Lecture Notes on Laplace Transforms-II

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Theorem (Linearity property): Let $f(t)$ and $g(t)$ functions whose Laplace Transform exists and α, β are any arbitrary constants then Laplace Transform of $\alpha f(t) + \beta g(t)$ exists and we have

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\}$$

Proof: We will prove this theorem for a piecewise continuous, growth restricted functions.
Given: Laplace Transforms of $f(t)$ and $g(t)$ exists, so we have

$$\alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\} = \alpha \int_0^\infty e^{-st} f(t)\, dt + \beta \int_0^\infty e^{-st} g(t)\, dt = \int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]\, dt \tag{1}$$

Now question is why this integral $\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]\, dt$ converges (finite) ? For the existence of this integral, first we will prove that $\alpha f(t) + \beta g(t)$ are piecewise continuous and growth restricted function. Clearly $\alpha f(t)+\beta g(t)$ is piecewise continuous function because we assumed that $f(t)$ and $g(t)$ are piecewise continuous functions. As $f(t)$ and $g(t)$ are growth restricted functions so $\exists\, k_1, k_2, M_1, M_2$ such that $|f(t)| \leq M_1\, e^{k_1 t}$ for $t > t_1$ and $|g(t)| \leq M_2\, e^{k_2 t}$ for $t > t_2$. So consider $|\alpha f(t) + \beta g(t)|$

$$|\alpha f(t)+\beta g(t)| \leq |\alpha||f(t)|+|\beta||g(t)| \implies |\alpha f(t)+\beta g(t)| \leq (|\alpha|M_1+|\beta|M_2)e^{\alpha t} \text{ where } \alpha = \max\{\alpha_1, \alpha_2\}$$

Hence $|\alpha f(t) + \beta g(t)| \leq (|\alpha|M_1 + |\beta|M_2)e^{\alpha t}$ for $t >$ $\max\{t_1, t_2\}$. Therefore by existence theorem gives us gurantee of the integral $\int_0^\infty e^{-st}[\alpha f(t)+\beta g(t)]\, dt$ exists and this integral is $\mathscr{L}\{\alpha f(t)+\beta g(t)\}$. Hence by (1) $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\}$ this proves the theorem.

Laplace Transforms of some elementry functions

1. $f(t) = 1$ for $t \geq 0$ so $\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\, dt = \dfrac{1}{s}$, when $s > 0$

2. $f(t) = t^\alpha$ where $\alpha > -1$ so $\mathscr{L}(t^\alpha) = \int_0^\infty e^{-st} t^\alpha\, dt$ For solving this integral we substitute $st = x$ which gives us $t = \dfrac{x}{s}$ and $dt = \dfrac{dx}{s}$

$$\mathscr{L}(t^\alpha) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^\alpha \frac{dx}{s} \implies \mathscr{L}(t^\alpha) = \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-x} x^\alpha\, dx \implies \mathscr{L}(t^\alpha) = \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-x} x^{(\alpha+1)-1}\, dx$$

We know gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ defined as $\Gamma(\alpha) = \int\limits_0^\infty e^{-x} x^{\alpha-1}\, dx$, for $\alpha > 0$ hence

$$\mathscr{L}(t^\alpha) = \frac{1}{s^{\alpha+1}} \int\limits_0^\infty e^{-x} x^{(\alpha+1)-1}\, dx \implies \mathscr{L}(t^\alpha) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \quad \text{when } s > 0$$

We also know following properties of gamma function. (They are useful for solving examples)

(i) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ (ii) $\Gamma(n+1) = n!$ (iii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

So we get $\mathscr{L}(t^n) = \dfrac{n!}{s^{\alpha+1}}$ for $s > 0$. where $n \in \mathbb{N}$. This result we can prove by using mathematical Induction also!

3. Laplace Transform of Exponential Function: $f(t) = e^{at}$ for $t \geq 0$ where a is constant

$$\mathscr{L}(e^{at}) = \int\limits_0^\infty e^{-st} e^{at}\, dt \implies \mathscr{L}(e^{at}) = \int\limits_0^\infty e^{-t(s-a)}\, dt \implies \mathscr{L}(e^{at}) = \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty \quad \text{when } s-a > 0$$

Hence $\mathscr{L}(e^{at}) = \dfrac{1}{s-a}$ when $s - a > 0$. Prove that $\mathscr{L}(e^{-at}) = \dfrac{1}{s+a}$ when $s + a > 0$.

4. Laplace Transform of Hyperbolic Function.

We know that hyperbolic sine and cosine function can be rewritten as the linear combination of e^{at} and e^{-at} as $\cos h(at) = \dfrac{e^{at} + e^{-at}}{2}$, and $\sin h(at) = \dfrac{e^{at} - e^{-at}}{2}$. So by using linearity property of Laplace transform we can find the Laplace Transform of hyperbolic sine and cosine function.

$$\mathscr{L}(\cos h(at)) = \mathscr{L}\left[\frac{1}{2}\left(e^{at} + e^{-at}\right)\right] \implies \mathscr{L}(\cos h(at)) = \frac{1}{2}\mathscr{L}(e^{at}) + \frac{1}{2}\mathscr{L}(e^{-at})$$

$$\mathscr{L}(\cos h(at)) = \frac{1/2}{s-a} + \frac{1/2}{s+a} \text{ when } s > a \text{ and } s > -a \implies \mathscr{L}(\cos h(at)) = \frac{s}{s^2 - a^2} \text{ when } s > |a|$$

$$\mathscr{L}(\sin h(at)) = \mathscr{L}\left[\frac{1}{2}\left(e^{at} - e^{-at}\right)\right] \implies \mathscr{L}(\sin h(at)) = \frac{1}{2}\mathscr{L}(e^{at}) - \frac{1}{2}\mathscr{L}L(e^{-at})$$

$$\mathscr{L}(\sin h(at)) = \frac{1/2}{s-a} - \frac{1/2}{s+a} \text{ when } s > a \text{ and } s > -a \implies \mathscr{L}(\sin h(at)) = \frac{a}{s^2 - a^2} \text{ when } s > |a|$$

5. Laplace Transform of Trigonometric Functions: $\sin t$ and $\cos t$ are nice functions (nice in the sense that they are continuous, differentiable, bounded and periodic functions) so we can easily check Laplace transform of sine and cosine functions exists (why?). To find the Laplace Transforms of sine and cosine functions we use following results

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) \text{ and } \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

If you don't know these result you can prove these results by using integration by parts.

$$\mathscr{L}(\sin(\omega t)) = \int\limits_0^\infty e^{-st} \sin \omega t\, dt \implies \mathscr{L}(\sin \omega t) = \left[\frac{e^{-st}(-s \sin \omega t - w\omega \cos \omega t)}{s^2 + \omega^2}\right]_0^\infty = \frac{\omega}{s^2 + \omega^2} \text{ when } s > 0$$

$$\mathscr{L}(\cos\omega t) = \int_0^\infty e^{-st}\cos\omega t\,dt \implies \mathscr{L}(\cos\omega t) = \left[\frac{e^{-st}(-s\cos\omega t + \omega\sin\omega t)}{s^2+\omega^2}\right]_0^\infty = \frac{s}{s^2+\omega^2} \text{ when } s>0$$

Let's summerize all the results here:

1. $\mathscr{L}\{1\} = \dfrac{1}{s}$, when $s > 0$
2. $\mathscr{L}\{t\} = \dfrac{1}{s^2}$; $n \in \mathbb{N}$, when $s > 0$
3. $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$; $n \in \mathbb{N}$, when $s > 0$
4. $\mathscr{L}\{t^\alpha\} = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$; $s > 0$
5. $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$, when $s - a > 0$
6. $\mathscr{L}\{e^{-at}\} = \dfrac{1}{s+a}$, when $s + a > 0$
7. $\mathscr{L}\{sinat\} = \dfrac{a}{s^2+a^2}$, when $s > 0$
8. $\mathscr{L}\{cosat\} = \dfrac{s}{s^2+a^2}$, when $s > 0$
9. $\mathscr{L}\{sinhat\} = \dfrac{a}{s^2-a^2}$, when $s > |a|$
10. $\mathscr{L}\{coshat\} = \dfrac{s}{s^2-a^2}$, when $s > |a|$

Note: Unfortunately, Laplace Transforms of the other trigonometric functions such as tan, cot, cosec and sec does not exist! (why?)

Uniqueness of Laplace Transforms:

Laplace Transform if it exists, is unique. The question is whether two distinct functions can have the same Laplace Transform? Answer of this question is yes! For example, $f(t) = \sin t$ and $g(t) = \begin{cases} \sin t & \text{for} \quad t \neq \pi \\ 1 & \text{for} \quad t = \pi \end{cases}$ have the same Laplace Transform. Observe that they differ only at $t = \pi$. Also note that function $h(t) = \begin{cases} \sin t & \text{for} \quad t \neq n\pi \\ 1 & \text{for} \quad t = n\pi \end{cases}$, $g(t)$ and $f(t)$ have the same Laplace Transform.

Theorem: Let $f(t)$ and $g(t)$ be two piecewise continuous functions of exponential order. If there exists a constant s_0, such that $\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t)\}$ for all $s > s_0$ then $f(t) = g(t)$ for all $t > 0$ except possibly at the points of discontinuity.

In short, if two piecewise continuous functions of exponential order have the same Laplace Transforms then they must be equal almost everywhere.

Another question is whether two continuous and growth restricted function have the same Laplace Transforms? Answer of this question is NO.

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If you found any mistake(s) please report me at `dng.maths@coep.ac.in` ■