National University of Singapore
MA2001 Linear Algebra

MATLAB Worksheet 4
Coordinate vectors, Row Space, Column Space, Nullspace

Type `format rat`. Throughout the entire worksheet, we will use the rational format to read the entries of matrices.

A. Coordinate Vectors

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a basis for a vector space V. Then every vector in V can be uniquely represented as a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. Precisely, for any $\boldsymbol{v} \in V$, there exist unique numbers $c_1, c_2, \ldots, c_k \in \mathbb{R}$ such that

$$\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k.$$

Then the column vector $(c_1, c_2, \ldots, c_k)$ is called the **coordinate vector** of $\boldsymbol{v}$ relative to S, denoted by $(\boldsymbol{v})_S$.
We shall use the example of V, S, T in Worksheet 3 Section D to find the coordinate vector of $\boldsymbol{h} = (-1, -3, 0, 1, -2)$ in V relative to the basis T.

Recall that $V = \text{span}(S)$, $S = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3, \boldsymbol{g}_4\}$, and $T = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3\}$ where

$$\boldsymbol{g}_1 = (1, 1, 1, 1, 1), \quad \boldsymbol{g}_2 = (1, -1, 2, 3, 0), \quad \boldsymbol{g}_3 = (-1, -3, 0, 1, -2), \quad \boldsymbol{g}_4 = (0, 1, 1, -1, -1)$$

and

$$\boldsymbol{h}_1 = (2, 0, 3, 4, 1), \quad \boldsymbol{h}_2 = (1, 0, 3, 2, -1), \quad \boldsymbol{h}_3 = (1, 2, 2, 0, 0).$$

(i) Input $\boldsymbol{h}$ as a column vector in MATLAB.

```
>>  h = [-1; -3; 0; 1; -2]
h =  -1
     -3
     0
     1
     -2
```

(ii) Solve the linear system $\boldsymbol{T}\boldsymbol{x} = \boldsymbol{h}$ (recall that $\boldsymbol{T} = \begin{pmatrix}\boldsymbol{h}_1 & \boldsymbol{h}_2 & \boldsymbol{h}_3\end{pmatrix}$).

```
>>  rref([T h])
ans =  1    0    0    -1/2
       0    1    0    3/2
       0    0    1    -3/2
       0    0    0    0
       0    0    0    0
```

Observing the entries in the column corresponding to $\boldsymbol{h}$, we obtain $\boldsymbol{h} = -\frac{1}{2}\boldsymbol{h}_1 + \frac{3}{2}\boldsymbol{h}_2 - \frac{3}{2}\boldsymbol{h}_3$. Hence, $(\boldsymbol{h})_T = \left(-\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}\right)$.

B. Row Space

Let $\boldsymbol{A} = (a_{ij})$ be an $m \times n$ matrix. Let $\boldsymbol{r}_i = \begin{pmatrix} a_{i1} & \cdots & a_{in} \end{pmatrix}$ be the i^{th} row of $\boldsymbol{A}$. Then $\boldsymbol{r}_i \in \mathbb{R}^n$ and

$$\operatorname{span}\{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_m\}$$

is a subspace of $\mathbb{R}^n$, called the **row space** of $\boldsymbol{A}$.

Recall that, if $\boldsymbol{R}$ is a row-echelon form of $\boldsymbol{A}$, then the nonzero rows of $\boldsymbol{R}$ form a basis for the row space of $\boldsymbol{A}$.

For example, let $\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 1 & 2 & 8 \\ 2 & 8 & 2 & 3 & 12 \\ 3 & 12 & 3 & -1 & -4 \\ 4 & 16 & -1 & -4 & -16 \end{pmatrix}$.

(i) Input $\boldsymbol{A}$ in MATLAB.

```
>> A = [1 4 1 2 8; 2 8 2 3 12; 3 12 3 -1 -4; 4 16 -1 -4 -16];
```

(ii) Find the reduced row-echelon form of $\boldsymbol{A}$.

```
>> rref(A)
ans =  1    4    0    0    0
       0    0    1    0    0
       0    0    0    1    4
       0    0    0    0    0
```

We conclude that the row space of $\boldsymbol{A}$ has a basis

$$\{(1,4,0,0,0), (0,0,1,0,0), (0,0,0,1,4)\}.$$

So the dimension of the row space of $\boldsymbol{A}$ is 3. This dimension is known as the **rank** of $\boldsymbol{A}$. In MATLAB, we can use a simple command `rank` to find the rank of a matrix directly:

```
>> rank(A)
ans =  3
```

Repeat the same procedure for $\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 3 & 4 & 8 & 16 & 32 \\ 4 & 1 & 7 & 7 & 19 & 31 \end{pmatrix}$.

(i) Input $\boldsymbol{B}$ in MATLAB.

```
>> B = [1 1 1 1 1 1; 1 -1 1 -1 1 -1; 1 3 4 8 16 32; 4 1 7 7 19 31];
```

(ii) Find the reduced row-echelon form of $\boldsymbol{B}$.

```
>> rref(B)
ans =  1    0    0    0    -4    -8
       0    1    0    0    0     0
       0    0    1    0    5     8
       0    0    0    1    0     1
```

So the row space of $\boldsymbol{B}$ has a basis

$$\{(1,0,0,0,-4,-8),(0,1,0,0,0,0),(0,0,1,0,5,8),(0,0,0,1,0,1)\}$$

and the rank of $\boldsymbol{B}$ is

```
>> rank(B)
ans =  4
```

In this case, there is no zero row in the RREF. So $\boldsymbol{B}$ is **full rank** and the (original) four rows of $\boldsymbol{B}$ also form a basis for the row space of $\boldsymbol{B}$.

C. Column Space

Let $\boldsymbol{A} = (a_{ij})$ be an $m \times n$ matrix. Let $\boldsymbol{c}_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$ be the j^{th} column of $\boldsymbol{A}$. Then $\boldsymbol{c}_j \in \mathbb{R}^m$, and

$$\text{span}\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$$

is a subspace of $\mathbb{R}^m$, called the **column space** of $\boldsymbol{A}$. Recall that, if $\boldsymbol{R}$ is a row-echelon form of $\boldsymbol{A}$, then the columns of $\boldsymbol{A}$ which correspond to the pivot columns of $\boldsymbol{R}$ form a basis for the column space of $\boldsymbol{A}$.

We use the same matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and their RREF's above for illustration.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 1 & 2 & 8 \\ 2 & 8 & 2 & 3 & 12 \\ 3 & 12 & 3 & -1 & -4 \\ 4 & 16 & -1 & -4 & -16 \end{pmatrix} \text{ has the RREF } \begin{pmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the reduced row-echelon form, the 1^{st}, 3^{rd} and 4^{th} columns are pivot columns. So the column space of $\boldsymbol{A}$ has a basis formed by the 1^{st}, 3^{rd} and 4^{th} columns of $\boldsymbol{A}$:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -1 \\ 4 \end{pmatrix} \right\}.$$

Note that the dimension of the column space of $\boldsymbol{A}$ is also given by the rank of $\boldsymbol{A}$, which is 3.

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 3 & 4 & 8 & 16 & 32 \\ 4 & 1 & 7 & 7 & 19 & 31 \end{pmatrix} \text{ has the RREF } \begin{pmatrix} 1 & 0 & 0 & 0 & -4 & -8 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 & 8 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

In the reduced row-echelon form, the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} columns are pivot. Then the column space of $\boldsymbol{B}$ has a basis formed by the 1^{st}, 2^{nd}, 3^{rd} and 4^{th} columns of $\boldsymbol{B}$:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 8 \\ 7 \end{pmatrix} \right\}.$$

D. Finding Basis for Vector Space

Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$. There are two methods to find a basis for $V = \text{span}(S)$.

Row space Method

View each $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ as a row vector. Then the nonzero rows of any row-echelon form of the matrix $\begin{pmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_k \end{pmatrix}$ form a basis for V.

For example, let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$, where

$$\boldsymbol{v}_1 = (1, 1, 1, 1, 1), \ \boldsymbol{v}_2 = (1, -1, 1, -1, 1), \ \boldsymbol{v}_3 = (2, 0, 2, 0, 2),$$

$$\boldsymbol{v}_4 = (1, -2, 4, -8, 16), \ \boldsymbol{v}_5 = (0, 4, -6, 16, -30).$$

(i) Input $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5$ into MATLAB as row vectors:

```
>> v1 = [1 1 1 1 1];  v2 = [1 -1 1 -1 1];  v3 = [2 0 2 0 2];
>> v4 = [1 -2 4 -8 16];  v5 = [0 4 -6 16 -30];
```

(ii) Find the reduced row-echelon form of the matrix $\begin{pmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_5 \end{pmatrix}$.

```
>> rref([v1; v2; v3; v4; v5])
ans =  1    0    0    2   -4
       0    1    0    1    0
       0    0    1   -2    5
       0    0    0    0    0
       0    0    0    0    0
```

Its nonzero rows $\{(1, 0, 0, 2, -4), (0, 1, 0, 1, 0), (0, 0, 1, -2, 5)\}$ form a basis for $V = \text{span}(S)$. Note that the vectors in the basis are not necessarily in S.

Column Space Method

View each $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ as column vectors. Find the pivot columns of any row-echelon form of the matrix $\begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k \end{pmatrix}$. Then the corresponding vectors in S form a basis S' for V. Note that $S' \subseteq S$.

(i) Input $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5$ into MATLAB as column vectors. In the previous section, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_5$ are defined as row vectors. Their transposes $\boldsymbol{v}_1^{\mathrm{T}}, \ldots, \boldsymbol{v}_5^{\mathrm{T}}$ (`v1'`, ..., `v5'`) are the required column vectors.

(ii) Find the reduced row-echelon form of the matrix $\begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_5 \end{pmatrix}$.

```
>> rref([v1' v2' v3' v4' v5'])
ans =  1    0    1    0    1
```

```
0    1    1    0    1
0    0    0    1    -2
0    0    0    0    0
0    0    0    0    0
```

Its 1st, 2nd and 4th columns are pivot. Then

$$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\} = \{(1,1,1,1,1), (1,-1,1,-1,1), (1,-2,4,-8,16)\}$$

form a basis for V. Note that every vector in this basis is taken from S.

E. Nullspace

Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then the solution set of the homogeneous linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ is always a subspace of $\mathbb{R}^n$, called the **nullspace** of $\boldsymbol{A}$.

We use the same matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and their RREF's as above.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 1 & 2 & 8 \\ 2 & 8 & 2 & 3 & 12 \\ 3 & 12 & 3 & -1 & -4 \\ 4 & 16 & -1 & -4 & -16 \end{pmatrix} \text{ has the RREF } \begin{pmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Assume that the variables are x_1, x_2, x_3, x_4, x_5. Since the 2nd and the 5th columns of the reduced row-echelon form are non-pivot, set $x_2 = s$ and $x_5 = t$ as arbitrary parameters. By separating parameters, we get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -4s \\ s \\ 0 \\ -4t \\ t \end{pmatrix} = s \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 1 \end{pmatrix}.$$

Then the nullspace of $\boldsymbol{A}$ has a basis

$$\left\{ \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

MATLAB can provide a basis for the nullspace of $\boldsymbol{A}$ directly using the command `null(A,'r')`.

```
>>  null(A, 'r')
ans =   -4     0
         1     0
         0     0
         0    -4
         0     1
```

The two columns of the answer above form a basis for the nullspace of $\boldsymbol{A}$.

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 3 & 4 & 8 & 16 & 32 \\ 4 & 1 & 7 & 7 & 19 & 31 \end{pmatrix} \text{ has the RREF } \begin{pmatrix} 1 & 0 & 0 & 0 & -4 & -8 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 & 8 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

We use the command `null(B,'r')` directly:

```
>>  null(B, 'r')
ans =  4       8
       0       0
       -5     -8
       0      -1
       1       0
       0       1
```

Then the nullspace of $\boldsymbol{B}$ has a basis

$$\left\{ \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \\ -8 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

F. Practices

Use MATLAB to solve Questions 4.1, 4.2, 4.3, 4.5, 4.7, 4.11, 4.16 in the textbook Exercise 4.