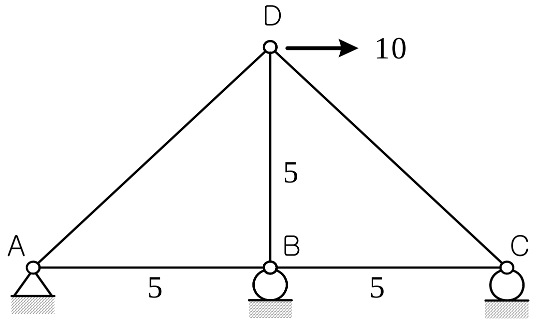
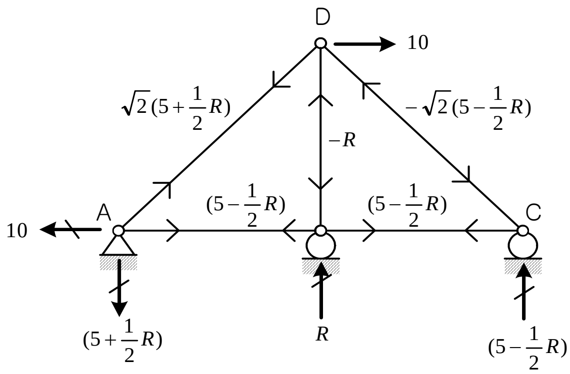
• The Principle of Least Work

The magnitudes of the redundants of an indeterminate structure must be such that the strain energy stored in the structure is a minimum.

EX4) Determine the reaction at B. Assume the members are pin connected at their ends. (AE=1)



sol) Assume that the reaction at B is R and get the member forces.



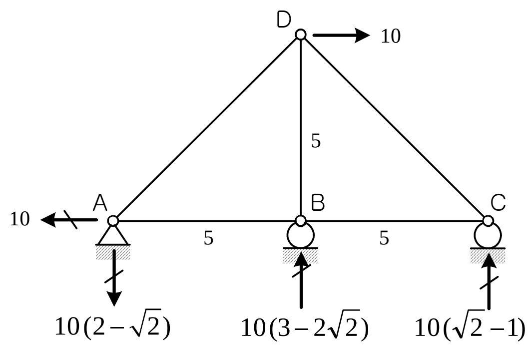
$$\frac{\partial U}{\partial R} = \frac{FL}{EA}\frac{\partial F}{\partial R}$$

$$= 5\sqrt{2}\,(5+\frac{1}{2}R-5+\frac{1}{2}R)+5\,(R-5+\frac{1}{2}R)$$

$$= (5\sqrt{2}+\frac{15}{2})R-25$$

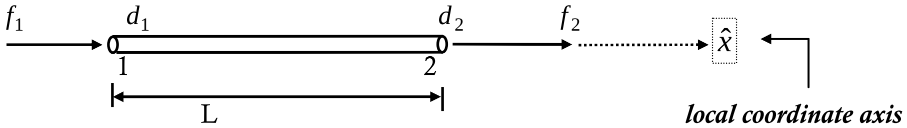
$$\therefore R = \frac{25}{5\sqrt{2}+\frac{15}{2}} = 10\,(3-2\sqrt{2})$$

$$\therefore \; reaction \;\; at \;\; B = R = 10\,(3-2\sqrt{2})$$



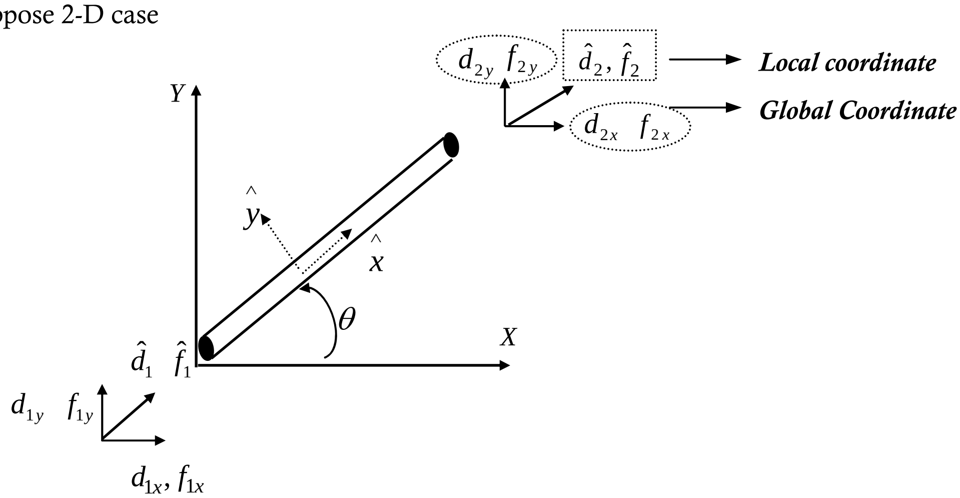
3.4. Coordinate Transformation

Recall that in 1-D bar element



$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} \Rightarrow \hat{\boldsymbol{f}}\boldsymbol{e} = \hat{\boldsymbol{K}}\boldsymbol{e}\hat{\boldsymbol{d}}\boldsymbol{e}$$

Now suppose 2-D case



Let $\hat{d} = \begin{Bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{Bmatrix}$: nodal displacement in local coordinate system

$d = \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$: nodal displacement in global coordinate system

then, $\hat{d}_1 = d_{1x}\cos\theta + d_{1y}\sin\theta \quad \Rightarrow \quad [\cos\theta \ \ \sin\theta]\begin{Bmatrix} d_{1x} \\ d_{1y} \end{Bmatrix}$

$\hat{d}_2 = d_{2x}\cos\theta + d_{2y}\sin\theta \quad \Rightarrow \quad [\cos\theta \ \ \sin\theta]\begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix}$

$$\begin{Bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

$\therefore \hat{\boldsymbol{d}}\boldsymbol{e} = \boldsymbol{T} \cdot \boldsymbol{de}$ where, $\boldsymbol{T}$: Transformation matrix (global $\Rightarrow$ local value)

As the same way

$$\hat{\boldsymbol{fe}} = \boldsymbol{T} \cdot \boldsymbol{fe}$$

Now suppose the global components

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \hat{f}_1 \qquad \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{Bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{Bmatrix}$$

$$\boldsymbol{fe} = \boldsymbol{T^T} \ \hat{\boldsymbol{fe}}$$

$$\begin{aligned} \boldsymbol{fe} &= \boldsymbol{T^T} \ \hat{\boldsymbol{fe}} \\ &= \boldsymbol{T^T} \ \hat{\boldsymbol{Ke}} \ \hat{\boldsymbol{de}} && \Leftarrow (\hat{\boldsymbol{fe}} = \hat{\boldsymbol{Ke}} \ \hat{\boldsymbol{de}}) \\ &= \boldsymbol{T^T} \ \hat{\boldsymbol{Ke}} \ \boldsymbol{T} \ \boldsymbol{de} && \Leftarrow (\hat{\boldsymbol{de}} = \boldsymbol{T} \ \boldsymbol{de}) \end{aligned}$$

Therefore, $\boldsymbol{Ke} = \boldsymbol{T^T} \ \hat{\boldsymbol{Ke}} \ \boldsymbol{T}$

By using the above results, we can express the element level equation in global coordinate system as,

$\boldsymbol{fe} = \boldsymbol{Ke} \ \boldsymbol{de}$ where $\boldsymbol{Ke}$: Element stiffness matrix in global coordinate system.

Global $\boldsymbol{Ke}$ in 2-D truss element

$$\begin{aligned} \boldsymbol{Ke} &= \boldsymbol{T^T} \ \hat{\boldsymbol{Ke}} \ \boldsymbol{T} \\ &= \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \end{aligned}$$

$$= \frac{AE}{L} \left[\begin{array}{cc:cc} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ \hdashline -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{array} \right]$$

$$= \frac{AE}{L} \left[\begin{array}{cc:cc} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ \hdashline -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{array} \right] \quad where, \begin{matrix} C = \cos\theta \\ S = \sin\theta \end{matrix}$$