Hint and Solution for Selected Exercises

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Conventions: In this course we take Laplace transform of function $f(t)$ which is defined for $t \geq 0$ as $\mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$ **for those s for which the integral exists.** Note that this Laplace transform is also known as unilateral Laplace transform or one-sided Laplace transform. So in this course **we say Laplace transform of function $f(t)$ then it is understood that this unilateral Laplace transform of function $f(t)$ which is defined for** $t \geq 0$. In the mathematics literature there is Bilateral Laplace transform (also known as two-sided Laplace transform) of function $g(t)$ which is defined for $t \in \mathbb{R}$ is defined as $\int_{-\infty}^{\infty} e^{-st} g(t)\,dt$. **Also note that$\mathscr{L}^{-1}\{F(s)\} = f(t)$ means by default it is understood function $f(t)$ is defined for $t \geq 0$.** For example $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\} = \cos(t)$ **(This means function $\cos(t)$ defined for $t \geq 0$)**

Note: Capital letters in bracket is the choice for the correct answer.

1 $\mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$ (B)

2 Observe that $\lim_{t\to\infty} e^{(t^2-kt)}$ is not finite (B)

3 $|\tan(t)| \to \infty$ as $t \to (2n+1)\pi/2$ where $n \in \mathbb{N}$
Hence $\tan(t)$ is not of exponential order. (D)

4 We know that $\mathscr{L}(e^{at}) = \dfrac{1}{s-a}$ when $s > a$
Hence $\mathscr{L}(e^{-t}) = \dfrac{1}{s+1}$ for $s > -1$ (C)

5 $\mathscr{L}(e^{-2t}) = \dfrac{1}{s+2}$ for $s > -2$ (C)

6 $\mathscr{L}\{1-e^{-2t}\} = \dfrac{1}{s} - \dfrac{1}{s+2} = \dfrac{2}{s(s+2)}$ for $s > 0$, because $\mathscr{L}\{1\} = \dfrac{1}{s}$ for $s > 0$ and $\mathscr{L}\{e^{-2t}\} = \dfrac{1}{s+2}$ for $s > -2$ so we take maximun value of s for which laplace transform exist. (A)

7 $\mathscr{L}\{\cos(\omega t)\} = \int\limits_0^{\infty} e^{-st}\cos\omega t\,dt$

$\mathscr{L}\{\cos(\omega t)\} = \left[\dfrac{e^{-st}}{s^2+\omega^2}(-s\cos\omega t + \omega\sin\omega t)\right]_0^{\infty}$

$\mathscr{L}\{\cos(\omega t)\} = \dfrac{s}{s^2+\omega^2}$ for $s > 0$ (A)

8 We $\mathscr{L}\{\cos h(at)\} = \mathscr{L}\left\{\dfrac{e^{at}+e^{-at}}{2}\right\}$

$\mathscr{L}(\cos h(at)) = \dfrac{\mathscr{L}\{e^{at}\} + \mathscr{L}\{e^{-at}\}}{2}$

$\mathscr{L}\{\cos h(at)\} = \dfrac{1}{2}\left(\dfrac{1}{s-a} + \dfrac{1}{s+a}\right)$ $s{>}a$ & $s{>}{-a}$

$\mathscr{L}\{\cos h(at)\} = \dfrac{s}{s^2-a^2}.$ for $s > |a|$ (A)

9 $\mathscr{L}\{2t+6\} = 2\mathscr{L}\{t\} + 6\mathscr{L}\{1\} = \dfrac{2}{s^2} + \dfrac{6}{s}$; $s > 0$(C)

10 $\mathscr{L}\{t^2+2t+1\} = \mathscr{L}\{t^2\} + 2\mathscr{L}\{t\} + \mathscr{L}\{1\}$

$\mathscr{L}\{t^2+2t+1\} = \dfrac{2}{s^3} + \dfrac{2}{s^2} + \dfrac{1}{s}$ for $s > 0$. (D)

Having undersood so far now we do not write the values of s for which Laplace transform exist.

11. $\mathscr{L}\{\sqrt{t}\} = \dfrac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}} = \dfrac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2s^{3/2}}$ (C)

12. $\int\limits_0^{\infty} \dfrac{e^{-st}}{\sqrt{t}}\,dt = \dfrac{1}{\sqrt{s}}\int\limits_0^{\infty} e^{-u}u^{\frac{1}{2}-1}\,du$ (take $st = u$)

$\int\limits_0^{\infty} \dfrac{e^{-st}}{\sqrt{t}}\,dt = \dfrac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\dfrac{\pi}{s}}$ for $s > 0$

Alternative way to solve above integral is use $st = w^2$ substitution in above integral

$\int\limits_0^{\infty} \dfrac{e^{-st}}{\sqrt{t}}dt = \dfrac{2}{\sqrt{s}}\int\limits_0^{\infty} e^{-w^2}\,dw = \sqrt{\dfrac{\pi}{s}}$ (A)

13. $\mathscr{L}\left\{\dfrac{1}{\sqrt{\pi t}}\right\} = \dfrac{1}{\sqrt{\pi}}\mathscr{L}\left\{\dfrac{1}{\sqrt{t}}\right\} = \dfrac{1}{\sqrt{s}}$ (B)

14 $\mathscr{L}\left\{6t^3 + 3\sin 4t\right\} = 6\mathscr{L}\left\{t^3\right\} + 3\mathscr{L}\left\{\sin 4t\right\}$

$\mathscr{L}\left\{6t^3 + 3\sin 4t\right\} = \dfrac{6\cdot 3!}{s^4} + \dfrac{3\cdot 4}{s^2+4^2}$

$\mathscr{L}\left\{6t^3 + 3\sin 4t\right\} = \dfrac{36}{s^4} + \dfrac{12}{s^2+4^2}$ (A)

15 We know that $\sinh 5t = \dfrac{e^{5t}-e^{-5t}}{2}$

$3e^{5t}\sinh 5t = \dfrac{3(e^{10t}-1)}{2} = \dfrac{3e^{10t}}{2} - \dfrac{3}{2}$

$\mathscr{L}\{3e^{5t}\sinh 5t\} = \mathscr{L}\left\{\dfrac{3e^{10t}}{2}\right\} - \mathscr{L}\left\{\dfrac{3}{2}\right\}$

$\mathscr{L}\{3e^{5t}\sinh 5t\} = \dfrac{3}{2(s-10)} - \dfrac{3}{2s} = \dfrac{15}{s^2-10s}$ (B)

16 $\mathscr{L}\{e^{-2t}\cos 4t\} = \dfrac{s+2}{(s+2)^2+16}$ (D)

17 $\mathscr{L}\{e^{2t}\sin 5t\} = \dfrac{5}{(s-2)^2+5^2} = \dfrac{5}{s^2-4s+29}$ (A)

18 $\mathscr{L}\{te^t\} = -\dfrac{d}{ds}\left(\dfrac{1}{s-1}\right) = \dfrac{1}{(s-1)^2}$ (B)

19 $\mathscr{L}\{t^n e^{-\alpha t}\} = \dfrac{n!}{(s+\alpha)^{n+1}}$ (B)

20 Let $f(t) = 2\cos 2t - \sin 2t$, $\mathscr{L}\{f(t)\} = F(s)$.
$\mathscr{L}\{e^{-t}f(t)\} = F(s+1) = \dfrac{2(s+1-1)}{(s+1)^2+4}$ So
$F(s) = \dfrac{2(s-1)}{s^2+4} \implies F(s-1) = \dfrac{2(s-2)}{(s-1)^2+4}$(A)

21 $\mathscr{L}\{t\cosh t\} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2-1}\right) = \dfrac{s^2+1}{(s^2-1)^2}$ (A)

22 Given that $\mathscr{L}\{f(t)\} = F(s) = \dfrac{1}{s^2+s+1}$
$\mathscr{L}\{tf(t)\} = -\dfrac{d}{ds}\left(\dfrac{1}{s^2+s+1}\right) = \dfrac{2s+1}{(s^2+s+1)^2}$
because $\mathscr{L}\{tf(t)\} = -F'(s)$ (D)

23 $\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \dfrac{F(s)}{s}$ (Standard Theorem)(A)

24 Given $f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 1 & \text{if } t > 1 \end{cases}$

$$f(t) = t[u(t) - u(t-1)] + [u(t-1)]$$
$$f(t) = tu(t) - (t-1)u(t-1)$$
$$\mathscr{L}\{f(t)\} = \mathscr{L}\{tu(t) - (t-1)u(t-1)\}$$
$$\mathscr{L}\{f(t)\} = \mathscr{L}\{tu(t)\} - \mathscr{L}\{(t-1)u(t-1)\}$$
$$\mathscr{L}\{f(t)\} = \frac{e^{-0s}}{s^2} - \frac{e^{-s}}{s^2} = \frac{1-e^{-s}}{s^2} \qquad (B)$$

25 Take $g(x) = x$ So $g(x-3) = x-3$ Using second shifting property$\mathscr{L}\{f(t-a)u(t-a)\} = e^{-st}F(s)$

$$g(x-3)u(x-3) = \begin{cases} 0 & \text{for } x \le 3 \\ x-3 & \text{for } x \ge 3 \end{cases}$$

$\mathscr{L}\{g(x-3)u(x-3)\} = e^{-3s}\mathscr{L}\{x\} = e^{-3s}s^{-2}$ (D)

26 $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2+k^2} \neq \dfrac{s}{s^2+k^2}$ (C)

27 Unit-Step function defined as
$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$ where $a \ge 0$

$$\mathscr{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt = \int_a^\infty e^{-st}\,dt$$
$$\mathscr{L}\{u(t-a)\} = \frac{e^{-as}}{s} \quad \text{for } s > 0 \qquad (C)$$

28

$$\mathscr{L}\{2[u(t)-u(t-1)]\} = 2\mathscr{L}\{u(t)\} - 2\mathscr{L}\{u(t-1)\}$$
$$\mathscr{L}\{2u(t) - 2u(t-1)\} = \frac{2e^{-0s}}{s} - \frac{2e^{-s}}{s} = \frac{2(1-e^{-s})}{s}$$
$$\mathscr{L}\{2[u(t)-u(t-1)]\} = \frac{2(1-e^{-s})}{s} \qquad (C)$$

29 Given function can be written as combination of unit step funcation as $f(t) = u(t-3) - u(t-5)$.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{u(t-3)\} - \mathscr{L}\{u(t-5)\}$$
$$\mathscr{L}\{f(t)\} = \frac{e^{-3s}}{s} - \frac{e^{-5s}}{s} = \frac{e^{-3s}-e^{-5s}}{s}$$

Alternative way is solve this question is use definition of Laplace transform $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt = \int_3^5 e^{-st}\,dt = \dfrac{e^{-3s}-e^{-5s}}{s}$ (A)

30 Let $f(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$ can be written as linear combination of unit step funcation as $f(t) = u(t) - u(t-1)$.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{u(t)\} - \mathscr{L}\{u(t-1)\}$$
$$\mathscr{L}\{f(t)\} = \frac{e^{-0s}}{s} - \frac{e^{-s}}{s}$$
$$\mathscr{L}\{f(t)\} = \frac{(1-e^{-s})}{s} \quad \text{for } s > 0$$
$$\mathscr{L}\{f\}\cdot\mathscr{L}\{f\} = \frac{(1-e^{-s})^2}{s^2} = \frac{1-2e^{-s}+e^{-2s}}{s^2}$$

Convolution Theorem $\mathscr{L}\{f*f\} = \mathscr{L}\{f\}\cdot\mathscr{L}\{f\}$

$$\mathscr{L}\{f*f\} = \frac{(1-e^{-s})^2}{s^2} = \frac{1-2e^{-s}+e^{-2s}}{s^2}$$
$$f*f = \mathscr{L}^{-1}\left\{\frac{1-2e^{-s}+e^{-2s}}{s^2}\right\}$$

$$f*f = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} - 2\mathscr{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\} + \mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$
$$f*f = t - 2u(t-1)(t-1) + u(t-2)(t-2)$$

$$f*f = \begin{cases} t & 0 \le t \le 1 \\ t-2(t-1) & 1 < t \le 2 \\ t-2(t-1)+(t-2) & 2 < t < \infty \end{cases}$$

$$f*f = \begin{cases} t & \text{for } 0 \le t \le 1 \\ 2-t & \text{for } 1 < t \le 2 \\ 0 & \text{for } 2 < t < \infty \end{cases}$$

$x(t) = f*f$ hence $\mathscr{L}\{x(t)\} = \mathscr{L}\{f*f\}$ (D)

31 $\mathscr{L}\{f(t)\} = \dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st}f(t)dt$

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-s}}\left(\int_0^{1/2} e^{-st}f(t)dt + \int_{1/2}^1 e^{-st}f(t)dt\right)$$

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-s}}\left(\int_0^{1/2} e^{-st}(1)dt + \int_{1/2}^{1} e^{-st}(0)dt\right)$$

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-s}}\left(\frac{e^{-st}}{-s}\right)_0^{1/2}$$

$$\mathscr{L}\{f(t)\} = \frac{1-e^{-s/2}}{s(1-e^{-s})}$$

$$\mathscr{L}\{f(t)\} = \frac{1-e^{-s/2}}{s(1-e^{s/2})(1+e^{-s/2})}$$

$$\mathscr{L}\{f(t)\} = \frac{1}{s(1+e^{-s/2})} \qquad (B)$$

32 $\mathscr{L}^{-1}\left\{\frac{s+10}{(s+2)(s+20)}\right\} = \mathscr{L}^{-1}\left\{\frac{4/9}{s+2} + \frac{5/9}{s+20}\right\}$

$\mathscr{L}^{-1}\left\{\frac{s+10}{(s+2)(s+20)}\right\} = \frac{4}{9}e^{-2t} + \frac{5}{9}e^{-20t}$ (C)

33 $\mathscr{L}^{-1}\left\{\frac{s+1}{(s+4)(s-3)}\right\} = \mathscr{L}^{-1}\left\{\frac{3/7}{s+4} + \frac{4/7}{s-3}\right\}$

$\mathscr{L}^{-1}\left\{\frac{s+1}{(s+4)(s-3)}\right\} = \frac{3}{7}e^{-4t} + \frac{4}{7}e^{3t}$ (B)

34 $F(s) = \frac{1}{s^2(s+1)} = \frac{1}{s}\left[\frac{1}{s(s+1)}\right] = \frac{1}{s}G(s)$

where $G(s) = \frac{1}{s(s+1)}$ so $g(t) = 1 - e^{-t}$

Using $\mathscr{L}\left\{\int_0^t g(t)dt\right\} = \frac{1}{s}G(s)$

$$\mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s}G(s)\right\}$$

$$\mathscr{L}^{-1}\{F(s)\} = \int_0^t (1-e^{-\tau})d\tau$$

$$\mathscr{L}^{-1}\{F(s)\} = -1 + t + e^{-t} \qquad (A)$$

35 $F(s) = \frac{1}{s(s+1)} = \frac{s+1-s}{s(s+1)} = \frac{1}{s} - \frac{1}{(s+1)}$

$\mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{(s+1)}\right\} = 1 - e^{-t}$ (C)

36 $\mathscr{L}^{-1}\left\{\frac{s+3}{s^2+2s+1}\right\} = \mathscr{L}^{-1}\left\{\frac{s+1+2}{(s+1)^2}\right\}$

$\mathscr{L}^{-1}\left\{\frac{s+3}{s^2+2s+1}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{(s+1)} + \frac{2}{(s+1)^2}\right\}$

$\mathscr{L}^{-1}\left\{\frac{s+3}{s^2+2s+1}\right\} = e^{-t} + 2te^{-t}$ (C)

37 $\frac{2s^2-4}{(s-3)(s^2-s-2)} = \frac{A}{(s-3)} + \frac{B}{(s-2)} + \frac{C}{(s+1)}$

$2s^2-4 = A(s-2)(s+1) + B(s-3)(s+1) + C(s-3)(s-2)$ For $s = 3,\ 2$ and -1 we get $A = \frac{7}{2}$ $B = \frac{-4}{3}$ and $C = \frac{-1}{6}$ respectively

$\mathscr{L}^{-1}\{F(s)\} = \frac{7}{2}\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{4}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = \frac{7}{2}e^{3t} - \frac{4}{3}e^{2t} - \frac{1}{6}e^{-t}$ (C)

38 $\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+(10)^2}\right\} = e^{-2t}\cos 10t = f(t)$

So $f(0) = 1$ (C)

39 $F(s) = \frac{2s+4-2}{s^2+4s+4+3} = \frac{2(s+2)-2}{(s+2)^2+3}$

$F(s) = \frac{2(s+2)}{(s+2)^2+3} - \frac{2}{(s+2)^2+3}$

$\mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{2(s+2)}{(s+2)^2+3} - \frac{2}{(s+2)^2+3}\right\}$

$f(t) = 2e^{-2t}\left[\cos(\sqrt{3}t) - \frac{\sin(\sqrt{3}t)}{\sqrt{3}}\right]$

$f(0) = 2$ and $\lim_{t\to\infty} f(t) = 0$ (B)

40 $F(s) = \frac{1/2}{s} + \frac{1/2}{s+2}$

$\mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{1/2}{s} + \frac{1/2}{s+2}\right\}$

$\mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\frac{1/2}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{1/2}{s+2}\right\}$

$f(t) = \frac{1}{2}(1) + \frac{1}{2}e^{-2t}$ Alternative way is as follow

$\frac{s+1}{s(s+2)} = \frac{s}{s(s+2)} + \frac{1}{s(s+2)}$

$\frac{s+1}{s(s+2)} = \frac{1}{(s+2)} + \frac{1}{s(s+2)}$

$\frac{s+1}{s(s+2)} = \frac{1}{(s+2)} + \frac{1}{2}\left(\frac{s+2-s}{s(s+2)}\right)$

$\frac{s+1}{s(s+2)} = \frac{1}{2(s+2)} + \frac{1}{2s}$

$\mathscr{L}^{-1}\left\{\frac{s+1}{s(s+2)}\right\} = \frac{1+e^{-2t}}{2}$

$f(0) = 1$ and $\lim_{t\to\infty} f(t) = 1/2$ (B)

41 Given $y'' + y = 6\cos(2x)$; $y(0) = 3,\ y'(0) = 1$

$$\mathscr{L}\{y'' + y\} = \mathscr{L}\{6\cos(2x)\}$$

$$\mathscr{L}\{y''\} + \mathscr{L}\{y\} = 6\mathscr{L}\{\cos(2x)\}$$

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + \mathscr{L}\{y\} = \frac{6s}{s^2+4}$$

$$(s^2+1)\mathscr{L}\{y\} - 3s - 1 = \frac{6s}{s^2+4}$$

$$(s^2+1)\mathscr{L}\{y\} = \frac{6s}{s^2+4} + (3s+1)$$

$$\mathscr{L}\{y\} = \frac{6s}{(s^2+4)(s^2+1)} + \frac{(3s+1)}{(s^2+1)}$$

$$Y(s) = \frac{6s + (3s+1)(s^2+4)}{(s^2+4)(s^2+1)}$$

$$Y(1) = \frac{13}{5} \qquad (B)$$

42 Given $u'' + 2u' + u = 1$; $u(0) = 0,\ u'(0) = 5$.

$$\mathscr{L}\{u'' + 2u' + u\} = \mathscr{L}\{1\}$$
$$\mathscr{L}\{u''\} + \mathscr{L}\{u'\} + \mathscr{L}\{u\} = \mathscr{L}\{1\}$$

$$(s^2 + 2s + 1)\mathscr{L}\{u\} - su(0) - u'(0) - 2u(0) = \frac{1}{s}$$
$$(s^2 + 2s + 1)\mathscr{L}\{u\} = \frac{1}{s} + 5$$
$$U(s) = \frac{1 + 5s}{s(s^2 + 2s + 1)} \qquad \text{(B)}$$

43 $y'' + 5y' + 6y = u(t) - u(t - 2)$

Taking Laplace Transform on both side

$\mathscr{L}\{y'' + 5y' + 6y\} = \mathscr{L}\{u(t) - u(t - 2)\}$

$$s^2\mathscr{L}\{y(t)\} + 5s\mathscr{L}\{y(t)\} + 6\mathscr{L}\{y(t)\} = \frac{e^{-0s}}{s} - \frac{e^{-2s}}{s}$$
$$(s^2 + 5s + 6)Y(s) = \frac{1 - e^{-2s}}{s}$$
$$Y(s) = \frac{1 - e^{-2s}}{s(s^2 + 5s + 6)} \qquad \text{(B)}$$

44 We can rewrite $f(x)$ as $f(x) = 2[u(x) - u(x - 1)]$

So $y' + y = 2u(x) - 2u(x - 1)$; $y(0) = 0$.

Take Laplace Transform on both side

$$\mathscr{L}\{y' + y\} = \mathscr{L}\{2u(x) - 2u(x - 1)\}$$
$$\mathscr{L}\{y'\} + \mathscr{L}\{y\} = 2\mathscr{L}\{u(x)\} - 2\mathscr{L}\{u(x - 1)\}$$
$$s\mathscr{L}\{y\} - y(0) + \mathscr{L}\{y\} = \frac{2e^{-0s}}{s} - \frac{2e^{-s}}{s}$$
$$(s + 1)Y(s) = \frac{2(1 - e^{-s})}{s}$$

$$Y(s) = \frac{2(1 - e^{-s})}{s(s + 1)}$$
$$Y(s) = 2\left(\frac{1}{s} - \frac{1}{s + 1}\right) - 2e^{-s}\left(\frac{1}{s} - \frac{1}{s + 1}\right)$$

Now take Inverse Laplace Transform on both side,

$$y(x) = \mathscr{L}^{-1}\left\{\frac{2}{s} - \frac{2}{s + 1} - \frac{2e^{-s}}{s} + \frac{2e^{-s}}{s + 1}\right\}$$
$$y(x) = 2 - 2e^{-x} - 2\ u(x - 1)(1) + 2u(x - 1)e^{-(x-1)}$$

$$y(x) = \begin{cases} 2 - 2e^{-x} & \text{for} \quad 0 \le x \le 1 \\ 2 - 2e^{-x} - 2 + 2e^{-(x-1)} & \text{for} \quad x \ge 1 \end{cases}$$

$$y(x) = \begin{cases} 2(1 - e^{-x}) & \text{for} \quad 0 \le x < 1 \\ (2e - 2)e^{-x} & \text{for} \quad x \ge 1 \end{cases} \qquad (A)$$

45 Given $y'' + 2y' + 5y = 3e^{-t}\sin t$; $y(0) = 0,\ y'(0) = 3$

$$\mathscr{L}\{y'' + 2y' + 5y\} = \mathscr{L}\{3e^{-t}\sin t\}$$
$$\mathscr{L}\{y''\} + 2\mathscr{L}\{y'\} + 5\mathscr{L}\{y\} = \mathscr{L}\{3e^{-t}\sin t\}$$
$$(s^2 + 2s + 5)\mathscr{L}\{y\} - 3 = \frac{3}{(s + 1)^2 + 1}$$

$$Y(s) = \frac{3[(s + 1)^2 + 1] + 3}{[(s + 1)^2 + 1][(s + 1)^2 + 2^2]}$$
$$Y(s) = \frac{3}{[(s + 1)^2 + 2^2]} + \frac{3}{[(s + 1)^2 + 2^2][(s + 1)^2 + 2^2]}$$

$$Y(s) = \frac{3}{[(s + 1)^2 + 2^2]} + \frac{[s^2 + 2s + 5] - [s^2 + 2s + 2]}{[(s + 1)^2 + 1][(s + 1)^2 + 2^2]}$$
$$Y(s) = \frac{2}{[(s + 1)^2 + 2^2]} + \frac{1}{[(s + 1)^2 + 1^2]}$$
$$y(t) = e^{-t}(\sin(2t) + \sin(t)) \qquad \text{(B)}$$

46 Note that $\sin(t - 2\pi) = \sin(t)$

$$\mathscr{L}\{y'' + y\} = \mathscr{L}\{\sin(t)u(t - 2\pi)\}$$
$$\mathscr{L}\{y'' + y\} = \mathscr{L}\{\sin(t - 2\pi)u(t - 2\pi)\}$$
$$(s^2 + 1)\mathscr{L}\{y\} - s - 1 = \frac{e^{-2\pi s}}{s^2 + 1}$$
$$\mathscr{L}\{y\} = \frac{e^{-2\pi s}}{(s^2 + 1)^2} + \frac{s + 1}{s^2 + 1} \qquad (?)$$

48 $\mathscr{L}\{y'(t) + 5y(t)\} = \mathscr{L}\{u(t)\}$

$$s\mathscr{L}\{y(t)\} - y(0) + 5\mathscr{L}\{y(t)\} = \frac{1}{s}$$
$$(s + 5)Y(s) = \frac{s + 1}{s}$$
$$Y(s) = \frac{s + 1}{s(s + 5)} = \frac{1/5}{s} + \frac{4/5}{s + 5} = \frac{0 \cdot 2}{s} + \frac{0 \cdot 8}{s + 5}$$
$$y(t) = \mathscr{L}^{-1}\left\{\frac{0 \cdot 2}{s} + \frac{0 \cdot 8}{s + 5}\right\} = 0.2 + 0.8e^{-5t} \quad \text{(A)}$$

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$$\mathscr{L}\left\{\frac{dy}{dt} + y(t)\right\} = \mathscr{L}\{\delta(t)\}$$
$$\mathscr{L}\left\{\frac{dy}{dt}\right\} + \mathscr{L}\{y(t)\} = \mathscr{L}\{\delta(t)\}$$
$$s\mathscr{L}\{y\} - y(0) + \mathscr{L}\{y\} = 1$$
$$(s + 1)Y(s) = 1$$
$$Y(s) = \frac{1}{s + 1}$$
$$\mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s + 1}\right\}$$
$$y(t) = e^{-t} \quad \text{for} \quad t \ge 0$$
$$y(t) = e^{-t}u(t) \qquad (D)$$

52 Given system of differential eq. can be written as $x' = -x,\ \ y' = -2y$ with $x(0) = 1,\ \ y(0) = -1$

$x' = -x;\ \ x(0) = 1$	$y' = -2y;\ \ y(0) = -1$
$\mathscr{L}\{x'\} = \mathscr{L}\{-x\}$	$\mathscr{L}\{y'\} = \mathscr{L}\{-2y\}$
$(s + 1)X(s) = x(0)$	$(s + 2)Y(s) = y(0)$
$X(s) = \frac{1}{s + 1}$	$Y(s) = \frac{-1}{s + 2}$
$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{s + 1}\right\}$	$y(t) = \mathscr{L}^{-1}\left\{\frac{-1}{s + 2}\right\}$
$x(t) = e^{-t}$	$y(t) = -e^{-2t}$ (C)

53 Given equation $y(x) = x^3 + \sin t * y(t)$

$$\mathscr{L}\{y(x)\} = \mathscr{L}\{x^3\} + \mathscr{L}\{\sin t * y(t)\}$$
$$Y(s) = \frac{3!}{s^4} + \mathscr{L}\{\sin t\} \cdot \mathscr{L}\{y(t)\}$$
$$Y(s) = \frac{6}{s^4} + \frac{Y(s)}{s^2+1}$$
$$\left(1 - \frac{1}{s^2+1}\right) Y(s) = \frac{6}{s^4}$$
$$\left(\frac{s^2}{s^2+1}\right) Y(s) = \frac{6}{s^4}$$
$$Y(s) = \frac{6}{s^4}\frac{s^2+1}{s^2}$$
$$Y(s) = 6\left(\frac{1}{s^4} + \frac{1}{s^6}\right)$$
$$y(t) = \mathscr{L}^{-1}\left\{6\left(\frac{1}{s^4} + \frac{1}{s^6}\right)\right\}$$
$$y(t) = 6\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\} + 6\mathscr{L}^{-1}\left\{\frac{1}{s^6}\right\}$$
$$y(t) = 6\left(\frac{t^3}{3!} + \frac{t^5}{5!}\right) = t^3 + \frac{t^5}{20}$$
$$y(1) = 1 + \frac{1}{20} = \frac{21}{20} \qquad (D)$$

54 Given equation $\varphi(x) = x + \varphi(x) * \sin(x)$

$$\mathscr{L}\{\varphi(x)\} = \mathscr{L}(x) + \mathscr{L}\{\varphi(x) * \sin(x)\}$$
$$\mathscr{L}\{\varphi(x)\} = \mathscr{L}(x) + \mathscr{L}\{\varphi(x)\} \cdot \mathscr{L}\{\sin(x)\}$$
$$\mathscr{L}\{\varphi(x)\} = \frac{1}{s^2} + \frac{1}{s^2+1}\mathscr{L}\{\varphi(x)\}$$
$$\mathscr{L}\{\varphi(x)\}\left(1 - \frac{1}{s^2+1}\right) = \frac{1}{s^2}$$
$$\mathscr{L}\{\varphi(x)\} = \frac{s^2+1}{s^4}$$
$$\varphi(x) = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\}$$
$$\varphi(x) = x + \frac{x^3}{3!} \qquad \text{(C)}$$

55 $f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau$

$f(t) = e^{-t}$ and $g(t) = e^{-2t}$ for $t > 0$

$$f * g = \int_0^t e^{-\tau}e^{-2(t-\tau)}\,d\tau$$
$$f * g = e^{-2t}\int_0^t e^{\tau}\,d\tau = e^{-t} - e^{-2t}$$

Alternative way to solve this problemis use convolution theorem

$$\mathscr{L}\{e^{-t} * e^{-2t}\} = \mathscr{L}\{e^{-t}\}\mathscr{L}\{e^{-2t}\}$$
$$\mathscr{L}\{e^{-t} * e^{-2t}\} = \frac{1}{(s+1)(s+2)}$$
$$e^{-t} * e^{-2t} = \mathscr{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$
$$e^{-t} * e^{-2t} = \mathscr{L}^{-1}\left\{\frac{(s+2)-(s+1)}{(s+1)(s+2)}\right\}$$
$$e^{-t} * e^{-2t} = \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}$$
$$e^{-t} * e^{-2t} = e^{-t} - e^{-2t} \qquad (A)$$

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$$\mathscr{L}\{e^{-t^2}\} = \int_0^\infty e^{-t^2}e^{-st}dt = \int_0^\infty e^{-(t^2+st)}dt$$
$$\mathscr{L}\{e^{-t^2}\} = \int_0^\infty e^{-(t^2+st+\frac{s^2}{4}-\frac{s^2}{4})}\,dt$$
$$\mathscr{L}\{e^{-t^2}\} = \int_0^\infty e^{(-t+\frac{s}{2})^2}e^{\frac{s^2}{4}}\,dt = e^{\frac{s^2}{4}}\int_0^\infty e^{-(t+\frac{s}{4})^2}\,dt$$

Take $t + \frac{s}{2} = u$ so $dt = du$

$$\mathscr{L}\{e^{-t^2}\} = e^{\frac{s^2}{4}}\int_{s/2}^\infty e^{-u^2}\,du$$
$$\mathscr{L}\{e^{-t^2}\} = e^{\frac{s^2}{4}}\left[\int_0^{s/2} e^{-u^2}\,du + \int_{s/2}^\infty e^{-u^2}\,du - \int_0^{\frac{s}{2}} e^{-u^2}\,du\right]$$
$$\mathscr{L}\{e^{-t^2}\} = e^{\frac{s^2}{4}}\left[\int_0^\infty e^{-u^2}\,du - \int_0^{\frac{s}{2}} e^{-u^2}\,du\right]$$

Recall: The Error function $erf(x)$ is defined by $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$. The complementry error function defined as $1 - erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$ Note that $erf(x) + erfc(x) = 1$.

since $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$

$$\mathscr{L}\{e^{-t^2}\} = e^{\frac{s^2}{4}}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}erf\left(\frac{s}{2}\right)\right]$$
$$\mathscr{L}\{e^{-t^2}\} = e^{\frac{s^2}{4}}\frac{\sqrt{\pi}}{2}\left[1 - erf\left(\frac{s}{2}\right)\right] = \frac{\sqrt{\pi}}{2}e^{\frac{s^2}{4}}erfc\left(\frac{s}{2}\right)$$

65 Please refer solution of question no: 30

If you found any mistake(s) please report me at `dng.maths@coep.ac.in` ■