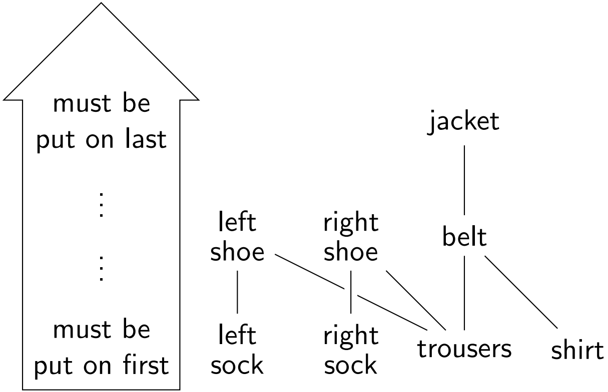
Partial orders

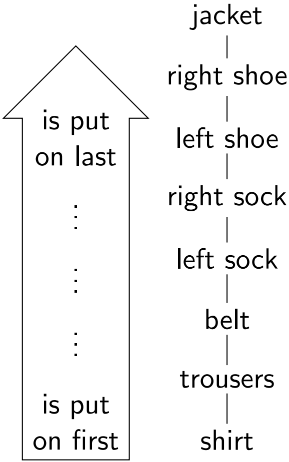
Marcel Erné (1985),
There must be order!

Motivating examples of partial orders

(1) the "*must be* done before (or at the same time as)" relation on the set of all tasks

(2) the "*is* done before (or at the same time as)" relation on the set of all tasks





Motivating examples of partial orders: a closer look

(1) the "*must be* done before (or at the same time as)" relation on the set of all tasks

(2) the "*is* done before (or at the same time as)" relation on the set of all tasks

- Each such relation has two versions: one with the parenthetical phrase, and one without. They have the same mathematical content. We focus on the former.
- So all such relations are reflexive and transitive.
- No multi-tasking is allowed, i.e., if R is one of the relations above, then

$$\forall x, y \; (x \, R \, y \land y \, R \, x \quad \Rightarrow \quad x = y). \qquad \text{(antisymmetry)}$$

- There may be x, y such that x need not be done before y, and y need not be done before x, i.e., maybe $\exists x, y \; (x \not R \, y \land y \not R \, x)$ if R is the relation in (1). (partiality)
- However, as time is linear and there is no multi-tasking, for all tasks x, y, either x is done before or at the same time as y, or y is done before or at the same time as x, i.e., $\forall x, y \; (x \, R \, y \lor y \, R \, x)$ if R is the relation in (2). (totality)
- Here "partiality" means "possibly partial", while "total" means "necessarily total".

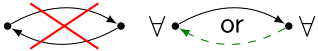
Partial orders

> Definition 6.2.4. (1) R is *reflexive* if $\forall x \in A \ \ x \, R \, x$.
> (3) R is *transitive* if $\forall x, y, z \in A \ (x \, R \, y \wedge y \, R \, z \ \Rightarrow \ x \, R \, z)$.

Definition 7.3.1

Let A be a set and R be a relation on A.

(1) R is *antisymmetric* if $\forall x, y \in A \ (x \, R \, y \wedge y \, R \, x \Rightarrow x = y)$.



(2) R is a *(non-strict) partial order* if R is reflexive, antisymmetric, and transitive.

(4) R is a *(non-strict)* $\underbrace{\textit{total order}}_{\textit{linear order}}$ if R is a partial order and $\underbrace{\forall x, y \in A \ \underbrace{(x \, R \, y \vee y \, R \, x)}_{x \text{ and } y \text{ are } \textit{comparable}}}_{\text{any two elements are comparable}}$.

(5) We say that the ordered pair (A, R) is a *partially ordered set*, or a *poset* for short, if R is a partial order on A.

Note 7.3.2

A total order is always a partial order.

Plan

- examples and non-examples
- min and max
- linearization (aka topological sorting)

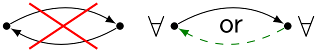
Inequalities

Definition 6.2.4. (1) R is *reflexive* if $\forall x \in A \;\; x \, R \, x$.
(3) R is *transitive* if $\forall x, y, z \in A \;\; (x \, R \, y \wedge y \, R \, z \;\Rightarrow\; x \, R \, z)$.

Definition 7.3.1

Let A be a set and R be a relation on A.

(1) R is *antisymmetric* if $\forall x, y \in A \;\; (x \, R \, y \wedge y \, R \, x \Rightarrow x = y)$.



(2) R is a *(non-strict) partial order* if R is reflexive, antisymmetric, and transitive.

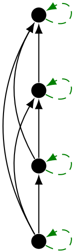
(4) R is a *(non-strict) total order* if R is a partial order and $\forall x, y \in A \;\; \underbrace{(x \, R \, y \vee y \, R \, x)}_{x \text{ and } y \text{ are } \textit{comparable}}$.

Example 7.3.3

Let R denote the non-strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x \, R \, y \quad \Leftrightarrow \quad x \leqslant y.$$

Then R is antisymmetric. In fact, it is a total order.



Example 7.3.4

Let R' denote the strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x \, R' \, y \quad \Leftrightarrow \quad x < y.$$

Then R' is antisymmetric because no $x, y \in \mathbb{Q}$ can make $x < y$ and $y < x$.
However, it is not a partial order because it is not reflexive.

Zoom poll

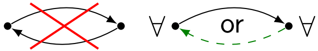
Equivalence relations

> Definition 6.2.4. (1) R is *reflexive* if $\forall x \in A\ \ x\,R\,x$.
> (3) R is *transitive* if $\forall x, y, z \in A\ \ (x\,R\,y \wedge y\,R\,z \ \Rightarrow\ x\,R\,z)$.

Definition 7.3.1

Let A be a set and R be a relation on A.

(1) R is *antisymmetric* if $\forall x, y \in A\ \ (x\,R\,y \wedge y\,R\,x \Rightarrow x = y)$.



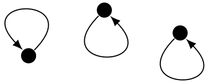
(2) R is a *(non-strict) partial order* if R is reflexive, antisymmetric, and transitive.

(4) R is a *(non-strict) total order* if R is a partial order and $\forall x, y \in A\ \ \underbrace{(x\,R\,y \vee y\,R\,x)}_{x \text{ and } y \text{ are } \textit{comparable}}$.

Example 7.3.5

Let R denote the equality relation on a set A, i.e., for all $x, y \in A$,

$$x\,R\,y \quad \Leftrightarrow \quad x = y.$$



Then R is antisymmetric. It is a partial order, but not a total order unless $|A| \leqslant 1$.

Example 7.3.6

Fix $n \in \mathbb{Z}^+$. Let R' denote the congruence-mod-n relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x\,R'\,y \quad \Leftrightarrow \quad x \equiv y \pmod{n} \quad \Leftrightarrow \quad \exists k \in \mathbb{Z}\ (x - y = nk).$$

Then R' is not antisymmetric because $0\,R'\,n$ and $n\,R'\,0$ but $0 \neq n$.

$\mathbb{Z}$

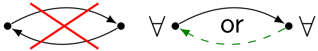
⋮	⋮	⋮	⋮
−8	−7	−6	−5
−4	−3	−2	−1
0	1	2	3
4	5	6	7
8	9	10	11
⋮	⋮	⋮	⋮

Divisibility

> Definition 6.2.4. (1) R is *reflexive* if $\forall x \in A \ \ x \, R \, x$.
> (3) R is *transitive* if $\forall x, y, z \in A \ (x \, R \, y \wedge y \, R \, z \ \Rightarrow \ x \, R \, z)$.

Definition 7.3.1

Let A be a set and R be a relation on A.



(1) R is *antisymmetric* if $\forall x, y \in A \ (x \, R \, y \wedge y \, R \, x \Rightarrow x = y)$.

(2) R is a *(non-strict) partial order* if R is reflexive, antisymmetric, and transitive.

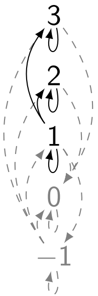
(4) R is a *(non-strict) total order* if R is a partial order and $\forall x, y \in A \ \underbrace{(x \, R \, y \vee y \, R \, x)}_{x \text{ and } y \text{ are } \textit{comparable}}$.

Example 7.3.7

Let R denote the divisibility relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x \, R \, y \quad \Leftrightarrow \quad x \mid y \quad \Leftrightarrow \quad \exists k \in \mathbb{Z} \ (y = xk).$$

Then R is not antisymmetric because $1 \mid -1$ and $-1 \mid 1$, but $1 \neq -1$.



Example 7.3.8

Let R' denote the divisibility relation on $\mathbb{Z}^+$, i.e., for all $x, y \in \mathbb{Z}^+$,

$$x \, R' \, y \quad \Leftrightarrow \quad x \mid y \quad \Leftrightarrow \quad \exists k \in \mathbb{Z} \ (y = xk).$$

✎ Then R' is antisymmetric. So it is a partial order by an argument similar to that in Example 6.2.12. It is not total because $2 \nmid 3$ and $3 \nmid 2$.

Zoom poll

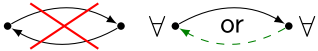
Inclusion

Definition 6.2.4. (1) R is *reflexive* if $\forall x \in A\ \ x\,R\,x$.
(3) R is *transitive* if $\forall x, y, z \in A\ \ (x\,R\,y \land y\,R\,z \ \Rightarrow\ x\,R\,z)$.

Definition 7.3.1

Let A be a set and R be a relation on A.

(1) R is *antisymmetric* if $\forall x, y \in A\ \ (x\,R\,y \land y\,R\,x \Rightarrow x = y)$.



(2) R is a *(non-strict) partial order* if R is reflexive, antisymmetric, and transitive.

(4) R is a *(non-strict) total order* if R is a partial order and $\forall x, y \in A\ \ \underbrace{(x\,R\,y \lor y\,R\,x)}_{x \text{ and } y \text{ are } \textit{comparable}}$.

Example 7.3.9

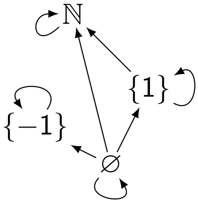
Let R denote the subset relation on a set U of sets, i.e., for all $x, y \in U$,

$$x\,R\,y \quad \Leftrightarrow \quad x \subseteq y.$$

Then R is antisymmetric. It is always a partial order, but it may not be a total order.

Remark 5.1.22(2). For all sets A, B,

$$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$



Hasse diagrams

Read $\preccurlyeq$ as "(curly) less than or equal to".

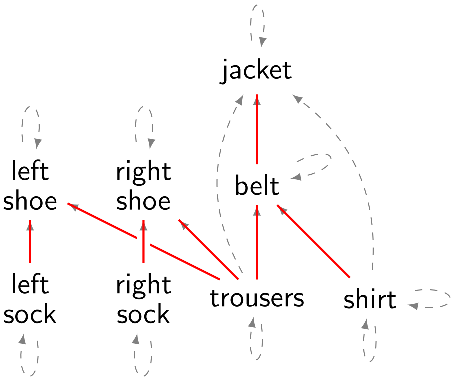
Notation 7.3.10

We often use $\preccurlyeq$ to denote a partial order. This symbol is often defined and redefined to mean different partial orders in different situations. If $\preccurlyeq$ denotes a partial order, then we write $x \prec y$ for $x \preccurlyeq y \wedge x \neq y$.

Definition 7.3.11

Let $\preccurlyeq$ be a partial order on a set A. A *Hasse diagram* of $\preccurlyeq$ satisfies the following condition for all $x, y \in A$:

> If $x \prec y$ and no $z \in A$ is such that $x \prec z \prec y$, then x is placed below y and there is a line joining x to y, else no line joins x to y.



Positive divisors of 30

Read $\preccurlyeq$ as "(curly) less than or equal to".

Notation 7.3.10

We often use $\preccurlyeq$ to denote a partial order. This symbol is often defined and redefined to mean different partial orders in different situations. If $\preccurlyeq$ denotes a partial order, then we write $x \prec y$ for $x \preccurlyeq y \wedge x \neq y$.

Definition 7.3.11

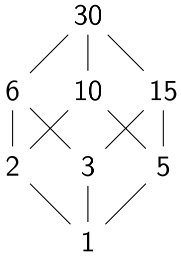
Let $\preccurlyeq$ be a partial order on a set A. A *Hasse diagram* of $\preccurlyeq$ satisfies the following condition for all $x, y \in A$:

> If $x \prec y$ and no $z \in A$ is such that $x \prec z \prec y$, then x is placed below y and there is a line joining x to y, else no line joins x to y.

Example 7.3.12

Consider $\{d \in \mathbb{Z}^+ : d \mid 30\}$ partially ordered by the divisibility relation $\mid$.

A Hasse diagram is as follows:



Subsets of $\{1, 2, 3\}$

Read $\preccurlyeq$ as "(curly) less than or equal to".

Notation 7.3.10

We often use $\preccurlyeq$ to denote a partial order. This symbol is often defined and redefined to mean different partial orders in different situations. If $\preccurlyeq$ denotes a partial order, then we write $x \prec y$ for $x \preccurlyeq y \wedge x \neq y$.

Definition 7.3.11

Let $\preccurlyeq$ be a partial order on a set A. A *Hasse diagram* of $\preccurlyeq$ satisfies the following condition for all $x, y \in A$:

> If $x \prec y$ and no $z \in A$ is such that $x \prec z \prec y$, then x is placed below y and there is a line joining x to y, else no line joins x to y.

Example 7.3.13

Consider $\mathcal{P}(\{1, 2, 3\})$ partially ordered by the inclusion relation $\subseteq$.

A Hasse diagram is as follows:

$$\begin{tikzcd}
& \{1,2,3\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{1,2\} \arrow[d, dash] \arrow[dr, dash] & \{1,3\} \arrow[dl, dash] \arrow[dr, dash] & \{2,3\} \arrow[dl, dash] \arrow[d, dash] \\
\{1\} \arrow[dr, dash] & \{2\} \arrow[d, dash] & \{3\} \arrow[dl, dash] \\
& \{\} &
\end{tikzcd}$$

The usual order on $\{1, 2, 3, 4\}$

Read $\preccurlyeq$ as "(curly) less than or equal to".

Notation 7.3.10

We often use $\preccurlyeq$ to denote a partial order. This symbol is often defined and redefined to mean different partial orders in different situations. If $\preccurlyeq$ denotes a partial order, then we write $x \prec y$ for $x \preccurlyeq y \wedge x \neq y$.

Definition 7.3.11

Let $\preccurlyeq$ be a partial order on a set A. A *Hasse diagram* of $\preccurlyeq$ satisfies the following condition for all $x, y \in A$:

> If $x \prec y$ and no $z \in A$ is such that $x \prec z \prec y$, then x is placed below y and there is a line joining x to y, else no line joins x to y.

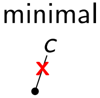
Example 7.3.14

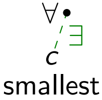
Consider $\{1, 2, 3, 4\}$ partially ordered by the non-strict less-than relation $\leqslant$.

A Hasse diagram is as follows:

$$\begin{tikzcd}
4 \arrow[d, no head] \\
3 \arrow[d, no head] \\
2 \arrow[d, no head] \\
1
\end{tikzcd}$$

Min and max





Let $\preccurlyeq$ be a partial order on a set A, and $c \in A$.

Definition 7.4.1

(1) c is a *minimal element* if no $x \in A$ is strictly $\preccurlyeq$-less than c, i.e.,

$$\forall x \in A \ (x \preccurlyeq c \Rightarrow c = x).$$

(2) c is a *maximal element* if no $x \in A$ is strictly $\preccurlyeq$-bigger than c, i.e.,

$$\forall x \in A \ (c \preccurlyeq x \Rightarrow c = x).$$

(3) c is the *smallest element* (or the *minimum element*) if all $x \in A$ are $\preccurlyeq$-bigger than or equal to c, i.e.,

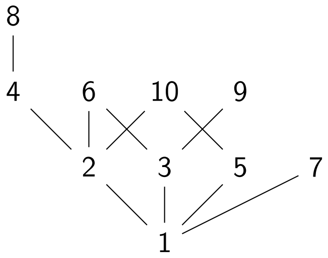
$$\forall x \in A \ (c \preccurlyeq x).$$

(4) c is the *largest element* (or the *maximum element*) if all $x \in A$ are $\preccurlyeq$-less than or equal to c, i.e.,

$$\forall x \in A \ (x \preccurlyeq c).$$

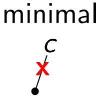
Example 7.4.2

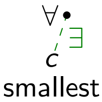
The divisibility relation $|$ on $\{1, 2, \ldots, 10\}$ is represented by the Hasse diagram



- The only minimal element is 1.
- The maximal elements are $6, 7, 8, 9, 10$.
- The smallest element is 1.
- There is no largest element.

No min/max





Definition 7.4.1

(1) c is a *minimal element* if no $x \in A$ is strictly $\preccurlyeq$-less than c, i.e.,

$$\forall x \in A \ (x \preccurlyeq c \Rightarrow c = x).$$

(2) c is a *maximal element* if no $x \in A$ is strictly $\preccurlyeq$-bigger than c, i.e.,

$$\forall x \in A \ (c \preccurlyeq x \Rightarrow c = x).$$

(3) c is the *smallest element* (or the *minimum element*) if all $x \in A$ are $\preccurlyeq$-bigger than or equal to c, i.e.,

$$\forall x \in A \ (c \preccurlyeq x).$$

(4) c is the *largest element* (or the *maximum element*) if all $x \in A$ are $\preccurlyeq$-less than or equal to c, i.e.,

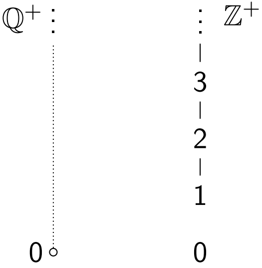
$$\forall x \in A \ (x \preccurlyeq c).$$

Let $\preccurlyeq$ be a partial order on a set A, and $c \in A$.

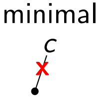
Example 7.4.3

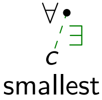
(1) $\mathbb{Q}^+$ under the non-strict less-than relation $\leqslant$ has neither a minimal element nor a maximal element.

(2) $\mathbb{Z}^+$ under the non-strict less-than relation $\leqslant$ has a smallest element but no maximal element.



Implication





Let $\preccurlyeq$ be a partial order on a set A, and $c \in A$.

Definition 7.4.1

(1) c is a *minimal element* if no $x \in A$ is strictly $\preccurlyeq$-less than c, i.e.,

$$\forall x \in A \ (x \preccurlyeq c \Rightarrow c = x).$$

(2) c is a *maximal element* if no $x \in A$ is strictly $\preccurlyeq$-bigger than c, i.e.,

$$\forall x \in A \ (c \preccurlyeq x \Rightarrow c = x).$$

(3) c is the *smallest element* (or the *minimum element*) if all $x \in A$ are $\preccurlyeq$-bigger than or equal to c, i.e.,

$$\forall x \in A \ (c \preccurlyeq x).$$

(4) c is the *largest element* (or the *maximum element*) if all $x \in A$ are $\preccurlyeq$-less than or equal to c, i.e.,

$$\forall x \in A \ (x \preccurlyeq c).$$

Proposition 7.4.4(1)

Consider a partial order $\preccurlyeq$ on a set A. Any smallest element is minimal.

smallest $\Leftrightarrow$ everything is above

$\Downarrow$

minimal $\Longleftrightarrow$ nothing is below

being above and below implies equality by antisymmetry

$\forall x, y \in A \left(\begin{array}{c} x \mathrel{R} y \wedge y \mathrel{R} x \\ \Rightarrow x = y \end{array} \right)$

Proof

1. Let c be a smallest element.
2. Take any $x \in A$ such that $x \preccurlyeq c$.
3. By smallestness, we know $c \preccurlyeq x$ too.
4. So $c = x$ by antisymmetry. $\square$

Existence of minimal elements

Definition 6.2.4(3). R is *transitive* if
$\forall x, y, z \in A \ (x \mathrel{R} y \wedge y \mathrel{R} z \ \Rightarrow \ x \mathrel{R} z)$.
Definition 7.3.1(1). R is *antisymmetric* if
$\forall x, y \in A \ (x \mathrel{R} y \wedge y \mathrel{R} x \Rightarrow x = y)$.

Proposition 7.4.6

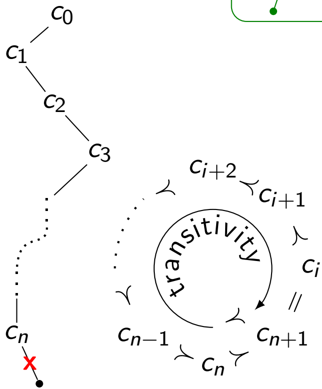
With respect to any partial order $\preccurlyeq$ on a finite set $A \neq \varnothing$, one can find a minimal element.

Keep picking smaller elements until minimal.

minimal

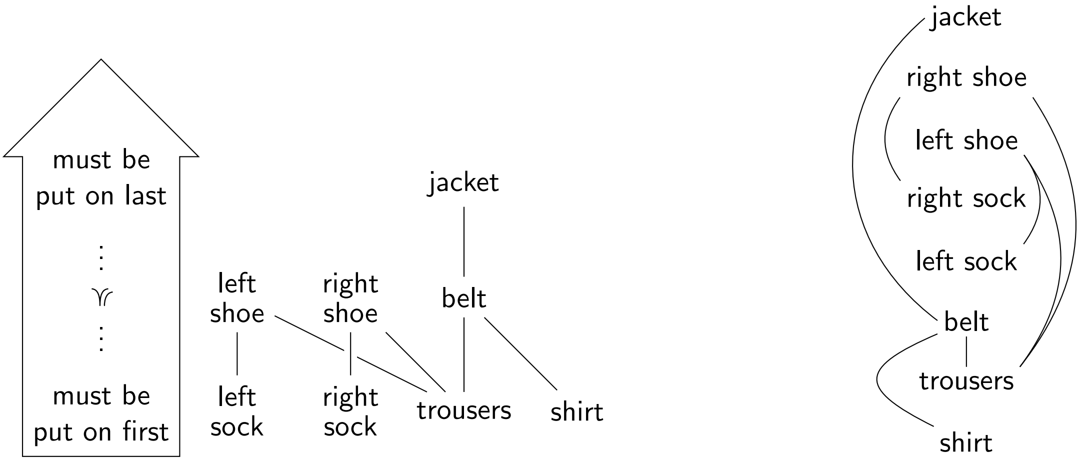
Proof (optional material)

1. Take any $c_0 \in A$. This is possible since $A \neq \varnothing$.
2. If c_0 is not minimal, then find $c_1 \in A$ such that $c_1 \prec c_0$.
3. Continue this process: if c_n is not minimal, then find $c_{n+1} \in A$ such that $c_{n+1} \prec c_n$.
4. Note that $c_{n+1} \neq c_i$ for any $i \in \{0, 1, \ldots, n\}$ because if $i \in \{0, 1, \ldots, n\}$ such that $c_{n+1} = c_i$, then
   - 4.1. $c_n \prec c_{n-1} \prec \cdots \prec c_i = c_{n+1}$;
   - 4.2. so $c_n \preccurlyeq c_{n+1}$ by transitivity;
   - 4.3. so $c_n = c_{n+1}$ by antisymmetry as $c_{n+1} \prec c_n$;
   - 4.4. so we have a contradiction with $c_{n+1} \prec c_n$.
5. Since A is finite, this process must end, say with c_n.
6. c_n must be minimal for this process to end. □



Linearization in terms of Hasse diagrams

A way to draw a Hasse diagram for the partial order
in which all the items are in one vertical line.

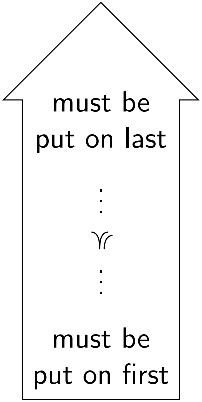


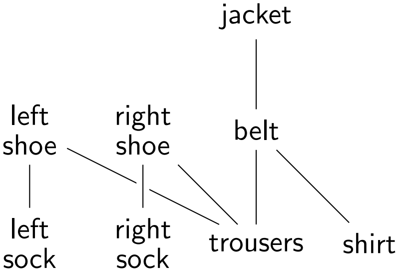
Linearization in terms of tasks and dependencies

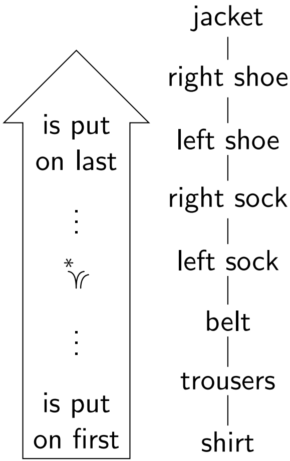
a total order $\preccurlyeq^*$

A sequential way to complete the tasks such that tasks that must be done first are done first.

$$\underbrace{\forall x, y\ (x \preccurlyeq y \ \Rightarrow\ x \preccurlyeq^* y)}_{\preccurlyeq \subseteq \preccurlyeq^*}$$





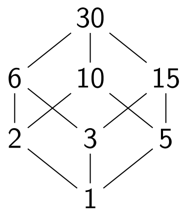


Linearization defined

Definition 7.4.8

Let A be a set and $\preccurlyeq$ be a partial order on A. A *linearization* of $\preccurlyeq$ is a total order $\preccurlyeq^*$ on A such that

$$\forall x, y \in A \ (x \preccurlyeq y \Rightarrow x \preccurlyeq^* y).$$



Question 7.4.9

Is the total order $\preccurlyeq^*$ represented by the bottom Hasse diagram a linearization of the partial order $\preccurlyeq$ represented by the top Hasse diagram?

Answer

No, because $2 \preccurlyeq 6$ but $2 \not\preccurlyeq^* 6$.

Remark

Swapping 2 and 6 in the bottom diagram gives a Hasse diagram of a linearization of $\preccurlyeq$.

Zoom poll

Linearizations exist

Keep collecting minimal elements.

Definition 7.4.8

Let A be a set and $\preccurlyeq$ be a partial order on A. A *linearization* of $\preccurlyeq$ is a total order $\preccurlyeq^*$ on A such that

$$\forall x, y \in A \ (x \preccurlyeq y \Rightarrow x \preccurlyeq^* y).$$

Theorem 7.4.10

Every partial order $\preccurlyeq$ has a linearization $\preccurlyeq^*$.

Proposition 7.4.6

With respect to any partial order $\preccurlyeq$ on a finite set $A \neq \varnothing$, one can find a minimal element.

Note 7.4.12

In step (2.1), there may be several minimal elements to choose from. Different choices give different linearizations.

Kahn's Algorithm (1962)

Input: a finite set A, a partial order $\preccurlyeq$ on A.

(1) Set $A_0 := A$ and $i := 0$.

(2) Repeat until $A_i = \varnothing$:

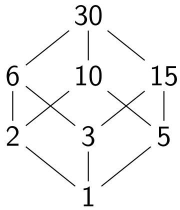
- (2.1) find a minimal element c_i of A_i wrt $\preccurlyeq$;
- (2.2) set $A_{i+1} := A_i \setminus \{c_i\}$;
- (2.3) set $i := i + 1$.

Output: a linearization $\preccurlyeq^*$ of $\preccurlyeq$ defined by setting, for all indices i, j,

$$c_i \preccurlyeq^* c_j \quad \Leftrightarrow \quad i \leqslant j.$$

A run of Kahn's Algorithm

Keep collecting minimal elements.



Example 7.4.13

Consider $\{d \in \mathbb{Z}^+ : d \mid 30\}$ partially ordered by the divisibility relation $|$.

- Set $A_0 := \{d \in \mathbb{Z}^+ : d \mid 30\}$.
- 1 is the only minimal element of A_0. Set $c_0 := 1$ and $A_1 := A_0 \setminus \{1\}$.
- $2, 3, 5$ are the minimal elements of A_1. Set $c_1 := 3$ and $A_2 := A_1 \setminus \{3\}$.
- $2, 5$ are the minimal elements of A_2. Set $c_2 := 2$ and $A_3 := A_2 \setminus \{2\}$.
- $5, 6$ are the minimal elements of A_3. Set $c_3 := 6$ and $A_4 := A_3 \setminus \{6\}$.
- 5 is the only minimal element of A_4. Set $c_4 := 5$ and $A_5 := A_4 \setminus \{5\}$.
- $10, 15$ are the minimal elements of A_5. Set $c_5 := 15$ and $A_6 := A_5 \setminus \{15\}$.
- 10 is the only minimal element of A_6. Set $c_6 := 10$ and $A_7 := A_6 \setminus \{10\}$.
- 30 is the only (minimal) element of A_7. Set $c_7 := 30$ and $A_8 := A_7 \setminus \{30\}$.
- $A_8 = \varnothing$ and so we stop.
- A linearization is given by $1 \preccurlyeq^* 3 \preccurlyeq^* 2 \preccurlyeq^* 6 \preccurlyeq^* 5 \preccurlyeq^* 15 \preccurlyeq^* 10 \preccurlyeq^* 30$.

Kahn's Algorithm stops

Keep collecting minimal elements.

- The input set A is finite.
- Each time the repeat-until loop is run, one element is taken out of A.
- So this loop is run exactly $|A|$ times.
- Then the set of remaining elements is empty, and the stopping condition is satisfied.

Kahn's Algorithm (1962)

Input: a finite set A, a partial order $\preccurlyeq$ on A.

(1) Set $A_0 := A$ and $i := 0$.

(2) Repeat until $A_i = \varnothing$:

(2.1) find a minimal element c_i of A_i wrt $\preccurlyeq$;

(2.2) set $A_{i+1} := A_i \setminus \{c_i\}$;

(2.3) set $i := i + 1$.

Output: a linearization $\preccurlyeq^*$ of $\preccurlyeq$ defined by setting, for all indices i, j,

$$c_i \preccurlyeq^* c_j \quad \Leftrightarrow \quad i \leqslant j.$$

Kahn's Algorithm is correct

Want $\preccurlyeq^*$ a total order and $\forall x, y \in A\ (x \preccurlyeq y \Rightarrow x \preccurlyeq^* y)$

Proof (optional material)

2. Suppose the run produces $A_0, A_1, \ldots, A_n, c_0, c_1, \ldots, c_{n-1}$ and $\preccurlyeq^*$.
3. Note $A = \{c_0, c_1, \ldots, c_{n-1}\}$ because the removal of $c_0, c_1, \ldots, c_{n-1}$ from A makes the set empty.
4. Note also that $\preccurlyeq^*$ is a total order on A because it is by definition only a renaming of the total order $\leqslant$ on $\{0, 1, \ldots, n-1\}$.
5.
   5.1. Take $x \in A$ and $c_j \in A$ such that $x \prec c_j$.
   5.2. Then $x \notin A_j$ as c_j is minimal in A_j.
   5.3. So $x \in A \setminus A_j = \{c_0, c_1, \ldots, c_{j-1}\}$ by the choices of $A_0, A_1, \ldots, A_j, c_0, c_1, \ldots, c_{j-1}$.
   5.4. Let $i \in \{0, 1, \ldots, j-1\}$ such that $x = c_i$.
   5.5. Then $x = c_i \preccurlyeq^* c_j$ by the definition of $\preccurlyeq^*$, as $i \leqslant j - 1 < j$. $\square$

Kahn's Algorithm (1962)

Input: a finite set A, a partial order $\preccurlyeq$ on A.

(1) Set $A_0 := A$ and $i := 0$.

(2) Repeat until $A_i = \varnothing$:

(2.1) find a minimal element c_i of A_i wrt $\preccurlyeq$;

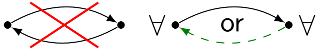
(2.2) set $A_{i+1} := A_i \setminus \{c_i\}$;

(2.3) set $i := i + 1$.

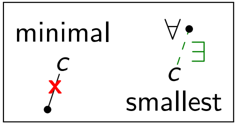
Output: a linearization $\preccurlyeq^*$ of $\preccurlyeq$ defined by setting, for all indices i, j,

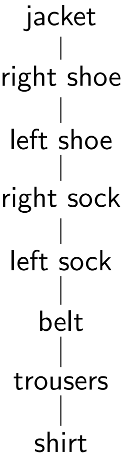
$$c_i \preccurlyeq^* c_j \quad \Leftrightarrow \quad i \leqslant j.$$

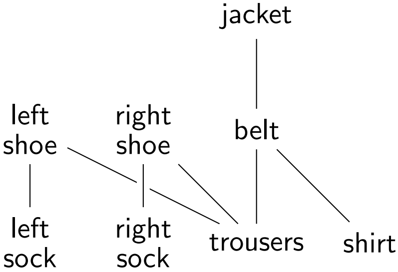
Checkpoint



- antisymmetric relations: if $x \mathrel{R} y$ and $y \mathrel{R} x$, then $x = y$.
- partial orders: reflexive, antisymmetric, transitive
- total orders: partial orders in which any two elements are comparable
- minimal/maximal vs smallest/largest
- linearization: a sequential way to complete the tasks such that tasks that must be done first are done first
- Kahn's Algorithm, with a proof of its termination and correctness







[T]he things themselves are not what [science] can reach, as the naive dogmatists think, but only the relations between things. Outside of these relations there is no knowable reality. Poincaré 1902

Next: induction