Lecture Notes on Laplace Transforms-III

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Change of Scale Property: If $\mathscr{L}\{f(t)\} = F(s)$ when $s > k$ for some k then $\mathscr{L}\{f(at)\} = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$.

Proof: The proof proceeds by using the definition of Laplace Transform

$\mathscr{L}\{f(t)\} = \displaystyle\int_0^\infty e^{-st}f(t)\,dt \implies \mathscr{L}\{f(at)\} = \int_0^\infty e^{-sat}f(at)\,dt$ take $at = u \implies t = \dfrac{u}{a}$ and $dt = \dfrac{du}{a}$ so

$\mathscr{L}\{f(at)\} = \displaystyle\int_0^\infty e^{-(s/a)u}f(u)\frac{du}{a} \implies \mathscr{L}\{f(at)\} = \frac{1}{a}\int_0^\infty e^{-(s/a)u}f(u)du \implies \mathscr{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right).$

Theorem (First Shifting Theorem): If $\mathscr{L}\{f(t)\} = F(s)$ when $s > k$ for some k then

$\mathscr{L}\{e^{at}f(t)\} = F(s-a) \quad \text{when} \quad s - a > k.$

Proof: We begin with the definition of Laplace Transform

$\mathscr{L}\{f(t)\} = F(s) = \displaystyle\int_0^\infty e^{-st}f(t)\,dt$ for $s > k \implies F(s-a) = \displaystyle\int_0^\infty e^{-(s-a)t}f(t)\,dt$ exists when $s - a > k$

$F(s-a) = \displaystyle\int_0^\infty e^{-st}[e^{at}f(t)]\,dt \implies F(s-a) = \mathscr{L}\{e^{at}f(t)\} \implies \mathscr{L}\{e^{at}f(t)\} = F(s-a)$ when $s-a > 0$

Prove that : $\mathscr{L}\{e^{-at}f(t)\} = F(s+a)$ when $s + a > 0$

Theorem (Laplace Transform of Derivatives): Let $f(t)$ be continuous function of exponential order and $f'(t)$ is piecewise continuous on every finite interval on the semi axis $t \geq 0$. The Laplace Transform of the first order derivatives of $f(t)$ satisfy $\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$.

Proof: We will prove this theorem under the assumption that $f'(t)$ is continuous.

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt \implies \mathscr{L}\{f'(t)\} = [e^{-st}f(t)]_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt$$

$$\mathscr{L}\{f'(t)\} = \lim_{t\to\infty} e^{-st}f(t) - f(0) + s\int_0^\infty e^{-st}f(t)\,dt$$

Given that $f(t)$ is of exponential order so we have $|f(t)| \leq Me^{kt}$ for sufficiently large t this condition is equivalent to $-Me^{kt} \leq f(t) \leq Me^{kt}$ and hence $\lim\limits_{t\to\infty} e^{-st}f(t) = 0$ (why? Hint is use squeeze (or sandwich) theorem) since $f(t)$ satisfies Existence Theorem of Laplace Transforms therefore $\displaystyle\int_0^\infty e^{-st}f(t)\,dt = \mathscr{L}\{f(t)\}$ hence we get $\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$. In the case of $f'(t)$ is piecewise continuous, interval of the integration of $f'(t)$ must be broken up into finite subparts such that $f'(t)$ is continuous in each intervals.

Theorem (Laplace Transform of n^{th} Order Derivatives): Let $f(t), f'(t), f''(t), \cdots, f^{n-1}(t)$ be continuous functions for $t \geq 0$ and satisfy growth restriction furthermore $f^n(t)$ be piecewise continuous on every finite interval on the semi axis then

$$\mathscr{L}\{f^n(t)\} = s^n \mathscr{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \quad \text{where } n \text{ is positive integer.}$$

Because of this theorem, The Laplace transform transform a linear differential equation with constant coefficients (along with appropriate initial conditions are given) transformed into algebraic equation in s. If we able to obtain the Inverse Laplace Transform, we may have solution of the differential equation.

Inverse Laplace Transform: If $f(t)$ has the Laplace Transform $F(s)$ that is $\mathscr{L}\{f(t)\} = F(s)$ then the Inverse Laplace Transform of $F(s)$ is denoted by $\mathscr{L}^{-1}\{F(s)\}$ and is defined as $\mathscr{L}^{-1}\{F(s)\} = f(t)$.

Inverse of Laplace Transform is NOT Unique. If $f(t)$ and $g(t)$ are identical except at a discrete set of points the values of the integral $\int_0^\infty e^{-st}f(t)dt$ and $\int_0^\infty e^{-st}g(t)dt$ are equal. That is essentially inverse of the $F(s)$ differ at most at their points of discontinuity. **If $F(s)$ has a continuous inverse $f(t)$ then $f(t)$ is the ONLY continuous inverse of $F(s)$.** So for continuous function $f(t)$ we have $\mathscr{L}^{-1}(\mathscr{L}(f)) = f$ and $\mathscr{L}(\mathscr{L}^{-1}(f)) = f$ (we have assumed this fact in the proof of following theorem)

Theorem: Inverse Laplace Transform is linear i.e If $\mathscr{L}^{-1}\{F(s)\} = f(t)$ and $\mathscr{L}^{-1}\{G(s)\} = g(t)$ then $\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathscr{L}^{-1}\{F(s)\} + \beta \mathscr{L}^{-1}\{G(s)\}$ where $\alpha, \beta \in \mathbb{R}$.

Proof: Since the Laplace Transform is linear we have $\mathscr{L}\{\alpha f(t)\} + \beta g(t) = \alpha F(s) = \beta G(s)$

Taking the Inverse Laplace Transform of above expression gives

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t) \implies \mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathscr{L}^{-1}\{F(s)\} + \beta \mathscr{L}^{-1}\{G(s)\}$$

Theorem: If $\mathscr{L}\{f(t)\} = F(s)$ then $F(s) \to 0$ as $s \to \infty$. Proof for a piecewise continuous function which is of exponential order follow from the existence theorem of Laplace Transform. But this theorem is valid for any function for which Laplace Transform exists. Note that above theorem say that function $F(s)$ without this behavior can not be Laplace Transform of any function. For example $F(s) = \dfrac{s}{s+1}$ is NOT Laplace Transform of any function (Try to find Inverse Laplace Transform of this function).

Partial Fractions Method of Finding Inverse Laplace Transform : Finding the inverse Laplace Transform is more challenging. We often need to obtain the Inverse Laplace Transform of a rational function $P(s)/Q(s)$ where numerator and denominator are polynomials in s and the degree of $Q(s)$ is larger than the degree of $P(s)$ in such case we use partial fractions to simplify rational functions. We use linearity property of the Inverse Laplace Transform and try to identify with Laplace Transform of $t, t^2, e^t, \sin t, \cosh t$ etc.

If you found any mistake(s) please report me at `dng.maths@coep.ac.in` ■