NATIONAL UNIVERSITY OF SINGAPORE

CS1231 - DISCRETE STRUCTURES

(Semester 1: AY2018/19)

Time Allowed: 2 Hours

SOLUTIONS

Part A: (30 marks) MCQ. Answer on the OAS form.

For each multiple choice question, choose the best answer and **shade** the corresponding choice on the OAS form. Remember to **shade** and **write** your **Student Number** (check that it is correct!) on the OAS form. Also write your **Tutorial Group** on the form. Each multiple choice question is worth 2 marks. No mark is deducted for wrong answers. You must use a **2B pencil**.

Q1. How many permutations are there for this 8-letter word "ICanDoIt"?

A. 7!

B. (8/2)!

C. 8!

D. 8!/2

E. 8×7

Q2. Which of the following arguments are valid?

(I) No mammals lay eggs.
Duck-billed platypus lays eggs.
Therefore, duck-billed platypus is not a mammal.

(II) You get A grade if and only if you get more than 90 marks.
If you fail your midterm test, you don't get more than 90 marks.
Therefore, if you don't fail your midterm test, you get A grade.

(III) Water is a necessary condition for air.
Water is a sufficient condition for ice.
Therefore air only if ice.

A. (I) only.

B. (I) and (II) only.

C. (I) and (III) only.

D. (II) and (III) only.

E. All of (I), (II) and (III).

Q3. How many integer solutions are there for the following equation, where each $x_i \geq 2$?

$$x_1 + x_2 + x_3 = 20.$$

A. 14

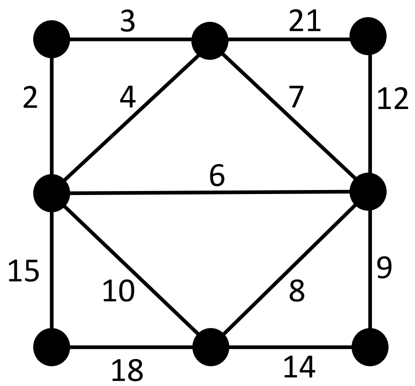
B. 120

C. 230

D. 680

E. None of the above.

Q4. What is the total weight of the minimum spanning tree of the graph shown below?



A. 39

B. 47

C. 50

D. 55

E. 60

Q5. Which of the following statements are TRUE about the graph in Q 4?

A. The graph contains an Euler circuit.

B. The graph contains an Euler trail but not an Euler circuit.

C. The graph does not contain an Euler trail nor an Euler circuit.

D. The graph does not contain a Hamiltonian circuit.

E. None of the above.

Q6. The preorder traversal and inorder traversal of a binary tree with vertices A, B, C, D, E, F, G, H, I, J, K and L are given below:

Preorder:	A	B	D	G	I	J	C	E	H	K	L	F
Inorder:	D	I	G	J	B	A	E	K	H	L	C	F

What is the postorder traveral of this binary tree?

A. A B C D E F G H I J K L

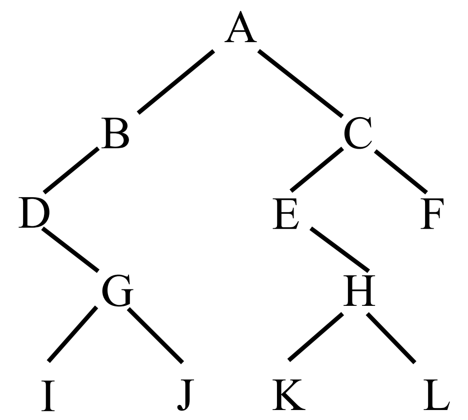
B. L K J I H G F E D C B A

C. I J G D B K L H E C F A

D. I J G D B K L H E F C A

E. None of the above.

Solution: The tree is as shown.



Q7. A sequence of integers u_n starts with $u_0 = 0$, and for all positive integers n, $u_n = u_{n-1} + n$. Determine u_{251}.

Solution: These are the Triangle numbers: $0, 1, 3, 6, 10, ...$
The closed form formula is $u_n = \binom{n+1}{2}$. Hence $u_{251} = \binom{252}{2} = 31626$.

A. 502.

B. 31626.

C. 251!.

D. 2^{251}.

E. None of the above.

The next six questions (Q8 – Q13) refer to the following definitions:

$$\text{Define} \quad U = \mathbb{Z} - \{0\},$$
$$A = \{-1, 0, 1\},$$
$$B = \{0, 1\}.$$

and also define the bitstring $S =$ the set of all non-empty strings over B. That is, S contains strings of the form: $0, 10, 1011, 0011010011, \ldots$, etc. By definition, ε, the empty string, is *not* in S. Further define:

$\mathcal{R}_1 \subseteq U \times U$ such that $\forall x, y \in U\ (x\ \mathcal{R}_1\ y\ \leftrightarrow\ \gcd(x, y) = 1)$.

$\mathcal{R}_2 \subseteq \mathbb{R} \times \mathbb{R}$ such that $\forall x, y \in \mathbb{R}\ (x\ \mathcal{R}_2\ y\ \leftrightarrow\ x^2 = y^2)$.

$\mathcal{R}_3 \subseteq A \times A$ such that $\forall x, y \in A\ (x\ \mathcal{R}_3\ y\ \leftrightarrow\ y \mid x)$.

$\mathcal{R}_4 \subseteq S \times S$ such that $\forall x, y \in S\ (x\ \mathcal{R}_4\ y\ \leftrightarrow\ x, y$ satisfy conditions $T1$ and $T2)$.

$T1$: both x, y have equal length n,

$T2$: if $x = x_1 x_2 \ldots x_n$, and $y = y_1 y_2 \ldots y_n$,
then each x_i is the "opposite letter" of y_i,
where 1 is the opposite of 0, and vice versa.

Examples: $100\ \mathcal{R}_4\ 011$, but $0101\ \not\mathcal{R}_4\ 1111$.

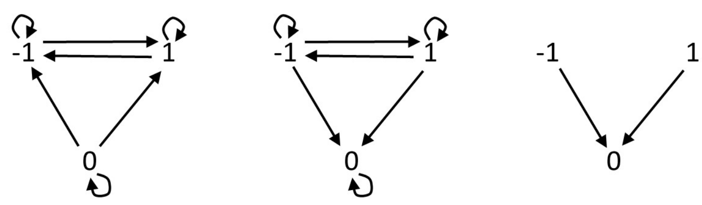
Final definition: given any binary relation $\mathcal{R} \subseteq X \times Y$, define its *relation complement*, denoted $\mathcal{R}^c$, by:

$$\mathcal{R}^c = (X \times Y) - \mathcal{R}.$$

Note that $\mathcal{R}^c \subseteq X \times Y$ is also a binary relation.

Solution: The table below summarizes the properties. Note that $\mathcal{R}_4$ is a bijection: it "flips the bits" of its input string, eg. 1011 becomes 0100.

Relation	Function?	Reflexive?	Symmetric?	Anti-symmetric?	Transitive?
$\mathcal{R}_1$, $\mathcal{R}_1^{-1}$ and $\mathcal{R}_1^c$	×	×	✓	×	×
$\mathcal{R}_2$ and $\mathcal{R}_2^{-1}$	×	✓	✓	×	✓
$\mathcal{R}_3$ and $\mathcal{R}_3^{-1}$	×	✓	×	×	✓
$\mathcal{R}_4$ and $\mathcal{R}_4^{-1}$	✓	×	✓	×	×
$\mathcal{R}_2^c$	×	×	✓	×	×
$\mathcal{R}_3^c$	×	×	×	✓	✓
$\mathcal{R}_4^c$	×	✓	✓	×	×



$\mathcal{R}_3$ $\qquad$ $\mathcal{R}_3^{-1}$ $\qquad$ $\mathcal{R}_3^c$

Q8. If 10 $\mathcal{R}_4$ y, then y could be:

A. 01.

B. 10.

C. 11.

D. 20.

E. 101.

Q9. Recall that the elements of a binary relation are ordered pairs. Which of the following binary relations has $(1, 0)$ as its element?

A. $\mathcal{R}_2^c$**.**

B. $\mathcal{R}_1^{-1}$.

C. $\mathcal{R}_3 \circ \mathcal{R}_3$.

D. $\mathcal{R}_4^c$.

E. All of the above.

Q10. Which of the following are functions?

A. $\mathcal{R}_3^c$.

B. $\mathcal{R}_4^c$.

C. $\mathcal{R}_4$ **and** $\mathcal{R}_4^{-1}$**.**

D. $\mathcal{R}_2^{-1}$ and $\mathcal{R}_3^{-1}$.

E. None of the above.

Q11. Which of the following are transitive?

(I) $\mathcal{R}_1^c$ (II)$\mathcal{R}_2$ (III) $\mathcal{R}_4^c$ (IV) $\mathcal{R}_3^{-1}$.

A. (II) only.

B. (II) and (IV) only.

C. (III) and (IV) only.

D. (I), (II) and (III) only.

E. None of the above.

Q12. Which of the following is a partial order?

A. $\mathcal{R}_3$.

B. $\mathcal{R}_2^c$.

C. $\mathcal{R}_1^{-1}$.

D. $\mathcal{R}_4^c$.

E. None of the above.

Q13. Which relation has the property that its composition with itself equals itself, ie. $\mathcal{R} \circ \mathcal{R} = \mathcal{R}$?

A. $\mathcal{R}_1$.

B. $\mathcal{R}_1^{-1}$.

C. $\mathcal{R}_3^c$.

D. $\mathcal{R}_3^{-1}$.

E. All of the above.

Q14. Let $A = \{1, 2, 3\}$, $D = \{1, \{1\}, 2, \{3, 2\}, \{\varnothing\}\}$, and $E = \{1, 2\}$. Which of the following statements are TRUE?

(I) $A \subseteq D$.

(II) $\mathcal{P}(E) \subseteq D$.

(III) $3 \in A - E$.

(IV) $\mathcal{P}(E) \cap D = \{\{1\}\}$.

A. (II) only.

B. (IV) only.

C. (III) and (IV) only.

D. (I), (II) and (III) only.

E. All of (I), (II), (III) and (IV).

Q15. Let $f : \mathbb{Z} \to \mathbb{Q}$ be defined by $f(z) = z/2$, and let $g : \mathbb{Q} \to \mathbb{R}$ be defined by $g(q) = \pi \cdot q$. Which of the following statements are FALSE?

(I) f is onto.

(II) $g \circ f$ is one-to-one.

(III) $(g \circ f)(6) = 3\pi$.

(IV) g^{-1} is not a function.

A. (I) only.

B. (II) only.

C. (IV) only.

D. (I) and (IV) only.

E. None of (I), (II), (III) and (IV).

Part B: (40 marks) Structured questions. Write your answer in the Answer Sheet.

Marks may be deducted for illegible handwriting and unnecessary statements in proofs.

Q16. Counting and Probability (14 marks)

Answer the following parts. Working is not required.

(a) (2 marks) What is the probability that on three rolls of a fair six-sided die, at least one 6 shows up? Leave your answer as a fraction or marks will be deducted.

> **Solution:** Let p be the probability that 6 shows up on a roll of a die.
> Then $p = 1/6$.
> P(6 does not show up on all three rolls) $= 5/6 \times 5/6 \times 5/6 = 125/216$.
> Therefore, P(at least one 6 shows up on three rolls) $= 1 - 125/216 = 91/216$.

(b) (2 marks) Figure 1 below shows a combination lock with 40 positions.

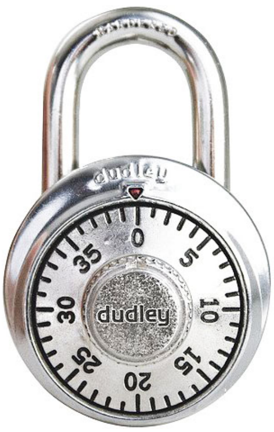


Figure 1: Combination Lock.

To open the lock, you rotate to a number in a clockwise direction, then to a second number in the counterclockwise direction, and finally to a third number in the clockwise direction. If consecutive numbers in the combination cannot be the same, how many combinations of three-number codes are there?

> **Solution:** $40 \times 39 \times 39 = 60840$

(c) (2 marks) There are 12 slips of paper in a bag. Some of the slips have a 2 written on them, and the rest have a 7 written on them. If the expected value of the number shown on a slip randomly drawn from the bag is 3.25, how many slips have a 2 written on them?

> **Solution:** Let x be the number of slips with a 2 written on them. Then there are $12 - x$ slips with a 7 on them.
> Expected value $= \frac{x}{12} \cdot 2 + \frac{12-x}{12} \cdot 7 = 3.25$
> Hence, $x = 9$.

(d) A bowl contains three coins. Two of them are normal coins and one of them is a two-headed coin.

i. (2 marks) You pick one coin at random and toss it. What is the probability that you get a head? Write your answer as a fraction.

> **Solution:** Let C_n be the event that you pick a normal coin and C_f be the event that you pick the two-headed coin. $P(H|C_n) = 0.5$ and $P(H|C_f) = 1$.
> $P(H) = P(H|C_n) \cdot P(C_n) + P(H|C_f) \cdot P(C_f) = \frac{1}{2} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{2}{3}$

ii. (2 marks) You pick one coin at random, toss it and get a head. What is the probability that the coin is the two-headed coin? Write your answer as a fraction.

> **Solution:** Using Bayes' rule, $P(C_f|H) = \frac{P(H|C_f) \cdot P(C_f)}{P(H)} = \frac{1 \cdot \frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$.

(e) (4 marks) A row of houses are randomly assigned distinct numbers between 1 and 50 inclusive. What is the minimum number of houses to ensure that there are 5 houses numbered consecutively?

To receive full credit, you must define the pigeon and pigeonholes.

> **Solution:** Split the numbers into 10 pigeon-holes: 1-5, 6-10, 11-15, ..., 46-50. Therefore, there must be at least $10 \times 4 + 1 = 41$ houses (pigeons).

Q17. Graphs (14 marks)

The *lazy caterer's sequence* describes the number of maximum pieces of a pancake (or pizza) that can be made with a given number of straight cuts.

For example, with three straight cuts, you get seven pieces as shown in Figure 2 below.

Figure 2: Pancake (Photo credit: Wikipedia).

Figure 3 below shows the first few values in the lazy caterer's sequence starting with $n = 0$ where n is the number of straight cuts. The sequence is 1, 2, 4, 7, 11, 16, ...

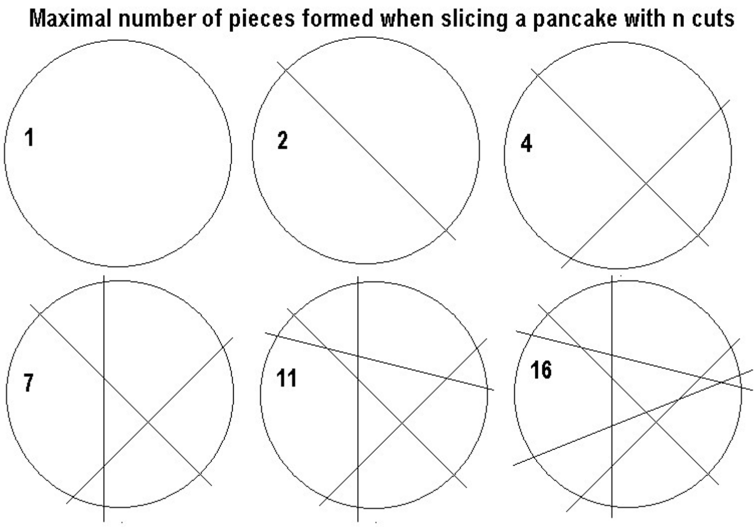


Figure 3: Lazy Caterer Sequence.

We may model this problem by using a graph. Figure 4 shows the graph corresponding to $n = 3$, where the vertices are the intersections among the cuts and the boundary of the pancake.

We may define the following functions:

$P(n)$: number of pieces of pancakes with n cuts

$V(n)$: number of vertices of a graph corresponding to a pancake with n cuts

$E(n)$: number of edges of a graph corresponding to a pancake with n cuts

In Figure 4, $P(3) = 7, V(3) = 9$ and $E(3) = 15$.

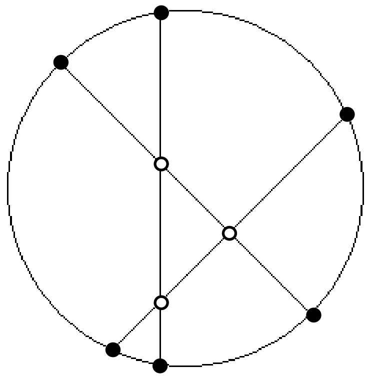


Figure 4: Graph representation.

The vertices of the graph are of two types: those with degree of three (which lie on the boundary of the circle, shown as black dots) and those with degree of four (which lie inside the circle, shown as white dots). Let's define two more functions:

$V_3(n)$: number of vertices with degree three

$V_4(n)$: number of vertices with degree four

In Figure 4, $V_3(3) = 6$ and $V_4(3) = 3$.

Answer the following parts. Working is not required.

Solution: The following summarizes the results for reference.

n	0	1	2	3	4	5	6	7
$P(n)$	1	2	4	7	11	16	22	29
$E(n)$	0	3	8	15	24	35	48	63
$V(n)$	0	2	5	9	14	20	27	35
$V_3(n)$	0	2	4	6	8	10	12	14
$V_4(n)$	0	0	1	3	6	10	15	21

$V_3(n) = 2n$ (every cut creates two new vertices on the boundary)

$V_4(n) = V_4(n-1) + n - 1$ (the n^{th} cut intersects the existing $n-1$ cuts)
$V_4(n) = n(n-1)/2$ for $n > 0$

$V(n) = V(n-1) + n + 1$ (the n^{th} cut cuts through the existing $n-1$ cuts and it also cuts the boundary at two places)
$V(n) = V_3(n) + V_4(n) = 2n + n(n-1)/2 = (n^2 + 3n)/2$

$E(n) = E(n-1) + 2n + 1$ (the n^{th} cut adds $n+1$ vertices as it cuts through the existing $n-1$ cuts and opposite sides of the circle, splitting each of the $n+1$ edges it cuts through into two, and also introducing n new edges, hence a total of $2n+1$ new edges)
$E(n) = (3V_3(n) + 4V_4(n))/2$ (sum of degrees in a graph is twice the number of edges)
$= (3(2n) + 4n(n-1)/2)/2 = n^2 + 2n$

$P(n) = E(n) - V(n) + 1$ (Using Euler's formula $v - e + f = 2$, where $f = P(n) + 1$ since the region outside the circle constitutes one face)
$P(n) = \binom{n+1}{2} + 1 = \binom{n}{2} + \binom{n}{1} + \binom{n}{0}$
$P(n) = P(n-1) + n$
All the above give the closed form formula $P(n) = (n^2 + n + 2)/2$.

(a) (2 marks) Express $E(n)$ in terms of $V_3(n)$ and $V_4(n)$.

Solution: Since the total sum of degrees in a graph is twice the number of edges, we have $E(n) = (3V_3(n) + 4V_4(n))/2$.

(b) (2 marks) Write the recurrence relation for $V(n)$. The base case is $V(0) = 0$.

Solution: $V(n) = V(n-1) + n + 1$, for $n > 0$.

(c) (2 marks) Write the closed form formula for $V(n)$.

Solution: $V(n) = (n^2 + 3n)/2$, for $n \geq 0$.

(d) (2 marks) Write the recurrence relation for $E(n)$. The base case is $E(0) = 0$.

Solution: $E(n) = E(n-1) + 2n + 1$, for $n > 0$.

(e) (2 marks) Write the closed form formula for $E(n)$.

Solution: $E(n) = n^2 + 2n$, for $n \geq 0$.

(f) (2 marks) Euler's formula is given as $v - e + f = 2$. Relate v, e and f with the functions defined in this question.

Solution: $v = V(n)$; $e = E(n)$, and $f = P(n) + 1$ (region outside the circle constitutes one face).

(g) (2 marks) From part (f), or otherwise, derive the closed form formula for $P(n)$.

Solution: $P(n) = (n^2 + n + 2)/2$, for $n \geq 0$.

Q18. Functions (12 marks)

Private cars in Singapore have license plates (see Figure 5) in the format: $S\alpha_1\alpha_2\ x_1x_2x_3x_4\ c$, where each α_1 and α_2 is a single letter taken from the usual English alphabet (excluding I and O), and each $x_1, \ldots, x_4$ is a single digit taken from $\{0, 1, \ldots, 9\}$. The last letter c is a checksum letter, ie. a function of the preceding letters and numbers. Its purpose is to serve as a quick check on the validity of the license plate.

Figure 5: A typical Singapore car license plate. (Photo Credit: Wikipedia)

Let $\mathcal{L}$ denote the set of all possible strings of the form: $\alpha_1\alpha_2x_1x_2x_3x_4$. The possible values of α_i and x_j are as described above. Also, let $\mathcal{K} = \{A, Z, Y, X, U, T, S, R, P, M, L, K, J, H, G, E, D, C, B\}$. Then the checksum function may be defined as $f : \mathcal{L} \to \mathcal{K}$, where $f(\alpha_1\alpha_2x_1x_2x_3x_4)$ is calculated in three steps:

F1. Let n_1 be the positional value of α_1 in the English alphabet, ie. $A = 1,\ B = 2,\ C = 3, \ldots,\ Z = 26$. And let n_2 be the positional value of α_2. (Note that since I and O are not allowed, $n_1, n_2 \notin \{9, 15\}$.)

F2. Compute $t = 9n_1 + 4n_2 + 5x_1 + 4x_2 + 3x_3 + 2x_4$, and $r = t\%19$. That is, r is the remainder of t modulo 19, which means $0 \le r < 19$.

F3. The checksum letter $c =$ the letter in $\mathcal{K}$ indexed by r, where $0 = A, 1 = Z, 2 = Y, \ldots, 18 = B$. (Here, we are treating $\mathcal{K}$ as an ordered set, in which its elements are indexed by position, starting from 0.)

Using the example in Figure 5:

F1. $\alpha_1 = D,\ \alpha_2 = N,\ x_1 = 7,\ x_2 = 4,\ x_3 = 8,\ x_4 = 4$; and so $n_1 = 4,\ n_2 = 14$.

F2. Then $t = 9 \cdot 4 + 4 \cdot 14 + 5 \cdot 7 + 4 \cdot 4 + 3 \cdot 8 + 2 \cdot 4 = 175$, and so $r = 175\%19 = 4$.

F3. Hence $c = U$.

(a) (2 marks) Determine the checksum letter for $CS1231$. (No working needed.)

Solution: Checksum letter $= H$.

(b) (2 marks) Show that f is not one-to-one by finding another $y \in \mathcal{L}$ such that $f(y)$ is the same checksum letter as that in Figure 5. (No working needed. Just state a suitable y.)

Solution: Many answers are possible. One easy way is to note that the weights of n_2 and x_2 are the same, which allows us to increase n_2 by the same amount as we decrease x_2, since that yields the same total t. Thus, $y = DP7284$ is a possible answer.

(c) (8 marks) (Difficult) Is f onto? Prove or disprove.

Solution: f is onto. Construct Table 1, which shows the values of a, b such that $3a + 2b \equiv r \pmod{19}$, for $r = 0, 1, 2, \ldots, 18$. Note that this table is a bijection between a, b and r.

$a \setminus b$	0	1	2	3	4
0	0	2	4		
1	3	5	7		
2	6	8	10	12	
3	9	11	13	15	17
4		14	16	18	1

Table 1: a indexes the row, b indexes the column.

Proof (by Construction).

1. Take any $w \in \mathcal{K}$.
2. From step F3, there is a unique r which is the positional index of w. Note that $0 \leq r < 19$.
3. (We will construct a string: $y = BE00ab$, where a, b will be derived from r, such that $f(y) = w$.)
4. Given the r from Line 2., lookup Table 1 for the unique pair a, b.
5. Let $y = BE00ab$. Clearly, $y \in \mathcal{L}$.
6. Now, $f(y)$ may be calculated as follows:
   - 6.1. From step F2: $t = 9 \cdot 2 + 4 \cdot 5 + 5 \cdot 0 + 4 \cdot 0 + 3a + 2b$.
   - 6.2. $\therefore\ t = 38 + 3a + 2b$.
   - 6.3. The remainder s is such that $s \equiv t \pmod{19}$.
   - 6.4. Thus $s \equiv 38 + 3a + 2b \pmod{19}$.
   - 6.5. Thus $s \equiv r \pmod{19}$, because $3a + 2b \equiv r \pmod{19}$.
   - 6.6. Thus $s = r$, which means $f(y) = w$.
7. Thus $\exists y \in \mathcal{L}$ such that $f(y) = w$.
8. Hence, f is onto. ■

END OF PAPER