Lecture Notes on Laplace Transforms-I

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Integral Transform: One class of transformation which are called integral transform may be defined as

$$T\{f(t)\} = \int_a^b k(s,t)f(t)\,dt$$

where $k(s,t)$ is a function of two variables s and t. $k(s,t)$ is called kernel of the transformation. Laplace Transform belongs to the family of Integral Transform.

Defintion (Laplace Transform): Let $f(t)$ be a function defined for all $t \geq 0$, its Laplace Transform is defined as follows

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt \qquad \text{for those } s \text{ for which the integral exists} \tag{1}$$

The notation $\mathscr{L}\{f(t)\}$ is used to denote the Laplace Transform of the function $f(t)$ (The letter $\mathscr{L}$ can be read as "The Laplace Transform of" in above expression). We also use $F(s)$ to denote Laplace Transform of the function $f(t)$. Note that original functions depend on t and their Laplace Transforms is depend on s. Also note that original functions are denoted by lowercase letters and their Laplace Transforms are denoted by same letters in uppercase.

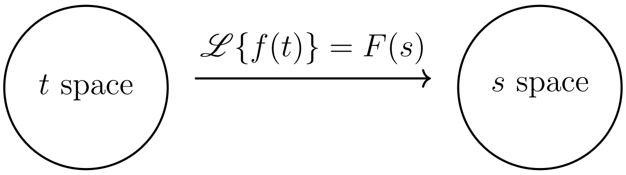


Figure-1: Laplace Transforms as a mapping

Comment-I: Laplace Transform is defined for complex valued function $f(t)$ also and the parameter s can also be complex. But we restrict our discussion only for the case in which $f(t)$ is real valued function and s is real.

Comment-II: Since the integral (1) is an improper integral, existance of Laplace Transform means that the following limit exists

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \lim_{R\to\infty}\int_0^R e^{-st}f(t)\,dt.$$

Comment-III: In this course we take Laplace transform of function $f(t)$ which is defined for $t \geq 0$ as $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$ **for those s for which the integral exists.** Note that this Laplace transform is also known as **unilateral Laplace transform or one-sided Laplace transform**. So in this course **we say Laplace transform of function $f(t)$ then it is understood that this is unilateral Laplace transform of function $f(t)$ which is defined for** $t \geq 0$. In the literature of mathematics there is Bilateral Laplace transform (also known as two-sided Laplace transform) of function $g(t)$ which is defined for $t \in \mathbb{R}$ is defined as $\int_{-\infty}^{\infty} e^{-st} g(t)\, dt$. Laplace transform is mapping from t space to s space, usually t is time (and time is never nagative) so Bilateral Laplace transform is not useful partically.

Motivation: Differential equations are language of nature and finding solution(s) (if it exists!) of differential equation is one of the diffiuclt problem in mathematics. Laplace Transform method is powerful method of solving ODEs. The Laplace transform transform a linear differential equation with constant coefficients (along with appropriate initial conditions are given) transformed into algebraic equation in s. If we able to obtain the Inverse Laplace Transform, we may have solution of the differential equation. Consider the second order ordinary differential equation $y'' + ay' + by = r(x)$ with initial conditions $y(0) = \alpha$ and $y'(0) = \beta$. Here $r(t)$ is the given input (driving force) applied to the mechanical or electrical system) and $y(t)$ is the output (responce of the input). To obtain the solution by Laplace's method we do following steps

1. The given ODE is transformed into an algebric equation (by taking Laplace Transforms)
2. Use linearity property of Laplace transform, use initial conditions and do algebric manipulation
3. Perform an inverse Laplace transform to give us the solution of given ODE.

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{ODEs}\\ \text{(IVP)}\end{array}} \arrow[r, "\mathscr{L}\{y''+ay'+by=r(t)\}"] \arrow[d, "\text{Classical Method}"'] & \boxed{\begin{array}{c}\text{use initial}\\ \text{conditions}\end{array}} \arrow[d, "\text{Do algebric manipulation}"] \\
\boxed{y(t)} & \boxed{Y(s)} \arrow[l, "\mathscr{L}^{-1}\{Y(s)\}"']
\end{tikzcd}
$$

Figure-2: Solution of Ordinary Differential Equation by using Laplace Transform

Advantages of this method over the classical methods that were discussed in first unit is no need to calculate complementry function and particular solution separately and it also work effectively even if $r(t)$ is piecewise continuous function, in such cases in classical methods we need to find separate particular function for each of continuous functions.

We define the concept of piecewise continuous function and growth restriction (also growth of exponential order). Then we can get sufficient condition for the existence of Laplace Transform.

Defintion (Piecewise Continuous Function): A function $f(t)$ is peicewise continuous on the interval $[a, b]$ if the interval $[a, b]$ can be broken into a finite number of subinternals $[a, t_1],\ [t_1, t_2],\ [t_2, t_3], \cdots$ $[t_n, b]$ with $a < t_1 < t_2 < \cdots < t_n < b$ such that $f(t)$ is continuous in each of subintervals but not necessarily at the end points of the subintervals and function $f(t)$ has finite limits as t approaches either end points of such a subinterval from the interior. **Intuitively piecewise contionuous if it consists of a finite number of continuous pieces with "finite discontinities" between them. Note that function need not be defined at the endpoints of the pieces only the limit with in the interval needs to exist and be finite!** Thus we call discontinuity of a function $f(t)$ at a point a is finite if both $\lim\limits_{t\to a^-} f(t)$ and $\lim\limits_{t\to a^+} f(t)$ are finite. Function is piecewise continuous on $[0, \infty)$ if it is piecewise continuous in $[0, r]$ for all $r > 0$. **Also note that continuous function is always piecewise continuous function but not conversely.**

Example: $f(t) = \cos t$ is piecewise continuous function since it is continuous function.

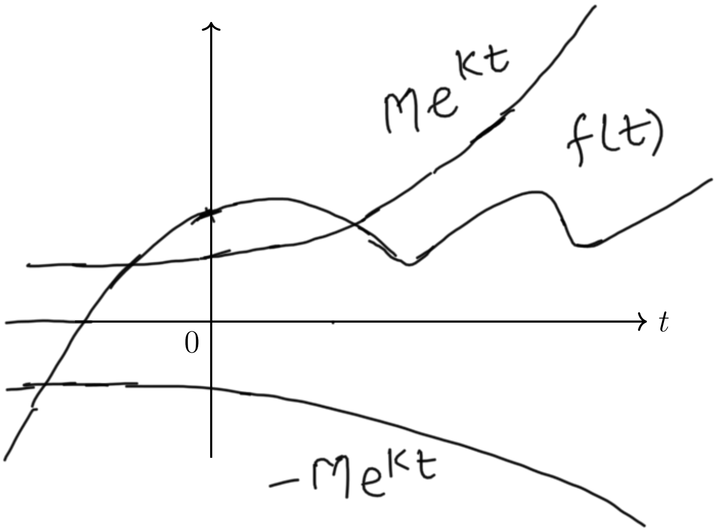
Example: Function $f(t) = \begin{cases} 1/t & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t = 0 \end{cases}$ is NOT piecewise continuous on $(0, \infty)$ since $f(t) \to \infty$ as $t \to 0$ (since this function has infinite jump at $t = 0$)

Example: Function $f(t) = \begin{cases} \sin t & \text{if} \quad 0 \le t \le \pi \\ 0 & \text{if} \quad \pi < t \le 2020 \\ 1 & \text{if} \quad t > 2020 \end{cases}$ is piecewise continuous function.

Defintion (Growth Restriction): We say that $f(t)$ is Growth Restricted function of exponential order if there exist $M, k \in \mathbb{R}$ such that $|f(t)| \le Me^{kt} \quad \forall \quad t \ge t_0$. Observe that $|f(t)| \le Me^{kt}$ for sufficiently large t, which is equivalent to $\lim\limits_{t\to\infty} e^{-kt} f(t) = C$ (finite quantity) why? because of $|f(t)| \le Me^{kt} \ \ \forall \ \ t \ge t_0 \implies -Me^{kt} \le f(t) \le Me^{kt} \ \ \forall \ \ t \ge t_0 \implies -M \le f(t)e^{-kt} \le M \ \ \forall \ \ t \ge t_0$ $-M \le f(t)e^{-kt} \le M \ \ \forall \ \ t \ge t_0 \implies -M \le \lim\limits_{t\to\infty} f(t)e^{-kt} \le M \ \ \forall \ \ t \ge t_0 \implies \lim\limits_{t\to\infty} f(t)e^{-kt} = C$ (finite quantity). Note that we have used squeeze (or sandwich) theorem to conclude (where?) (what are the squeeze (or sandwich) theorem? squeeze (or sandwich) theorem say that Let $D \subseteq \mathbb{R}$, $f, g, h : D \to \mathbb{R}$ such that $f(x) \le g(x) \le h(x)$ for all $x \in D$ and $\lim\limits_{x\to a} f(x) = l = \lim\limits_{x\to a} h(x)$ where $a \in D$ Then we also have $\lim\limits_{x\to a} g(x) = l$).

Example: All bounded functions are of exponential order. So $f(t) = \sin t,\ g(t) = \cos t,\ h(t) = 2020$ are the examples of functions of exponential order.

Example: Show that $f(t) = e^{t^2}$ is not of exponential order since $\lim\limits_{t\to\infty} e^{t^2} e^{-kt} = \lim\limits_{t\to\infty} e^{t^2 - kt} = \infty$. We can say that growth rate of e^{t^2} is more rapid than exponential function.



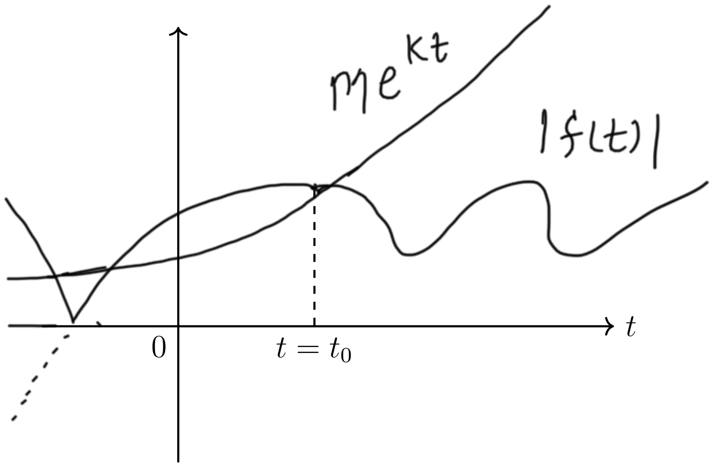


Figure-3: Geometric Interpretation of Growth Restriction

Example: Function $f(t) = t^2$ is of exponential order. This is clear from the fact that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \geq \frac{t^n}{n!}$ which implies that $t^n \leq n!e^t \implies |t^n| \leq n!e^t$. Moreover any polynomial $P(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ are of exponential order as

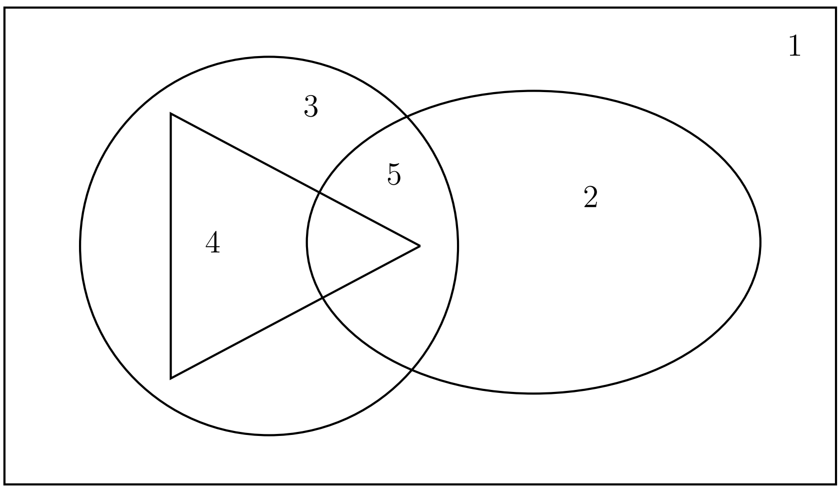
$$|P(t)| = |a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n| \leq |a_0| + |a_1 t| + |a_2 t^2| + \cdots + |a_n t^n| = |a_0| + |a_1||t| + |a_2||t^2| + \cdots + |a_n||t^n|$$

Take $a = \max\{|a_0|, |a_1|, \cdots, |a_n|\}$ so we get

$$|P(t)| \leq a(1 + |t| + |t^2| + \cdots + |t^n|) \leq a(|t^n| + |t^n| + |t^n| + \cdots + |t^n|) \implies |P(t)| \leq a(n+1)|t^n|$$

$$|P(t)| \leq a(n+1)|t^n| \implies |P(t)| \leq a(n+1)n!e^t = a(n+1)!e^t \implies |P(t)| \leq a(n+1)!e^t$$

Exercise : Show that $f(t) = e^{-t^2}$ is a function of exponential order.



1. Set of all functions
2. Functions of exponential order
3. Piecewise continuous functions
4. Continuous functions
5. Piecewise continuous function of exponential order

Venn Diagram-1: Relation between function of exponential order and piecewise continuous functions

Theorem-1 (Existence Theorem for Laplace Transform): If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and satisfies Growth Restriction then the Laplace Transform $\mathscr{L}\{f(t)\}$ exists for all $s > k$, where k is real number that depends on $f(t)$.

Proof: Given that $f(t)$ is piecewise continuous so $f(t)$ is integrable over any finite interval on the t-axis. Also give that $f(t)$ Growth Restricted function so $|f(t)| \leq Me^{kt}$ for sufficiently large t.

$$\left|\int_0^\infty e^{-st} f(t)\, dt\right| \leq \int_0^\infty |e^{-st} f(t)|\, dt \implies |\mathscr{L}\{f(t)\}| \leq \int_0^\infty |e^{-st}||f(t)|\, dt \implies |\mathscr{L}\{f(t)\}| \leq \int_0^\infty |e^{-st}| M e^{kt}\, dt$$

$$|\mathscr{L}\{f(t)\}| \leq \int_0^\infty |e^{-st}| M e^{kt}\, dt \implies |\mathscr{L}\{f(t)\}| \leq M \frac{e^{-(s-k)t}}{-(s-k)}\bigg|_0^\infty \implies |\mathscr{L}\{f(t)\}| \leq \frac{M}{s-k} \text{ ; as } s > k$$

Hence $\mathscr{L}\{f(t)\}$ exists for $s - k > 0$.

Now we are happy, why? because now atleast we know that functions $f(t) = 2020$, $g(t) = \sin t$, $h(t) = e^{-t^2}$, $p(t) = 2t^2 + 5t + 9$ have Laplace transform, how to find Laplace transform is different question. Also note sometimes finding the Laplace transform is vary difficult since definition of Laplace transform contain improper integral.

Comment-IV: Theorem 1 gives us **sufficient condition** for the existance of Laplace transform. If conditions given in theorem-1 are satisfied then Laplace Transform must exist. If any of these there conditions are not satisfied then Laplace Transform may or may not exist (That is both these conditions are not necessary). For example consider the following function $f(t) = \dfrac{1}{\sqrt{t}}$ which is NOT piecewise continuous in $(0, \infty)$ (why?) and is of exponential order (why? think geometrically) but $\mathscr{L}\{f(t)\} = \displaystyle\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\, dt = \frac{1}{\sqrt{s}} \int_0^\infty e^{-u} u^{\frac{1}{2}-1}\, du = \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$ for $s > 0$ ($st = u$ substitution used in this integral). Alternative way to find Laplace Transform of function $f(t)$ is use substitution $st = w^2$

$$\mathscr{L}\{f(t)\} = \int_0^\infty \frac{e^{-st}}{\sqrt{t}}\, dt = \int_0^\infty \frac{e^{-w^2}}{2s} \frac{\sqrt{s}}{w} w\, dw \implies \mathscr{L}\{f(t)\} = \frac{2}{\sqrt{s}} \int_0^\infty e^{-w^2}\, dw \implies \mathscr{L}\{f(t)\} = \sqrt{\frac{\pi}{s}},\ s > 0.$$

Exercise : Does every function have a Laplace Transform ?

Exercise : Find a function which is not of exponential order and its Laplace Transform exists.

Exercise : Find a function which is not of exponential order and its Laplace Transform doesn't exists.

If you found any mistake(s) please report me at `dng.maths@coep.ac.in` ■