Chapter 6: Equivalence relations

CS1231S Discrete Structures

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2021/22 Semester 1

Which of the following is/are correct?

1. 0.5 is $1/2$.
2. 0.5 is equal to $1/2$.
3. 0.5 is the same as $1/2$.
4. 0.5 and $1/2$ represent the same number.
5. 0.5 means $1/2$.
6. 0.5 is equivalent to $1/2$.
7. 0.5 is identical to $1/2$.

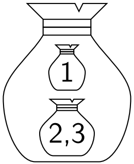
Share with us what you think in the Zoom poll when the lecture begins.

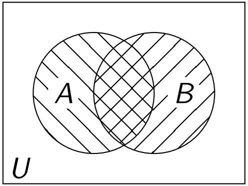
Zoom poll

What we saw

Sets

- membership, inclusion, and equality of sets
- power sets and Cartesian products
- union, intersections, complements
- set identities and their proofs
- Venn diagrams
- cardinalities of finite sets



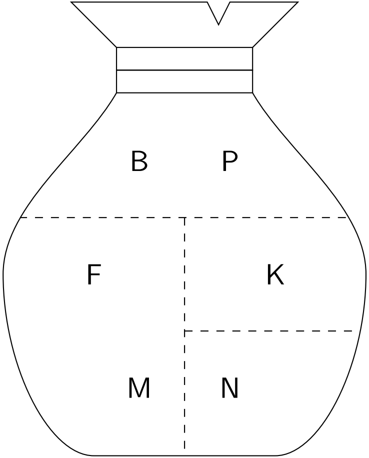


Next

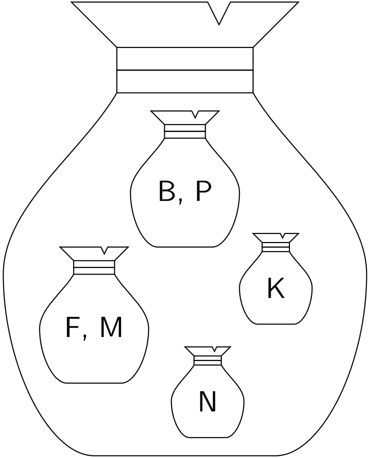
how sets can represent mathematical objects:

- partitions — degrees of computation, classifying pictures, ...
- predicates

How one can represent a partition by a set



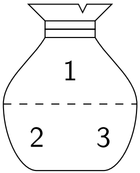
$A = \{\mathsf{B}, \mathsf{P}, \mathsf{F}, \mathsf{M}, \mathsf{K}, \mathsf{N}\}$



$\{\mathsf{B}, \mathsf{P}\}, \{\mathsf{F}, \mathsf{M}\}, \{\mathsf{K}\}, \{\mathsf{N}\}$

Partitions as sets

a set of mutually disjoint nonempty subsets of A whose union is A



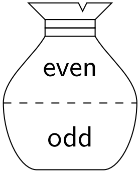
Definition 6.1.1

Call $\mathscr{C}$ a *partition* of a set A if

(1) $\mathscr{C}$ is a set of which all elements are *nonempty* subsets of A; and

(2) every element of A is in *exactly* one element of $\mathscr{C}$.

Elements of a partition are called *components* of the partition.



Remark 6.1.2

One can rewrite the two conditions above respectively as follows:

(1) $\varnothing \neq S \subseteq A$ for all $S \in \mathscr{C}$;

(2) $\forall x \in A\ \exists S \in \mathscr{C}\ (x \in S)$ and $\underbrace{\forall x \in A\ \forall S_1, S_2 \in \mathscr{C}\ \big(x \in S_1 \ \wedge\ x \in S_2 \ \Rightarrow\ S_1 = S_2\big)}_{\forall S_1, S_2 \in \mathscr{C}\ (S_1 \neq S_2 \Rightarrow S_1 \cap S_2 = \varnothing)}$.

Example 6.1.3

One partition of the set $A = \{1, 2, 3\}$ is $\{\{1\}, \{2, 3\}\}$. The others are

$$\{\{1\}, \{2\}, \{3\}\}, \quad \{\{2\}, \{1, 3\}\}, \quad \{\{3\}, \{1, 2\}\}, \quad \{\{1, 2, 3\}\}.$$

Example 6.1.4

One partition of $\mathbb{Z}$ is $\{\{2k : k \in \mathbb{Z}\}, \{2k + 1 : k \in \mathbb{Z}\}\}$.

How one can represent a predicate by a set

is enrolled in	
student	module
Peter	CS1231
Paul	CS1231
Mary	CS1231
Simon	MA1521
Garfunkel	MA1521
Tom	MA2001
Jerry	MA2001
Ben	MA2002
Jerry	MA2002
⋮	⋮

$$
\{\ (\text{Peter}, \text{CS1231}),\ (\text{Paul}, \text{CS1231}),\ (\text{Mary}, \text{CS1231}),\ (\text{Simon}, \text{MA1521}),\ (\text{Garfunkel}, \text{MA1521}),\ (\text{Tom}, \text{MA2001}),\ (\text{Jerry}, \text{MA2001}),\ (\text{Ben}, \text{MA2002}),\ (\text{Jerry}, \text{MA2002}),\ \ldots\ \}
$$

Predicates as sets

Let A, B be sets.

Definition 6.1.5

(1) A *relation* from A to B is a subset of $A \times B$.

(2) Let R be a relation from A to B and $(x, y) \in A \times B$. Then we may write

$$x \mathrel{R} y \text{ for } (x, y) \in R \quad \text{and} \quad x \not\mathrel{R} y \text{ for } (x, y) \notin R.$$

We read "$x \mathrel{R} y$" as "x is *R-related* to y" or simply "x is *related* to y".

Example 6.1.6

Let S be the set of all NUS students and M be the set of all modules offered by the NUS. Then the predicate "is enrolled in" is represented by the relation

$$\{(x, y) \in S \times M : x \text{ is enrolled in } y\}$$

from S to M.

Example 6.1.7

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3, 4\}$. Define the relation R from A to B by setting

$$x \mathrel{R} y \quad \Leftrightarrow \quad x < y.$$

Then $0 \mathrel{R} 1$ and $0 \mathrel{R} 2$, but $2 \not\mathrel{R} 1$. Thus

$$R = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}.$$

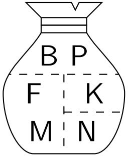
Checkpoint

What we saw

how sets can represent partitions and predicates

Next

partitions as relations



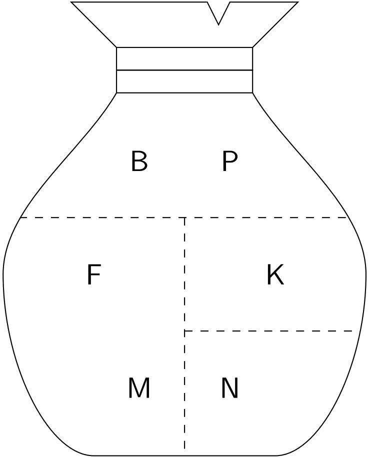
$$\{\{B, P\}, \{F, M\}, \{K\}, \{N\}\}$$

is enrolled in	
student	module
Peter	CS1231
Paul	CS1231
Mary	CS1231
Simon	MA1521
Garfunkel	MA1521
Tom	MA2001
Jerry	MA2001
Ben	MA2002
Jerry	MA2002
$\vdots$	$\vdots$

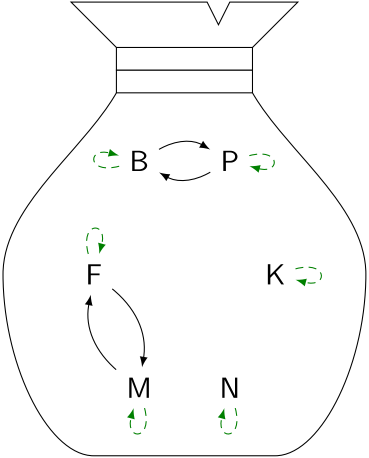
$$\{ (\text{Peter}, \text{CS1231}), (\text{Paul}, \text{CS1231}), \ldots \}$$

> [S]ometimes there really are two different approaches to a problem. This should not be disturbing, but should instead be seen as a great opportunity. After all, two approaches to the same idea indicates that there are some new mathematics to be investigated and some new connections to be found and exploited, which hopefully will uncover a wealth of new results. Kaye (2007)

Partitions as relations



$A = \{\mathsf{B}, \mathsf{P}, \mathsf{F}, \mathsf{M}, \mathsf{K}, \mathsf{N}\}$



the "is in the same component as" relation

Relations on a set

subset of $A \times A$

Definition 6.2.1

A *(binary) relation on a set A* is a relation from A to A.

Arrow diagrams (for relations on a set)

One can draw an arrow diagram representing a relation R on a set A as follows.

(1) Draw all the elements of A.

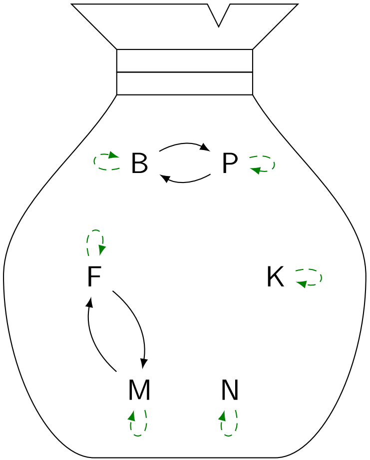
(2) For all $x, y \in A$ which may or may not be distinct, draw an arrow from x to y if and only if $x \mathrel{R} y$.

Example 6.2.3

The diagram on the right represents the relation

$$\{(\mathrm{B},\mathrm{P}),(\mathrm{P},\mathrm{B}),(\mathrm{F},\mathrm{M}),(\mathrm{M},\mathrm{F}),$$
$$(\mathrm{B},\mathrm{B}),(\mathrm{P},\mathrm{P}),(\mathrm{F},\mathrm{F}),(\mathrm{M},\mathrm{M}),(\mathrm{K},\mathrm{K}),(\mathrm{N},\mathrm{N})\}$$

on the set $\{\mathrm{B},\mathrm{P},\mathrm{F},\mathrm{M},\mathrm{K},\mathrm{N}\}$.



the "is in the same component as" relation

What does the same-component relation satisfy?

(1) Every element is in the same component as itself.

(2) If x is in the same component as y, then y is in the same component as x.

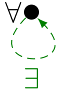
(3) If x is in the same component as y, and y is in the same component as z, then x is in the same component as z.

Definition 6.2.4

Let A be a set and R be a relation on A.

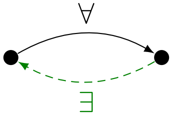
(1) R is *reflexive* if every element of A is R-related to itself, i.e.,

$$\forall x \in A \ (x \, R \, x).$$



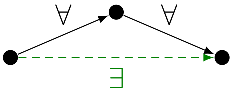
(2) R is *symmetric* if x is R-related to y implies y is R-related to x, for all $x, y \in A$, i.e.,

$$\forall x, y \in A \ (x \, R \, y \Rightarrow y \, R \, x).$$

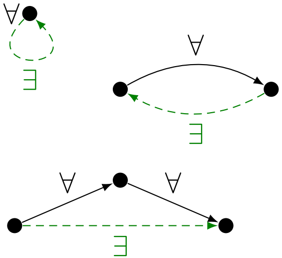


(3) R is *transitive* if x is R-related to y and y is R-related to z imply x is R-related to z, for all $x, y, z \in A$, i.e.,

$$\forall x, y, z \in A \ (x \, R \, y \wedge y \, R \, z \Rightarrow x \, R \, z).$$



Reflexivity, symmetry, and transitivity



Definition 6.2.4. Let A be a set and R be a relation on A.

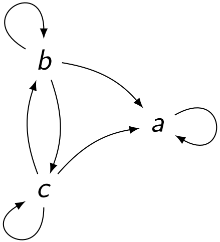
(1) R is *reflexive* if $\forall x \in A \; (x \mathrel{R} x)$.

(2) R is *symmetric* if $\forall x, y \in A \; (x \mathrel{R} y \Rightarrow y \mathrel{R} x)$.

(3) R is *transitive* if $\forall x, y, z \in A \; (x \mathrel{R} y \wedge y \mathrel{R} z \Rightarrow x \mathrel{R} z)$.

Example 6.2.5

Let R be the relation represented by the following arrow diagram.



- Then R is reflexive.
- It is not symmetric because $b \mathrel{R} a$ but $a \not\mathrel{R} b$.
- It is transitive, as one can show by exhaustion:

$$\begin{gathered} a \mathrel{R} a \wedge a \mathrel{R} a \Rightarrow a \mathrel{R} a; \\ a \mathrel{R} a \wedge a \mathrel{R} b \Rightarrow a \mathrel{R} b; \\ \vdots \\ c \mathrel{R} c \wedge c \mathrel{R} c \Rightarrow c \mathrel{R} c. \end{gathered}$$

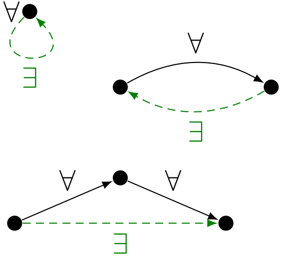
Equality and inclusion

Definition 6.2.4. Let A be a set and R be a relation on A.

(1) R is *reflexive* if $\forall x \in A \ (x \, R \, x)$.

(2) R is *symmetric* if $\forall x, y \in A \ (x \, R \, y \Rightarrow y \, R \, x)$.

(3) R is *transitive* if $\forall x, y, z \in A \ (x \, R \, y \wedge y \, R \, z \Rightarrow x \, R \, z)$.

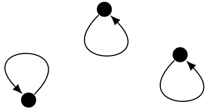


Example 6.2.6

Let R denote the equality relation on a set A, i.e., for all $x, y \in A$,

$$x \, R \, y \quad \Leftrightarrow \quad x = y.$$

Then R is reflexive, symmetric, and transitive.

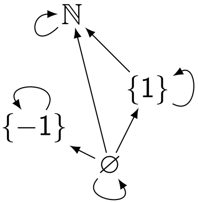


Example 6.2.7

Let R' denote the subset relation on a set U of sets, i.e., for all $x, y \in U$,

$$x \, R' \, y \quad \Leftrightarrow \quad x \subseteq y.$$

Then R' is reflexive, may not be symmetric, but is transitive.



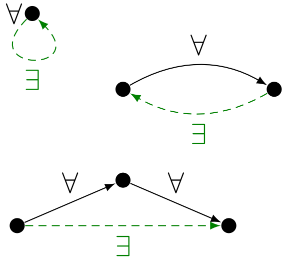
Inequalities

Definition 6.2.4. Let A be a set and R be a relation on A.

(1) R is *reflexive* if $\forall x \in A \ (x \, R \, x)$.

(2) R is *symmetric* if $\forall x, y \in A \ (x \, R \, y \Rightarrow y \, R \, x)$.

(3) R is *transitive* if $\forall x, y, z \in A \ (x \, R \, y \wedge y \, R \, z \Rightarrow x \, R \, z)$.

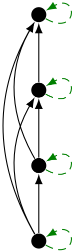


Example 6.2.9

Let R denote the non-strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x \, R \, y \quad \Leftrightarrow \quad x \leqslant y.$$

Then R is reflexive, not symmetric, but transitive. $\boxed{0 \leqslant 1 \text{ but } 1 \not\leqslant 0.}$



Example 6.2.10

Let R' denote the strict less-than relation on $\mathbb{Q}$, i.e., for all $x, y \in \mathbb{Q}$,

$$x \, R' \, y \quad \Leftrightarrow \quad x < y.$$

Then R' is not reflexive, not symmetric, but transitive. $\boxed{0 \not< 0}$ $\boxed{0 < 1 \text{ but } 1 \not< 0.}$

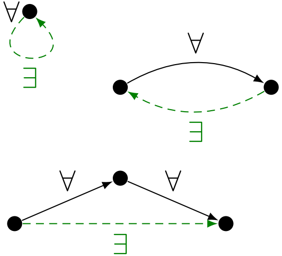
Divisibility

Definition 6.2.4. Let A be a set and R be a relation on A.

(1) R is *reflexive* if $\forall x \in A \; (x \, R \, x)$.

(2) R is *symmetric* if $\forall x, y \in A \; (x \, R \, y \Rightarrow y \, R \, x)$.

(3) R is *transitive* if $\forall x, y, z \in A \; (x \, R \, y \wedge y \, R \, z \Rightarrow x \, R \, z)$.



Definition 6.2.11

Let $n, d \in \mathbb{Z}$. Then d is said to *divide* n if

$$n = dk \quad \text{for some } k \in \mathbb{Z}.$$

"n is *divisible* by d" or "n is a *multiple* of d" or "d is a *factor/divisor* of n"

We write $d \mid n$ for "d divides n", and $d \nmid n$ for "d does not divide n".

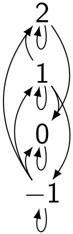
Example 6.2.12

Let R denote the divisibility relation on $\mathbb{Z}$, i.e., for all $x, y \in \mathbb{Z}$,

$$x \, R \, y \quad \Leftrightarrow \quad x \mid y.$$

Then R is reflexive, not symmetric, but transitive.

$1 \mid 2$ but $2 \nmid 1$.



Zoom poll

Equivalence relations

Proposition 6.2.16

Let $\mathscr{C}$ be a partition of a set A. Denote by $\sim_{\mathscr{C}}$ the same-component relation with respect to $\mathscr{C}$, i.e., for all $x, y \in A$,

$$\begin{aligned} x \sim_{\mathscr{C}} y \quad &\Leftrightarrow \quad x \text{ is in the same component of } \mathscr{C} \text{ as } y \\ &\Leftrightarrow \quad x, y \in S \text{ for some } S \in \mathscr{C}. \end{aligned}$$

Then $\sim_{\mathscr{C}}$ is reflexive, symmetric and transitive. So $\sim_{\mathscr{C}}$ is an equivalence relation.

Definition 6.2.13

An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

Convention 6.2.14

People usually use equality-like symbols such as $\sim$, $\approx$, $\simeq$, $\cong$, and $\equiv$ to denote equivalence relations. These symbols are often defined and redefined to mean different equivalence relations in different situations. We may read $\sim$ as "is equivalent to".

Example 6.2.15

The equality relation on a set, as defined in Example 6.2.6, is an equivalence relation.

A finite equivalence relation

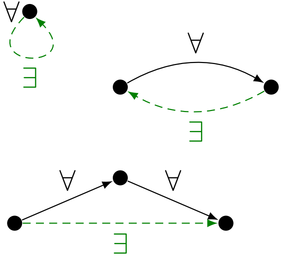
Definition 6.2.4 and Definition 6.2.13. Let A be a set and R be a relation on A.

(1) R is *reflexive* if $\forall x \in A \ (x \, R \, x)$.

(2) R is *symmetric* if $\forall x, y \in A \ (x \, R \, y \Rightarrow y \, R \, x)$.

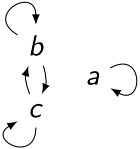
(3) R is *transitive* if $\forall x, y, z \in A \ (x \, R \, y \wedge y \, R \, z \Rightarrow x \, R \, z)$.

An *equivalence relation* is a relation that is reflexive, symmetric and transitive.



Example 6.2.17

Let R be the relation represented by the arrow diagram below.



- Then R is reflexive, symmetric and transitive.
- So it is an equivalence relation.

Same distance from 0

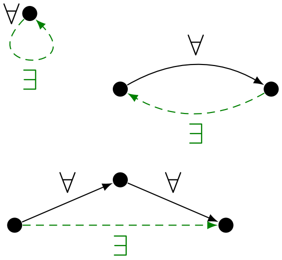
Definition 6.2.4 and Definition 6.2.13. Let A be a set and R be a relation on A.

(1) R is *reflexive* if $\forall x \in A \ (x \, R \, x)$.

(2) R is *symmetric* if $\forall x, y \in A \ (x \, R \, y \Rightarrow y \, R \, x)$.

(3) R is *transitive* if $\forall x, y, z \in A \ (x \, R \, y \wedge y \, R \, z \Rightarrow x \, R \, z)$.

An *equivalence relation* is a relation that is reflexive, symmetric and transitive.



Exercise 6.2.18

Define a relation $\sim$ on $\mathbb{Z}$ by setting, for all $x, y \in \mathbb{Z}$,

$$x \sim y \quad \Leftrightarrow \quad x = y \text{ or } x = -y.$$

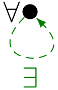
Verify that $\sim$ is an equivalence relation on $\mathbb{Z}$.

Checkpoint

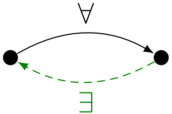
Definition 6.2.4 and Definition 6.2.13

Let A be a set and R be a relation on A.

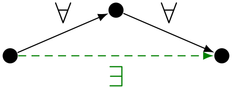
(1) R is *reflexive* if $\forall x \in A\ (x\,R\,x)$.



(2) R is *symmetric* if $\forall x, y \in A\ (x\,R\,y \Rightarrow y\,R\,x)$.



(3) R is *transitive* if $\forall x, y, z \in A\ (x\,R\,y \wedge y\,R\,z \Rightarrow x\,R\,z)$.



An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

> It is difficult to overstate the importance and ubiquity of the equivalence relation concept in mathematics. This notion arises in nearly every area of pure mathematics and should be seen as a general conceptual tool. Hamkins 2020

Next

- a partition that is naturally represented by an equivalence relation
- Prove that reflexivity, symmetry, and transitivity are *precisely* the properties that the same-component relation with respect to a partition needs to have.

Quick check

Definition 6.2.4 and Definition 6.2.13

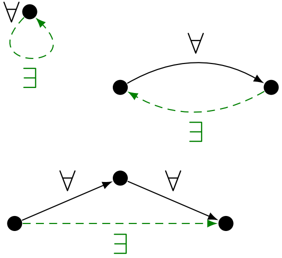
Let A be a set and R be a relation on A.

(1) R is *reflexive* if $\forall x \in A \ (x \, R \, x)$.

(2) R is *symmetric* if $\forall x, y \in A \ (x \, R \, y \Rightarrow y \, R \, x)$.

(3) R is *transitive* if $\forall x, y, z \in A \ (x \, R \, y \wedge y \, R \, z \Rightarrow x \, R \, z)$.

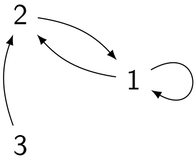
An *equivalence relation* is a relation that is reflexive, symmetric and transitive.



Exercise 6.2.19

Let $A = \{1, 2, 3\}$ and $R = \{(1,1), (1,2), (2,1), (3,2)\}$. Consider R as a relation on A. Is R reflexive? Is R symmetric? Is R transitive?

- Then R is not reflexive because $2 \not R \, 2$.
- It is not symmetric because $3 \, R \, 2$ but $2 \not R \, 3$.
- It is not transitive because $2 \, R \, 1$ and $1 \, R \, 2$ but $2 \not R \, 2$.



Zoom poll

Generalizing the even–odd partition in Example 6.1.4

x is in the same component as y
$\Leftrightarrow \quad x - y = 3k$ for some $k \in \mathbb{Z}$

$\{\{3k : k \in \mathbb{Z}\}, \{3k+1 : k \in \mathbb{Z}\}, \{3k+2 : k \in \mathbb{Z}\}\}$

$\mathbb{Z}$

$\vdots$	$\vdots$
-4	-3
-2	-1
0	1
2	3
4	5
$\vdots$	$\vdots$

$\mathbb{Z}$

$\vdots$	$\vdots$	$\vdots$
-6	-5	-4
-3	-2	-1
0	1	2
3	4	5
6	7	8
$\vdots$	$\vdots$	$\vdots$

$\mathbb{Z}$

$\vdots$	$\vdots$	$\vdots$	$\vdots$
-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\{\{2k : k \in \mathbb{Z}\}, \{2k+1 : k \in \mathbb{Z}\}\}$

$\{\{4k : k \in \mathbb{Z}\}, \{4k+1 : k \in \mathbb{Z}\},$
$\{4k+2 : k \in \mathbb{Z}\}, \{4k+3 : k \in \mathbb{Z}\}\}$

x is in the same component as y
$\Leftrightarrow \quad x - y = 2k$ for some $k \in \mathbb{Z}$

x is in the same component as y
$\Leftrightarrow \quad x - y = 4k$ for some $k \in \mathbb{Z}$

Congruence

Definition 6.3.1

Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Then *a is congruent to b modulo n* if $a - b = nk$ for some $k \in \mathbb{Z}$. In this case, we write $a \equiv b \pmod{n}$.

> Definition 6.2.11. Let $n, d \in \mathbb{Z}$. Then $d \mid n$ if $n = dk$ for some $k \in \mathbb{Z}$.

Remark 6.3.2

In terms of divisibility, for all $a, b \in \mathbb{Z}$ and all $n \in \mathbb{Z}^+$,

$$a \equiv b \pmod{n} \quad \Leftrightarrow \quad n \mid (a - b).$$

Example 6.3.3

(1) $5 \equiv 1 \pmod{2}$ because $5 - 1 = 4 = 2 \times 2$.

(2) $-2 \equiv 4 \pmod{3}$ because $-2 - 4 = -6 = 3 \times (-2)$.

(3) $-4 \not\equiv 5 \pmod{7}$ because $-4 - 5 = -9 \neq 7k$ for any $k \in \mathbb{Z}$.

Congruence is an equivalence relation

Definition 6.3.1

Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Then *a is congruent to b modulo n* if $a - b = nk$ for some $k \in \mathbb{Z}$. In this case, we write $a \equiv b \pmod{n}$.

Proposition 6.3.4

Congruence-mod-n is an equivalence relation on $\mathbb{Z}$ for every $n \in \mathbb{Z}^+$.

Definition 6.2.4(1). A relation R on a set A is *reflexive* if $\forall x \in A\ (x\ R\ x)$.

Proof

1. (Reflexivity.) For all $a \in \mathbb{Z}$,
   1.1. we know $a - a = 0 = n \times 0$, and
   1.2. so $a \equiv a \pmod{n}$ by the definition of congruence.
2. (Symmetry.) ...
3. (Transitivity.) ...

Definition 6.2.13. An *equivalence relation* is a relation that is reflexive, symmetric and transitive.

Congruence is an equivalence relation

Definition 6.3.1

Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Then *a is congruent to b modulo n* if $a - b = nk$ for some $k \in \mathbb{Z}$. In this case, we write $a \equiv b \pmod{n}$.

Proposition 6.3.4

Congruence-mod-n is an equivalence relation on $\mathbb{Z}^+$ for every $n \in \mathbb{Z}$.

Proof

Definition 6.2.4(2). A relation R on a set A is *symmetric* if $\forall x, y \in A \ (x \mathrel{R} y \Rightarrow y \mathrel{R} x)$.

1. (Reflexivity.) . . .
2. (Symmetry.)
   - 2.1. Let $a, b \in \mathbb{Z}$ such that $a \equiv b \pmod{n}$.
   - 2.2. Use the definition of congruence to find $k \in \mathbb{Z}$ such that $a - b = nk$.
   - 2.3. Then $b - a = -(a - b) = -nk = n(-k)$.
   - 2.4. Note that $-k \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under $-$.
   - 2.5. So $b \equiv a \pmod{n}$ by the definition of congruence.
3. (Transitivity.) . . .

Congruence is an equivalence relation

Definition 6.3.1

Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Then *a is congruent to b modulo n* if $a - b = nk$ for some $k \in \mathbb{Z}$. In this case, we write $a \equiv b \pmod{n}$.

Proposition 6.3.4

Congruence-mod-n is an equivalence relation on $\mathbb{Z}$ for every $n \in \mathbb{Z}^+$.

Definition 6.2.4(3). A relation R on a set A is *transitive* if $\forall x, y, z \in A \ (x \mathrel{R} y \wedge y \mathrel{R} z \Rightarrow x \mathrel{R} z)$.

Proof

1. (Reflexivity.) ...
2. (Symmetry.) ...
3. (Transitivity.) ...
   - 3.1. Let $a, b, c \in \mathbb{Z}$ such that $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$.
   - 3.2. Use the definition of congruence to find $k, \ell \in \mathbb{Z}$ such that $a - b = nk$ and $b - c = n\ell$.
   - 3.3. Then $a - c = (a - b) + (b - c) = nk + n\ell = n(k + \ell)$.
   - 3.4. Note that $k + \ell \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under $+$.
   - 3.5. So $a \equiv c \pmod{n}$ by the definition of congruence. $\square$

Checkpoint

$\mathbb{Z}$

$\vdots$	$\vdots$
-4	-3
-2	-1
0	1
2	3
4	5
$\vdots$	$\vdots$

$\mathbb{Z}$

$\vdots$	$\vdots$	$\vdots$
-6	-5	-4
-3	-2	-1
0	1	2
3	4	5
6	7	8
$\vdots$	$\vdots$	$\vdots$

$\mathbb{Z}$

$\vdots$	$\vdots$	$\vdots$	$\vdots$
-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11
$\vdots$	$\vdots$	$\vdots$	$\vdots$

What we saw

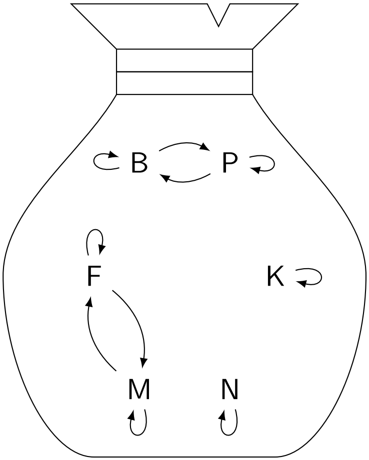
natural ways to represent the partitions above as sets and as equivalence relations

Next

Prove that equivalence relations are *precisely* the same-component relations with respect to partitions.

- We already saw that same-component relations are equivalence relations.
- So it remains to prove that every equivalence relation is the same-component relation with respect to some partition.

How to recover the components from the same-component relation



$A = \{\mathsf{B}, \mathsf{P}, \mathsf{F}, \mathsf{M}, \mathsf{K}, \mathsf{N}\}$

$$\begin{aligned}
[\mathsf{B}] &= \text{the component containing B} \\
&= \{x \in A : \text{B is in the same component as } x\} \\
&= \{\mathsf{B}, \mathsf{P}\}. \\
[\mathsf{P}] &= \text{the component containing P} \\
&= \{x \in A : \text{P is in the same component as } x\} \\
&= \{\mathsf{P}, \mathsf{B}\} = \text{the component containing B.} \\
[\mathsf{F}] &= \text{the component containing F} \\
&= \{x \in A : \text{F is in the same component as } x\} \\
&= \{\mathsf{F}, \mathsf{M}\}. \\
[\mathsf{M}] &= \text{the component containing M} \\
&= \cdots = \{\mathsf{M}, \mathsf{F}\} = \text{the component containing F.} \\
[\mathsf{K}] &= \text{the component containing K} = \cdots = \{\mathsf{K}\}. \\
[\mathsf{N}] &= \text{the component containing N} = \cdots = \{\mathsf{N}\}.
\end{aligned}$$

Equivalence classes: equality

When there is no risk of confusion, we may drop the subscript.

Definition 6.4.1

Let $\sim$ be an equivalence relation on a set A. For each $x \in A$, the *equivalence class* of x with respect to $\sim$, denoted $[x]_\sim$, is defined by

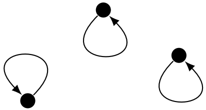
$$[x]_\sim = \{y \in A : x \sim y\}.$$

the set of all elements of A that are $\sim$-related to x

Example 6.4.2

Let A be a set. The equivalence classes with respect to the equality relation on A are of the form

$$[x] = \{y \in A : x = y\} = \{x\}, \qquad \text{where } x \in A.$$



Equivalence classes: congruence

When there is no risk of confusion, we may drop the subscript.

Definition 6.4.1

Let $\sim$ be an equivalence relation on a set A. For each $x \in A$, the *equivalence class* of x with respect to $\sim$, denoted $[x]_\sim$, is defined by

$$[x]_\sim = \{y \in A : x \sim y\}.$$

the set of all elements of A that are $\sim$-related to x

Example 6.4.3

Let $n \in \mathbb{Z}^+$. The equivalence classes with respect to the congruence-mod-n relation on $\mathbb{Z}$ are of the form

$$\begin{aligned}[x] &= \{y \in \mathbb{Z} : x \equiv y \pmod{n}\}\\ &= \{y \in \mathbb{Z} : x - y = nk \text{ for some } k \in \mathbb{Z}\}\\ &= \{nk + x : k \in \mathbb{Z}\}\\ &= \{\dots, x - 2n, x - n, x, x + n, x + 2n, \dots\},\end{aligned}$$

where $x \in \mathbb{Z}$. Note that for all $x \in \mathbb{Z}$,

$$[x + n] = \{\dots, x - n, x, x + n, x + 2n, x + 3n, \dots\} = [x].$$

So if $n = 2$, then $\dots = [-2] = [0] = [2] = [4] = \cdots$ and $\cdots = [-1] = [1] = [3] = \cdots$.

$\mathbb{Z}$

$\vdots$	$\vdots$	$\vdots$	$\vdots$
-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Equivalence classes are either equal or disjoint

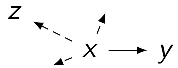
$$\begin{tikzcd} & \text{(ii)} \arrow[dr, Rightarrow] & \\ \text{(i)} \arrow[ur, Rightarrow] & & \text{(iii)} \arrow[ll, Rightarrow] \end{tikzcd}$$

Lemma 6.4.4

Let $\sim$ be an equivalence relation on a set A. The following are equivalent for all $x, y \in A$.

(i) $x \sim y$. (ii) $[x] = [y]$. (iii) $[x] \cap [y] \neq \varnothing$.

Proof



1. ((i) $\Rightarrow$ (ii))
   1.1. Suppose $x \sim y$.
   1.2. Then $y \sim x$ by symmetry.
   1.3. For every $z \in [x]$,
      1.3.1. $x \sim z$ by the definition of $[x]$;
      1.3.2. $\therefore$ $y \sim z$ by transitivity, as $y \sim x$;
      1.3.3. $\therefore$ $z \in [y]$ by the definition of $[y]$.
   1.4. This shows $[x] \subseteq [y]$.
   1.5. Switching the roles of x and y, we see also that $[y] \subseteq [x]$.
   1.6. So $[x] = [y]$.

Definition 6.2.4

(1) R is *reflexive* if $\forall x \in A$
$$x \mathrel{R} x.$$
(2) R is *symmetric* if $\forall x, y \in A$
$$x \mathrel{R} y \;\Rightarrow\; y \mathrel{R} x.$$
(3) R is *transitive* if $\forall x, y, z \in A$
$$x \mathrel{R} y \wedge y \mathrel{R} z \;\Rightarrow\; x \mathrel{R} z.$$

Definition 6.4.1. $[x]_\sim = \{y \in A : x \sim y\}$.

Equivalence classes are either equal or disjoint

$$\begin{tikzcd} & \text{(ii)} \arrow[dr, Rightarrow] & \\ \text{(i)} \arrow[ur, Rightarrow] & & \text{(iii)} \arrow[ll, Rightarrow] \end{tikzcd}$$

Lemma 6.4.4

Let $\sim$ be an equivalence relation on a set A. The following are equivalent for all $x, y \in A$.

(i) $x \sim y$. (ii) $[x] = [y]$. (iii) $[x] \cap [y] \neq \varnothing$.

Proof

2. ((ii) $\Rightarrow$ (iii))
   - 2.1. Suppose $[x] = [y]$.
   - 2.2. Then $[x] \cap [y] = [x]$ by the Idempotent Law for $\cap$.
   - 2.3. However, we know $x \sim x$ by the reflexivity of $\sim$.
   - 2.4. So the definition of $[x]$ and line 2.2 tell us $x \in [x] = [x] \cap [y]$.
   - 2.5. Hence $[x] \cap [y] \neq \varnothing$.

Definition 6.2.4

(1) R is *reflexive* if $\forall x \in A$
$x \, R \, x$.

(2) R is *symmetric* if $\forall x, y \in A$
$x \, R \, y \;\Rightarrow\; y \, R \, x$.

(3) R is *transitive* if $\forall x, y, z \in A$
$x \, R \, y \wedge y \, R \, z \;\Rightarrow\; x \, R \, z$.

Definition 6.4.1. $[x]_\sim = \{y \in A : x \sim y\}$.

Equivalence classes are either equal or disjoint

$$\begin{tikzcd} & \text{(ii)} \arrow[dr, Rightarrow] & \\ \text{(i)} \arrow[ur, Rightarrow] & & \text{(iii)} \arrow[ll, Rightarrow] \end{tikzcd}$$

Lemma 6.4.4

Let $\sim$ be an equivalence relation on a set A. The following are equivalent for all $x, y \in A$.

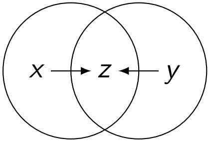
(i) $x \sim y$. (ii) $[x] = [y]$. (iii) $[x] \cap [y] \neq \varnothing$.

Proof

3. ((iii) $\Rightarrow$ (i))
   - 3.1. Suppose $[x] \cap [y] \neq \varnothing$.
   - 3.2. Take $z \in [x] \cap [y]$.
   - 3.3. Then $x \sim z$ and $y \sim z$.
   - 3.4. The latter implies $z \sim y$ by symmetry.
   - 3.5. So $x \sim y$ by transitivity. $\square$

Definition 6.2.4

(1) R is *reflexive* if $\forall x \in A$
$$x \mathrel{R} x.$$
(2) R is *symmetric* if $\forall x, y \in A$
$$x \mathrel{R} y \;\Rightarrow\; y \mathrel{R} x.$$
(3) R is *transitive* if $\forall x, y, z \in A$
$$x \mathrel{R} y \wedge y \mathrel{R} z \;\Rightarrow\; x \mathrel{R} z.$$

$[x]$  $[y]$

Definition 6.4.1. $[x]_\sim = \{y \in A : x \sim y\}$.

Quick check

Lemma 6.4.4

Let $\sim$ be an equivalence relation on a set A. The following are equivalent for all $x, y \in A$.

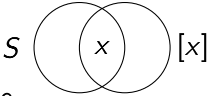
(i) $x \sim y$. (ii) $[x] = [y]$. (iii) $[x] \cap [y] \neq \varnothing$.

Question 6.4.5

Consider an equivalence relation. Is it true that if x is an element of an equivalence class S, then $S = [x]$?

Answer

Yes, as shown below.



1. We know $x \sim x$ as $\sim$ is reflexive.
2. So $x \in [x]$ by the definition of $[x]$.
3. Hence $x \in S \cap [x]$ by the hypothesis.
4. This implies $S \cap [x] \neq \varnothing$.
5. Thus $S = [x]$ by Lemma 6.4.4. $\square$

Definition 6.2.4

(1) R is *reflexive* if $\forall x \in A$
$$x\,R\,x.$$

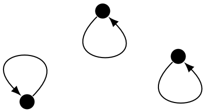
(2) R is *symmetric* if $\forall x, y \in A$
$$x\,R\,y \Rightarrow y\,R\,x.$$

(3) R is *transitive* if $\forall x, y, z \in A$
$$x\,R\,y \wedge y\,R\,z \Rightarrow x\,R\,z.$$

Definition 6.4.1. $[x]_\sim = \{y \in A : x \sim y\}$.

Zoom poll

Dividing a set by an equivalence relation



Definition 6.4.6

Let A be a set and $\sim$ be an equivalence relation on A. Denote by $A/\!\sim$ the set of all equivalence classes with respect to $\sim$, i.e.,

$$A/\!\sim\; = \{[x]_\sim : x \in A\}.$$

We may read $A/\!\sim$ as "the quotient of A by $\sim$".

Example 6.4.7

Let A be a set. Then $A/\!=$ is equal to $\{\{x\} : x \in A\}$.

Example 6.4.8

Let $n \in \mathbb{Z}^+$. If $\sim_n$ denotes the congruence-mod-n relation on $\mathbb{Z}$, then

$$\begin{aligned} \mathbb{Z}/\!\sim_n &= \{[x] : x \in \mathbb{Z}\} \\ &= \{\{nk : k \in \mathbb{Z}\}, \{nk+1 : k \in \mathbb{Z}\}, \dots, \{nk+(n-1) : k \in \mathbb{Z}\}\}. \end{aligned}$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$
-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\mathbb{Z}$

Definition 6.4.1. $[x]_\sim = \{y \in A : x \sim y\}$.

Equivalence classes form a partition

$A/\sim = \{[x] : x \in A\}$,
where $[x] = \{y \in A : x \sim y\}$.

Theorem 6.4.9

Let $\sim$ be an equivalence relation on a set A. Then $A/\sim$ is a partition of A.

Definition 6.1.1

Call $\mathscr{C}$ a *partition* of a set A if

(1) $\mathscr{C}$ is a set of which all elements are *nonempty* subsets of A; and

(2) every element of A is in *exactly* one element of $\mathscr{C}$.

Lemma 6.4.4

TFAE.

(i) $x \sim y$.

(ii) $[x] = [y]$.

(iii) $[x] \cap [y] \neq \varnothing$.

Proof

1. $A/\sim$ is by definition a set.
2. We show that every element of $A/\sim$ is a nonempty subset of A.
   - 2.1. Let $S \in A/\sim$.
   - 2.2. Use the definition of $A/\sim$ to find $x \in A$ such that $S = [x]$.
   - 2.3. Then $S = [x] \subseteq A$ in view of the definition of equivalence classes.
   - 2.4. Note that the reflexivity of $\sim$ implies $x \sim x$.
   - 2.5. Hence $x \in [x] = S$ by the definition of $[x]$ and the choice of x.
   - 2.6. In particular, we know S is nonempty.

Equivalence classes form a partition

$A/{\sim} = \{[x] : x \in A\}$,
where $[x] = \{y \in A : x \sim y\}$.

Theorem 6.4.9

Let $\sim$ be an equivalence relation on a set A. Then $A/{\sim}$ is a partition of A.

Definition 6.1.1

Call $\mathscr{C}$ a *partition* of a set A if

(1) $\mathscr{C}$ is a set of which all elements are *nonempty* subsets of A; and

(2) every element of A is in *exactly* one element of $\mathscr{C}$.

Lemma 6.4.4

TFAE.

(i) $x \sim y$.

(ii) $[x] = [y]$.

(iii) $[x] \cap [y] \neq \varnothing$.

Proof

3. We show that every element of A is in at least one element of $A/{\sim}$.
   - 3.1. Let $x \in A$.
   - 3.2. Then $x \sim x$ by reflexivity.
   - 3.3. So $x \in [x] \in A/{\sim}$.
4. We show that every element of A is in at most one element of $A/{\sim}$. . . .

Equivalence classes form a partition

$A/{\sim} = \{[x] : x \in A\}$,
where $[x] = \{y \in A : x \sim y\}$.

Theorem 6.4.9

Let $\sim$ be an equivalence relation on a set A. Then $A/{\sim}$ is a partition of A.

Definition 6.1.1

Call $\mathscr{C}$ a *partition* of a set A if

(1) $\mathscr{C}$ is a set of which all elements are *nonempty* subsets of A; and

(2) every element of A is in *exactly* one element of $\mathscr{C}$.

Lemma 6.4.4

TFAE.

(i) $x \sim y$.

(ii) $[x] = [y]$.

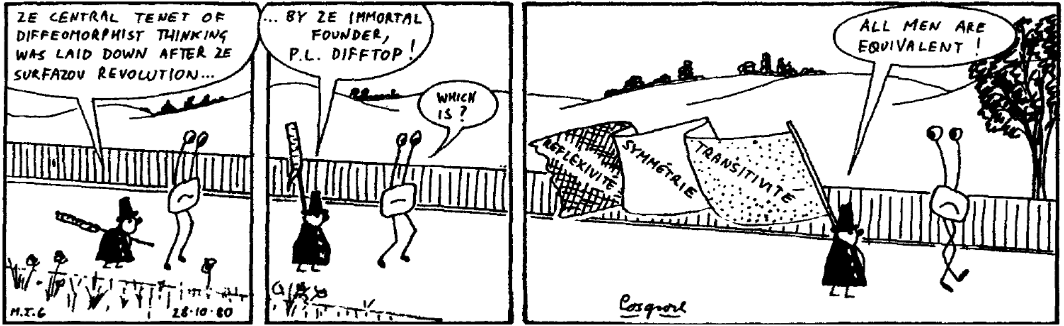
(iii) $[x] \cap [y] \neq \varnothing$.

Proof

3. We show that every element of A is in at least one element of $A/{\sim}$. ...
4. We show that every element of A is in at most one element of $A/{\sim}$.
   - 4.1. Let $x \in A$ that is in two elements of $A/{\sim}$, say S_1 and S_2.
   - 4.2. Use the definition of $A/{\sim}$ to find $y_1, y_2 \in A$ such that $S_1 = [y_1]$ and $S_2 = [y_2]$.
   - 4.3. Line 4.1 tells us $x \in [y_1] \cap [y_2]$.
   - 4.4. So $[y_1] \cap [y_2] \neq \varnothing$.
   - 4.5. Lemma 6.4.4 then implies $S_1 = [y_1] = [y_2] = S_2$. $\square$

Checkpoint

- Reflexivity, symmetry, and transitivity are really *exactly* the properties that the same-component relation with respect to a partition satisfies!
- We have two equivalent ways of representing partitions: using sets, and using equivalence relations.



The Mathematical Intelligencer, Vol. 3(3), page 132, 1981

Next

operations on equivalence classes