CS1231S Chapter 6

Equivalence relations

6.1 Representation

Definition 6.1.1. Call $\mathscr{C}$ a *partition* of a set A if

(1) $\mathscr{C}$ is a set of which all elements are *nonempty* subsets of A; and

(2) every element of A is in *exactly* one element of $\mathscr{C}$.

Elements of a partition are called *components* of the partition.

Remark 6.1.2. One can rewrite the two conditions in the definition of partitions respectively as follows:

(1) $\varnothing \neq S \subseteq A$ for all $S \in \mathscr{C}$;

(2) $\forall x \in A\ \exists S \in \mathscr{C}\ \ (x \in S)$ and $\forall x \in A\ \forall S_1, S_2 \in \mathscr{C}\ \ \big(x \in S_1\ \wedge\ x \in S_2\ \Rightarrow\ S_1 = S_2\big)$.

Yet another way to formulate this is to say that $\mathscr{C}$ is a set of mutually disjoint nonempty subsets of A whose union is A.

Example 6.1.3. One partition of the set $A = \{1, 2, 3\}$ is $\{\{1\}, \{2, 3\}\}$. The others are

$$\{\{1\}, \{2\}, \{3\}\}, \quad \{\{2\}, \{1, 3\}\}, \quad \{\{3\}, \{1, 2\}\}, \quad \{\{1, 2, 3\}\}.$$

Example 6.1.4. One partition of $\mathbb{Z}$ is

$$\{\{2k : k \in \mathbb{Z}\}, \{2k + 1 : k \in \mathbb{Z}\}\}.$$

Definition 6.1.5. Let A, B be sets.

(1) A *relation* from A to B is a subset of $A \times B$.

(2) Let R be a relation from A to B and $(x, y) \in A \times B$. Then we may write

$$x\ R\ y \text{ for } (x, y) \in R \quad \text{and} \quad x \not R\ y \text{ for } (x, y) \notin R.$$

We read "$x\ R\ y$" as "x is *R-related* to y" or simply "x is *related* to y".

Example 6.1.6. Let S be the set of all NUS students and M be the set of all modules offered by the NUS. Then the predicate "is enrolled in" is represented by the relation

$$\{(x, y) \in S \times M : x \text{ is enrolled in } y\}$$

from S to M.

Example 6.1.7. Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3, 4\}$. Define the relation R from A to B by setting

$$x\ R\ y \quad \Leftrightarrow \quad x < y.$$

Then $0\ R\ 1$ and $0\ R\ 2$, but $2 \not R\ 1$. Thus

$$R = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}.$$

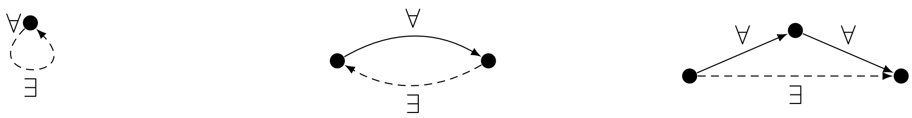


Figure 6.1: Reflexivity, symmetry, and transitivity

6.2 Reflexivity, symmetry, and transitivity

Definition 6.2.1. A *(binary) relation on a set* A is a relation from A to A.

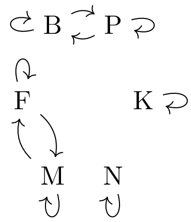
Remark 6.2.2. It follows from Definition 6.1.5 and Definition 6.2.1 that the relations on a set A are precisely the subsets of $A \times A$.

Arrow diagrams (for relations on a set). One can draw an arrow diagram representing a relation R on a set A as follows.

(1) Draw all the elements of A.

(2) For all $x, y \in A$, draw an arrow from x to y if and only if $x \mathrel{R} y$.

Example 6.2.3. The arrow diagram



represents the relation

$$\{(\mathrm{B}, \mathrm{P}), (\mathrm{P}, \mathrm{B}), (\mathrm{F}, \mathrm{M}), (\mathrm{M}, \mathrm{F}), (\mathrm{B}, \mathrm{B}), (\mathrm{P}, \mathrm{P}), (\mathrm{F}, \mathrm{F}), (\mathrm{M}, \mathrm{M}), (\mathrm{K}, \mathrm{K}), (\mathrm{E}, \mathrm{E})\}$$

on the set $\{\mathrm{B}, \mathrm{P}, \mathrm{F}, \mathrm{M}, \mathrm{K}, \mathrm{E}\}$.

Definition 6.2.4. Let A be a set and R be a relation on A.

(1) R is *reflexive* if every element of A is R-related to itself, i.e.,

$$\forall x \in A \quad (x \mathrel{R} x).$$

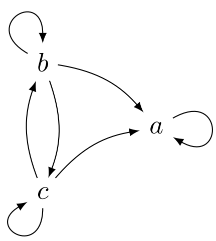
(2) R is *symmetric* if x is R-related to y implies y is R-related to x, for all $x, y \in A$, i.e.,

$$\forall x, y \in A \quad (x \mathrel{R} y \Rightarrow y \mathrel{R} x).$$

(3) R is *transitive* if x is R-related to y and y is R-related to z imply x is R-related to z, for all $x, y, z \in A$, i.e.,

$$\forall x, y, z \in A \quad (x \mathrel{R} y \wedge y \mathrel{R} z \Rightarrow x \mathrel{R} z).$$

Example 6.2.5. Let R be the relation represented by the following arrow diagram.



Then R is reflexive. It is not symmetric because $b \mathrel{R} a$ but $a \not\mathrel{R} b$. It is transitive, as one can show by exhaustion:

$$
\begin{gathered}
a \mathrel{R} a \wedge a \mathrel{R} a \Rightarrow a \mathrel{R} a; \\
a \mathrel{R} a \wedge a \mathrel{R} b \Rightarrow a \mathrel{R} b; \\
a \mathrel{R} a \wedge a \mathrel{R} c \Rightarrow a \mathrel{R} c; \\
a \mathrel{R} b \wedge b \mathrel{R} a \Rightarrow a \mathrel{R} a; \\
\vdots \\
c \mathrel{R} c \wedge c \mathrel{R} b \Rightarrow c \mathrel{R} b; \\
c \mathrel{R} c \wedge c \mathrel{R} c \Rightarrow c \mathrel{R} c.
\end{gathered}
$$

Example 6.2.6. Let R denote the equality relation on a set A, i.e., for all $x, y \in A$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x = y.$$

Then R is reflexive, symmetric, and transitive.

Example 6.2.7. Let R' denote the subset relation on a set U of sets, i.e., for all $x, y \in U$,

$$x \mathrel{R'} y \quad \Leftrightarrow \quad x \subseteq y.$$

Then R' is reflexive, may not be symmetric (when U contains x, y such that $x \subsetneq y$), but is transitive.

Exercise 6.2.8. Write down a proof of the transitivity claim in Example 6.2.7. ✎ 6a

Example 6.2.9. Let R denote the non-strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x \leqslant y.$$

Then R is reflexive, not symmetric, but transitive.

Example 6.2.10. Let R' denote the strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x \mathrel{R'} y \quad \Leftrightarrow \quad x < y.$$

Then R' is not reflexive as $0 \not< 0$. It is not symmetric because $0 < 1$ but $1 \not< 0$. It is transitive.

Definition 6.2.11. Let $n, d \in \mathbb{Z}$. Then d is said to *divide* n if

$$n = dk \quad \text{for some } k \in \mathbb{Z}.$$

We write $d \mid n$ for "d divides n", and $d \nmid n$ for "d does not divide n". We also say

"n is *divisible* by d" or "n is a *multiple* of d" or "d is a *factor/divisor* of n"

for "d divides n".

Example 6.2.12. Let R denote the divisibility relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x \mathrel{R} y \quad \Leftrightarrow \quad x \mid y.$$

Is R reflexive? Is R symmetric? Is R transitive? ✎ 6b

Definition 6.2.13. An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

Convention 6.2.14. People usually use equality-like symbols such as $\sim$, $\approx$, $\simeq$, $\cong$, and $\equiv$ to denote equivalence relations. These symbols are often defined and redefined to mean different equivalence relations in different situations. We may read $\sim$ as "is equivalent to".

Example 6.2.15. The equality relation on a set, as defined in Example 6.2.6, is an equivalence relation.

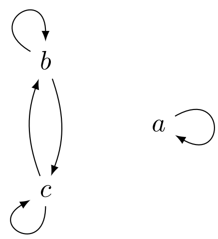
Proposition 6.2.16. Let $\mathscr{C}$ be a partition of a set A. Denote by $\sim_{\mathscr{C}}$ the same-component relation with respect to $\mathscr{C}$, i.e., for all $x, y \in A$,

$$\begin{aligned} x \sim_{\mathscr{C}} y \quad &\Leftrightarrow \quad x \text{ is in the same component of } \mathscr{C} \text{ as } y \\ &\Leftrightarrow \quad x, y \in S \text{ for some } S \in \mathscr{C}. \end{aligned}$$

Then $\sim_{\mathscr{C}}$ is an equivalence relation on A.

Proof. 1. (Reflexivity.) Every element is in the same component as itself.
2. (Symmetry.) If x is in the same component as y, then y is in the same component as x.
3. (Transitivity.) If x is in the same component as y, and y is in the same component as z, then x is in the same component as z. $\square$

Example 6.2.17. Let R be the relation represented by the following arrow diagram.



Then R is reflexive, symmetric, and transitive. So it is an equivalence relation on $\{a, b, c\}$.

Exercise 6.2.18. Let $A = \{1, 2, 3\}$ and $R = \{(1,1), (1,2), (2,1), (3,2)\}$. Is R reflexive? Is R symmetric? Is R transitive? ✎ 6c

6.3 Congruence

Definition 6.3.1. Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Then a *is congruent to* b *modulo* n if $a - b = nk$ for some $k \in \mathbb{Z}$. In this case, we write $a \equiv b \pmod{n}$.

Remark 6.3.2. In terms of divisibility, for all $a, b \in \mathbb{Z}$ and all $n \in \mathbb{Z}^+$,

$$a \equiv b \pmod{n} \quad \Leftrightarrow \quad n \mid (a - b).$$

Example 6.3.3. (1) $5 \equiv 1 \pmod{2}$ because $5 - 1 = 4 = 2 \times 2$.

(2) $-2 \equiv 4 \pmod{3}$ because $-2 - 4 = -6 = 3 \times (-2)$.

(3) $-4 \not\equiv 5 \pmod{7}$ because $-4 - 5 = -9 \neq 7k$ for any $k \in \mathbb{Z}$.

Proposition 6.3.4. Let $n \in \mathbb{Z}^+$ and $\sim_n$ denote the congruence-mod-n relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x \sim_n y \quad \Leftrightarrow \quad x \equiv y \pmod{n}.$$

Then $\sim_n$ is an equivalence relation.

Proof. 1. (Reflexivity.) For all $a \in \mathbb{Z}$, we know $a - a = 0 = n \times 0$ and so $a \equiv a \pmod{n}$ by the definition of congruence.

2. (Symmetry.)
    2.1. Let $a, b \in \mathbb{Z}$ such that $a \equiv b \pmod{n}$.
    2.2. Use the definition of congruence to find $k \in \mathbb{Z}$ such that $a - b = nk$.
    2.3. Then $b - a = -(a - b) = -nk = n(-k)$.
    2.4. Note that $-k \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under $-$.
    2.5. So $b \equiv a \pmod{n}$ by the definition of congruence.
3. (Transitivity.)
    3.1. Let $a, b, c \in \mathbb{Z}$ such that $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$.
    3.2. Use the definition of congruence to find $k, \ell \in \mathbb{Z}$ such that $a-b = nk$ and $b-c = n\ell$.
    3.3. Then $a - c = (a - b) + (b - c) = nk + n\ell = n(k + \ell)$.
    3.4. Note that $k + \ell \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under $+$.
    3.5. So $a \equiv c \pmod{n}$ by the definition of congruence. $\square$

6.4 Equivalence classes

Definition 6.4.1. Let $\sim$ be an equivalence relation on a set A. For each $x \in A$, the *equivalence class* of x with respect to $\sim$, denoted $[x]_\sim$, is defined to be the set of all elements of A that are $\sim$-related to x, i.e.,

$$[x]_\sim = \{y \in A : x \sim y\}.$$

When there is no risk of confusion, we may drop the subscript and write simply $[x]$.

Example 6.4.2. Let A be a set. The equivalence classes with respect to the equality relation on A are of the form

$$[x] = \{y \in A : x = y\} = \{x\},$$

where $x \in A$.

Example 6.4.3. Let $n \in \mathbb{Z}^+$. The equivalence classes with respect to the congruence-mod-n relation on $\mathbb{Z}$ are of the form

$$[x] = \{y \in \mathbb{Z} : x \equiv y \pmod{n}\} = \{nk + x : k \in \mathbb{Z}\},$$

where $x \in \mathbb{Z}$. If $n = 2$, then there are two equivalence classes:

$$\{2k : k \in \mathbb{Z}\} \quad \text{and} \quad \{2k + 1 : k \in \mathbb{Z}\}.$$

Lemma 6.4.4. Let $\sim$ be an equivalence relation on a set A. The following are equivalent for all $x, y \in A$.

(i) $x \sim y$.

(ii) $[x] = [y]$.

(iii) $[x] \cap [y] \neq \varnothing$.

Proof. 1. ((i) $\Rightarrow$ (ii))
    1.1. Suppose $x \sim y$.
    1.2. Then $y \sim x$ by symmetry.
    1.3. For every $z \in [x]$,

1.3.1.		$x \sim z$	by the definition of $[x]$;
1.3.2.	$\therefore$	$y \sim z$	by transitivity, as $y \sim x$;
1.3.3.	$\therefore$	$z \in [y]$	by the definition of $[y]$.

    1.4. This shows $[x] \subseteq [y]$.
    1.5. Switching the roles of x and y, we see also that $[y] \subseteq [x]$.

1.6. So $[x] = [y]$.
2. ((ii) $\Rightarrow$ (iii))
   2.1. Suppose $[x] = [y]$.
   2.2. Then $[x] \cap [y] = [x]$ by the Idempotent Law for $\cap$.
   2.3. However, we know $x \sim x$ by the reflexivity of $\sim$.
   2.4. So the definition of $[x]$ and line 2.2 tell us $x \in [x] = [x] \cap [y]$.
   2.5. Hence $[x] \cap [y] \neq \varnothing$.
3. ((iii) $\Rightarrow$ (i))
   3.1. Suppose $[x] \cap [y] \neq \varnothing$.
   3.2. Take $z \in [x] \cap [y]$.
   3.3. Then $x \sim z$ and $y \sim z$.
   3.4. The latter implies $z \sim y$ by symmetry.
   3.5. So $x \sim y$ by transitivity. $\square$

Question 6.4.5. Consider an equivalence relation. Is it true that if x is an element of an equivalence class S, then $S = [x]$? ✐ 6d

Definition 6.4.6. Let A be a set and $\sim$ be an equivalence relation on A. Denote by $A/{\sim}$ the set of all equivalence classes with respect to $\sim$, i.e.,

$$A/{\sim} = \{[x]_\sim : x \in A\}.$$

We may read $A/{\sim}$ as "the quotient of A by $\sim$".

Example 6.4.7. Let A be a set. Then from Example 6.4.2 we know $A/{=}$ is equal to $\{\{x\} : x \in A\}$.

Example 6.4.8. Let $n \in \mathbb{Z}^+$. If $\sim_n$ denotes the congruence-mod-n relation on $\mathbb{Z}$, then from Example 6.4.3 we know

$$\mathbb{Z}/{\sim_n} = \{[x] : x \in \mathbb{Z}\} = \big\{\{nk : k \in \mathbb{Z}\}, \{nk+1 : k \in \mathbb{Z}\}, \ldots, \{nk+(n-1) : k \in \mathbb{Z}\}\big\}.$$

Theorem 6.4.9. Let $\sim$ be an equivalence relation on a set A. Then $A/{\sim}$ is a partition of A.

Proof. 1. $A/{\sim}$ is by definition a set.
2. We show that every element of $A/{\sim}$ is a nonempty subset of A.
   2.1. Let $S \in A/{\sim}$.
   2.2. Use the definition of $A/{\sim}$ to find $x \in A$ such that $S = [x]$.
   2.3. Then $S = [x] \subseteq A$ in view of the definition of equivalence classes.
   2.4. Note that the reflexivity of $\sim$ implies $x \sim x$.
   2.5. Hence $x \in [x] = S$ by the definition of $[x]$ and the choice of x.
   2.6. In particular, we know S is nonempty.
3. We show that every element of A is in at least one element of $A/{\sim}$.
   3.1. Let $x \in A$.
   3.2. Then $x \sim x$ by reflexivity.
   3.3. So $x \in [x] \in A/{\sim}$.
4. We show that every element of A is in at most one element of $A/{\sim}$.
   4.1. Let $x \in A$ that is in two elements of $A/{\sim}$, say S_1 and S_2.
   4.2. Use the definition of $A/{\sim}$ to find $y_1, y_2 \in A$ such that $S_1 = [y_1]$ and $S_2 = [y_2]$.
   4.3. Line 4.1 tells us $x \in [y_1] \cap [y_2]$.
   4.4. So $[y_1] \cap [y_2] \neq \varnothing$.
   4.5. Lemma 6.4.4 then implies $S_1 = [y_1] = [y_2] = S_2$. $\square$